ON A GENERALIZED FISHER EQUATION

BY

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To my parents, wife and my children and to my brothers and sisters

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ABSTRACT

Name: Radwan Ali Ali Al-Rubaee.

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We consider a reaction diffusion equation known as the Fisher equation which models problems in genetics, population growth and mathematical biology among others. A generalized non-linear form of this equation in cylindrical coordinates with radial symmetry is studied from Lie symmetry point of view. The diffusivity and the reaction terms are assumed to be functions of the dependent variable. An attempt to classify the diffusivity function is made and exact solutions are obtained in some cases. The known results in case of diffusivity being proportional to the dependent variable are shown to be a special case of our analysis. It is found that the power law dependence of diffusivity function leads to exact solution in the form of the respective root of a linear combination of the Bessel function of order zero of first and the second kind. However, the reaction term can be classified if the diffusivity is a linear function. The study can lead to a classification in the most general settings in which no radial symmetry is present.

ملخص الرسالة

الأســـــــــــم: رضوان علي علي الرباعي

عنوان الرسالة: تعمـيم معادلة فيشـــر

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ندرس في هذه الرسالة معادلة رد فعل الانتشار المعروفة بمعادلة فيشر والتي تصف كثيراً من المشاكل في مجال علم الوراثة والنمو السكاني والأحياء وغيرها. تركز الدراسة على تعميم غير خطي لهذه المعادلة باستخدام المحاور الاسطوانية بتماثل قطري من منظور "تناظر لي". نفترض أن حدي الانتشارية والتفاعل هما دوال في المتغير التابع. تمت محاولة تصنيف دالة الانتشار وحصلنا على حلول تامة لبعض الحالات. وجدنا أن الحل المعروف في حالة اعتماد الانتشارية على المتغير التابع يكون حالة خاصة من حلنا العام في هذه الدراسة. وجدنا أن قانون القوة الذي يحكم الانتشارية يؤدي إلى حل تام على شكل الجذر المناسب لتركيب خطي لداله "بيسل" من الدرجة الصفرية من النوع الأول والثاني. ومع ذلك يمكن تصنيف حد التفاعل إذا كانت الانتشارية عبارة عن دالة خطية. هذه الدراسة يمكن أن تقود إلى تصنيف بشكل أعم في حالة عدم وجود تناظر.

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Chapter 1

INTRODUCTION

Most models in physics, engineering, social and biological sciences are described by partial differential equations (PDEs) [26,27,28]. In most real life situations, these PDEs are nonlinear in nature. In many cases the nonlinearity may be due to a non-homogeneous source function of dependent variable [16] such as in the sine-Gorden equation or the classical Fisher equation. In case of practical interest however, this may occur as the properties of medium depend upon independent variables. This phenomenon is exhibited, for example, in gases in which case the thermal diffusivity is found to be proportional to the temperature. There are interesting physical processes that also lead us to nonlinear partial differential equations such as in Burgers equation and KDV equation. In such cases finding exact solutions of these PDEs is a formidable task. More often approximate or numerical methods [3, 9, 10, 12, 13, 30] are employed to obtain approximate solutions. Bokhari et.al. have employed an analytic method to obtain certain series solutions of a nonlinear heat equation [4, 5]. Over the last two decades a lot of attention has been given to the use of symmetry methods due to Sophus Lie [22,23,24]. These methods exploit the invariance properties of the PDEs under the transformations known as Lie symmetry transformations. This approach reduces the nonlinear PDEs into one with less number of independent variables and/or to an ordinary differential equation. A systematic description of this method can be found in [18, 29, 31] and the method as well as some interesting applications to fluid dynamics problems in [6] Clarkson and Mansfield [9] and Aijaz Ahmad et.al. [2] have performed symmetry analysis of some nonlinear diffusion / heat equations.

In this thesis we are interested in employing the Lie symmetry methods to the so-called Fisher equation. The Fisher equation was first studied by Fisher [11, 12] in its simplest form given by

$$u_t - u_{xx} = u(1-u) \tag{1.1}$$

where $u(x,t)$ denotes the concentration of fluid or bacteria or a particular biological cell depending upon the nature of the model. The term on the right hand side of (1.1) corresponds to the reaction or growth term. A more general form of equation (1.1) is known as Kolmogorov-Petrovskii-Piscounov equation [21] given by

$$u_t - u_{xx} = f(u) \tag{1.2}$$

where f is a sufficiently smooth function of u. Equation (1.2) reduces to the well known reaction-diffusion equation when $f(u)$ is a polynomial in u of order three. There have been a considerable interest in this class of equations. For example the Huxley equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u) \tag{1.3}$$

has been studied for neural model by Hodgkin and Huxley [20, 21] who were awarded nobel prize for their model. Another important equation of this class is Fitzhugh-

Nagumo equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(1 - u^2) \tag{1.4}$$

which arises in the study of nerve cells [14].

Newell-Whitehead equation given below has been studied by various authors [9]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u^2). \tag{1.5}$$

A more general form of Fitzhugh-Nagumo equation is given by [14, 15]

$$u_t - u_{xx} = \alpha u^3 + \beta u^2 + \gamma u, \tag{1.6}$$

where α, β, γ are arbitrary constants and the equation occurs in various situations such as population genetics etc. [25]. In [12], the Fisher equation is studied in terms of its traveling wave solutions while numerical solutions are given in [10]. In [17, 19, 27, 28] Lie symmetry analysis is used to study a generalized Fisher equation of the type

$$u_t = \frac{\partial}{\partial x}(g(u)u_x) + f(u) \tag{1.7}$$

where f and g are sufficiently smooth functions. It is shown in [18] that equation (1.7) possesses a minimal 'two dimensional' algebra which extends to larger algebras in special cases [18]. This generalized version of the Fisher equation is used to model heat and reaction-diffusion problems with reference to their applications in mathematical biology, chemistry, genetics and bacterial growth problem [25].

Whereas equation (1.1) - (1.7) have been widely studied in literature, in most cases the modeling is based on constant diffusivity requiring the Fisher equation to be of the form

$$\frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(d \, \bar{\nabla} u \right) = f(u). \tag{1.8}$$

For a constant d, equation (1.8), in (2+1) dimensional space, becomes

$$\frac{\partial u}{\partial t} - d \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f(u) \tag{1.9}$$

Realistically, diffusion coefficient is not generally constant. For an example in gases the coefficient is proportional to u^α, where α is some real constant. This fact motivates one to generalize the Fisher equation in such a way that it incorporates variable diffusivity $g(u)$ there. In the latter setting (1.8) takes the more general form

$$\frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(g(u) \bar{\nabla}(u) \right) = f(u) \tag{1.10}$$

where $g(u)$ is the diffusively of the medium. The focus of thesis is to study the generalized nonlinear Fisher equation (1.10) in cylindrical coordinates. The motivation behind using this coordinate system is that many engineering situations require use of the cylindrical coordinates due to the inherent model. In order to write (1.10) in cylindrical coordinates, we first transform the operator $\bar{\nabla}$ to cylindrical coordinate, and compute $\bar{\nabla} \cdot \left(\bar{\nabla} u \right)$ in cylindrical coordinates and we use x for radial, y for polar angle and z for height. In this setting $\bar{\nabla}\cdot\big(g(u)\bar{\nabla}u\big)$ takes the from:

$$\begin{aligned}\bar{\nabla}\cdot\big(g(u)\bar{\nabla}(u)\big) &= \frac{\partial}{\partial x}\left(g(u)\frac{\partial u}{\partial x}\right)+\frac{1}{x}\left(g(u)\frac{\partial u}{\partial y}\right)+\frac{1}{x^2}g(u)\frac{\partial^2 u}{\partial y^2}+g(u)\frac{\partial^2 u}{\partial z^2}\\ &= \frac{1}{x}\frac{\partial}{\partial x}\left(xg(u)\frac{\partial u}{\partial x}\right)+\frac{1}{x^2}g(u)\frac{\partial^2 u}{\partial y^2}+g(u)\frac{\partial^2 u}{\partial z^2} \qquad (1.11)\end{aligned}$$

For the present work we assume radial symmetry and restrict u to depend only on radius which is denoted by x. Thus, equation (1.11) reads

$$\bar{\nabla}\cdot\big(g(u)\bar{\nabla}(u)\big)=\frac{1}{x}\frac{\partial}{\partial x}\left(xg(u)\frac{\partial u}{\partial x}\right). \qquad (1.12)$$

Therefore (1.10), reduces to

$$\frac{\partial u}{\partial t}-\frac{1}{x}\frac{\partial}{\partial x}\left(xg(u)\frac{\partial u}{\partial x}\right)=f(u). \qquad (1.13)$$

The objective is now to perform a symmetry analysis of equation (1.13). This requires classifications of both $f(u)$ and $g(u)$. The symmetry generators are then to be obtained and used to reduce the resulting partial differential equation to an ordinary differential equation. Since all resulting nonlinear ordinary differential equations can not be solved, we give solutions in those cases where a solution is possible and leave others as they are.

This thesis is organized as follows:

In chapter two, we present some basic definitions and results of the Lie symmetry method. In particular we give procedure for finding Lie point symmetries of the

PDEs and show how these symmetries are used to reduce the nonlinear PDEs to ODEs. In chapter three, a simple form of (1.13) with $g(u) = u$ and $f(u) = u(1-u)$ is considered and solved [4] to illustrates the Lie symmetry method for this class of evolution equations. Chapter four deals with a classification of solutions of the general problem (1.13). We classify $f(u)$ and $g(u)$ and reduce the generalized Fisher equation (1.13) to an ODE. Moreover, exact solutions in some cases are obtained. Some recommendations for future work are addressed in chapter five.

Chapter 2

PRELIMINARIES

2.1 Introduction

It is well known that the exact solutions of nonlinear PDEs play a pivotal role in understanding several physical phenomena. However, finding exact solutions of PDEs is not an easy task. The problem is even more difficult in case of nonlinear PDEs. Over the past few decades Lie symmetry methods have been widely used and developed. These methods are commonly known as Lie group theoretic methods and provide powerful tool for dealing with nonlinear PDEs which admit certain Lie point symmetries. Under the action of such symmetries the PDEs and their solutions remain invariant. In this chapter we give certain results which form a basis of the Lie symmetry methods and wherever possible use examples to illustrate the methods.

2.2 Group

Consider $(G, *)$ to be a non-empty set with a binary operation $*$ that assigns to every ordered pair of elements of G a unique element with the following properties:

1. **Closure property**

   For all x, y in G, $x * y$ is also in G.

2. **Associative property**

   For all x, y, z in G,

$$(x * y) * z = x * (y * z) \tag{2.1}$$

3. **Identity property**

   In G there exists an element 'e' known as the identity such that $x * e = e * x$, for all x in G.

4. **Inverse property**

   For every x in G there exists an element y in G known as inverse of x such that

$$x * y = e = y * x \tag{2.2}$$

   where e is the identity element of G with respect to the binary operation $*$.

Example 1

Group of integers with binary operation addition.

Example 2

Group of all invertible matrices with binary operation defined as matrix multiplication.

Definition 2.1

A group G is called Abelian if in addition to the above properties it satisfies:

$$x * y = y * x \tag{2.3}$$

for all x, y in G.

2.3 Lie Group

A Lie group, also called an infinitesimal group, is the one in which the group operations (multiplication and inversion) are smooth maps possessing derivatives of all order.

Example 3

Consider the set of transformations such that

$$T_\epsilon : (x, y) \rightarrow (\bar{x}, \bar{y}) = (x + \epsilon, y)$$

we show that $\{T_\epsilon\}$ form a group

let $T_{\epsilon_1}, T_{\epsilon_2} \in \{T_\epsilon\}$ then

$$T_{\epsilon_2} T_{\epsilon_1}(x, y) = T_{\epsilon_2}(x + \epsilon_1, y) = (x + \epsilon_1 + \epsilon_2, y) = (x + \epsilon, y) \in \{T_\epsilon\}$$

also, there exist an identity transformation $\{T_0\}$ such that

$$T_0(x, y) = (\bar{x}, \bar{y}) = (x + 0, y) = (x, y)$$

we also, said that T_{ϵ_2} is an inverse of T_{ϵ_1} if $\epsilon_1 + \epsilon_2 = 0 \Rightarrow \epsilon_1 = -\epsilon_2 \Rightarrow \epsilon_1^{-1} = -\epsilon_2$ and the converse is also true, then

$T_0 \in \{T_\epsilon\}$, $T_{\epsilon_1}T_{\epsilon_2} \in \{T_\epsilon\}$ and $\forall\ T_\epsilon\ \exists\ T_{\epsilon^{-1}} \in \{T_\epsilon\}$ therefore, $\{T_\epsilon\}$ forms a group

2.4 Group of Transformations

Definition 2.2

The set of transformations given by

$$\bar{x} = \chi(x, \epsilon), \tag{2.4}$$

where $x = (x_1, x_2, ..., x_n)$ lie in region D$\subset \mathbb{R}^n$ is defined for each ϵ in set $S \subset \mathbb{R}$ with the law of composition $\psi(\epsilon, \delta)$, forms a one-parameter group of transformation on D if the following hold:

1. For all $\epsilon \in S$ the transformations are one-to-one onto D.

2. S with ψ forms a group G.

3. For all $x \in$ D, $\bar{x}$ $=x$ when $\epsilon = \epsilon_0$ corresponding to the identity e, i.e.,

$$\chi(x, \epsilon_0) = x.$$

4. If $\bar{x} = \chi(x, \epsilon)$,then

$$\bar{\bar{x}} = \chi(\bar{x}, \delta) = \chi(x; \psi(\epsilon, \delta)).$$

2.5 Lie Groups of Transformations

Definition 2.3

A group G of transformations with composition law 'ψ' is said to be a one-parameter Lie group of transformation if :

1. ϵ is a continuous parameter i.e, the set S is an interval in $\mathbb{R}$.
2. χ is infinitely differentiable function with respect to x in D.
3. the composition function $\psi(\epsilon, \delta)$ is an analytic function.

Example 4

The transformation defined by

$$G_i : (x, y) \longrightarrow (\bar{x}, \bar{y})$$

such that,

$$\bar{x} = \alpha x \qquad \text{and} \qquad \bar{y} = \alpha^2 y$$

where $0 < \alpha < \infty$ is called group of scalings in the xy-plane. Here $\psi(\alpha, \beta) = \alpha\beta$, and the identity element corresponds to $\alpha = 1$. This group of transformations can also be re-parameterized in terms of $\epsilon = \alpha - 1$ as

$$\bar{x} = (1+\epsilon)x, \qquad \bar{y} = (1+\epsilon)^2 y, \ -1 < \epsilon < \infty$$

where the identity element corresponds to $\epsilon = 0$ and the law of composition of parameters is given by

$$\psi(\alpha, \beta) = \epsilon + \delta + \epsilon\delta.$$

Example 5

Consider the reflection transformation [1] $\bar{x} = -x$, $\bar{y} = -y$.

Since

$$\bar{\bar{x}} = -\bar{x} = -(-x) = x$$
$$\bar{\bar{y}} = -\bar{y} = -(-y) = y,$$

which shows that it is not invertible hence does not form a Lie group of transformation.

2.6 Infinitesimal Transformations

Consider a one parameter 'ϵ' Lie group of transformation

$$\bar{x} = G_i(x, \epsilon), \tag{2.5}$$

with the identity $\epsilon = 0$ and law of composition ψ. Expanding (2.5) about $\epsilon = 0$, one gets,

$$\bar{x} = x + \epsilon \frac{\partial \bar{x}}{\partial \epsilon}\bigg|_{\epsilon=0} + O(\epsilon^2) \tag{2.6}$$

where $\frac{\partial \bar{x}}{\partial \epsilon}\Big|_{\epsilon=0} = \xi(x)$. The transformation $\bar{x} = x + \epsilon\ \xi(x)$ is called the infinitesimal transformation of the Lie group of transformation and the component $\xi(x)$ is called the infinitesimal of the transformation.

Theorem 2.1: (First Fundamental Theorem of Lie[7,9])

There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations $\bar{x} = G_i(x, \epsilon)$ is equivalent to the solution of an initial value problem for the system of first order ordinary differential equations

$$\frac{\partial \bar{x}}{\partial \tau} = \xi(\bar{x}) \tag{2.7}$$

with

$$\bar{x} = x \quad \text{when} \quad \tau = 0.$$

Example 6

Consider the transformation $\bar{x} = G_i(x, \epsilon)$ where $\bar{x} = x + \epsilon$, $\bar{y} = y$, the law of composition $\psi(x, y) = x + y$ and $\epsilon^{-1} = \epsilon$. Such a G_i defines the group of translations.

Here,

$$\xi(x) = \left.\frac{\partial\chi(x,\ \epsilon)}{\partial\epsilon}\right|_{\epsilon=0} = 1 \quad \text{and} \quad \frac{d\bar{x}}{d\epsilon} = 1, \frac{d\bar{y}}{d\epsilon} = 0$$

with

$$\bar{x} = x, \quad \bar{y} = y \quad \text{at} \quad \epsilon = 0$$

2.7 Infinitesimal Generator

Consider the transformation

$$\bar{x} = G_i(x, \epsilon) \tag{2.8}$$

where $x = (x_1, x_{2,} x_3, ..., x_n) \in \mathbb{R}^n$. Then the operator defined by

$$\mathbf{X} = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \tag{2.9}$$

is called an infinitesimal generator of the one parameter Lie group of transformation (2.8), where $\xi_i = \left.\frac{\partial \bar{x_i}}{\partial \epsilon}\right|_{\epsilon=0}$ are components of the tangent vector χ. In particular, if a point $p = (x, y) \in \mathbb{R}^2$, the above symmetry generator becomes

$$\mathbf{X} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}, \tag{2.10}$$

where

$$\xi(x, y) = \left.\frac{d\bar{x}}{d\epsilon}\right|_{\epsilon=0}, \quad \eta(x, y) = \left.\frac{d\bar{y}}{d\epsilon}\right|_{\epsilon=0}.$$

We can determine the transformation (2.8) with the help of infinitesimal generator **X** by integrating

$$\xi_i(\bar{x}) = \frac{\partial \bar{x}_i}{\partial \epsilon} \tag{2.11}$$

with initial condition

$$\bar{x}_i\Big|_{\epsilon=0} = x_i.$$

Theorem 2.2. [14]

The one-parameter Lie group of transformations $\bar{x} = G_i(x, \epsilon)$ is equivalent to :

$$\begin{aligned} \bar{x} &= e^{\epsilon \mathbf{X}} x \\ &= x + \epsilon \mathbf{X} x + \frac{\epsilon^2}{2} \mathbf{X}^2 x + \cdots \\ &= [1 + \epsilon \mathbf{X} + \frac{\epsilon^2}{2} \mathbf{X}^2 + \cdots] x \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathbf{X}^k x, \end{aligned} \tag{2.12}$$

where the operator **X** is given by (2.9).

Example 7

Consider the rotation group:

$$\bar{x} = x \cos\epsilon + y \sin\epsilon, \qquad \bar{y} = -x \sin\epsilon + y \cos\epsilon, \tag{2.13}$$

the infinitesimals $\xi(x, y) = \frac{\partial \bar{x}}{\partial \epsilon}\Big|_{\epsilon=0} = y$ and $\eta(x, y) = \frac{\partial \bar{y}}{\partial \epsilon}\Big|_{\epsilon=0} = -x$ defines the symmetry generator associated with (2.13) as

$$\mathbf{X} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{2.14}$$

Alternatively, given the symmetry generator, one can find the transformation associated with that generator. This can achieved as follows:

Consider the Lie series corresponding to the generator (2.14) given by

$$(\bar{x}, \bar{y}) = (e^{\epsilon\mathbf{X}}x, e^{\epsilon\mathbf{X}}y), \tag{2.15}$$

where $\mathbf{X}x = y$, $\mathbf{X}^2x = -x$ and $\mathbf{X}^3x = -y$ etc. Then (2.15) can be re-cast in the form of rotations (2.13) as follows:

$$\begin{aligned} \bar{x} &= e^{\epsilon\mathbf{X}}x \\ &= \sum_{k=0}^{\infty}\frac{\epsilon^k}{k¡}\mathbf{X}^k x \\ &= x + \epsilon\mathbf{X}x + \frac{\epsilon^2}{2}\mathbf{X}^2x + \cdots \\ &= (1 - \frac{\epsilon^2}{2} + \frac{\epsilon^4}{4} - \cdots)x + (\epsilon - \frac{\epsilon^3}{3} + \frac{\epsilon^5}{5} - \cdots)y \\ &= x\cos\epsilon + y\sin\epsilon. \end{aligned} \tag{2.16}$$

Similarly

$$\bar{y} = e^{\epsilon\mathbf{X}}y = \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\mathbf{X}^k y = -x\sin\epsilon + y\cos\epsilon. \tag{2.17}$$

In matrix notation, the rotation given by (2.16) and (2.17) is written as

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.18}$$

The matrix $\begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$ in the above expression is known as the rotation matrix.

2.8 Lie Algebras

Lie algebra is a vector space, equipped with bilinear product [,]: $V \times V \rightarrow$V satisfying (for all vector fields $\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k$ belonging to V) with the following properties:

1. $[\mathbf{X}_i, \mathbf{X}_j] = 0,$ i=j
2. $[\mathbf{X}_i, \mathbf{X}_j] = -[\mathbf{X}_j, \mathbf{X}_i],$
3. Any three infinitesimal symmetry generators $\mathbf{X}_i, \mathbf{X}_j$ and $\mathbf{X}_k$, satisfy the Jacobi's identity,

$$[\mathbf{X}_i, [\mathbf{X}_j, \mathbf{X}_k] + [\mathbf{X}_k, [\mathbf{X}_i, \mathbf{X}_j] + [\mathbf{X}_j, [\mathbf{X}_k, \mathbf{X}_i] = 0.$$

where the commutator operator [,] for any two symmetry generators $\mathbf{X}_i, \mathbf{X}_j$ is defined, as in [29], by

$$[\mathbf{X}_i, \mathbf{X}_j] = \mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i. \tag{2.19}$$

Definition 2.4

Let G be an r-parameter Lie group of transformations with basis $\{\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_r\}$, where $\mathbf{X}_i$ is an infinitesimal symmetry generator corresponding to the parameter ϵ_i. Then the Lie group G of transformations forms an r-dimensional Lie algebra G^r over the field F=$\mathbb{R}$ with respect to commutation law [8].

Thus, the Lie algebra is a vector space 'G' together with the commutator operator which is bilinear skew symmetric and satisfies the Jacobi identity.

Example 8

The group of rigid motions in $\mathbb{R}^2$ that preserve distances between any two points in $\mathbb{R}^2$ is the three-parameter Lie group of transformations of rotations and translations in $\mathbb{R}^2$ given by

$$\bar{x} = x\cos\epsilon_1 - y\sin\epsilon_1 + \epsilon_2 \tag{2.20}$$

$$\bar{y} = x\sin\epsilon_1 + y\cos\epsilon_1 + \epsilon_3$$

The corresponding infinitesimal generators are given by

$$\mathbf{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{2.21}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x},$$

$$\mathbf{X}_3 = \frac{\partial}{\partial y}.$$

The commutator table of the above Lie point symmetries is as follows:

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0

Table 1: commutator table

Example 9

The similitude group in $\mathbb{R}^2$ consists of uniform scalings and rigid motions in $\mathbb{R}^2$. It is the four-parameter Lie group of transformations given by

$$\bar{x} = e^{\epsilon_4}(x\cos\epsilon_1 - y\sin\epsilon_1) + \epsilon_2 \tag{2.22}$$

$$\bar{y} = e^{\epsilon_4}(x\sin\epsilon_1 + y\sin\epsilon_1) + \epsilon_3$$

The corresponding infinitesimal generators $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ and $\mathbf{X}_4$ are,

$$\mathbf{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \tag{2.23}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial x},$$

$$\mathbf{X}_3 = \frac{\partial}{\partial y},$$

$$\mathbf{X}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The corresponding relation are given as table:

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$
$\mathbf{X}_1$	0	$-\mathbf{X}_3$	$\mathbf{X}_2$	0
$\mathbf{X}_2$	$\mathbf{X}_3$	0	0	$\mathbf{X}_2$
$\mathbf{X}_3$	$-\mathbf{X}_2$	0	0	$\mathbf{X}_3$
$\mathbf{X}_4$	0	$-\mathbf{X}_2$	$-\mathbf{X}_3$	0

Table 2: commutator table

Definition 2.5

A subset A of Lie algebra G is called a subalgebra of G if it is closed under the commutation operator, i.e for all $\mathbf{X}_\alpha, \mathbf{X}_\beta \in A$, $[\mathbf{X}_\alpha, \mathbf{X}_\beta] \in A$.

2.9 Solvable Lie Algebras

The order of an n^{th} order ordinary differential equation (ODE) can be reduced constructively by two if it admits a Lie algebra of transformations of two parameters. But for an r-parameter Lie algebra (r$\geq$ 3) the order of the differential equation can be reduced constructively by p, if there exist a p-dimensional solvable subalgebra.

Definition 2.6

A subalgebra $A \subset G$ is called an ideal or normal subalgebra of G if $[g, a] \in A$ for all $a \in A$, $g \in G$.

Definition 2.7

A^p is p-dimensional solvable Lie algebra if there exists a chain of subalgebras, $A^1 \subset A^2 \subset A^3 \subset \cdots \subset A^{p-1} \subset A^p$ such that A^{i-1} is an ideal of A^i for all $i = 2, 3, \cdots, p$.

Definition 2.8

An algebra G is called an abelian Lie algebra if $[\mathbf{X}_\alpha, \mathbf{X}_\beta] = 0$ for all $\mathbf{X}_\alpha$, $\mathbf{X}_\beta \in G$.

Theorem 2.2. [8]

Every two-dimensional Lie algebra and every Abelian Lie algebra is a solvable Lie algebra.

2.10 Structure Constants

Theorem 2.3:(Second fundamental Theorem of Lie [8])

The commutator of any two infinitesimal generator of an r- parameter Lie group of transformations is also an infinitesimal generator. In particular,

$$[\mathbf{X}_\alpha, \mathbf{X}_\beta] = \sum_{r=1}^{r} C^r_{\alpha\beta} \mathbf{X}_r \in G \tag{2.24}$$

where $C^{\gamma}_{\alpha\beta}$ are the structure constants.

Definition 2.9 (Commutation Relations)

For an r-parameter Lie group of transformations with basis $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_r$ the relations defined by equation (2.24) are called commutation relations.

Theorem 2.4 (Third Fundamental Theorem of Lie [31])

The structure constants, defined by commutation (2.24), satisfy the relations:

1.$C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha}$ (**skew symmetry**).

2.$C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma} + C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha} + C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta} = 0$ (**Jacobi identity**).

2.11 Prolongation

In order to apply the transformations (2.5) to an n^{th} order partial differential equation (PDE), one needs to extend the infinitesimal symmetry generator (2.9) to include all derivatives of the dependent variables. In this section we discuss the prolongation formula for a PDE which consists of 'p' dependent and 'q' independent variables. Since we will be dealing with a PDE of order two, we later deduce a prolongation formula for a second order PDE in which there is only one dependent variable.

Let

$$F(x; u, u^{(1)}, u^{(2)}, \cdots, u^{(n)}) = 0, \tag{2.25}$$

be an n^{th} order PDE with q independent variables $x = (x_1, x_2, x_3, \cdots, x_q)$, p dependent variables $u = (u^1, u^2, \cdots, u^p)$ and the derivatives of dependent up to order n. In this case the infinitesimal symmetry generator associated with this equation becomes

$$\mathbf{X} = \sum_{i=1}^{q} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{k=1}^{p} \phi^k(x,u)\frac{\partial}{\partial u^k}. \tag{2.26}$$

Then the prolongation of the generator (2.26) is obtained by extending it to include all the derivatives [28] as

$$\mathbf{X}^p = \mathbf{X} + \sum_{k=1}^{p}\sum_{j} \phi^{jk}(x,u,u^{(n)})\frac{\partial}{\partial u_{jk}}, \tag{2.27}$$

where ϕ^j and ϕ^{jk} are given [31] by

$$\phi^j = D_j(\phi - \xi^i u_{,i}) + \xi^i u_{,ji} \tag{2.28}$$

$$\phi^{jk} = D_k D_j(\phi - \xi^i u_{,i}) + \xi^i u_{,jki}, \tag{2.29}$$

in which D_i represents the total derivative given by the formula

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j} u_{j,i}\frac{\partial}{\partial u_j}. \tag{2.30}$$

2.12 Invariance

A Lie group of transformations can have invariant functions, surfaces, curves, and invariant points. The invariance can transform the complicated nonlinear conditions into simpler linear conditions under the corresponding infinitesimal generator of the symmetry group. The symmetry group of the system transforms its solutions to other solutions giving new invariant solutions of the system.

2.12.1 Invariance of a function

Definition 2.10

Let $\bar{x} = G_i(x, \epsilon)$ be the Lie group of transformations of one parameter ϵ and let $f(u)$ be an infinitely differentiable function. The function $f(u)$ is said to be an invariant function if and only if

$$f(\bar{x}) = f(x) \tag{2.31}$$

Theorem 2.5. [8]

A function $f(u)$ is an invariant of the Lie group of transformation $\bar{x} = G_i(x, \epsilon)$ if and only if

$$\mathbf{X}f(x) = 0 \tag{2.32}$$

where $\mathbf{X}$ is the infinitesimal generator of the symmetry transformation.

Theorem 2.6. [8]

Given a Lie group of transformation $\bar{x} = G_i(x, \epsilon)$ with a symmetry generator $\mathbf{X}$, the identity,

$$f(\bar{x}) = f(x) + \epsilon \tag{2.33}$$

holds if

$$\mathbf{X} f(x) = 1 \tag{2.34}$$

and conversely.

2.12.2 Invariance of a surface

Theorem 2.7. [8]

Let $f(x) = 0$ be a surface and let $\bar{x} = G_i(x, \epsilon)$ be a one-parameter Lie group of transformations. The surface $f(x) = 0$ is said to be an invariant surface under the symmetry transformation if and only if

$$\mathbf{X} f(\bar{x}) = 0 \qquad when \qquad f(x) = 0 \tag{2.35}$$

2.12.3 Invariance of a Partial Differential Equation

Consider a system of partial differential equation of order n with q independent $x = (x_1, x_2, ..., x_q)$ and p dependent variable $u = (u^1, u^2, ..., u^p)$, given by

$$\mathbf{F}_\mu(x, u, \partial u, \partial^2 u, ..., \partial^n u) = 0 \tag{2.36}$$

where

$$\mu = 1, 2, 3, ..., k$$

The derivative of order m is denoted as,

$$u_j^\alpha = \frac{\partial^m u^\alpha}{\partial x_{j1} \partial x_{j2} ... \partial x_{jm}} \tag{2.37}$$

where $1 \leq j_i \leq p$ for all $i = 1, 2, ..., m$ and the order of m-tuple of integers $j = (j_1, j_2, ..., j_m)$ indicates the order of the derivative.

Theorem 2.8 (Invariance Criterion of PDEs [1])

Let the system (2.36) of k differential equations be of maximal rank. If G is group of transformations and

$$\mathbf{X}^{(n)}\{\mathbf{F}_\mu(x, u, \partial u, \partial^2 u, ..., \partial^n u)\} = 0 \tag{2.38}$$

whenever

$$\mathbf{F}_\mu(x, u, \partial u, \partial^2 u, ..., \partial^n u) = 0,$$

for every infinitesimal symmetry generator $\mathbf{X}$ of the group G, then G is a symmetry group of the system.

Example 10:(Diffusion Equation [8])

We demonstrate how a PDE remains invariant under a group of transformation. Consider the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{2.39}$$

and the transformation

$$\bar{x} = e^a x \tag{2.40}$$

$$\bar{t} = e^a t \tag{2.41}$$

$$\bar{u} = e^a u \tag{2.42}$$

Different values of a, b, c give different elements of the group of transformations ($a = b = c = 0$) gives the identity transformation. It is straightforward to see

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{t}}(e^c u) = \frac{\partial}{\partial t}(e^c u)\frac{\partial t}{\partial \bar{t}} = e^{c-b}\frac{\partial u}{\partial t} \tag{2.43}$$

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = e^{c-2a}\frac{\partial^2 u}{\partial x^2}. \tag{2.44}$$

The diffusion equation transforms into

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = e^{c-b}\frac{\partial u}{\partial t} - e^{c-2a}\frac{\partial^2 u}{\partial x^2}. \tag{2.45}$$

which gives invariance iff b=2a, as

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = e^{c-2a}(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}. \tag{2.46}$$

2.13 Procedure to calculate symmetries

The infinitesimal transformations and the infinitesimal symmetry generators of the Lie group of a partial differential equation can be calculated by a systematic computational procedure in the light of Theorem 2.8 and used the prolongation formula(2.27). The first step is to find the one-parameter symmetry generator $\mathbf{X}$. The coefficients $\xi^i(x, u)$ and $\phi_\alpha(x, u)$ of the symmetry generator $\mathbf{X}$ will be the functions of x, u. The symmetry generator $\mathbf{X}$ will be prolonged to the order equivalent to the order of partial differential equation.

Application of a prolonged symmetry generator to the partial differential equation using the theorem 2.8 of the infinitesimal criterion for the invariance of PDE gives a general equation that involves x, u and the derivatives of u with respect to x, as well as, $\xi^i(x, u)$, $\phi_\alpha(x, u)$ and their partial derivatives with respect to x and u. By comparing the coefficients of the partial derivatives of u we get a system of equations known as determining equations for the coefficients functions $\xi^i(x, u)$ and $\phi_\alpha(x, u)$. The general solution of this system of determining equations determines the most general expressions for $\xi^i(x, u)$ and $\phi_\alpha(x, u)$; thus giving the general infinitesimal symmetry generator $\mathbf{X}$.

Chapter 3

A QUASILINEAR FISHER EQUATION

As we noted in chapter 1, the Fisher equation arises naturally in a number of situations such as reaction diffusion processes, genetics, and biology. We remarked that in many situations of interest, the diffusivity of the medium is not constant but may depend upon the dependent variable. It was further envisaged that we may need to study this equation in other coordinate systems to suit a particular model. In this regard we present here the symmetry analysis of the Fisher equation in which we have the nonlinear reaction diffusion term $g(u) = u(1-u)$ and the diffusivity is assumed proportional to u, or simply, $f(u) = u$. Bokhari, Mustafa and Zaman [4] have studied this model with the radial symmetry taken in two account. In this case the Fisher equation has the form

$$\frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}(xu\frac{\partial u}{\partial x}) = u(1-u). \tag{3.1}$$

To lay a basis of a more general model which is studied in chapter 4, we present the symmetry analysis performed recently by Bokhari, Mustafa and Zaman [4] who have given a complete analysis and some reductions in case of interest. Since the PDE (3.1) is of order two we prolong the symmetry generator (2.30) to second order [4].

$$\mathbf{X}^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}, \tag{3.2}$$

where $\phi^x, \phi^t, \phi^{tt}, \phi^{xt}$ and ϕ^{xx} are given by (2.28)-(2.29),

$$\phi^x = \phi_x + (\phi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t \tag{3.3}$$

$$\phi^t = \phi_t + (\phi_u - \tau_t)u_t - \xi_t u_x - \tau_u u_t^2 - \xi_u u_x u_t \tag{3.4}$$

$$\phi^{tx} = \phi_{tx} + \phi_{xu}u_t + \phi_{tu}u_x + \phi_{uu}u_tu_x + \phi_u u_{tx} - u_{tt}(\tau_x + \tau_u u_x) - u_{tx}(\xi_x + \xi_u u_x)$$

$$-u_{tx}(\tau_t + \tau_u u_t) - u_{xx}(\xi_t + \xi_u u_x) - u_t(\tau_{tx} + \tau_{xu}ut + \tau_{tu}u_x + \tau_{uu}u_tux + \tau_u u_{tx})$$

$$-u_x(\xi_{tx} + \xi_{xu}u_t + \xi_{tu}u_x + \xi_{uu}u_tu_x + \xi_u u_{tx}) = 0 \tag{3.5}$$

$$\phi^{xx} = \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3 \tag{3.6}$$

$$-\tau_{uu}u_x^2u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_x u_{tx} - 3\xi_u u_x u_{xx} - \tau_u u_t u_{xx} - 2\tau_u u_x u_{xt}$$

$$\phi_{tt} = \phi_{tt} + 2\phi_{tu}u_t + \phi_{uu}u_t^2 + \phi_u u_{tt} - 2u_{tt}(\tau_t + \tau_u u_t) - u_t(\tau_{tt} + 2\tau_{tu}u_t \tag{3.7}$$

$$+\tau_{uu}u_t^2 + \tau_u u_{tt}) - 2u_{tx}(\xi_t + \xi_u^2 u_t) - u_x(\xi_{tt} + 2\xi_{tu}u_t + \xi_{uu}u_t^2 + \xi_u u_{tt})$$

Now the Lie symmetry criterion (2.10) for equation (3.1) takes the form:

$$\mathbf{X}^{(2)}(PDE)\big|_{PDE=0} = 0 \tag{3.8}$$

Writing equation (3.8) in the expanded form,

$$\mathbf{X}^{(2)}(xu_t - uu_x - xu_x^2 - xuu_{xx} - xu + xu^2)\big|_{PDE=0} = 0$$

Which leads to

$$\xi uu_x - x\phi u_x - x^2\phi u_{xx} - x^2\phi + 2x^2\phi u - x\phi^x u - 2x^2\phi^x u_x + x^2\phi^t - x^2u\phi^{xx} = 0. \tag{3.9}$$

At this stage we substitute values of ϕ^x, ϕ^t and ϕ^{xx} obtained using (3.4),(3.4)and (3.6) to get

$$\xi u u_x - x\phi u_x - x^2\phi u_{xx} - x^2\phi + 2x^2\phi u - xu\phi_x - xu\phi_u u_x + xu\xi_x u_x + xu\tau_x u_t + xu\xi_u u_x^2$$
$$+xu\tau_u u_x u_t - 2x^2\phi_x u_x - 2x^2 u_x\phi_u + 2x^2\xi_x u_x^2 + 2x^2\tau_x u_x u_t + 2x^2\xi_u u_x^3 + 2x^2\tau_u u_x^2 u_t + x^2\phi_t$$
$$+x^2\phi_u u_t - x^2\tau_t u_t - x^2 u\xi_t u_x - x^2\tau_u u_t^2 - x^2\xi_u u_x u_t - x^2 u\phi_{xx} - 2x^2 u\phi_{ux} u_x + x^2 u\xi_{xx} u_x$$
$$+x^2 u\tau_{xx} u_t - x^2 u\phi_{uu} u_x^2 + 2x^2 u\xi_{xu} u_x^2 + 2x^2 u\tau_{xu} u_x u_t + x^2 u u_x^3 - x^2 u\phi_u u_{xx} + 2x^2 u\xi_x u_{xx}$$
$$+2x^2 u\tau_x u_{tx} + 3x^2 u\xi_u u_x u_{xx} + x^2 u\tau_u u_{xx} u_t + 2x^2 u\tau_u u_x u_{xt} + 2x^2 u\tau_u u_x u_{xt} = 0 \tag{3.10}$$

Equation (3.10) is an algebraic equation in u, its derivatives and powers and products of derivatives. In order to proceed with the analysis we compare like terms to obtain the following determining equations:

$$\xi_u = 0 \tag{3.11}$$

$$\tau_u = 0 \tag{3.12}$$

$$\tau_x = 0 \tag{3.13}$$

$$-u\xi + x\phi + x^2\xi_t - xu\xi_x - xu\tau_t + 2x^2\phi_x - ux^2\xi_{xx} + 2ux^2\phi_{xu} = 0 \tag{3.14}$$

$$-x\phi + 2xu\phi - xu\tau_t + xu^2\tau_t + x\phi_t + xu\phi_u - xu^2\phi_u - u\phi_x - xu\phi_{xx} = 0 \tag{3.15}$$

$$\phi - 2u\xi_x + u\tau_t = 0 \tag{3.16}$$

$$-2\xi_x + \tau_t + \phi_u + u\phi_{uu} = 0 \tag{3.17}$$

Solving the above system we obtain its solution given by

$$\xi = 0 \tag{3.18}$$

$$\tau = -e^{-t}k_1 + k_2 \tag{3.19}$$

$$\phi = e^{-t}k_1 u \tag{3.20}$$

From above we notice that the system admits two symmetries which in generator form are written as,

$$\mathbf{X}_1 = -e^{-t}\frac{\partial}{\partial t} - e^{-t}u\frac{\partial}{\partial u}$$
$$\mathbf{X}_2 = \frac{\partial}{\partial t}$$

The commutation relations for the above symmetry generators are listed in the following table:

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$
$\mathbf{X}_1$	0	0
$\mathbf{X}_2$	$-\mathbf{X}_1$	0

Table 3: Commutation Relation

We present procedure to achieve reduction with respect to one symmetry generator '$\mathbf{X_1}$' and subsequently obtain one exact solution. Picking the generator

$$\mathbf{X}_1 = -e^{-t}\frac{\partial}{\partial t} - e^{-t}u\frac{\partial}{\partial u}, \tag{3.21}$$

we re-write it in the form,

$$\mathbf{X}_1 = 0\frac{\partial}{\partial x} - e^{-t}\frac{\partial}{\partial t} - e^{-t}u\frac{\partial}{\partial u} \tag{3.22}$$

The characteristic equations for the above equation is,

$$\frac{dx}{0} = \frac{dt}{-e^{-t}} = \frac{du}{-e^{-t}u} \tag{3.23}$$

The first two terms give:

$$\frac{dx}{0} = \frac{dt}{-e^{-t}}, \tag{3.24}$$

which can be solved to obtain,

$$x = z(t). \tag{3.25}$$

Similarly solving the last two terms in (3.23)

$$\frac{dt}{1} = \frac{du}{u}, \tag{3.26}$$

giving us,

$$u = e^{t}V(z). \tag{3.27}$$

Now we use (3.25) and (3.27) to re-cast Equation (3.1) in new coordinates. For this purpose we take first and second derivatives of u (using chain rule) to obtain:

$$u_t = e^{t}V(z), \tag{3.28}$$

$$u_x = e^{t}\frac{dV}{dz}, \tag{3.29}$$

$$u_{xx} = e^t \frac{d^2V}{dz^2}. \tag{3.30}$$

Using these expressions the Fisher equation (3.1) transforms into a second order ODE,

$$zV(z)\frac{d^2V}{dz^2} + z(\frac{dV}{dz})^2 + V(z)\frac{dV}{dz} - zV^2(z) = 0. \tag{3.31}$$

Further, it is easy to see that the equation (3.31) is again difficult to solve so we try to reduce it further by finding its symmetries. Take the infinitesimal generator of its symmetry algebra of ordinary differential equation (3.31) of the form

$$\mathbf{X} = \xi(z,V)\frac{\partial}{\partial z} + \phi(z,V)\frac{\partial}{\partial V} \tag{3.32}$$

$$V\xi + 3V^2z^2\xi_\nu - Vz\xi_z - 2z^2\phi_z + Vz^2\xi_{zz} - 2Vz^2\phi_{z\nu} = 0 \tag{3.33}$$

$$-z\phi - 2Vz\xi_z - Vz\phi_\nu + \phi_z + z\phi_{zz} = 0 \tag{3.34}$$

$$z\phi - 2V^2\xi_\nu - Vz\phi_\nu + 2V^2z\xi_{z\nu} - V^2z\phi_{\nu\nu} = 0 \tag{3.35}$$

$$-\xi_\nu + V\xi_{\nu\nu} = 0 \tag{3.36}$$

These equations are again nontrivial to solve. However, by observation, we have

$$\xi = 0 \tag{3.37}$$

$$\phi = V \tag{3.38}$$

satisfy the above system of equations. Hence,

$$X = V\frac{\partial}{\partial V} \tag{3.39}$$

is a symmetry of the ordinary differential equation (3.31) which reduces it to a 1^{st}order ordinary differential equation. The prolongation is

$$\mathbf{X} = 0\frac{\partial}{\partial z} + V\frac{\partial}{\partial V} + V'\frac{\partial}{\partial V'} + V''\frac{\partial}{\partial V''} \tag{3.40}$$

we then write characteristic equation

$$\frac{dz}{0} = \frac{dV}{V} = \frac{dV'}{V'} = \frac{dV''}{V''} \tag{3.41}$$

By solving the characteristic system gives differential invariants

$$z = s, \quad w(s) = \frac{dV}{V} \tag{3.42}$$

Hence

$$V'' = \frac{dV'}{dz} = \frac{d(Vw)}{dz} = V\frac{dw}{ds} + V'w \tag{3.43}$$

Putting the new variables in the ordinary differential equation (3.31) reduces it to the first order equation

$$\frac{dw}{ds} = -2w^2 - \frac{1}{s}w + 1 \tag{3.44}$$

The reduced 1^{st}order ordinary differential equation (3.44) is the Riccati equation having the solution given by

$$w(s) = \frac{1}{2}\{\sqrt{2}\frac{C_1 I_1(\sqrt{2}s) - K_1(\sqrt{2}s)}{C_0 I_0(\sqrt{2}s) - K_0(\sqrt{2}s)} \tag{3.45}$$

where

I_0, I_1, K_0 and K_1 are modified Bessel functions.

Now, using the substitutions given by $x = z(x,t), w(s) = \frac{V'}{V}$ from above gives

$$V(z) = e^{\int w(z)dz + C} \tag{3.46}$$

Finally $z = x$ and $u(x,t) = e^t V(z)$will gives an exact solution of the Fisher equation in cylindrical coordinates, given by

$$u(x,t) = C_2 e^t \sqrt{C_1 I_0(\sqrt{2}x) + k_0(\sqrt{2}x)} \tag{3.47}$$

We have used here $\mathbf{X}_1$to obtain the solution. If $\mathbf{X}_2 = \frac{\partial}{\partial t}$ is used, this symmetry generator will lead to the traveling wave solutions as translation in time only. We can re-write the generator as

$$\mathbf{X}_2 = 0\frac{\partial}{\partial x} + 1\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u} \tag{3.48}$$

The characteristic equation is given by

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} \tag{3.49}$$

$$\alpha = x, \quad u = w(\alpha) \tag{3.50}$$

Now, we prepare to recast equation (3.1) in new coordinates we get,

$$u_t = 0, \quad u_x = w_\alpha \quad \texttt{and} \quad u_{\alpha\alpha} = w_{\alpha\alpha} \tag{3.51}$$

By substituting in equation (3.1), we reduce it to the following ordinary differential equation

$$\frac{d^2 w}{d\alpha^2} + \frac{1}{w}(\frac{dw}{d\alpha})^2 + \frac{1}{\alpha}\frac{dw}{d\alpha} - w + 1 = 0. \tag{3.52}$$

This is second order nonlinear ordinary differential equation which needs further investigations but have not been pursued.

Chapter 4

A Generalized Nonlinear Fisher Equation

In this chapter we classify the symmetries of a generalized Fisher equation. We use these symmetries to reduce the Fisher equation to second order ordinary differential equations and solve the reduced ordinary differential equations in some cases. Since it is not possible to find analytical solutions of the reduced equations in all cases, we present solutions in cases where solutions are possible. As discussed in chapter 1, the generalized Fisher equation in cylindrical coordinates with radial symmetry is given by

$$\frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}(xg(u)\frac{\partial u}{\partial x}) = f(u) \tag{4.1}$$

Expanding second term in above equation (4.1) and re-writing it gives,

$$xu_t - gu_x - xg_u u_x^2 - xgu_{xx} - xf = 0 \tag{4.2}$$

Our aim here is to find the Lie point symmetries of the generalized nonlinear Fisher equation (4.1). For this purpose we will use its form given by (4.2). The symmetry generator associated with the above equation is

$$\mathbf{X} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}, \tag{4.3}$$

where

$$\xi = \xi(x,t,u), \quad \tau = \tau(x,t,u) \quad \text{and} \quad \phi = \phi(x,t,u).$$

Since equation (4.1) is a second order partial differential equation, the symmetry generator (4.3) is to be prolonged to second order. The expression for this prolonged symmetry generator is given by

$$\mathbf{X}^{(2)} = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \quad (4.4)$$

At this stage one requires that the partial differential equation (4.2) should satisfy the symmetry criterion [6]

$$\mathbf{X}^{(2)}\left[xu_t - g(u)u_x - xg_u(u)u_x^2 - xg(u)u_{xx} - xf(u)\right]\Big|_{PDE(4.2)=0} = 0 \quad (4.5)$$

The above requirement leads to an equation given by,

$$\begin{aligned} &\xi u_t - \xi g_u(u)u_x^2 - \xi g(u)u_{xx} - \xi f(u) - \phi g_u(u)u_x - x\phi g_{uu}(u)u_x^2 - x\phi g_{uu}(u)u_{xx} \\ &-x\phi g_u(u)u_{xx} - x\phi f_u(u) - g(u)\phi^x - 2x\phi^x g_u(u)u_x + x\phi^t - xg(u)\phi^{xx} = 0. \end{aligned} \quad (4.6)$$

In order to proceed further, we now use the expression of ϕ^x, ϕ^t and ϕ^{xx} [7] and replace u_t using (4.2) in the above equation (4.6) to get,

$$\begin{aligned} &\tfrac{\xi g u_x}{x} - \phi g_u u_x - x\phi g_{uu}u_x^2 - x\phi g_u u_{xx} - x\phi f_u - g\phi_x - g\phi_u u_x + g\xi_x u_x - 2xg_u\phi_x u_x \\ &-2xg_u\phi_x u_x^2 + 2x\xi_x g_u u_x^2 + x\phi_t + \phi_u g_u u_x + x\phi_u g_u u_x^2 + xg\phi_u u_{xx} + x\phi_u f - g\tau_t u_x \\ &-x\tau_t g_u u_x^2 - x\tau_t g u_{xx} - xf\tau_t - x\xi_t u_x - xg\phi_{xx} - 2xg\phi_{ux}u_x + xg\xi_{xx}u_x - xg\phi_{uu}u_x^2 \\ &-xg\phi_u u_{xx} + 2xg\xi_x u_{xx} + 2xg\tau_x u_{tx} + 2xg\xi_u u_x u_{xx} + 2xg\tau_u u_x u_{xt} = 0. \end{aligned} \quad (4.7)$$

The equation (4.7) can be seen as an algebraic equation in derivatives of u. To proceed further we now compare coefficients of all derivatives to obtain a system which leads to the following determining equations

$$\tau_x = 0, \tag{4.8}$$

$$\tau_u = 0, \tag{4.9}$$

$$\xi_u = 0, \tag{4.10}$$

$$-2g_u\xi_x + g_u\tau_t + g_u\phi_u + \phi g_{uu} + g\phi_{uu} = 0, \tag{4.11}$$

$$g\xi - x\phi g_u - x^2\xi_t + xg\xi_x - xg\tau_t - 2x^2g_u\phi_x + x^2g\xi_{xx} - 2x^2g\phi_{xu} = 0, \tag{4.12}$$

$$x\phi f_u + xf\tau_t - x\phi_t - xf\phi_u + g\phi_x + xg\phi_{xx} = 0, \tag{4.13}$$

$$\phi g_u - 2g\xi_x + g\tau_t = 0 \tag{4.14}$$

In order to solve the above system for ξ, τ and ϕ, we proceed as follows:

First differentiating equation (4.14) twice we get

$$\phi_u g_u + \phi g_{uu} - 2g_u\xi_x + g_u\tau_t = 0, \tag{4.15}$$

$$\phi_{uu}g_u + 2\phi_u g_{uu} + \phi g_{uuu} - 2g_{uu}\xi_x + g_{uu}\tau_t = 0. \tag{4.16}$$

In order to classify the generalized Fisher equation equation (4.1) and hence its solutions in terms of $g(u)$, we start by assuming that $g_{uu} = 0$, that is

$$g = \alpha_1 + \alpha_2 u$$

where α_1 and α_2 are some non-zero constants of integration. Using this condition in (4.16) implies that,

$$\phi_{uu} = 0$$

which gives,

$$\phi(x,t,u) = A(x,t)u + B(x,t) \tag{4.17}$$

where $A(x,t)$ and $B(x,t)$ are functions of integration to be determined in the process of solving above system. Using the above conditions in (4.15) it becomes,

$$(A - 2\xi_x + \tau_t) = 0. \tag{4.18}$$

To proceed further we consider following cases:

Case I

$$A - 2\xi_x + \tau_t = 0, \quad \phi(x,t,u) = A(x,t)u + B(x,t), \quad g = \alpha_1 + \alpha_2 u.$$

Notice that equation (4.15) is identically satisfied by the conditions that arise in this case. Substituting the above conditions in (4.14) we obtain $\alpha_2 B = \alpha_1 A$ implying that

$$\phi = A(u + \frac{\alpha_1}{\alpha_2}) \tag{4.19}$$

Substituting (4.19) in (4.13), we get

$$\begin{aligned} &\alpha_2(xAuf_u + xf\tau_t - xA_t u - xfA + \alpha_2 A_x u^2 - x\alpha_2 A_{xx} u^2) + \\ &\alpha_1(xAf_u - xA_t + \alpha_1 A_x + \alpha_1 x A_{xx}) + 2\alpha_1\alpha_2 u(A_x + xA_{xx}) = 0 \end{aligned} \tag{4.20}$$

From the above equation two possibilities arise namely:

(a) $\alpha_2 = 0$ and (b) $\alpha_1 = 0$. We first consider possibility (a):

Case (Ia)

From equation (4.19) it is observed that $A = 0$ which implies that $\phi = 0$. We also observe that (4.15) and (4.14) are identically satisfied. Substituting $\phi = 0$ and $\alpha_2 = 0$ in (4.13) we obtain that

$$xf\tau_t = 0$$

$$\tau_t = 0$$

Above results implies that

$$\tau = c$$

Since

$$2\xi_x - \tau_t = 0,$$

then

$$\xi_x = 0.$$

so that $\xi = \xi(t)$; here ξ is an arbitrary function of t only. To proceed further, we assume that $\xi(t) = c_1 t$ and substitute in (4.12) to get

$$c_1(\alpha_1 t - x^2) = 0 \qquad \text{which implies} \quad c_1 = 0$$

So that, we obtain the following system:

$$\xi = 0, \tag{4.21}$$

$$\tau = c,$$

$$\phi = 0.$$

Corresponding to the above system (4.21) there exists one infinitesimal symmetry generator [8] which is given by

$$\mathbf{X} = \frac{\partial}{\partial t} \tag{4.22}$$

The infinitesimal symmetry generator reduces the number of the independent variables by one in the partial differential equation [8, 19]. We find reduction of the generalized Fisher equation under the infinitesimal symmetry generator **X**. The detailed calculations for the reduction under **X** are given below.

Consider the generator (4.22) and re-write it as

$$\mathbf{X} = 0\frac{\partial}{\partial x} + 1\frac{\partial}{\partial t} + 0\frac{\partial}{\partial u}.$$

The characteristic equations for this generator are

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}.$$

The similarity variables for the above generator are obtained as follows:

From the first two terms we obtain

$$\frac{dx}{0} = \frac{dt}{1}, \tag{4.23}$$

which implies that,

$$x = \alpha$$

Similarly, from the relation

$$\frac{dt}{1} = \frac{du}{0} \tag{4.24}$$

we get

$$u = w(\alpha)$$

The generalized Fisher equation (4.1) can be transformed into these similarity variables α, w as follows

$$u_t = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial t} = w_\alpha.0 = 0$$

$$u_x = \frac{\partial w}{\partial x} = \frac{\partial w}{\partial \alpha}\frac{\partial \alpha}{\partial x} = w_\alpha.1 = w_\alpha$$

$$u_{xx} = \frac{\partial w_\alpha}{\partial x} = \frac{\partial w_\alpha}{\partial \alpha}\frac{\partial \alpha}{\partial x} = w_{\alpha\alpha}.1 = w_{\alpha\alpha}$$

leading to order ordinary differential equation

$$\frac{d^2w}{d\alpha^2} + \frac{1}{\alpha}\frac{dw}{d\alpha} + \frac{1}{\alpha_1}f(w) = 0. \tag{4.25}$$

(i) If we choose $f(w) = w$ the equation (4.25) becomes,

$$\frac{d^2w}{d\alpha^2} + \frac{1}{\alpha}\frac{dw}{d\alpha} + \frac{1}{\alpha_1}w = 0. \tag{4.26}$$

which is the Bessel equation of order zero and has the following solution

$$\begin{aligned} w(\alpha) &= C_1 J(0, \frac{1}{\sqrt{\alpha_1}}\alpha) + C_2 Y(0, \frac{1}{\sqrt{\alpha_1}}\alpha) \\ u(x,t) &= C_1 J(0, \frac{1}{\sqrt{\alpha_1}}x) + C_2 Y(0, \frac{1}{\sqrt{\alpha_1}}x) \end{aligned} \tag{4.27}$$

Where J and Y are Bessel functions.

(ii) If we choose $\alpha_1 = 1, f(w) = w^2$, the equation (4.25) becomes

$$\frac{d^2w}{d\alpha^2} + \frac{1}{\alpha}\frac{dw}{d\alpha} + w^2 = 0. \tag{4.28}$$

in which the independent variable is α and dependent variable is w. Let

$$\mathbf{X} = \xi(\alpha, w)\frac{\partial}{\partial \alpha} + \phi(\alpha, w)\frac{\partial}{\partial w}.$$

The second prolongation of $\mathbf{X}$ is

$$\mathbf{X}^{(\mathbf{2})} = \xi\frac{\partial}{\partial \alpha} + \phi\frac{\partial}{\partial w} + \phi^{\alpha}\frac{\partial}{\partial w_\alpha} + \phi^{\alpha\alpha}\frac{\partial}{\partial w_{\alpha\alpha}}. \tag{4.29}$$

The symmetry criterion gives

$$\mathbf{X}^{(\mathbf{2})}(\alpha w_{\alpha\alpha} + w_\alpha + \alpha w^2)\big|_{\alpha w_{\alpha\alpha}+w_\alpha+\alpha w^2=0} = 0. \tag{4.30}$$

We need only the expansions of ϕ^{α} and $\phi^{\alpha\alpha}$. The determining equations, using the procedure described before are

$$\xi_w = 0, \tag{4.31}$$

$$\phi_{ww} = 0,$$

$$-\xi_\alpha + \phi_w - \alpha\xi_{\alpha\alpha} + 2\alpha\phi_{\alpha w}$$

$$w^2\xi + 2\alpha w\phi + \phi_\alpha + \alpha\phi_{\alpha\alpha} = 0$$

$$\xi - 2\alpha\xi_\alpha + \alpha\phi_w = 0$$

The general solutions of the determining equations are

$$\xi = K_1 \alpha, \tag{4.32}$$

$$\phi = -2K_1 w.$$

where K_1 is constant. The infinitesimal symmetry generator is

$$\mathbf{X} = \alpha \frac{\partial}{\partial \alpha} - 2w \frac{\partial}{\partial w}. \tag{4.33}$$

The prolongation of $\mathbf{X}$ is given by

$$\mathbf{X} = \alpha \frac{\partial}{\partial \alpha} - 2w \frac{\partial}{\partial w} - 3w_\alpha \frac{\partial}{\partial w_\alpha} - 4w_{\alpha\alpha} \frac{\partial}{\partial w_{\alpha\alpha}}. \tag{4.34}$$

A part of the characteristic equation corresponding to the above generator is:

$$\frac{d\alpha}{\alpha} = \frac{dw}{-2w} = \frac{dw_\alpha}{-3w_\alpha}. \tag{4.35}$$

Solving the characteristic system gives differential invariants:

$$\alpha^2 w = u, \tag{4.36}$$

$$\alpha^3 w_\alpha = v.$$

This implies that

$$\begin{aligned} \frac{dv}{du} &= \frac{3\alpha^2 w_\alpha + \alpha^3 w_{\alpha\alpha}}{2\alpha w + \alpha^2 w_\alpha} \\ &= \frac{2v - u^2}{2u + v} \end{aligned} \tag{4.37}$$

Let $2u + v = s$. So, $v = s - 2u$ then

$$\frac{dy}{dv} = \frac{ds}{dv} - 2. \tag{4.38}$$

Substituting in equation (4.37), we obtain the following ordinary differential equation

$$s\frac{ds}{du} = 4s - 4u - u^2 \tag{4.39}$$

which is **Abel's** equation of second kind and is not generally solvable.

(iii) if we choose $\alpha_1 = 1,\ f(w) = \exp(w) = e^w$ then equation (4.25) becomes

$$\frac{d^2w}{d\alpha^2} + \frac{1}{\alpha}\frac{dw}{d\alpha} + e^w = 0 \tag{4.40}$$

we consider the operator

$$\mathbf{X} = \xi(x;w)\frac{\partial}{\partial x} + \phi(x;w)\frac{\partial}{\partial w}. \tag{4.41}$$

The second prolongation is given by

$$\mathbf{X^{(2)}} = \xi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial w} + \phi^{\alpha}\frac{\partial}{\partial w_\alpha} + \phi^{\alpha\alpha}\frac{\partial}{\partial w_{\alpha\alpha}}. \tag{4.42}$$

The symmetry criterion is

$$\mathbf{X^{(2)}}(\alpha w_{\alpha\alpha} + w_\alpha + e^w)|_{\alpha w_{\alpha\alpha}+w_\alpha+e^w=0} = 0, \tag{4.43}$$

This results in

$$\phi^{\alpha\alpha} + \frac{1}{\alpha}\phi^{\alpha} + \phi e^w - \frac{w_\alpha}{\alpha^2}\xi = 0.$$

Simplifying the above after expanding $\phi^\alpha, \phi^{\alpha\alpha}$ and replacing $w_{\alpha\alpha}$, with $-e^w - \frac{1}{\alpha}w_\alpha$, yields

$$\begin{aligned}&\phi_{\alpha\alpha} + 2w_\alpha\phi_{\alpha w} + w_\alpha^2\phi_{ww} - (\tfrac{1}{\alpha}w_\alpha + e^w)\phi_w - w_\alpha\xi_{xx} - 2w_\alpha^2\xi_{xw} - w_\alpha^3\xi_{ww}\\ &2(\tfrac{1}{\alpha}w_\alpha + e^w)(\xi_\alpha + \xi_w w_\alpha) + \tfrac{1}{\alpha}(\phi_\alpha + w_\alpha\phi_w - \xi_\alpha w_\alpha - w^2\xi_w) + \phi e^w - \tfrac{1}{\alpha^2}w_\alpha\xi = 0.\end{aligned} \tag{4.44}$$

This is a cubic equation in w_α. It splits into four equations and after one compares the coefficients of power of w_α

$$\begin{aligned} &w_\alpha^3 : \xi_{ww} = 0, \\ &w_\alpha^2 : \phi_{ww} - 2\xi_{\alpha w} + \tfrac{2}{\alpha}\xi_w = 0, \\ &w_\alpha : 2\phi_{\alpha w} + 3e^w\xi_w - \xi_{xx} + \tfrac{1}{\alpha}\xi_\alpha - \tfrac{1}{\alpha^2}\xi = 0, \\ &1 : \phi_{\alpha\alpha} - e^w\phi_w + 2e^w\xi_\alpha + \tfrac{1}{\alpha}\phi_\alpha + e^w\phi = 0. \end{aligned} \tag{4.45}$$

The general solutions of the determining equations are

$$\begin{aligned} &\xi = C_2\alpha\ln\alpha - C_2\alpha + \tfrac{1}{2}C_1\alpha, \\ &\phi = -2C_2\ln\alpha - C_1. \end{aligned} \tag{4.46}$$

where C_1 and C_2 are constant, $\alpha > 0$. Since the determining equations are linear homogeneous, the general solutions can be represented as linear combination of two independent solutions.

$$\begin{aligned} &\xi_1 = \tfrac{1}{2}\alpha, \quad \phi_1 = -1 \\ &\xi_2 = \alpha\ln\alpha - \alpha, \quad \phi_2 = -2\ln\alpha. \end{aligned} \tag{4.47}$$

This admits two linearly independent operators and the corresponding symmetry generators are

$$\begin{aligned} &\mathbf{X_1} = \tfrac{\alpha}{2}\tfrac{\partial}{\partial\alpha} - \tfrac{\partial}{\partial w}, \\ &\mathbf{X_2} = (\alpha\ln\alpha - \alpha)\tfrac{\partial}{\partial\alpha} - 2\ln\alpha\tfrac{\partial}{\partial w}. \end{aligned} \tag{4.48}$$

This Lie algebra is spanned by $\mathbf{X_1}$ and $\mathbf{X_2}$ since $[\mathbf{X_1}, \mathbf{X_2}] = \mathbf{X_1}$. The commutation relations for these generators are given in the form of table 4. The commutator table describes the structure of the associated Lie algebra in a convenient way [8].

$[\mathbf{X}_i, \mathbf{X}_j]$	$\mathbf{X}_1$	$\mathbf{X}_2$
$\mathbf{X}_1$	0	$\mathbf{X}_1$
$\mathbf{X}_2$	$\mathbf{X}_1$	0

Table 4: Commutator Relations

Since $[\mathbf{X_1}, \mathbf{X_2}] = \mathbf{X_1}$, we begin the reduction using $\mathbf{X_1}$. Now

$$\mathbf{X_1}^{(\mathbf{1})} = \frac{\alpha}{2}\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial w} - \frac{w'}{2}\frac{\partial}{\partial w'} \tag{4.49}$$

The invariant of the group generated by $\mathbf{X_1}^{(\mathbf{1})}$ are

$$u = \alpha^2 e^w \quad and \quad \upsilon = \alpha w' \tag{4.50}$$

A second-order differential invariant by use of a theorem of Lie can be expressed in terms of υ,u and $\frac{du}{d\upsilon}$. This reduces the ODE (4.40) to the following ODE

$$\frac{d\upsilon}{du} = \frac{-1}{2+\upsilon} \tag{4.51}$$

that admits $\mathbf{X_2}$ written in (u, υ) coordinates. The solution of the first-order equation (4.51) is easily seen to be:

$$2\upsilon + \frac{1}{2}\upsilon^2 + u = A \tag{4.52}$$

where A is a constant. The Substitution of $u = \alpha^2 e^w$ and $\upsilon = \alpha w'$ results in

$$2\alpha w' + \frac{1}{2}\alpha^2 w'^2 + \alpha^2 e^w = A \tag{4.53}$$

We use an invariant to integrate this equation. This is readily seen to be a variable in the form $u^2 = \alpha^2 e^w$. We get

$$\alpha^2 u'^2 = -2u^3 + 4u^2 + 2u^2 A \tag{4.54}$$

which is variables separable.

case (Ib)

$$\alpha_1 = 0, g = \alpha_2 u$$

implies that $\phi = Au$. Also, from equation (4.20) we observe that

$$\alpha_2(xAuf_u + xf\tau_t - xA_t u - xfA + \alpha_2 A_x u - x\alpha_2 A_{xx} u^2) = 0 \tag{4.55}$$

Differentiating (4.55) three times with respect to 'u' we obtain

$$\frac{uf_{uuuu}}{f_{uuu}} = \frac{-2A + \tau_t}{A} = C_1. \tag{4.56}$$

From (4.56) we get

$$f(u) = \frac{\beta u^{c_1+3}}{(c_1+1)(c_1+2)(c_1+3)} + \frac{c_2 u^2}{2} + c_3 u + c_4, \tag{4.57}$$

Comparison With Special Case [4]:

We note that if $\beta = 0, C_2 = 2, C_3 = -1, C_4 = 0$, and $\alpha_2 = 1$ then,

$$f(u) = u(u-1) \tag{4.58}$$

$$g(u) = u \tag{4.59}$$

This is the case discussed in (Bokhari, Mustafa and Zaman) [4]. We find that the determining equations (4.8-4.14) reduce to those in [4]. The solution is given by

$$u(x,t) = C_2 e^t \sqrt{C_1 I_0(\sqrt{2}x) + k_0(\sqrt{2}x)} \tag{4.60}$$

Let us now continue with the general case we are considering, we note that

$$\tau_t = -A(C_1 + 2), \tag{4.61}$$

Differentiating (4.61) with respect to x we observe that $A_x = 0$ implies that $A = A(t)$ only. With this in mind, we have

$$\tau = -(C_1 + 2)\int A(t)dt + C_5, \tag{4.62}$$

Substituting (4.62) and (4.57) in (4.55) we obtain,

$$A\left(\frac{\beta u^{C_1+3}}{(C_1+1)(C_1+2)} + C_2u^2 + C_3u\right) - A(C_1+2)\left(\frac{\beta u^{C_1+3}}{(C_1+1)(C_1+2)(C_1+3)} + \frac{C_2u^2}{2} + C_3u + C_4\right)$$
$$-A_t u - A\left(\frac{\beta u^{C_1+3}}{(C_1+1)(C_1+2)(C_1+3)} + \frac{C_2u^2}{2} + C_3u + C_4\right) = 0. \tag{4.63}$$

At this stage we compare the coefficient of u^2, u and 1 to get:

$u^2 : C_1C_2 + C_2 = 0$ implies that $C_2 = 0$ (since $C_1 \neq -1$)

$u : 2AC_3 + AC_1C_3 + A_t = 0$,

$1 : (C_1 + 3)C_4 = 0$. implies that $C_4 = 0$ (since $C_1 \neq -3$)

In the light of above we find that

$$A = C_5 e^{-(C_1+2)C_3 t} \tag{4.64}$$

$$f(u) = \frac{\beta u^{C_1+3}}{(C_1+1)(C_1+2)(C_1+3)} + C_3 u \tag{4.65}$$

Substituting in (4.15) we get

$$\xi = -\frac{1}{2}C_5(C_1+1)e^{-(C_1+2)C_3 t}x + C_6. \tag{4.66}$$

Using above in (4.13) we obtain $(C_1 + 2)C_3 = 0$ which implies that $C_3 = 0$.(since $C_1 \neq -2$). Thus,

$$A = C_5 \tag{4.67}$$

$$\xi = -\tfrac{1}{2}C_5(C_1+1)x + C_6$$

$$\tau_t = -C_5(C_1+1)$$

$$\phi = C_5 u$$

Substituting in (4.12) we observe that $C_5 = 0$. Therefore, we obtain the following system:

$$\xi = 0 \tag{4.68}$$

$$\tau = c$$

$$\phi = 0$$

where

$$f(u) = \frac{\beta u^{C_1+3}}{((C_1+1)(C_1+2)(C_1+3))}$$

and C_1 is an arbitrary constant not equal to -1, -2 and -3. The symmetry generator with the above system (4.68) is given by

$$\mathbf{X} = \frac{\partial}{\partial t}. \tag{4.69}$$

This is the same symmetry generator (4.22) which has the new similarity variables α and w where

$$\alpha = x, \quad w(\alpha) = u \tag{4.70}$$

The new similarity variables transform the generalized Fisher equation (4.1) to the following ordinary differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{u}(\frac{du}{dx})^2 + \frac{1}{x}\frac{du}{dx} + \frac{1}{\alpha_2 u}f(u) = 0 \tag{4.71}$$

where

$$f(u) = \frac{\beta u^{C_1+3}}{(C_1+1)(C_1+2)(C_1+3)}. \tag{4.72}$$

In principle we anticipate that $f(u)$ may also be given functional values such as a constant, u or u^2. However, from equation (4.72) we note that these values of $f(u)$ are not permissible because at $c \neq -1, -2$ and -3 the equation (4.72) is not valid. In order to discuss the solution of our generalized Fisher equation for these three values of $f(u)$, the only route available is to start solving the original determining equations with (1): $g = \alpha_2 u$, and $f(u) = constant$, (2): $g = \alpha_2 u$, and $f(u) = u$ and (3): $g = \alpha_2 u$, and $f = u^2$. Using the procedure followed earlier the solution of the determining equations can be easily found in all three cases. The solution of the third case is given by

$$u(x,t) = (C_1 J(0, \sqrt{\frac{2}{\alpha_2}} x) + C_2 Y(0, \sqrt{\frac{2}{\alpha_2}} x))^{\frac{1}{2}}$$

Case II

In order to proceed with the classification and make a general statement we consider equation (4.16) and the above conditions to obtain

$$(2A - 2\xi_x + \tau_t) g_{uu} + (Au + B) g_{uuu} = 0 \tag{4.73}$$

and assume $g_{uuu} = 0$. This requirement yields $g = \alpha_1 + \alpha_2 u + \alpha_3 u^2$. From equation (4.73) and using this condition, we get

$$2A - 2\xi_x + \tau_t = 0.$$

From above conditions, we observe that equation (4.73) is identically satisfied. If we substitute the above in (4.15) we obtain that $B = \frac{\alpha_2 A}{3\alpha_3}$ implying

$$\phi = A(u + \frac{\alpha_2}{3\alpha_3}). \tag{4.74}$$

We note that (4.15) is now identically satisfied. Substituting these values in equation (4.14) results in the expression $(\alpha_2 - 4\alpha_1\alpha_3)A = 0$. This leads to the following three cases

(a) $A = 0,$ (b) $\alpha_2 - 4\alpha_1\alpha_3 = 0,$ (c) both.

Case (IIa)

In this case (4.73) becomes $\xi_x - \frac{1}{2}\tau_t = 0$ while (4.74) gives $\phi = 0$. It can be easily seen that (4.18), (4.15) and (4.14) are identically satisfied. Substituting the above conditions in (4.13), we get $\tau_t = 0$. This implies that $\xi_x = 0$ and $\tau = C$. Substituting these results in (4.12) we obtain

$$\xi = C_1 e^{\frac{gt}{x^2}}. \tag{4.75}$$

Since $\xi_x = 0$ then $\xi = 0$. Therefore, we obtain the following system:

$$\begin{aligned} \xi &= 0 \\ \tau &= C \\ \phi &= 0. \end{aligned} \tag{4.76}$$

We construct one infinitesimal symmetry $\mathbf{X}$ from the system (4.76) given by $\mathbf{X} = \frac{\partial}{\partial t}$. The corresponding characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} \tag{4.77}$$

Using relation $\frac{dx}{0} = \frac{dt}{1}$ gives new similarity variable $\alpha = x$, while the relation $\frac{dt}{1} = \frac{du}{0}$ gives the second similarity variable $u = w(\alpha)$. In these similarity variables the partial differential equation (4.1) reduces to:

$$\frac{d^2w}{d\alpha^2} + (\frac{\alpha_2 + 2\alpha_3 w}{\alpha_1 + \alpha_2 w + \alpha_3 w^2})(\frac{dw}{d\alpha})^2 + \frac{1}{\alpha}\frac{dw}{d\alpha} + (\frac{1}{\alpha_1 + \alpha_2 w + \alpha_3 w^2})f(w) = 0 \qquad (4.78)$$

Since $u = w(\alpha)$, $x = \alpha$ and $f(u) = f(w)$, then (4.78) takes the form

$$\frac{d^2u}{dx^2} + (\frac{\alpha_2 + 2\alpha_3 u}{\alpha_1 + \alpha_2 w + \alpha_3 u^2})(\frac{du}{dx})^2 + \frac{1}{x}\frac{du}{dx} + (\frac{1}{\alpha_1 + \alpha_2 u + \alpha_3 u^2})f(u) = 0 \qquad (4.79)$$

If we choose $\alpha_1 = \alpha_2 = 0$ and $f(u) = u^3$ and multiply by u the equation (4.79) becomes

$$u\frac{d^2u}{dx^2} + 2(\frac{du}{dx})^2 + \frac{u}{x}\frac{du}{dx} + \frac{1}{\alpha_3}u^2 = 0 \qquad (4.80)$$

Let $W = u^3$ then,

$$\frac{dW}{dx}\frac{1}{3u} = u\frac{du}{dx} \qquad (4.81)$$

$$\frac{d^2W}{dx^2}\frac{1}{3u} = u\frac{d^2u}{dx^2} + 2(\frac{du}{dx})^2 \qquad (4.82)$$

Substituting in the above equation we obtain the following equation

$$\frac{d^2W}{dx^2} + \frac{1}{x}\frac{dW}{dx} + \frac{3}{\alpha_3}W = 0 \qquad (4.83)$$

which is the Bessel equation of order zero and has the following solution

$$W(x) = C_1 J(0, \frac{\sqrt{3}}{\sqrt{\alpha_3}}x) + C_2 Y(0, \frac{\sqrt{3}}{\sqrt{\alpha_3}}x) \qquad (4.84)$$

Since $W = u^3$ then,

$$u(x,t) = (C_1 J(0, \frac{\sqrt{3}}{\sqrt{\alpha_3}}x) + C_2 Y(0, \frac{\sqrt{3}}{\sqrt{\alpha_3}}x))^{\frac{1}{3}} \tag{4.85}$$

Case (IIb)

$A = \xi_x - \frac{1}{2}\tau_t$, $g = \alpha_1 + \alpha_2 u + \alpha_3 u^2$, $\alpha_2^2 - 4\alpha_1\alpha_3 = 0$, $\phi = A(u + \frac{\alpha_2}{3\alpha_3})$. Here we note that equations (4.18) and (4.15) are identically satisfied. Substituting the above conditions in (4.14) we get:

$$Ag = 0 \tag{4.86}$$

From the above equation there arise three cases as before, namely,

(L) $A = 0$, (m) $g = 0$, or (n) both.

Case (IIbL)

In this case, $A = 0$, we have the system given by (4.76) and gives the same solution as in case (IIa).

Case (IIbm)

For $g = 0$, the partial differential equation (4.1) degenerates to a first order equation which is of no interest for our work. Similarly, the case case (**(IIbLn)**) is same as the above and is of no interest again.

Case III

In order to make more general classification we differentiate (4.73) once again with respect to u and obtain

$$(3A - 2\xi_x + \tau_t)g_{uuu} + (Au + B)g_{uuuu} = 0 \tag{4.87}$$

and assume $g_{uuuu} = 0$ implying that $g = \alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3 = 0$. From equation (4.87) and using this conditions, we get

$$(3A - 2\xi_x + \tau_t) = 0.$$

From equations (4.14 - 4.15), and using above conditions, we get

$$-2\xi_x + \tau_t = 0 \tag{4.88}$$

$$\phi = 0 \tag{4.89}$$

Substituting the above results in (4.12-4.13), we get $\xi = 0$ and $\tau = C$. Therefore, we obtain the following system

$$\xi = 0 \tag{4.90}$$

$$\tau = C$$

$$\phi = 0.$$

As we discussed before this system gives us new similarity variable $\alpha = x$ and $u = w(\alpha)$. In these similarity variables the partial differential equation (4.1) reduces to

$$\frac{d^2u}{dx^2} + (\frac{\alpha_2 + 2\alpha_3 u + 3\alpha_4 u^2}{\alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3})(\frac{du}{dx})^2 + \frac{1}{x}\frac{du}{dx} + (\frac{1}{\alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3})f(u) = 0 \tag{4.91}$$

If we choose $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $f(u) = u^4$ and multiply by u the equation becomes

$$u\frac{d^2u}{dx^2} + 3(\frac{du}{dx})^2 + \frac{u}{x}\frac{du}{dx} + \frac{1}{\alpha_4}u^2 = 0 \tag{4.92}$$

Let $W = u^4$ then,

$$\frac{dW}{dx}\frac{1}{4u^2} = u\frac{du}{dx} \tag{4.93}$$

$$\frac{d^2W}{dx^2}\frac{1}{4u^2} = u\frac{d^2u}{dx^2} + 3(\frac{du}{dx})^2 \tag{4.94}$$

Substituting (4.93-4.94) in (4.92) we obtain the following equation

$$\frac{d^2W}{dx^2} + \frac{1}{x}\frac{dW}{dx} + \frac{4}{\alpha_4}W = 0 \tag{4.95}$$

which is a Bessel equation of order zero and has the following solution

$$W(x) = C_1 J(0, \frac{2}{\sqrt{\alpha_4}}x) + C_2 Y(0, \frac{2}{\sqrt{\alpha_4}}x) \tag{4.96}$$

Since $W = u^4$ the solution becomes

$$u(x,t) = (C_1 J(0, \frac{2}{\sqrt{\alpha_4}}x) + C_2 Y(0, \frac{2}{\sqrt{\alpha_4}}x))^{\frac{1}{4}} \tag{4.97}$$

where J and Y are Bessel functions.

Case IV

If we proceed the manner above, we obtain

$$((n-1)A - 2\xi_x + \tau_t)g_{\underbrace{uuu...u}} + (Au + B)g_{\underbrace{uuu...uu}} = 0 \tag{4.98}$$

and assume $g_{uuu...uu} = 0$. This requirement yields $g = \alpha_1 + \alpha_2 u + \alpha_3 u^2 + ... + \alpha_n u^{n-1}$. From equation (4.98) and using this condition, we obtain

$$(n-1)A - 2\xi_x + \tau_t = 0$$

Substituting the above conditions in (4.15-4.14-4.13-4.12) we get the same system (4.90) which reduces the PDE to the following ordinary differential equation

$$\frac{d^2u}{dx^2} + (\frac{\alpha_2 + 2\alpha_3 u + 3\alpha_4 u^2 + ... + (n-1)\alpha_n u^{n-2}}{\alpha_1 + \alpha_2 u + \alpha_3 u^2 + \alpha_4 u^3 + ... + \alpha_n u^{n-1}})(\frac{du}{dx})^2 + \frac{1}{x}\frac{du}{dx} \tag{4.99}$$

$$+ (\frac{1}{\alpha_1 + \alpha_2 u + \alpha_3 u^2 + ... + \alpha_n u^{n-1}})f(u) = 0$$

If we choose $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = 0$, $f(u) = u^n$ and multiply by u, the equation (4.99) becomes

$$u\frac{d^2u}{dx^2} + (n-1)(\frac{du}{dx})^2 + \frac{u}{x}\frac{du}{dx} + \frac{1}{\alpha_n}u^2 = 0 \tag{4.100}$$

Let $W = u^n$ then,

$$\frac{dW}{dx}\frac{1}{nu^{n-2}} = u\frac{du}{dx} \tag{4.101}$$

$$\frac{d^2W}{dx^2}\frac{1}{nu^{n-2}} = u\frac{d^2u}{dx^2} + (n-1)(\frac{du}{dx})^2 \tag{4.102}$$

Substituting equations (4.101-4.102) in equation (4.100) we obtain the following equation

$$\frac{d^2W}{dx^2} + \frac{1}{x}\frac{dW}{dx} + \frac{n}{\alpha_n}W = 0 \tag{4.103}$$

which is a Bessel equation of order zero and has the following solution

$$W(x) = C_1 J(0, \frac{\sqrt{n}}{\sqrt{\alpha_n}}x) + C_2 Y(0, \frac{\sqrt{n}}{\sqrt{\alpha_n}}x) \tag{4.104}$$

Since $W = u^n$ the solution becomes

$$u(x,t) = (C_1 J(0, \frac{\sqrt{n}}{\sqrt{\alpha_n}}x) + C_2 Y(0, \frac{\sqrt{n}}{\sqrt{\alpha_n}}x))^{\frac{1}{n}} \tag{4.105}$$

This is the general solution where $g(u)$ of the form $g(u) = \alpha_n u^{n-1}$ and $f(u)$ of the form $f(u) = u^n$

In the light of above results we conclude our work in the form of the following theorem:

Theorem 4.1

In the classification of generalized Fisher equation (4.1) with an in-homogenous term '$f(u)$' on its right hand side, the classification of '$f(u)$' appears as follows:

(a). A complete classification of '$f(u)$' in terms of explicit function of 'u' in the generalized Fisher equation can be achieved when 'g' is a linear function of 'u'.

(b). when $g(u) = \alpha_n u^{n-1}$ and $f(u) = u^n$ the solution is given by

$$u(x,t) = (C_1 J(0, \sqrt{\frac{n}{\alpha_n}}x) + C_2 Y(0, \sqrt{\frac{n}{\alpha_n}}x))^{\frac{1}{n}}$$

where J and Y are Bessel functions of order zero of first and second kind respectively.

Chapter 5

Conclusion and Future Work

We have studied a generalized Fisher equation in cylindrical polar coordinates

$$\frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}(xg(u)\frac{\partial u}{\partial x}) = f(u)$$

to solve from the Lie symmetry point of view. In order to perform the Lie symmetry analysis of the equation, we have assumed that the equation possesses a radial symmetry so that the Fisher equation remains a $(1+1)$ partial differential equation. We conclude that the $g(u)$ representing diffusivity can be classified in terms of powers of u. For a quadratic inhomogeneous term the $g(u)$ turns out to be an arbitrary function of u leading to a nonlinear ordinary differential equation.

In future study of the model it is recommended that a complete symmetry analysis of the generalized Fisher equation without radial and azimuthal symmetry is performed. It is hoped that this will allow the equation to admit additional symmetries and study of solutions under these symmetries may result in further insights in the whole reaction-diffusion process.

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