Duals and Transforms of Ideals in PVMDs
Compact and Coprime Packedness With Respect to Star Operations

BY

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To my mother and my father

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All praise and glory to ALLAH almightily who taught man through the pen. And peace and blessing of ALLAH be upon his last messenger Mohammad (Sallallah-Alihe-Waalehi-Wasallam), who brought the absolute truth and wisdom to mankind.

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DISSERTATION ABSTRACT

Name: Abdulaziz Abdullah Bin-Obaid

Title of study: Duals and Transforms of Ideals in PVMDs

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This Ph.D. thesis traverses two chapters which contribute to the study of star operations in integral domains. In Chapter 1, we study when the dual of a t-ideal in a PVMD is a ring and we treat the question of when it coincides with its endomorphism ring. We also study particular classes of overrings of PVMDs. Specifically, we investigate the Nagata transform and the endomorphism ring of ideals in PVMDs in an attempt to establish analogues for well-known results on overrings of Prüfer domains. In Chapter 2, we study the notions of compactly packed ring and coprimely packed ring with respect to a star operation of finite type. We extend well-known results and investigate more properties of these notions in different contexts such as Prüfer-like settings, Noetherian-like settings, and pullbacks. Particular attention is paid to the t-operation as the often usual star operation of finite type. Examples are constructed in order to illustrate the scopes and limits of the results.

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Key Words and Phrases: dual of an ideal, endomorphism ring, t-ideal, t-operation, ideal transform, overring, PVMD, star operation, compactly packed, coprimely packed, Prüfer domain, GCD domain, pullback.

ملخص بحث

درجة الدكتوراه في الفلسفة

الاســــــــــــم: عبد العزيز عبد الله صالح بن عبيد

عنوان الرسالة: ثنويات و محولات مثالي في حلقة شبة بروفرية و تعبئة متقاربة و كوبريملي متقارب بالنسبة لعمليات النجمة .

التخصـــــــص: الريــــاضيــــات

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هذه الرسالة قسمت إلى جزئيين و ذلك إسهاما و تعمق في دراسة عملية النجمة المعرفة للنطاق صحيح. في الجزء الأول درسنا فيه بعض الحالات التي يكون فيها الثنوي لـ t-مثالي (t-ideal) حلقة فوقية لنطاق صحيح شبه بروفري (PVMD) و كذلك عالجنا بعض الحالات التي يكون فيها الثنوي لـ t-مثالي متطابق مع الحلقة الإندومورفيزمية لـ t-مثالي في نطاق شبه بروفري و اختتمنا الجزء الأول بدراسة بعض أصناف الحلقات الفوقية لنطاق شبه بروفري مثل محول ناجاتا و الحلقة الإندومورفيزمية لـ t-مثالي في نطاق شبه بروفري. بدأنا الجزء الثاني بتعريف المصطلحات تعبئة متقاربة و كوبريملي متقارب بالنسبة لعملية نجمة منتهية التوليد. و بذلك نكون عممنا المصطلحات الأساسية و بالتالي درسنا بعض خصائص هذه المصطلحات لأنواع مختلفة من حالات النطاقات مثل حالات الشبه بروفرية و حالات الشبه نوثيرنية و أخيرا في نطاقات التركيبات الخلفية. كذلك أنتجنا أمثلة لتوضيح مجالات و محدودية هذه النتائج. و قد أولينا اهتمام خاص لعملية النجمة t.

Introduction

Star operations such as the t-closure, the v-closure and the w-closure are essential tools in modern multiplicative ideal theory for characterization and investigation of several classes of integral domains. During the last decades, the t-operation (known as the largest star operation of finite type) has been intensively studied, probably for its ability to classify many classes of integral domains as generalizations of well-known domains such as Bezout domains (i.e., every f. g. ideal is principal) to GCD domains (i.e., for every f. g. ideal I, I_t is principal), Dedekind domains (i.e., every ideal is invertible) to Krull domains (i.e., every ideal is t-invertible), and Prüfer domains (i.e., every f. g. ideal is invertible) to PVMDs (i.e., every f. g. ideal is t-invertible).

Many authors studied the structure of particular overrings of an integral domain, specially overrings of a Prüfer domain. In 1968, Brewer [4] gave a representation theorem for the Nagata transform $T(I)$, when I is a finitely generated ideal (which coincides in this case with the Kaplansky transform $\Omega(I)$) and in 1974, Kaplansky [36] gave a complete description of $\Omega(I)$ for each ideal I in an integral domain R, where these two special overrings are defined as $T(I) = \bigcup_{n=1}^{\infty}(R : I^n)$ and $\Omega(I) = \{u \in K :$ for each $a \in I$ there is a positive integer $n(a)$ with $ua^{n(a)} \in R\}$. In [21], Fontana, Huckaba and Papick described some relations between the above constructions in the case of Prüfer domains.

While $(I : I)$ is an overring of R which is isomorphic to the endomorphism ring $\mathrm{End}_R(I)$ of I, $I^{-1} := (R : I)$ is an R-submodule of L containing $(I : I)$ which is not, in general, a ring. Many papers in the literature deal with the fractional ideal I^{-1}. The main problem is to examine settings in which I^{-1} is a ring and then when it coincides with $(I:I)$. In 1982, Huckaba and Papick [32] examined settings in which I^{-1} is a ring where I is an ideal of a Prüfer domain. Later, in 1983, D. F. Anderson [1], using pullbacks, constructed an example of a domain R and an ideal I of R for which I^{-1} is a ring but $(I:I) \subsetneqq I^{-1}$. In [27], Heinzer and Papick gave a necessary and sufficient condition for I^{-1}, when it is a ring, to be equal to $(I:I)$ for an ideal I in a Prüfer domain with Noetherian spectrum. In 1993, Fontana, Huckaba, Papick and Roitman [22] provided various representations of the endomorphism ring $(I:I)$ of an ideal I in a Prüfer domain as intersections of localizations. Finally in 2000, Houston, Kabbaj, Lucas and Mimouni [29] established several characterizations for I^{-1} to be a ring for a nonzero ideal I in an integrally closed domain.

The notion of compactly packed ring (or *CP*-ring for short) was introduced by Reis and Viswanathan in [41], where Noetherian *CP*-rings were characterized by the property that prime ideals are radicals of principal ideals. The notion of coprimely packed ring was introduced by Erdoğdu in 1988 [12] and intensively studied in a series of papers, for instance see [12, 13, 14, 15, 16], [18], [7] and [42]. Erdogdu studied the notion of coprimely packed rings in many contexts such as Notherian domains, Bezout domains, Nagata rings, polynomial extensions, and *QR*-domains. For instance, he proved that "*a Dedekind domain R is coprimely packed if and only if R has a torsion class group*." He studied the relation between the compact packedness and the coprime packedness. However the most important part in his study is the correlation between the coprime packedness and the set theoretic intersection of ideals in polynomial rings [16].

This thesis contributes to the investigation of the dual and the Nagata and Kaplansky transforms of an ideal in PVMDs. Also we extend the notions of compact and coprime packedness to an integral domain with respect to a star operation of finite type and study some algebraic properties of these notions in various settings. The thesis is divided into two chapters. The first part of Chapter 1 deals with the question of when the dual of an ideal is a ring for a t-ideal in a PVMD, and then when I^{-1} coincides with the endomorphism ring of I. Our first main contribution, Theorem1.2.3 and Theorem1.2.6, is a generalization of two well-known theorems established by Huckaba-Papick [32, Theorem 3.8] and Heinzer-Papick [27, Theorem 2.5]. The second main contribution, Theorem1.2.14, is a complete description of the endomorphism ring of a t-ideal in a tQR-domain which generalizes a well-known result by Fontana et al., [22, Corrollary 4.4 and Theorem 4.11]. The second part of Chapter 1 is devoted to Kaplansky and Nagata transforms of an ideal in a PVMD, in an attempt to establish analogues for well-known results on overrings of Prüfer domains. Specifically, we prove t-analogues for many results collected in Fontana-Huckaba-Papick's book [21, Section 3.3] for t-linked overrings of PVMDs. The first main theorem, Theorem 1.3.2, generalizes [21, Theorem 3.3.7] to the case of t-prime ideals in a PVMD. The second main theorem, Theorem 1.3.6, is a satisfactory t-analogue for [21, Theorem 3.3.10].

Chapter 2 extends the compact and the coprime notions to a domain R endowed with an arbitrary star operation $*$ of finite type. In the particular case where $* = d$ is the trivial operation on R, we obtain the so-called compactly and coprimely packed rings. We study various aspects of these notions in many different classes of integral domains, including Nagata rings, Prüfer-like rings, polynomial rings, and pullbacks. In Section 2.2, we define the notions of $*$-compact and $*$-coprime packedness with respect to a star operation of finite type (Definitions 2.2.1 and 2.2.2) and then examine the possible transfer of these notions to Nagata rings, Theorem 2.2.10, which stands as a t-analogue of Erdogdu's result [15, Theorem 3.1] and polynomial rings, Theorem 2.2.12. Section 2.3 focuses on the t-coprime packedness. Our objective is to seek generalizations or t-analogues of well-known results in the classical case. The first main theorem of this section deals with the context of GCD domains, Theorem 2.3.10, and provides a satisfactory analogue for [16, Theorem 2.5]. We also characterize t- coprimely packed generalized Krull domains, Theorem

2.3.2, and t-coprimely packed t-almost Dedekind domains, Theorem 2.3.6, as a satisfactory analogue of [16, Theorem 2.1]. The last section of Chapter 2 deals with the transfer of the aforementioned notions to special types of pullback constructions in order to provide original examples. Precisely, we characterize the compact and coprime packedness in pullbacks issued from local rings, Theorem 2.4.2. Also, we study the t-compact and t-coprime packedness in pullback constructions, Theorem 2.4.3. Finally, we give an example to illustrate the correlation between (t)-compact and (t)-coprime packedness of integral domains, Example 2.4.4.

Chapter 1

Duals and transforms of ideals in PVMDs

This chapter* studies when the dual of a t-ideal in a PVMD is a ring and treats the question of when it coincides with its endomorphism ring. Also this chapter studies the structure of particular classes of overrings of PVMDs.

1.1 Introduction

Let R be an integral domain and K its quotient field. For nonzero fractional ideals I and J of R, we define the fractional ideal $(I:J)=\{x\in K|xJ\subseteq I\}$. We denote $(R:I)$ by I^{-1} and we call it the dual of an ideal I since it is isomorphic, as an R-module, to $Hom_R(I,R)$. The Nagata transform (or ideal transform) of I is defined as $T(I)=\bigcup_{n=1}^{\infty}(R:I^n)$ and the Kaplansky transform of I is defined as $\Omega(I)=\{u\in K:$ for each $a\in I$ there is a positive integer $n(a)$ with $ua^{n(a)}\in R\}$. The zero cohomology of I over R is defined by $R^I=\bigcup_{n=1}^{\infty}(I^n:I^n)$. It is clear that $(I:I)\subseteq R^I\subseteq T(I)\subseteq\Omega(I)$ and $(I:I)\subseteq I^{-1}\subseteq T(I)\subseteq\Omega(I)$. Also we notice that $\Omega(I)$ is a variant of the Nagata transform $T(I)$, and useful in the case when I is not finitely generated, but if I is a finitely generated ideal of R, then $\Omega(I)=T(I)$. It is

*This work is accepted for publication in Communications in Algebra (in collaboration with A. Mimouni).

worthwhile noting that $\Omega(I)$, $T(I)$, $(I:I)$ and R^I are overrings of R for each ideal I in a domain R, while I^{-1} is not, in general, a ring. Moreover, $(I:I)$ is the largest subring of K in which I is an ideal and it is isomorphic to the endomorphism ring of I.

In 1968, Brewer [4] proved a representation theorem for the Nagata transform $T(I)$, when I is a finitely generated ideal (which coincides in this case with $\Omega(I)$) and in 1974, Kaplansky [36] gave the complete description of the Kaplansky transform $\Omega(I)$ for each ideal I in an integral domain R. He proved that "*if I is a nonzero ideal of R, then $\Omega(I) = \bigcap R_P$, where P varies over the set of prime ideals that do not contain I*" (this result was also obtained independently by Hays [26]). In [24, Exercise 11, page 331] Gilmer described $T(I)$ for an ideal I which is contained in a finite number of minimal prime ideals in a Prüfer domain R, specifically, "*let R be a Prüfer domain, I a nonzero ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, and $\{M_\beta\}$ the set of maximal ideals that do not contain I. Then $T(I) \subseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$, where Q_α is the unique prime ideal determined by $\bigcap_{n=1}^{\infty} I^n R_{P_\alpha} = Q_\alpha R_{P_\alpha}$. Moreover, if the set $\{P_\alpha\}$ is finite, equality holds*" (see also [21, Theorem 3.2.5]). In [21], Fontana, Huckaba and Papick described some relations between the above overrings in the case of Prüfer domains. For instance, they showed that "*if P is a nonzero non-invertible prime ideal of a Prüfer domain R, then there is no proper overring between P^{-1} and $\Omega(P)$*" ([21, Theorem 3.3.7]). In 1986, Houston [28] studied the divisorial prime ideals in PVMDs, and among others, he proved that "*if P is a nonzero, non-t-maximal t-prime ideal of a PVMD R, then $P^{-1} = R_P \cap \mathscr{C}_t(I)$, where $\mathscr{C}_t(I) = \bigcap_{I \not\subseteq M_\beta \in Max_t(R)} R_{M_\beta}$, and $T(P) = R_{P_0} \cap \mathscr{C}_t(I)$, where $P_0 = (\bigcap_n P^n R_P) \cap R$* and $Max_t(R)$ is the set of all t-maximal ideals of R" ([28, Proposition 1.1 and Proposition 1.5]).

Many papers in the literature deal with the fractional ideal I^{-1}. The main problem is to examine settings in which I^{-1} is a ring. In 1982, Huckaba and Papick [32] stated the following: "*let R be a Prüfer domain, I a nonzero ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, and $\{M_\beta\}$ the set of maximal ideals that do not contain I. Then $I^{-1} \supseteq$*

$(\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$. *If* I^{-1} *is a ring, equality holds*" ([32, Theorem 3.2 and Lemma 3.3]). They also proved that "*for a radical ideal* I *of a Prüfer domain* R*, let* $\{P_\alpha\}$ *be the set of minimal prime ideals of* I *and assume that* $\bigcap P_\alpha$ *is irredundant. Then* I^{-1} *is a subring of* K *if and only if for each* α, P_α *is not invertible*" ([32, Theorem 3.8]). In [27], Heinzer and Papick gave a necessary and sufficient condition for I^{-1}, when it is a ring, to be equal to $(I:I)$ for an ideal I in a Prüfer domain with Noetherian spectrum . Namely, they proved that "*for a Prüfer domain* R *with* $\mathrm{Spec}(R)$ *Noetherian, let* I *be a nonzero ideal of* R *and assume that* I^{-1} *is a ring. Then* $I^{-1} = (I:I)$ *if and only if* $I = \sqrt{I}$ *(i.e.,* I *is a radical ideal) if and only if the minimal prime ideals of* I *in* $(I:I)$ *are all maximal ideals*" ([27, Theorem 2.5]). In 1993, Fontana, Huckaba, Papick and Roitman [22] studied the endomorphism ring of an ideal in a Prüfer domain. One of their main results asserted that "*for a nonzero ideal* I *of a Prüfer domain* R*, let* $\{Q_\alpha\}$ *be the set of maximal prime ideals of* $\mathscr{Z}(R,I)$ *and* $\{M_\beta\}$ *be the set of maximal ideals that do not contain* I*. Then* $(I:I) \supseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$*. Moreover, if* R *is a QR-domain, equality holds*" ([22, Theorem 4.11 and Corollary 4.4]). Finally in 2000, Houston, Kabbaj, Lucas and Mimouni [29], gave several characterizations of when I^{-1} is a ring for a nonzero ideal I in an integrally closed domain. For instance they generalized [22, Theorem 4.11] to the PVMD's case. Namely they proved that "*if* I *is an ideal of a PVMD with no embedded primes, then* I^{-1} *is a ring if and only if* $I^{-1} = (I:I) = R_{\mathfrak{N}} \cap \mathscr{C}_t(I)$*, where* $\mathfrak{N}$ *the complement in* R *of the set of zero divisors on* R/I" ([29, Theorem 4.7]).

The purpose of this chapter is to continue the investigation of when the dual of an ideal in a PVMD is a ring and when it coincides with its endomorphism ring. We also aim at giving a full description of the Nagata and Kaplansky transforms of ideals in a PVMD, seeking generalizations or t-analogues of well-known results.

In Section 1.2, we deal with the dual of a t-ideal in a PVMD. We give a generalization of the above mentioned results of Huckaba-Papick and Heinzer-Papick. Precisely, we prove that "*for a radical* t*-ideal* I *of a PVMD* R*, let* $\{P_\alpha\}$ *be the set of minimal prime ideals of*

I and assume that $\bigcap P_\alpha$ is irredundant. Then I^{-1} is a subring of K if and only if P_α is not t-invertible for each α" (Theorem 1.2.3). We also prove that "*if R is a PVMD with $Spec_t(R)$ Noetherian, and I is a t-ideal of R such that I^{-1} is a ring, then $I^{-1} = (I : I)$ if and only if $I = \sqrt{I}$ if and only if the minimal prime ideals of I in $(I : I)$ are all t-maximal ideals*" (Theorem 1.2.6). In the particular case where R is a Prüfer domain we obtain the aforementioned results of Huckaba-Papick and Heinzer-Papick simply by remarking that a Prüfer domain is just a PVMD in which the t-operation is trivial, that is, $t = d$. We close this section with a description of the endomorphism ring of a t-ideal in a tQR-domain, that is, a PVMD R such that each t-linked overring of R is a quotient ring of R (recall that an overring T of R is t-linked over R if for every finitely generated ideal I of R, $I^{-1} = R$ implies that $(T : IT) = T$). Particularly we give a generalization of a well-known result by Fontana et al., [22, Corrollary 4.4 and Theorem 4.11], that is, "*let I be a t-ideal of a PVMD R, $\{Q_\alpha\}$ be the set of all maximal prime ideals of $Z(R,I)$ and $\{M_\beta\}$ be the set of t-maximal ideals of R that do not contain I. Then $(I : I) \supseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$, and if R is a tQR-domain then equality holds*" (Theorem 1.2.14).

Section 1.3 deals with Kaplansky and Nagata transforms of an ideal in a PVMD. Our aim is to give the t-analogues for many results of Fontana-Huckaba-Papick [21, Section 3.3] for t-linked overrings of PVMDs. Our first main theorem generalizes [21, Theorem 3.3.7] to the case of t-prime ideals in a PVMD. For instance we prove that "*if P is a non-t-invertible t-prime ideal of a PVMD R, then there is no proper overring between P^{-1} and $\Omega(P)$*" (Theorem 1.3.2). The second main theorem is a satisfactory t-analogue for [21, Theorem 3.3.10], that is, "*let R be a PVMD and P a t-prime ideal of R. Then $T(P) \subsetneqq \Omega(P)$ if and only if $T(P) = R_P \cap \Omega(P)$ and $\Omega(P) \nsubseteq R_P$. Moreover, $(P\Omega(P))_{t_1} = \Omega(P)$ if and only if $\Omega(P) \nsubseteq R_P$ if and only if $P = \sqrt{I}$ for some t-invertible ideal where t_1 is the t-operation with respect to $\Omega(P)$*" (Theorem 1.3.6). Other applications of the obtained results are given.

Throughout this chapter R is an integral domain with quotient field K. By a fractional ideal, we mean a nonzero R-submodule I of of K such that $dI \subseteq R$ for some nonzero element d of R and by a proper ideal we mean a nonzero ideal I such that $I \subsetneq R$. Recall that for a fractional ideal I of R, the v-closure of an ideal I is the fractional ideal $I_v = (I^{-1})^{-1}$ and the t-closure of an ideal I is the ideal $I_t = \bigcup J_v$, where J ranges over the set of all finitely generated subideals of I. A fractional ideal I is said to be a v-ideal (or divisorial) (resp. t-ideal , resp. t-invertible) if $I = I_v$ (resp. $I = I_t$, resp. $(II^{-1})_t = R$). A t-prime ideal t-prime ideal is a t-ideal which is prime and a t-maximal ideal is a t-prime ideal which is maximal in the set of t-ideals. The set of all t-prime ideals is denoted by $Spec_t(R)$ and the set of all t-maximal ideal is denoted by $Max_t(R)$. A domain R is said to be a PVMD (for Prüfer v-multiplication domain) if every nonzero finitely generated ideal is t-invertible (equivalently, R_M is a valuation domain for every t-maximal ideal M of R). For more basic details about star operations, we refer the reader to [24, sections 32, 34]. Also it is worth noting that many of our results are inspired from the Prüfer case, and some proofs are dense and use a lot of techniques of the t-operation. We are grateful to the huge work on the t-move (from Prüfer to PVMD) done during the last decades.

1.2 Duals of ideals in a PVMD

We start this section by noticing that for a fractional ideal I of a domain R, $I^{-1} = (I_t)^{-1} = (I_v)^{-1}$, I is t-invertible if and only if I_t is t-invertible and if $I_t = R$, then $I^{-1} = (I : I) = R$. In this regard, we will focus on the case where I is a proper t-ideal of R.

Before giving the first main theorem of this section, we begin with the following two results on necessary and sufficient conditions for I^{-1} to be a ring. The first one is a generalization of [32, Lemma 2.0] (since invertible ideals are t-invertible t-ideals) and the second one is a t-analogue of [29, Proposition 2.2].

Lemma 1.2.1. *Let R be a domain and I a t-ideal of R. If I is t-invertible, then I^{-1} is not a* *ring.*

Proof. Deny, assume that I^{-1} is a ring. Let M be a t-maximal ideal of R containing I. Since I is t-invertible, then II^{-1} is not contained in any t-maximal ideal of R. Hence $(II^{-1})_M = R_M$. So IR_M is an invertible ideal of R_M and hence principal. Since I is t-invertible, then I is v-finite. Hence there is a finitely generated ideal A of R such that $A \subseteq I$ and $I = A_t = A_v$. Since A is a finitely generated ideal of R, by [43, Lemma 4], $(AR_M)_{v_1} = (A_vR_M)_{v_1}$, where v_1 is the v-operation with respect to R_M. So $(IR_M)^{-1} = (A_vR_M)^{-1} = (AR_M)^{-1} = A^{-1}R_M = (A_v)^{-1}R_M = I^{-1}R_M$. Since I^{-1} is a ring, $(IR_M)^{-1}$ is also a ring, which contradicts the fact that IR_M is principal in R_M. □

Corollary 1.2.2. *Let I be a t-ideal of a domain R. Then I^{-1} is a ring if and only if I is not t-invertible and $(M : I)$ is a ring for each t-maximal ideal M of R containing I.*

Proof. If I^{-1} is a ring, then I is not t-invertible by Lemma 1.2.1. By [29, Proposition 2.1], $(M : I)$ is a ring for each t-maximal ideal M containing I. Conversely, if I is not t-invertible, then $II^{-1} \subseteq M$ for some t-maximal ideal M of R and hence $I^{-1} = (M : I)$. So I^{-1} is a ring. □

Now, we turn our attention to the duals of ideals in a PVMD. Our approach is similar to that of Huckaba-Papick done in [32] for Prüfer domains. Let R be a PVMD. We divide $Spec_t(R)$, that is, the set of all nonzero t-prime ideals of R, into three disjoint sets:

$S_1 = \{P \in Spec_t(R) : P \text{ is } t \text{ -invertible}\}$

$S_2 = \{P \in Spec_t(R) : P \text{ is a non- } t \text{ -invertible } t \text{ -maximal ideal and } PR_P \text{ is principal}\}$

$S_3 = \{P \in Spec_t(R) : P \notin S_1 \cup S_2\}$. Our first main theorem is a generalization of [32, Theorem 3.8] to PVMDs.

Theorem 1.2.3. *Let I be a radical t-ideal of a PVMD R, $\{P_\alpha\}$ the set of all minimal prime ideals of I and assume that $\bigcap P_\alpha$ is irredundant. Then I^{-1} is a subring of K if and only if P_α is not t-invertible for each α.*

Proof. ($\Rightarrow$) If I^{-1} is a ring, by [29, Proposition 2.1(2)], $(P_\alpha)^{-1}$ is a ring for each α. So, by Lemma 1.2.1, P_α is not t-invertible for each α. Whence $\{P_\alpha\} \subseteq S_2 \cup S_3$.

($\Leftarrow$) By [29, Lemma 4.3], it is enough to prove that $I^{-1} \subseteq (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$ where $\{M_\beta\}$ is the set of all t-maximal ideals of R that do not contain I. Clearly $I^{-1} \subseteq \bigcap R_{M_\beta}$ (for if $x \in I^{-1}$ and $a \in I \backslash M_\beta$, then $x = \frac{xa}{a} \in R_{M_\beta}$). Now we show that $I^{-1} \subseteq \bigcap R_{P_\alpha}$. Let P_α be any minimal prime over I. Since P_α is not t-invertible, $P_\alpha \in S_2 \cup S_3$. If $P_\alpha \in S_2$, set $J := \bigcap_{\gamma \neq \alpha} P_\gamma$. Then $I = J \cap P_\alpha$ and since $\bigcap P_\alpha$ is irredundant, $J \nsubseteq P_\alpha$. But since P_α is a non-t-invertible t-maximal ideal of a PVMD R, $(J + P_\alpha)_t = R$ and $(P_\alpha)^{-1} = R$.

Lemma 1.2.4. *Let R be a PVMD and A and B nonzero ideals of R such that $(A+B)_t = R$. Then $(A \cap B)_t = (AB)_t$.*

Proof. By [35] it suffices to check that $(A \cap B)_t R_M = (AB)_t R_M$ for every t-maximal ideal M of R. Let M be a t-maximal ideal of R. Since A and B are t-comaximal, then either $A \nsubseteq M$ or $B \nsubseteq M$. Without loss of generality, we may assume that $A \nsubseteq M$. Hence, by [33, Lemma 3.3] $(A \cap B)_t R_M = (A \cap B) R_M = AR_M \cap BR_M = R_M \cap BR_M = BR_M = ABR_M = (AB)_t R_M$, as desired. □

Now, by the previous lemma, $I = J \cap P_\alpha = (J \cap P_\alpha)_t = (JP_\alpha)_t$. So $I^{-1} = (JP_\alpha)^{-1} = (R : P_\alpha J) = ((R : P_\alpha) : J) = (R : J) = J^{-1}$. But since $J \nsubseteq P_\alpha$, $I^{-1} = J^{-1} \subseteq R_{P_\alpha}$. Assume that $P_\alpha \in S_3$ and let N be a t-maximal ideal of R properly containing P_α. Since I is a radical ideal of R, $IR_N = P_\alpha R_N$. Since $P_\alpha R_N$ is a nonmaximal prime ideal of the valuation domain R_N, it is not invertible. Hence $I^{-1} \subseteq (I^{-1})_{R \backslash N} \subseteq (R_N : IR_N) = (R_N : P_\alpha R_N) = R_{P_\alpha}$ ([32, Corollary 3.6]), as desired. □

The following example shows that the irredundancy condition in Theorem 1.2.3 cannot be removed. This example is a slight modification of [29, Example 5.1], where the authors constructed an example of a Bezout domain R with a principal ideal I (so I^{-1} is not a ring) such that P^{-1} is a ring for each minimal prime ideal P of I. Our example is just an adjunct of an indeterminate Y to the domain R to get outside the Prüfer situation but keeping us in the context of PVMDs.

Example 1.2.5. *Let $\mathbb{Q}$ be the field of rational numbers and set $T = \mathbb{Q}[\{X^n : n \in \mathbb{Q}^+\}]$ and $J = (X-1)T$. By ([29, Example 5.1]), T is a Bezout domain, J is a principal radical ideal of T (so J^{-1} is not a ring) and P^{-1} is a ring for each minimal P over J in T. Also, by [32, Theorem 3.8], the intersection of the minimal primes of J is not an irredundant intersection. Now let $R = T[Y]$, $I = J[Y]$. Clearly R is a PVMD (which is not Prüfer), and I is a radical principal ideal of R (so $I^{-1} = J^{-1}[Y]$ is not a ring). Since $J = I \cap T \subseteq Q \cap T = P$, it is easy to check that every minimal prime ideal Q of R over I is of the form $Q = P[Y]$, where P is a minimal prime ideal of T over J. Hence $Q^{-1} = P^{-1}[Y]$ is a ring for each Q. Finally $I = J[Y] = (\bigcap P)[Y] = \bigcap P[Y]$ is not an irredundant intersection.*

Let T be an overring of an integral domain R. According to [8], T is said to be t-linked over R if for each finitely generated ideal I of R with $I^{-1} = R$, we have $(IT)^{-1} = T$. Also we say that T is t-flat over R if $T_M = R_P$ for each t-maximal ideal M of T, where $P = R \cap M$ (cf. [38]). Finally, we say that R has a Noetherian t-spectrum ($Spec_t(R)$ is Noetherian) if R satisfies the a.c.c. condition on radical t-ideals.

Our second main theorem generalizes Heinzer-Papick's theorem [27, Theorem 2.5].

Theorem 1.2.6. *Let R be a PVMD with $Spec_t(R)$ Noetherian, and let I be a t-ideal of R. Assume that I^{-1} is a ring. Then the following conditions are equivalent:*

(1) $I^{-1} = (I : I)$;

(2) $I = \sqrt{I}$;

(3) *The minimal prime ideals of I in $(I : I)$ are all t-maximal ideals.*

The proof of this theorem involves several lemmas of independent interest, some of them are t-analogues of well-known results.

Lemma 1.2.7. *Let T be a t-flat overring of a domain R. The following equivalent conditions hold:*

(1) $I_t \subseteq (IT)_{t_1}$ *for each $I \in F(R)$, where t_1 is the t-operation with respect to T.*

(2) *If J is a t-ideal of T and $J \cap R \neq 0$, then $J \cap R$ is a t-ideal of R.*

(3) $I_v T \subseteqq (IT)_{v_1}$ *for each $I \in f(R)$, where v_1 is the v-operation with respect to T.*

(4) $(IT)_{v_1} = (I_v T)_{v_1}$ *for each $I \in f(R)$.*

(5) $(IT)_{t_1} = (I_t T)_{t_1}$ *for each $I \in F(R)$.*

(6) $(IT)_{v_1} = (I_t T)_{v_1}$ *for each $I \in F(R)$.*

Proof. The six conditions are equivalent for an arbitrary overring T of R by [2, Proposition 1.1]. To prove (i) , let $x \in I_t$. Then there is a finitely generated ideal J of R such that $J \subseteq I$ and $x(R:J) \subseteq R$. Now, let N be a t-maximal ideal of T and set $M = N \cap R$. Since T is t-flat over R, $T_N = R_M$. Since J is finitely generated, $x(T:JT)T_N = x(T_N:JT_N) = x(R_M:JR_M) = x(R:J)R_M \subseteq R_M = T_N$. Hence $x(T:JT) \subseteq T$ and so $x \in (JT)_{v_1} \subseteq (IT)_{t_1}$, as desired. □

The next lemma is crucial and it is a generalization of [24, Theorem 26.1]. We will often use it whenever we want to prove that an overring T of a PVMD R is contained in R_Q for some t-prime ideal Q of R.

Lemma 1.2.8. *Let R be a PVMD and T a t-linked overring of R. Then:*

(1) *If M is a t-prime ideal of T, then $T_M = R_P$ and $M = PR_P \cap T$, where $P = M \cap R$.*

(2) *If P is a nonzero t-prime ideal of R, then $(PT)_{t_1} \neq T$ if and only if $R_P \supseteq T$, where t_1 is the t-operation with respect to T.*

(3) *If J is a t-ideal of T and $I = J \cap R$, then $J = (IT)_{t_1}$.*

(4) *$\{(PT)_{t_1}\}_{P\in\Delta}$ is the set of all t-prime ideals of T, where $\Delta = \{P \in Spec_t(R) : (PT)_{t_1} \neq T\}$.*

Proof. (i) Since T is a t-linked overring of a PVMD R, T is a t-flat overring of R ([38, Proposition 2.10]). Hence $R_P = T_M$ where $P = M \cap R$ ([10, Theorem 2.6]). Therefore $M = MT_M \cap T = PR_P \cap T$. (ii) If $(PT)_{t_1} \subsetneqq T$, then there is a t-maximal ideal M of T such that $M \supseteq (PT)_{t_1}$. Since $M \cap R \supseteq (PT)_{t_1} \cap R \supseteq PT \cap R \supseteq P$, $R_P \supseteq R_{M\cap R} = T_M \supseteq T$, as desired. Conversely, if $R_P \supseteq T \supseteq R$, then $T_{R\setminus P} = R_P$. Hence R_P is t-linked over T. So, by Lemma 1.2.7, $(PT)_{t_1} \subseteq (PR_P)_{t_2} = PR_P \subsetneqq R_P$ (here t_2 is the t-operation with respect to R_P and it is trivial since R_P is valuation). Since $T_{R\setminus P} = R_P$ is a valuation overring of a PVMD T, $J_{t_1}T_{R\setminus P} = JT_{R\setminus P}$ for each ideal J of T. If $(PT)_{t_1} = T$, then $R_P = T_{R\setminus P} = (PT)_{t_1}T_{R\setminus P} = PT_{R\setminus P} = PR_P$, a contradiction. Therefore $(PT)_{t_1} \subsetneqq T$.

(iii) Clearly $(IT)_{t_1} \subseteq J$. It suffices to show that $J \subseteq (IT)_{t_1}$. Let $\{M_\alpha\}$ be the set of all t-maximal ideals of T. Since T is a t-linked overring of R, T is a PVMD. Hence $J = \bigcap JT_{M_\alpha}$. Set $P_\alpha = M_\alpha \cap R$ for each α and let $x \in JR_{M_\alpha} = JR_{P_\alpha}$. Then $x = \frac{a}{t}$, where $a \in J$ and $t \in R \setminus P_\alpha$. Since $J \subseteq T \subseteq T_{M_\alpha} = R_{P_\alpha}$, then $a = \frac{b}{s}$, where $b \in R$ and $s \in R \setminus P_\alpha$. Hence $b = as \in J \cap R = I$. So $x = \frac{b}{st} \in IR_{P_\alpha} \subseteq (IT)R_{P_\alpha} = (IT)T_{M_\alpha}$. Therefore $J \subseteq (IT)_{t_1}$, as desired.

(iv) By (iii) , each t-prime ideal of T is of the form $(PT)_{t_1}$ for some $P \in \Delta$. Conversely, if $P \in \Delta$, then $P_tR_P = PR_P$ is a t-prime ideal of R_P ([33, Lemma 3.3] and R_P is a valuation domain) and $T \subseteq R_P$ (by part(ii)). So $R_P = T_{R\setminus P}$ and then R_P is t-linked over T. Hence $PR_P \cap T$ is a t-prime ideal of T (Lemma 1.2.7) and $PR_P \cap T = (((PR_P \cap T) \cap R)T)_{t_1} = (PT)_{t_1}$ by (iii) . □

The next lemma is a generalization of [27, Lemma 2.4] and it relates the condition I^{-1} not being a ring to a kind of “separation property” for a minimal prime ideal over a t-ideal of a PVMD.

Lemma 1.2.9. *Let R be a PVMD, I a t-ideal of R and P a minimal prime ideal of I in R. If* *there is a finitely generated ideal J of R such that $I \subseteq J \subseteq P$, then I^{-1} is not a ring.*

Proof. By way of contradiction, assume that I^{-1} is a ring. Then by [29, Theorem 4.5] and [35, Theorem 2.22], $I^{-1} \subseteq R_P$ and I^{-1} is a t-linked overring of R. So R_P is t-linked over I^{-1}. Since $J^{-1} \subseteq I^{-1}$, $R = (JJ^{-1})_t \subseteq (JI^{-1})_{t_1}$ where t_1 is the t-operation with respect to I^{-1} (Lemma 1.2.7). Also by Lemma 1.2.7, $(PI^{-1})_{t_1} \subseteq (PR_P)_{t_2} = PR_P$ (where t_2 is the t-operation with respect to R_P, so it is trivial). Therefore $1 \in R = (JJ^{-1})_t \subseteq (JI^{-1})_{t_1} \subseteq (PI^{-1})_{t_1} \subseteq PR_P$, which is a contradiction. □

Lemma 1.2.10. ([34, Lemma 2.8]) *Let R be a PVMD and I a t-ideal of R. Then I is a t-ideal of $(I : I)$.*

Lemma 1.2.11. ([10, Lemma 3.7)] *Let R be an integral domain. The following conditions are equivalent.*

(i) *Each t-prime ideal is the radical of a v-finite ideal.*

(ii) *Each radical t-ideal is the radical of a v-finite ideal.*

(iii) *$Spec_t(R)$ is Noetherian.*

Proof of Theorem 1.2.6 $(ii) \Rightarrow (i)$ Follows from [1, Proposition 3.3] without any more conditions.

$(i) \Rightarrow (ii)$ Deny, assume that $I \subsetneq \sqrt{I}$. Then there is a t-maximal ideal M of R such that IR_M is not a radical ideal. Moreover, there is a prime ideal P contained in M and minimal over I with $IR_M \subsetneq PR_M$ and $\sqrt{IR_M} = PR_M$. Note that P is a t-prime ideal of R (as a minimal prime over a t-ideal).

Claim 1. $IR_P = PR_P$.

Deny. Let $b \in P$ such that $IR_P \subsetneq bR_P \subseteq PR_P$. Since $Spec_t(R)$ is Noetherian, $P = \sqrt{(a_1, \ldots, a_r)_v}$ for some $a_1, \ldots, a_r \in P$. Set $J := (a_1, \ldots, a_r, b)$. Note that $P = \sqrt{J_v}$ ($P =$

$\sqrt{(a_1,\ldots,a_r)_v} \subseteq \sqrt{(a_1,\ldots,a_r,b)_v} \subseteq P$). Now, we prove that $I \subseteq J \subseteq P$, which contradicts the assumption that I^{-1} is a ring by Lemma 1.2.9. Let N be a t-maximal ideal of R. If $P \nsubseteq N$, then $R_N = PR_N = \sqrt{J_vR_N} = \sqrt{J_tR_N} = \sqrt{JR_N}$ ([33, Lemma 3.3]). Hence $JR_N = R_N \supseteq IR_N$. Assume that $P \subseteq N$. Then $PR_P = PR_N$ since R_P is an overring of the valuation domain R_N. Since $IR_P \subsetneqq bR_P$, $b^{-1}I \subsetneqq R_P$ and so $b^{-1}I \subseteq PR_P = PR_N \subseteq R_N$. Hence $IR_N \subseteq bR_N \subseteq JR_N$ as desired.

Now since R_M is a valuation domain, $Z(R_M, IR_M) = QR_M$ for some t-prime ideal $Q \subseteq M$. Since R is a PVMD and P and Q are t-primes contained in M, Q and P are comparable under inclusion. Moreover, let $x \in PR_M \setminus IR_M$. Since $PR_M = PR_P = IR_P$ (Claim 1), there exists $y \in R \setminus P$ such that $yx \in I$. Hence $y \in Z(R_M, IR_M) \cap R = Q$ and therefore $P \subsetneqq Q$.

Claim 2. $(QI^{-1})_{t_1} = I^{-1}$.

Note that $I^{-1} = (I : I)$ is a subintersection of R ([29, Theorem 4.5]) and so I^{-1} is t-linked over R ([35, Theorem 2.22]). Since $Spec_t(R)$ is Noetherian, $Q = \sqrt{A_v}$ for some finitely generated ideal A of R. Say $A = \sum_{n=1}^{n=m} b_nR$. Since $P \subsetneqq Q$, $P \subsetneqq A_v$. Indeed, let N be a t-maximal ideal of R. If $Q \nsubseteq N$, then $PR_N \subseteq R_N = QR_N = AR_N$. If $Q \subseteq N$, then AR_N and PR_N are comparable as ideals of the valuation domain R_N. But if $AR_N \subseteq PR_N$, then $QR_N = \sqrt{A_vR_N} = \sqrt{A_tR_N} = \sqrt{AR_N} \subseteq PR_N$ and so $Q \subseteq P$, which is absurd. Hence $PR_N \subsetneqq AR_N$ and therefore $P \subsetneqq A_t = A_v$. Now since $I \subseteq P \subseteq A_v$, $A^{-1} \subseteq I^{-1}$. So $1 \in R = (AA^{-1})_t \subseteq (AI^{-1})_t \subseteq (AI^{-1})_{t_1} \subseteq (QI^{-1})_{t_1}$ (Lemma 1.2.7). Hence $(QI^{-1})_{t_1} = I^{-1}$, as desired.

Finally, by Lemma 1.2.8, $I^{-1} \nsubseteq R_Q$. On the other hand $(I : I) \subseteq (I : I)R_M \subseteq (IR_M : IR_M) = (R_M)_{QR_M} = R_Q$ by [21, Lemma 3.1.9], which is absurd. It follows that I is a radical ideal of R.

$(iii) \Rightarrow (ii)$ Assume that all minimal prime ideals of I in $(I : I)$ are t-maximal ideals. If $I \subsetneqq \sqrt{I}$, as in the proof of $(i) \Rightarrow (ii)$, there exist two t-prime ideals P and Q of R such that $I \subseteq P \subsetneqq Q$ and $(I : I) \subseteq R_Q$. Then $(I : I)_{R \setminus Q} = R_Q$ and so R_Q is t-linked over $(I : I)$. Hence

$QR_Q \cap (I:I)$ and $PR_Q \cap (I:I)$ are t-prime ideals of $(I:I)$ with $I \subseteq PR_Q \cap (I:I) \subsetneqq QR_Q \cap (I:I)$ which is absurd.

$(i) \Rightarrow (iii)$ Assume that $I^{-1} = (I:I)$ and let P be a prime of $(I:I)$ minimal over I. By Lemma 1.2.10, I is a t-ideal of $(I:I)$ and so P is a t-prime ideal of $(I:I)$ (as a prime minimal over a t-ideal). Now by a way of contradiction, assume that there is a t-prime ideal Q of $(I:I)$ such that $P \subsetneqq Q$. Since $(I:I)$ is a t-linked overring of R, $P = (P'(I:I))_{t_1}$ and $Q = (Q'(I:I))_{t_1}$ for some t-prime ideals P' and Q' of R with $I \subseteq P' \subsetneqq Q'$ (Lemma 1.2.8(iv)). Set $Q' = \sqrt{A}$ for some finitely generated ideal A of R. As in the proof of Claim 2, $I \subseteq P' \subseteq A_t$. So $A^{-1} \subseteq I^{-1} = (I:I)$. Hence $1 \in R = (AA^{-1})_t \subseteq (A(I:I))_{t_1} \subseteq (Q'(I:I))_{t_1} = Q$, which is absurd. It follows that P is a t-maximal ideal of $(I:I)$, completing the proof. □

The next two results deal with the duals of primary t-ideals in a PVMD.

Proposition 1.2.12. (cf. [20, Lemma 4.4]) *Let R be a PVMD and I a primary t-ideal of R. If I^{-1} is a ring, then $I^{-1} = (I:I)$.*

Proof. Deny, assume that there is $x \in I^{-1} \backslash (I:I)$. Since I is a t-ideal of R, there is $a \in I$ and a t-maximal ideal M of R containing I such that $xa \notin IR_M$. Since I^{-1} is a ring, $I^{-1} = (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$ where $\{P_\alpha\}$ and $\{M_\beta\}$ are respectively the sets of all prime minimal ideals of I and t-maximal ideals do not containing I ([29, Theorem 4.5]). Let P_α be a minimal prime of I with $P_\alpha \subseteq M$. Then $x \in R_{P_\alpha}$. Write $x = \frac{b}{s}$ where $b \in R$ and $s \in R \backslash P_\alpha$. If $t = \frac{s}{a} \in R_M$, then $s = ta \in P_\alpha R_M \cap R = P_\alpha$, which is a contradiction. If $\frac{a}{s} \in R_M$, since I is a primary ideal of R, $ax = a\frac{b}{s} = b\frac{a}{s} \in IR_{P_\alpha} \cap R_M = IR_M$, which is a contradiction too. It follows that $I^{-1} = (I:I)$. □

Corollary 1.2.13. (cf. [21, Proposition 3.1.14]) *Let R be a PVMD with $Spec_t(R)$ Noetherian and I a t-ideal of R. If I is a primary ideal which is not prime, then I^{-1} is not a ring .*

Proof. Deny, assume that I^{-1} is a ring. Then $I^{-1} = (I:I)$ by Proposition 1.2.12. Therefore I is a radical ideal (and so prime) by Theorem 1.2.6, which is absurd. □

According to [24, Section 27], a Prüfer domain R is called a QR-domain if each overring of R is a quotient ring of R. In [9] the authors defined tQR-domains as PVMDs R such that each t-linked overring of R is a quotient ring of R and they characterized tQR-domains as follows: "*Let R be a PVMD. Then R is a tQR-domain if and only if for each finitely generated ideal A of R, there is $n \geq 1$ and $b \in R$ such that $A^n \subseteq bR \subseteq A_v$*" [9, Theorem 1.3].
We close this section with a third main theorem. It generalizes well-known results by Fontana et al. [22, Corrollary 4.4 and Theorem 4.11] and gives a description of $(I : I)$ for a t-ideal I in a PVMD that is a tQR-domain.

Theorem 1.2.14. *Let I be a t-ideal of a PVMD R, $\{Q_\alpha\}$ be the set of all maximal prime ideals of $Z(R,I)$ and $\{M_\beta\}$ be the set of t-maximal ideals of R that do not contain I. Then:*

(1) $(I : I) \supseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$;

(2) *If R is a tQR-domain, then equality holds.*

Before proving this theorem, we need the following lemma.

Lemma 1.2.15. *Let I be a t-ideal of a PVMD R. Then $Z(R,I) = \bigcup Q$ where Q ranges over the set of all t-prime ideals contained in $Z(R,I)$. Q's are called the primes of $Z(R,I)$ and the primes of $Z(R,I)$ that are maximal for the inclusion are called the maximal primes of $Z(R,I)$.*

Proof. First we claim that $Z(R,I) = \bigcup_{M \in M_t(R,I)} Z(R_M, IR_M) \cap R$. Indeed, let $x \in Z(R,I)$. Then there is $a \in R \backslash I$ such that $ax \in I$. Since I is a t-ideal, there is a t-maximal ideal M containing I such that $a \in R_M \backslash IR_M$. Since $ax \in IR_M$, $x \in Z(R_M, IR_M) \cap R$. Conversely, let $M \in Max_t(R,I)$ and let $z \in Z(R_M, IR_M) \cap R$. Then there is $\frac{c}{t} \in R_M \backslash IR_M$ such that $\frac{zc}{t} \in IR_M$ with $c \in R \backslash I$ and $t \in R \backslash M$. Hence $szc \in I$ for some $s \in R \backslash M$. If $cs \in I$, then $c = \frac{i}{s} \in IR_M$. Thus $\frac{c}{t} \in IR_M$, a contradiction. Hence $cs \notin I$ and then $z \in Z(R,I)$, as desired. Now, clearly $Z(R,I) \supseteq \bigcup Q$. Conversely, if $x \in Z(R,I)$, then $x \in Z(R_M, IR_M) \cap R$ for some t-maximal ideal M containing

I. Set $Q = Z(R_M, IR_M) \cap R$. Then Q is a t-prime ideal of R ([35, Corollary 2.47]), $x \in Q$ and $Q \subseteq Z(R, I)$, as desired. Finally, note that Q's are exactly $(Z(R_M, IR_M) \cap R)$'s, where M ranges over the set of all t-maximal ideals of R containing I. □

Proof of Theorem 1.2.14. (i) Let $u \in (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$ and $a \in I$. It is enough to prove that $ua \in I$. Since $u \in \bigcap R_{M_\beta}$, it suffices to show that $ua \in IR_{N_\gamma}$ for each γ, where $\{N_\gamma\}$ is the set of t-maximal ideals of R containing I. By [24, Corollary 4.6], $\bigcap R_{Q_\alpha} = R_{R \setminus \cup Q_\alpha}$. Write $u = \frac{r}{s}$, where $r \in R$ and $s \in R \setminus \cup Q_\alpha$. Fix γ and set $Q = Z(R_{N_\gamma}, IR_{N_\gamma}) \cap R$. Then Q is a prime of $Z(R, I)$ by Lemma 1.2.15 and $I \subseteq Q \subseteq N_\gamma$. Let Q_{α_0} be a maximal prime of $Z(R, I)$ containing Q. We claim that $\frac{a}{s} \in R_{N_\gamma}$. For if not, then $\frac{s}{a} = t \in R_{N_\gamma}$ and thus $s = at \in IR_{N_\gamma} \cap R \subseteq QR_{N_\gamma} \cap R = Q \subseteq Q_{\alpha_0}$, a contradiction. Hence $\frac{a}{s} \in R_{N_\gamma}$ and so $ua = a\frac{r}{s} = r\frac{a}{s} \in R_{N_\gamma}$. Thus if $ua \notin IR_{N_\gamma}$, then $sua = ra \in I \subseteq IR_{N_\gamma}$, and so $s \in Z(R_{N_\gamma}, IR_{N_\gamma}) \cap R = Q \subseteq Q_{\alpha_0}$, a contradiction. Therefore $ua \in IR_{N_\gamma}$, as desired.

(ii) Set $T := (I : I)$. Clearly $T \subseteq \bigcap R_{M_\beta}$. Now we will prove that $T \subseteq \bigcap R_{Q_\alpha}$. Since R is a PVMD and I is a t-ideal, T is t-linked over R . Hence $T = R_S$ for some multiplicative closed set S of R since R is a tQR-domain. By Lemma 1.2.8(ii), it suffices to show that $(Q_\alpha T)_{t_1} \neq T$ for each α. By way of contradiction, assume that $(QT)_{t_1} = T$ where $Q = Q_\alpha$ for some α. Then there exists a finitely generated ideal B such that $B_{v_1} = T$ and $B \subseteq QT$. Say $B = \sum_{i=1}^{i=r} a_n T$ with $a_i \in QT$ and write $a_i = \sum_{s=1}^{s=m_i} q_{is} t_{is}$ with $q_{is} \in Q$ and $t_{is} \in T$ for each $i = 1, \dots, n$ and $s = 1, \dots, m_i$. Now let A be the finitely generated ideal of R generated by all $q'_{is}s$. Then $A \subseteq Q$ and $B \subseteq AT$. Hence $T = B_{v_1} \subseteq (AT)_{v_1} \subseteq (A_v T)_{v_1} \subseteq T$ and therefore $(AT)_{v_1} = (A_v T)_{v_1} = T$. Since R is a tQR-domain and T is t-linked over R, by [8, Proposition 2.17], $A_v T = T$. But since $A_v = A_t \subseteq Q$ (here Q is a t-prime ideal by Lemma 1.2.15), $QT = T$. Hence $1 = \sum_{i=1}^{i=n} q_i a_i$ where $q_i \in Q$ and $a_i \in T$. Set $J = \sum_{i=1}^{i=n} q_i R$. Clearly $JT = T$ and by induction $J^s T = T$ for all positive integer s. Since R is a tQR-domain, there is a positive integer N and $d \in R$ such that $J^N \subseteq dR \subseteq J_v = J_t \subseteq Q$. Since $J^N T = T$, then

$1 = \sum_{i=1}^{i=s} \lambda_i y_i$ where $\lambda_i \in J^N$ and $y_i \in T$, and since $J^N \subseteq dR$, there exists $\mu_i \in R$ such that $\lambda_i = d\mu_i$ for each i. Now, since $d \in Q \subseteq Z(R,I)$, there exists $r \in R \setminus I$ such that $rd \in I$. Hence $r = \sum_{i=1}^{i=s} r\lambda_i y_i = \sum_{i=1}^{i=s} rdy_i\mu_i \in IT = I$, a contradiction. Hence $(QT)_{t_1} \subsetneqq T$ and by Lemma 1.2.8, $T \subseteq R_Q$, completing the proof. □

1.3 Ideal transform overrings of a PVMD

We start this section with the following theorem which is a generalization of [21, Theorem 3.2.5]. As the proof is similar to that of [21, Theorem 3.2.5] simply by replacing maximal ideals by t-maximal ideals, we omit it here.

Theorem 1.3.1. *Let R be a PVMD, I a t-ideal of R, $\{P_\alpha\}$ the set of minimal prime ideals of I, and $\{M_\beta\}$ the set of t-maximal ideals of R that do not contain I. Then:*

(1) *$T(I) \subseteq (\bigcap R_{Q_\alpha}) \cap (\bigcap R_{M_\beta})$, where Q_β is the unique prime ideal determined by $\bigcap_{n=1}^{\infty} I^n R_{P_\alpha}$;*

(2) *The equality holds, if I has a finitely many minimal primes.*

Our next theorem generalizes [21, Theorem 3.3.7] to PVMDs.

Theorem 1.3.2. *Let P be a non-t-invertible t-prime ideal of a PVMD R. Then there is no proper overring of R between P^{-1} and $\Omega(P)$.*

The proof of this theorem involves the following lemmas.

Lemma 1.3.3. *Let R be a PVMD, I a t-ideal of R and let T be a t-linked overring of R contained in $\Omega(I)$. Then there is one-to-one correspondence between the sets $S_1 = \{P \in Spec_t(R) : P \not\supseteq I\}$ and $S_2 = \{Q \in Spec_t(T) : Q \not\supseteq IT\}$.*

Proof. Define $\Psi : S_1 \to S_2$ by $\Psi(P) = PR_P \cap T = Q$ for each $P \in S_1$. Then Ψ is well-defined. Indeed, let $P \in S_1$. Since $T \subseteq \Omega(I)$, $T \subseteq R_P$. So $T_{R\setminus P} = R_P$ and then R_P is a

t-linked overring of T. Hence $PR_P \cap T$ is a t-prime of T. Also, if $x \in I \setminus P$, then $x \in IT \setminus Q$ and the injectivity of Ψ is clear.

Now, let $Q \in S_2$ and set $P := R \cap Q$. Then $P \not\supseteq I$, and since $R_P = T_Q$, $PR_P = QT_Q$. Hence $\Psi(P) = PR_P \cap T = QT_Q \cap T = Q$. □

Lemma 1.3.4. *Under the same notation as Lemma 1.3.3, if $(IT)_{t_1} = T$, then $T = \Omega(I)$.*

Proof. Assume that $(IT)_{t_1} = T$. Then IT is not contained in any t-prime ideal of T. Since R is a PVMD and T is a t-linked overring of R, T is a PVMD. By Lemma 1.3.3, $T = \bigcap_{Q \in Spec_t(T)} T_Q = \bigcap_{P \in Spec_t(R), P \not\supseteq I} R_P \supseteq \Omega(I)$. Hence $T = \Omega(I)$. □

Proof of Theorem 1.3.2. Let T be an overring of R such that $P^{-1} \subsetneqq T \subseteq \Omega(P)$ and let $\{M_\beta\}$ be the set of all t-maximal ideals of R that do not contain P. By [21, Theorem 3.2.2], $T \subseteq \Omega(P) \subseteq \bigcap R_{M_\beta}$. If $(PT)_{t_1} \neq T$, then $T \subseteq R_P$ (Lemma 1.2.8(ii)). So $P^{-1} \subsetneqq T \subseteq R_P \cap (\bigcap R_{M_\beta}) = P^{-1}$ ([28, Proposition 1.2]), which is a contradiction. Hence $(PT)_{t_1} = T$, and so $T = \Omega(P)$ by Lemma 1.3.4.□

Corollary 1.3.5. (cf. [21, Corollary 3.3.8]) *Let P be a non t-invertible t-prime ideal of a PVMD R. Then:*

(1) *$P^{-1} = T(P)$ or $T(P) = \Omega(P)$;*

(2) *If $P \neq (P^2)_t$, then $T(P) = \Omega(P)$;*

(3) *If $P = (P^2)_t$, then $P^{-1} = T(P)$;*

(4) *If P is unbranched, then $P^{-1} = T(P) = \Omega(P)$.*

Proof. (i) Follows from Theorem 1.3.2.

(ii) If $P \neq (P^2)_t$, then there is a prime ideal Q of R such that $\bigcap (P^n)_t R_P = QR_P$. Note that $P \nsubseteq Q$ (otherwise, if $P = Q$, then $PR_P = QR_P$. But $QR_P \subseteq (P^2)_t R_P = P^2 R_P \subsetneqq PR_P$, a contradiction). Hence $T(P) \supseteq R_Q \cap (\bigcap R_{M_\beta}) \supseteq \Omega(P)$, where $\{M_\beta\}$ is the set of all t-maximal ideals of R that do not contain I. Since $T(P) \subseteq \Omega(P)$, $T(P) = \Omega(P)$.

(iii) If $P = (P^2)_t$, then $P = (P^n)_t$ for each $n \geq 1$. Hence $(R : P^n) = (R : (P^n)_t) = (R : P)$. So $T(P) = P^{-1}$ by the definition of $T(P)$.

(iv) Since P is unbranched and $(P^2)_t$ is a P-primary ([28, Proposition 1.3]), $P = (P^2)_t$. Hence $T(P) = P^{-1}$ by (iii) . It is clear that $\Omega(P) \supseteq T(P)$. By [11, Proposition 1.2], $P = \bigcup P_\gamma$ where $\{P_\gamma\}$ is the set of primes ideal of R properly contained in P, and we may assume that they are maximal with this property. Then by [24, Corollary 4.6], $R_P = \bigcap R_{P_\gamma}$. Hence by [21, Theorem 3.2.2], $\Omega(P) \subseteq R_P$. Since $\Omega(P) \subseteq \bigcap R_{M_\beta}$, $\Omega(P) \cap R_P \subseteq \bigcap R_{M_\beta} \cap R_P$. It follows that $\Omega(P) \subseteq P^{-1} = T(P)$. Therefore $T(P) = \Omega(P)$. □

Our last theorem generalizes [21, Theorem 3.3.10].

Theorem 1.3.6. *Let R be a PVMD and P a t-prime ideal of R. Then:*

(1) $T(P) \subsetneq \Omega(P)$ *if and only if* $T(P) = R_P \cap \Omega(P)$ *and* $\Omega(P) \nsubseteq R_P$.

(2) *The following conditions are equivalent:*

(i) $(P\Omega(P))_{t_1} = \Omega(P)$*;*

(ii) $\Omega(P) \nsubseteq R_P$*;*

(iii) $P = \sqrt{I}$ *for some t-invertible ideal I.*

The proof of this theorem involves the following lemmas. First we notice that in [24], Gilmer mentioned that $IT(I) = T(I)$ for any invertible ideal I of an arbitrary domain R. Our first lemma provides a t-analogue result in the class of PVMDs. Note that one can replace the condition “PVMD” on R by assuming that $T(I)$ is a t-flat overring of R.

Lemma 1.3.7. *Let I be an ideal of a domain R.*

(i) *If I is t-invertible and R is a PVMD, then $(IT(I))_{t_1} = T(I)$ where t_1 is the t-operation with respect to $T(I)$.*

(ii) *If I and J are two ideals of a domain R such that $\sqrt{I} = \sqrt{J}$, then $\Omega(I) = \Omega(J)$.*

Proof. (i) Since I is t-invertible, then there is a finitely generated ideal A of R such that $A \subseteq I_t$ and $A_t = I_t$. Then $T(I) = T(I_t) = T(A_t) = T(A) = \Omega(A)$ and hence $T(I)$ is a t-linked overring of R. Since I is t-invertible, then $(II^{-1})_t = R$ and hence $(I(R:I^n))_t = (R:I^{n-1})$ for each $n \geq 2$. Since $I(R:I^n) \subseteq (I(R:I^n))T(I)$ for each n, then $(I(R:I^n))_t \subseteq (IT(I))_{t_1}$ for each n (Lemma 1.2.7). Hence $\bigcup(I(R:I^n))_t \subseteq (IT(I))_{t_1}$. So $T(I) = \bigcup(I(R:I^n))_t \subseteq (IT(I))_{t_1} \subseteq T(I)$ and therefore $(IT)_{t_1} = T(I)$, as desired. (ii) Straightforward via [21, Theorem 3.2.2]. □

Lemma 1.3.8. (cf. [24, Proposition 25.4]) *Let R be a PVMD and $A_1, \ldots., A_n, B$ and C be nonzero fractional ideals of R. Then:*

(1) *If for each i, A_i is t-finite, then $\bigcap_{i=1}^{n} A_i$ is t-finite.*

(2) *If B is t-finite, then $(C:B) = (CB^{-1})_t$.*

(3) *If B and C are t-finite, then $(C:_R B)$ is t-finite.*

Proof. (1) It suffices to prove it for $n = 2$. We have $((A_1 \cap A_2)(A_1 + A_2))_t = (A_1A_2)_t$ ([25, Theorem 5]). Since A_1 and A_2 are t-invertible, A_1A_2 is t-invertible and therefore $A_1 \cap A_2$ is t-invertible and so t-finite.

(2) If $x \in (R:B)C$, then $x = \sum_{i=1}^{n} b_i c_i$ where $b_iB \subseteq R$ and $c_i \in C$. Hence $xB = \sum c_i b_i B \subseteq RC \subseteq C$. So $(R:B)C \subseteq (C:B)$. Therefore $((R:B)C)_t \subseteq (C:B)_t = (C:B)$. Conversely, we have $B(C:B) \subseteq C$. Then $(C:B) = (C:B)_t = ((C:B)BB^{-1})_t \subseteq (CB^{-1})_t$.

(3) By definition, $(C:_R B) = (C:_R B)_t = ((C:B) \cap R)_t = ((CB^{-1})_t \cap R)_t$. Since C and B are t-finite, $(CB^{-1})_t$ is t-finite. So by (1), $(C:_R B)$ is t-finite. □

Proof of Theorem 1.3.6 (1) Assume that $T(P) \subsetneqq \Omega(P)$. Then P is a non-t-invertible t-prime ideal of R (otherwise, if P is t-invertible, then P is t-finite, i.e., there is a finitely generated ideal A of R such that $P = A_t$. Hence $\Omega(P) = \Omega(A_t) = \Omega(A) = T(A) = T(A_t) = T(P)$, a contradiction). If $P = M$ is a non-t-invertible t-maximal ideal of R, then $M^{-1} = R$ and so $(R:M^n) = R$ for all positive integers n. Hence $T(M) = R = R_M \cap \Omega(M)$. Also if $\Omega(M) \subseteq$

R_M, then $\Omega(M) = R = T(M)$, a contradiction. Hence $\Omega(M) \nsubseteq R_M$. Assume that P is a non-t-maximal t-prime ideal. By Theorem 1.3.2, $P^{-1} = T(P)$. Hence $T(P) = R_P \cap \Omega(P)$ by [28, Proposition 1.1] and [21, Theorem 3.2.2]. Therefore $\Omega(P) \nsubseteq R_P$.

The converse is trivial.

(2) $(i) \Rightarrow (ii)$ By [21, Theorem 3.2.2] and [5, Proposition 4], $\Omega(P)$ is a t-linked overring of R. Since $(P\Omega(P))_{t_1} = \Omega(P)$, $\Omega(P) \nsubseteq R_P$ by Lemma 1.2.8(ii).

$(ii) \Rightarrow (iii)$ Let $\{Q_\alpha\}$ be the set of all t-prime ideals of R that do not contain P. Choose $x \in \Omega(P) \setminus R_P$. Write $x = \frac{a}{b}$ where $a, b \in R$. If $I = (bR :_R aR)$, then $I \nsubseteq Q_\alpha$ for each α and $I \subseteq P$. By Lemma 1.3.8, I is t-finite and $\sqrt{I} = P$. For this if $z \notin \sqrt{I}$, then $z^n \notin A_v$ for each finitely generated ideal A of R such that $A \subseteq I$. Hence $z^n ab^{-1} \notin R$ for each n. Since $ab^{-1} \in \Omega(P)$, $z \notin P$.

$(iii) \Rightarrow (i)$ Since $P = \sqrt{I}$, $\Omega(P) = \Omega(I)$ by Lemma 1.3.7(ii). Since I is t-invertible, by Lemma 1.3.7 $(IT(I))_{t_1} = T(I)$. Also since I is t-invertible, there is a finitely generated ideal A of R such that $A \subseteq I$ and $I_t = A_t$. Hence $T(I) = T(I_t) = T(A_t) = T(A) = \Omega(A) = \Omega(A_t) = \Omega(I_t) = \Omega(I)$ by [19, Proposition 3.4]. So $\Omega(P) = \Omega(I) = (I\Omega(I))_{t_1} \subseteq (P\Omega(I))_{t_1} = (P\Omega(P))_{t_1} \subseteq \Omega(P)$.□

Corollary 1.3.9. (cf. [21, Corollary 3.3.11]) *Let R be a PVMD and P a non-t-maximal t-prime ideal of R. Then $T(P) \subsetneqq \Omega(P)$ if and only if $P = (P^2)_t$ and $P = \sqrt{I}$ for some t-invertible ideal I of R*

Proof. $\Rightarrow$) Since $T(P) \subsetneqq \Omega(P)$, $P = (P^2)_t$ (Corollary 1.3.5(ii)) and $\Omega(P) \nsubseteq R_P$ (Theorem 1.3.6). Hence there is a t-invertible ideal I of R with $P = \sqrt{I}$ (Theorem 1.3.6).

$\Leftarrow$) $P = (P^2)_t$ implies that $P^{-1} = T(P)$ by Corollary 1.3.5(iii). Since $P = \sqrt{I}$ for some t-invertible ideal I of R, $\Omega(P) \nsubseteq R_P$ by Theorem 1.3.6. By [29, Theorem 4.5] $P^{-1} = R_P \cap (\bigcap R_{M_\beta})$, where $\{M_\beta\}$ is the set of all t-maximal ideals of R that do not contain$P$. By [21, Theorem 3.2.2], $T(P) = P^{-1} = R_P \cap \Omega(P)$. By Theorem 1.3.6, $T(P) \subsetneqq \Omega(P)$. □

Corollary 1.3.10. (cf. [21, Corollary 3.3.12]) *Let R be a PVMD and P a non-t-invertible t-prime ideal of R. Then:*
$(PT(P))_{t_1} \neq T(P)$ and $(P\Omega(P))_{t_2} = \Omega(P)$ if and only if $P^{-1} = T(P) \subsetneq \Omega(P)$ where t_1 (resp. t_2) is the t-operation with respect to $T(I)$ (resp. $\Omega(I)$).

Proof. If $(PT(P))_{t_1} \neq T(P)$ and $(P\Omega(P))_{t_2} = \Omega(P)$, then clearly $T(P) \subsetneq \Omega(P)$. Hence $P^{-1} = T(P)$ by Theorem 1.3.2. Conversely, if $P^{-1} = T(P) \subsetneq \Omega(P)$, then $(PT(P))_{t_1} \neq T(P)$ by Lemma 1.3.4. Moreover $P = \sqrt{I}$ for some t-invertible ideal I of R by Corollary 1.3.9. Therefore $(P\Omega(P))_{t_2} = \Omega(P)$ by Theorem 1.3.6. □

Chapter 2

Compact and coprime packedness with respect to star operations

This chapter* studies the notions of compactly packed ring and coprimely packed ring with respect to a star operation of finite type.

2.1 Introduction

Let R be a commutative ring. An ideal I of R is said to be compactly packed (resp. coprimely packed) by prime ideals of R if whenever $I \subseteq \bigcup_{\alpha\in\Omega} P_\alpha$, where $\{P_\alpha\}_{\alpha\in\Omega}$ is a family of prime ideals of R, I is actually contained in P_α (resp. $I + P_\alpha \subsetneq R$) for some $\alpha \in \Omega$; and R is said to be a compactly packed domain (resp. a coprimely packed domain) if every ideal of R is compactly (resp. coprimely) packed. The notions of compactly packed (or *CP*-ring for short) was introduced by Reis and Viswanathan, [41], where Noetherian *CP*-rings were characterized by the property that prime ideals are radicals of principal ideals. The notion of coprimely packed rings was introduced by Erdoğdu in 1988 [12], and intensively studied

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in a series of papers, for instance see [12, 13, 14, 15, 16], [18], [7] and [42]. In this chapter we extend these notions to a domain with an arbitrary star operation of finite type in the following way: Let R be a domain and $*$ be a star operation of finite type on R. A $*$-ideal I of R is said to be a $*$-compactly packed ideal (resp. $*$-coprimely packed ideal) if whenever $I \subseteq \bigcup_{\alpha\in\Omega} P_\alpha$, where $\{P_\alpha\}_{\alpha\in\Omega}$ is a family of $*$-prime ideals of R, I is actually contained in P_α (resp. $(I+P_\alpha)_* \subsetneq R$) for some $\alpha \in \Omega$; and R is said to be $*$-compactly (resp. $*$-coprimely) packed if every $*$-ideal of R is $*$-compactly (resp. $*$-coprimely) packed. In the particular case where $* = d$ is the trivial operation on R, we obtained the so-called compactly and coprimely packed rings. We also study various aspects of these notions in many different classes of integral domains such as Nagata rings, polynomial rings, Prüfer-like rings, pullbacks etc.

In Section 2.2, we define the notions of $*$-coprime and $*$-compact packedness with respect to a star operation of finite type and we give a diagram summarizing different implications between these notions. We then concentrate on the transfer of the above notions to Nagata rings and polynomial rings. Our first main result states that given a star operation $*$ of finite type, if R is $*$-coprimely packed, then its Nagata ring $Na(R,*)$ with respect to $*$ is coprimely packed (Theorem 2.2.10). The second main result establishes a connection between the $[*]$-compact packedness of the polynomial ring $R[X]$ and $\tilde{*}$-compact packedness of R where $[*]$ is the extension to $R[X]$ of a star operation of finite type $*$ on R and $\tilde{*}$ its spectral star operation. Precisely we prove that for an integral domain R and given a star operation of finite type $*$ on R, the polynomial ring $R[X]$ is a $[*]$-compactly packed domain if and only if R is a $\tilde{*}$-compactly packed domain and each $[*]$-prime ideal of $R[X]$ is either an upper to zero or extended, and each upper to zero is a set theoretic complete intersection (Theorem 2.2.12).

During the last decades, the t-operation (as the largest star operation of finite type) has been intensively studied, probably for its ability to classify many classes of integral domains as a generalizations of well-known domains. For instance, Bezout domains (i.e., every f. g. ideal is principal) to GCD domains (i.e., for every f. g. ideal I, I_t is principal), Dedekind domains (i.e., every ideal is invertible) to Krull domains (i.e., every ideal is t-invertible), Prüfer domains (i.e., every f. g. ideal is invertible) to PVMD (i.e., every f. g. ideal is t-invertible) etc. In this regard, the third section focuses on the t-coprime packedness. Our objective is to seek for generalizations or t-analogues of well-known results in the classical case. The first main theorem of this section is a satisfactory analogue of [16, Theorem 2.1]. Precisely we prove that for a t-almost Dedekind domain R (i.e., R_M is Dedekind for every t-maximal ideal M), R is t-coprimely packed if and only if $R < X >$ is t-coprimely packed if and only if $R[X]$ is t-coprimely packed if and only if each t-prime ideal of $R[X]$ is a set theoretic complete intersection if and only if R is a Krull domain with torsion class group (Theorem 2.3.6). The second main theorem is a generalization of [16, Theorem 2.5]. Thus, for a GCD domain R, consider the following statements:

(i) Every t-prime ideal of $R[X]$ is a set theoretic complete intersection.

(ii) $R[X]$ is t-compactly packed.

(iii) $R[X]$ is t-coprimely packed.

(iv) R is t-coprimely packed.

Then $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$. Moreover, if $t-dimR=1$, then the statements are equivalent (Theorem 2.3.10).

The last section deals with the transfer of the pre-mentioned notions to special type of pullback constructions in order to provide original examples.

Throughout R is an integral domain with quotient field L, $F(R)$ is the set of all nonzero fractional ideals of R, i.e., all R-submodules A of L such that $dA \subseteq R$ for some nonzero $d \in R$, and $f(R)$ is the set of all nonzero finitely generated R-submodules of L. Then $f(R) \subseteq F(R)$. A mapping $F(R) \to F(R)$, $E \mapsto E^*$ is called a star operation on R if for all nonzero $x \in L$ and $E, F \in F(R)$, the following properties are satisfied:

$(*_1)$ $(xE)^* = xE^*$ and $R^* = R$

$(*_2)$ $E \subseteq E^*$, and $E \subseteq F$ implies $E^* \subseteq F^*$

$(*_3)$ $E^{**} = E^*$.

A star operation $*$ on R is said to be of finite type (or of finite character) if $E^* = \cup\{F^* | F \in f(R), F \subseteq E\}$ for each $E \in F(R)$. For each star operation $*$ on R, we associate a star operation of finite type $*_f$ defined by $E^{*_f}| = \cup\{F^* : F \in f(R), F \subseteq E\}$ for each $E \in F(R)$. Obviously, a star operation $*$ is of finite type if and only if $* = *_f$. An ideal I is said to be a $*$-ideal if $I = I^*$. A $*$-prime ideal is a prime ideal that is a $*$-ideal and a $*$-maximal ideal is a (prime) $*$-ideal which is maximal in the set of all $*$-ideals. Notice that if $*$ is of finite type, then every $*$-ideal is contained in a $*$-maximal ideal and every minimal prime of a $*$-ideal is $*$-prime.

Finally, let $SFc(R)$ the set of all star operations of finite type on R, and for $* \in SFc(R)$, let $\text{Spec}_*(R)$ be the set of all $*$-prime ideals of R, $\text{Max}_*(R)$ the set of all $*$-maximal ideals of R and $X^1(R)$ the set of all height-one prime ideals of R. Also for a $*$-ideal I of R, $\text{Max}_*(R,I)$ will denote the set of all $*$-maximal ideals of R containing I, $\overline{\text{Max}_*(R,I)}$, the set of all $*$-maximal ideals do not containing I and if $I = aR$ is a principal ideal, we use the notation $\text{Max}_*(R,a)$ for $\text{Max}_*(R,aR)$. Unreferenced material is standard, typically as in [24] or [36].

2.2 General results

Definition 2.2.1. *Let R be an integral domain and $*$ be a star operation of finite type on R. The following statements are equivalent.*

(i) *For every $*$-ideal I of R and $\{P_\alpha\}_{\alpha\in\Omega}$ a family of $*$-prime ideals of R, $I \subseteq \bigcup P_\alpha$ implies that $I \subseteq P_\alpha$ for some $\alpha \in \Omega$.*

(ii) *For every $*$-ideal I of R, $\sqrt{I} = \sqrt{aR}$ for some $a \in I$.*

(iii) *Every $*$-prime ideal of R is the radical of a principal ideal.*

A domain R is said to be a $*$-compactly packed domain if R satisfies one of the above equivalent conditions.

Proof. Similar to [40, Theorem 1], but for the convenience of the reader we include a brief proof here.

$(i) \Longrightarrow (ii)$. Let I be a $*$-ideal and $\{Q_\alpha\}_\alpha$ be the set of all $*$-prime ideals of R do not containing I. By (i), $I \nsubseteq \bigcup Q_\alpha$. Let $a \in I \setminus \bigcup Q_\alpha$. It is easy to check that $Min(I) = Min(aR)$ and therefore $\sqrt{I} = \sqrt{aR}$.

$(ii) \Longrightarrow (iii)$ Trivial.

$(iii) \Longrightarrow (i)$ Assume that $I \subseteq \bigcup P_\alpha$. Let Q be a minimal prime ideal of I such that $Q \subseteq \bigcup P_\alpha$ (this is always possible since $S = R \setminus \bigcup P_\alpha$ is a multiplicative set of R with $I \cap S = \emptyset$. Thus there is a prime ideal Q containing I such that $Q \cap S = \emptyset$. Then shrink Q to a minimal prime over I). Since Q is $*$-prime, by (iii) $Q = \sqrt{aR}$. But $Q \subseteq \bigcup P_\alpha$ implies that $a \in P_\alpha$ for some P_α. Hence $I \subseteq Q = \sqrt{aR} \subseteq P_\alpha$, as desired. □

Definition 2.2.2. *Let R be an integral domain and $*$ be a star operation of finite type on R. The following statements are equivalent.*

(i) *For every $*$-ideal I of R and $\{P_\alpha\}_{\alpha\in\Omega}$ a family of $*$-prime ideals of R, $I \subseteq \bigcup P_\alpha$ implies that $(I+P_\alpha)_* \subsetneq R$ for some $\alpha \in \Omega$.*

(ii) *For every $*$-prime ideal P of R and $\{P_\alpha\}_{\alpha\in\Omega}$ a family of $*$-prime ideals of R, $P \subseteq \bigcup P_\alpha$ implies that $(P+P_\alpha)_* \subsetneq R$ for some $\alpha \in \Omega$.*

(iii) *For every $*$-ideal I of R and $\{M_\alpha\}_{\alpha\in\Omega}$ a family of $*$-maximal ideals of R, $I \subseteq \bigcup M_\alpha$ implies that $I \subseteq M_\alpha$ for some $\alpha \in \Omega$.*

(iv) *For every $*$-ideal I of R, $I \nsubseteq \bigcup\{M | M \in \overline{\mathrm{Max}_*(R,I)}\}$.*

(v) *For every $*$-prime P of R and $\{M_\alpha\}_{\alpha\in\Omega}$ a family of $*$-maximal ideals of R, $P \subseteq \bigcup M_\alpha$ implies that $P \subseteq M_\alpha$ for some $\alpha \in \Omega$.*

(vi) *For every $*$-prime ideal P of R, there exists $b \in P$ such that $j-rad_*(P) = j-rad_*(bR)$,* *where* $j-rad_*(I)=\bigcap\{M|M\in \mathrm{Max}_*(R,I)\}$.

A domain R is said to be a $*$*-coprimely packed domain if R satisfies one of the above equivalent conditions.*

Proof. $(i)\Longrightarrow(ii)$ is trivial and for $(ii)\Longrightarrow(i)$, let I be a $*$-ideal of R and $\{P_\alpha\}_{\alpha\in\Omega}$ a family of $*$-prime ideals of R such that $I\subseteq\bigcup P_\alpha$. Set $S=R\setminus\bigcup P_\alpha$. Then S is a multiplicative set of R and $I\cap S=\emptyset$. Let P be a minimal prime ideal of I such that $P\cap S=\emptyset$. Then P is a $*$-prime ideal and $P\subseteq\bigcup P_\alpha$. By (ii), $(I+P_\alpha)_*\subseteq(P+P_\alpha)_*\subsetneq R$, as desired. The proof of the other assertions is similar to [18, Lemma 2]. □

Definition 2.2.3. *Let R be an integral domain and* $*$ *be a star operation of finite type on R. The following statements are equivalent.*

(i) *For every ideal I of R and* $\{P_\alpha\}_{\alpha\in\Omega}$ *a family of* $*$*-prime ideals of R,* $I\subseteq\bigcup P_\alpha$ *implies that* $(I+P_\alpha)_*\subsetneq R$ *for some* $\alpha\in\Omega$.

(ii) *For every prime ideal P of R and* $\{P_\alpha\}_{\alpha\in\Omega}$ *a family of* $*$*-prime ideals of R,* $P\subseteq\bigcup P_\alpha$ *implies that* $(P+P_\alpha)_*\subsetneq R$ *for some* $\alpha\in\Omega$.

(iii) *For every ideal I of R and* $\{M_\alpha\}_{\alpha\in\Omega}$ *a family of* $*$*-maximal ideals of R,* $I\subseteq\bigcup M_\alpha$ *implies that* $I\subseteq M_\alpha$ *for some* $\alpha\in\Omega$.

(iv) *For every ideal I of R,* $I\nsubseteq\bigcup\{M|M\in\overline{\mathrm{Max}_*(R,I)}\}$.

(v) *For every prime P of R and* $\{M_\alpha\}_{\alpha\in\Omega}$ *a family of* $*$*-maximal ideals of R,* $P\subseteq\bigcup M_\alpha$ *implies that* $P\subseteq M_\alpha$ *for some* $\alpha\in\Omega$.

A domain R is said to be a $(d,*)$*-domain if R satisfies the above statements are equivalent.*

Proof. $(i)\Longrightarrow(ii)$ Trivial.

$(ii)\Longrightarrow(iii)$ Set $S=R\setminus\bigcup_\alpha M_\alpha$. Then S is a multiplicative set of R and $S\cap I=\emptyset$. Let P be a prime ideal of R such that $P\cap S=\emptyset$ and $I\subseteq P$. Then $P\subseteq\bigcup_\alpha M_\alpha$ and by (ii), $(P+M_\beta)_*\subsetneq R$ for some β. Since M_β is $*$-maximal, $I\subseteq P\subseteq M_\beta$.

$(iii)\Longrightarrow(iv)$ Trivial.

$(iv) \Longrightarrow (v)$ Suppose that $P \nsubseteq M_\alpha$ for each α. Then $\{M_\alpha\}_{\alpha\in\Omega} \subseteq \overline{\mathrm{Max}_*(R,P)}$. So $P \subseteq \bigcup M_\alpha \subseteq \bigcup\{M|M \in \overline{\mathrm{Max}_*(R,P)}\}$, which contradicts (iv).

$(v) \Longrightarrow (i)$ Suppose that $I \subseteq \bigcup P_\alpha$ and for each α, let M_α be a $*$-maximal ideal such that $P_\alpha \subseteq M_\alpha$. Set $S = R \setminus \bigcup_\alpha M_\alpha$. Then S is a multiplicative of R and $S \cap I = \emptyset$. Let P be a prime ideal of R such that $P \cap S = \emptyset$ and $I \subseteq P$. Then $P \subseteq \bigcup_\alpha M_\alpha$ and by (v) $I \subseteq P \subseteq M_\beta$ for some β. Since $P_\beta \subseteq M_\beta$, $I + P_\beta \subseteq M_\beta$ and therefore $(I+P_\beta)_* \subseteq M_\beta \subsetneq R$. □

Remark 2.2.4. (1) *Let $*_1 \leq *_2$ be two star operations of finite type on R. If R is $*_1$-compactly packed, then R is $*_2$-compactly packed. The converse is not true. Indeed, let k be a field and X and Y indeterminates over k. Set $R = k[X,Y]$. Clearly $Spect_t(R) = X^1(R)$ and since R is a UFD, every t-prime of R is principal. Hence R is t-compactly packed. However R is not compactly packed since R is two-dimensional Noetherian domain [13, Proposition 1].*

(2) *If $* = d$, then (d,d)-domains are exactly the coprimely packed domains.*

(3) *If $* - dimR = 1$, then $*$-compact and $*$-coprime packedness coincide.*

Proposition 2.2.5. *Let R be a domain and $*$ be a star operation of finite type on R. Then R is a $(d,*)$-domain if and only if R is coprimely packed and* $\mathrm{Max}(R) = \mathrm{Max}_*(R)$.

Proof. Assume that R is a $(d,*)$-domain and let M be a maximal ideal of R. Then $M \subseteq \bigcup_{m\in M} M_m$, where M_m is a $*$-maximal ideal of R containing m. Thus, $M \subseteq M_{m_0}$ for some $m_0 \in M$ and therefore $M = M_{m_0}$. Hence $\mathrm{Max}(R) \subseteq \mathrm{Max}_*(R)$. On the other hand, if $Q \in \mathrm{Max}_*(R)$, then $Q \subseteq M$ for some maximal ideal M of R. But since M is a $*$-maximal ideal of R, $M = Q$ and therefore $\mathrm{Max}(R) = \mathrm{Max}_*(R)$. Now the coprime packedness and the converse are clear. □

The diagram in Figure 1 summarizes the relations between all these classes of integral domains where the implications are, in general, irreversible. Note that straight arrows for implications and arcs for irreverences.

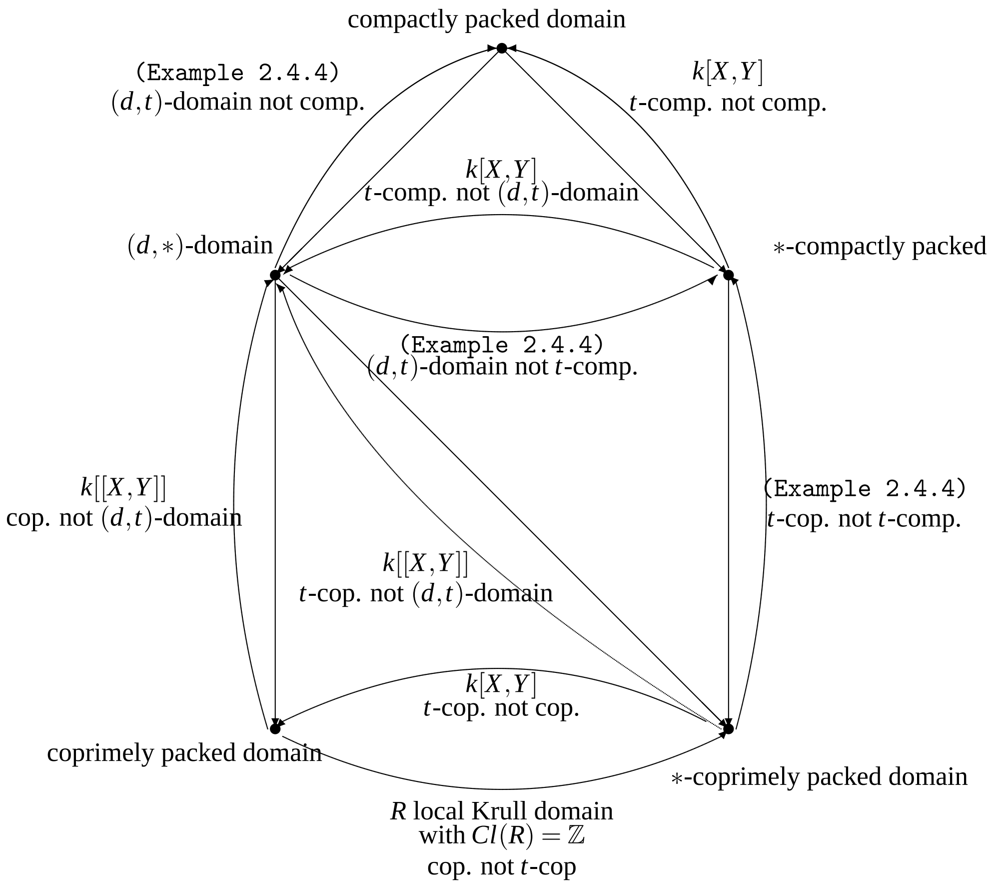

Figure 2.1: Relations between $*$-compact, $*$-coprime packedness and $(d,*)$-domains.

Now we turn our attention to the ascent and descent. Let R be a domain, S a multiplicative closed set of R and $*$ a star operation of finite type on R. In [30], the authors defined a star operation of finite type $*_S$ on R_S as follows: For every nonzero fractional ideal F of R_S, if $F = ER_S$ for some fractional ideal E of R, $F^{*_S} = (ER_S)^{*_S} = E^*R_S$ (notice that $*_S$ does not depend on the choice of E).

Lemma 2.2.6. *Assume $R, S, *$ as above. Then:*

(1) $(ER_S)^{*_S} = (E^*R_S)^{*_S}$.

(2) *If E is a $*$-ideal of R, then ER_S is a $*_S$-ideal of R_S.*

(3) *If ER_S is a $*_S$-ideal of R_S, then $ER_S \cap R$ is a $*$-ideal of R.*

(4) *Let P be a $*$-prime ideal of R which is disjoint from S. Then PR_S is a $*_S$-prime ideal of*

R_S.

(5) *If M is a $*$-maximal ideal of R which is disjoint from S, then MR_S is a $*_S$-maximal ideal of R_S.*

Proposition 2.2.7. *Assume $R, S, *$ as above and assume that $Spec_*(R)$ is a tree. If R is a $*$-coprimely packed domain, then R_S is a $*_S$-coprimely packed domain.*

Proof. Straightforward via Lemma 2.2.6. □

Proposition 2.2.8. *Assume $R, *$ as in Lemma 2.2.6 and assume that S is the complement of the union of a set of $*$-maximal ideals of R. If R is a $*$-coprimely packed domain, then R_S is a $*_S$-coprimely packed domain.*

Proof. Say $S = R \setminus \bigcup N_\beta$. Let PR_S be a $*_S$-prime ideal of R_S and $\{M_\alpha R_S\} \subseteq Max_{*_S}(R_S)$ such that $PR_S \subseteq \bigcup M_\alpha R_S$. By Lemma 2.2.6, P is a $*$-prime ideal of R, $\{M_\alpha\} \subseteq Spec_*(R)$ and $M_\alpha \cap S = \phi$ for each α. Hence for each α, $M_\alpha \subseteq \bigcup N_\beta$ and so $M_\alpha \subseteq N_\beta$ for some β since R is a $*$-coprimely packed domain. Thus $M_\alpha R_S \subseteq N_\beta R_S$ and since $M_\alpha R_S$ is a $*_S$-maximal ideal of R_S (Lemma 2.2.6), $M_\alpha R_S = N_\beta R_S$. Hence $M_\alpha = N_\beta$ is a $*$-maximal ideal of R for each α. Now, $PR_S \subseteq \bigcup M_\alpha R_S$ implies that $P \subseteq \bigcup M_\alpha$. So $P \subseteq M_{\alpha_0}$ for some α_0 since R is $*$-coprimely packed. Therefore $PR_S \subseteq M_{\alpha_0} R_S$, as desired. □

Proposition 2.2.9. *Assume $R, S, *$ as in Lemma 2.2.6. If R is a $*$-compactly packed domain, then R_S is a $*_S$-compactly packed domain.*

Proof. Straightforward. □

Let R be a domain and $* \in SF(R)$. According to [23] the Nagata ring of R with respect to $*$ (or the $*$-Nagata ring of R) is the ring defined by $Na(R, *) := R[X]_{N^*}$ where $N^* = \{f \in R[X] : f \neq 0 \text{ and } c(f)^* = R\}$. In the particular case where $* = d$ is the trivial star operation, $Na(R, d)$ coincides with the classical Nagata domain $R(X)$ as defined in ([39, Chapter I, §6, p.18] and [24, Section 33]). Our first main theorem deals with the transfer of the $*$-coprime packedness from R to its $*$-Nagata ring.

Theorem 2.2.10. *Let R be a domain and $* \in SF(R)$. If R is $*$-coprimely packed, then $Na(R,*)$ is coprimely packed.*

Proof. Let P' be a prime ideal of $Na(R,*)$ and $\{M'_\alpha\} \subseteq \mathrm{Max}(Na(R,*))$ such that $P' \subseteq \bigcup M'_\alpha$. Then there is a prime ideal P of $R[X]$ such that $P' = P_{N^*}$ and for each α there is a $*$-maximal ideal M_α of R such that $M'_\alpha = M_\alpha[X]_{N^*}$ ([23, Proposition 3.1]. Now, let $f \in P$. Then there is α_0 such that $f \in M'_{\alpha_0}$. So there is $g \in N^*$ such that $fg \in M_{\alpha_0}[X]$. Since $g \notin M_{\alpha_0}[X]$, $f \in M_{\alpha_0}[X]$. Therefore $P \subseteq \bigcup M_\alpha[X]$. We claim that $(c(P))^* \subseteq \bigcup M_\alpha$. Indeed, let $a \in (c(P))^*$. Then there is a finitely generated ideal $A = (a_1, a_2, ..., a_r) \subseteq c(P)$ such that $a \in A^*$. So, for each $1 \leq i \leq r$, a_i is a linear combination of coefficients of polynomials $f_{i,1}, f_{i,2}, ..., f_{i,s_i}$ of P of degree $q_{i,1}, q_{i,2}, ..., q_{i,s_i}$ respectively. Set $f_i = f_{i,1} + X^{q_{i,1}+1} f_{i,2} + X^{q_{i,1}+q_{i,2}+2} f_{i,3} + ... + X^{q_{i,1}+....+q_{i,s_i}+s_i-1} f_{i,s_i}$ and assume that f_i is of degree p_i. Then $f_i \in P$ and $a_i \in c(f_i)$ for each i. Now set $f = f_1 + X^{p_1+1} f_2 + ... + X^{p_1+...+p_r+r-1} f_r$. Then $f \in P$ and clearly $a \in A^* \subseteq (c(f))^*$. But since $f \in P \subseteq \bigcup M_\alpha[X]$, $f \in M_\beta[X]$ for some β and thus $c(f) \subseteq M_\beta$. Therefore $a \in (c(f))^* \subseteq M^*_\beta = M_\beta$, as claimed. Now since R is $*$-coprimely packed, $(c(P))^* \subseteq M_\alpha$ for some α. Therefore $P \subseteq c(P)[X] \subseteq M_\alpha[X]$ and hence $P' \subseteq M'_\alpha$, as desired. □

Let R be a domain, L its quotient field, X and Y indeterminates over R and S a multiplicative set of $R[X]$. In [6, Theorem 2.1], Chang and Fontana defined a stable semistar operation of finite type $\circlearrowleft_S$ on R as follows: $E^{\circlearrowleft_S} := ER[X]_S \cap K$ for each $E \in F(R)$. If $S \subseteq N_t$, then $\circlearrowleft_S$ is a star operation of finite type on R and if S is extended, that is, $S = R[X] \setminus \bigcup\{P[X] : P \in \text{ Spec } (D) \text{ and } P[X] \cap S = 0\}$, then $Na(R, \circlearrowleft_S) = R[X]_S$. More generally, given a (semi)star operation $*$ on R, the authors defined a (semi)star operation $[*]$ on $R[X]$ as follows: Set $D_1 := R[X], K_1 := L(X)$ and take the following subset of

Spec(D_1): $\Delta_1^* := \{Q_1 \in \text{Spec}(D_1) | Q_1 \cap R = 0$ or $Q_1 = (Q_1 \cap R)[X]$ and $(Q_1 \cap R)^{*_f} \subsetneq R\}$. Set $S_1^* := D_1[Y] \setminus (\bigcup\{Q_1[Y] | Q_1 \in \Delta_1^*\})$. Then take $[*] = \circlearrowleft_{S_1^*}$ ([6, Theorem 2.3]). Our second main theorem examines the (decent) $*$-compact packedness between R and $R[X]$.

Corollary 2.2.11. *Let R be a domain and $S \subseteq N_t$ a multiplicative closed set of $R[X]$. If R is a $\circlearrowleft_S$-coprimely packed domain, then $Na(R, \circlearrowleft_S)$ is a coprimely packed domain.*

Recall that an ideal I is said to be a set theoretic complete intersection ideal if $\sqrt{I} = \sqrt{(a_1, \ldots, a_n)}$ where $n = htI$ (htI is the height of I, i. e., the infinimum of the heights of prime divisors of I).

Theorem 2.2.12. *Let R be a domain and $*$ be a star operation of finite type on R. Then $R[X]$ is a $[*]$-compactly packed domain if and only if R is a $\tilde{*}$-compactly packed domain and each $[*]$-prime ideal of $R[X]$ is either an upper to zero or extended, and each upper to zero is a set theoretic complete intersection.*

Proof. Let P be a $\tilde{*}$-prime ideal of R. Then $P[X]$ is a $[*]$-prime ideal of $R[X]$ ([6, Theorem 2.3(d)]). Hence there is $f \in P[X]$ such that $P[X] = \sqrt{fR[X]}$. Let $0 \neq a \in P$. Then there is an integer n and $g \in R[X]$ such that $a^n = fg$. Thus $f = c$ would be a constant in P and hence $P = \sqrt{cR}$. Therefore R is $\tilde{*}$-compactly packed. Now, let Q be a $[*]$-prime ideal of $R[X]$ such that $0 \neq P = Q \cap R$. Then, $Q = \sqrt{fR[X]}$ for some $f \in Q$, and as above, $f = c \in P$ and hence $Q = P[X]$, as desired.

Conversely, let Q be a $[*]$-prime ideal of $R[X]$. If $Q \cap R = 0$, we are done. If $0 \neq P = Q \cap R$, then $Q = P[X]$ and P is a $\tilde{*}$-prime ideal of R ([6, Theorem 2.3(d)]). Since R is a $\tilde{*}$- compactly packed domain, then there is $a \in P$ such that $P = \sqrt{aR}$. Therefore $Q = \sqrt{aR}[X] = \sqrt{aR[X]}$, as desired. □

2.3 Compact and coprime packedness with respect to the t-operation

We start this section with a characterization of Krull domains that are t-compactly packed.

Proposition 2.3.1. *Let R be a Krull domain. Then R is t-compactly packed if and only if the class group of R is torsion.*

Proof. Let R be a Krull t-compactly packed domain. By [35, Theorem 6.8], it suffices to prove that each t-maximal ideal of R has a principal t-power. Let M be a t-maximal ideal of R. Then $M = \sqrt{xR}$ for some $x \in M$. Hence xR_M is an MR_M-primary ideal in R_M. Since R_M is a DVR, then $xR_M = (MR_M)^n = M^n R_M$ for some positive integer n. Therefore $(M^n)_t = xR$ since $(M^n)_t R_M = M^n R_M$ and $(M^n)_t$ is M-primary in the PVMD R ([28, Proposition 1.3]). So the class group of R is torsion.
Conversely, let P be a t-prime ideal of R. Since $t - dimR = 1$, $P = M$ is a t-maximal ideal of R. Thus $P = M = \sqrt{(M^n)_t} = \sqrt{xR}$, as desired. □

We recall that an overring T of R is t-linked over R if for every finitely generated ideal I of R, $(R : I) = R$ implies that $(T : IT) = T$. A domain R has Noetherian t-spectrum if it satisfies the *acc* on radical t-ideals. Generalized Krull domains (or GK-domain for short) as defined in [10] are particular classes of PVMD with Noetherian t-spectrum. Finally, according to [9], a PVMD R is a tQR-domain if each t-linked overring of R is a quotient ring of R. Our next result characterizes t-compactly packed domains in the context of PVMDs with Noetherian t-spectrum.

Theorem 2.3.2. *Let R be a Generalized Krull domain. Then R is t-compactly packed if and only if R is a tQR-domain.*

Proof. Let I be a finitely generated ideal of R. If $I_t = R$, we are done. Assume that $I_t \subsetneq R$. Then $Min(I_t)$ is finite since R is a GK-domain ([10, Theorem 3.9]), say $Min(I_t) = \{P_1, ..., P_n\}$. We note that P_i is a t-prime ideal of R for each i. Since R is t-compactly packed, then there is $x_i \in P_i$ such that $P_i = \sqrt{x_i R}$, for each i. Set $x = x_1 ... x_n$. Then $Min(xR) = Min(I_t)$. Indeed, if P is a minimal prime ideal of xR, then $P_1 \cap .... \cap P_n = \sqrt{x_1 R} \cap .... \cap \sqrt{x_n R} = \sqrt{x_1 ... x_n R} = \sqrt{xR} \subseteq P$. Hence $P_i \subseteq P$ for some i. Therefore $P = P_i$, since P is minimal over xR and $x \in P_i$. Therefore $\sqrt{I_t} = \sqrt{xR}$ and hence R is a tQR-domain.

Conversely, Let P be a t-prime ideal of R. Since R is a GK-domain, then there is a finitely generated ideal I of R such that $P = \sqrt{I_t}$ ([10, Theorem 3.5]). Since R is a tQR-domain, then $P = \sqrt{I_t} = \sqrt{xR}$ for some $x \in I_t$, as desired ([9, Theorem 1.3]). □

The next corollary is an immediate consequence of Theorem 2.2.12.

Corollary 2.3.3. *Let R be an integral domain and X an indeterminate over R. The following are equivalent:*

(i) *$R[X]$ is t-compactly packed.*

(ii) *R is t-compactly packed, every t-prime of $R[X]$ is either an upper to zero or extended from R and every upper to zero is a set theoretic complete intersection.*

Proof. Follows immediately from Theorem 2.2.12 since w-compact packedness implies t-compact packedness and $w - \text{Max}(A) = t - \text{Max}(A)$ for any integral domain A. □

The next proposition deals with the t-coprime packedness of the set $Max_t(R)$.

Proposition 2.3.4. *(cf. [15, Proposition 2.2]) For a PVMD R, the following are equivalent.*

(i) *$Max_t(R)$ is t-coprimely packed.*

(ii) *Each t-maximal ideal M of R contains a principal ideal I such that $\sqrt{I}$ is a t-prime ideal contained only in M.*

Proof. Let M be a t-maximal ideal of R and let $\{N_\alpha\}_{\alpha \in \Omega}$ be the set of all t-maximal ideals of R distinct than M.

$(i) \Longrightarrow (ii)$ Let $m \in M \setminus \bigcup_{\alpha\in\Omega} N_\alpha$ and set $I = mR$. Clearly M is the unique t-maximal ideal of R containing I and every minimal prime ideal P of I is a t-prime ideal and contained in M since it contains m. But since R is a PVMD, the prime ideals under a t-maximal ideal form a chain. Hence $\sqrt{I}$ is prime, as desired.

$(ii) \Longrightarrow (i)$ Let M be a t-maximal ideal of R and $I = aR$ a principal ideal such that $\sqrt{I}$ is prime and $Max_t(R,I) = \{M\}$. If $M \subseteq \bigcup_{\alpha\in\Omega} N_\alpha$, then $aR = I \subseteq \bigcup_{\alpha\in\Omega} N_\alpha$ and then $a \in N_\alpha$ for some $\alpha \in \Omega$. Hence $N_\alpha \in Max_t(R,I) = \{M\}$ and therefore $M = N_\alpha$, absurd. It follows that R is t-coprimely packed. □

Corollary 2.3.5. *Let R be a domain. If R is t-coprimely packed, then $Na(R,t)$ is both coprimely and t-coprimely packed.*

Proof. Follows from Theorem 2.2.10 since $\text{Max}(Na(R,t)) = Max_t(Na(R,t))$. □

Recall that a domain R is t-almost Dedekind domain (t-ADD for short) if R_M is a DVR for each t-maximal ideal M of R ([35]). Our next theorem is a satisfactory analogue of [16, Theorem 2.1].

Theorem 2.3.6. *Let R be a t-ADD domain. Then the following conditions are equivalent:*

(1) *R is a t-coprimely packed domain;*

(2) *$R < X >$ is a t-coprimely packed domain;*

(3) *$R[X]$ is a t-coprimely packed domain;*

(4) *Each t-prime ideal of $R[X]$ is a set theoretic complete intersection;*

(5) *R is a Krull domain with torsion class group.*

The proof of this theorem requires the following lemma which is a t-analogue of [15, Theorem 2.1].

Lemma 2.3.7. *Let R be a t-ADD domain. Then $Max_t(R)$ is t-coprimely packed if and only if R is a Krull domain with torsion class group.*

Proof. ($\Rightarrow$) Assume that R is a t-ADD domain. By [35, Theorem 2.54], R is a PVMD and $t-dim(R) = 1$. Let $M \in Max_t(R)$. Since $Max_t(R)$ is t-coprimely packed, there is $x \in M$ with $x \notin N$ for each $N \in Max_t(R) \setminus \{M\}$. Hence $M = \sqrt{xR}$ and thus $MR_M = \sqrt{xR_M}$. Since MR_M is a maximal ideal of R_M, xR_M is an MR_M-primary ideal and since R_M is a DVR, there is a positive integer n such that $xR_M = M^nR_M$. Thus $(M^n)_tR_M = M^nR_M$ and $(M^n)_t$ is M-primary ([28, Proposition 1.3]). Hence $xR = (M^n)_t$ and then M is a t-invertible ideal of R. Therefore R is a Krull domain and has torsion class group by Proposition 2.3.1.

($\Leftarrow$) Follows immediately from Proposition 2.3.1. □

Proof of Theorem 2.3.6. ($i \Leftrightarrow iv$) If R is a t-coprimely packed domain, then R is a Krull domain with torsion class group (Lemma 2.3.7). Let P' be a t-prime ideal of $R[X]$. Since R is a Krull domain, then so is $R[X]$ and hence P' is a t-maximal ideal of $R[X]$. If $0 \neq P = P' \cap R$, then $P' = P[X]$ ([31, Proposition 1.1]). But $P = \sqrt{aR}$ for some $a \in P$ since R is t-coprimely packed. Hence $P' = P[X] = \sqrt{aR}R[X] = \sqrt{aR[X]}$. Since $htP' = 1$, then P' is a set theoretic complete intersection. If P' is an upper to zero, then, $P' = fK[X] \cap R[X]$ for some polynomial $f \in P'$ ([31, Corollary 1.5]) and P' is t-invertible. By [31, Proposition 2.6, Lemma 2.5], $P' = f(c(f)^{-1})R[X]$, since R is integrally closed. Set $J = c(f)^{-1}$. Since R is a PVMD, then J is a t-invertible fractional t-ideal of R. Since R has torsion class group, then there is a positive integer n such that $(J^n)_t = cR$ for some $c \in J$. So $((P')^n)_{t_1} = ((fJR[X])^n)_{t_1} = (f^n(J^nR[X])_{t_1})_{t_1} = (f^n(J^n)_tR[X])_{t_1} = (f^ncR[X])_{t_1} = f^ncR[X]$. Therefore $P' = \sqrt{((P')^n)_{t_1}} = \sqrt{cf^nR[X]}$, as desired.

Conversely, if $P \in Spec_t(R)$, then P is a t-maximal ideal of R of height one. Hence $P[X]$ is a t-prime ideal of $R[X]$ of height one. So $P[X] = \sqrt{fR[X]}$ for some polynomial $f \in P[X]$ with $c(f) \subseteq P$ and say that a is the leading coefficient of f. Then $P = \sqrt{aR}$, as desired.

($i \Leftrightarrow ii$) Assume that $R < X >$ is t-coprimely packed. Since R is a t-ADD domain, $R[X]$ and $R < X >$ are also t-ADD domains ([35, Theorem 2.51, Theorem 2.52]) and hence PVMDs. So for each t-ideal J of $R[X]$, $J < X >$ is a t-ideal of $R < X >$, by the fact that "if R is a v-coherent domain, I is a t-ideal of R and S is a multiplicative closed set of R, then I_S is a t-ideal of R_S" and $R[X]$ is a v-coherent domain. Note that $t-dim(R < X >) = t-dim(R[X]) = 1$ since they are t-ADD domains. Now let P be a t-maximal ideal of R. Then $P[X]$ is a t-maximal ideal of $R[X]$. Since $R < X >$ is a t-coprimely packed domain, there is $f \in P[X]$ such that $P < X >= \sqrt{fR < X >}$. Let a be the leading coefficient of f and let $c \in P$. Then $c \in P < X >$ and hence there is a positive integer n such that $c^n = \frac{fg}{h}$ for some $g \in R[X]$ and a monic polynomial $h \in R[X]$. So $c^n h = gf$ and thus $c^n = ad$ where d is the leading coefficient of h. So $c \in \sqrt{aR}$. Therefore $P = \sqrt{aR}$, as desired.

Conversely, if Q is a t-prime ideal of $R < X >$, then $Q = (P')_U$ for some t-prime ideal of $R[X]$. As in the proof of ($i \Leftrightarrow iv$), the t-coprime packedness of R implies that $P' = \sqrt{fR[X]}$ for some polynomial $f \in P'$. Therefore $Q = (P')_U = (\sqrt{fR[X]})_U = \sqrt{fR < X >}$, as desired.

($i \Leftrightarrow v$) Follows from Lemma 2.3.7.

($iv \Rightarrow iii$) Trivial since $t-dim(R[X]) = 1$.

($iii \Rightarrow i$) Since $t-dim(R[X]) = 1$, $R[X]$ is a t-compactly packed domain. By Corollary 2.3.3, R is t-compactly packed and hence t-coprimely packed. □

Recall that a domain R is said to be of finite t-character if every nonzero nonunit $x \in R$ is contained in only finitely many t-maximal ideals.

Proposition 2.3.8. *Let R be a GCD domain. If R is of finite t-character, then R is t-coprimely packed.*

Proof. Let P be a t-prime ideal of R. Then $Max_t(R,P) = \{M_1, M_2, ..., M_n\}$, since R is of finite t-character. Pick $0 \neq c \in P$. If $Max_t(R,P) = Max_t(R,cR)$, then $j-rad_t(P) = j-rad_t(cR)$. If not, then $Max_t(R,c) = \{M_1, ..., M_n, M_{n+1}, ..., M_{n+s}\}$ and we can choose an element $y \in P$ with $y \notin \bigcup M_{n+i}$ for $i = 1,2,...,s$. So $j - rad_t(P) = j - rad_t((c,y)_t) = j - rad_t(bR)$ for some $b \in P$ since R is a GCD domain, as desired. □

The converse is not true. For instance, let $R = \mathbb{Z}[Y] + X\mathbb{Q}(Y)[[X]]$. Then R is a GCD ([3, Theorem 3.13]) t-coprimely packed domain (Theorem 2.4.3) of t-dimension 2 ([37, Theorem 2.4]), but not of finite t-character since each nonzero element of M is contained in all t-maximal ideals of the form $p[Y] + M$ where p is a prime positive integer.

Proposition 2.3.9. *Let R be a Noetherian domain containing a field of characteristic zero. Then each t-prime ideal of $R[X]$ is a set theoretic complete intersection if and only if R is a Krull domain with torsion class group.*

Proof. Since height-one prime ideals are t-primes, by ([17, Theorem 2.2]), R is integrally closed. Hence R is a Krull domain and so is $R[X]$. Let P be a t-prime ideal of R. Then $P[X]$ is a t-maximal ideal of $R[X]$ of height one. By assumption, there is a polynomial $f \in P[X]$ such that $P[X] = \sqrt{fR[X]}$. Hence $P = \sqrt{aR}$ where a is the leading coefficient of f. So R is t-compactly packed. Hence R has torsion class group (Proposition 2.3.1). The converse follows from Theorem 2.3.6. □

Our second main result is a satisfactory analogue of [16, Theorem 2.5]. Before stating the result, we recall that a domain is said to be a UMT-domain if every upper to zero is a t-maximal ideal ([31, Definition in page 1962]).

Theorem 2.3.10. *Let R be a GCD domain and consider the following statements:*

(i) *Every t-prime ideal of $R[X]$ is a set theoretic complete intersection.*

(ii) *$R[X]$ is t-compactly packed.*

(iii) *$R[X]$ is t-coprimely packed.*

(iv) *R is t-coprimely packed.*

Then $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$. Moreover, if $t - dimR = 1$, then the statements are equivalent.

Proof. $(i) \Longrightarrow (ii)$ Let Q be a t-prime ideal of $R[X]$ and set $htQ = r$. By (i), there exist polynomials $f_1, \dots, f_r$ such that $Q = \sqrt{(f_1, \dots, f_r)}$. Since $(f_1, \dots, f_r) \subseteq Q$, then $(f_1, \dots, f_r)_t \subseteq Q$ and therefore $Q = \sqrt{(f_1, \dots, f_r)_t}$. But since R is a GCD domain, then so is $R[X]$ ([24, Theorem 34.10]). Hence $Q = \sqrt{(f_1, \dots, f_r)_t} = \sqrt{(h)}$, as desired.

$(ii) \Longrightarrow (iii)$ Trivial.

$(iii) \Longrightarrow (iv)$. Let P be a t-prime ideal of R. Then $P[X]$ is a t-prime ideal of $R[X]$. By (iii), there exists a polynomial $f \in P[X]$ such that $j - rad_t(P[X]) = j - rad_t(fR[X])$. Since $f \in P[X]$, then $c(f) \subseteq P$. But since R is a GCD domain, then $I = (c(f))_t = aR$. We claim that $j - rad_t(P) = j - rad_t(aR)$. Indeed, let Q be a t-maximal ideal of R containing P. Since $aR \subseteq P \subseteq Q$, then $j - rad_t(aR) \subseteq Q$ and therefore $j - rad_t(aR) \subseteq j - rad_t(P)$. On the other hand, let Q be a t-maximal ideal of R containing aR. Clearly $Q[X]$ is a t-maximal ideal of $R[X]$. Since $fR[X] \subseteq c(f)[X] \subseteq aR[X] \subseteq Q[X]$, then $P[X] \subseteq j - rad_t(P[X]) = j - rad_t(fR[X]) \subseteq Q[X]$. Hence $P \subseteq Q$ and therefore $P \subseteq j - rad_t(aR)$. It follows that $j - rad_t(P) = j - rad_t(aR)$, as desired.

$(iv) \Longrightarrow (i)$ Assume that $t - dimR = 1$. Let Q be a t-prime ideal of $R[X]$ and set $P = Q \cap R$. If $P = (0)$, then Q is an upper to zero. Since R is a GCD domain, then R is a UMT-domain and so Q is a t-maximal ideal of $R[X]$ ([31, Proposition 3.2]). Also by [31, Corollary 1.5], $Q = (f,g)_v$ where $Q = fK[X] \cap R[X]$ and $(c(g))_v = R$. But since R is a GCD domain, then so is $R[X]$ ([24, Theorem 34.10]). Hence $Q = (f,g)_v = (h)$ and so Q is a set theoretic complete intersection since $htQ = 1$. Assume that $P \neq (0)$. Since $t - dimR = 1$, then P is a t-maximal ideal of R. Hence $Q = P[X]$ ([31, Proposition 1.1]). Since R is t-coprimely packed, then there exists $a \in P$ such that $P = j - rad_t(aR)$. Note that $P = \sqrt{aR}$. Indeed, if M is a minimal prime over aR, then M is a t-prime ideal. But since $t - dimR = 1$, then M is a t-maximal ideal of R. Hence $P = j - rad_t(aR) \subseteq M$ and therefore $M = P$ (by t-maximality). Hence $P = \sqrt{aR}$. Now it is easy to see that $Q = P[X] = \sqrt{(aR[X])}$ and hence Q is a set theoretic complete intersection since $htQ = htP[X] = 1$ and this completes the proof. □

Example 2.3.11. *A one-dimensional Noetherian local domain R, so coprimely packed, such that $R[X]$ is not t-coprimely packed.*

Let $\mathbb{Q}$ be the field of rational numbers and Y an indeterminate over $\mathbb{Q}$. Set $R = \mathbb{Q}[[Y^3, Y^5]]$. Clearly R is a one-dimensional Noetherian local domain with maximal ideal $M = (Y^3, Y^5)$, and so R is (t)-coprimely packed. Since $J(R) = M \subsetneq J(R') = Y\mathbb{Q}[[Y]]$, by [18, Corollary 13], there is a height-one maximal ideal Q of $R[X]$ such that $\{Q\} = \mathrm{Max}(R[X], Q) \neq \mathrm{Max}(R[X], f)$ for every polynomial $f \in Q$. Now, suppose that $Max_t(R[X], g) = Max_t(R[X], Q) = \{Q\}$ for some $g \in Q$. Let $N \in \mathrm{Max}(R[X], g)$ and let P be a minimal prime of $gR[X]$ with $P \subseteq N$. Then P is a t-prime ideal of $R[X]$ and since $t - dim(R[X]) = 1$, then P is a t-maximal ideal of $R[X]$. Hence $Q = P \subseteq N$ and by maximality of Q, $Q = N$, a contradiction. It follows that $Max_t(R[X], g) \neq Max_t(R[X], Q) = \{Q\}$ for all $g \in Q$ and therefore $R[X]$ is not t-coprimely packed.

2.4 Pullbacks

The purpose of this section is to investigate the transfer of the notions of compactly (t-compactly) packed and coprimely (t-coprimely) packed rings to the pullbacks to generate new families and examples.

Let us fix the notation for the rest of this section. Let T be an integral domain, M a maximal ideal of T, K its residue field, $\phi : T \longrightarrow K$ the canonical surjection, D a proper subring of K, and $k := \mathrm{qf}(D)$. Let $R := \phi^{-1}(D)$ be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

We shall refer to this diagram as a diagram of type $(\square)$. Also we recall that M is a prime ideal of R and if T is local, then every ideal of R is comparable to M.

Lemma 2.4.1. *For the diagram of type* $(\square)$*, if* R *is compactly (resp. coprimely) packed, then so are* T *and* D*.*

Proof. (1) Compact packedness. Clearly D is compactly packed since if J is an ideal of D and $\{P'_\alpha\}$ is a family of prime ideals of D such that $J \subseteq \bigcup P'_\alpha$, then $\phi^{-1}(J)$ is an ideal of R and $\{\phi^{-1}(P'_\alpha)\}$ is a family of prime ideals of R such that $\phi^{-1}(J) \subseteq \bigcup \phi^{-1}(P'_\alpha)$. So $\phi^{-1}(J) \subseteq \phi^{-1}(P'_\alpha)$ for some α_0 and therefore $J \subseteq P'_{\alpha_0}$, as desired.
Let J be an ideal of T and $\{Q_\alpha\}$ a family of prime ideals of T such that $J \subseteq \bigcup Q_\alpha$. Set $I = J \cap R$ and $P_\alpha = Q_\alpha \cap R$ for each α. Then $I \subseteq \bigcup P_\alpha$ and so $I \subseteq P_{\alpha_0}$ for some α_0. Now, if $J + M \subsetneqq T$, then $J \subseteq M$, and so $J = I \subseteq P_{\alpha_0} \subseteq Q_{\alpha_0}$. If $J + M = T$, then $I + M = R$. Thus $JM = J \cap M = I \cap M = IM \subseteq I \subseteq P_{\alpha_0} \subseteq Q_{\alpha_0}$ and therefore $J \subseteq Q_{\alpha_0}$ (since $M \nsubseteq Q_{\alpha_0}$), as desired.
(2) Coprimely packedness. Similar to (1) by assuming that $I = P$ is a prime ideal of R and $\{Q_\alpha\}$ is a family of maximal ideals of T. $\square$

Theorem 2.4.2. *For the diagram of type* $(\square)$*, assume that* T *is local. Then*
(1) *R is compactly packed if and only if D and T are compactly packed.*
(2) *R is coprimely packed if and only if D is coprimely packed.*

Proof. $(\Longrightarrow)$ Follows from Lemma 2.4.1.

$(\Longleftarrow)$ (1) Assume that D and T are compactly packed. Let I be a nonzero ideal of R and $\{P_\alpha\}_{\alpha \in \Omega}$ a family of prime ideals of R such that $I \subseteq \bigcup P_\alpha$. Let $\Omega_1 = \{\alpha \in \Omega | P_\alpha \subseteq M\}$ and $\Omega_2 = \{\alpha \in \Omega | M \subsetneq P_\alpha\}$. Since T is local, each ideal of R is comparable to M. Three cases are then possible:
Case 1 $\Omega_1 = \emptyset$. Then $M \subsetneq P_\alpha$ for each α and hence $P_\alpha = \phi^{-1}(Q_\alpha)$ for some prime ideal Q_α of D. In this case, if $I \subseteq M$, then $I \subseteq P_\alpha$ for all α. If $M \subsetneq I$, then $I = \phi^{-1}(J)$ for some

nonzero ideal J of D. Then $J \subseteq \bigcup Q_\alpha$ and since D is compactly packed, $J \subseteq Q_{\alpha_0}$ for some α_0. Therefore $I \subseteq P_{\alpha_0}$, as desired.

Case 2 $\Omega_2 = \emptyset$. Then $P_\alpha \subseteq M$ and so P_α is a prime ideal of T for each α. Also $I \subseteq \bigcup P_\alpha \subseteq M$. Now, if $P_{\alpha_0} = M$ for some α_0, then $I \subseteq M = P_{\alpha_0}$ and we are done. Assume that $P_\alpha \subsetneq M$ for each α. Since IM is an ideal of T and $IM \subseteq I \subseteq \bigcup P_\alpha$, $IM \subseteq P_{\alpha_0}$ for some α_0. But since $P_{\alpha_0} \subsetneq M$, $I \subseteq P_{\alpha_0}$, as desired.

Case 3 $\Omega_1 \neq \emptyset$ and $\Omega_2 \neq \emptyset$. Hence $P_\alpha \subseteq M \subseteq P_\beta$ for each $\alpha \in \Omega_1$ and $\beta \in \Omega_2$. Hence $I \subseteq \bigcup_{\beta \in \Omega_2} P_\beta$. Set $P_\beta = \phi^{-1}(Q_\beta)$ for some prime ideal Q_β of D. If $I \subseteq M$, then $I \subseteq P_\beta$ for each $\beta \in \Omega_2$ and we are done. If $M \subsetneq I$, then $I = \phi^{-1}(J)$ for some nonzero ideal J of D. As in case 1, $I \subseteq P_\beta$ for some $\beta \in \Omega_2$. It follows that R is compactly packed.

(2) Assume that D is coprimely packed. Let P be a prime ideal of R and $\{M_\alpha\}_{\alpha \in \Omega}$ be a family of maximal ideals of R such that $P \subseteq \bigcup M_\alpha$. Since each M_α is comparable to M, and by maximality, $M \subseteq M_\alpha$ for each α. Hence, for each α, $M_\alpha = \phi^{-1}(Q_\alpha)$ for some maximal ideal Q_α of D. Now, if $P \subseteq M$, then $P \subseteq M_\alpha$ for each α and we are done. If $M \subsetneq P$, then $P = \phi^{-1}(Q)$ for some prime ideal Q of D. But $P \subseteq \bigcup M_\alpha$ implies that $Q \subseteq \bigcup Q_\alpha$ and thus $Q \subseteq Q_{\alpha_0}$ for some α_0 since D is coprimely packed. Hence $P \subseteq M_{\alpha_0}$, as desired. □

Now, we turn our attention to the t-compact and t-coprime packedness. Recall that an overring S of R is said to be t-flat over R if $T_N = R_{N \cap R}$ for each t-maximal ideal N of T ([38]).

Theorem 2.4.3. *For the diagram of type* (□)*:*

(1) *If R is t-compactly (resp. t-coprimely) packed, then so is D.*

(2) *If T is t-flat over R and R is t-coprimely packed, then so is T.*

(3) *If T is local, then R is t-coprimely packed if and only if so is D.*

Proof. (1) Clearly D is t-compactly (resp. t-coprimely) packed since if P is a t-prime ideal of D and $\{Q_\alpha\}$ is a family of t-prime (resp. t-maximal) ideals of D such that $P \subseteq \bigcup Q_\alpha$, then

$Q=\phi^{-1}(P)$ is a t-prime ideal of R and $\{\phi^{-1}(Q_\alpha)\}$ is a family of t-prime (resp. t-maximal) ideals of R such that $Q=\phi^{-1}(P)\subseteq\bigcup\phi^{-1}(Q_\alpha)$. Thus $Q\subseteq\phi^{-1}(Q_{\alpha_0})$ for some α_0 and therefore $P\subseteq Q_{\alpha_0}$, as desired.

(2) Let Q be a t-prime ideal of T and $\{Q_\alpha\}\subseteq Spec_t(T)$ such that $Q\subseteq\bigcup Q_\alpha$. Since T is a t-flat overring of R, $Q\cap R$ and $Q_\alpha\cap R$ are t-prime ideals of R (Lemma 1.2.7) and we have $Q\cap R\subseteq\bigcup Q_\alpha\cap R$. Since R is t-compactly packed, then $Q\cap R\subseteq Q_\alpha\cap R$. Hence, by [10, Proposition 2.4], $Q=((Q\cap R)T)_t\subseteq((Q_\alpha\cap R)T)_t=Q_\alpha$,as desired.

(3) Similar to Theorem 2.4.2 (2) by substituting t-prime to prime and t-maximal to maximal. □

Example 2.4.4. *Let $T=\mathbb{Q}(\sqrt{2})[[X,Y]]=\mathbb{Q}(\sqrt{2})+M$ where $M=(X,Y)T$. Set $R=\mathbb{Q}+M$. Then T is a t-compactly packed domain since it is a Krull local domain. However, R is not t-compactly packed since R is Noetherian of t-dimension two.*

- *This example shows that the assertion* (1) *of Theorem 2.4.2 is not true for t-compact packedness (even if T is local).*

- *R is a t-coprimely packed domain which is not t-compactly packed.*

- *R is a (d,t)-domain since it is coprimely packed and* $\mathrm{Max}(R)=Max_t(R)=\{M\}$.

- *R is not a compactly packed domain, since T is not compactly packed.*

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Vita

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