Two Operations for Stable Structures of Elementary Regions

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Abstract

The set of *regions* of a *condition/event transition system* represents all the possible local states of a *net system* the behaviour of which is specified by the transition system. This set can be endowed with a structure, so as to form an *orthomodular partial order*. Given such a structure, one can then define another condition/event transition system. We study cases in which this second transition system has the same collection of regions as the first one. When it is so, the structure of regions is called stable. We propose, to this aim, a composition operation, and a refinement operation for stable orthomodular partial orders, the results of which are stable.

1 Introduction

This work studies the interrelations between local states, locally observable properties, and events of distributed systems. To this aim, its framework is set in the relation between elementary Petri net systems, and the labelled transition systems expressing their behaviour, their case graphs. Indeed, labelled transition systems are commonly used models for verification of properties of the specified system [8].

We focus on the particular case in which local states act like Boolean variables, forbidding executions by carrying true or false values. Thus, the framework is narrowed down to elementary and condition/event models, and their close relation to the theory of regions. In [10, 11], regions were shown to be the key to solving the synthesis problem: Given a labelled transition system, determine whether it is the case graph of some elementary net system. The extension of a local state of a net system in its case graph is the set of global states at which it holds the value true. In a labelled transition system, a region is a set of global states having a consistent incidence with respect to the actions, so as to be the extension of a local state. Given an elementary, or condition/event transition system, a net system can be constructed with regions as local states, such that its case graph will be precisely the specified transition system. It was further shown [9] that given an elementary (condition/event) system, it suffices to consider the sub-collection of regions that is minimal with respect to set inclusion [1]. The remaining regions correspond to local states which are redundant in the net system.

Regions, as subsets of global states of a system, carry an algebraic structure [2]. They can be ordered by set inclusion, in a relation that expresses implication. Negation, disjunction, and conjunction can be defined as set operations, only partially in the latter two cases. The resulting structure was shown to be a prime and coherent orthomodular partial order [2], OMP for short.

Orthomodular partial orders typically represent, in an algebraic fashion, systems about which the acquisition of information depends on the point of view, or experiment. They were first introduced by Birkhoff and Von Neumann [6] in order to formalise the logic of properties of quantum systems. Foulis and Randall [12, 18] extended their approach to provide a generalized model.

In the view that OMP's are a suitable specification for the observable properties of a system, we are concerned with a second synthesis problem: Given an OMP determine whether it is the regional structure of some elementary transition system. This problem was first posed in [2], and a solution was pointed at. When an OMP is prime, one can make use of a representation theorem to find a set of states such that the elements of the OMP are subsets of it. By considering a complete graph whose vertices are these states, one can label each edge so that all elements of the OMP can be found as regions of this new transition system. When the structure of regions of this new system is isomorphic to the specified OMP, then the latter is called *stable*.

Not every OMP is stable, and the full characterization of the class of stable OMP's is an open problem which represents a long term objective. It is conjectured that every regional OMP, the structured collection of regions of some elementary transition system, is stable. The present work includes itself in a series of papers [2, 3, 4, 5] the scope of which is to study this stability of OMP's, in an attempt to prove the conjecture.

The present work extends [4], in which some classes of stable OMP's were presented. Here, composition and refinement operations are defined, which preserve stability, thus proving the stability of regional OMP's for a wide range of systems.

2 Background

This work implicitly deals with elementary net systems (ENS for short) [15]. Even though these models will not be formalised here, they underlie the main ideas in the presented results. ENS are a class of pure, and simple Petri net systems with a particular firing rule which ensures that the system remains 1-safe in case of contact. *Conditions* of ENS can only be in one of two states, marked or un-marked, and as such they can be seen as Boolean variables. Moreover, out of Petri's extensionality principle, two events having the same incidence on the set of conditions, and viceversa for conditions in relation to events, should be considered the same [17].

Condition/Event net systems (CENS for short) are structurally identical to ENS with the addition of a backward firing rule which provides symmetry to the reachability relation amongst markings; reachability becomes an equivalence relation, and reachability classes substitute the initial marking required for ENS.

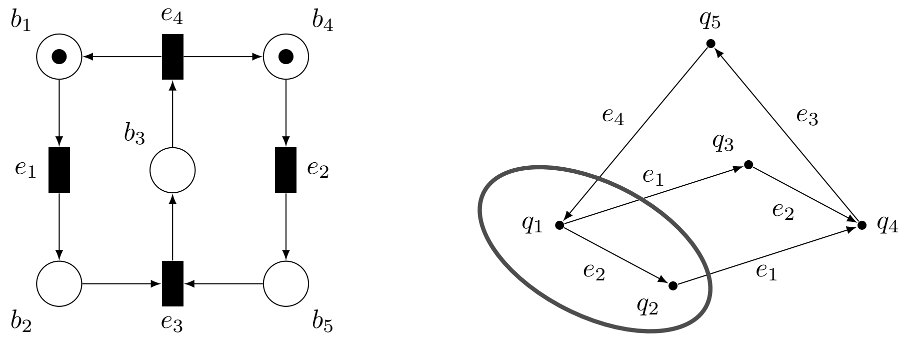


Figure 1: To the left, a CENS. To the right, its sequential case graph. The state q_1 corresponds to the marking $\{b_1, b_4\}$, at which e_1 and e_2 are concurrently enabled. To the right, the subset of states $\{q_1, q_2\}$ is in evidence; it corresponds to the extension of b_1, namely the set of markings containing b_1.

2.1 Transition Systems and Regions

The behaviour of CENS can be expressed by means of a labelled transition system. In such a model, states correspond to the reachable markings of the net system. A marking enabling an event will correspond to a state from which an arrow, labelled with this event, points to the state which is reached when the event is fired. Such a labelled transition system is the *sequential case graph* of the underlying net system.

Definition 1 (Transition System)**.** *A* transition system *is a structure* $A = (Q, E, T)$*, where* Q *is a set of states,* E *is a set of* events *and* $T \subseteq Q \times E \times Q$ *is a set of* transitions *carrying labels in* E*, such that:*

1. *the underlying graph of the* transition system *is connected;*
2. $\forall (q_1, e, q_2) \in T \quad q_1 \neq q_2$*;*
3. $\forall (q, e_1, q_1)(q, e_2, q_2) \in T \quad q_1 = q_2 \Rightarrow e_1 = e_2$*;*
4. $\forall e \in E \quad \exists\, (q_1, e, q_2) \in T$*.*

All the structures we consider are finite. A fundamental notion used in this contribution is the one of *region.* Regions were introduced in [10, 11] as the fundamental tool for solving the so called *synthesis problem* for Petri Nets.

Definition 2 (Region)**.** *A* region *of a transition system* $A = (Q, E, T)$ *is a subset* r *of* Q *such that:* $\forall e \in E, \forall (q_1, e, q_2), (q_3, e, q_4) \in T$*:*

1. *(*$q_1 \in r$ *and* $q_2 \notin r$*) implies (*$q_3 \in r$ *and* $q_4 \notin r$*)* and
2. *(*$q_1 \notin r$ *and* $q_2 \in r$*) implies (*$q_3 \notin r$ *and* $q_4 \in r$*).*

Regions are subsets of states which have a consistent orientation with the labelling of transitions. As such, they can be assigned an incidence with respect to each event, and interpreted as Boolean conditions which hold at the states composing them.

Example 1. *With reference to Figure 1, the extension of b_1 is a region. As a subset of states, all occurrences of e_1 exit it and e_4 enters it. All occurrences of the remaining events do not cross the border of b_1.*

The set of regions of a transition system A will be denoted by $\mathcal{R}(A)$ and, given a state q of A, R_q will denote the subset of the regions in $\mathcal{R}(A)$ which contains the state q. Given a transition system A and its set of regions $\mathcal{R}(A)$, the *pre-set* and *post-set* operators can be defined:

Definition 3 (Pre- and Post-sets)**.** $^\bullet(\,.\,)$*, the* pre-set *operator, and* $(\,.\,)^\bullet$*, the* post-set *operator, are defined on the events E and on the regions $\mathcal{R}(A)$ of the transition system A as follows. Let r be a region in $\mathcal{R}(A)$*

1. $^\bullet r = \{e \in E \,|\, \exists\, (q_1, e, q_2) \in T$ *such that* $q_1 \notin r$ *and* $q_2 \in r\}$;
2. $r^\bullet = \{e \in E \,|\, \exists\, (q_1, e, q_2) \in T$ *such that* $q_1 \in r$ *and* $q_2 \notin r\}$;
3. $^\bullet e = \{r \in \mathcal{R}(A) \,|\, e \in r^\bullet\}$;
4. $e^\bullet = \{r \in \mathcal{R}(A) \,|\, e \in {}^\bullet r\}$.

We are concerned with a specific class of transition systems, the *condition/ event transition systems* (CETS for short). CETS can be defined as those labelled transition systems which are the sequential case graph of some CENS. An alternative definition, in terms of regions and the *separation axioms*, follows from the results of [10, 11].

Definition 4 (CETS – Condition/Event Transition System)**.** *A* CETS *is a transition system in which the following conditions hold:*

1. $\forall\, q_1, q_2 \in Q \quad R_{q_1} = R_{q_2} \Rightarrow q_1 = q_2$;
2. $\forall\, q_1 \in Q\, \forall\, e \in E \quad {}^\bullet e \subseteq R_{q_1} \Rightarrow \exists\, q_2 \in Q$ *such that* $(q_1, e, q_2) \in T$;
3. $\forall\, q_1 \in Q\, \forall\, e \in E \quad e^\bullet \subseteq R_{q_1} \Rightarrow \exists\, q_2 \in Q$ *such that* $(q_2, e, q_1) \in T$.

This definition requires of a transition system, that its set of regions is sufficient to fully determine the incidence of each event with respect to each state. Under these conditions, a CENS can be constructed, which has the set of regions for conditions, and the transition labels as events, and such that its case graph is isomorphic to the given transition system. This tight relation between CENS and CETS was thoroughly formalised in [15], in the equivalent framework of elementary systems.

The following basic properties of the regions $\mathcal{R}(A)$ of a CETS A are proven in [1]:

1. $\emptyset, Q \in \mathcal{R}(A)$;
2. $r \in \mathcal{R}(A) \;\Rightarrow\; Q \setminus r \in \mathcal{R}(A)$;
3. $r_1, r_2 \in \mathcal{R}(A) \;\Rightarrow\; (r_1 \cap r_2 \in \mathcal{R}(A) \Leftrightarrow r_1 \cup r_2 \in \mathcal{R}(A))$.

2.2 Orthomodular Partially Ordered Sets

Orthomodular partially ordered sets, indicated as OMP in what follows, are well known algebraic structures in the literature on *Quantum Logics*. The following definition can be found as the finite version of Definition 1.1.1 in [16]. Note that $\wedge$ and $\vee$ are the usual meet and join operations on a partial order, denoting, respectively, the *greatest lower bound*, and the *least upper bound*.

Definition 5 (OMP - Orthomodular Partially Ordered Set)**.** *An* orthomodular partially ordered set $\langle L, \leq, (.)', 0, 1\rangle$ *is a set* L *endowed with a partial order* $\leq$ *and a unary operation* $(\,.\,)'$ *(called orthocomplement), such that the following conditions are satisfied:*

1. *L has a least and a greatest element (respectively* 0 *and* 1*) and* $0 \neq 1$*;*
2. $\forall\, x, y \in L \quad x \leq y \Rightarrow y' \leq x'$*;*
3. $\forall\, x \in L \quad (x')' = x$*;*
4. *for any finite sequence* $\{x_i \,|\, i \in I\}$ *of elements of* L *such that* $i \neq j \Rightarrow x_i \leq x_j'$*, then* $\bigvee_{i \in I} x_i$ *exists in* L*;*
5. $\forall\, x, y \in L \quad x \leq y \;\Rightarrow\; y = x \vee (x' \wedge y)$*.*

This latter condition is sometimes referred to as orthomodular law*.*

We will often denote $\langle L, \leq, (.)', 0, 1\rangle$ simply by L, and assume that an OMP L_i has underlying structure $\langle L_i, \leq_i, (.)', 0_i, 1_i\rangle$.

In general, $\wedge$ and $\vee$ operations can be undefined for some pairs of elements of L. Two elements x and y in L are said to be *orthogonal*, noted by $x \perp y$, when $x \leq y'$. As a consequence of their basic properties, as listed at the end of Section 2.1 above, the set of regions of a CETS is an OMP. 0 and 1 are, respectively, $\emptyset$ and Q, the order is given by set containment and orthocomplement by set complement; moreover, two regions are *orthogonal* whenever their intersection is empty.

Since all the structures we consider are finite, an OMP can be equivalently specified by its collection of *atoms* $\mathcal{A}(L) := \{x \in L \setminus \{0\} \mid \forall y \in L : (y \leq x) \rightarrow (y = x \text{ or } y = 0)\}$ together with their orthogonality relation. Each element of L can then be retrieved as the join of a collection of pairwise orthogonal atoms. The reader is referred to [5] for the details, and to [12, 18] for a full characterisation of this representation of OMP's. A subset M of an OMP L with the order relation and the orthocomplement operation inherited from L is a sub-OMP of L in case M is an OMP ([16], definition 1.2.2).

Definition 6 (OMP-morphism)**.** *([16], finite case of definition 1.2.7) Let* L_1 *and* L_2 *be* OMP*. A mapping* $f : L_1 \to L_2$ *is a* morphism of OMP's *if the following conditions are satisfied:*

1. $f(0) = 0$*;*
2. $\forall x \in L_1 \; f(x') = f(x)'$*;*
3. *for any finite sequence* $\{x_i \,|\, i \in I\}$ *of mutually orthogonal elements in* L_1*,*

$$f(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} f(x_i)$$

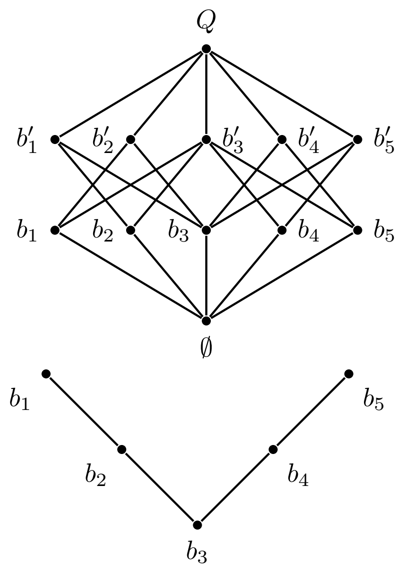


Figure 2: Two representations of the OMP obtained by ordering the regions of the transition system in Figure 1. To the left, all regions are represented in a *Hasse diagram.* To the right, only the atoms are represented; each of the two solid lines is a maximal clique of orthogonality. This last representation is called *Greechie diagram.*

A morphism $f : L_1 \to L_2$ is an isomorphism *if f is injective, maps L_1 onto L_2 and f^{-1} is a morphism. An isomorphism $f : L_1 \to M$, when M is a sub-*OMP *of L_2, is called* embedding.

Morphisms of OMP's preserve *compatibility*, order and orthogonality [2]. Two elements x, y of an OMP L are said to be *compatible* if they admit a common orthogonal base. Formally, $x \ \$ \ y \Leftrightarrow \exists a, b, c \in L : (a \perp b \perp c \perp a)$ and ($x = a \vee b$ and $y = b \vee c$) ([16], definition 1.2.1). This orthogonal base generates a Boolean algebra which contains both elements, and is a sub-OMP of L. When considered as conditions of a net system, two compatible elements will induce sequential behaviour of their neighbouring events, imposed by the orthogonal base. Indeed, orthogonality is to be interpreted as mutual exclusion of the corresponding conditions, and in fact, when restricted to atoms, the compatibility relation reduces to orthogonality [5]. We say two elements are incompatible, denoted $x \not\$ \, y$, when they are not compatible. Compatible elements x and y will be denoted by $x \ \$ \ y$. An embedding that preserves and reflects compatibility is called *strong* ([4], Definition 15).

A *two-valued state* of an OMP is a well-known concept in the literature on *Quantum Logics* ([16], Chapter 2). We will use here a specific definition of *two-valued state*, *state* for short, apt to the OMP's of the *regions* of CETS.

Definition 7 (State of an OMP)**.** *([2], theorem 41) Let $\mathcal{A}(L)$ be the set of* atoms *of an* OMP *L. Let E be the set of the* maximal cliques *of $\perp$ in L restricted to $\mathcal{A}(L)$; let C be the set of the maximal cliques of $\not\$$ in L restricted to $\mathcal{A}(L)$; let $S = \{s \in C \mid \forall e \in E \ |s \cap e| = 1\}$, then the up-closure of s, $\uparrow(s)$, in L is a* state *in L.*

Since *states* of L are uniquely defined by means of up-closures of maximal cliques of $\not\$$, we will frequently refer to these maximal cliques as the *states* of the OMP L when clear from the context.

The collection of all states of an OMP L will be denoted by $S(L)$. It will also be useful to consider the collection of states which contain a given element. Given an element $x \in L$, let us define the notation $S_x := \{s \in S(L) \mid x \in s\}$.

The following section is concerned with the representation of an OMP as a collection of subsets of its states. The order in L will then be set inclusion, and the orthocomplement will simply be given by set complement. When this is the case, the OMP is said to admit a concrete representation.

2.3 Properties of Regional OMP's

As explained in more detail in section 2.4 below, OMPs can be constructed from the regions of a CETS. These OMPs will, from now on, be referred to as *regional* OMP*s*. They have been shown, in [2], to satisfy the following properties. We shall not get in the details, and the reader is referred to [2] for a full explanation of these, and the implications they provide in the interpretation of OMPs for system specification.

- *Regularity:* L is regular whenever every set of pairwise compatible elements is a compatible set [16]
- *Richness:* L is rich whenever it admits a concrete representation [16]

There is some inconsistency in the terminology referring to these concepts in the literature, and it has led to some confusion in the axiomatic definition of these properties. Indeed *regularity* is the term used in [16], whereas [14] refers to this property as *coherence*. The concept of *coherence* is however slightly more general in [12, 18].

Richness is axiomatically defined in [16], and assumed through [2, 3, 4, 5] to be equivalent to the notion of *primeness* defined in [14]. The two definitions however, differ enough to raise a doubt, as their equivalence might not seem straightforward. This equivalence is, in what follows, proven, for the sake of mathematical rigour.

An OMP L is called unital when for every element $x \in L \setminus \{0\}$ there is a state $s \in S(L)$ such that $x \in s$.

Definition 8 (Rich OMP)**.** *[16] An* OMP *L is rich iff*
$$\forall x, y \in L : S_x \subseteq S_y \Rightarrow x \leq y$$

In Chapter 2 of [16], a few equivalent characterisations of richness are shown. In particular we are interested in the following:

Theorem 1. *[16] An* OMP *L is rich iff it is unital and*
$$\forall x, y \in L : x \not\$ y \Rightarrow \exists s \in S(L) : x \in s \text{ and } y \in s$$

Rich OMP's where shown in Chapter 2 of [16] to also be unital.

Definition 9 (Prime OMP)**.** *[14] An* OMP *L is prime iff*
$$\forall x, y \in L : x \neq y \Rightarrow \exists s \in S(L) : x \in s \Leftrightarrow y \notin s$$

We now prove two lemmas that will be used in the proof of Theorem 2 below:

Lemma 1. *In an* OMP *L we have that:*
$\forall x, y \in L : x \not\$ y \Rightarrow \exists s \in S(L) : x \notin s$ *and* $y \notin s$ *if and only if:*
$\forall x, y \in L : x \not\$ y \Rightarrow \exists s \in S(L) : x \in s$ *and* $y \in s$

Proof. Suppose that L is an OMP such that $\forall x, y \in L : x \not\$ y \Rightarrow \exists s \in S(L) : x \in s$ and $y \in s$ and consider two elements $x, y \in L : x \not\$ y$; then it is also true that $x' \not\$ y'$. By hypothesis we have that $\exists s \in S(L) : x' \in s$ and $y' \in s$ and neither x nor y belong to s, which means that $\exists s \in S(L) : x \notin s$ and $y \notin s$ for every pair of incompatible elements in L.

Conversely, suppose that L is an OMP such that $\forall x, y \in L : x \not\$ y \Rightarrow \exists s \in S(L) : x \notin s$ and $y \notin s$ and consider two elements $x, y \in L : x \not\$ y$; again it is true that $x' \not\$ y'$. By hypothesis we have that $\exists s \in S(L) : x' \notin s$ and $y' \notin s$ and both x and y belong to s, which means that $\exists s \in S(L) : x \in s$ and $y \in s$ for every pair of incompatible elements in L. □

Lemma 2. *A prime* OMP *L is also unital.*

Proof. For every element $x \in L$ distinct from 0 we have that $\exists s \in S(L) : x \in s \Leftrightarrow 0 \notin s$ and since 0 doesn't belong to any state of L then x belongs to s. □

Theorem 2. *An* OMP *L is rich iff it is prime.*

Proof. Suppose that L is rich and consider two elements $x, y \in L : x \neq y$. There are two cases:

$x \$ y$: in this case there exists a Boolean subalgebra of L that contains both x and y. For both of them we calculate the set of atoms below them, $\mathcal{A}_{\downarrow}(x) = \{a \in \mathcal{A}(L) \mid a \leq x\}$ and $\mathcal{A}_{\downarrow}(y) = \{a \in \mathcal{A}(L) \mid a \leq y\}$. Such sets differs in at least an element z, because otherwise it would mean, out of Axiom 4 in Definition 5, that $x = y$. If we consider a state $s \in S(L)$ such that $z \in s$ then s will contain only one element between x and y because z belongs to to only one of the two sets $\mathcal{A}_{\downarrow}(x)$ and $\mathcal{A}_{\downarrow}(y)$. The existence of s is ensured by the fact that L is unital, which means that $\exists s \in S(L) : x \in s \Leftrightarrow y \notin s$.

$x \not\$ y$: if $x \not\$ y$ then we also have that $x \not\$ y'$. From the hypothesis we have that $\exists s \in S(L) : x \in s$ and $y' \in s$ which means that $\exists s \in S(L) : x \in s \Leftrightarrow y \notin s$.

Now, suppose that L is prime, for Lemma 2 L is also unital. Consider two elements $x, y \in L : x \not\$ y$. That means that $x \not\$ y'$ and in particular $x \neq y'$. By Definition 9 of prime OMP we have that $\exists s \in S(L) : x \in s \Leftrightarrow y' \notin s$ which means that $\exists s \in S(L) : x \in s \Leftrightarrow y \in s$. Again, we consider two cases:

$x \in s$: then $\exists s \in S(L) : x \in s$ and $y \in s$

$x \notin s$: then $\exists s \in S(L) : x \notin s$ and $y \notin s$

which, as shown in Lemma 1, is equivalent to $\exists s' \in S(L) : x \in s'$ and $y \in s'$. $\square$

2.4 Saturated Transition System and the Stability Problem

As anticipated in section 2.3 above, we know that, given a CETS $A = (S, E, T)$, $H(A) = (\mathcal{R}(A), \subseteq, \emptyset, S, (\,.\,)')$ with $(\,.\,)'$ the *set complement* operation, is an OMP. Moreover, this *regional* OMP is *regular* and *rich*.

We know as well that, given a *regular* and *rich* OMP L, $J(L) = (S(L), E(L), T(L))$ is a CETS; where $E(L)$ and $T(L)$, the sets of, respectively, *events* and *transitions*, are constructed by exploiting symmetric differences between states as defined in [2] and [5]. Formally, a state is a collection of elements of L, and each event $e \in E(L)$ will be of the form $e = [s, s'] = (s \setminus s', s' \setminus s)$, as such, it is characterised by its sets of pre-conditions, and post-conditions seen as elements of L. The underlying graph of $J(L)$ is complete, presenting a transition $(s, [s, s'], s')$ between each ordered pair of distinct states (s, s'), and is therefore called the *saturated transition system* of L.

In [5] it is shown how the construction of $J(L)$ can be done by considering exclusively the *atoms* of L. In particular, states are determined by their atoms, and so are events, as symmetric differences of the atoms of states. A natural question concerns the full axiomatic definition of the *regional* OMP*s*. We know that, in the general case, given a *rich* and *regular* logic L, L embeds in $H(J(L))$. We consequently define as *stable* an OMP L isomorphic to $H(J(L))$.

It is our long-term goal to fully characterise the class of stable OMP's, and it is conjectured that it coincides with the class of regional OMP's. In the next section, we will present a compositional approach which will allow us to prove the stability of a wide class of regional OMPs.

3 Composition of OMPs and their Stability

This section will start introducing a rather general composition operation for OMP's. The result of this composition will not always be an OMP but we show

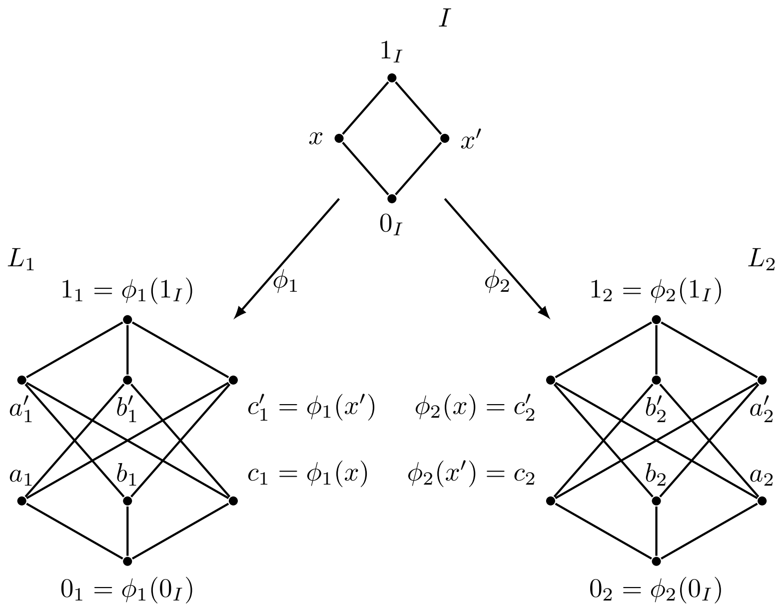


Figure 3: A V-formation

that it is the case in particular instances.

3.1 Composition of OMP's

We here present a composition operation for OMP's, in the fashion of those presented in [7] for modular ortholattices and orthomodular lattices. We focus on the class of orthomodular posets, of which orthomodular lattices are a subclass. Note that, whereas the cases treated in [7] are classes of algebras, our class is not, since in OMP's joins and meets are only partially defined.

Definition 10 (V-formation)**.** *A* V-formation *of* OMP*'s is a tuple* $(I, L_1, L_2, \phi_1, \phi_2)$*, such that* $I, L_1,$ *and* L_2 *are* OMP*'s, and* $\phi_i : I \to L_i (i = 1, 2)$ *are morphisms of* OMP*'s.*

A V-formation serves as specification for composing L_1 and L_2 on the common interface I. In categorical terms, it is simply a diagram in the category of OMP's. Strictly speaking, the interface would only be $\phi_1^{-1}(L_1) \cap \phi_2^{-1}(L_2)$, so in order to simplify notation, we here consider V-formations in which $\phi_i(i = 1, 2)$ are embeddings, and so for each $i = 1, 2,$ $\phi_i(I)$ is a sub-OMP of L_i isomorphic to I.

Given a V-formation of OMP'embeddingss, we propose a straightforward composition operation. This construction is inspired by co-equalisers [13], however, universality of the construction remains an open problem. As a matter of fact, the nature of the proposed composition interprets the interface I as a sub-OMP of each component, so as to identify the two copies element-wise. In this sense, it requires the morphisms of the V-formation to be actual embeddings.

Definition 11 (Equivalence induced by a V-formation)**.** *Let* $V = (I, L_1, L_2, \phi_1, \phi_2)$ *be a V-formation of* OMP*'s. Consider* $\tilde{L}$*, the disjoint union of* L_1 *and* L_2 *such* *that* $\phi_i : I \to \tilde{L}\,(i = 1, 2)$*, with* $\phi_1(I) \cap \phi_2(I) = \emptyset$*. The* equivalence relation induced by V *is the binary relation* $\sim_V := \{(x, x) \in \tilde{L}^2 \mid x \in \tilde{L}\} \cup \{(\phi_1(x), \phi_2(x)) \in \tilde{L}^2 \mid x \in I\} \cup \{(\phi_2(x), \phi_1(x)) \in \tilde{L}^2 \mid x \in I\}$.

It is straightforward to verify that $\sim_V$ is an equivalence relation. It is reflexive, and symmetric by construction. If $x, y, z \in \tilde{L}$ are such that $x \sim_V y$ and $y \sim_V z$ then it must be either $z = x$ or $z = y$, so $\sim_V$ is transitive. We will denote the equivalence class of an element $x \in \tilde{L}$ by $[x]$. This equivalence relation satisfies these additional properties:

Proposition 1. *Let* $V = (I, L_1, L_2, \phi_1, \phi_2)$ *be a V-formation of* OMP*'s, and* $\sim_V$ *as in Definition 11. Then:*

1. $\forall i \in \{1, 2\} : \forall x \in L_i : [x] \cap L_i = \{x\}$
2. $\forall x, y \in \tilde{L} : x \sim_v y \Leftrightarrow x' \sim_V y'$
3. $[0_1] = [0_2] = \{0_1, 0_2\}$ *and* $[1_1] = [1_2] = \{1_1, 1_2\}$
4. $\forall x, y \in \tilde{L} : (x \leq y) \Rightarrow \neg(\exists \tilde{x} \in [x], \tilde{y} \in [y] : \tilde{y} < \tilde{x})$.

Proof.

1. By construction.
2. Let $x, y \in \tilde{L}$ be two distinct elements such that $x \sim y$. From the definition of $\sim_V$, it follows that $\exists z \in I : \phi_1(z) = x$ and $\phi_2(z) = y$ (up to swapping of x and y). Since ϕ_1 and ϕ_2 are OMP-morphisms, it follows that $\phi_1(z') = x'$ and $\phi_2(z') = y'$, hence $x' \sim y'$.
3. It is a direct consequence of Axioms 1 and 2 in Definition 6.
4. Let $x \leq y$ then $\exists i \in \{1, 2\} : x, y \in L_i$, and $x \leq_i y$. Analogously, $\exists j \in \{1, 2\} : \tilde{x}, \tilde{y} \in L_j$, and $\tilde{y} <_j \tilde{x}$. Clearly it must be that $i \neq j$. Now, $x \sim_V \tilde{x}$, and $y \sim_V \tilde{y} \Rightarrow \exists a, b \in I : \phi_i(a) = x, \phi_j(a) = \tilde{x}, \phi_i(a) = y$, and $\phi_j(b) = \tilde{y}$. Since ϕ_i is an OMP-embedding, it must reflect the order, yielding $a \leq b$, but ϕ_j preserves the order, and so $\tilde{x} \leq \tilde{y}$.

□

These results allow for endowing the quotient $\tilde{L}/\sim_V$ with a structure.

Definition 12 (I-pasting of OMP's)**.** *Consider the setting of Definition 11, and define:*

1. $L = \tilde{L}/\sim_V$,
2. $0 = [0_1] = [0_2]$ *and* $1 = [1_1] = [1_2]$,
3. $\forall [x] \in L : [x]' = [x']$,
4. $[x] \prec [y] \Leftrightarrow \exists x \in [x] : \exists y \in [y] : x \leq y$ *in* $\tilde{L}$*, and*
5. $\leq\ \subseteq L \times L$ *as the transitive closure of* $\prec$.

Then the I-pasting *of* L_1 *and* L_2 *induced by* V *is the structure* $L_1|I|L_2 = \langle L, \leq, (\cdot)', 0, 1\rangle$.

It follows immediately from Proposition 1 (4.) that $\leq$ is an order relation. Proposition 1 (2.), states that $\sim_V$ is congruence for complementation, and so $(\cdot)'$ is well-defined on L. It is furthermore trivial to verify that 0 and 1 are respectively the minimal and maximal elements in L. Note that whenever $x \perp y$, then $[x] \perp [y]$. So defined, the composition of two OMP's over an interface is simply obtained by identifying the elements whose pre-images through ϕ_1 and ϕ_2 coincide. In general, such an I-pasting will be an *orthocomplemented partial order* [14]. This is, however, not sufficient to guarantee that it is in fact an OMP. The following example shows that it can fail to be.

Example 2. *With reference to Figure 3. Let $I = \{0, 1, x, x'\}$ be an* OMP *with $0 \leq x, x' \leq 1$. For $i = 1, 2$, let L_i be Boolean algebras with three atoms each $\{a_i, b_i, c_i\}$. Since ϕ_i are* OMP*-morphisms, $\phi_i(0) = 0_i$, and $\phi_i(1) = 1_i$. Now let $\phi_1(x) = c_1$, so that $\phi_1(x') = c_1' = a_1 \vee b_1$; and $\phi_2(x') = c_2$ so that $\phi_2(x) = c_2' = a_2 \vee b_2$. In this case, $\sim$ is the reflexive and symmetric closure of $\{(0_1, 0_2), (1_1, 1_2), (c_1, c_2'), (c_1', c_2)\}$, let $[x]$ denote its equivalence classes. In $L_1|I|L_2$, we have that $[a_1] \leq [c_1'] = [c_2] \leq [a_2']$. Hence, $[a_1]$ and $[a_2]$ are orthogonal, but they have no least upper bound. Indeed, $[a_1], [a_2] \leq [b_1'], [b_2']$ and $[b_1'] \wedge [b_2'] = [0]$.*

In what follows, we will study cases in which this composition is actually an OMP.

3.2 Extending a system with a sequential component

It is a known result [16] that whenever L_1 and L_2 are OMP's, and $I = \{0, 1\}$ is the trivial Boolean algebra, then $L_1|I|L_2$, as defined in the last subsection, is an OMP. It was further shown in [4] that in this case, if L_1 and L_2 are stable, then so must be $L_1|I|L_2$. This composition operation corresponds to the parallel composition of the corresponding operand systems. Indeed, the two systems are simply considered as a whole, although they remain independent, they do not synchronise or exchange information. Since the result of $L_1|I|L_2$ is stable, it can be itself an operand for a further composition, and so this composition can be iterated, in order to generate a wide class of stable systems. As an operation, it is associative and commutative.

We consider now the case in which the interface is a non-trivial Boolean algebra $I = \{0, 1, y, y'\}$ whose atoms are: $\mathcal{A}(I) = \{y, y'\}$. With such an interface, we impose that the corresponding saturated transition systems synchronise according to the specified embeddings. One of the components will be a finite Boolean algebra, and will be denoted B. Boolean algebras considered as OMP's were shown to be stable in [4]. The proof of stability will require the notion of *free atom*. An atom is *free* in an OMP if it belongs to just one maximal Boolean subalgebra. When seen as the region of a CETS, a free atom is a minimal region belonging to a single regional partition.

Example 3. *With reference to Figure 2, b_1, b_2, b_4, b_5 and b_3 are all atoms, however b_1, b_2, b_4, b_5 are free, but b_3 is not. Indeed, b_3 belongs to two maximal Boolean algebras.*

The considered embeddings will then identify a free atom of an OMP with an atom of B.

Theorem 3. *Let L be an* OMP*, and $x \in \mathcal{A}(L)$ be an atom. Let B be a finite Boolean algebra, and $a \in \mathcal{A}(B)$. Let $I = \{0, 1, y, y'\}$, and define $\phi_L : I \to L$, and $\phi_B : I \to B$ such that $\phi_L(y) = x$, and $\phi_B(y) = a$. Then $L|I|B$ induced by the V-formation $(I, L, B, \phi_L, \phi_B)$ is an* OMP*.*

Proof. After Proposition 1, it suffices to show that orthogonal joins are well defined, and that the orthomodular law holds. First note that in this case, the only identifications are $[0] = \{0_L, 0_B\}, [1] = \{1_L, 1_B\}, [x] = \{x, a\}$ and $[x'] = \{x', a'\}$, all other equivalence classes being singletons. Since both x and a are atoms, we have that $\leq = \prec$, in the setting of Definition 12. Furthermore, for each pair of orthogonal elements $[c] \perp [d]$, there must be $c \in L' \cap [c]$ and $d \in L' \cap [d]$, where $L' \in \{L, B\}$ such that $c \perp d$ in L'. If this holds for both $L' = L$ and $L' = B$, then the only possibility is $[c] = [x]$ and $[d] = [x']$, for which the join must be $[1]$, and is well defined. Now, from the Definition 12 (4.) of $\prec$, it follows that for every pair of ordered elements $[c] \leq [f]$, there must be one $L' \in \{L, B\}$ such that $c \in L' \cap [c]$ and $f \in L' \cap [f]$, with $c \leq f$. Now, this ensures, on one hand, that orthogonal joins (and meets) are well defined in the $I-pasting$, whenever they are well-defined on L and B. Indeed if $c \in L' \cap [c]$ and $f \in L' \cap [f]$, with $c \leq f$ holds for both $L' = L$ and $L' = B$, ϕ'_L preserving order, it must be either $c = 0_{L'}$ or $f = 1_{L'}$.

On the other hand, since L' is an OMP, then $c \leq f$ implies that $f = c \vee (f \wedge c')$, hence $[f] = [c] \vee ([f] \wedge [c]')$. ☐

In the following, $L|I|B$ will refer to this particular construction, and L will be assumed to be stable. Furthermore, we will suppose that $\phi_L(y) = x$ is a free atom, and show that $L|I|B$ is stable whenever L is.

We start defining $J'(L) = (Q'_L, E'_L, T'_L)$ in the following way:

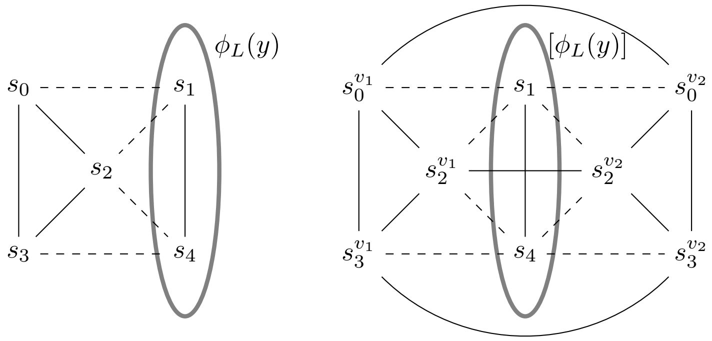
$$\begin{aligned} Q'_L &= S_y \cup \{s \cup \{v_i\} | s \in S_{y'}, \phi_B(y) \neq v_i \in \mathcal{A}(B)\} \\ E'_L &= \{[s, s'] | s, s' \in Q'_L, s \neq s'\} \\ T'_L &= \{(s, [s, s'], s') | s, s' \in Q'_L, [s, s'] \in E'_L, s \neq s'\}. \end{aligned}$$

Lemma 3. *$J'(L)$ is isomorphic to $J(L|I|B)$. Furthermore, for every $v_i \in \mathcal{A}(B)$ such that $\phi_B(y) \neq v_i$, the subgraph of $J'(L)$ induced by $S(y) \cup S(v_i)$ is isomorphic to $J(L)$.*

Proof. Every state $q \in Q'_L$ contains one, and only one, atom of B and since $\phi_L(y)$ is a free atom, it has one, and only one, atom for every Boolean algebra of L. Hence the elements of Q'_L coincide with the elements of $S(L|I|B)$. Since the construction of $J'(L)$ is completely determined by the set of states as in the construction of $J(L|I|B)$, the two transition systems $J(L|I|B)$ and $J'(L)$ are isomorphic.

We also observe that for every atom $v_i \in \mathcal{A}(B)$ distinct from $\phi_B(y)$ the elements in $S(y) \cup S(v_i)$ coincide with the elements of $S(y) \cup S(y')$, which is the set of states of $J(L)$. From this observation it is easy to see that there is an isomorphism between $J(L)$ and any subgraph of $J'(L)$ induced by a set of states in the form $S(y) \cup S(v_i)$. ☐

This last lemma will permit to consider $J'(L)$ instead of $J(L|I|B)$.



$J(L)$ $\qquad$ $J'(L) \cong J(L|I|B)$

Figure 4: Construction of $J'(L)$, where L is the OMP of Figure 2, I is as in Theorem 3, and B is a Boolean algebra with three atoms: $\mathcal{A}(B) = \{\phi_B(y), v_1, v_2\}$. Lines represent transitions in both directions. Dashed lines have an incidence with respect to $\phi_L(y)$ or $[\phi_L(y)]$, whereas solid lines are independent from them.

Lemma 4. *If a region $H \in \mathcal{R}(J'(L))$ contains a state $s \cup \{v_i\} \in Q'_L$ and $S_{v_i} \not\subseteq H$ then $\forall s \cup \{v_j\} \in Q'_L : s \cup \{v_j\} \in H$.*

Proof. Since $S_{v_i} \not\subseteq H$ there must be a state $s' \cup \{v_i\} \notin H$, which means that the event $[s, s']$ is an event labeling a transition exiting H. Now suppose that there is a state $s \cup \{v_j\} \in Q'_L$ that doesn't belong to H. This means that the transition from $s \cup \{v_j\}$ to $s' \cup \{v_j\} \in Q'_L$ does not exit H, but such a transition is also labeled $[s, s']$, which is not possible since that would mean that H is not a region. $\square$

Lemma 5. *Every atomic region of $J'(L)$ is in the form S_x for $x \in \mathcal{A}(L|I|B)$.*

Proof. Assume that there is an atomic region $H \notin \{S_x \mid x \in \mathcal{A}(L|I|B)\}$. Consider the subgraphs induced by $S_y \cup S_{v_i}$ for all the $\phi_B(y) \neq v_i \in B$ and call $H_{v_i} \subset H$ the sets $H \cap (S_y \cup S_{v_i})$. All the H_{v_i} are atomic regions in every subgraph of $J'(L)$ induced by $S_y \cup S_{v_i}$, since they are all isomorphic to $J(L)$. The regions H_{v_i} can be atomic or not. If they are atomic, then they must coincide with an atomic region S_x, with $y \neq x \in L_n$. Hence, after Lemma 4, $H \in \{S_x\}_{x \in L_{n+1}}$. If they are not atomic, then there are $H'_{v_1} \subset H_{v_1}$,...,$H'_{v_k} \subset H_{v_k}$ from which we can make the region $\bigcup_{i \in \{1,..,k\}} H'_{v_i} \subset H$, hence H is not atomic. $\square$

Theorem 4. *Let L be a stable* OMP*, and $x \in \mathcal{A}(L)$ be a free atom. Let B be a finite Boolean algebra, and $a \in \mathcal{A}(B)$. Let $I = \{0, 1, y, y'\}$, and define $\phi_L : I \to L$, and $\phi_B : I \to B$ such that $\phi_L(y) = x$, and $\phi_B(y) = a$. Then $L|I|B$ induced by the V-formation $(I, L, B, \phi_L, \phi_B)$ is stable.*

Proof. We wish to show that $H(J(L|I|B)) \simeq L|I|B$. With Lemma 3, we reduce it to showing that $H(J'(L)) \simeq L|I|B$. Since $H(J'(L))$ is a finite OMP, it is characterised by the orthogonality relation among its atoms. Now, Lemma 5 states that each atom of $H(J'(L))$ corresponds to an atom of $L|I|B$, and it was shown in [4] that every atom of $L|I|B$ must be an atom of $H(J'(L))$.

For each pair of elements $x \in \mathcal{A}(L)$, $y \in \mathcal{A}(B)$, there is a state in $s \in J(L|I|B)$ such that $[x] \in s$ and $[y] \in s$. Hence, the pasting must preserve incompatibility. Since the pasting also preserves orthogonality, we have that $L|I|B$, and $H(J(L|I|B))$ have same collection of atoms, with identical orthogonality relations. As it was shown in [5], this is sufficient to state that $H(J(L|I|B)) \simeq L|I|B$ □

3.3 Stability of Atom Refinement

The operation of refining an atom of an OMP into two new atoms preserves stability.

Theorem 5. *Let L be a stable* OMP*. Let $x \in \mathcal{A}(L)$. Consider $M_a = (\mathcal{A}(L) \setminus \{x\}) \cup \{y, z\}$, in which all orthogonal atoms of L remain orthogonal in M_a, and all atoms orthogonal to x in L are orthogonal to both y and z in M_a. Then the* OMP *M generated by M_a is stable.*

Proof. We will consider only the atoms of L and M, and states as represented by maximal cliques of $\not\perp$ as in Definition 7. Let $S_{x'}$ be the set of states of L not containing x. By construction of M_a, for each state $s \in S_x$ of L there are two states of M, in S_y and S_z respectively. Furthermore, the states of $S_{x'}$ all contain an atom orthogonal to x in L, and it will be orthogonal to both y and z in M. Thus, $S_{x'}, S_y$ and S_z constitute a partition of the states of M.

Starting from the states of M as partitioned above, it is possible to define the following sets of events: $E_{x'} = \{[s, s'] \mid s, s' \in S_{x'}, s \neq s'\}$, $E_{y,z} = \{[s, s'] | s \in S_y, s' \in S_z\}$, $E_y = \{[s, s'] | s, s' \in S_y\}$, $E_z = \{[s, s'] | s, s' \in S_z\}$, $E_{x',y} = \{[s, s'] \mid s \in S_{x'}, s' \in S_y\}$ and $E_{x',z} = \{[s, s'] \mid s \in S_{x'}, s' \in S_z\}$.

Let A_y be the transition system with the following sets of states and events: $S_{x'} \cup S_y$ and $E_{x'} \cup E_y \cup E_{x',y}$, let, symmetrically, A_z be the transition system whose states are $S_{x'} \cup S_z$ and whose events are $E_{x'} \cup E_z \cup E_{x',z}$. We note that both $\mathcal{R}(A_y)$ and $\mathcal{R}(A_z)$ are isomorphic to the regions of the saturated system $J(L)$ since in both cases of $\mathcal{R}(A_y)$ and $\mathcal{R}(A_z)$, atoms y and z replace uniformly x. Moreover, since states S_y and S_z are disjoint, it is possible to construct the CETS $A = A_y \cup A_z$ endowed by all the new events in $E_{y,z}$. We note that $\mathcal{R}(A)$ must contain $\mathcal{R}(J(M))$ since CETS A, having less events than $J(M)$, can have less constraints in the construction of its regions. We want to show that $\mathcal{R}(A) = \mathcal{R}(J(M))$. Let, by contradiction, r be a region in $\mathcal{R}(A)$ not belonging to $\mathcal{R}(J(M))$.

If $r \subseteq S_{x'}$, then $r \in \mathcal{R}(J(M))$ since all the labels in $E_{x'}$ belong to both CETS A and $J(M)$ and the new events in $E_{x',y}$ and $E_{x',z}$ are distinct copies of the original events $E_{x',x}$ in $J(L)$, so they do not create new regions. If $r \subseteq S_y$ and, symmetrically, for $r \subseteq S_z$ then r must be a region in $\mathcal{R}(J(M))$ since all the labels in E_y and $E_{x',y}$, and symmetrically E_z and $E_{x',z}$ are, by construction, isomorphic to the labels $E_{x',x}$ in $J(L)$ and all the new labels $E_{y,z}$ are exiting from or, respectively, entering in r. The only remaining case could be for r being a minimal region in $\mathcal{R}(A)$ and a non minimal region in $\mathcal{R}(J(M))$ but this would be in contradiction with y and z being atoms in M. □

Example 4. *Consider three Boolean algebras B_1, B_2 and B_3 with three atoms each, $\mathcal{A}(B_i) = \{a_i, b_i, c_i\}$ for $i = 1, 2, 3$. Let $I = \{0, x, x', 1\}$ and consider the two* OMP*-morphisms $\phi_i : I \to B_i$ $(i = 1, 2)$, such that $\phi_i(x) = c_i$. B_1 is a stable*

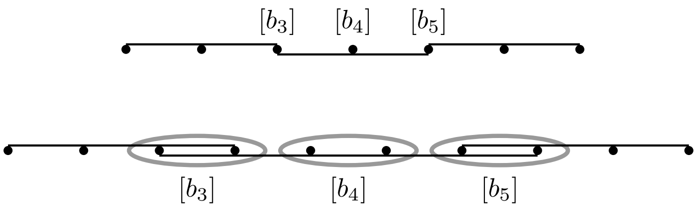


Figure 5: Greechie diagrams of two OMPs. The OMP depicted below, is obtained from the one depicted above, by refining the atoms as shown. Since the OMP above is stable, so is the one below.

logic, and c_1 is clearly a free atom, so after Theorem 4, $L = B_1|I|B_2$ is a stable OMP. *L is isomorphic to the* OMP *in Figure 2, by considering $b_3 = [\phi_1(x)] = [\phi_2(x)]$. Since L is stable, and b_5 is a free atom, we can compose it with B_3, by means of the morphisms ϕ_L and ϕ_3, provided by $\phi_L(x) = b_5$ and $\phi_3(x) = c_3$. The Greechie diagram of $L|I|B_3$ is depicted at the top of Figure 5. Thanks to Theorem 5, we can now split any atom of $L|I|B_3$, obtaining, for example, the stable* OMP *depicted at the bottom of Figure 5.*

A free atom of the operands can be refined both after and before the composition operation, in this second way, only one of the two refining atoms will be used as interface with the appended sequential component, the other one remaining free for further composition. With this method, one can iterate the composition operation without worrying about exhaustion of available free atoms of the original system.

Example 5. *Consider a system made of two sequential components each of which can get to a state for which they require the same resource. If each of these components can be in two additional states, the regional* OMP *for this system is represented as L_1 in Figure 6. B_1 and B_2 represent the two sequential components, and x_1, x_2 correspond to their mutually exclusive states. B_c represents the state of the resource c, it can be in state x_i, indicating that B_i holds c, or in state y_1, indicating that c is available. When c is available, no other component is involved in the coresponding local state of B_c, and so y_1 is a free atom. L_1 is isomorphic to the* OMP *at the top of Figure 5, and was shown to be stable in Example 4. We may want to make the resource c available for a third sequential component B_3, so we can use Theorem 4 to compose L_1 and B_3 on y_1, obtaining the stable $L_2 = L_1|I|B_3$ a shown in Figure 6. However, in this new compound system, the resource must be held by one of the three components B_i, as B_c has no more free atoms. No additional component can be added to the system, to compete for c. Instead of performing the composition $L_1|I|B_3$ directly, we can first make use of Theorem 5, and refine y_1 in L_1 into two new free atoms x_3, y_2, thus obtaining the* OMP *L_3 of Figure 6. We can now compose it with B_3 on x_3, thus obtaining $L_4 = L_3|I|B_3$, which is stable. One can see that y_2 remains a free atom, a local state representing that the resource is available. This process can be iterated, to obtain a system with n sequential components competing for the same resource.*

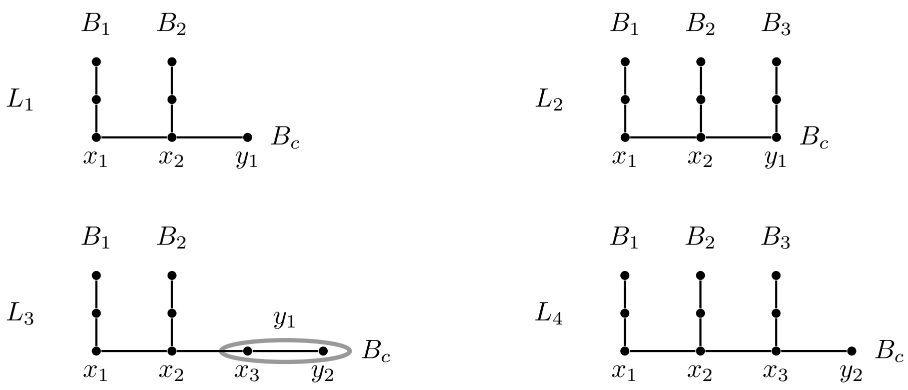


Figure 6: Refining a free atom

4 Conclusion

In [4], a collection of regional OMPs were shown to be stable. Furthermore, the parallel composition operation was shown to preserve stability. In the present work, we have formalised two additional operations which preserve stability. One corresponds to the refinement of a local state into two. The second corresponds to the extension of a system with a sequential component which synchronises with it over a local state which is not already a synchronisation. With these elements, we can define an algebra of system OMPs, such that all its elements are stable.

However, not every regional OMP can be obtained with the defined operations. For instance, a strong limitation is the restriction of the composition operation to interfacing on free atoms. Another limitation of this operation is that it does not allow for extending a system with a sequential component that would synchronise with the system at more than one state. A clear goal in our approach is to extend the composition operation so as to generate every regional OMP. In this way, by showing that we can generate all regional OMPs with stability preserving operations, we would prove the conjecture that all regional OMPs are stable.

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