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Abstract

The complexity of the 3D-Delaunay triangulation (tetrahedralization) of n points distributed on a surface ranges from linear to quadratic. When the points are a deterministic good sample of a smooth compact generic surface, the size of the Delaunay triangulation is $O(n \log n)$ [2]. Using this result, we prove that when points are Poisson distributed on a surface under the same hypothesis, whose expected number of vertices is λ, the expected size is $O(\lambda \log^2 \lambda)$.

1 Introduction

While the complexity of the Delaunay triangulation of n points is strictly controlled in two dimensions to be between n and $2n$ triangles (depending on the size of the convex hull) the gap between the lower and upper bound ranges from linear to quadratic in dimension 3. The worst case is obtained using points on the moment curve[1] and the best case by using the center of spheres defining a packing.[2]

To get a more precise result on the size of the 3D Delaunay triangulation, it is possible to make different kinds of hypotheses on the point set. A first possibility is to assume a random distribution in 3D and if the points are evenly distributed in a sphere [6], (resp. in a cube [3]), Dwyer (resp. Bienkoswki et al.) proved that the expected size is $\Theta(n)$. But this hypothesis of random distribution is not relevant for all applications, for example when dealing with 3D reconstruction the Delaunay triangulation is an essential tool and it is much more natural to assume that the points are not distributed in space but on a surface [4]. If the points are evenly distributed on the boundary of a polyhedron, the expected size was proved to be $\Theta(n)$ in the convex case [9] and between $\Omega(n)$ and $\tilde{O}(n)$ in the non convex case by Golin and Na [8].

Instead of using probabilistic hypotheses one can assume that the points are a good sampling of the surface, namely an (ϵ, η)-sample where any ball of radius ϵ centered on the surface contains at least one and at most η points of the point-set. Under such hypothesis Attali and Boissonnat proved that the complexity of the Delaunay triangulation of a polyhedron is linear [1]. Attali, Boissonnat, and Lieutier extend this result to smooth surfaces verifying some genericity hypotheses with an upper bound of $O(n \log n)$ [2]. The genericity hypothesis is crucial since Erickson proved that there exists good sample of a cylinder with a triangulation of size $\Omega(n\sqrt{n})$ [7]. In the example by Erickson the point set is placed in a very special position on an helix, nevertheless, even with an unstructured point set it is possible to reach a supra-linear triangulation since Erickson, Devillers, and Goaoc proved that the triangulation of points evenly distributed on a cylinder has expected size $\Theta(n \log n)$ [5].

[1] The moment curve is parameterized by (t, t^2, t^3). When computing the Delaunay triangulation of points on this curve, any pair of points define a Delaunay edge.
[2] The kissing number in 3D is 12, thus in such a point set, the number of edges is almost $6n$.

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Contribution

In this paper we prove that a Poisson sample of parameter λ on a smooth surface of finite area is an (ε,η)-sample for $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ and $\eta = 1000\log\lambda$ with high probability. Using the result of Attali, Boissonnat, and Lieutier, it yields that the complexity of the Delaunay triangulation of a Poisson sample of a generic surface is $O(\lambda\log^2\lambda)$ losing an extra logarithmic factor with respect to the case of good sampling (see Section 3).

2 Notation, definitions, previous results

We consider a surface Σ embedded in $\mathbb{R}^3$, compact, smooth, oriented and without boundary. At a point $p \in \Sigma$, for a given orientation, we denote by $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures at p with $\kappa_1(p) > \kappa_2(p)$. We assume that the curvature is bounded and define $\kappa_{\sup} = \sup_{p\in\Sigma}\max(|\kappa_1(p)|,|\kappa_2(p)|)$. We denote by $\sigma(p,R)$ the sphere of center p and radius R. We denote by $B(\sigma)$ the closed ball whose boundary is the sphere σ, by $\mathring{E}$ the interior of a set E and, for $p \in \Sigma$, by $D(p,R)$ the intersection between Σ and the $\mathring{B}(\sigma(p,R))$. Abusively we call $D(p,R)$ a disk. For a discrete set X, we denote $\sharp\,(X)$ the cardinality of X. If X is a set of points, $\mathrm{Del}\,(X)$ denotes the Delaunay triangulation of X. In the $3D$ case, $\sharp\,(\mathrm{Del}\,(X))$ is the sum of the number of tetrahedra, triangles, edges and vertices belonging to the Delaunay triangulation.

Without loss of generality, we assume that $Area(\Sigma) = 1$ and consider that the set of points X is a Poisson point process with parameter $\lambda > 0$ over Σ.

We recall classical properties of a Poisson sample:

▶ **Observation 2.1.** *For two regions R and R' of Σ,*

- $\mathbb{P}\left[\sharp\,(X\cap R) = k\right] = \frac{(\lambda\,\mathrm{Area}(R))^k}{k!}e^{-\lambda\,\mathrm{Area}(R)}$,
- $\mathbb{E}\left[\sharp\,(X\cap R)\right] = \lambda\,\mathrm{Area}\,(R)$,
- $R\cap R' = \emptyset \Rightarrow \sharp\,(X\cap R)$ *and* $\sharp\,(X\cap R')$ *are independent random variables.*

In particular, we have $\mathbb{P}\left[\sharp\,(X\cap R) = 0\right] = e^{-\lambda\,\mathrm{Area}(R)}$ and $\mathbb{E}\left[\sharp\,(X)\right] = \lambda$.

We consider the same definition of genericity as Attali, Boissonnat and Lieutier, roughly: the set of points where one of the principal curvatures is locally maximal is a finite set of curves whose total length is bounded and, the number of contacts of any medial ball with the surface is finite.

Then we define what is a good-sampling of a surface and precise the result by Attali, Boissonnat and Lieutier.

▶ **Definition 2.2** (Good sample). A point-set on a surface is an (ε,η)-sample if any ball of radius ε centered on the surface contains at least one and at most η points of the sample.

▶ **Theorem 2.3** ([2]). *The 3D Delaunay triangulation of an (ε,η)-sample of a generic smooth surface has complexity $O\left(\frac{\eta^2}{\varepsilon^2}\log\frac{1}{\varepsilon}\right)$.*

While the result of Attali et al. provides a bound $O(N\ln N)$ on complexity of the Delaunay triangulation of an (ε,η)-sample of N points and a constant η, by looking more carefully at the result [2, Eq.(14)], we notice that the actual complexity can be expressed by $C(\frac{\eta}{\varepsilon})^2\log(\varepsilon^{-1})$ for C being a constant of the surface.

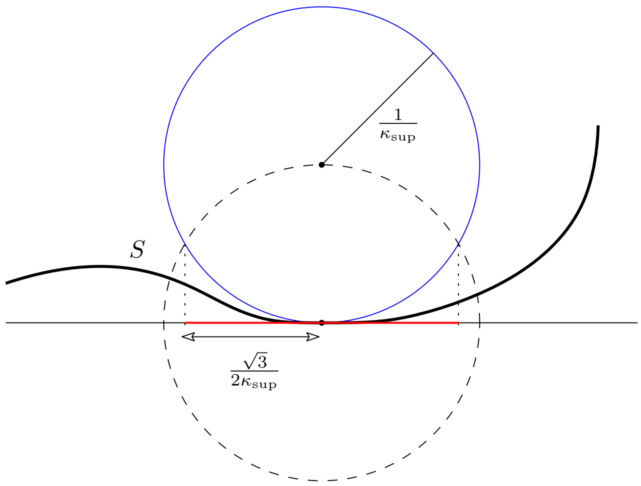

Figure 1 Illustration of the proof of Lemma 3.1 for the 2D case.

3 Is a random sample a good sample?

In a Poisson sampling of parameter λ on the surface, a disk of radius $\varepsilon = \frac{1}{\sqrt{\lambda}}$ is expected to contain π points, but with constant probability it can be empty or contains more than η points. Thus with high probability there will be such disks even if their number is limited. Thus such a sample is likely not to be a good sample with $\varepsilon^2 = \frac{1}{\lambda}$ and η constant. Nevertheless, it is possible to not consider η as a constant, namely, we take $\eta = \Theta(\log(\lambda))$. In a first Lemma, we bound the area of $D(p, R)$, for any $p \in \Sigma$ and $R > 0$ sufficiently small.

▶ **Lemma 3.1.** *Let Σ be a smooth surface of curvature bounded by κ_{sup}, and consider $p \in \Sigma$ and $R > 0$ smaller than $\frac{1}{\kappa_{\text{sup}}}$. The area of $D(p, R)$ is greater than $\frac{3}{4}\pi R^2$.*

Proof. The bound is obtained by considering the fact that the surface must stay in between the two tangent spheres of curvature κ_{sup} tangent to the surface at p. The tangent disk at p of radius $\frac{\sqrt{3}}{2}R > \frac{\sqrt{3}}{2}\frac{1}{\kappa_{\text{sup}}}$ is included in the projection of $D(p, r)$ on the tangent plane and thus has a smaller area than $D(p, R)$. ◀

▶ **Lemma 3.2.** *Let Σ be a $\mathcal{C}^3$ surface of curvature bounded by κ_{sup}.*
For R small enough, $Area(D(p, R)) < \frac{5}{4}\pi R^2$.

Proof. Let $z = f(x, y) := \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + O(x^3 + y^3)$ be the Monge of Σ patch [10] at a point p. We denote by $d\sigma$ an element of surface and by $A(p, R)$ the projection of $D(p, R)$ on the xy-plane. Since on $D(p, R)$ the slope of the normal to Σ is bounded, we have:

$$Area\,(D(p, R)) = \int_{D(p,R)} d\sigma = \int\int_{A(p,R)} \sqrt{1 + (\frac{\partial f}{\partial x}(x, y))^2 + (\frac{\partial f}{\partial y}(x, y))^2 dxdy}$$

That is smaller than $\int\int_{x^2+y^2 \le R^2} \sqrt{1 + (\frac{\partial f}{\partial x}(x, y))^2 + (\frac{\partial f}{\partial y}(x, y))^2} dxdy$, since $D(p, R) \subset B(p, R)$.

Since f is in $\mathcal{C}^3$ and $\frac{\partial f}{\partial x}(x, y) \sim \kappa_1 x$, we can say that there exists a neighborhood of p on which $|\frac{\partial f}{\partial x}| \le \sqrt{2}\kappa_1 |x| \le \sqrt{2}\kappa_{sup}|x|$, i.e., $\left(\frac{\partial f}{\partial x}\right)^2 \le 2(\kappa_{sup} x)^2$. Applying the same for y, and turning to polar coordinates, we get:

$$Area(D(p, R)) \le \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r\sqrt{1 + 2(r\kappa_{\text{sup}})^2} dr d\theta = \frac{\pi}{3}\frac{(2(R\kappa_{\text{sup}})^2 + 1)^{\frac{3}{2}} - 1}{\kappa_{\text{sup}}^2}$$

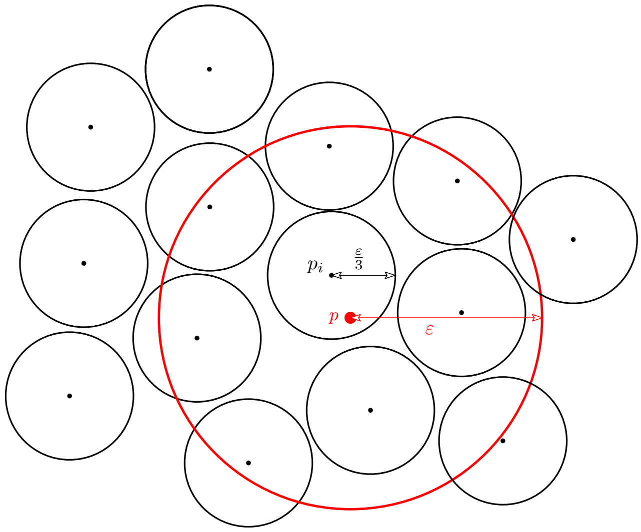

Figure 2 A disk of radius ε always contains a disk of a maximal set of disks of radius $\frac{\varepsilon}{3}$,

Noticing that $(a+1)^{\frac{3}{2}} - 1 = a\frac{a+\sqrt{a+1}+2}{\sqrt{a+1}+1} \leq \frac{15}{8}a$ for $a < 1$, we can conclude that for any R small enough,

$$Area(D(p,R)) \leq \frac{\pi}{3}\frac{\frac{15}{4}(R\kappa_{\text{sup}})^2}{\kappa_{\text{sup}}^2} = \frac{5}{4}\pi R^2.$$

◀

▶ **Lemma 3.3.** *Let Σ be a $\mathcal{C}^3$ surface with $Area(\Sigma) = 1$. Let M_R be a maximal set of k_R disjoint disks $D(p_i, R)$ on Σ. If R is small enough then $k_R \leq \frac{4}{3\pi R^2}$.*

Proof. By Lemma 3.1, for R small enough, we have $D(p,R) \geq \frac{3}{4}\pi R^2$. Thus:

$$k_R \cdot \frac{3}{4}\pi R^2 \leq \sum_{i=1}^{i=k_R} \text{Area}\,(D(p_i,R)) \leq \text{Area}\,(\Sigma) = 1,$$

and we can deduce the following bound: $k_R \leq \frac{4}{3\pi R^2}$. ◀

▶ **Lemma 3.4.** *Let X be a Poisson sample of parameter λ distributed on a $\mathcal{C}^3$ smooth closed surface Σ of area 1. If λ is large enough, the probability that there exists $p \in \Sigma$ such that $D\left(p, 3\sqrt{\frac{\log\lambda}{\lambda}}\right)$ does not contain any point of X is $O(\lambda^{-1})$.*

Proof. We prove that a Poisson sample has no empty disk of radius $3\sqrt{\frac{\log\lambda}{\lambda}}$ with probability $O(\lambda^{-1})$. In a first part we use a packing argument. On the one hand, for any $\varepsilon > 0$ small enough and given a maximal set $M_{\varepsilon/3}$ and any point $p \in \Sigma$, the disk $D(p,\varepsilon)$ contains entirely one of the disks $D(p_i, \frac{\varepsilon}{3})$ belonging to $M_{\varepsilon/3}$. Indeed, by maximality of $M_{\varepsilon/3}$, the disk $D(p,\varepsilon/3)$ intersects a disk of $M_{\varepsilon/3}$ whose diameter is $2\varepsilon/3$ so $D(p,\varepsilon)$ contains it entirely. On the other hand, remember from Lemma 3.1 that if ε is small enough then $Area(D(p,\varepsilon)) \geq \frac{3}{4}\pi\varepsilon^2$. Then we can bound the probability of existence of an empty disk for ε small enough:

$$\begin{aligned}\mathbb{P}\left[\exists p\in\Sigma,\ \sharp\left(X\cap D(p,\varepsilon)\right)=0\right] &\leq \mathbb{P}\left[\exists i<k_{\varepsilon/3},\ \sharp\left(X\cap D(p_i,\varepsilon/3)\right)=0\right]\\ &\leq k_{\varepsilon/3}\,\mathbb{P}\left[\sharp\left(X\cap D(c,\varepsilon/3)\right)=0\right] \text{ for a point } c \text{ on } \Sigma\\ &\leq \tfrac{4}{3\pi(\varepsilon/3)^2}e^{-\lambda\frac{3}{4}\pi(\frac{\varepsilon}{3})^2}=\tfrac{12}{\pi\varepsilon^2}e^{-\lambda\frac{1\pi\varepsilon^2}{12}}.\end{aligned}$$

By taking $\varepsilon=3\sqrt{\frac{\log\lambda}{\lambda}}$ we get:

$$\mathbb{P}\left[\exists p\in\Sigma,\ \sharp\left(X\cap D(p,3\sqrt{\tfrac{\log\lambda}{\lambda}})\right)=0\right]\leq\tfrac{4\lambda}{3\pi\log\lambda}e^{-\frac{3\pi\log\lambda}{4}}=O(\lambda^{-1}).$$

◀

We have proved that when a Poisson sample is distributed on a surface, the points sufficiently cover the surface, i.e., there is no large empty disk on the surface with high probability. Now we have to verify the other property of a good sample, namely, a Poisson sample does not create large concentration of points in a small area.

▶ **Lemma 3.5.** *Let X be a Poisson sample of parameter λ distributed on a $\mathcal{C}^3$ closed surface of area 1. If λ is large enough, the probability that there exists $p\in\Sigma$ such that $D(p,3\sqrt{\frac{\log\lambda}{\lambda}})$ contains more than $1000\log(\lambda)$ points of X is $O(\lambda^{-2})$.*

Proof. Consider an M_ε maximal set, we can notice that for any $p\in\Sigma$, the disk $D(p,\varepsilon)$ with $p\in\Sigma$ is entirely contained in one disk $D(p_i,3\varepsilon)$ that is an augmented disk of M_ε. Indeed, by maximality of M_ε, the disk $D(p,\varepsilon)$ intersects a disk from M_ε say $D(p_j,\varepsilon)$ so $D(p_j,3\varepsilon)$ contains entirely $D(p,\varepsilon)$.

Then we can bound the probability of existence of a disk containing more than η points:

$$\begin{aligned}\mathbb{P}\left[\exists p\in\Sigma,\ \sharp\left(X\cap D(p,\varepsilon)\right)>\eta\right] &\leq \mathbb{P}\left[\exists i<k_{\varepsilon},\ \sharp\left(X\cap D(p_i,3\varepsilon)\right)>\eta\right]\\ &\leq k_\varepsilon\,\mathbb{P}\left[\sharp\left(X\cap D(c,3\varepsilon)\right)>\eta\right] \text{ for a point } c \text{ on } \Sigma\\ &\leq \frac{4}{3\pi\varepsilon^2}\,\mathbb{P}\left[\sharp\left(X\cap D(c,3\varepsilon)\right)>\eta\right]\end{aligned}$$

We use a Chernoff inequality [11] to bound $\mathbb{P}\left[\sharp\left(X\cap D(c,3\varepsilon)\right)>\eta\right]$: If V follows a Poisson law of mean v_0, then $\forall v>v_0$,

$$P(V>v)\leq e^{v-v_0}(\frac{v_0}{v})^v.$$

From Lemmas 3.1 and 3.2, we have that: $\frac{27}{4}\pi\varepsilon^2\leq Area(D(c,3\varepsilon))\leq\frac{45}{4}\pi\varepsilon^2$ for ε small enough. Consequently we can say that the expected number of points v_0 in $D(c,3\varepsilon)$ verifies $\frac{27}{4}\lambda\pi\varepsilon^2\leq v_0\leq\frac{45}{4}\lambda\pi\varepsilon^2$.

Then we apply the above Chernoff bound with $v=\frac{45}{4}e\pi\lambda\varepsilon^2$ (chosen for the convenience of the calculus)

$$\begin{aligned}\mathbb{P}\left[\sharp\left(X\cap D(c,3\varepsilon)\right)>\frac{45}{4}e\pi\lambda\varepsilon^2\right] &\leq e^{\frac{45}{4}e\pi\lambda\varepsilon^2-v_0}\left(\tfrac{v_0}{\frac{45}{4}e\pi\lambda\varepsilon^2}\right)^{\frac{45}{4}e\pi\lambda\varepsilon^2}\\ &\leq e^{\frac{45}{4}e\pi\lambda\varepsilon^2-\frac{27}{4}\pi\lambda\varepsilon^2}\left(\tfrac{\frac{45}{4}\pi\lambda\varepsilon^2}{\frac{45}{4}e\pi\lambda\varepsilon^2}\right)^{\frac{45}{4}e\pi\lambda\varepsilon^2}=e^{-\frac{27}{4}\pi\lambda\varepsilon^2}\end{aligned}$$

So for $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ and $\eta = \frac{45}{4}e\pi\lambda\varepsilon^2 = \frac{405}{4}e\pi\log\lambda$, we have:

$$\mathbb{P}\left[\exists p \in \Sigma,\ \sharp\left(X \cap D(p, 3\sqrt{\tfrac{\log\lambda}{\lambda}})\right) > \frac{405}{4}e\pi\log\lambda\right] \le \tfrac{4\lambda}{27\pi\log\lambda}e^{-\frac{243}{4}\pi\log\lambda} = O(\lambda^{-189})$$

Since $\frac{405}{4}e\pi < 1000$, it is sufficient for our purpose to say:

$$\mathbb{P}\left[\exists p \in \Sigma,\ \sharp\left(X \cap D(p, 3\sqrt{\tfrac{\log\lambda}{\lambda}})\right) > 1000\log(\lambda)\right] = O(\lambda^{-2})$$

◀

▶ **Theorem 3.6.** *On a $\mathcal{C}^3$ closed surface, a Poisson sample of parameter λ large enough is a $(3\sqrt{\frac{\log\lambda}{\lambda}}, 1000\log\lambda)$-sample with probability $1 - O(\lambda^{-1})$.*

Proof. From Lemmas 3.4 and 3.5, we have that a Poisson sample is not a $(3\sqrt{\frac{\log\lambda}{\lambda}}, 1000\log\lambda)$-sample with probability $O(\lambda^{-1})$. ◀

▶ **Theorem 3.7.** *For λ large enough, the Delaunay triangulation of a point set Poisson distributed with parameter λ on a closed smooth generic surface of area 1 has $O(\lambda\log^2\lambda)$ expected size.*

Proof. Given a Poisson sample X we distinguish two cases:

- If X is a good sample, i.e., an (ε, η)-sample with $\varepsilon = 3\sqrt{\frac{\log\lambda}{\lambda}}$ and $\eta = 1000\log\lambda$, we apply the $O((\frac{\eta}{\varepsilon})^2\log(\varepsilon^{-1}))$ bound from the paper by Attali et al., that is $O(\lambda\log^2\lambda)$.
- If X is not a good sample, which arises with small probability by Lemma 3.6, we bound the triangulation size by the quadratic bound:

$$\sum_{k\in\mathbb{N}} k^2\,\mathbb{P}\left[\sharp\,(X) = k\right] = \sum_{k\in\mathbb{N}} k^2\frac{\lambda^k}{k!}e^{-\lambda} = \lambda(\lambda+1) = O(\lambda^2)$$

Combining the two results, we get

$$\begin{aligned}\mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\right] &= \mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\,|X \text{ good sample}\right]\mathbb{P}\left[X \text{ good sample}\right] \\ &\quad + \mathbb{E}\left[\sharp\,(\mathrm{Del}\,(X))\,|X \text{ not good sample}\right]\mathbb{P}\left[X \text{ not good sample}\right] \\ &\le O\left(\lambda\log^2\lambda\right)\times 1 + O(\lambda^2)\times O\left(\lambda^{-1}\right) = O(\lambda\log^2\lambda)\end{aligned}$$

◀

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