Multi-phase averaging of time-optimal low-thrust transfers

L. Dell'Elce[1], J.-B. Caillau[2], J.-B. Pomet[1]

An increasing interest in optimal low-thrust orbital transfers was triggered in the last decade by technological progress in electric propulsion and by the ambition of efficiently leveraging on orbital perturbations to enhance the maneuverability of small satellites.

The assessment of a control sequence that is capable of steering a satellite from a prescribed initial to a desired final state while minimizing a figure of interest is referred to as maneuver planning. From the dynamical point of view, the necessary conditions for optimality outlined by the infamous Pontryagin maximum principle (PMP) reveal the Hamiltonian nature of the system governing the joint motion of state and control variables.

Solving the control problem via so-called indirect techniques, e.g., shooting method, requires the integration of several trajectories of the aforementioned Hamiltonian. In addition, PMP conditions exhibit very high sensitivity with respect to boundary values of the satellite longitude owing to the fast-oscillating nature of orbital motion. Hence, using perturbation theory to facilitate the numerical solution of the planning problem is appealing. In particular, averaging techniques were used since the early space age to gain understanding into the long-term evolution of perturbed satellite trajectories. However, it is not generally possible to treat low-thrust as any other perturbation (whose spectral content is well defined and predictable) because the control variables may introduce additional frequencies in the system.

The talk focuses on time optimal maneuvers in a perturbed orbital environment, and it addresses two questions: (1) *Is it possible to average the vector field of this problem?* Optimal control Hamiltonians are not in the classical form of fast-oscillating systems. However, we demonstrate that averaged trajectories well approximate the original system if the adjoint variables of the PMP (i.e., conjugate momenta associated to the enforcement of the equations of motion) are adequately transformed before integrating the averaged trajectory. We discuss this transformation in detail, and we emphasize fundamental differences with respect to well-known mean-to-osculating transformations of uncontrolled motion. (2) *What is the impact of orbital perturbations and their frequencies on the controlled trajectory?* We show that control variables are highly sensitive to small exogenous forces. Hence, even the crossing of a high-order resonance may trigger a dramatic divergence between trajectories of the averaged and original system. We then discuss how averaged resonant forms may be used to avoid this divergence.

The methodology is finally applied to a deorbiting maneuver leveraging on solar radiation pressure. The presence of eclipses make the original planning problem highly challenging. Averaging with respect to satellite and Sun longitudes drastically simplifies the extremal flow yielding an averaged counterpart of the PMP conditions, which is reasonably easy to solve.

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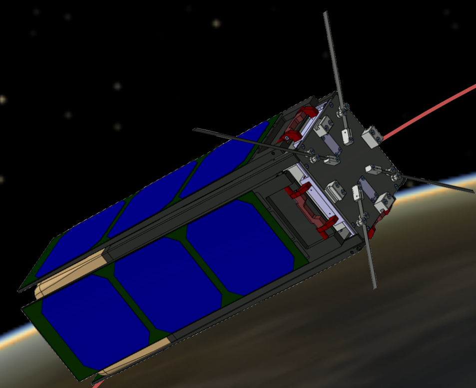
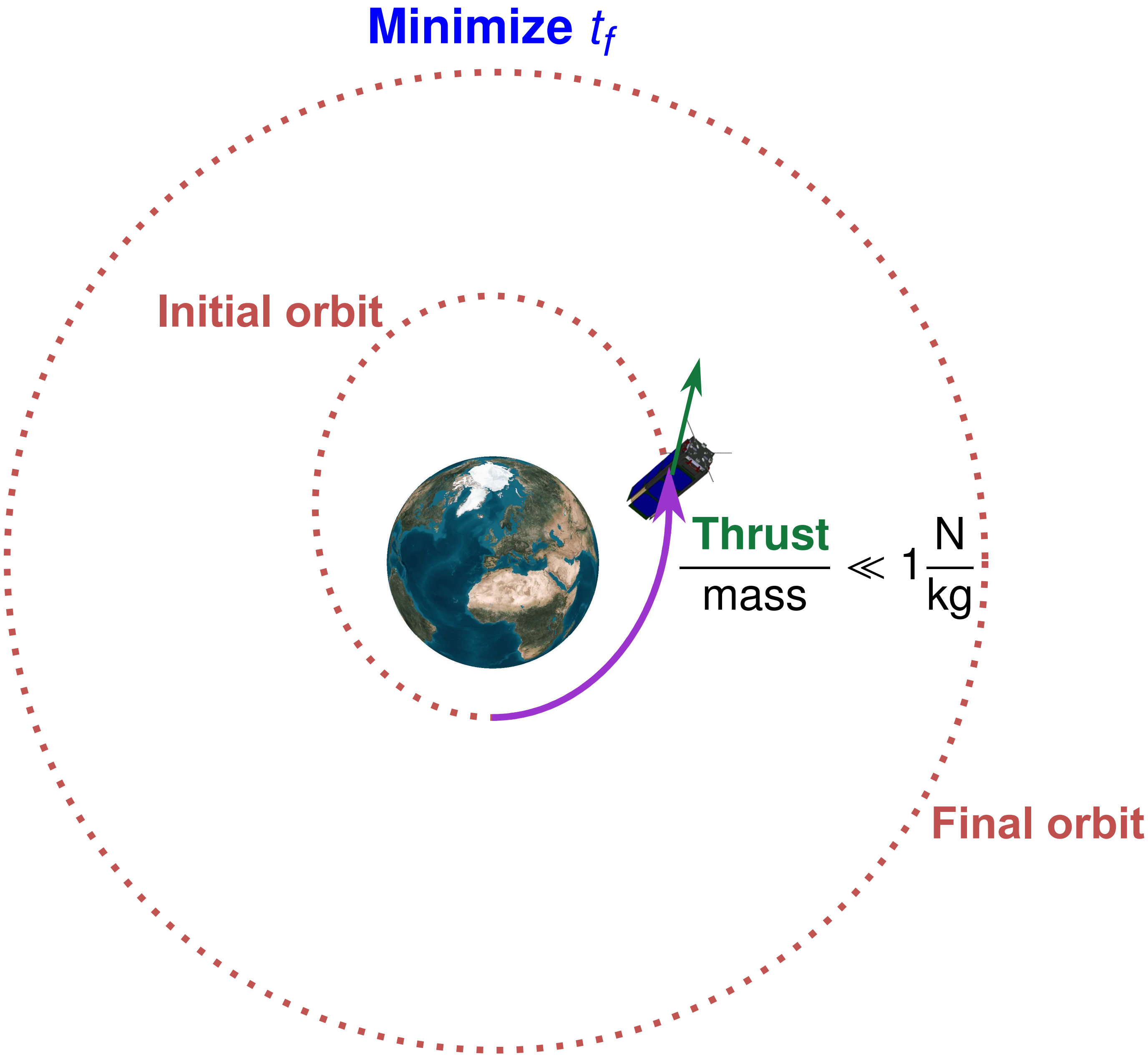


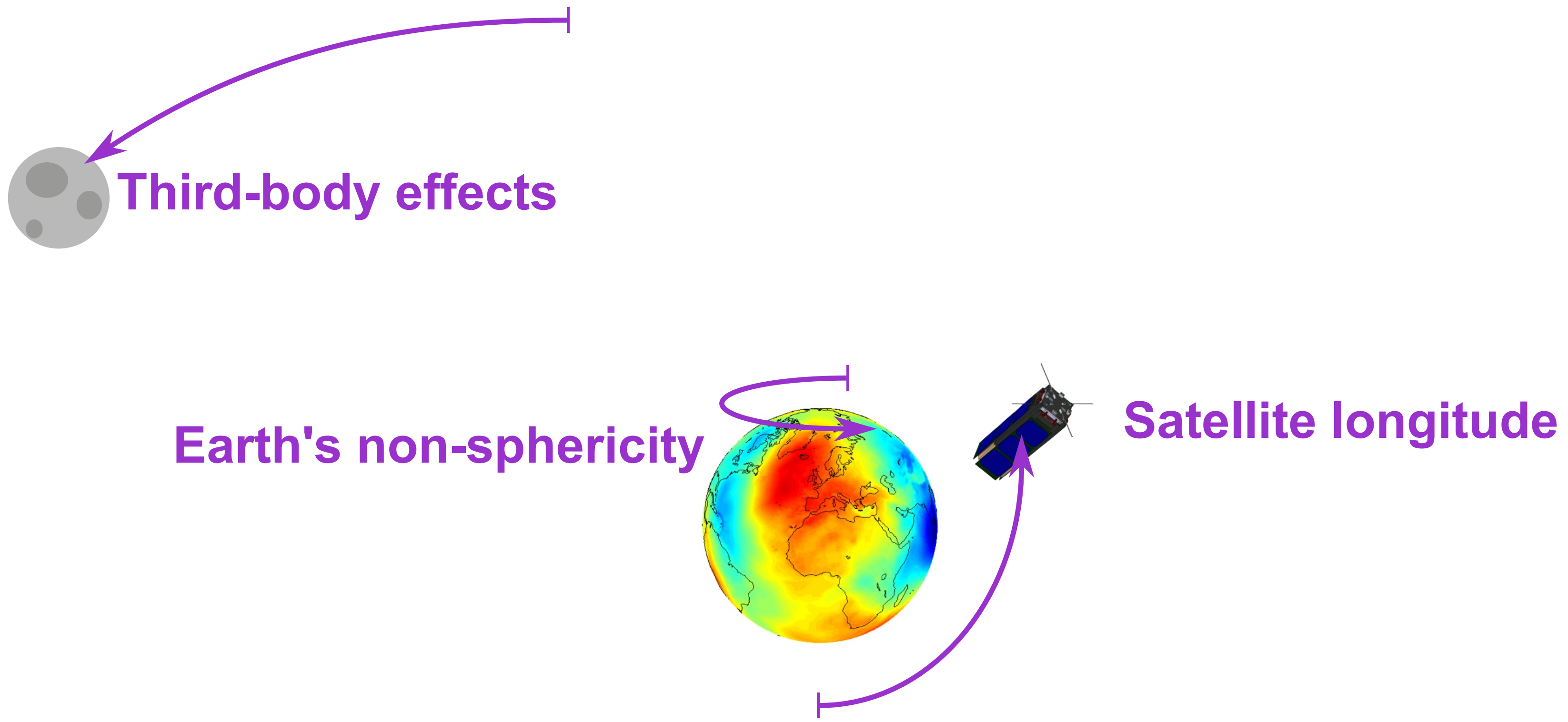
Logrono, 25/04/2019

INVENTEURS DU MONDE NUMÉRIQUE

Low-thrust transfer: a fast-oscillating control problem



Orbital perturbations may introduce new frequencies

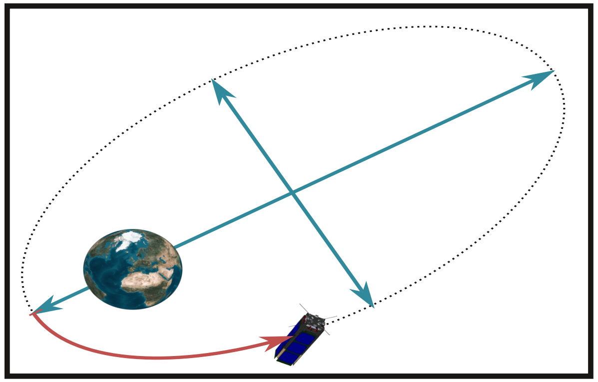


Objective: Simplify dynamics by averaging

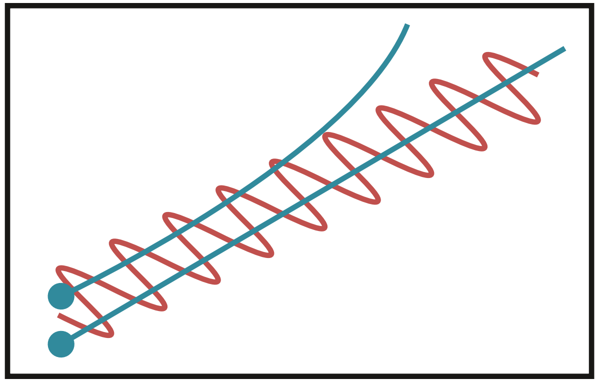
Motivation: Initial guess to shooting algorithms

Challenges: Do adjoint variables introduce additional fast dynamics?
What happens when resonances are crossed?

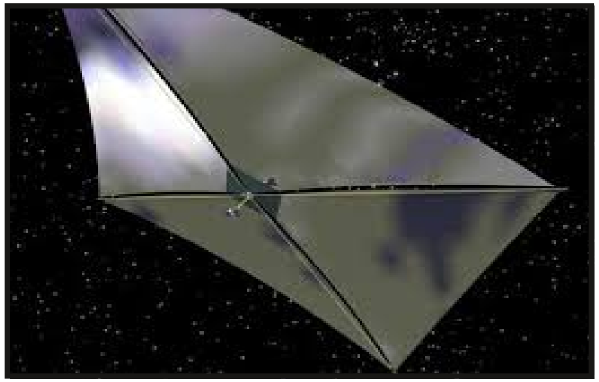
Outline



1. Minimum time control of fast oscillating systems

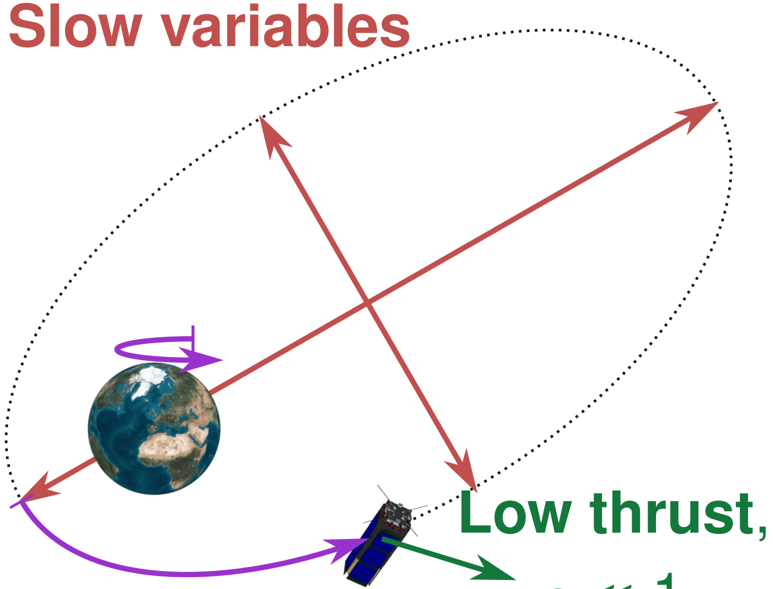


2. Averaging the optimal control Hamiltonian



3. Time optimal deorbiting of a solar sail

1. Minimum time control of fast oscillating systems



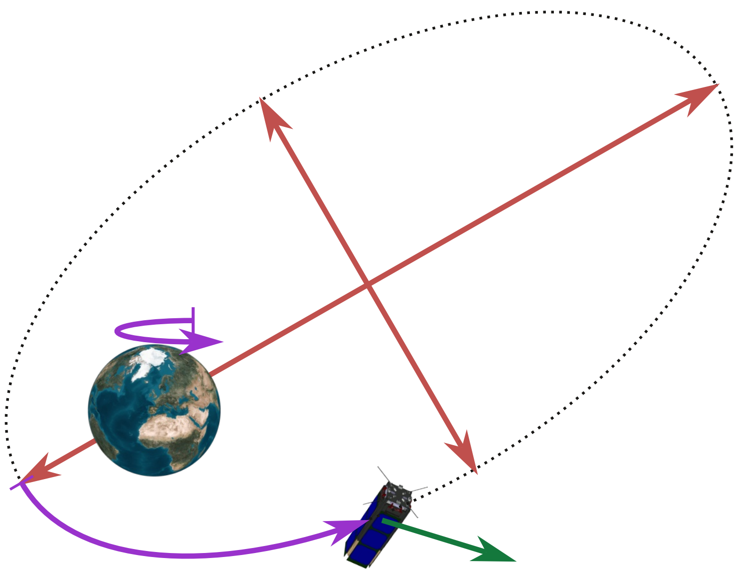
$$\min_{||\boldsymbol{u}||\leq 1} t_f \text{ subject to:}$$

$$\frac{d\boldsymbol{I}}{dt} = \epsilon \left[\boldsymbol{f}_0(\boldsymbol{I}, \phi) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{I}, \phi) u_i \right]$$

$$\frac{d\phi}{dt} = \omega(\boldsymbol{I})$$

$$\boldsymbol{I}(0) = \boldsymbol{I}_0$$
$$\boldsymbol{I}(t_f) = \boldsymbol{I}_f$$

1. Hamiltonian of the extremal flow

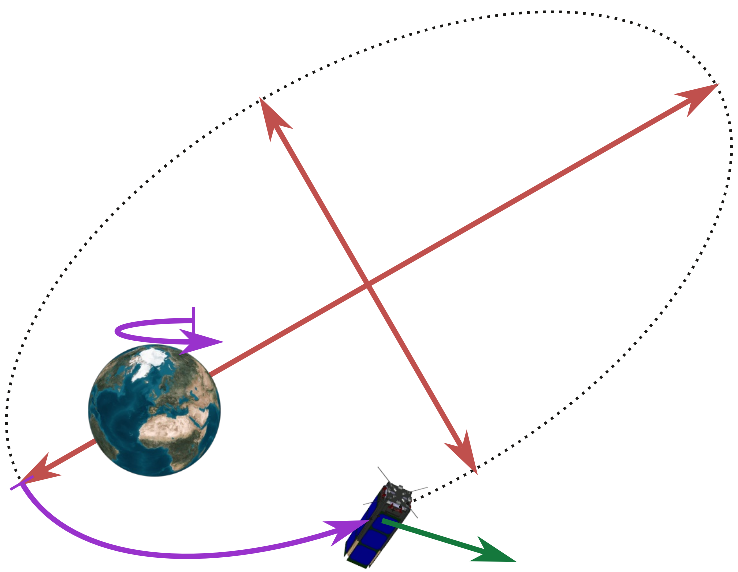


Denote by $\boldsymbol{p_I}$ and $\boldsymbol{p_\phi}$ the adjoints to $\boldsymbol{I}$ and ϕ

Define the pre-Hamiltonian

$$\mathcal{H}' = \omega(\boldsymbol{I}) \cdot \boldsymbol{p_\phi} + \epsilon \left[\boldsymbol{f}_0(\boldsymbol{I}, \phi) + \sum_{i=1}^{m} \boldsymbol{f}_i(\boldsymbol{I}, \phi) u_i \right] \cdot \boldsymbol{p_I}$$

1. Hamiltonian of the extremal flow



Denote by $\boldsymbol{p_I}$ and $\boldsymbol{p_\phi}$ the adjoints to I and ϕ

Define the pre-Hamiltonian

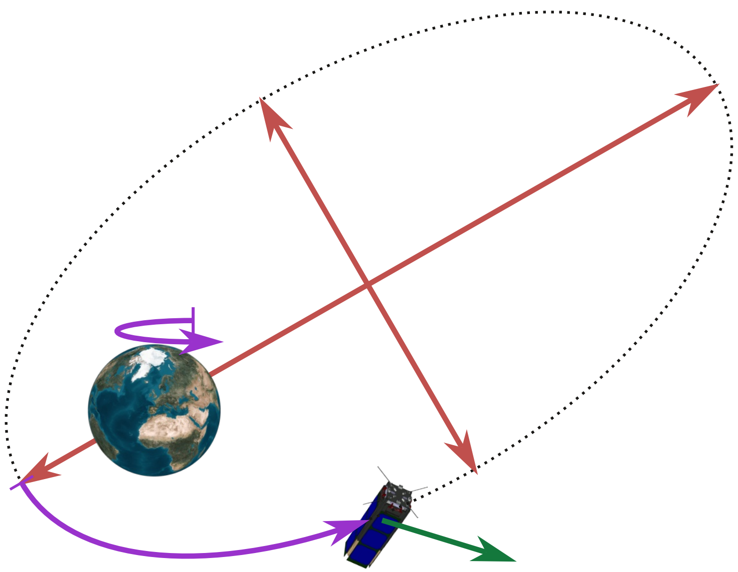
$$\mathcal{H}' = \omega(I)\cdot\boldsymbol{p_\phi} + \epsilon\left[\boldsymbol{f}_0(I,\phi) + \sum_{i=1}^{m}\boldsymbol{f}_i(I,\phi)u_i\right]\cdot\boldsymbol{p_I}$$

Apply Pontryagin maximum principle

$$\mathcal{H} = \max_{\|\boldsymbol{u}\|\leq 1}\mathcal{H}'(I,\phi,\boldsymbol{p_I},\boldsymbol{p_\phi},\boldsymbol{u})$$

$$= \omega(I)\cdot\boldsymbol{p_\phi} + \epsilon\underbrace{\left[\boldsymbol{f}_0(I,\phi)\cdot\boldsymbol{p_I} + \sqrt{\sum_{i=1}^{m}(\boldsymbol{f}_i(I,\phi)\cdot\boldsymbol{p_I})^2}\right]}_{\doteqdot K}$$

1. Necessary conditions for optimality



Boundary conditions

$$\boldsymbol{l}(0) = \boldsymbol{l_0} \qquad \boldsymbol{p_\phi}(0) = \boldsymbol{0}$$

$$\boldsymbol{l}(t_f) = \boldsymbol{l_f} \qquad \boldsymbol{p_\phi}(t_f) = \boldsymbol{0}$$

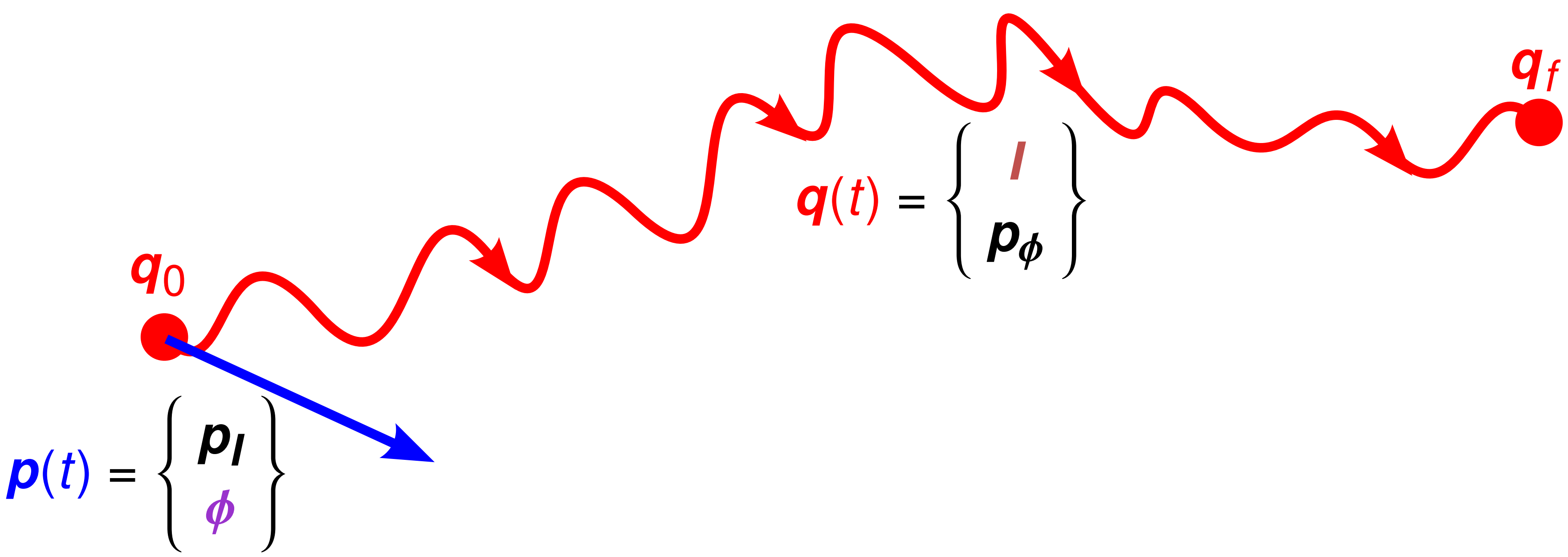
Equations of motion

$$\frac{\mathrm{d}\,\boldsymbol{l}}{\mathrm{d}\,t} = \frac{\partial\,\mathcal{H}}{\partial \boldsymbol{p_l}} \qquad \frac{\mathrm{d}\,\boldsymbol{p_l}}{\mathrm{d}\,t} = -\frac{\partial\,\mathcal{H}}{\partial\,\boldsymbol{l}}$$

$$\frac{\mathrm{d}\,\boldsymbol{\phi}}{\mathrm{d}\,t} = \frac{\partial\,\mathcal{H}}{\partial \boldsymbol{p_\phi}} \qquad \frac{\mathrm{d}\,\boldsymbol{p_\phi}}{\mathrm{d}\,t} = -\frac{\partial\,\mathcal{H}}{\partial \boldsymbol{\phi}}$$

1. Solution of the problem via shooting

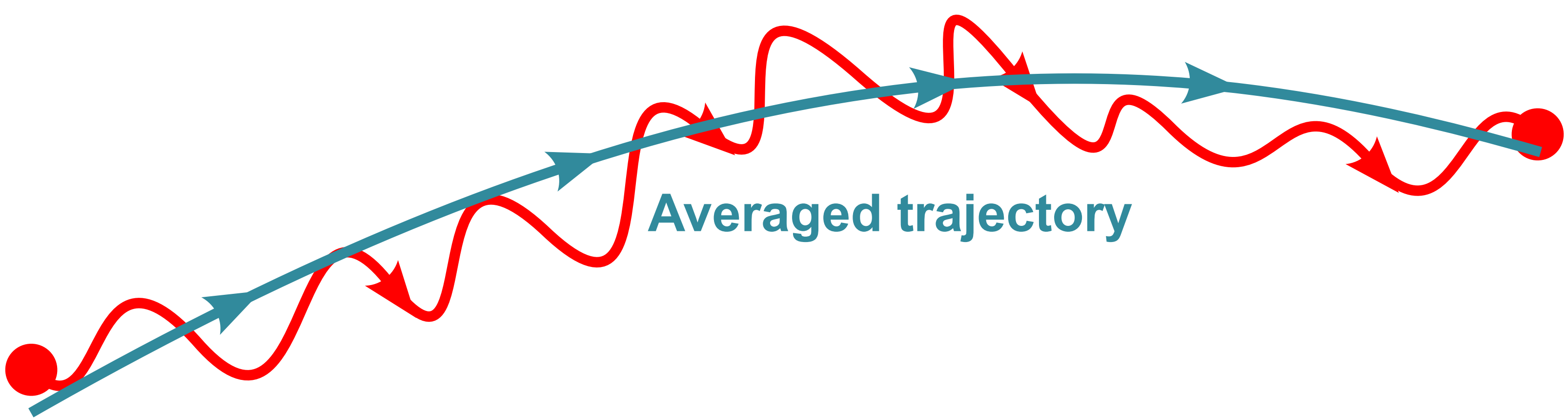
Find t_f and $\boldsymbol{p}(0)$ such that $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$



1. How averaging can facilitate the solution via shooting?

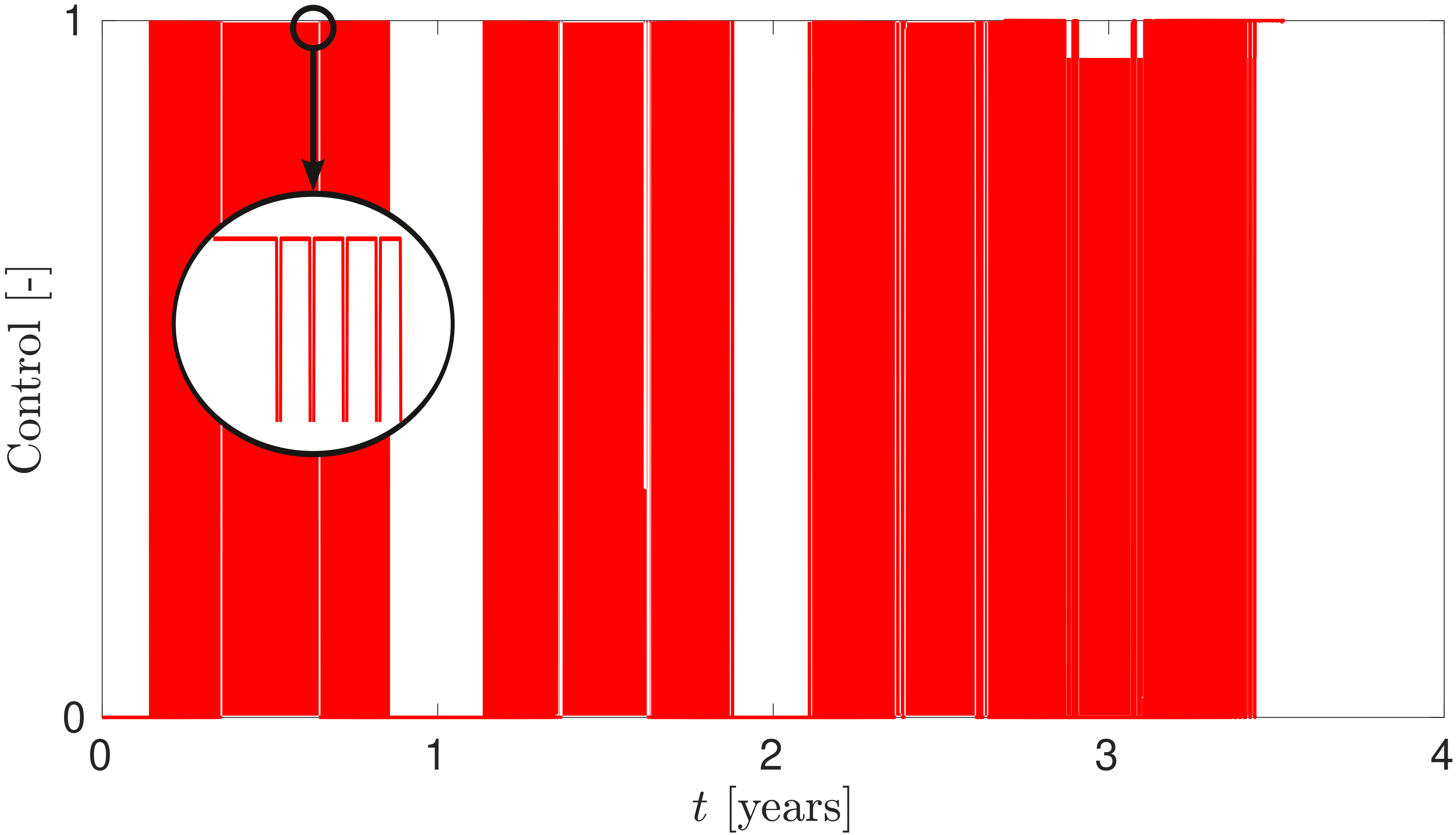
Smoothing: Less local minima, facilitates convergence

Reduced system: Independent of $\boldsymbol{\phi}$, $\boldsymbol{p_\phi}$ is constant

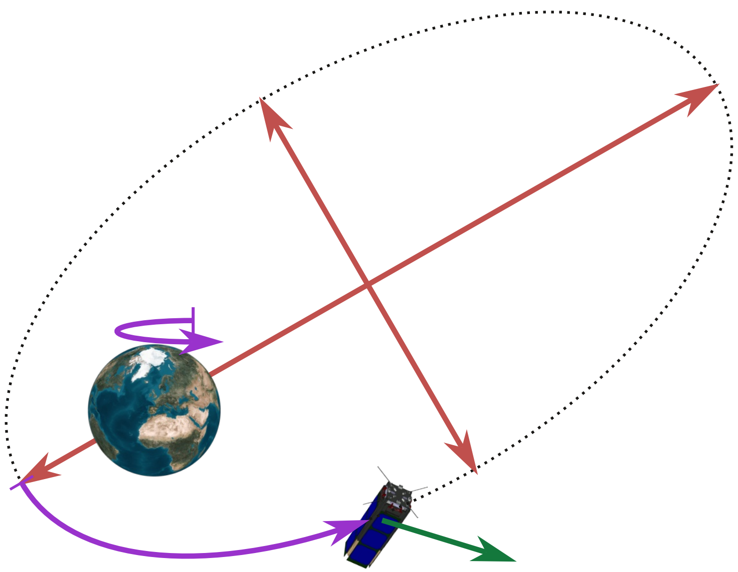


1. How averaging can facilitate the solution via shooting?

***A priori* knowledge of the control structure is not needed!**



2. Can we use averaging? Are adjoints **slow** or **fast**?



Hamiltonian:

$$\mathcal{H} = \boldsymbol{p}_{\boldsymbol{\phi}} \overset{?}{\cdot} \omega(\boldsymbol{I}) + \epsilon K\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p}_{\boldsymbol{I}}, \boldsymbol{p}_{\boldsymbol{\phi}}\right)$$

Equations of motion of the adjoints:

$$\frac{\mathrm{d}\,\boldsymbol{p}_{\boldsymbol{I}}}{\mathrm{d}\,t} = -\epsilon \frac{\partial K}{\partial \boldsymbol{I}} - \boldsymbol{p}_{\boldsymbol{\phi}} \overset{?}{} \frac{\partial \omega}{\partial \boldsymbol{I}}$$

$$\frac{\mathrm{d}\,\boldsymbol{p}_{\boldsymbol{\phi}}}{\mathrm{d}\,t} = -\epsilon \frac{\partial K}{\partial \boldsymbol{\phi}}$$

2. The averaged control system

Assume that $\boldsymbol{I}$ is in a non-resonant zone (i.e., incommensurate frequencies $\omega(\boldsymbol{I})$)

Averaged Hamiltonian

$$\overline{\mathcal{H}} = \int_{\mathbb{T}^r} \mathcal{H}\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p_I}, \boldsymbol{p_\phi}\right) \, \mathrm{d}\boldsymbol{\phi}$$
$$= \int_{\mathbb{T}^r} \left[\omega(\boldsymbol{I}) \cdot \boldsymbol{p_\phi} + \epsilon K\left(\boldsymbol{I}, \boldsymbol{\phi}, \boldsymbol{p_I}, \boldsymbol{p_\phi}\right)\right] \mathrm{d}\boldsymbol{\phi}$$

?

For trajectories of interest: $\boldsymbol{p_\phi}(t)$ is **ϵ-slow and ϵ-small** (not proven here)

2. A "non-conventional" fast-oscillating problem

Classical fast-oscillating system

$$\frac{\mathrm{d}\,\boldsymbol{x}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{f}(\boldsymbol{x}, \phi)$$
$$\frac{\mathrm{d}\,\phi}{\mathrm{d}\,t} = \omega(\boldsymbol{x})$$

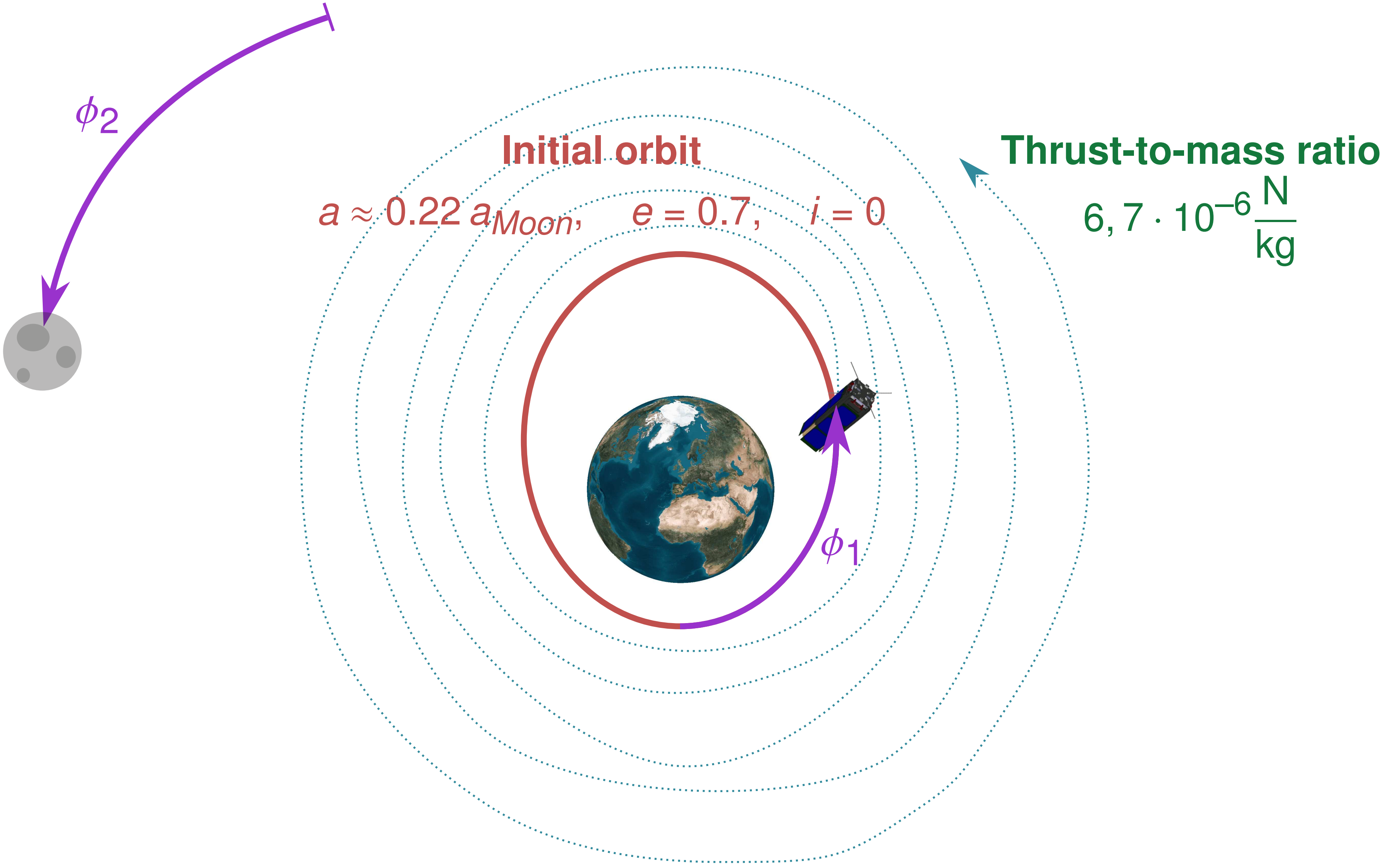
Problem studied in this talk

$$\frac{\mathrm{d}\,\boldsymbol{x}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{f}(\boldsymbol{x}, \phi, \boldsymbol{\eta}) + \boldsymbol{g}(\boldsymbol{x})\, \boldsymbol{\eta}$$
$$\frac{\mathrm{d}\,\boldsymbol{\eta}}{\mathrm{d}\,t} = \epsilon\, \boldsymbol{h}(\boldsymbol{x}, \phi, \boldsymbol{\eta})$$
$$\frac{\mathrm{d}\,\phi}{\mathrm{d}\,t} = \omega(\boldsymbol{x})$$

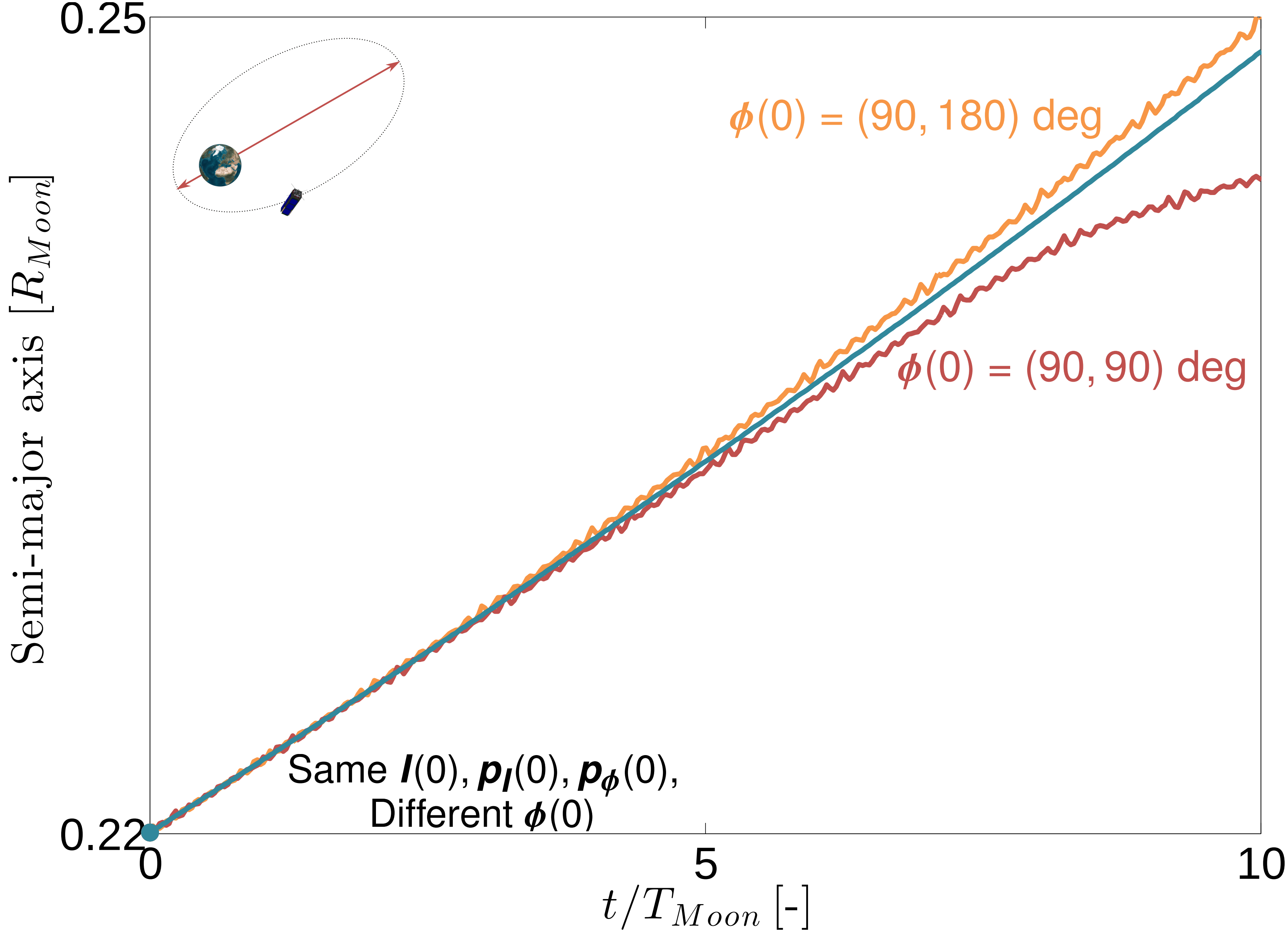
Initial conditions such that

$$\boldsymbol{\eta}(t) = \mathcal{O}(\epsilon) \quad \forall\ t \in [0, t_f]$$

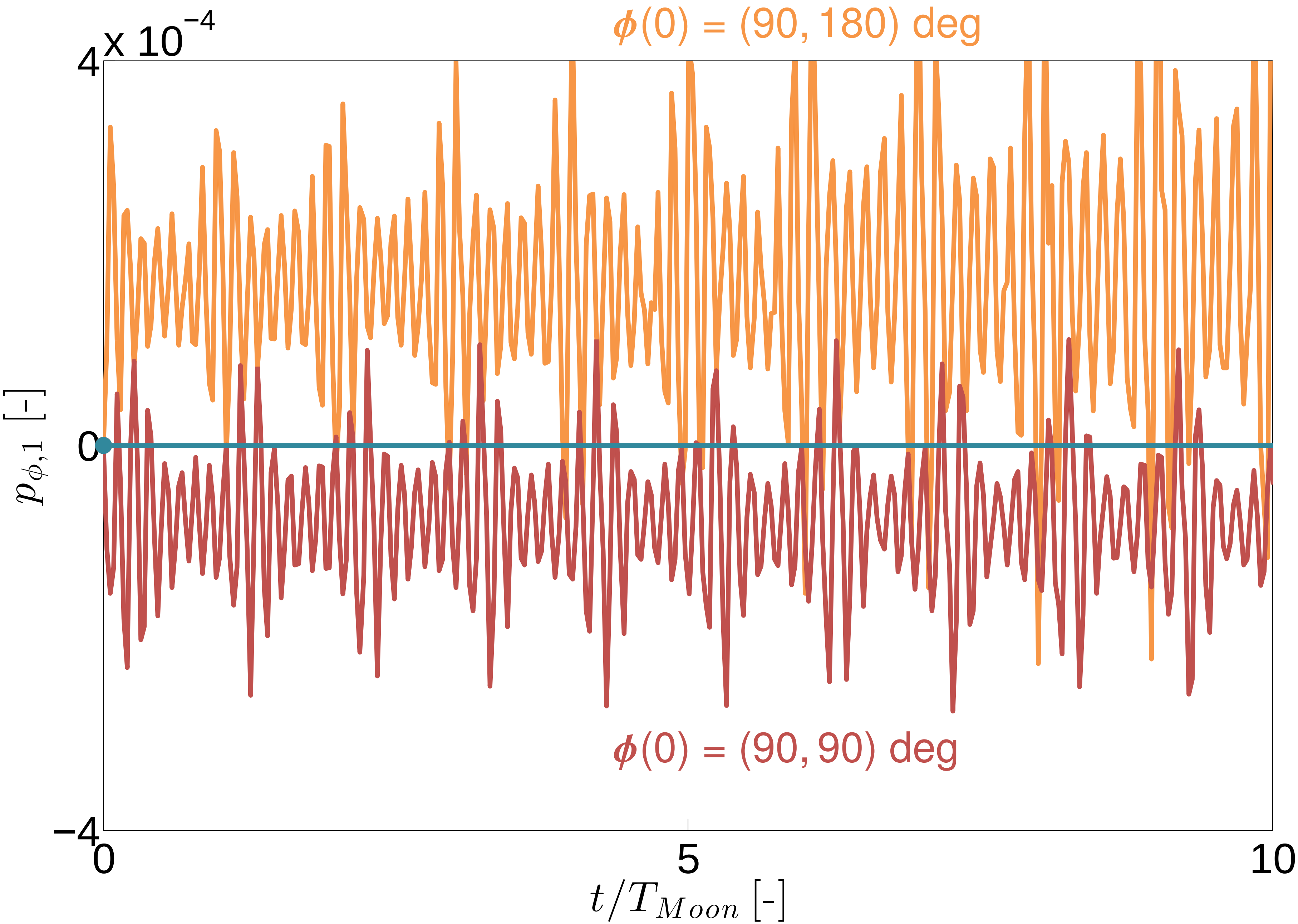
2. Case study: transfer in the Earth-Moon system



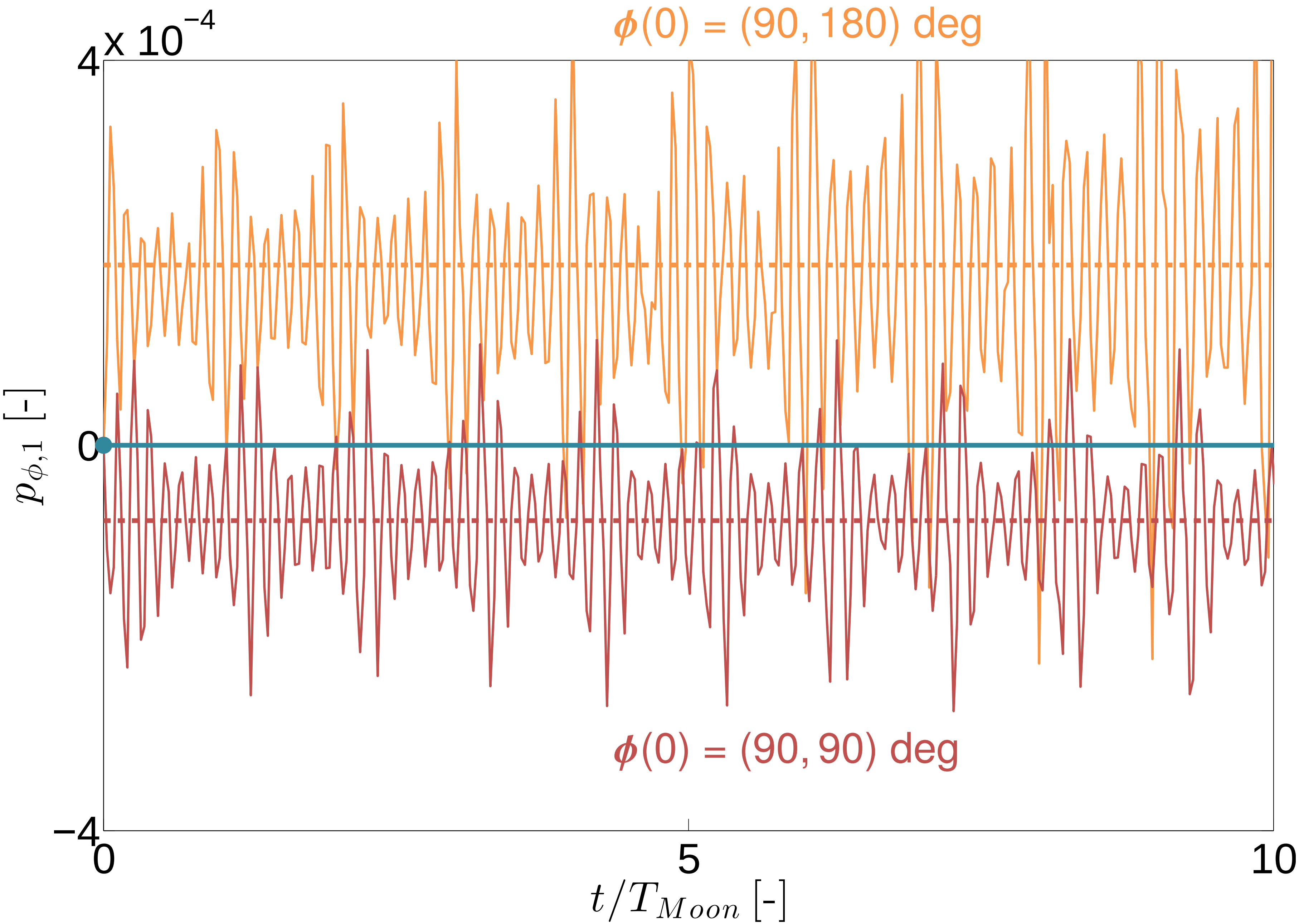
2. How to generate "reliable" averaged trajectories?



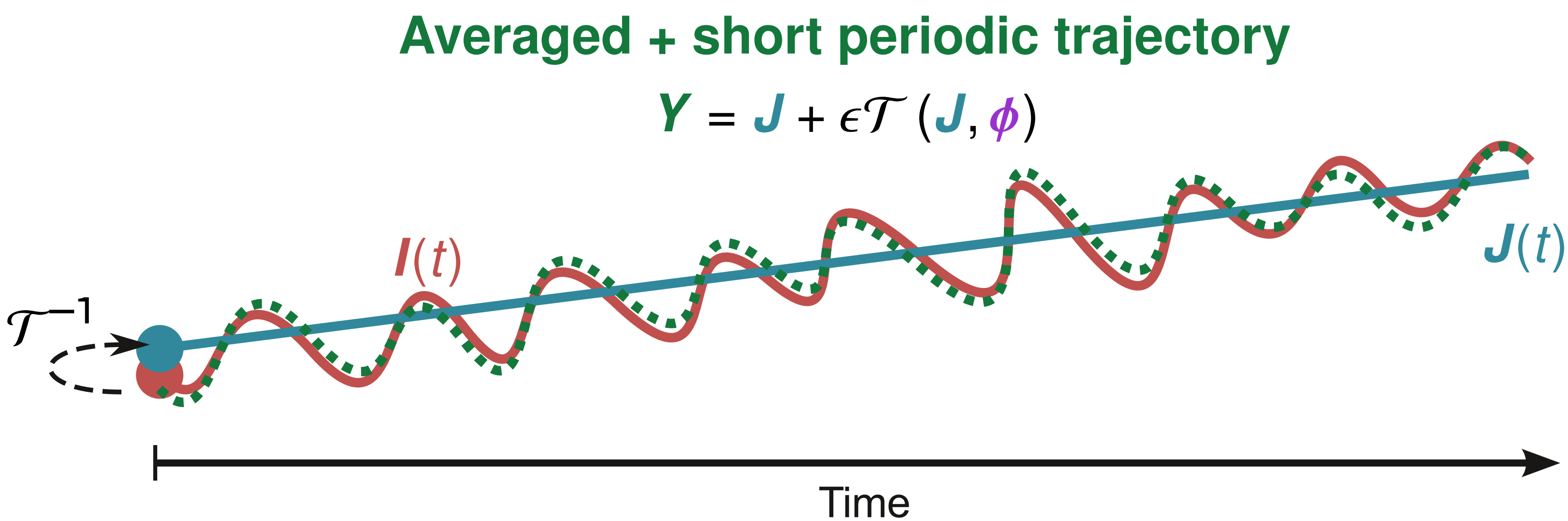
2. What makes the averaged control system different?



2. This is because $\boldsymbol{p_\phi}$ is constant and $\frac{\mathrm{d}\,\boldsymbol{p_I}}{\mathrm{d}\,t} = -\epsilon\frac{\partial K}{\partial \boldsymbol{I}} - \boldsymbol{p_\phi}\frac{\partial \omega}{\partial \boldsymbol{I}}$



2. Short-periodic variations

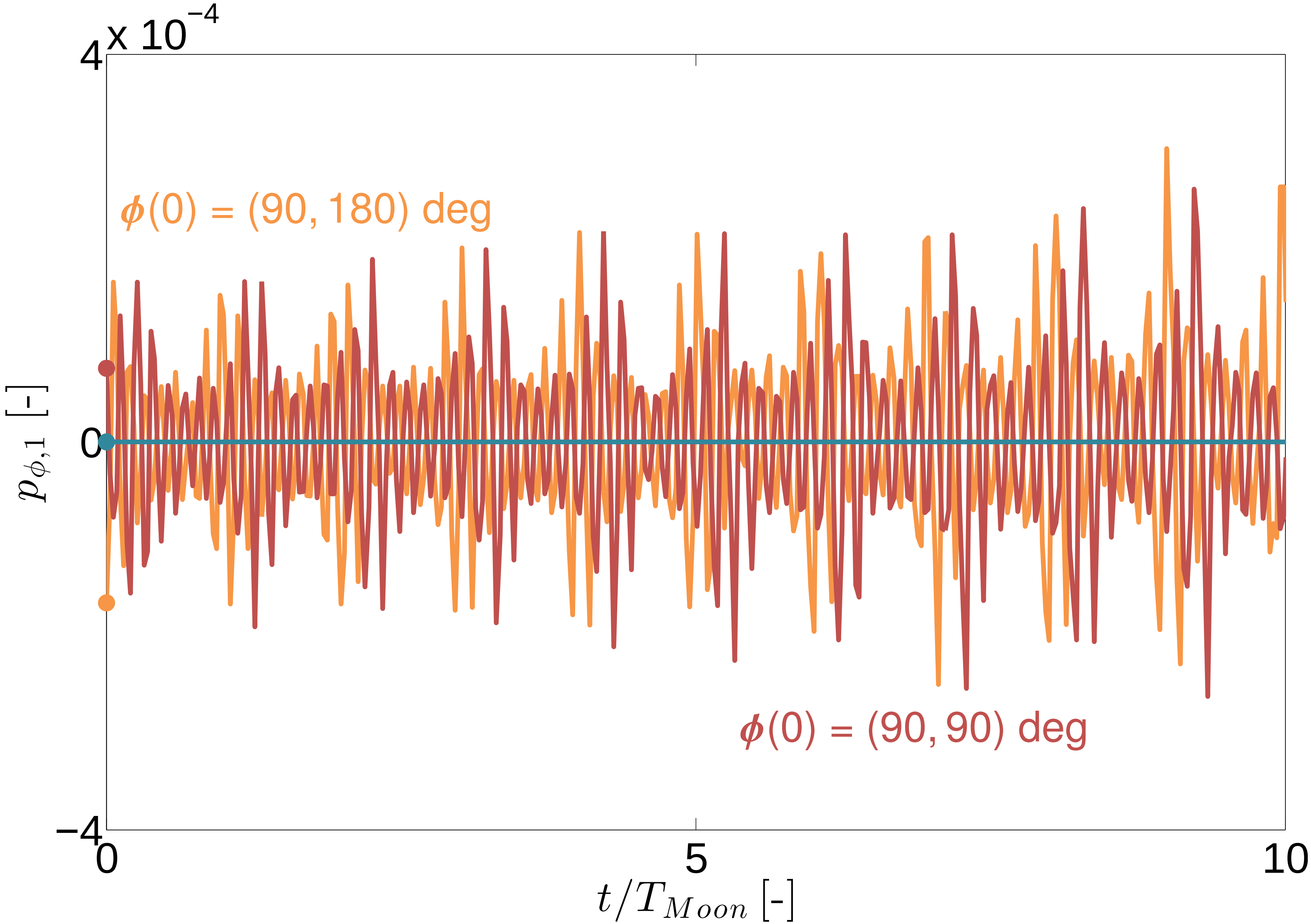


Near identity transformation:

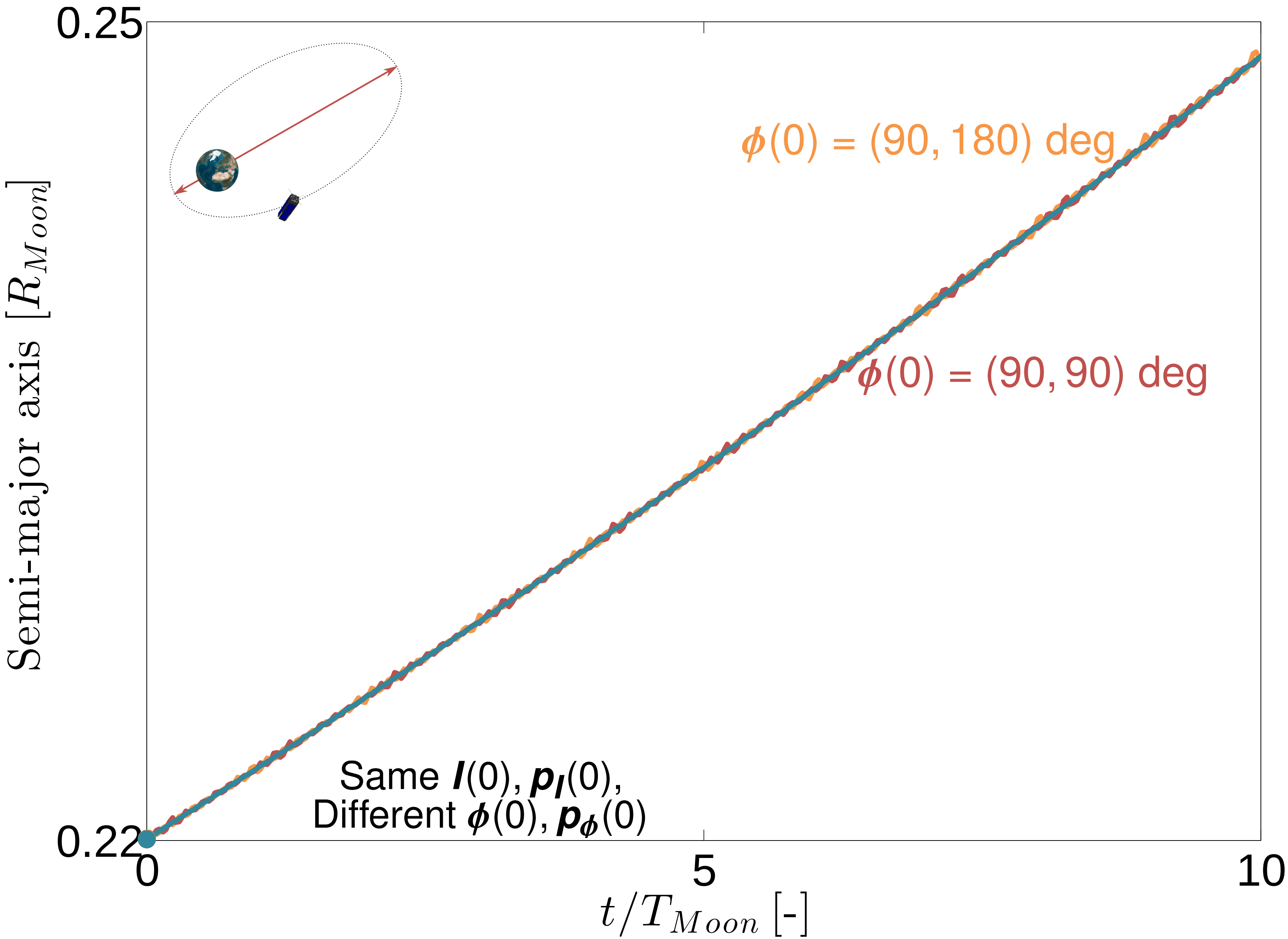
$$\mathcal{T}(J, \phi) = -i \sum_{0<|k|\leq N} \frac{c_k}{k \cdot \omega(J)} \exp(i k \cdot \phi)$$

Where c_k are Fourier coefficients of $\frac{\mathrm{d}\, I}{\mathrm{d}\, t}$

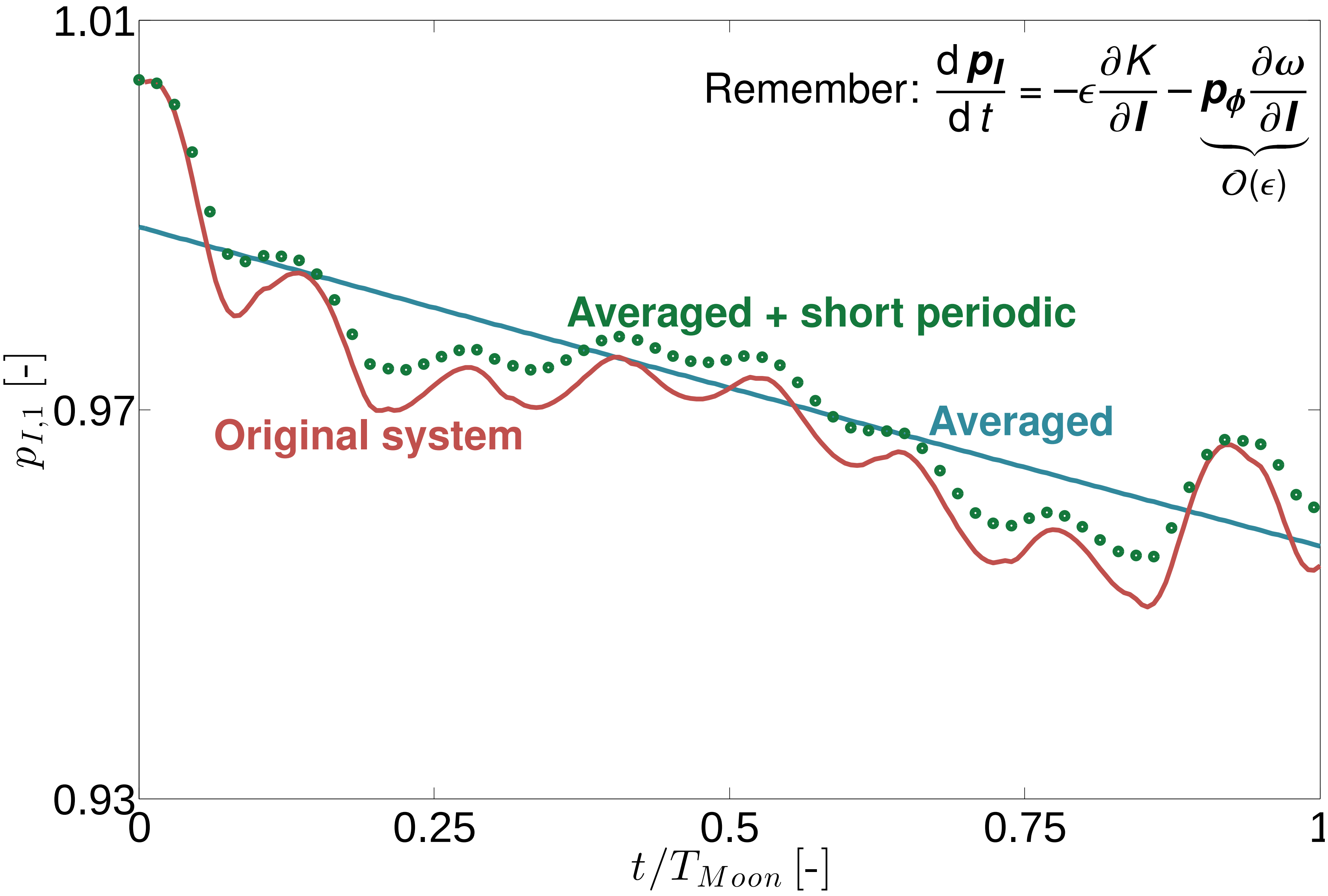
2. Transforming $\boldsymbol{p}_{\phi}(0)$ is the key



2. Transforming $\boldsymbol{p}_{\boldsymbol{\phi}}(0)$ is the key



2. The classical transformation is not adequate for $\boldsymbol{p_I}$



2. Nested transform for the short-periodic variations of $\boldsymbol{p_I}$

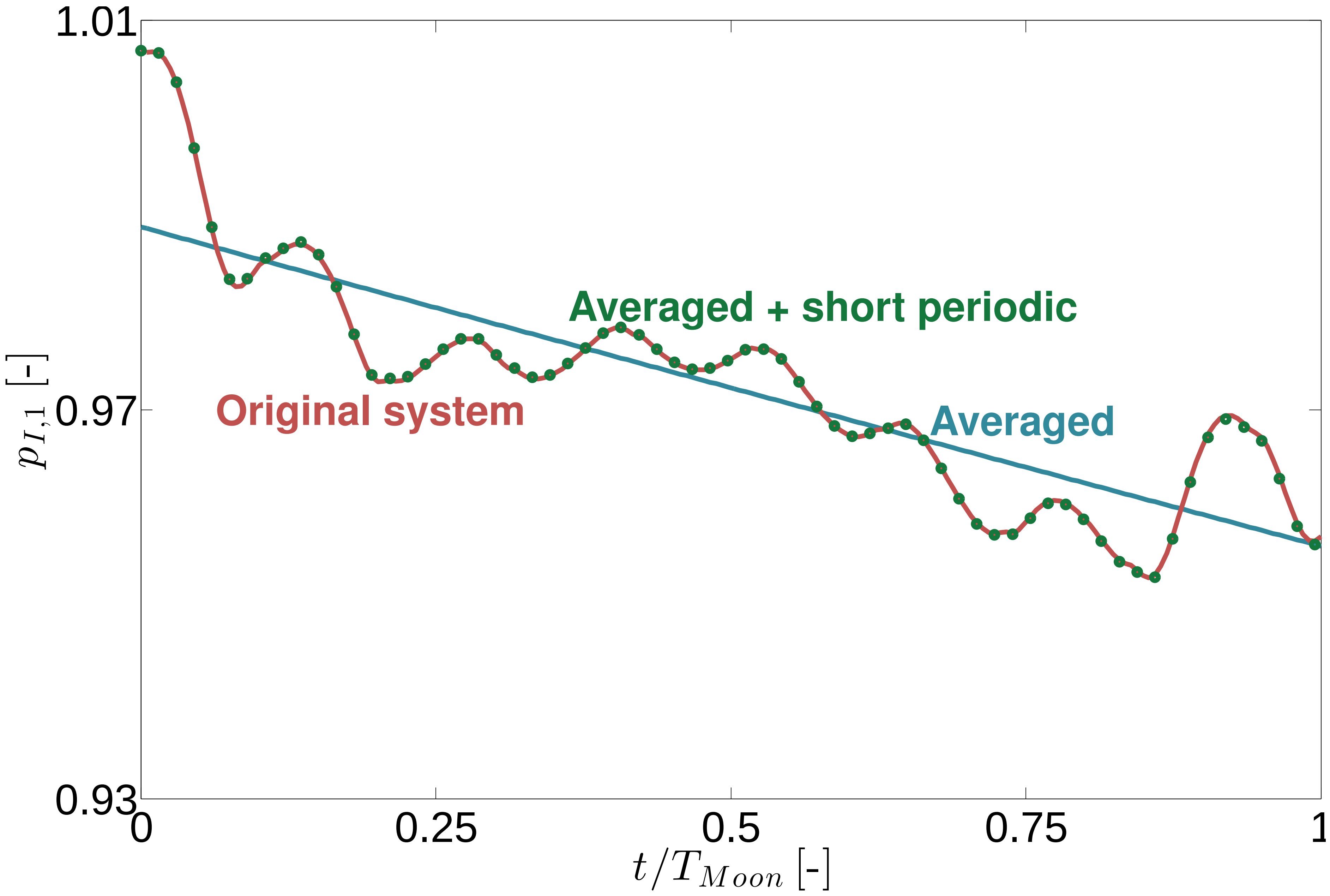
First, build the transformation of $\boldsymbol{p_\phi}$:

$$\hat{\boldsymbol{p}}_{\boldsymbol{\phi}}(\boldsymbol{\phi}) = \boldsymbol{p_\phi} + \mathcal{T}_{\boldsymbol{p_\phi}}(\boldsymbol{J}, \boldsymbol{\phi}, \boldsymbol{p_J}, \boldsymbol{p_\phi})$$

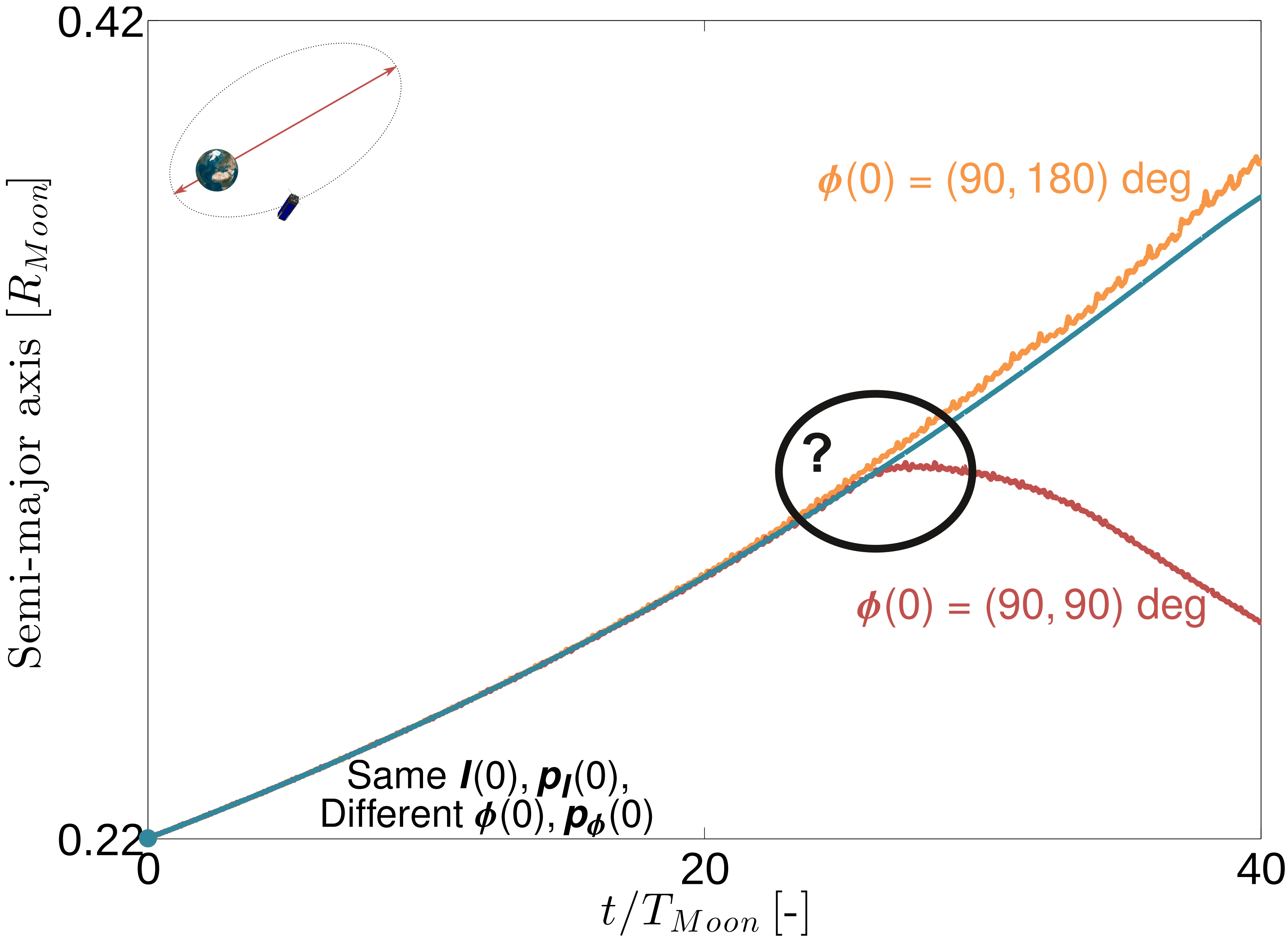
Then, use this information to evaluate the Fourier coefficients of:

$$\frac{\mathrm{d}\,\boldsymbol{p_I}}{\mathrm{d}\,t} = -\epsilon\left[\frac{\partial K}{\partial \boldsymbol{I}} - \hat{\boldsymbol{p}}_{\boldsymbol{\phi}}(\boldsymbol{\phi})\frac{\partial \omega}{\partial \boldsymbol{I}}\right]$$

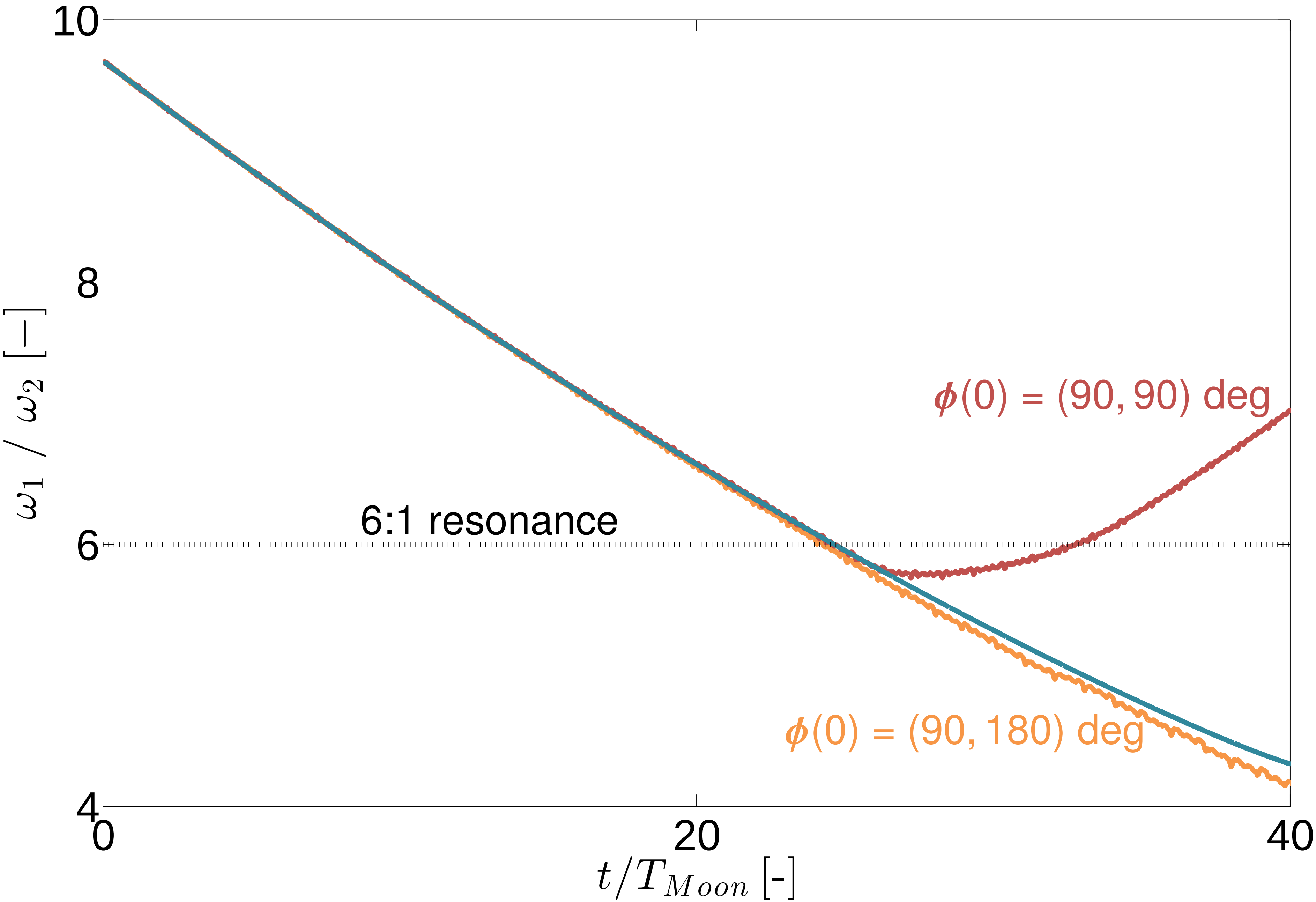
2. Short-periodic variations of p_I are accurately evaluated



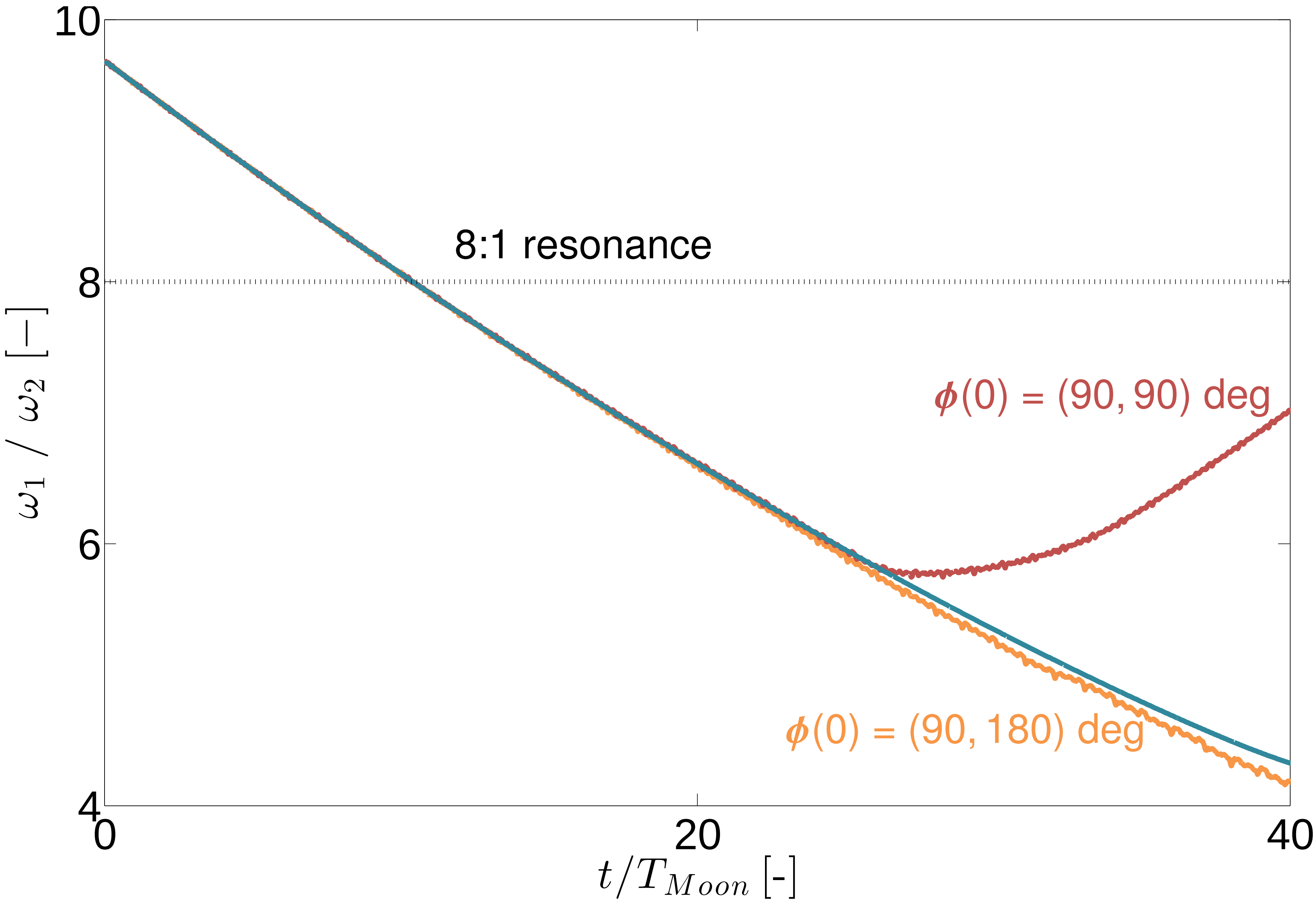
2. Transforming initial conditions is not yet enough



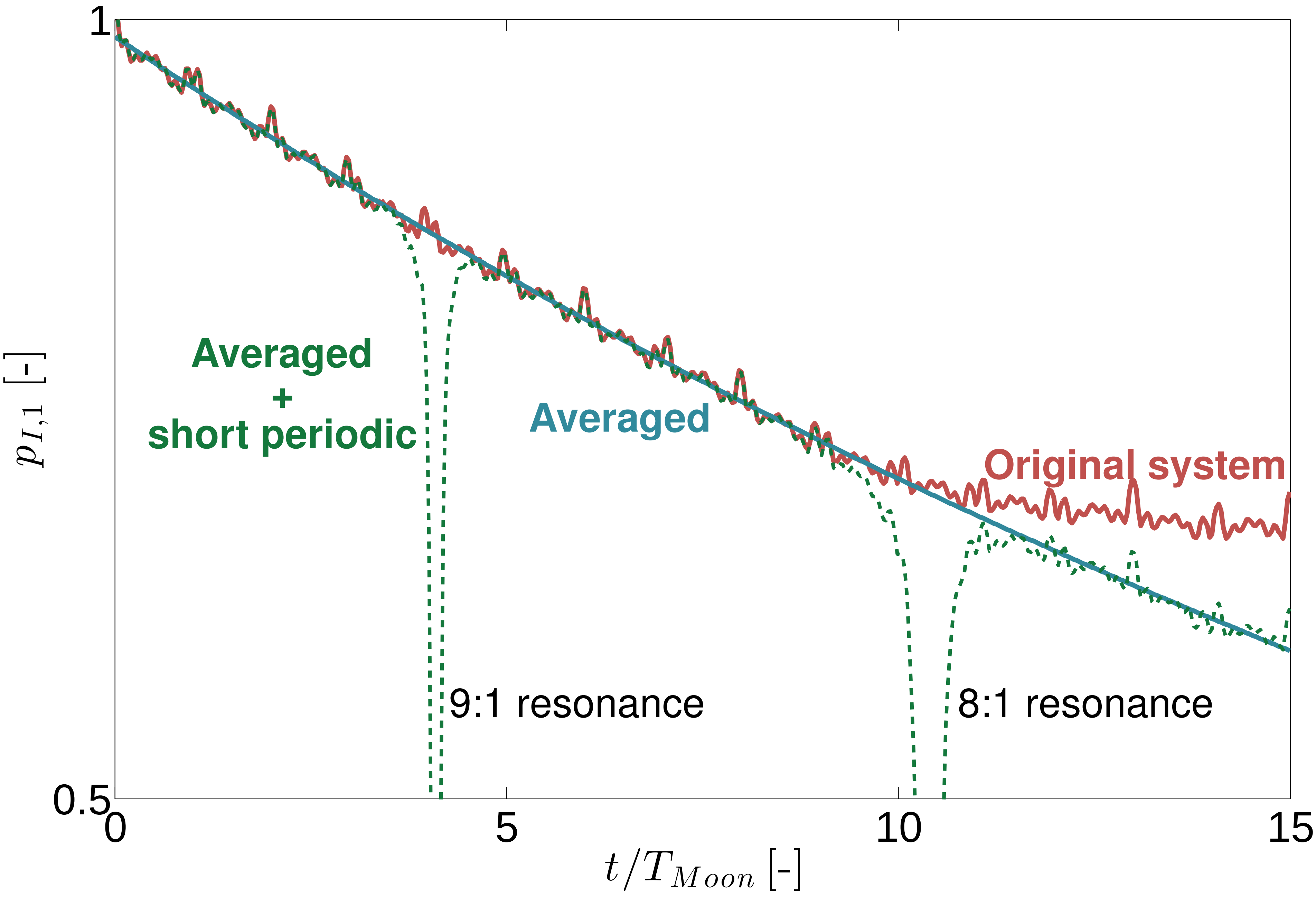
2. Is it a resonance effect?



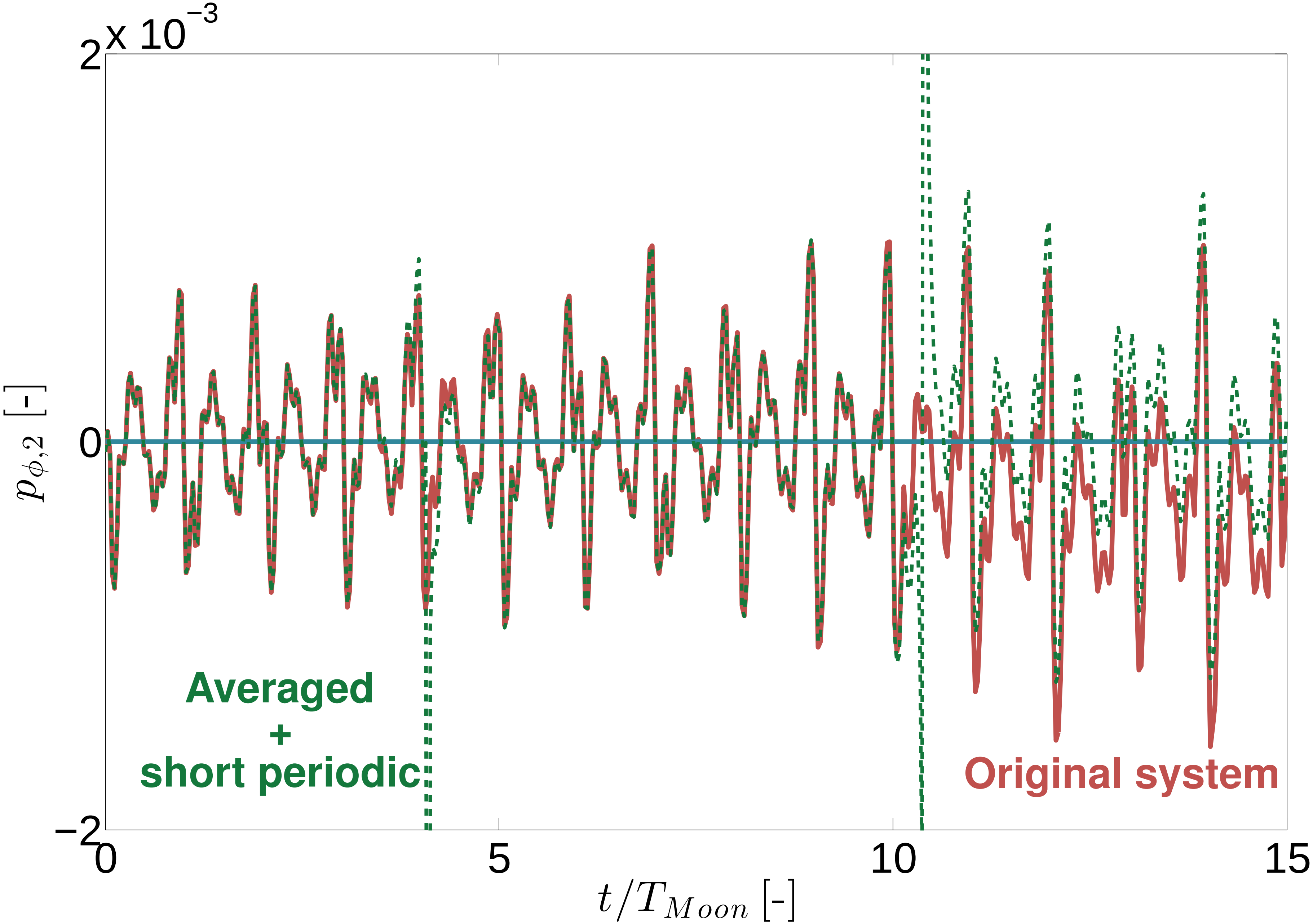
2. Yes, but divergence happened much earlier!



2. What happens when resonances are crossed?



2. Resonance crossing induces small jumps of p_ϕ



2. Resonant averaged form

Assume that there is $\boldsymbol{k}$ such that:

$$|\omega(\boldsymbol{J})\cdot\boldsymbol{k}| \leq c\sqrt{\epsilon}$$

Perform the change of variables:

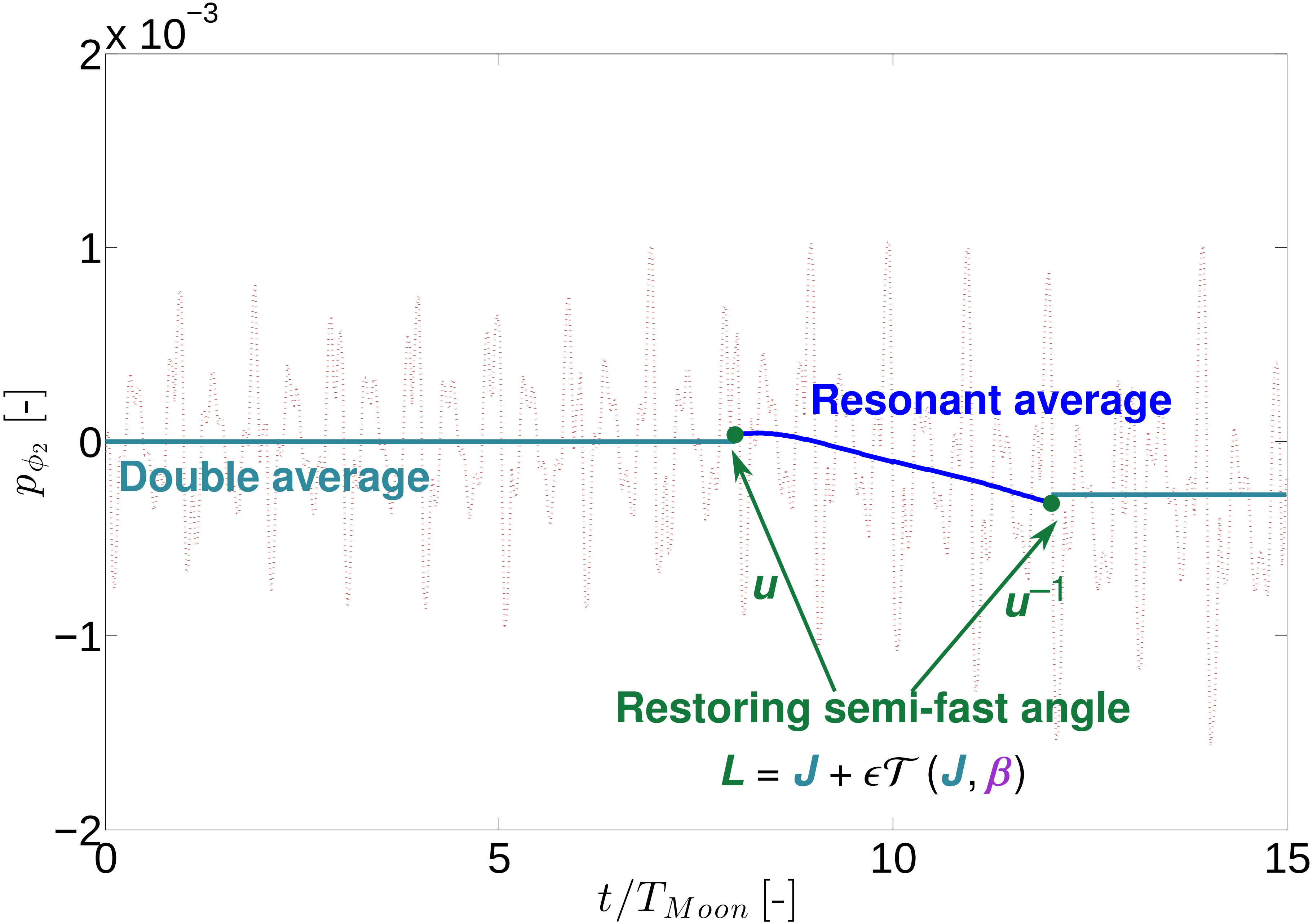
$$\boldsymbol{L} = \boldsymbol{J}, \qquad \beta = \frac{\boldsymbol{k}\cdot\boldsymbol{\phi}}{||\boldsymbol{k}||^2}, \qquad \alpha = \frac{\boldsymbol{k}^{\perp}\cdot\boldsymbol{\phi}}{||\boldsymbol{k}||^2}$$

$$\boldsymbol{p_L} = \boldsymbol{p_J}, \qquad p_\beta = \frac{\boldsymbol{k}\cdot\boldsymbol{p_\phi}}{||\boldsymbol{k}||^2}, \qquad p_\alpha = \frac{\boldsymbol{k}^{\perp}\cdot\boldsymbol{p_\phi}}{||\boldsymbol{k}||^2}$$

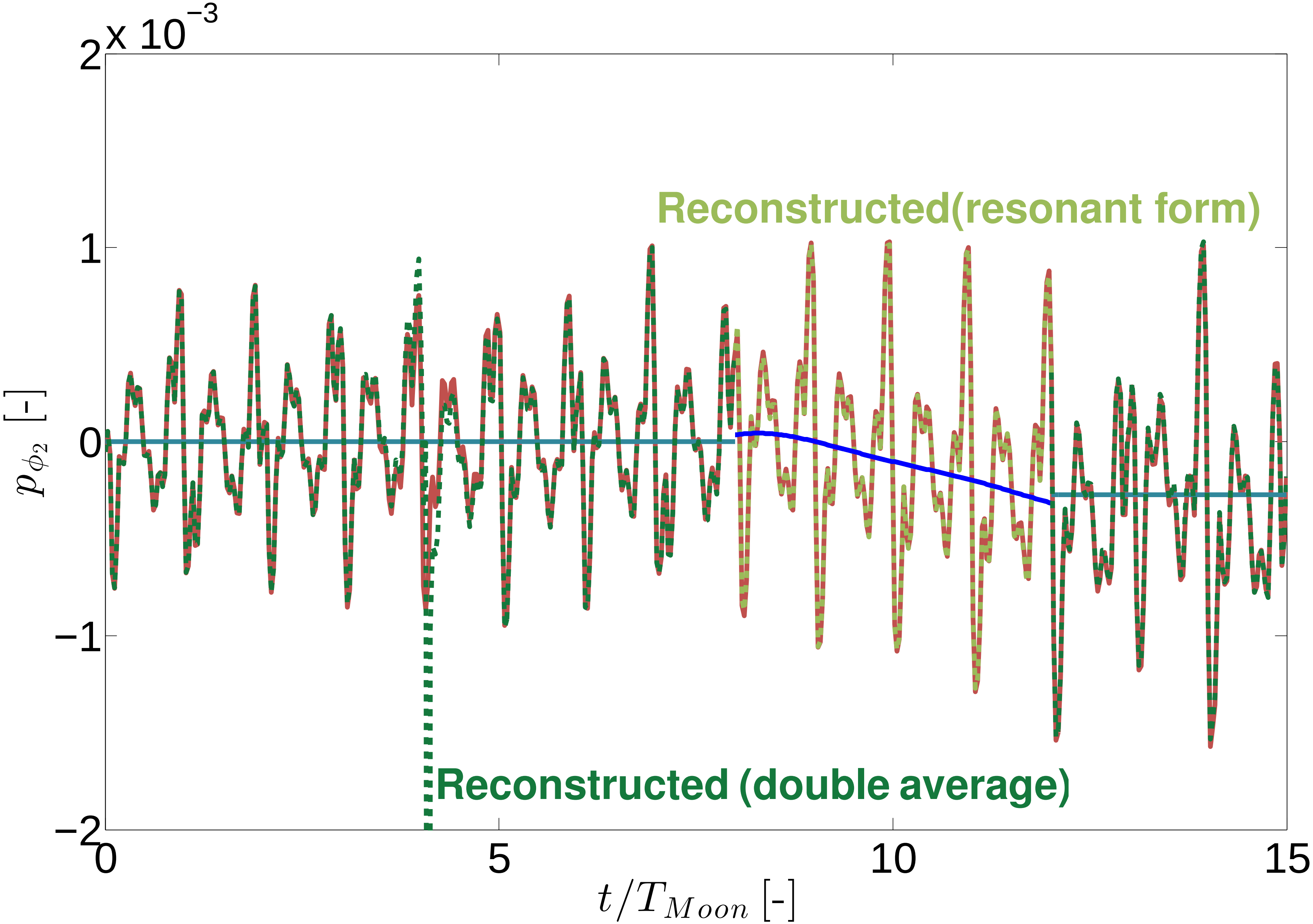
Average with respect to α

$$\overline{\mathcal{H}}_{\boldsymbol{k}} = \int_0^{2\pi} \mathcal{H}\left(\boldsymbol{L},\ \boldsymbol{p_L},\ \left[\frac{\boldsymbol{k}}{||\boldsymbol{k}||^2}\ \frac{\boldsymbol{k}^{\perp}}{||\boldsymbol{k}||^2}\right]\begin{Bmatrix}\beta\\ \alpha\end{Bmatrix},\ \left[\frac{\boldsymbol{k}}{||\boldsymbol{k}||^2}\ \frac{\boldsymbol{k}^{\perp}}{||\boldsymbol{k}||^2}\right]\begin{Bmatrix}p_\beta\\ p_\alpha\end{Bmatrix},\ 0\right)\mathrm{d}\alpha$$

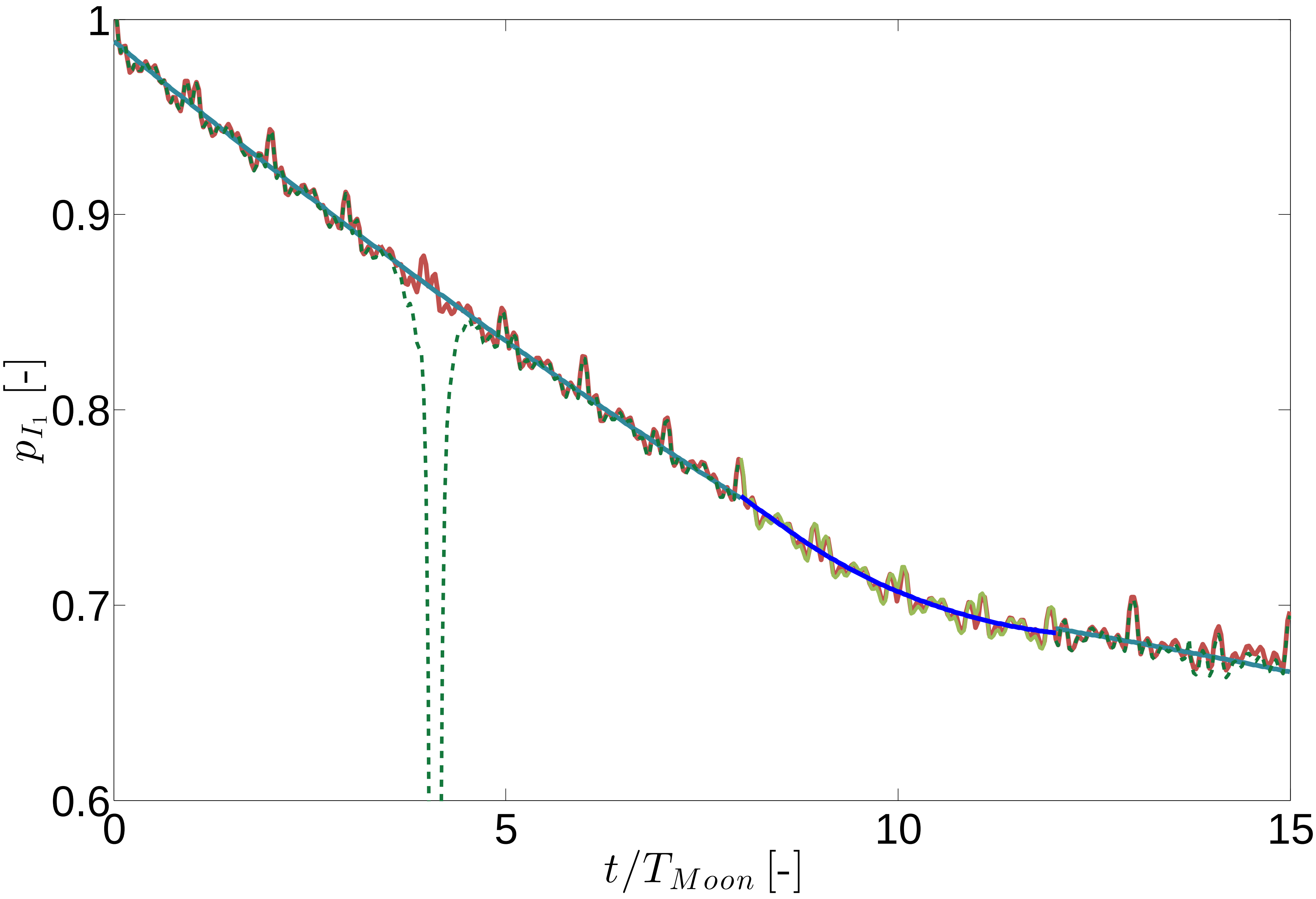
2. Transformation to interface averaged forms



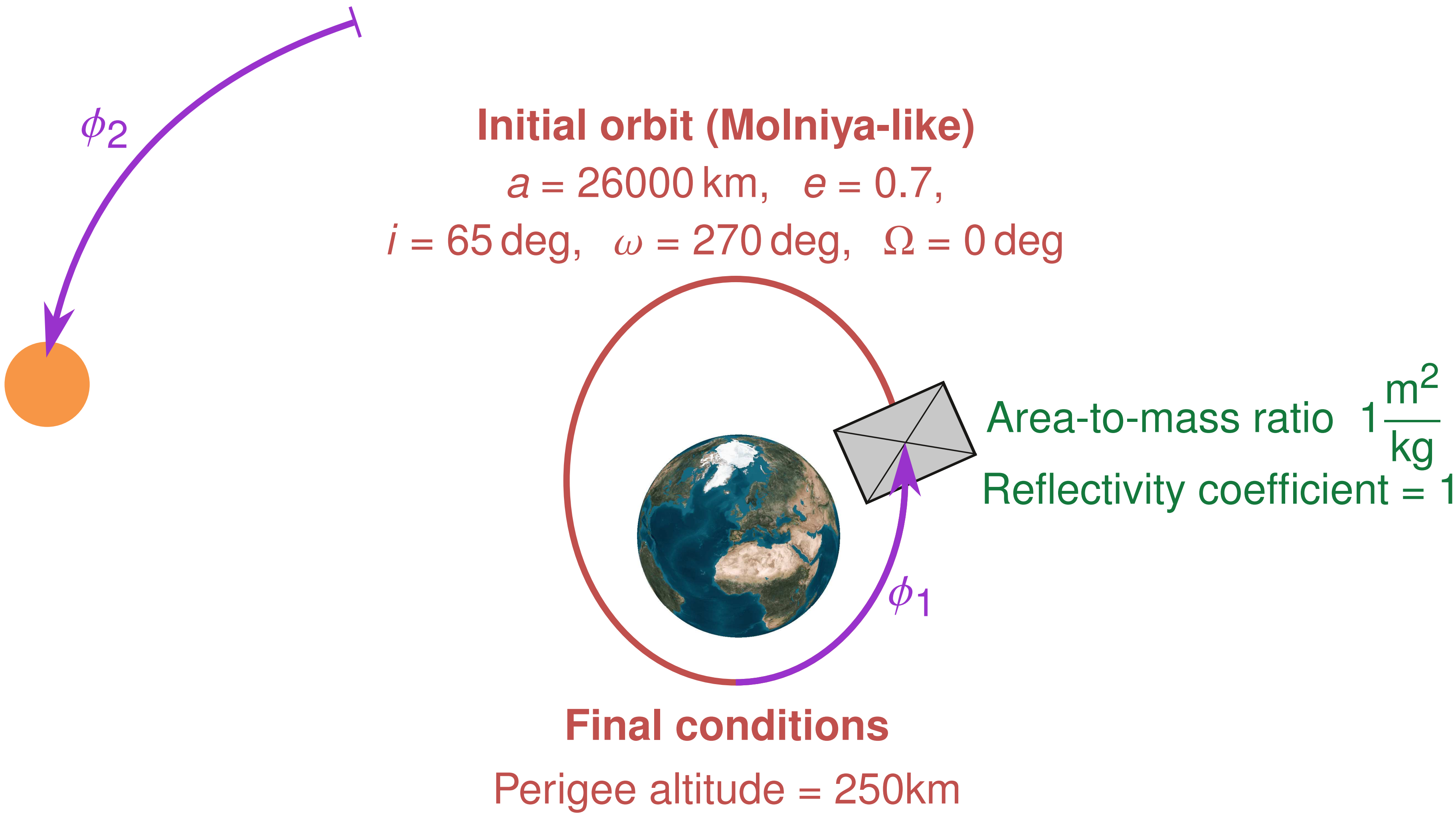
2. Jumps of adjoints to fast variables are properly



2. The transform enable 'gluing' of different forms



3. De-orbiting leveraging on solar radiation pressure



3. Mathematical modeling

Assumptions

- SRP is the only perturbation
- "Cannonball" model (SRP toward Sun direction)
- "Perfect sail" (SRP is negligible when u = 0)
- Attitude dynamics is neglected

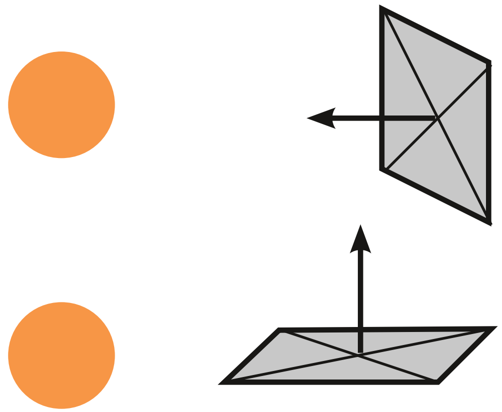
Optimal control

- Switching function

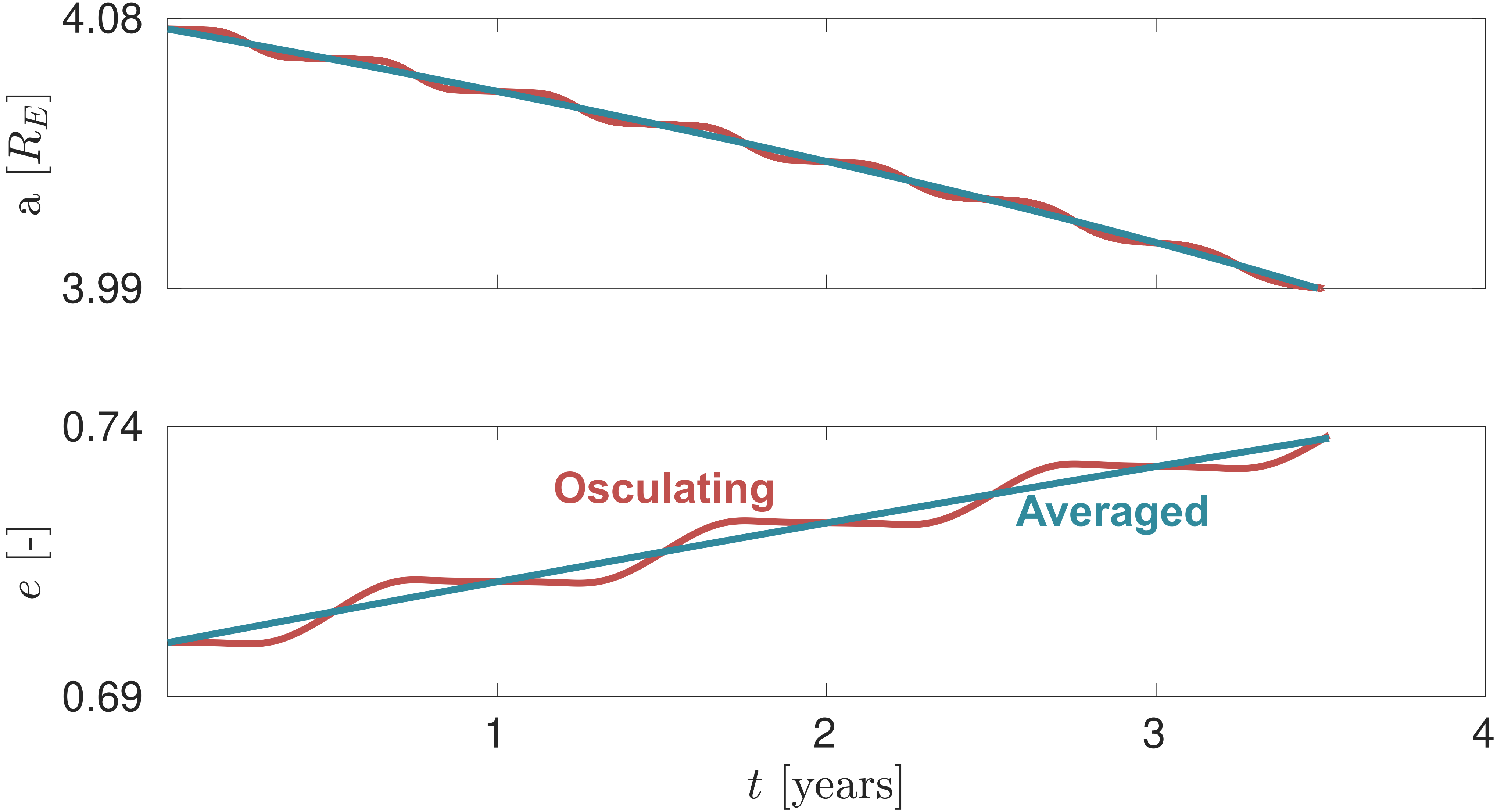
$$s = \boldsymbol{f_1}(l, \phi) \cdot \boldsymbol{p_l} + \boldsymbol{g_1}(l, \phi) \cdot \boldsymbol{p_\phi}$$

- Control

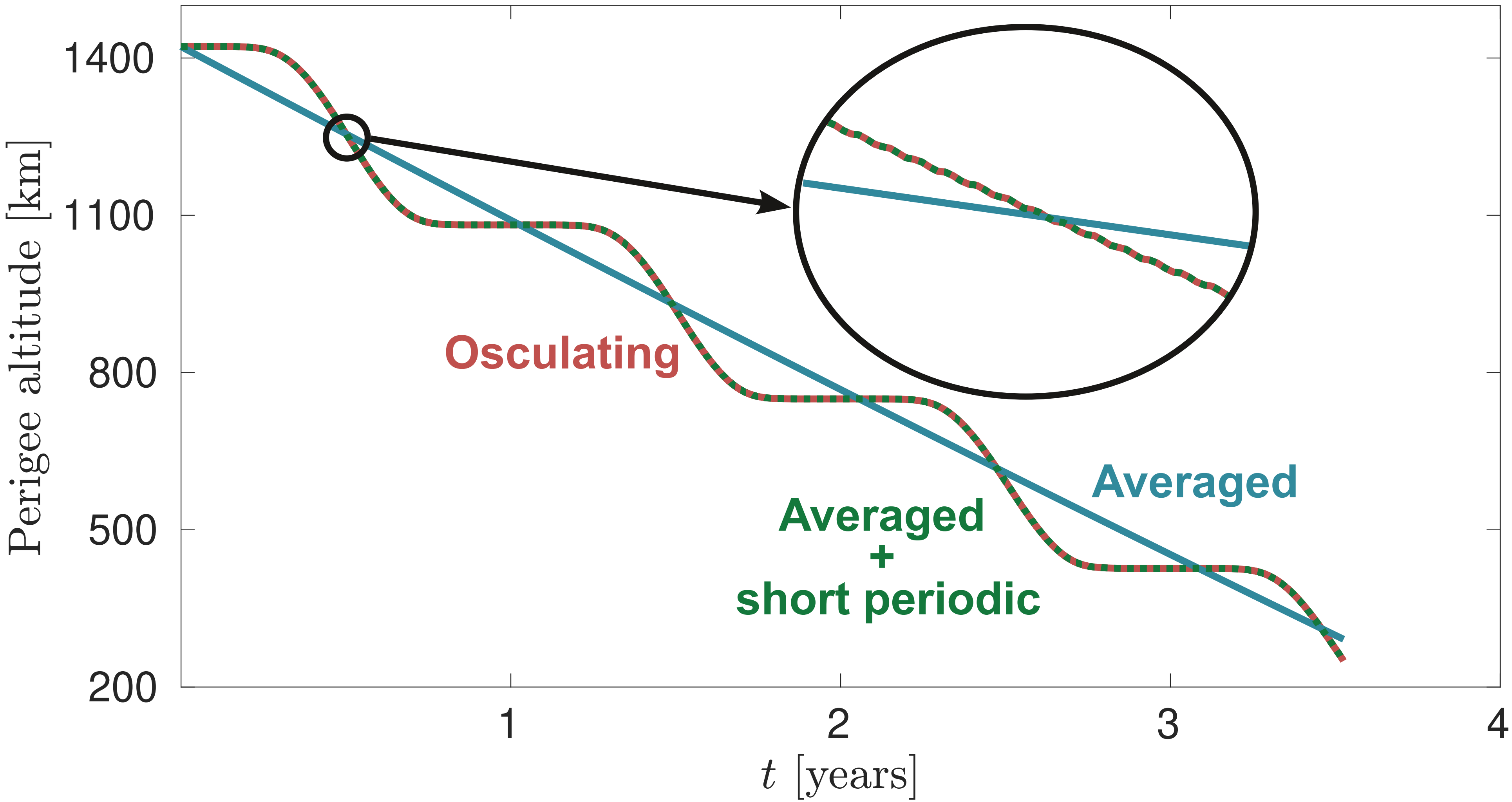
$$u = \begin{cases} 1 & \text{if} \quad s(l, \phi, \boldsymbol{p_l}, \boldsymbol{p_\phi}) > 0 \\ 0 & \text{otherwise} \end{cases}$$



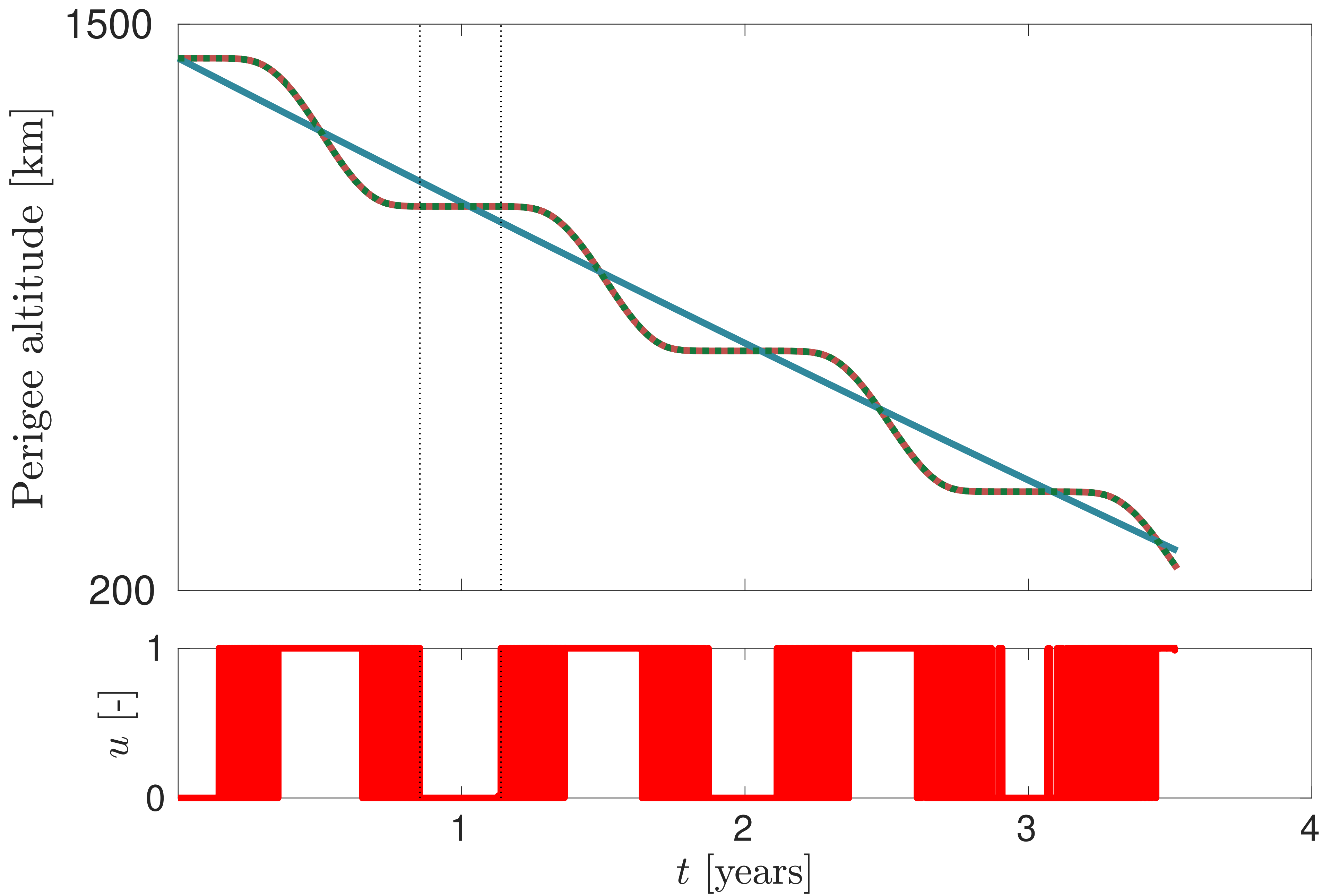
3. Semi-major axis and eccentricity



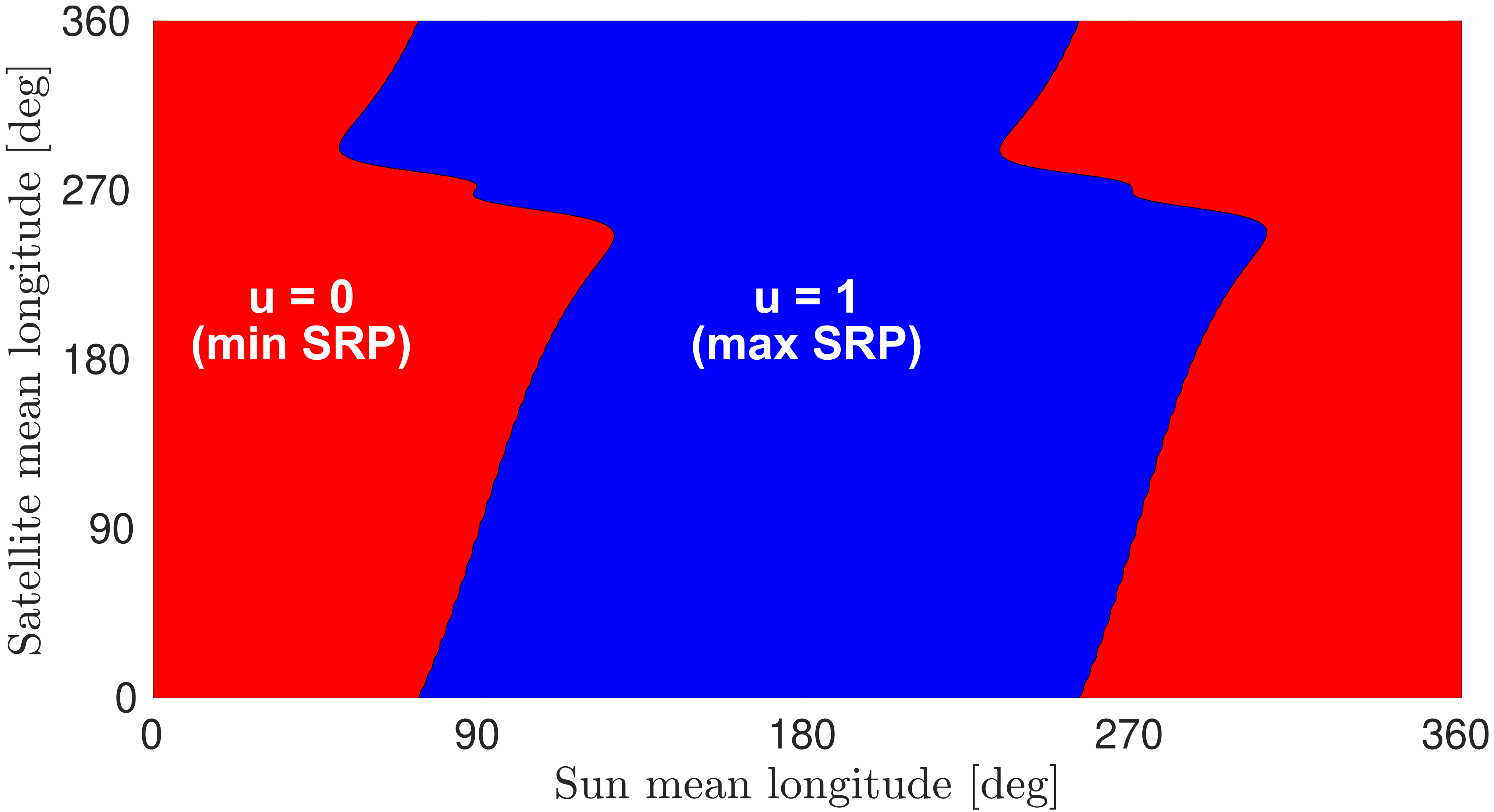
3. Trajectory of the perigee altitude



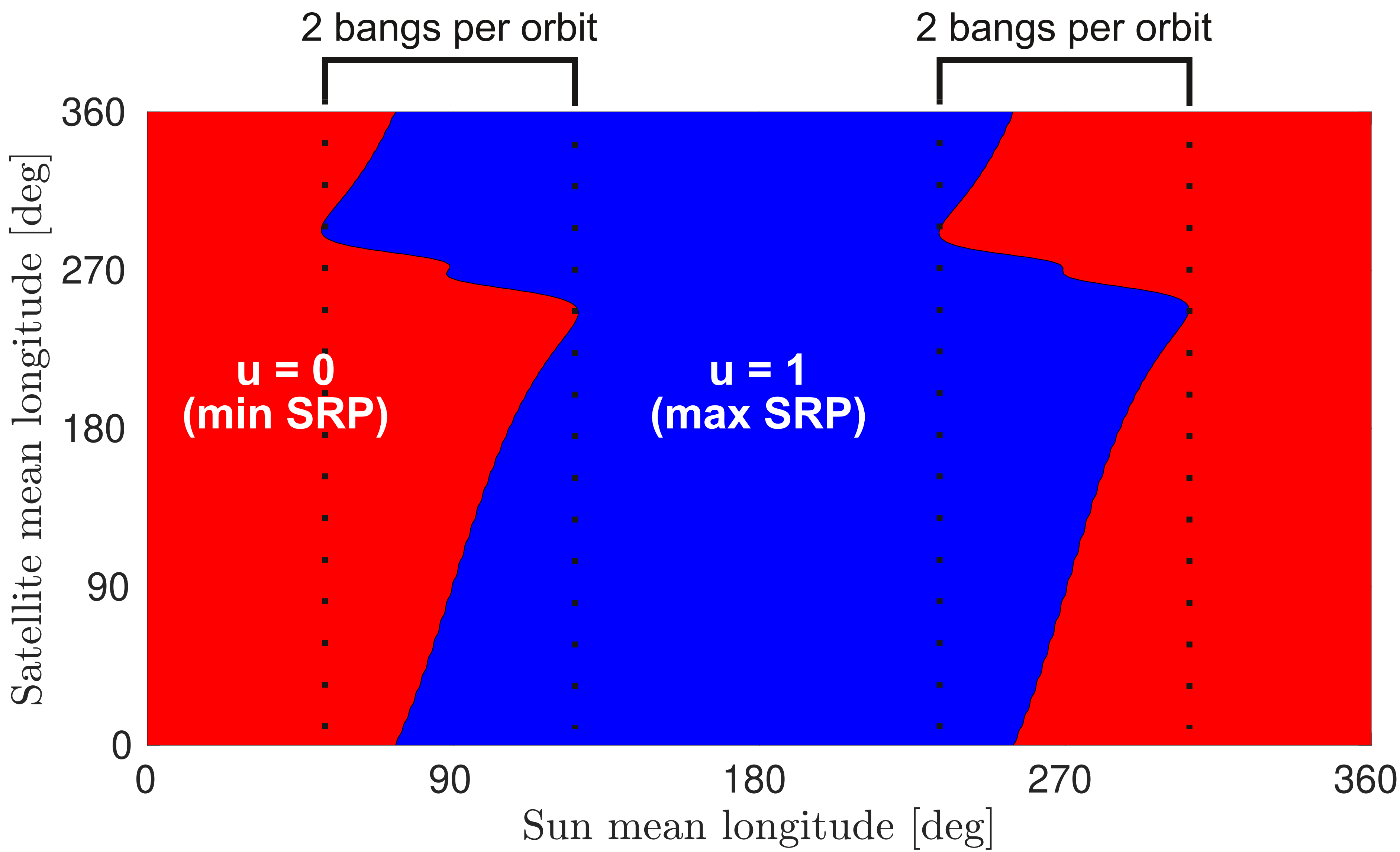
3. Short-periodic oscillations include the control structure



3. Control as a function of the phases at initial time



3. "Four-seasons" control structure



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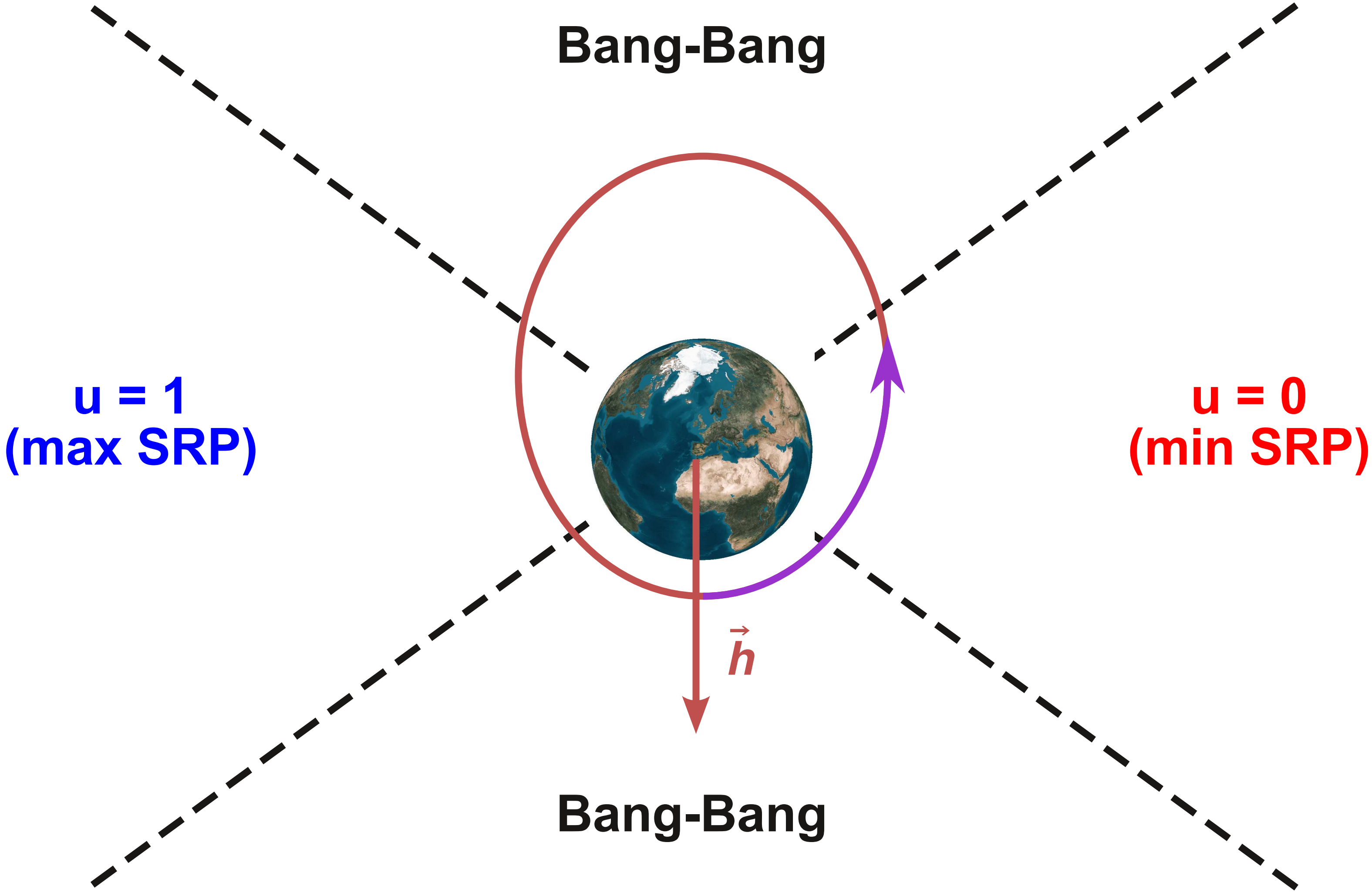

Ecliptic plane

3. Way forward

Complexity of the model

Orbital perturbations ► The second fast angle is: $l_{Sun} - \Omega$

Eclipses ► Similar treatment of bang-bang (regularization)

Singular arcs

Conclusion

Non-conventional fast-oscillating dynamical problem

Analogies with others problems in space mechanics (e.g., **quasi-satellite orbits**)

Key role of the transformation of the **adjoints to fast variables**

Benefits of averaged control system:

- Reduced set of unknown
- Smoothed trajectories
- Control structure is not required

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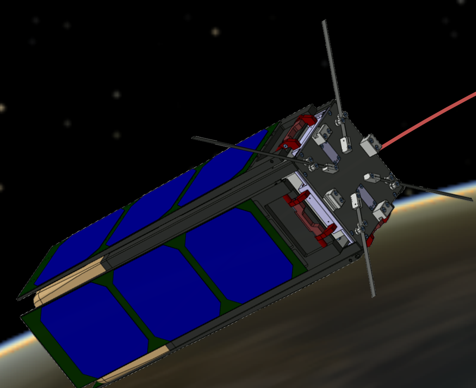

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