Spectral Gap for Measure-Valued Diffusion Processes *

Panpan Ren[b,c], **Feng-Yu Wang**[a,b]

[a] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

[b] Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, United Kingdom

[c] Mathematical Institute,Woodstock Road, OX2 6GG, University of Oxford

673788@swansea.ac.uk, Panpan.ren@maths.ox.ac.uk wangfy@tju.edu.cn, F.-Y.Wang@swansea.ac.uk

Abstract

The spectral gap is estimated for some measure-valued processes, which are induced by the intrinsic/extrinsic derivatives on the space of finite measures over a Riemannian manifold. These processes are symmetric with respect to the Dirichlet and Gamma distributions arising from population genetics. In addition to the evolution of allelic frequencies investigated in the literature, they also describe stochastic movements of individuals.

AMS subject Classification: 60J60, 58J65.
Keywords: Extrinsic derivative, weighted Gamma distribution, Poincaré inequality, weak Poincaré inequality, super Poincaré inequality.

1 Introduction

The Dirichlet distribution arises naturally in Bayesian inference as conjugate priors for categorical distribution and infinite non-parametric discrete distributions respectively. In population genetics, it describes the distribution of allelic frequencies (see for instance [3, 10, 14]). To simulate the Dirichlet distribution using stochastic dynamic systems, some diffusion processes generalized from the Wright-Fisher diffusion have been considered, see [4, 5, 6, 7, 20] and references within. In this paper, we investigate diffusion processes induced by the Dirichlet distribution and the intrinsic/extrinsic derivatives, where the extrinsic derivative term determines the evolution of allelic frequencies, and the intrinsic derivative drives the movement of individuals.

*Supported in part by NNSFC (11771326, 11831014, 11726627).

In the following three subsections, we introduce the reference measures, intrinsic and extrinsic derivatives, and the main results of the paper respectively. We will take the notation $\mu(f) = \int_E f \mathrm{d}\mu$ for a measurable space $(E, \mathscr{B}, \mu)$ and $f \in L^1(E, \mu)$.

1.1 Reference measures

Let $(M, \langle \cdot, \cdot \rangle_M)$ be a complete Riemannian manifold. Consider the space $\mathbb{M}$ of all nonnegative finite measures on M, and let $\mathbb{M}_1 := \{\mu \in \mathbb{M} : \mu(M) = 1\}$ be the set of all probability measures on M. According to [13, Theorem 3.2], both spaces are Polish under the weak topology. In general, for an ergodic Markov process X_t with stationary distribution μ, one may simulate μ by using the empirical measures $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s$, where δ_{X_s} is the Dirac measure at point X_s. In practice, one may also approximate μ using the discrete time empirical measures

$$\tilde{\mu}_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}, \quad n \geq 1.$$

See for instance [8] and references within for the study of the convergence rate.

For $0 \neq \theta \in \mathbb{M}$, the Dirichlet distribution $\mathbb{D}_\theta$ with shape θ is the unique probability measure on $\mathbb{M}_1$ such that for any measurable partition $\{A_i\}_{1=1}^n$ of M,

$$\mathbb{M}_1 \ni \mu \mapsto (\mu(A_1), \cdots, \mu(A_n))$$

obeys the Dirichlet distribution with parameter $(\theta(A_1), \cdots, \theta(A_n))$. Recall that for any $0 \neq \alpha = (\alpha_1, \cdots, \alpha_n) \in [0, \infty)^n$, the Dirichlet distribution with parameter α is the following probability measure on the simplex $\{s = (s_1, \cdots, s_n) : s_i \geq 0, \sum_{i=1}^n s_i = 1\}$:

$$\mathbb{D}_\alpha(\mathrm{d}s_1, \cdots, \mathrm{d}s_n) := \frac{\Gamma(\alpha_1 + \cdots + \alpha_n)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)} s_n^{\alpha_n - 1} \delta_{1 - \sum_{1 \leq i \leq n-1} s_i}(\mathrm{d}s_n) \prod_{i=1}^{n-1} s_i^{\alpha_i - 1} \mathrm{d}s_i,$$

where in case $\alpha_i = 0$ we set $\frac{1}{\Gamma(0)} s_i^{-1} \mathrm{d}s_i = \delta_0$, and δ_x denotes the Dirac measure at point x in a measurable space. If $\mathbb{D}_\theta$ refers to the distribution of population on M, then under the state $\mu \in \mathbb{M}_1$, $\mu(A_1), \cdots, \mu(A_n)$ stand for the proportions of population located in the areas $A_1, \cdots, A_n$ respectively.

We will also consider the Gamma distribution $\mathbb{G}_\theta$ on $\mathbb{M}$ with shape θ, whose marginal distribution on $\mathbb{M}_1$ coincides with the Dirichlet distribution $\mathbb{D}_\theta$. Recall that $\mathbb{G}_\theta$ is the unique probability measure on $\mathbb{M}$ such that for any finitely many disjoint measurable subsets $\{A_1, \cdots, A_n\}$ of M,

$$\mathbb{M} \ni \eta \mapsto \eta(A_i), \quad 1 \leq i \leq n$$

are independent Gamma random variables with shape parameters $\{\theta(A_i)\}_{1 \leq i \leq n}$ and scale parameter 1; that is,

$$\int_{\mathbb{M}} f(\eta(A_1), \cdots, \eta(A_n)) \mathbb{G}_\theta(\mathrm{d}\eta) = \int_{[0,\infty)^n} f(s_1, \cdots, s_n) \prod_{i=1}^n \gamma_{\theta(A_i)}(\mathrm{d}s_i) \tag{1.1}$$

holds for any $f \in \mathscr{B}_b(\mathbb{R}^n)$, where for a constant $r > 0$,

$$\gamma_r(\mathrm{d}s) := 1_{[0,\infty)}(s)\frac{s^{r-1}\mathrm{e}^{-s}}{\Gamma(r)}\,\mathrm{d}s, \quad \Gamma(r) := \int_0^\infty s^{r-1}\mathrm{e}^{-s}\mathrm{d}s, \tag{1.2}$$

and we set $\gamma_0 = \delta_0$, the Dirac measure at point 0.

The Gamma distribution $\mathbb{G}_\theta$ is supported on the class of finite discrete measures

$$\mathbb{M}_{dis} := \Big\{ \sum_{i=1}^\infty s_i\delta_{x_i} : \ s_i \ge 0, x_i \in M, \sum_{i=1}^\infty s_i < \infty \Big\}.$$

Moreover, under $\mathbb{G}_\theta$ the random variables $\eta(M) \in (0,\infty)$ and $\Psi(\eta) := \frac{\eta}{\eta(M)} \in \mathbb{M}_1$ are independent with

$$\mathbb{G}_\theta\big(\eta(M) < r, \Psi(\eta) \in \mathbf{A}\big) = \frac{\mathbb{D}_\theta(\mathbf{A})}{\Gamma(\theta(M))}\int_0^r s^{\theta(M)-1}\mathrm{e}^{-s}\mathrm{d}s, \quad r > 0, \mathbf{A} \in \mathscr{B}(\mathbb{M}_1). \tag{1.3}$$

Consequently,

$$\mathbb{D}_\theta = \mathbb{G}_\theta \circ \Psi^{-1}, \quad \Psi(\eta) := \frac{\eta}{\eta(M)}, \ \eta \in \mathbb{M}\setminus\{0\}. \tag{1.4}$$

Both $\mathbb{D}_\theta$ and $\mathbb{G}_\theta$ are images of the Poisson measure $\pi_{\hat\theta}$ with intensity

$$\hat\theta(\mathrm{d}x, \mathrm{d}s) := s^{-1}\mathrm{e}^{-s}\theta(\mathrm{d}x)\mathrm{d}s \tag{1.5}$$

on the product manifold $\hat M := M \times (0,\infty)$. Recall that $\pi_{\hat\theta}$ is the unique probability measure on the configuration space

$$\Gamma_{\hat M} := \Big\{\gamma = \sum_{i=1}^\infty \delta_{(x_i,s_i)} : \ (x_i,s_i) \in \hat M, \gamma(K) < \infty \text{ for any compact } K \subset \hat M\Big\}$$

equipped with the vague topology, such that for any disjoint compact subsets $\{K_i\}_{1\le i\le n}$ of $\hat M$, $\gamma \mapsto \gamma(K_i)$ are independent Poisson random variables of parameters $\hat\theta(K_i)_{1\le i\le n}$. By [9, Theorem 6.2], we have

$$\mathbf{s}(\gamma) := \sum_{i=1}^\infty s_i < \infty, \quad \text{for } \pi_{\hat\theta}\text{-a.s. } \gamma = \sum_{i=1}^\infty \delta_{(x_i,s_i)} \in \Gamma_{\hat M}, \tag{1.6}$$

and

$$\mathbb{G}_\theta = \pi_{\hat\theta} \circ \Phi^{-1}, \tag{1.7}$$

where

$$\Phi(\gamma) := \sum_{i=1}^\infty s_i\delta_{x_i} \in \mathbb{M}, \quad \gamma := \sum_{i=1}^\infty \delta_{(x_i,s_i)} \in \Gamma_{\hat M} \text{ with } \sum_{i=1}^\infty s_i < \infty.$$

Combining (1.4) with (1.7), we obtain

$$\mathbb{D}_\theta = \mathbb{G}_\theta \circ \Psi^{-1} = \pi_{\hat\theta}(\Psi \circ \Phi)^{-1}. \tag{1.8}$$

1.2 Intrinsic and extrinsic derivatives

These derivatives were introduced in [1] and [15] on the configuration space and the space of probability measures respectively, which can be extended to $\mathbb{M}$ under the map $\Phi : \Gamma_{\hat{M}} \to \mathbb{M}$, see for instance [11].

To introduce the intrinsic derivative for a function on $\mathbb{M}$ (or $\mathbb{M}_1$), we let $\mathscr{V}_0(M)$ be the class of smooth vector fields with compact supports on M. For any $v \in \mathscr{V}_0(M)$, let

$$\phi_v(x) = \exp_x[v(x)], \quad x \in M,$$

where exp is the exponential map on M. Then $\phi_v \in C^\infty(M \to M)$. For a function F on $\mathbb{M}$, we define its directional derivative along v by

$$\nabla_v^{int} F(\eta) := \lim_{\varepsilon \downarrow 0} \frac{F(\eta \circ \phi_{\varepsilon v}^{-1}) - F(\eta)}{\varepsilon}$$

if it exists. Let $L^2(\mathscr{V}(M), \eta)$ be the space of all measurable vector fields v on M with $\eta(|v|^2) < \infty$. When $\nabla_v^{int} F(\eta)$ exists for all $v \in \mathscr{V}_0(M)$ such that

$$|\nabla_v^{int} F(\eta)| \le c\|v\|_{L^2(\eta)}, \quad v \in \mathscr{V}_0(M)$$

holds for some constant $c \in (0, \infty)$, then by Riesz representation theorem there exists a unique $\nabla^{int} F(\eta) \in L^2(\mathscr{V}(M), \eta)$ such that

$$\nabla_v^{int} F(\eta) = \langle \nabla^{int} F(\eta), v \rangle_{L^2(\eta)} := \int_M \langle \nabla^{int} F(\eta), v \rangle_M \, \mathrm{d}\eta, \quad v \in \mathscr{V}_0(M). \tag{1.9}$$

In this case, we call F intrinsically differentiable at η with derivative $\nabla^{int} F(\eta)$. If F is intrinsically differentiable at all $\eta \in \mathbb{M}$ (or $\mathbb{M}_1$), we call it intrinsically differentiable on $\mathbb{M}$ (or $\mathbb{M}_1$).

Next, a measurable real function F on $\mathbb{M}$ is called extrinsically differentiable at $\eta \in \mathbb{M}$, if

$$\nabla^{ext} F(\eta)(x) := \frac{\mathrm{d}}{\mathrm{d}s} F(\eta + s\delta_x)\Big|_{s=0} \text{ exists for all } x \in M,$$

such that

$$\|\nabla^{ext} F(\eta)\| := \|\nabla^{ext} F(\eta)(\cdot)\|_{L^2(\eta)} < \infty.$$

When a function F on $\mathbb{M}_1$ is considered, it is called intrinsically differentiable if

$$\tilde{\nabla}^{ext} F(\mu)(x) := \frac{\mathrm{d}}{\mathrm{d}s} F((1-s)\mu + s\delta_x)\Big|_{s=0} \text{ exists for all } x \in M, \mu \in \mathbb{M}_1$$

and $\tilde{\nabla}^{ext} F(\mu) \in L^2(\mu)$. If F is extrinsically differentiable at all $\eta \in \mathbb{M}$ (or $\mathbb{M}_1$), we call it extrinsically differentiable on $\mathbb{M}$ (or $\mathbb{M}_1$). Let $\mathscr{D}(\mathbb{M})$ (respectively $\mathscr{D}(\mathbb{M}_1)$) denote the set of functions which are intrinsically and extrinsically differentiable on $\mathbb{M}$ (respectively $\mathbb{M}_1$).

A typical subclass of $\mathscr{D}(\mathbb{M})$ and $\mathscr{D}(\mathbb{M}_1)$ is the set of cylindrical functions

$$\mathscr{F}C_b^\infty(M) := \big\{\eta \mapsto f(\langle h_1, \eta\rangle, \cdots, \langle h_n, \eta\rangle) : \ n \ge 1, f \in C_b^\infty(\mathbb{R}^n), h_i \in C_0^\infty(M)\big\}, \tag{1.10}$$

where $\langle h_i, \eta\rangle := \eta(h_i) = \int_M h_i \mathrm{d}\eta$. This class is dense in $L^2(\mathbb{M}_1, \mathbb{D}_\theta)$ and $L^2(\mathbb{M}, \mathbb{G}_\theta)$, and the cylindrical function $F := f(\langle h_1, \cdot\rangle, \cdots, \langle h_n, \cdot\rangle)$ is differentiable with

$$\begin{aligned} \nabla^{int} F(\eta) &= \sum_{i=1}^n (\partial_i f)(\langle h_1, \eta\rangle, \cdots, \langle h_n, \eta\rangle)\nabla h_i, \\ \nabla^{ext} F(\eta) &= \sum_{i=1}^n (\partial_i f)(\langle h_1, \eta\rangle, \cdots, \langle h_n, \eta\rangle) h_i, \quad \eta \in \mathbb{M}, \end{aligned} \tag{1.11}$$

where ∇ is the gradient operator on M. Restricting on $\mathbb{M}_1$ we will consider

$$\tilde\nabla^{ext} F(\mu) := \sum_{i=1}^n (\partial_i f)(\langle h_1, \mu\rangle, \cdots, \langle h_n, \mu\rangle)(h_i - \mu(h_i)), \quad \mu \in \mathbb{M}_1,$$

which is the centered extrinsic derivative of F at μ since

$$\mu\big(\tilde\nabla^{ext} F(\mu)\big) = 0, \quad \mu \in \mathbb{M}_1. \tag{1.12}$$

See [16] for general results on the relations of ∇^{int}, ∇^{ext} and $\tilde\nabla^{ext}$.

1.3 The main result

Now, for any $\lambda > 0$, we consider the square fields for $F, G \in \mathscr{F}C_b^\infty(M)$:

(1.13)
$$\begin{aligned} \Gamma^\lambda(F,G)(\eta) &:= \int_M \Big\{\langle \nabla^{int} F(\eta), \nabla^{int} G(\eta)\rangle_M + \lambda(\nabla^{ext} F(\eta))\nabla^{ext} G(\eta)\Big\}(x)\,\eta(\mathrm{d}x), \quad \eta \in \mathbb{M}, \\ \tilde\Gamma^\lambda(F,G)(\mu) &:= \int_M \Big\{\langle \nabla^{int} F(\mu), \nabla^{int} G(\mu)\rangle_M + \lambda(\tilde\nabla^{ext} F(\mu))\tilde\nabla^{ext} G(\mu)\Big\}(x)\,\mu(\mathrm{d}x), \quad \mu \in \mathbb{M}_1, \end{aligned}$$

which lead to the following bilinear forms on $L^2(\mathbb{M}_1, \mathbb{D}_\theta)$ and $L^2(\mathbb{M}, \mathbb{G}_\theta)$ respectively:

$$\begin{aligned} \mathscr{E}^\lambda_{\mathbb{D}_\theta}(F,G) &:= \int_{\mathbb{M}_1} \tilde\Gamma^\lambda(F,G)(\mu)\,\mathbb{D}_\theta(\mathrm{d}\mu), \\ \mathscr{E}^\lambda_{\mathbb{G}_\theta}(F,G) &:= \int_{\mathbb{M}} \Gamma^\lambda(F,G)(\eta)\,\mathbb{G}_\theta(\mathrm{d}\eta), \quad F, G \in \mathscr{F}C_b^\infty(M). \end{aligned} \tag{1.14}$$

To ensure the closability of these bilinear forms, we take

$$\theta(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{vol}(\mathrm{d}x) \text{ for some } V \in W^{1,1}_{loc}(M), \quad \theta(M) < \infty, \tag{1.15}$$

where vol is the Riemannian volume measure. Then the integration by parts formula gives

$$\mathscr{E}_\theta(h_1, h_2) := \int_M \langle \nabla h_1, \nabla h_2\rangle_M(x)\,\theta(\mathrm{d}x) = -\int_M h_1(\Delta + \nabla V)h_2 \mathrm{d}\theta, \quad h_1, h_2 \in C_0^\infty(M). \tag{1.16}$$

So, the bilinear form is closable in $L^2(M,\theta)$, and the closure $(\mathscr{E}_\theta, \mathscr{D}(\mathscr{E}_\theta))$ is a Dirichlet form.

We will prove the closability of $(\mathscr{E}^\lambda_{\mathbb{G}_\theta}, \mathscr{F}C_b^\infty(M))$ and $(\mathscr{E}^\lambda_{\mathbb{D}_\theta}, \mathscr{F}C_b^\infty(M))$, and calculate the spectral gaps for the corresponding Dirichlet forms.

Recall that for a probability space $(E, \mathscr{B}, \mu)$ and a symmetric Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ on $L^2(E,\mu)$ with $1 \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(1,1) = 0$, the spectral gap of the Dirichlet form is given by

$$\mathrm{gap}(\mathscr{E}) = \inf\Big\{\mathscr{E}(F,F):\ F \in \mathscr{D}(\mathscr{E}), \mu(F) = 0, \mu(F^2) = 1\Big\}.$$

By the spectral theorem, $\mathrm{gap}(\mathscr{E})$ is the exponential convergence rate of the associated Markov semigroup $(P_t)_{t\geq 0}$, i.e.

$$\|P_t - \mu\|_{L^2(\mu)} := \sup_{\mu(F^2)\leq 1} \|P_t F - \mu(F)\|_{L^2(\mu)} = \mathrm{e}^{-\mathrm{gap}(\mathscr{E})t}, \quad t \geq 0.$$

Let

$$\lambda_\theta = \mathrm{gap}(\mathscr{E}_\theta) := \inf\big\{\theta(|\nabla f|^2):\ f \in C_b^1(M), \theta(f) = 0, \theta(f^2) = 1\big\}$$

be the spectral gap of the Dirichlet form $(\mathscr{E}_\theta, \mathscr{D}(\mathscr{E}_\theta))$ in $L^2(M,\theta)$. The main result of this paper is the following.

Theorem 1.1. *Let θ be in* (1.15) *and let $F_0(\eta) = \eta(M), \eta \in \mathbb{M}$.*

(1) $(\mathscr{E}^\lambda_{\mathbb{G}_\theta}, \mathscr{F}C_b^\infty(M))$ *is closable in $L^2(\mathbb{M}, \mathbb{G}_\theta)$ and the closure $(\mathscr{E}^\lambda_{\mathbb{G}_\theta}, \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta}))$ is a quasi-regular symmetric Dirichlet form with spectral gap* $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{G}_\theta}) = \lambda$.

(2) $(\mathscr{E}^\lambda_{\mathbb{D}_\theta}, \mathscr{F}C_b^\infty(M))$ *is closable in $L^2(\mathbb{M}_1, \mathbb{D}_\theta)$ and the closure $(\mathscr{E}^\lambda_{\mathbb{D}_\theta}, \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{D}_\theta}))$ is a quasi-regular symmetric Dirichlet form with spectral gap satisfying*

$$\lambda\theta(M) \leq \mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{D}_\theta}) \leq \lambda\theta(M) + \lambda_\theta(\theta(M) + 1).$$

Consequently, if $\lambda_\theta = 0$, then $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{D}_\theta}) = \lambda\theta(M)$. Moreover, $F_0(F \circ \Psi) \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ for $F \in \mathscr{F}C_b^\infty(M)$ and for any $F, G \in \mathscr{F}C_b^\infty(M)$,

$$\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_0(F \circ \Psi), F_0(G \circ \Psi)) = \theta(M)\mathscr{E}^\lambda_{\mathbb{D}_\theta}(F,G) + \lambda\theta(M)\mathbb{D}_\theta(F,G). \tag{1.17}$$

The formula (1.17) will play a crucial role in the proof of the closability of $(\mathscr{E}^\lambda_{\mathbb{D}_\theta}, \mathscr{F}C_b^\infty(M))$. When $\lambda_\theta > 0$, the exact value of $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{D}_\theta})$ is unknown. Since in this case the intrinsic derivative part will play a non-trivial role, we believe that $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{D}_\theta})$ is strictly larger than $\lambda\theta(M)$, hopefully our upper bound could be sharp.

When $\lambda = 1$ and without the intrinsic derivative part, the Dirichlet form $\mathscr{E}^\lambda_{\mathbb{D}_\theta}$ reduces to

$$\mathscr{E}^{FV}_{\mathbb{D}_\theta}(F,G) := \int_{\mathbb{M}_1} \mathbb{D}_\theta(\mathrm{d}\mu) \int_M \big\{(\tilde{\nabla}^{ext} F(\mu))\tilde{\nabla}^{ext} G(\mu)\big\}\mathrm{d}\mu, \tag{1.18}$$

which is associated with the Fleming-Viot process. It has been derived in [18] that

$$\mathrm{gap}(\mathscr{E}^{FV}_{\mathbb{D}_\theta}) = \theta(M), \tag{1.19}$$

see [20] for the study of log-Sobolev inequality in finite-dimensions as well as [5, 21, 22] for functional inequalities of a modified Fleming-Viot process.

When $\lambda = 1$ and $V = 0$, [11, Theorem 14] presents an integration by parts formula for $\mathscr{E}_{\mathbb{G}_\theta}^\lambda$ in the class

$$\tilde{\mathscr{F}}C_b^\infty(M) := \big\{F \circ \psi_\varepsilon : \ F \in \mathscr{F}C_b^\infty(M), \varepsilon > 0\big\},$$

where

$$\psi_\varepsilon(\eta) = \sum_{\eta(\{x\}) \geq \varepsilon} \eta(\{x\})\delta_x, \quad \varepsilon > 0, \eta \in \mathbb{M}.$$

Therefore, letting $(\mathscr{L}_{\mathbb{G}_\theta}^\lambda, \mathscr{D}(\mathscr{L}_{\mathbb{G}_\theta}^\lambda))$ be the generator of the Dirichlet form $(\mathscr{E}_{\mathbb{G}_\theta}^\lambda, \mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^\lambda))$, we have $\mathscr{D}(\mathscr{L}_{\mathbb{G}_\theta}^\lambda) \supset \tilde{\mathscr{F}}C_b^\infty(M)$. However, in general $\mathscr{F}C_b^\infty(M)$ is not included in $\mathscr{D}(\mathscr{L}_{\mathbb{G}_\theta}^\lambda)$. Indeed, according to [11] we have the integration by parts formula

$$\int_{\mathbb{M}} \nabla_v^{int}(F \circ \psi_\varepsilon)\, \mathrm{d}\mathbb{G}_\theta = -\int_{\mathbb{M}} (F \circ \psi_\varepsilon) B_{\varepsilon,v}\, \mathbb{G}_\theta(\mathrm{d}\eta), \quad v \in \mathscr{V}_0(M), F \in \mathscr{F}C_b^\infty(M) \tag{1.20}$$

for $B_{\varepsilon,v}(\eta) := \sum_{\eta(\{x\}) \geq \varepsilon} \big\{\mathrm{div}(v) + \langle v, \nabla V\rangle\big\}(x)$, where div is the divergence operator in M. This formula makes sense because

$$|\mathbb{G}_\theta(B_{\varepsilon,v})| = |\theta(\mathrm{div} + \langle v, \nabla V(x)\rangle_M)| \int_\varepsilon^\infty s^{-1}\mathrm{e}^{-s}\mathrm{d}s < \infty.$$

So, $F \circ \psi_\varepsilon \in \mathscr{D}(\mathscr{L}_{\mathbb{G}_\theta}^\lambda)$ for any $\varepsilon > 0$ and $F \in \mathscr{F}C_b^\infty(M)$. However, since $\int_0^\infty s^{-1}\mathrm{e}^{-s}\mathrm{d}s = \infty$, (1.20) does not make sense for $\varepsilon = 0$.

To prove Theorem 1.1, we will formulate the bilinear form $\mathscr{E}_{\mathbb{D}_\theta}$ as the image of the Dirichlet form on the configuration space constructed in [1], for which the (weak) Poincaré inequality has been established in [17, Section 7]. To this end, we first recall in Section 2 some known results on the configuration space, then prove Theorem 1.1(1) in Section 3 by transforming these results to the Gamma process on $\mathbb{M}$, and finally prove Theorem 1.1(2) in Section 4 by mapping the Gamma process to the subclass $\mathbb{M}_1$.

2 Analysis on the configuration space

In this section, we first recall the diffusion process on the configuration space constructed in [1, 2], then calculate the spectral gap.

For $F := f(\langle \hat{h}_1, \cdot\rangle, \cdots, \langle \hat{h}_n, \cdot\rangle) \in \mathscr{F}C_b^\infty(\hat{M})$, where $f \in C_b^\infty(\mathbb{R}^n)$ and $\hat{h}_i \in C_0^\infty(\hat{M})$, let

$$\nabla^\Gamma F(\gamma) = \sum_{i=1}^n (\partial_i f)(\langle \hat{h}_1, \gamma\rangle, \cdots, \langle \hat{h}_n, \gamma\rangle)\hat{\nabla}\hat{h}_i, \quad \gamma \in \Gamma_{\hat{M}}. \tag{2.1}$$

For $\lambda > 0$, we take the following Riemannian metric on the manifold $\hat{M} := M \times (0, \infty)$:

$$\langle a_1\partial_s + v_1, a_2\partial_s + v_2\rangle_{\hat{M}} := (\lambda s)^{-1}a_1a_2 + s\langle v_1, v_2\rangle_M, \quad a_1, a_2 \in \mathbb{R}, v_1, v_2 \in TM. \tag{2.2}$$

Let $\hat{\Delta}, \hat{\nabla}$ and $\hat{\mathrm{vol}}$ be the Laplacian, gradient and volume measure on $\hat{M}$ respectively.

Consider the bilinear form

$$\mathscr{E}_{\hat{\theta}}^{\Gamma}(F,G) := \int_{\Gamma_{\hat{M}}} \pi_{\hat{\theta}}(\mathrm{d}\gamma) \int_{\hat{M}} \langle \nabla^{\Gamma}(F\circ\Phi)(\gamma), \nabla^{\Gamma}(G\circ\Phi)(\gamma)\rangle_{\hat{M}} \mathrm{d}\gamma, \quad F,G \in \mathscr{F}C_b^{\infty}(M). \tag{2.3}$$

To formulate the integration by parts formula of this form, we intend to find out a function W on $\hat{M}$ such that

$$\mathrm{e}^{W(s,x)}\hat{\mathrm{vol}}(\mathrm{d}s,\mathrm{d}x) = \hat{\theta}(\mathrm{d}s,\mathrm{d}x), \tag{2.4}$$

where $\hat{\theta}(\mathrm{d}s,\mathrm{d}x) := s^{-1}\mathrm{e}^{-s}\mathrm{d}s\theta(\mathrm{d}x)$ is given by (1.5). So, for

$$\hat{L}f := \hat{\Delta} f + \langle \hat{\nabla} W, \hat{\nabla} f\rangle_{\hat{M}}, \quad f \in C^2(\hat{M}), \tag{2.5}$$

we have the integration by parts formula

$$\mathscr{E}_{\hat{\theta}}(\hat{h}_1,\hat{h}_2) := \int_{\hat{M}} \langle \hat{\nabla}\hat{h}_1, \hat{\nabla}\hat{h}_2\rangle_{\hat{M}} \mathrm{d}\hat{\theta} = -\int_{\hat{M}} (\hat{h}_1\hat{L}\hat{h}_2)\mathrm{d}\hat{\theta}, \quad \hat{h}_1,\hat{h}_2 \in C_0^{\infty}(\hat{M}). \tag{2.6}$$

Therefore, letting

$$\begin{aligned} \mathscr{L}_{\hat{\theta}}^{\Gamma} F(\gamma) = &\sum_{i,j=1}^{n} (\partial_i\partial_j f)(\langle \hat{h}_1,\gamma\rangle, \cdots, \langle \hat{h}_n,\gamma\rangle)\gamma(\langle \hat{\nabla}\hat{h}_i, \hat{\nabla}\hat{h}_j\rangle_{\hat{M}}) \\ &+ \sum_{i=1}^{n} (\partial_i f)(\langle \hat{h}_1,\gamma\rangle, \cdots, \langle \hat{h}_n,\gamma\rangle)\gamma(\hat{L}\hat{h}_i), \quad \gamma \in \Gamma_{\hat{M}}, \end{aligned} \tag{2.7}$$

we have (see [2, Theorem 4.3])

$$\mathscr{E}_{\hat{\theta}}^{\Gamma}(F,G) = -\int_{\Gamma_{\hat{M}}} (G\mathscr{L}_{\hat{\theta}}^{\Gamma} F)\,\mathrm{d}\pi_{\hat{\theta}}, \quad F,G \in \mathscr{F}C_b^{\infty}(\hat{M}), \tag{2.8}$$

which implies the closability of $(\mathscr{E}_{\hat{\theta}}^{\Gamma}, \mathscr{F}C_b^{\infty}(\hat{M}))$, so that the closure $(\mathscr{E}_{\hat{\theta}}^{\Gamma}, \mathscr{D}(\mathscr{E}_{\hat{\theta}}^{\Gamma}))$ is a symmetric Dirichlet form in $L^2(\Gamma_{\hat{M}}, \pi_{\hat{\theta}})$. Moreover, as explained in [13, Section 4.5.1], the result [13, Corollary 4.9] applies to this situation, so that the Dirichlet form $(\mathscr{E}_{\hat{\theta}}^{\Gamma}, \mathscr{D}(\mathscr{E}_{\hat{\theta}}^{\Gamma}))$ is quasi-regular and local, and hence is associated with a diffusion process on $\Gamma_{\hat{M}}$. Recall that $\Gamma_{\hat{M}}$ is equipped with the vague topology.

To calculate the generator $\mathscr{L}_{\hat{\theta}}^{\Gamma}$ defined in (2.7), it suffices to figure out the operator $\hat{L}$. To this end, for a fixed point $z := (\bar{s},\bar{x}) \in \hat{M}$, we take the normal coordinates $(s, x_1, \cdots, x_t)$ in a neighbourhood $\mathscr{O}(z)$ of z such that

$$U_1 := \partial_s, \quad U_{i+1} := \partial_{x_i}, \ 1 \le i \le d$$

satisfy

$$g(z) = \mathrm{diag}\{(\lambda\bar{s})^{-1}, \bar{s}, \cdots, \bar{s}\}, \quad \hat{\nabla} U_i(z) = 0, \ 1 \le i \le d+1. \tag{2.9}$$

Let

$$g(s,x) := (\langle U_i, U_j\rangle_{\hat M})_{1\le i,j\le d+1}(s,x),\quad (s,x)\in \mathscr{O}(z).$$

Then the Riemannian volume measure on $\mathscr{O}(z)$ is

$$\hat{\text{vol}}(\text{d}s,\text{d}x) = \sqrt{\det g(s,x)}\,\text{d}s\text{d}x,$$

so that (2.4) holds for

$$W(s,x) := \log\Big[\frac{\hat\theta(\text{d}s,\text{d}x)}{\hat{\text{vol}}(\text{d}s,\text{d}x)}\Big] = -s-\log s + V(x) - \frac{1}{2}\log\big[\det g(s,x)\big],\quad (s,x)\in\mathscr{O}(z).$$

Letting $(g^{ij})_{1\le i,j\le d+1} = g^{-1}$, we derive from (2.9) that $\nabla g^{ij}(z)=0$. So,

$$\hat\nabla W(z) := \sum_{i=1}^{d+1}\big\{g^{ii}(U_iW)U_i\big\}(z) = \bar s^{-1}\nabla V(\bar x) - \lambda(1+\bar s)\partial_s. \tag{2.10}$$

By the same reason, for any $\hat h\in C^\infty(\hat M)$ we have

$$\begin{aligned}\hat\nabla\hat h(z) &:= \sum_{i=1}^{d+1}\big\{g^{ii}(U_i\hat h)U_i\big\}(z) = \lambda\bar s(\partial_s\hat h)(\bar s,\bar x)\partial_s + \bar s^{-1}\sum_{i=1}^{d}(\partial_{x_i}\hat h(\bar s,\bar x))\partial_{x_i},\\ \hat\Delta\hat h(z) &:= \frac{1}{\sqrt{\det g(z)}}\sum_{i=1}^{d+1}U_i\Big(\sqrt{\det g}\,g^{ii}U_i\hat h\Big)(z) = \sum_{i=1}^{d+1}\big\{g^{ii}U_i^2\hat h\big\}(z)\\ &= \lambda\bar s\partial_s^2\hat h(\bar s,\bar x) + \bar s^{-1}\Delta\hat h(\bar s,\cdot)(\bar x).\end{aligned}$$

This together with (2.10) implies that at point z,

$$\hat L := \hat\Delta + \hat\nabla W = \lambda s(\partial_s^2-\partial_s) - \lambda\partial_s + s^{-1}(\Delta+\nabla V). \tag{2.11}$$

Since $z\in\hat M$ is arbitrary, this formula holds for all points $(s,x)\in\hat M$.

Theorem 2.1. *Let $\lambda>0$ and $\theta\in\mathbb{M}$ be as in* (1.15)*. Then* $\text{gap}(\mathscr{E}^\Gamma_{\hat\theta})=\lambda$.

Proof. According to [19], see also [17, Theorem 7.1], we have

$$\text{gap}(\mathscr{E}^\Gamma_{\hat\theta}) = \lambda_{\hat\theta} := \inf\big\{\mathscr{E}_{\hat\theta}(\hat h,\hat h) : \hat h\in\mathscr{D}(\mathscr{E}_{\hat\theta}), \hat\theta(\hat h^2)=1\big\}, \tag{2.12}$$

where $\mathscr{D}(\mathscr{E}_{\hat\theta})$ is the closure of $C_0^\infty(\hat M)$ under the Sobolev norm $\|\hat h\|_1 := \sqrt{\hat\theta(|\hat h|^2)+\mathscr{E}_{\hat\theta}(\hat h,\hat h)}$. Let

$$\hat h(x,s) := s+1,\quad (x,s)\in\hat M.$$

By (2.11) we have

$$\hat L\hat h(x,s) = -\lambda\hat h(x,s),\quad (x,s)\in\hat M. \tag{2.13}$$

Combining this with (2.6), for any $g \in C_0^\infty(\hat{M})$ we have

$$\begin{aligned}\lambda\hat{\theta}(g^2) &= -\int_{\hat{M}} \frac{g^2}{\hat{h}} \hat{L}\hat{h}\, \mathrm{d}\hat{\theta} = \int_{\hat{M}} \langle \hat{\nabla}(g^2/\hat{h}), \hat{\nabla}\hat{h}\rangle_{\hat{M}} \mathrm{d}\hat{\theta} \\ &= \int_{\hat{M}} \big\{2g\langle \hat{\nabla} g, \hat{\nabla}\log \hat{h}\rangle_{\hat{M}} - g^2 \langle \hat{\nabla}\log\hat{h}, \hat{\nabla}\log\hat{h}\rangle_{\hat{M}}\big\}\mathrm{d}\hat{\theta} \\ &\le \int_{\hat{M}} \langle \hat{\nabla} g, \hat{\nabla} g\rangle_{\hat{M}} \mathrm{d}\hat{\theta} = \mathscr{E}_{\hat{\theta}}(g,g).\end{aligned}$$

Therefore, $\lambda_{\hat{\theta}} \ge \lambda$.

On the other hand, since M is complete, there exists a sequence $\{h_n\}_{n\ge 1} \subset C_0^\infty(M)$ such that

$$0 \le h_n \uparrow 1 \text{ as } n\uparrow\infty, \text{ and } \|\nabla h_n\|_\infty \le \frac{1}{n}, \quad n\ge 1. \tag{2.14}$$

For any $\varepsilon \in (0,1)$ let

$$\hat{h}_{n,\varepsilon}(x,s) = (s-\varepsilon)^+ h_n(x), \quad \hat{h}_\varepsilon(x,s) = (s-\varepsilon)^+, \quad (x,s)\in\hat{M}, n\ge 1.$$

Then $\{\hat{h}_{n,\varepsilon}\}_{n\ge 1, \varepsilon\in(0,1)} \subset \mathscr{D}(\mathscr{E}_{\hat{\theta}})$ with $0\le \hat{h}_{n,\varepsilon} \le \hat{h}_\varepsilon$. So, for fixed ε, by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{\hat{M}} |\hat{h}_{n,\varepsilon} - \hat{h}_\varepsilon|^2 \mathrm{d}\hat{\theta} = \int_{\hat{M}} \Big(\lim_{n\to\infty} |\hat{h}_{n,\varepsilon} - \hat{h}_\varepsilon|^2\Big)\mathrm{d}\hat{\theta} = 0,$$

and due to (2.6) and (2.14),

$$\begin{aligned}&\lim_{n,m\to\infty} \mathscr{E}_{\hat{\theta}}(\hat{h}_{n,\varepsilon} - \hat{h}_{m,\varepsilon}, \hat{h}_{n,\varepsilon} - \hat{h}_{m,\varepsilon}) \\ &= \lim_{n,m\to\infty} \int_{\hat{M}} \big\{\lambda s 1_{\{s\ge\varepsilon\}} |h_n - h_m|^2(x) + s^{-1}|(s-\varepsilon)^+|^2 |\nabla(h_n-h_m)|^2(x)\big\} s^{-1}\mathrm{e}^{-s}\mathrm{d}s\theta(\mathrm{d}x) \\ &\le \lim_{n,m\to\infty} \Big(\lambda\theta(|h_n-h_m|^2) + \frac{\theta(M)}{n^2\wedge m^2}\Big) = 0.\end{aligned}$$

Thus, $\hat{h}_\varepsilon \in \mathscr{D}(\mathscr{E}_{\hat{\theta}})$ with

$$\mathscr{E}_{\hat{\theta}}(\hat{h}_\varepsilon, \hat{h}_\varepsilon) = \lim_{n\to\infty} \mathscr{E}_{\hat{\theta}}(\hat{h}_{n,\varepsilon}, \hat{h}_{n,\varepsilon}) = \lambda\theta(M)\int_\varepsilon^\infty \mathrm{e}^{-s}\mathrm{d}s, \quad \varepsilon\in(0,1). \tag{2.15}$$

Combining this with

$$\lim_{\varepsilon\to 0} \hat{\theta}(h_\varepsilon^2) = \theta(M)\int_0^\infty s\mathrm{e}^{-s}\mathrm{d}s = \theta(M),$$

we obtain

$$\lambda_{\hat{\theta}} \le \lim_{\varepsilon\downarrow 0} \frac{\mathscr{E}_{\hat{\theta}}(\hat{h}_\varepsilon, \hat{h}_\varepsilon)}{\hat{\theta}(h_\varepsilon^2)} = \lambda.$$

This together with $\lambda_{\hat{\theta}} \ge \lambda$ derived above gives $\lambda_{\hat{\theta}} = \lambda$. So, the proof is finished by (2.12). $\square$

3 Proof of Theorem 1.1(1)

Theorem 3.1. *Let θ be as in* (1.15). *Then* $(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}, \mathscr{F}C_b^\infty(M))$ *is closable in* $L^2(\mathbb{M}, \mathbb{G}_\theta)$ *and the closure* $(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}, \mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}))$ *is a quasi-regular symmetric Dirichlet form. Moreover,* $\mathrm{gap}(\mathscr{E}_{\mathbb{G}_\theta}) = \lambda$.

Proof. Due to (1.7), we may prove this result by using (2.8) and Theorem 2.1. For any $h \in C_0^\infty(M)$, let $\hat{h}(x,s) = sh(x), (x,s) \in \hat{M}$. Then

$$\Phi(\gamma)(h) = \gamma(\hat{h}), \quad \gamma \in \Gamma_{\hat{M}}. \tag{3.1}$$

Let $F = f(\langle h_1, \cdot\rangle, \cdots, \langle h_n, \cdot\rangle), G = g(\langle h_1, \cdot\rangle, \cdots, \langle h_n, \cdot\rangle) \in \mathscr{F}C_b^\infty(M)$. We have

$$F \circ \Phi = f(\langle \hat{h}_1, \cdot\rangle, \cdots, \langle \hat{h}_n, \cdot\rangle),\ G \circ \Phi = g(\langle \hat{h}_1, \cdot\rangle, \cdots, \langle \hat{h}_n, \cdot\rangle) \in \mathscr{D}(\mathscr{E}_{\hat{\theta}}^{\Gamma})$$

such that

$$\langle \nabla^\Gamma(F\circ\Phi), \nabla^\Gamma(G\circ\Phi)\rangle_{\hat{M}}(\gamma) = \sum_{i,j=1}^n \big\{(\partial_i f)(\partial_j g)\big\}(\langle \hat{h}_1, \gamma\rangle, \cdots, \langle \hat{h}_n, \gamma\rangle)\gamma\big(\langle \hat{\nabla}\hat{h}_i, \hat{\nabla}\hat{h}_j\rangle_{\hat{M}}\big).$$

Because of (2.2) and (3.1), we have

$$\begin{aligned} \gamma\big(\langle \hat{\nabla}\hat{h}_i, \hat{\nabla}\hat{h}_j\rangle_{\hat{M}}\big) &= \int_{\hat{M}} \big\{\lambda s(h_i h_j)(x) + s\langle \nabla h_i, \nabla h_j\rangle_M(x)\big\}\gamma(\mathrm{d}x, \mathrm{d}s) \\ &= \Phi(\gamma)\big(\langle \nabla h_i, \nabla h_j\rangle_M + \lambda h_i h_j\big). \end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned} &\int_{\hat{M}} \langle \nabla^\Gamma(F\circ\Phi)(\gamma), \nabla^\Gamma(G\circ\Phi)(\gamma)\rangle_{\hat{M}} \mathrm{d}\gamma \\ &= \sum_{i,j=1}^n \big\{(\partial_i f)(\partial_j g)\big\}(\langle h_1, \Phi(\gamma)\rangle, \cdots, \langle h_n, \Phi(\gamma)\rangle)\Phi(\gamma)\big(\langle \nabla h_i, \nabla h_j\rangle_M + \lambda h_i h_j\big) \\ &= \Gamma^\lambda(F,G)(\Phi(\gamma)), \quad \gamma \in \Gamma_{\hat{M}}. \end{aligned}$$

Combining this with (1.7), (1.14) and (2.3), we obtain

$$\mathscr{E}_{\hat{\theta}}^{\Gamma}(F\circ\Phi, G\circ\Phi) = \int_{\mathbb{M}} \Gamma^\lambda(F,G)\,\mathrm{d}\mathbb{G}_\theta = \mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F,G). \tag{3.3}$$

Below we prove the closability, quasi-regularity, and the spectral gap bounds respectively.

(a) The closability. Let $\{F_n\}_{n\ge1} \subset \mathscr{F}C_b^\infty(M)$ such that

$$\lim_{n\to\infty} \mathbb{G}_\theta(F_n^2) = 0 \text{ and } \lim_{n,m\to\infty} \mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F_n - F_m, F_n - F_m) = 0. \tag{3.4}$$

It remains to show that $\lim_{n\to\infty} \mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F_n, F_n) = 0$. By (1.7) and (3.3), (3.4) implies

$$\lim_{n\to\infty} \pi_{\hat{\theta}}(|F_n\circ\Phi|^2) + \lim_{n,m\to\infty} \mathscr{E}_{\hat{\theta}}^{\Gamma}(F_n\circ\Phi - F_m\circ\Phi, F_n\circ\Phi - F_m\circ\Phi) = 0,$$

so that the closability of $(\mathscr{E}_{\hat\theta}^{\Gamma}, \mathscr{D}(\mathscr{E}_{\hat\theta}^{\Gamma}))$ and (3.4) imply

$$\lim_{n\to\infty} \mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F_n, F_n) = \lim_{n\to\infty} \mathscr{E}_{\hat\theta}^{\Gamma}(F_n\circ\Phi, F_n\circ\Phi) = 0.$$

(b) The quasi-regularity. According to [12, Chap. IV, Def. 3.1], the Dirichlet form $(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}, \mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}))$ is quasi-regular if and only if there exist a sequences of compact subsets $\{K_n\}_{n\ge1}$ of $\mathbb{M}$ such that the class

$$\mathscr{D}_K := \big\{F\in\mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}) : F|_{\mathbb{M}\setminus K_n} = 0 \text{ for some } n\ge1\big\} \tag{3.5}$$

is dense in $\mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda})$ under the Sobolev norm

$$\|F\|_{L^2(\mathbb{G}_\theta)} + \sqrt{\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F,F)},\quad F\in\mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}).$$

In this case, the sequence $\{K_n\}$ is called a $\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}$-nest.

To construct such a $\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}$-nest, we choose a function $\psi\in C^\infty([0,\infty))$ with $1\le\psi(r)\uparrow\infty$ as $r\uparrow\infty$ and $0\le\psi'\le1$, such that $\theta(|\psi\circ\rho_o|^2)<\infty$, where ρ_o is the Riemannian distance from a fixed point $o\in M$. Thus,

$$f := \psi\circ\rho_o\in\mathscr{D}(\mathscr{E}_\theta). \tag{3.6}$$

Since M is complete and $\psi\circ\rho_o\uparrow\infty$ as $\rho_o\uparrow\infty$, the level sets

$$K_n := \{\eta\in\mathbb{M} : \eta(\psi\circ\rho_o)\le n\},\quad n\ge1$$

are compact in $\mathbb{M}$. It remains to show that $\{K_n\}_{n\ge1}$ is a $\mathscr{E}_{\mathbb{G}_\theta}^{\lambda}$-nest. Since $\mathscr{F}C_b^\infty(M)$ is dense in $\mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda})$, it suffices to show that for any $F\in\mathscr{F}C_b^\infty(M)$, there exist a sequence $\{F_n\}_{n\ge1}\subset\mathscr{D}_K$, where $\mathscr{D}_K$ is defined in (4.3) for the present $\{K_n\}_{n\ge1}$, such that

$$\lim_{n\to\infty}\Big\{\mathbb{G}_\theta(|F_n-F|^2) + \mathscr{E}_{\mathbb{G}_\theta}^{\lambda}(F_n-F, F_n-F)\Big\} = 0. \tag{3.7}$$

We will prove this formula for

$$F_n := F\cdot\big\{1\wedge(n+1-F_0)^+\big\},\quad n\ge1, F_0(\eta) := \eta(\psi\circ\rho_o). \tag{3.8}$$

To this end, we first confirm that $F_0\in\mathscr{D}(\mathscr{E}_{\mathbb{G}_\theta}^{\lambda})$, so that $\{F_n\}_{n\ge1}\subset\mathscr{D}_K$ by the definitions of K_n and F_n. By (3.6), there exist functions $\{f_n\}_{n\ge1}\subset C_0^\infty(M)$ such that

$$0\le f_n\le f := \psi\circ\rho_o,\quad \lim_{n\to\infty}\int_M \big\{|f_n-f|^2 + |\nabla(f_n-f)|^2\big\}\mathrm{d}\theta = 0. \tag{3.9}$$

Noting that $\eta\mapsto\eta(\psi\circ\rho_o)$ obeys the Gamma-distribution with parameter $\delta := \theta(\psi\circ\rho_o)<\infty$, we have

$$\int_{\mathbb{M}} F_0^2\,\mathrm{d}\mathbb{G}_\theta = \int_{\mathbb{M}} \eta(\psi\circ\rho_o)^2\,\mathbb{G}_\theta(\mathrm{d}\eta) = \delta(\delta+1)<\infty.$$

By the dominated convergence theorem and (3.9), the functions $G_n(\eta) := \eta(f_n), n \geq 1$ satisfy

$$\lim_{n\to\infty} \int_{\mathbb{M}} |F_0 - G_n|^2 \mathrm{d}\mathbb{G}_\theta = 0.$$

Moreover, (1.11), (1.13), (1.14) and (3.9) imply

$$\begin{aligned}
&\lim_{n,m\to\infty} \mathscr{E}^\lambda_{\mathbb{G}_\theta}(G_n - G_m, G_n - G_m) = \lim_{n,m\to\infty} \int_{\mathbb{M}} \Gamma^\lambda(G_n - F_0, G_n - F_0)\mathrm{d}\mathbb{G}_\theta \\
&= \lim_{n,m\to\infty} \int_{\mathbb{M}} \eta(|f_n - f)|^2 + \lambda|f - f_n|^2)\mathbb{G}_\theta(\mathrm{d}\eta) \\
&= \lim_{n,m\to\infty} \int_{M} (|f_n - f)|^2 + \lambda|f - f_n|^2)\mathrm{d}\theta = 0.
\end{aligned}$$

Therefore, $F_0 \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$. Now, let F_n be defined in (3.8). Then $F_n \to F$ in $L^2(\mathbb{G}_\theta)$, and (1.11), (1.13) and (1.14) imply

$$\begin{aligned}
&\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_n - F, F_n - F) \\
&\leq 2\int_{\mathbb{M}} \Big\{ \|F\|_\infty \Gamma^\lambda((F_0 - n)^+ \wedge 1, (F_0 - n)^+ \wedge 1) + \|\Gamma^\lambda(F,F)\|_\infty |(F_0 - n)^+ \wedge 1|^2 \Big\} \mathrm{d}\mathbb{G}_\theta \\
&\leq C \int_{\mathbb{M}} \Big\{ \Gamma^\lambda(F_0, F_0) 1_{\{n \leq F_0 \leq n+1\}} + 1_{\{F_0 \geq n\}} \Big\} \mathrm{d}\mathbb{G}_\theta \to 0 \text{ as } n \to \infty,
\end{aligned}$$

where $C := 2(\|F\|_\infty + \|\Gamma^\lambda(F,F)\|_\infty) < \infty$. Moreover, the dominated convergence theorem implies that $\|F_n - F\|_{L^2(\mathbb{G}_\theta} \to 0$ as $n \to \infty$. Therefore, (3.7) holds as desired.

(c) Spectral gap estimates. By Theorem 2.1 and (3.3), for any $F \in \mathscr{F}C_b^\infty(M)$ with $\mathbb{G}_\theta(F) = 0$, we have

$$\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F,F) = \mathscr{E}^\Gamma_{\hat\theta}(F \circ \Phi, F \circ \Phi) \geq \lambda \pi_{\hat\theta}(|F \circ \Phi|^2) = \lambda \mathbb{G}_\theta(F^2).$$

This implies $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{G}_\theta}) \geq \lambda$. On the other hand, we take $F_0(\eta) = \eta(M)$ and intend to show that $F_0 \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ with

$$\frac{\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_0, F_0)}{\mathbb{G}_\theta(F_0^2) - \mathbb{G}_\theta(F_0)^2} = \lambda, \tag{3.10}$$

so that by definition $\mathrm{gap}(\mathscr{E}^\lambda_{\mathbb{G}_\theta}) \leq \lambda$, and hence the proof is finished.

Let $\xi \in C_0^\infty([0,\infty))$ such that $0 \leq \xi \leq 1, \xi(s) = 1$ for $s \leq 1$. For h_n in (2.14), define

$$F_n(\eta) = \int_0^{\eta(h_n)} \xi(s/n)\mathrm{d}s, \quad \eta \in \mathbb{M}. \tag{3.11}$$

Then $F_n \in \mathscr{F}C_b^\infty(M)$ and $0 \leq F_n \uparrow F_0$ as $n \uparrow \infty$. By (1.3) we have

$$\begin{aligned}
&\int_{\mathbb{M}} F_0 \mathrm{d}\mathbb{G}_\theta = \frac{1}{\Gamma(\theta(M))} \int_0^\infty s^{\theta(M)} \mathrm{e}^{-s} \mathrm{d}s = \theta(M), \\
&\int_{\mathbb{M}} F_0^2 \mathrm{d}\mathbb{G}_\theta = \frac{1}{\Gamma(\theta(M))} \int_0^\infty s^{\theta(M)+1} \mathrm{e}^{-s} \mathrm{d}s = \theta(M)^2 + \theta(M).
\end{aligned} \tag{3.12}$$

So, the dominated convergence theorem implies $\mathbb{G}_\theta(|F_n - F_0|^2) \to 0$ as $n \to \infty$, and

$$\begin{aligned}
&\lim_{n,m\to\infty} \mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_n - F_m, F_n - F_m)\\
&= \lim_{n,m\to\infty} \int_{\mathbb{M}} \Big\{\eta\big(\lambda\big|\xi(\eta(h_n)/n)h_n - \xi(\eta(h_m)/m)h_m\big|^2\big)\\
&\qquad\qquad + \big(\big|\xi(\eta(h_n)/n)\nabla h_n - \xi(\eta(h_m)/m)\nabla h_m\big|^2\big)\Big\}\mathbb{G}_\theta(\mathrm{d}\eta) = 0.
\end{aligned}$$

Therefor, $F_0 \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ with

$$\begin{aligned}
\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_0, F_0) &= \lim_{n\to\infty} \mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_n, F_n) = \lambda \int_{\mathbb{M}} \eta\big(\lambda\big|\xi(\eta(h_n)/n)h_n\big|^2 + \big|\xi(\eta(h_n)/n)\nabla h_n\big|^2\big)\mathbb{G}_\theta(\mathrm{d}\eta)\\
&= \lambda\mathbb{G}_\theta(F_0) = \lambda\theta(M).
\end{aligned}$$

Combining this with (3.12) we derive (3.10) and hence finish the proof.

□

4 Proof of Theorem 1.1(2)

The quasi-regularity can be proved by the same means as in the step (b) of the proof of Theorem 1.1(1). So, we need only to prove the closability, the formula (1.17), and the claimed spectral gap bounds.

(1) Closability and (1.17). Let $F_0(\eta) = \eta(M)$. It suffices to prove that for any $F \in \mathscr{F}C_b^\infty(M)$, one has $F_0 \cdot (F \circ \Psi) \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ such that (1.17) holds. Indeed, since $(\mathscr{E}^\lambda_{\mathbb{G}_\theta}, \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta}))$ is closed, (1.17) implies the closability of $(\mathscr{E}^\lambda_{\mathbb{D}_\theta}, \mathscr{F}C_b^\infty(M))$.

Similarly to (b) in the proof of Theorem 3.1, and noting that $\nabla^{int} F_0 = 0, \nabla^{ext} F_0 = 1$, we see that for any $G \in \mathscr{F}C_b^\infty(M)$, $GF_n \to GF_0$ in $\mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ with

$$\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_0 G, F_0 G) = \int_{\mathbb{M}} \eta\big(\lambda\big|F_0\nabla^{ext} G + G\big|^2 + \big|F_0 \nabla^{int} G\big|^2\big)\mathbb{G}_\theta(\mathrm{d}\eta). \tag{4.1}$$

However, since for general $F \in \mathscr{F}C_b^\infty(M)$ we do not have $F\circ\Psi \in \mathscr{F}C_b^\infty(M)$, this does not imply $F_0 \cdot (F \circ \Psi) \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$ as desired.

To approximate $F\circ\Psi$ using functions in $\mathscr{F}C_b^\infty(M)$, we write $F = f(\langle h_1, \cdot\rangle, \cdots, \langle h_k, \cdot\rangle)$ for some $f \in C_b^\infty(\mathbb{R}^k)$ and $h_1, \cdots, h_k \in C_0^\infty(M)$. Since the Riemannian manifold M is complete, we may construct $\{\phi_n\}_{n\ge1}^\infty \subset C_0^\infty(M)$ such that

$$0 \le \phi_n \uparrow 1, \quad |\nabla\phi_n| \le \mathrm{e}^{-n}, \quad \cup_{i=1}^k \mathrm{supp} h_i \subset \{\phi_n = 1\}, \quad n \ge 1. \tag{4.2}$$

For any $n \ge 1$, let

$$\tilde{F}_n(\eta) = F_0 F_n, \quad F_n(\eta) := f\Big(\frac{\langle h_1, \eta\rangle}{\eta(\phi_n) + n^{-1}}, \cdots, \frac{\langle h_k, \eta\rangle}{\eta(\phi_n) + n^{-1}}\Big), \quad \eta \in \mathbb{M}.$$

So, (4.1) implies $\tilde{F}_n \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$. Obviously, $\mathbb{G}_\theta(|\tilde{F}_n - F_0(F\circ\Psi)|^2) \to 0$ $(n\to\infty)$. By (4.2) and noting that

$$\nabla^{ext} F_n(\eta) = \sum_{i=1}^k (\partial_i f)\Big(\frac{\langle h_1,\eta\rangle}{\eta(\phi_n)+n^{-1}},\cdots,\frac{\langle h_k,\eta\rangle}{\eta(\phi_n)+n^{-1}}\Big)\Big(\frac{h_i}{\eta(\phi_n)+n^{-1}} - \frac{\eta(h_i)\phi_n}{(\eta(\phi_n)+n^{-1})^2}\Big),$$
$$\nabla^{int} F_n(\eta) = \sum_{i=1}^k (\partial_i f)\Big(\frac{\langle h_1,\eta\rangle}{\eta(\phi_n)+n^{-1}},\cdots,\frac{\langle h_k,\eta\rangle}{\eta(\phi_n)+n^{-1}}\Big)\Big(\frac{\nabla h_i}{\eta(\phi_n)+n^{-1}} - \frac{\eta(h_i)\nabla\phi_n}{(\eta(\phi_n)+n^{-1})^2}\Big),$$

we may find out a constant $c>0$ such that

$$\begin{aligned} I_{n,m}(\eta) &:= \eta\Big(\lambda\big|\eta(M)\big(\nabla^{ext}(F_n-F_m)(\eta) + (F_n-F_m)(\eta)\big|^2 + \big|\eta(M)\nabla^{int}(F_n-F_m)(\eta)\big|^2\Big)\\ &\le c(1+\eta(M)), \quad \eta\in\mathbb{M}, \quad n,m\ge 1. \end{aligned}$$

Since $F_0 \in L^1(\mathbb{G}_\theta)$, by (4.1) and Fatou's lemma, we arrive at

$$\begin{aligned} \limsup_{n,m\to\infty} \mathscr{E}^\lambda_{\mathbb{G}_\theta}(\tilde{F}_n-\tilde{F}_m,\tilde{F}_n-\tilde{F}_m) &= \limsup_{n,m\to\infty}\int_{\mathbb{M}} I_{n,m}(\eta)\mathbb{G}_\theta(\mathrm{d}\eta)\\ &\le \int_{\mathbb{M}}\Big(\limsup_{n,m\to\infty} I_{n,m}(\eta)\Big)\mathbb{G}_\theta(\mathrm{d}\eta) = 0. \end{aligned}$$

Then $F_0(F\circ\Psi) = \lim_{n\to\infty}\tilde{F}_n \in \mathscr{D}(\mathscr{E}^\lambda_{\mathbb{G}_\theta})$. Similarly, for $G = g(\langle h_1,\cdot\rangle,\cdots,\langle h_k,\cdot\rangle) \in \mathscr{F}C_b^\infty(M)$, we define $\tilde{G}_n = F_0 G_n$ in the same way. By the above formulas of intrinsic and extrinsic derivatives for F_n and G_n, it is easy to see that

$$\lim_{n\to\infty}\nabla^{ext}F_n(\eta) = \frac{1}{\eta(M)}(\tilde{\nabla}^{ext}F)(\Psi(\eta)), \quad \lim_{n\to\infty}\nabla^{int}F_n(\eta) = \frac{1}{\eta(M)}(\nabla^{int}F)(\Psi(\eta))$$

and the same holds for (G_n, G) replacing (F_n, F). Therefore, by the dominated convergence theorem and using (1.3) and (1.4), we obtain

$$\begin{aligned} &\mathscr{E}^\lambda_{\mathbb{G}_\theta}(F_0(F\circ\Psi), F_0(G\circ\Psi)) = \lim_{n\to\infty}\mathscr{E}^\lambda_{\mathbb{G}_\theta}(\tilde{F}_n,\tilde{G}_n)\\ &= \lim_{n\to\infty}\int_{\mathbb{M}}\Big[\eta\big(\lambda\{\eta(M)(\nabla^{ext}F_n)(\eta)+F_n(\eta)\}\cdot\{\eta(M)(\nabla^{ext}G_n)(\eta)+G_n(\eta)\}\big)\\ &\qquad\qquad + \eta(M)^2\eta\big(\big\langle(\nabla^{int}F_n)(\eta),(\nabla^{int}G_n)(\eta)\big\rangle_M\big)\Big]\mathbb{G}_\theta(\mathrm{d}\eta)\\ &= \int_{\mathbb{M}}\eta(M)\Big[\lambda\Psi(\eta)\big(\{\tilde{\nabla}^{ext}F(\Psi(\eta))\}\cdot\{\tilde{\nabla}^{ext}G(\Psi(\eta))\}\big) + \lambda(FG)(\Psi(\eta))\\ &\qquad\qquad + \Psi(\eta)\big(\langle\nabla^{int}F(\Psi(\eta)),\nabla^{int}G(\Psi(\eta))\rangle_M\big)\Big]\mathbb{G}_\theta(\mathrm{d}\eta)\\ &= \theta(M)\int_{\mathbb{M}_1}\eta\Big(\lambda\{\tilde{\nabla}^{ext}F(\eta)\}\cdot\{\tilde{\nabla}^{ext}G(\eta)\} + \lambda(FG)(\eta) + \langle\nabla^{int}F(\eta),\nabla^{int}G(\eta)\rangle_M\Big)\mathbb{D}_\theta(\mathrm{d}\eta)\\ &= \lambda\theta(M)\mathbb{D}_\theta(FG) + \theta(M)\mathscr{E}^\lambda_{\mathbb{D}_\theta}(F,G). \end{aligned}$$

Therefore, (1.17) holds.

(2) Spectral gap estimates. Since

$$\mathscr{E}^{\lambda}_{\mathbb{D}_\theta}(F,F) \geq \lambda \mathscr{E}^{FV}_{\mathbb{D}_\theta}(F,F), \quad F \in \mathscr{F}C_b^\infty(M),$$

where $\mathscr{E}^{FV}_{\mathbb{D}_\theta}$ is given in (1.18) as the Dirichlet form of the Fleming-Viot process, it follows from (1.19) that

$$\mathrm{gap}(\mathscr{E}^{\lambda}_{\mathbb{D}_\theta}) \geq \lambda \mathrm{gap}(\mathscr{E}^{FV}_{\mathbb{D}_\theta}) = \lambda\theta(M).$$

To verify the spectral gap upper bound, for any $h \in C_b^\infty(M)$ with $\theta(h) = 0$ and $\theta(h^2) = 1$, let $F_h(\eta) = \eta(h)$. Then $F_h \in \mathscr{D}(\mathscr{E}^{\lambda}_{\mathbb{D}_\theta})$. It suffices to show that

$$0 < \mathscr{E}^{\lambda}_{\mathbb{D}_\theta}(F_h, F_h) \leq \big\{\lambda\theta(M) + \theta(|\nabla h|^2)(\theta(M)+1)\big\} \cdot \{\mathbb{D}_\theta(F_h^2) - \mathbb{D}_\theta(F_h)^2\}. \tag{4.3}$$

To this end, we recall that (see for instance the proof of [17, Lemma 7.2])

$$\pi_{\hat\theta}(\langle \hat h, \cdot\rangle) = \hat\theta(\hat h), \quad \pi_{\hat\theta}(\langle \hat h, \cdot\rangle^2) = \hat\theta(\hat h^2) + \hat\theta(\hat h)^2, \quad \hat h \in L^2(\hat\theta). \tag{4.4}$$

Letting $\hat g(x,s) = sg(x)$ for $g \in L^2(M,\theta)$ and applying (1.3), (1.4), and (1.7), we deduce from (4.4) that

$$\begin{aligned}\theta(M)\mathbb{D}_\theta(F_g) &= \int_{\mathbb{M}} \eta(M)\langle g, \Psi(\eta)\rangle \mathbb{G}_\theta(\mathrm{d}\eta) = \int_{\mathbb{M}} \eta(g)\mathbb{G}_\theta(\mathrm{d}\eta)\\ &= \pi_{\hat\theta}(\langle \hat g, \cdot\rangle) = \theta(g), \quad g \in L^2(M,\theta),\end{aligned}$$

and similarly,

$$\begin{aligned}\theta(M)(\theta(M)+1)\mathbb{D}_\theta(|F_g|^2) &= \int_{\mathbb{M}} \eta(M)^2\langle g, \Psi(\eta)\rangle^2 \mathbb{G}_\theta(\mathrm{d}\eta) = \int_{\mathbb{M}} |\eta(g)|^2\mathbb{G}_\theta(\mathrm{d}\eta)\\ &= \pi_{\hat\theta}(\langle \hat g, \cdot\rangle^2) = \theta(g^2) + \theta(g)^2, \quad g \in L^2(M,\theta).\end{aligned}$$

Thus, for $h \in C_b^\infty(M)$ with $\theta(h) = 0$ and $\theta(h^2) = 1$, we have

$$\mathbb{D}_\theta(F_h^2) - \mathbb{D}_\theta(F_h)^2 = \frac{\theta(h^2) - \theta(h)^2}{\theta(M)(\theta(M)+1)} - \frac{\theta(h)^2}{\theta(M)^2} = \frac{1}{\theta(M)(\theta(M)+1)},$$

and

$$\begin{aligned}\mathscr{E}^{\lambda}_{\mathbb{D}_\theta}(F_h, F_h) &= \mathbb{D}_\theta\big(\lambda\big\{\langle h^2, \cdot\rangle - \langle h, \cdot\rangle^2\big\} + \langle |\nabla h|^2, \cdot\rangle\big)\\ &= \lambda\Big(\frac{\theta(h^2)}{\theta(M)} - \frac{\theta(h^2) + \theta(h)^2}{\theta(M)(\theta(M)+1)}\Big) + \frac{\theta(|\nabla h|^2)}{\theta(M)} = \frac{\lambda}{\theta(M)+1} + \frac{\theta(|\nabla h|^2)}{\theta(M)}.\end{aligned}$$

Therefore, (4.3) holds.

Acknowledgement. We would like to thank the referees for helpful comments on an earlier version of the paper.

References

[1] S. Albeverio,Y. G. Kondratiev, M. Röckner, *Differential geometry of Poisson spaces,* C. R. Acad. Sci. Paris Sér. I Math. 323(1996), 1129–1134.

[2] S. Albeverio,Y. G. Kondratiev, M. Röckner, *Analysis and geometry on configuration spaces,* J. Funct. Anal. 154(1998), 444–500.

[3] R. J. Connor, J. E. Mosimann, *Concepts of independence for proportions with a generalization of the Dirichlet distribution,* J. Amer. Statist. Assoc. 64(1969), 194-206.

[4] C. L. Epstein, R. Mazzeo, *Wright-Fisher diffusion in one dimension,* SIAM J. Math. Anal. 42(2010), 568–608.

[5] S. Feng, L. Miclo, F.-Y. Wang, *Poincaré inequality for Dirichlet distributions and infinite-dimensional generalizations,* Lat. Am. J. Probab. Math. Stat. 14(2017), 361–380.

[6] S. Feng, F.-Y. Wang, *A class of infinite-dimensional diffusion processes with connection to population genetics,* J. Appl. Probab. 44(2007), 938–949.

[7] S. Feng, F.-Y. Wang, *Harnack inequality and applications for infinite-dimensional GEM processes,* Potential Anal. 44(2016), 137–153

[8] F. Q. Gao, *Long time asymptotics of unbounded additive functionals of Markov processes,* Electron J. Probab. 22(2017), 1–21.

[9] D. Hagedorn, Y. Kondratiev, T. Pasurek, M. Röckner, *Gibbs states over the cone of discrete measures,* J. Funct. Anal. 264(2013), 2550–2583.

[10] N. L. Johnson, *An approximation to the multinomial distribution, some properties and applications,* Biometrika, 47(1960), 93–102.

[11] Y. Kondratiev, E. Lytvynov, A. Vershik, *Laplace operators on the cone of Radon measures,* J. Funct. Anal. 269(2015), 2947–2976.

[12] Z.-M. Ma, M. Röckner, *An Introduction to the Theory of (Non-Symmetric) Dirichlet Forms,* Springer, 1992.

[13] Z.-M. Ma, M. Röckner, *Construction of diffusion on configuration spaces,* Osaka J. Math. 37(2000), 273–314.

[14] J. E. Mosimann, *On the compound multinomial distribution, the multivariate-distribution, and correlations among proportions,* Biometrika, 49(1962), 65–82.

[15] L. Overbeck, M. Röckner, B. Schmuland, *An analytic approach to Fleming-Viot processes with interactive selection,* Ann. Probab. 23(1995), 1–36.

[16] P. Ren, F.-Y. Wang, *Derivative formulas in measure on Riemannian manifolds,* arXiv:1908.03711.

[17] M. Röckner, F.-Y. Wang, *Weak Poincaré inequalities and convergence rates of Markov semigroups,* J. Funct. Anal. 185(2001), 564–603.

[18] N. Shimakura, *Equations différentielles provenant de la génetique des populations,* Tôhoka Math. J. 29(1977), 287–318.

[19] B. Simon, *The* $P(\Phi)_2$*-Euclidean (Quantum) Field Theory,* Princeton Univ. Press, Princeton, NJ. 1974.

[20] W. Stannat, *On validity of the log-Sobolev inequality for symmetric Fleming-Viot operators,* Ann. Probab. 28(2000), 667–684.

[21] F.-Y. Wang, *Functional inequalities for weighted Gamma distribution on the space of finite measures,* arXiv:1810.10387.

[22] F.-Y. Wang, W. Zhang, *Nash inequality for diffusion processes associated with Dirichlet distributions,* arXiv:1801.09209v1.