FIXED POINT RESULTS OF SOME NONLINEAR MAPS WITH APPLICATIONS

BY

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A Dissertation Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

Mathematics

May 2006

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
DHAHRAN, SAUDI ARABIA

DEAN SHIP OF GRADUATE STUDIES

This dissertation, written by Abdul-Aziz Mustafa Domlo under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS.

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Acknowledgments

All glory and praise be to the Almighty ALLAH, most gracious, most merciful, who has favoured the human beings with wisdom to explore the universe, and glories for His Prophet Muhammad (Peace be upon him).

Acknowledgement is due to King Fahd University of Petroleum & Minerals for supporting this research; in particular, to the Department of Mathematical Sciences.

I wish to express my appreciation and deepest gratitude to my supervisor Dr. Abdul Rahim Khan for his guidance, encouragement, valuable suggestions and giving me all the necessary support I needed to complete this research project. My sincere gratitude goes to my dissertation committee members:

- Dr. Abdelkader Boucherif
- Dr. Mohammad Aslam Chaudhry
- Dr. Muhammad Ashfaq Bokhari
- Dr. Qamrul Hasan Ansari

for their valuable guidance and suggestions to accomplish this work.

I would like to thank all my teachers in the Department of Mathematical Sciences, King Fahd University of Petroleum & Minerals for their constant encouragement during my stay in the Department.

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Abstract

Name:	Abdul-Aziz Mustafa Domlo
Title:	Fixed Point Results of Some Nonlinear Maps with Applications
Major Field:	Mathematics
Date of Degree:	May 2006

The main purpose of this work is to establish new coincidence and common fixed point theorems using contractive and Lipschitz type conditions for nonself single-valued and multivalued mappings (not necesarily continuous) on a metric space and cite their applications in approximation theory and eigenvalue problems. A general iteration scheme for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces is introduced and its convergence to a common fixed point of the family is studied. Random versions of these results are presented. A deep result concerning the existence of random fixed point of an inward multivalued random operator on a separable Banach space with characteristic of noncompact convexity less than 1 is also proved.

Doctor of Philosophy Degree

King Fahd University of Petroleum & Minerals

Dhahran, Saudi Arabia

May 2006

الخلاصة

الاسم : عبدالعزيز مصطفى نجاة الدين دوملو
العنوان : نتائج النقطة الثابتة لبعض الرواسم اللاخطية مع تطبيقات
التخصص الرئيسي : رياضيات
تاريخ نيل الدرجة : ربيع الثاني ١٤٢٧هـ - مايو ٢٠٠٦م

في هذه الرسالة أثبتنا نظريات جديدة حول النقطة المتطابقة والنقطة الثابتة المشتركة للرواسم الأحادية القيمة والمتعددة القيمة (والتي ليست بالضرورة متصلة) وذلك باستخدام شروط تقاص أو شروط من نوع "ليبشتز" في الفضاءات المترية ومن ثم إعطاء تطبيقات على هذه النتائج في نظرية التقريب ومسائل القيمة المميزة "آيجن" . كما قمنا باستنباط مخطط تكراري عام لأي عائلة منتهية من الرواسم شبه اللامتمددة التقاربية في فضاءات "باناخ"، ثم درسنا تقارب هذا المخطط التكراري إلى نقطة ثابتة مشتركة لهذه العائلة من الرواسم . أيضاً قدمنا نتائج مناظرة للنتائج التي حصلنا عليها وذلك في الحالة العشوائية . كما أثبتنا وجود نقطة ثابتة عشوائية للرواسم العشوائية الداخلية متعددة القيمة في فضاءات "باناخ" القابلة للفصل والتي لها مميز عدم إندماج وتحدب أقل من الواحد.

درجة الدكتوراه
جامعة الملك فهد للبترول والمعادن
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ربيع الثاني ١٤٢٧ هـ - مايو ٢٠٠٦ م

PREFACE

The abstract fixed point theory of single-valued mappings has evolved as a natural extension of the corresponding classical theory on Euclidean spaces considered by Brouwer. This theory has been a popular area of research and has applications in various fields. The best known result in this theory is the Banach contraction mapping principle: "Every contraction selfmapping of a complete metric space has a unique fixed point". It has become a vigorous tool for studying nonlinear volterra integral equations and nonlinear functional differential equations in Banach spaces. In the last several decades a number of generalizations of Banach principle have appeared in the literature. Browder [22] and Göhde [46], in 1965, proved fixed point theorems for nonexpansive mappings in a uniformly convex Banach space.

In 1969, the study of fixed points of multivalued nonexpansive mappings was initiated by Nadler [85] and using the concept of Hausdorff metric, he established the multivalued contraction principle containing the Banach principle as a special case. Since then, the fixed point theory of such mappings has received attention of many researchers. For a survey of this subject and more useful references, see, Singh et al. [114]. In recent past, more interest has developed in finding coincidence and common fixed points for many classes of single-valued and multivalued mappings (see, De Marr [31], Dotson [35], Itoh [56], Khan [66], Khan and Hussain [67], Khan et al. [70], Petryshyn [93], Tarafdar [128] and Xu [136-137]).

Approximation theory is an important subject, which has applications in analysis, artificial neural networks, wavelets and engineering. The application of fixed point theorems to approximation theory was initiated by Meinardus [84] in 1968. Brosowski [21], in 1969, obtained the following generalization of a theorem of Meinradus [84]: " Let T be a linear nonexpansive selfmapping of a normed space X and M be a T-invariant subset of X. If u is a fixed point of T and the set of best approximations to u from M is nonempty compact convex, then there exists a best approximation y which is also a fixed point of T". Singh [113] observed that the linearity of the operator and the convexity of the set of best approximations can be relaxed. Since then, many results have been obtained in this direction (see, Habinaik [48], Hicks and Humphries [49], Sahab et al. [100], Smoluk [115]). All the above mentioned results are summarized and extended by Al-Thagafi [3]. Recently, various interesting papers have appeared in this area (see, e.g. Hussain and Khan [50-51], Khan [65], Shahzad [106]).

Approximating fixed points by successive iteration has been one of the central problems of fixed point theory ever since the introduction of one-step Mann iteration process and two-step Ishikawa iteration process (see, Lions [80] Schu [103], Tan and Xu [125-126]). In recent years, the iteration processes have been studied extensively by various authors for:

(a) approximating fixed points of nonlinear mappings,

(b) finding solutions of nonlinear operator equations,

(c) investigating variational inequalities,

(d) seeking convergence of the iterates to a common fixed points of mappings,

in Hilbert and Banach spaces (see, [25, 33, 37, 38, 72, 86, 124, 132, 138]).

The Prague School of probabilists, in 1950s, initiated systematic study of random operator equations by employing methods of functional analysis. The study of random fixed point theory is the core around which the theory of random operators has been developed. It is also worth mentioning that as applications of random fixed point theorems, a number of existence theorems for random approximation theory, random nonlinear Hammerstein equations and stochastic partial differential equations have been given by many authors. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Spacek [116]; the survey article by

Bharucha - Reid [19] attracted the attention of several mathematicians in this area of investigations. Itoh [55] extended Spacek's theorem to multivalued random contraction mappings. Now a days, this theory has become full fledged research area. Many authors have studied the existence of random fixed points of various classes of random operators; see, Beg [8-10], Beg et al. [12], Beg and Shahzad [13-14], Khan and Hussain [68-69], Khan et al. [71], Liu [81], Papageorgiou [92], Ramirez [98], Xu [134-135], Yuan and Yu [139].

We divide this thesis into five chapters.
Chapter 1 summarizes some basic definitions and known results about fixed points, coincidence points, best approximations, iterative procedures and random operators.

In Chapter 2, we prove some coincidence and common fixed point theorems for nonself mappings (not necessarily continuous) satisfying different contractive conditions on an arbitrary nonempty subset of a metric space. Applications of these results are given in the best approximation theory and eigenvalue problems. Our work improves, unifies and sets analogues of the earlier results by Aamri and El Moutawakil [1], Baskaran and Subrahmanyam [7], Ćirić [28], Hussain and Khan [51] and Al-Thagafi [3].

Chapter 3 concerns general iteration schemes for a finite family of asymptotically quasi-nonexpansive mappings; here we study weak and strong convergence of these schemes to a common fixed point of the family of mappings. Our results are generalizations and refinement of the results of Ghosh and Debnath [40], Kuhfitting [77], Petryshyn and Williamson [94], Qihou [96], Rhoades [99], Shahzad and Udomene [110], Tan and Xu [126] and Xu and Noor [133].

Chapter 4 deals with multivalued mappings. We obtain coincidence and common fixed point theorems in a metric space by using Lipschitz type conditions for hybrid mappings which are not necessarily continuous. Some applications to best approximation theory and eigenvalue problems are also included. Our work extends the results of Kamran [64], Pant [90], Shastry and Murthy [101] and Singh and Hashim [111].

In Chapter 5, rand om versions of some results obtained in chapters 2-4 are established; in particular, we give random analogues of Theorems 2.2.1, 2.2.4, 3.2.2, 3.3.2, 3.3.3 and 4.2.1. We also study random fixed points of inward random multivalued operators on a separable Banach space with characteristic of noncompact convexity less than 1.

CHAPTER 1

PRELIMINARIES

1.1 Introduction

The purpose of this chapter is to recall some relevant basic definitions and some known fundamental results from the existing literature for the convenience of later references. In addition, necessary notations and terminology used in the sequal are also fixed (for more details, see [5, 6, 29, 43, 45, 47, 54, 122, 129, 130, 140]).

1.2 Single-Valued Mappings

Let C be a nonempty subset of a metric space (X, d) and $f, g : C \to X$. A point $x \in C$ is called a fixed point of f if $fx = x$. A coincidence (respectively, common fixed) point of f and g is an $x \in C$ such that $fx = gx$ (respectively, $x = fx = gx$).

Definition 1.2.1 Let f and g be selfmappings of a metric space X. The mappings f and g are

(1) commuting if

$$fgx = gfx, \quad \text{for all} \quad x \in X,$$

(2) weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx), \quad \text{for all} \quad x \in X,$$

(3) compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some} \quad t \in X,$$

(4) weakly compatible if they commute at their coincidence points; i.e., if $fu = gu$ for some u in X, then $fgu = gfu$,

(5) satisfying the property $(E \cdot A)$ if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some} \quad t \in X.$$

Note that weakly commuting mappings are compatible and compatible mappings are weakly compatible but the converse in each case does not hold (for examples and counter-examples, see [1], [59] and [61]). It is easy to see that two noncompatible mappings satisfy the property $(E \cdot A)$ (see [1], Remark 1). Some fixed point results for noncompatible mappings are obtained in [89].

Definition 1.2.2 Let C be a subset of a metric space X and $f, g : C \to X$. Then f is:

(i) nonexpansive if

$$d(fx, fy) \leq d(x, y), \quad \text{for all} \quad x, y \in X,$$

(ii) g-nonexpansive if

$$d(fx, fy) \leq d(gx, gy), \quad \text{for all} \quad x, y \in X,$$

Definition 1.2.3 Let M be a subset of a metric space X and $u \in X$. We denote by $P_M(u)$, the set of best approximations to u from M; that is,

$$P_M(u) = \{y \in M : d(y, u) = d(u, M)\},$$

where $d(u, M) = \inf\{d(u, m) : m \in M\}$.

For $f : M \to X$, we follow Al-Thagafi [3] to define:

$$C_M^f(u) = \{x \in M : fx \in P_M(u)\}$$

and

$$D_M^f(u) = P_M(u) \bigcap C_M^f(u).$$

Remark 1.2.4 By [3, Proposition 3.1], we have

(i) $D_M^f(u) = P_M(u) = C_M^f(u)$ whenever f is the identity mapping on M;

(ii) if $f(P_M(u)) \subseteq P_M(u)$, then $P_M(u) \subseteq C_M^f(u)$ and hence $D_M^f(u) = P_M(u)$;

(iii) if $f(C_M^f(u)) \subseteq C_M^f(u)$, then $f(D_M^f(u)) \subseteq f(C_M^f(u)) \subseteq D_M^f(u)$.

The existence of common fixed points in $P_M(u)$ and $D_M^f(u)$ has been studied by various authors; see Al-Thagafi [3], Hussain and Khan [51], Jungck and Sessa [63], Kamran [64], O'Regan and Shahzad [87], Sahab et al. [100] and Shahzad [106].

For solutions of eigenvalue problems of nonself mappings on closed balls, we need the following result.

Theorem 1.2.5 ([30], p. 92). Let X be a reflexive Banach space, $B = \{x \in X : \|x\| \leq r\}$ and $f : B \to X$ be a weakly continuous mapping. Suppose that for each $x \in \partial B$ (boundary of B), one of the following conditions holds:

(i) $\|fx\| \leq \max\{\|fx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|fx - x\|^p + \|x\|^p$.

Then f has a fixed point in B.

Now, we recall definitions of some important classes of Banach spaces.

Definition 1.2.6 A Banach space X (or the norm $\|\cdot\|$) is said to be strictly convex if the following implication holds for all $x, y \in X$:

$$\|x\| \leq 1, \|y\| \leq 1 \text{ and } \|x-y\| > 0 \text{ imply that } \left\|\frac{x+y}{2}\right\| < 1.$$

This is equivalent to the condition that the unit sphere (or any sphere) contains no line segments. In such a space, any three points x, y, z satisfying $\|x-z\| + \|z-y\| = \|x-y\|$ must lie on a line.

A strong form of the above definition was introduced by Clarkson, in 1936, which turned out to be very useful in Banach space and operator theory.

Definition 1.2.7 A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for all $x, y \in X$, the following holds:

$$\|x\| \leq 1, \ \|y\| \leq 1 \ \text{ and } \ \|x-y\| > \epsilon \ \text{ imply that } \left\|\frac{x+y}{2}\right\| \leq \delta.$$

Obviously, uniformly convex spaces are strictly convex. Moreover, the two concepts are equivalent in finite dimensional spaces (since balls in such spaces are compact).

The simplest example of uniformly convex space is a Hilbert space. It is well-known that every uniformly convex Banach space is reflexive (see [43, 45].

Lemma 1.2.8 [23, Theorem 8.4]. Let C be a bounded closed convex subset of a uniformly convex Banach space X, and $T : C \to X$ a nonexpansive mapping. Then

(i) If $\{x_n\}$ is a weakly convergent sequence in C with weak limit x_0 and if $(I - T)x_n$ converges strongly to ω in X (here I is the identity mapping), then $(I - T)x_0 = \omega$.

(ii) $(I - T)(C)$ is a closed subset of X.

1.3 Iterative Algorithms

Let C be a nonempty subset of a real Banach space X and T a selfmapping of C. Denote by $F(T)$, the set of fixed points of T. Throughout this section, we assume that $F(T) \neq \phi$.

Definition 1.3.1 The mapping T is said to be

(i) quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|, \quad \text{for all} \quad x \in C \quad \text{and} \quad p \in F(T);$$

(ii) asymptotically nonexpansvie if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim_{n\to\infty} u_n = 0$ and

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|,$$

for all $x, y \in C$ and $n = 1, 2, \ldots;$

(iii) asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\}$ in $[0, +\infty)$ with $\lim\limits_{n\to\infty} u_n = 0$ and

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\|,$$

for all $x \in C,\ p \in F(T)$ and $n = 1, 2, \ldots$;

(iv) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(v) $(L-\gamma)$ uniform Lipschitz if there exist constants $L > 0$ and $\gamma > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|^{\gamma},$$

for all $x, y \in C$ and $n = 1, 2, 3, \ldots$ (cf. Qihou [97], p. 468);

(vi) semi-compact if for a sequence $\{x_n\}$ in C with $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

Remark 1.3.2 From the above definitions, it follows that:

(i) a nonexpansive mapping must be quasi-nonexpansive and asymptotically nonexpansive;

(ii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive;

(iii) a uniformly L-Lipschitzian mapping is $(L-1)$ uniform Lipschitz.

However, the converse of these statements are not true, in general (see [96-97]).

Definition 1.3.3 The mapping $T : C \to X$ is said to be demiclosed at 0 if for each sequence $\{x_n\}$ in C converging weakly to x and $\{Tx_n\}$ converging strongly to 0, we have $Tx = 0$.

Definition 1.3.4 A Banach space X is said to satisfy Opial's property if for each $x \in X$ and each sequence $\{x_n\}$ weakly convergent to x, the following condition holds for all $x \neq y$:

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

If $\lim\limits_{n\to\infty} \inf \|x_n - x\| \leq \lim\limits_{n\to\infty} \inf \|x_n - y\|$ holds, then X satisfies the nonstrict Opial's property

It is well known that all Hilbert spaces and $\ell_p (1 < p < \infty)$ spaces are Opial spaces while L_p spaces $(p \neq 2)$ are not (cf. [119], [126]).

Let C be a convex set and $x_1 \in C$. Mann [83], in 1953, defined an iterative procedure as:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \tag{1.3.1}$$

where $\alpha_n \in [0, 1]$, $\quad n = 1, 2, 3, \ldots$.

In 1973, Petryshyn and Williamson [94] proved a necessary and sufficient condition for the Mann iteration to converge to a fixed point of a quasi-nonexpansive mapping.

Ishikawa [52], in 1974, devised an iteration scheme:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.3.2}$$

where $\alpha_n, \beta_n \in [0,1], \quad n = 1,2,3,\ldots$; to establish convergence of a Lipschitzian pseudocontractive mapping in the context of a Hilbert space where the Mann iteration process failed to converge.

If $\beta_n = 0$ for all n, then (1.3.2) becomes (1.3.1).

In 1997, Ghosh and Debnath [40] extended the results of Petryshyn and Williamson [94] to the Ishikawa iteration.

Let $\{T_i : i = 1,2,\ldots,k\}$ be a family of selfmappings of C. Kuhfittig [77], in 1981, defined the following iteration: Let $x_1 \in C,\ U_0 = I$ (identity mapping on C), $\quad \alpha \in (0,1)$,

$$\begin{aligned} U_1 &= (1-\alpha)I + \alpha T_1 U_0, \\ U_2 &= (1-\alpha)I + \alpha T_2 U_1, \\ \ldots & \cdots \ldots\ldots\ldots\ldots\ldots \\ U_k &= (1-\alpha)I + \alpha T_k U_{k-1}, \\ x_{n+1} &= (1-\alpha)x_n + \alpha T_k U_{k-1}\, x_n, \quad n = 1,2,3,\ldots \end{aligned} \tag{1.3.3}$$

Let S and T be two selfmappings of a convex set C. In 1998, Atsushiba and Takahashi [4] introduced the following Mann's type iteration:

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)\frac{1}{n^2}\sum_{i,j=0}^{n} S^i T^j x_n, \tag{1.3.4}$$

where $x_1 \in C$ and $0 \leq \alpha_n \leq a < 1,\ n = 1, 2, \ldots$.

The iteration scheme (1.3.4) has been studied by many authors to approximate common fixed points of S and T (see, for example, Atsushiba and Takahashi [4], Suzuki [118-119] and Suzuki and Takahashi [120]).

Rhoades [99] noted that the iteration scheme (1.3.4) is much more complicated than (1.3.3), and the results obtained on the basis of it generally need commutativity of S and T and certain conditions on their domain such as the Opial condition (see, for example, Theorem 1 in [4], and Theorem 4 of Suzuki in [118] and [119]).

Schu [103], in 1991, considered the following modified Mann iteration process:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n = 1, 2, 3, \ldots \tag{1.3.5}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1, i.e., $a \leq \alpha_n \leq b$ for all n and some $0 < a \leq b < 1$. In 1994, Tan and Xu [126] studied the modified Ishikawa iteration process:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n((1 - \beta_n)x_n + \beta_n T^n x_n), \quad n = 1, 2, 3, \ldots \tag{1.3.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ such that $\{\alpha_n\}$ is bounded away from 0 and 1 and $\{\beta_n\}$ is bounded away from 1.

Xu and Noor [133], in 2002, introduced a three-step iteration as follows:

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n\, y_n, \quad n = 1,2,3,\ldots, \end{aligned} \tag{1.3.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real numbers in $[0,1]$.

We need the following useful known lemmas for the development of our convergence results.

Lemma 1.3.5 (cf. [117, Lemma 2.2]). Let the sequences $\{a_n\}$ and $\{u_n\}$ of real numbers satisfy:

$$a_{n+1} \le (1+u_n)a_n, \text{ where } a_n \ge 0, u_n \ge 0, \text{ for all } n = 1,2,3,\ldots$$

and $\sum_{n=1}^{\infty} u_n < +\infty$. Then

(i) $\lim_{n\to\infty} a_n$ exists;

(ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3.6 [103, Lemma 1.3]. Let X be a uniformly convex Banach space. Assume that $0 < b \le t_n \le c < 1$, $n = 1,2,3,\ldots$ Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that

(i) $\limsup_{n\to\infty} \|x_n\| \le a$,

(ii) $\limsup_{n\to\infty} \|y_n\| \le a$,

(iii) $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$,

where $a \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

We now state two useful conditions:

Condition A ([53]). A real sequence $\{\alpha_n\}$ is said to satisfy Condition A if $0 \le \alpha_n \le b < 1$, $n = 0, 1, 2, \ldots$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Condition B ([82], Condition A]). The mapping $T : C \to C$ with $F(T) \ne \phi$ is said to satisfy Condition B if there exists a nondecreasing function f : $[0, \infty) \to [0,\ \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - Ty\| \ge f(d(x, F(T)))$ for $x \in C$ and all corresponding $y = (1-t)x + tTx$, where $0 \le t \le \beta < 1$.

Note that if $t = 0$, the Condition B reduces to the Condition I of Senter and Dotson, Jr. [104] and the Condition A of Tan and Xu [125].

We need the following known results:

Theorem 1.3.7 [53, Theorem 1]. Let C be a closed subset of a Banach space X, and T a nonexpansive mapping from C into a compact subset of X. Suppose that there exists $\{\alpha_n\}$ satisfying the Condition A. If $\{x_n\}$ is defined by (1.3.1) with $x_n \in C$ for all n, then T has a fixed point in C and $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.3.8 [96, Corollary 1]. Let C be a nonempty closed convex subset of a Banach space X, and T a quasi-nonexpansive selfmapping of C. Assume that $F(T) \neq \phi$. Then the Ishikawa iteration, defined by (1.3.2), converges strongly to $y \in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Theorem 1.3.9 [82, Theorem 1]. Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and T a quasi-nonexpansive selfmapping of C satisfying the Condition B. Then, the Ishikawa iteration scheme (1.3.2), with $0 < a \leq \alpha_n \leq b < 1$ and $0 \leq \beta_n \leq \beta < 1$, converges strongly to a fixed point of T.

Lemma 1.3.10 (cf.[103, Lemma 1.6]). Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying Opial's property, and $T : C \to C$ be asymptotically nonexpansive. Then $I - T$ is demiclosed at 0.

Theorem 1.3.11 [34, Theorem 4]. If C is a weakly compact convex subset of a strictly convex normed linear space, and $\{T_\alpha\}$ is a commutative family of quasi-nonexpansive selfmappings of C, then $\bigcap_\alpha F(T_\alpha) \neq \phi$.

Theorem 1.3.12 [44, Theorem 3.1] Let C be a nonempty compact convex subset of a Banach space X, and τ a commutative semigroup of asymptotically nonexpansive selfmappings of C. Then there exists a point $x \in C$ such that $Tx = x$ for each $T \in \tau$.

1.4 Multivalued Mappings

Throughout this section, (X, d) denotes a metric space. Suppose that $x \in X$ and $A \subseteq X$. We denote by 2^X(respectively, $C(X), CB(X), K(X), KC(X)$) the class of all nonempty subsets (respectively, nonempty closed, nonempty closed bound, nonempty compact, nonempty compact convex subsets) of X. Let H be the Hausdorff metric with respect to d; that is,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \text{ for every } A, B \in C(X).$$

Definition 1.4.1 Let $f : X \to X$ and $S : X \to C(X)$. A point $u \in X$ is a coincidence (respectively, common fixed) point of f and S if $fu \in Su$ (respectively, $u = fu \in Su$). The mappings f and S are:

(1) weakly commuting if $fSx \in C(X)$ for all $x \in X$, and

$$H(Sfx, fSx) \leq d(fx, Sx);$$

(2) compatible if

$$\lim_{n\to\infty} H(fSx_n, Sfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = A \in C(X) \text{ and } \lim_{n\to\infty} fx_n = t \in A;$$

(3) weakly compatible if they commute at their coincidence points; i.e., if $fu \in Su$ for some u in X, then $fSu = Sfu$;

(4) R-weakly commuting if $fSx \in C(X)$ for all $x \in X$, and there exists a real number $R > 0$ such that

$$H(Sfx, fSx) \leq Rd(fx, Sx);$$

(5) pointwise R-weakly commuting if $fSx \in C(X)$ for all $x \in X$, and for a given x in X, there exists a real number $R > 0$ such that

$$H(Sfx, fSx) \leq Rd(fx, Sx);$$

(6) (IT)-commuting at $x \in X$ if $fSx \subset Sfx$ (cf. [111]);

(7) f is S-weakly commuting at $x \in X$ if $ffx \in Sfx$ (see [64]);

(8) satisfying the property (E.A) (called tangential mappings by Sastry and Murthy [101])if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = A \in C(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n = t \in A.$$

Remark 1.4.2 Note that

(i) weakly commuting mappings are R-weakly commuting and compatible;

(ii) compatible mappings are weakly compatible;

(iii) weakly compatible mappings are R-weakly commuting;

(iv) R-weakly commuting mappings are pointwise R-weakly commuting;

(v) weakly compatible mappings are (IT)-commuting at their coincidence points;

(vi) f and S are (IT)-commuting at the coincidence points implies that f is S-weakly commuting;

but the converse in each case does not hold (for examples and counter examples, see [59-61], [64], [91], [111] and [112]). We remark that commutativity, compatibility, R-weak commutativity and weak compatibility of f and S are equivalent at their coincidence points (cf. [111]).

Definition 1.4.3 Let E be a normed space. A real number λ is said to be an eigenvalue of a mapping $S : E \to 2^E$ (respectively, $S : E \to E$) if there exists a point $x \neq 0$ in E such that $\lambda x \in Sx$ (respectively, $\lambda x = Sx$).

Solutions of nonlinear eigenvalue problems for single-valued mappings on a Banach space have been obtained by many authors (see Baskaran and Subrahmanyam [7] and Kim [74]).

Definition 1.4.4 Let C be a nonempty closed subset of a Banach space X. A multivalued mapping $T : C \to 2^X$ is said to be :

(i) upper semicontinuous if $\{x \in C : Tx \subset V\}$ is open whenever $V \subset X$ is open;

(ii) lower semicontinuous if $T^{-1}(V) = \{x \in C : Tx \bigcap V \neq \phi\}$ is open whenever $V \subset X$ is open;

(iii) continuous if it is both upper and lower semicontinuous (see [16, 29]);

(iv) Lipschitz if there exists a constant $k \geq 0$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \text{ for all } x, y \in C;$$

If $0 \leq k < 1$ (respectively, $k = 1$), then T is called contractive (respectively, nonexpansive);

(v) demiclosed at 0 if the following implication holds:
$\{x_n\}$ in $C, \{x_n\}$ converges weakly to x, $y_n \in Tx_n$ and $y_n \to 0$ imply that $0 \in Tx$.

There is another different kind of continuity for multivalued operators:

$T : C \to CB(X)$ is said to be continuous (with respect to the Hausdorff metric H) if

$$H(Tx_n, Tx) \to 0 \text{ whenever } x_n \to x.$$

It is not hard to see that both definitions of continuity are equivalent if Tx is compact for every $x \in C$ (see[6, 16]).

Let C be a convex subset of a linear space E. Then $f : C \to \mathbb{R}$ is convex if for any $x, y \in C$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Lemma 1.4.5 [122, Lemma 1.3.9] Let C be a closed convex subset of a Banach space X and T be a convex function of C into $(-\infty, +\infty)$. Then, T is lower semicontinuous if and only if T is lower semicontinuous in the weak topology.

Let X be a metric space. The diameter of a subset B of X is

$$\operatorname{diam} B = \sup\{d(x, y) : x, y \in B\},$$

with $\operatorname{diam} \phi = 0$. It is known that $\operatorname{diam} B = 0$ if and only if B is an empty set or consists of exactly one point. Some other important properties of the diameter are the following:

(i) If $B_1 \subset B_2$ then $\operatorname{diam} B_1 \leq \operatorname{diam} B_2$;

(ii) $\operatorname{diam}\overline{B} = \operatorname{diam} B$; here – stands for the closure;

(iii) Contor's intersection theorem: If $\{B_n\}$ is a decreasing sequence of nonempty closed bounded subsets of X and $\lim\limits_{n\to\infty} \operatorname{diam} B_n = 0$, then $\bigcap\limits_{n=1}^{\infty} B_n$ is nonempty and consists of exactly one point (cf. [130]).

The set $B \subset X$ is said to be r-separated (or,r-separation of X) if

$$d(x,y) \geq r, \text{ for all } x \neq y \text{ in } B \text{ and some } r > 0.$$

In recent years measures of noncompactness have been utilized to define new geometrical properties of Banach spaces which are interesting for fixed point theory. The notion of a measure of noncompactness was originally introduced in metric spaces as in the following:

Definition 1.4.6 The Kuratowski, Hausdorff and separation measures of noncompactness of a nonempty bounded subset B of X are, respectively, defined as the numbers

$$\begin{aligned}
\alpha(B) &= \inf\{r > 0 : B \text{ can be covered by finitely many sets of diameter } \leq r\}, \\
\chi(B) &= \inf\{r > 0 : B \text{ can be covered by finitely many balls of radius} \leq r\}, \\
\beta(B) &= \sup\{r > 0 : B \text{ has an infinite } r - \text{separation}\}.
\end{aligned}$$

Definition 1.4.7 The mapping $T : C \to 2^X$ is called γ-condensing (respectively, 1- γ-contractive) where $\gamma = \alpha(.)$ or $\chi(.)$ if, for each bounded subset B of C with $\gamma(B) > 0$, the following holds:

$$\gamma(T(B)) < \gamma(B) \text{ (respectively, } \gamma(T(B)) \leq \gamma(B)),$$

where $T(B) = \bigcup_{x \in B} Tx$.

Definition 1.4.8 Let X be a Banach space and $\phi = \alpha, \beta \,\text{or}\, \chi$. Then:

(i) the modulus of noncompact convexity associated to ϕ is defined by

$$\Delta_{X,\phi}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex}, \phi(A) \geq \epsilon\},$$

where B_X is the unit ball of X,

(ii) the characteristic of noncompact convexity of X associated to the measure of noncompactness ϕ is defined by

$$\epsilon_\phi(X) = \sup\{\epsilon \geq 0 : \Delta_{X,\phi}(\epsilon) = 0\},$$

(iii) if $\epsilon_\phi(X) = 0$, then the space X is called nearly uniformly convex.

The following relationships among the different moduli are easy to obtain (see[130])

$$\Delta_{X,\alpha}(\epsilon) \leq \Delta_{X,\beta}(\epsilon) \leq \Delta_{X,\chi}(\epsilon),$$

and consequently

$$\epsilon_\alpha(X) \geq \epsilon_\beta(X) \geq \epsilon_\chi(X).$$

When X is a reflexive Banach space we have some alternative expressions for the moduli of noncompact convexity associated to β and χ :

$$\begin{aligned}
\Delta_{X,\beta}(\epsilon) &= \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \operatorname{sep}(\{x_n\}) \geq \epsilon\}, \\
\Delta_{X,\chi}(\epsilon) &= \inf\{1 - \|x\| : \{x_n\} \subset B_X, x = w - \lim x_n, \chi(\{x_n\}) \geq \epsilon\}.
\end{aligned}$$

Note that if $\epsilon_\alpha(X) < 1$, then X is reflexive (this is true also if $\epsilon_\phi(X) < 1$ where $\phi = \chi$ or β) (see Theorem 5.1.7 and Remark 5.1.7 in [130]).

In order to study the fixed point theory for nonself mappings, we must introduce some terminology.

Definition 1.4.9 The inward set of C at $x \in C$ is defined by

$$I_C(x) = \{x + \lambda(y - x) : \lambda \geq 0, y \in C\}.$$

Clearly, $C \subset I_C(x)$ and it is not hard to show that $I_C(x)$ is a convex set as C is so. A multivalued mapping $T : C \to 2^X$ is said to be inward if

$$Tx \subset I_C(x), \forall x \in C.$$

Let $\bar{I}_C(x) = x + \{\lambda(z - x) : z \in C, \lambda \geq 1\}$. Note that for a convex set C, we have $\bar{I}_C(x) = \overline{I_C(x)}$, and T is said to be weakly inward on C if

$$Tx \subset \bar{I}_C(x), \forall x \in C.$$

Lemma 1.4.10 (cf.[16]) Let X be a Banach space and $\phi \neq C \subset X$ be closed bounded convex. Let $T : C \to 2^X$ be upper semicontinuous γ-condensing with closed convex values, where $\gamma(.) = \alpha(.)$ or $\chi(.)$. If $Tx \bigcap \overline{I_C(x)} \neq \phi$ for all $x \in C$, then T has a fixed point.

Lemma 1.4.11 [79] Let C be a nonempty closed convex subset of a Banach space X and $T : C \to K(X)$ a contraction. If T satisfies

$$Tx \subset \overline{I_C(x)}, \text{ for all } x \in C,$$

then T has a fixed point.

Definition 1.4.12 Let C be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X, we use $r(C, \{x_n\})$ and $A(C, \{x_n\})$ to denote the asymptotic radius and the asymptotic center of $\{x_n\}$ in C, respectively, i.e.

$$\begin{aligned} r(C, \{x_n\}) &= \inf\left\{\limsup_{n\to\infty} \|x_n - x\| : x \in C\right\}, \\ A(C, \{x_n\}) &= \left\{x \in C : \limsup_{n\to\infty} \|x_n - x\| = r(C, \{x_n\})\right\}. \end{aligned}$$

If D is a bounded subset of X, the Chebyshev radius of D relative to C is defined by

$$r_C(D) = \inf\{\sup\{\|x - y\| : y \in D\} : x \in C\}.$$

Remark 1.4.13 The convexity of C implies that $A(C, \{x_n\})$ is convex. The set $A(C, \{x_n\})$ is nonempty weakly compact if C is weakly compact, or C is a closed convex subset of a reflexive Banach space X (see [17]).

Definition 1.4.14 Let C be a nonempty bounded closed subset of a Banach space X. Then a sequence $\{x_n\}$ in X is called regular with respect to C if $r(C, \{x_n\}) = r(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$; while $\{x_n\}$ is called asymptotically uniform with respect to C if $A(C, \{x_n\}) = A(C, \{x_{n_i}\})$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$.

Lemma 1.4.15 [41, 78] Let $\{x_n\}$ and C be as in Definition 1.4.14. Then:

(i) there always exists a subsequence of $\{x_n\}$ which is regular with respect to C;

(ii) if C is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to C.

Theorem 1.4.16 [16, Theorem 3.4] Let C be a closed convex subset of a reflexive Banach space X, and let $\{x_n\}$ be a bounded sequence in C which is regular with respect to C. Then

$$r_C(A(C, x_n)) \leq (1 - \Delta_{X,\beta}(1^-))r(C, \{x_n\}).$$

Moreover, if X satisfies the nonstrict Opial's property then

$$r_C(A(C, x_n)) \leq (1 - \Delta_{X,\chi}(1^-))r(C, \{x_n\}).$$

Let A be a set and $B \subset A$. A net $\{x_\alpha : \alpha \in D(\text{directed set})\}$ in A is eventually in B if there exists $\alpha_0 \in D$ such that $x_\alpha \in B$ for all $\alpha \geq \alpha_0$. A net $\{x_\alpha : \alpha \in D\}$ in a set A is called an ultranet if either $\{x_\alpha : \alpha \in D\}$ is eventually in B or $\{x_\alpha : \alpha \in D\}$ is eventually in $A \backslash B$, for each subset B of A. It is well-known that every net in a set has a subnet which is an ultranet (cf. [17]).

Theorem 1.4.17 [17, Theorem 3.2] Let C be a closed convex subset of a reflexive Banach space X, and let $\{x_\beta : \beta \in D\}$ be a bounded ultranet. Then

$$r_C(A(C, x_\beta)) \leq (1 - \Delta_{X,\alpha}(1^-))r(C, \{x_\beta\}).$$

A sequence $\{x_n\}$ is called asymptotically T-regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 1.4.18 Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \to KC(X)$. The mapping T is called subsequentially limit-contractive (SL) if for every asymptotically T-regular sequence $\{x_n\}$ in C, we have

$$\limsup_{n\to\infty} H(Tx_n, Tx) \leq \limsup_{n\to\infty} \|x_n - x\|,$$

for all $x \in A(C, \{x_n\})$.

It is clear that every nonexpansive mapping is an SL mapping. The converse does not hold (for example and counter example see Shahzad and Lone [109]).

1.5 Random Operators

Let $(\Omega, \sum)$ be a measurable space ($\sum$ denotes σ-algebra of subsets of Ω) and C be a nonempty subset of a Banach space X.

Definition 1.5.1 Let $\xi : \Omega \to C$ and $S, T : \Omega \times C \to X$. Then

(i) ξ is measurable if $\xi^{-1}(U) \in \sum$, for each open subset U of X,

(ii) T is a random operator if for each fixed $x \in C$, the mapping $T(., x) : \Omega \to X$ is measurable,

(iii) ξ is a deterministic fixed point of the random operator T if

$$T(\omega, \xi(\omega)) = \xi(\omega), \text{ for each } \omega \in \Omega,$$

(iv) ξ is a random fixed point of the random operator T if ξ is measurable and

$$T(\omega, \xi(\omega)) = \xi(\omega), \text{ for each } \omega \in \Omega,$$

(v) ξ is a random coincidence (respectively, common fixed) point of S and T if ξ is measurable and for each $\omega \in \Omega$,

$$S(\omega, \xi(\omega)) = T(\omega, \xi(\omega))$$

$$(\text{respectively, } \xi(\omega) = S(\omega, \xi(\omega)) = T(\omega, \xi(\omega))).$$

Definition 1.5.2 Let C be a subset of a separable metric space X and $S, T : \Omega \times C \to X$. Then S and T are:

(i) compatible if $S(\omega, .)$ and $T(\omega, .)$ are compatible for each $\omega \in \Omega$ [14];

(ii) weakly compatible if

$$T(\omega, S(\omega, \xi(\omega))) = S(\omega, T(\omega, \xi(\omega))),$$

for every $\omega \in \Omega$ whenever $T(\omega, \xi(\omega)) = S(\omega, \xi(\omega))$ where $\xi : \Omega \to C$ is a measurable mapping;

(iii) said to satisfy the random property (E.A) if there exists a sequence $\{\xi_n\}$ of measurable mappings from Ω to C such that for every $\omega \in \Omega$,

$$\lim_{n\to\infty} S(\omega, \xi_n(\omega)) = \lim_{n\to\infty} T(\omega, \xi_n(\omega)) = \xi(\omega),$$

where $\xi : \Omega \to X$ is a measurable mapping.

The set of random fixed points of T will be denoted by $RF(T)$. For a mapping T, the nth iterate $T(\omega, T(\omega, T(\omega, ..., T(\omega, x))))$ will be written as $T^n(\omega, x)$, and T° stands for the random operator $I : \Omega \times C \to C$ defined by $I(\omega, x) = x$.

Definition 1.5.3 A random operator $T : \Omega \times C \to X$ is called:

(i) continuous (respectively, demiclosed, nonexpansive, contractive, SL, uniformly L-Lipschitzian, $(L - \gamma)$ uniform Lipschitz) if the mapping $T(\omega, .)$ is continous (respectively, demiclosed, nonexpansive, contractive, SL, uniformly L-Lipschitzian, $(L - \gamma)$ uniform Lipschitz);

(ii) quasi-nonexpansive random operator if

$$\|T(\omega, \eta(\omega)) - \xi(\omega)\| \leq \|\eta(\omega) - \xi(\omega)\|,$$

for each $\omega \in \Omega$ where $\xi : \Omega \to C$ is a random fixed point of T and $\eta : \Omega \to C$ is any measurable mapping;

(iii) asymptotically nonexpansive random operator if there exists a sequence of measurable mappings $u_n : \Omega \to [0, \infty)$ with $\lim\limits_{n\to\infty} u_n(\omega) = 0$, for each $\omega \in \Omega$, such that for arbitrary $x, y \in C$, we have

$$\|T^n(\omega, x) - T^n(\omega, y)\| \leq (1 + u_n(\omega))\|x - y\|, \text{ for each}\, \omega \in \Omega;$$

(iv) asymptotically quasi-nonexpansive random operator if there exists a sequence of measurable mappings $u_n : \Omega \to [0, \infty)$ with $\lim\limits_{n\to\infty} u_n(\omega) = 0$, for each $\omega \in \Omega$, such that

$$\|T^n(\omega, \eta(\omega)) - \xi(\omega)\| \leq (1 + u_n(\omega))\|\eta(\omega) - \xi(\omega)\|,$$

for each $\omega \in \Omega$, where $\xi : \Omega \to C$ is a random fixed point of T and $\eta : \Omega \to C$ is any measurable mapping;

(v) semi-compact random operator if for a sequence of measurable mappings $\{\xi_n\}$ from Ω to C with $\lim\limits_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0$, for every $\omega \in \Omega$, there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ such that

$$\xi_{n_i}(\omega) \to \xi(\omega), \text{ for each}\, \omega \in \Omega,$$

where ξ is a measurable mapping from Ω to C.

Definition 1.5.4 (Random Mann iteration). Let $T : \Omega \times C \to C$ be a random operator, where C is a nonempty convex subset of a separable Banach space X. The random Mann iteration is a sequence of mappings $\{\xi_n\}$ defined by

$$\xi_{n+1}(\omega) = (1 - \alpha_n)\xi_n(\omega) + \alpha_n T(\omega, \xi_n(\omega)), \tag{1.5.1}$$

for each $\omega \in \Omega, n = 1, 2, 3, ...,$ where $0 \leq \alpha_n \leq 1$ and $\xi_1 : \Omega \to C$ is an arbitrary measurable mapping.

Note that the convexity of C implies that ξ_n is a mapping from Ω to C for each n.

Remark 1.5.5 Let C be a closed convex subset of a separable Banach space X, and the sequence of mappings $\{\xi_n\}$, defined as in the above definition, is pointwise convergent; that is, $\xi_n(\omega) \to q = \xi(\omega)$, for each $\omega \in \Omega$. Then closedness of C implies that ξ is a mapping from Ω to C. Since C is a subset of a separable Banach space X, so, if T is a continuous random operator then by [6, Lemma 8.2.3], the mapping $\omega \to T(\omega, f(\omega))$ is a measurable mapping for any measurable mapping f from Ω to C. Thus $\{\xi_n\}$ is a sequence of measurable mappings. Hence $\xi : \Omega \to C$, being the limit of the sequence of measurable mappings, is also measurable (see [11]).

Definition 1.5.6 (Random Ishikawa iteration). Let $T : \Omega \times C \to C$ be a random operator, where C is a nonempty closed convex subset of a separable Banach space X. The random Ishikawa iteration scheme is defined by

$$\begin{aligned} \xi_{n+1}(\omega) &= (1-\alpha_n)\xi_n(\omega) + \alpha_n T(\omega, \eta_n(\omega)), \\ \eta_n(\omega) &= (1-\beta_n)\xi_n(\omega) + \beta_n T(\omega, \xi_n(\omega)), \end{aligned} \tag{1.5.2}$$

for each $\omega \in \Omega, n = 1, 2, 3, \ldots,$ where $0 \leq \alpha_n, \beta_n \leq 1,$ $\xi_1 : \Omega \to C$ is an arbitrary measurable mapping and $\{\xi_n\}$ and $\{\eta_n\}$ are sequences of mappings from $\Omega \to C$.

It is remarked that if $\beta_n = 0$, for all n in (1.5.2), then the random Ishikawa iteration reduces to the random Mann iteration (1.5.1).

We shall need the following result:

Proposition 1.5.7 [9, Proposition 3.4]. Let C be a nonempty bounded closed convex subset of a separable Banach space X, and $T : \Omega \times C \to C$ a nonexpansive random operator. Suppose that $\{\xi_n\}$ is a sequence of mappings from Ω to C defined by

$$\xi_{n+1}(\omega) = (1-\alpha)\xi_n(\omega) + \alpha T(\omega, \xi_n(\omega)), \text{ for each}\, \omega \in \Omega, \tag{1.5.3}$$

where $0 < \alpha < 1, n = 1, 2, 3, \ldots,$ and $\xi_1 : \Omega \to C$ is an arbitrary measurable mapping. Then for each $\omega \in \Omega$,

$$\lim_{n\to\infty} \|\xi_n(\omega) - T(\omega, \xi_n(\omega))\| = 0.$$

Now we present some definitions and results for multivalued mappings.

Definition 1.5.8 Let C be a subset of a metric space X :

(i) a multivalued operator $T : \Omega \to 2^X$ is measurable if, for any open subset B of X, then $T^{-1}(B) \in \sum$ where

$$T^{-1}(B) = (\omega \in \Omega : T(\omega) \bigcap B \neq \phi\};$$

(ii) a mapping $x : \Omega \to X$ is said to be a measurable selector of a measurable multivalued operator $T : \Omega \to 2^X$ if $x(.)$ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$;

(iii) an operator $T : \Omega \times C \to 2^X$ is called a random operator if, for each fixed $x \in C$, the operator $T(., x) : \Omega \to 2^X$ is measurable;

(iv) a mapping $x : \Omega \to C$ is said to be a random fixed point of $T : \Omega \times C \to 2^X$ if x is measurable and

$$x(\omega) \in T(\omega, x(\omega)), \quad \text{for each} \quad \omega \in \Omega;$$

(v) a mapping $x : \Omega \to C$ is a random coincidence (respectively, random common fixed) point of random operators $T : \Omega \times C \to 2^X$ and $f : \Omega \times C \to X$ if x is measurable and

$$f(\omega, x(\omega)) \in T(\omega, x(\omega))$$

$$(\text{respectively}, x(\omega) = f(\omega, x(\omega)) \in T(\omega, x(\omega)));$$

(vi) a random operator $T : \Omega \times C \to 2^X$ is continuous (respectively, contractive, nonexpansive etc.) if for each $\omega \in \Omega, T(\omega, .)$ is continuous (respectively, contractive, nonexpansive etc.).

We will denote by $F(\omega)$, the fixed point set of $T(\omega, .)$, i.e.,

$$F(\omega) = \{x \in C : x \in T(\omega, x)\}.$$

Note that if we do not assume the existence of fixed point for the deterministic mapping $T(\omega, .) : C \to 2^X, F(\omega)$ may be empty.

For later convenience, we list the following results.

Lemma 1.5.9 [131] Let (X, d) be a complete separable metric space and $T : \Omega \to C(X)$ a measurable operator. Then T has a measurable selector.

Lemma 1.5.10 [55] Suppose $\{T_n\}$ is a sequence of measurable multivalued operators from Ω to $CB(X)$ and $T : \Omega \to CB(X)$ is an operator. If, for each $\omega \in \Omega$,

$$H(T_n(\omega), T(\omega)) \to 0,$$

then T is measurable.

Lemma 1.5.11 [127] Let X be a separable metric space and Y a metric space. If $f : \Omega \times X \to Y$ is measurable in $\omega \in \Omega$ and continuous in $x \in X$, and if $x : \Omega \to X$ is measurable, then $f(., x(.)) : \Omega \to Y$ is measurable.

As an application of Proposition 3 of Itoh [55], we have the following result.

Lemma 1.5.12 (cf. [95]) Let C be a closed separable subset of a Banach space $X, T : \Omega \times C \to C$ a random continuous operator and $F : \Omega \to 2^C$ a measurable closed-valued operator. Then for any $s > 0$, the operator $G : \Omega \to 2^C$ given by

$$G(\omega) = \{x \in F(\omega) : \|x - T(\omega, x)\| < s\},\ \omega \in \Omega$$

is measurable and so is the operator $cl\{G(\omega)\}$ (the closure of $G(\omega)$).

Lemma 1.5.13 [18] Suppose that C is a weakly closed nonempty separable subset of a Banach space $X, F : \Omega \to 2^X$ is measurable with weakly compact values, $f : \Omega \times C \to \mathbb{R}$ is a measurable, continuous and weakly lower semicontinuous function. Then the marginal function $r : \Omega \to \mathbb{R}$ defined by

$$r(\omega) = \inf_{x \in F(x)} f(\omega, x)$$

and the marginal mapping $R : \Omega \to X$ defined by

$$R(\omega) = \{x \in F(x) : f(\omega, x) = r(\omega)\}$$

are measurable.

CHAPTER 2

COINCIDENCE AND FIXED POINTS OF NONSELF CONTRACTIVE MAPPINGS

2.1 Introduction

Common fixed point theorems for families of commuting contraction mappings have been a popular area of research (see, e.g. Al-Thagafi [3], Belluce and Kirk [15] and Jungck and Sessa [63]). In 1982, Sessa [105] introduced the concept of weakly commuting mappings to generalize commutativity. Jungck [59], in 1986, generalized weak commutativity to the notion of compatible mappings. In 1996, Jungck [61] further weakened compatibility to the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible mappings under various contractive conditions have been obtained by a number of authors (see, for example, Aamri and El Moutawakil [1], Djoudi and Khemis [32], Jachymski [57], Jungck [58-62], and Pant [89]).

In [58], Jungck generalized the Banach contraction principle to the case of two commuting selfmappings on a metric space. Baskaran and Subrahmanyam [7] noted that the commutativity of the mappings in Jungck's theorem can be replaced by weak commutativity and then they obtained some common fixed point theorems for two mappings on the closed ball of a Banach space. They also provided a solution to a nonlinear eigenvalue problem for operators on the closed ball of a Banach space. The existence of fixed points of mappings defined on closed balls has been studied by several authors; for example, see Delbosco [30] and Liu [81].

Aamri and El Moutawakil [1], in 2002, defined the property (E.A) for selfmappings (need not be continuous) on a metric space, and extended the theorem of Jungck [58] to the case of weakly compatible mappings satisfying the property (E.A) and certain contractive conditions. Very recently, Ćirić [28] has established fixed point theorems for continuous nonself mappings satisfying certain contractive conditions on a nonempty closed subset of a metric space of hyperbolic type (Takahashi [121] uses the term "convex metric space").

In this chapter, we establish new results related to coincidence and common fixed points of weakly compatible nonself mappings satisfying the property $(E \cdot A)$ and strict contractive conditions on an arbitrary nonempty subset of a metric space. Applications of our results to best approximation and eigenvalue problems will also be given.

2.2 Coincidence and Fixed Point Results

Throughout this section, B denotes an arbitrary nonempty subset of a metric space X. We obtain some coincidence and common fixed point theorems for weakly compatible nonself mappings (which need not be continuous) satisfying the property $(E \cdot A)$ and strict contractive conditions. We begin with an extension of Theorem 1 of Aamri and El Moutawakil [1] for nonself mappings on $B \subseteq X$; our result is also an improvement of Theorem 2.2 of Ćirić [28] in the sense that continuity of the mapping and compactness of the domain are removed.

Theorem 2.2.1 Let $f, g : B \to X$ be such that:

(i) f and g satisfy the property $(E \cdot A)$,

(ii) gB is complete or fB is complete with $fB \subseteq gB$,

(iii) for all $x \neq y$ in B, the following contractive condition holds:

$$\begin{aligned} d(fx, fy) &< \max\Big\{d(gx, gy), rd(fx, gx) + \alpha d(fy, gy), \\ &\quad \frac{1}{2}[d(fx, gy) + d(fy, gx)]\Big\} \end{aligned} \tag{2.2.1}$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then f and g have a coincidence point in B. Further, if a is a coincidence point of f and g such that $fa \in B$ and f and g are weakly compatible, then f and g have a unique common fixed point in B.

Proof. By (i), there exists a sequence $\{x_n\}$ in B such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X.$$

If gB is complete, then

$$\lim_{n\to\infty} gx_n = t = ga, \text{ for some } a \in B.$$

Now, we show that $fa = ga$. By (iii), we have

$$\begin{aligned} d(fx_n, fa) &< \max\{d(gx_n, ga), rd(fx_n, gx_n) + \alpha d(fa, ga) , \\ &\quad \frac{1}{2}[d(fx_n, ga) + d(fa, gx_n)]\Big\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} d(ga, fa) &\leq \max\Big\{d(ga, ga), rd(ga, ga) + \alpha d(fa, ga), \\ &\quad \frac{1}{2}[d(ga, ga) + d(fa, ga)]\Big\} \\ &= \max\left\{\alpha d(fa, ga), \frac{1}{2}d(fa, ga)\right\}. \end{aligned}$$

This is possible only if $d(ga, fa) = 0$; that is, $fa = ga$.

Now, if $fa \in B$ and f and g are weakly compatible, then $ffa = fga = gfa = gga$. We prove that fa is a common fixed point. Suppose not; then

$$\begin{aligned} d(fa, ffa) &< \max\left\{d(ga, gfa), rd(fa, ga) + \alpha d(ffa, gfa),\right. \\ &\qquad \left.\frac{1}{2}[d(fa, gfa) + d(ffa, ga)]\right\} \\ &= d(fa, ffa) \end{aligned}$$

a contradiction. Thus

$$ffa = gfa = fa.$$

Similarly, we can prove the case fB is complete and $fB \subseteq gB$. Finally, assume that $a \neq b$ are two common fixed points of f and g. Then by (iii), we get

$$\begin{aligned} d(a, b) = d(fa, fb) &< \max\left\{d(ga, gb), rd(fa, ga) + \alpha d(fb, gb),\right. \\ &\qquad \left.\frac{1}{2}[d(fa, gb) + d(fb, ga)]\right\} \\ &= d(a, b) \end{aligned}$$

a contradiction. Hence $a = b$. ∎

The following example shows that our theorem works where Theorem 1 of Aamri and El Moutawakil [1] is not applicable.

Example 2.2.2 Let X be the usual space of reals. Define

$$f(x) = x^2 \text{ and } g(x) = x^4.$$

It is easy to verify that f and g satisfy the property $(E \cdot A)$ for the sequence $\left\{1 + \frac{1}{n}\right\}$, $n = 1, 2, 3, \ldots$ Note that the contractive condition of Theorem 1 in [1] is not satisfied (take $x = 1$ and $y = 0$). Now, if $f, g : B \to X$ where $B = [1, 2]$, then for all $x \neq y$ in B, (2.2.1) holds because

$$|f(x) - f(y)| = |x^2 - y^2| < |x^2 - y^2|\,|x^2 + y^2| = |x^4 - y^4| = |g(x) - g(y)|.$$

Thus all the conditions of Theorem 2.2.1 are satisfied and 1 is the common fixed point of f and g in $[1, 2]$.

In the following result, we replace the property $(E \cdot A)$ in Theorem 2.2.1 by a mapping ϕ satisfying a contractive condition. The proof is similar to that of Corollary 2 in [1] and hence is omitted.

Corollary 2.2.3 Let $f, g : B \to X$ be such that:

(i) there exists a mapping $\phi : B \to \mathbb{R}^+$ (the set of all nonnegative reals) such that

$$d(fx, gx) < \phi(gx) - \phi(fx), \text{ for all } x \text{ in } B,$$

(ii) gB is complete or fB is complete with $fB \subseteq gB$,

(iii) for all $x \neq y$ in B, (2.2.1) holds.

Then f and g have a coincidence point in B. Further, if a is a coincidence point of f and g such that $fa \in B$ and f and g are weakly compatible, then f and g have a unique common fixed point.

Suppose that $F : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following conditions:

(i) F is nondecreasing,

(ii) $0 < F(t) < t, \quad \text{for all} \quad t \in (0, \infty)$.

The next theorem deals with four nonself mappings under a contractive condition in terms of the function F; this theorem is a considerable improvement of Theorem 2 of Aamri and El Moutawakil [1] for nonself mappings on an arbitrary subset of a metric space (compare our result also with Theorem 2.3 in [28]).

Theorem 2.2.4 Let $f, g, p, q : B \to X$ be such that:

(i) the pair (f, p) or (g, q) satisfies the property $(E \cdot A)$,

(ii) the range of one of the mappings f, g, p or q is complete, $fB \subseteq qB$ and $gB \subseteq pB$,

(iii) for all x, y in B, the following condition holds:

$$d(fx, gy) \leq F(\max\{d(px, qy), d(px, gy), d(qy, gy)\}). \tag{2.2.2}$$

Then:

(a) f and p have a coincidence point, and g and q have a coincidence point,

(b) if a is a coincidence point of f and p such that $fa \in B$ and f and p are weakly compatible, then they have a common fixed point,

(c) if b is a coincidence point of g and q such that $gb \in B$ and g and q are weakly compatible, then they have a common fixed point,

(d) f, g, p and q have a unique common fixed point provided (b) and (c) hold.

Proof. (a) Assume that g and q satisfy the property $(E \cdot A)$; that is, there exists a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} qx_n = t$, for some $t \in X$. Since $gB \subseteq pB$, there exists a sequence $\{y_n\}$ in B with $gx_n = py_n$, for all n. So, $\lim_{n\to\infty} py_n = t$. By (iii), we have

$$\begin{aligned} d(fy_n, gx_n) &\leq F(\max\{d(py_n, qx_n), d(py_n, gx_n), d(qx_n, gx_n)\}) \\ &\leq F(d(gx_n, qx_n)) \\ &< d(gx_n, qx_n). \end{aligned}$$

Thus $\lim_{n\to\infty} d(fy_n, t) = 0$ and so, $\lim_{n\to\infty} fy_n = t$. Let pB be complete. Then $t = pa$, for some $a \in B$. By (iii), we get

$$d(fa, gx_n) \leq F(\max\{d(pa, qx_n), d(pa, gx_n), d(qx_n, gx_n)\}).$$

Taking the limit as $n \to \infty$, it follows that $fa = pa$. Also $fB \subseteq qB$ implies that $fa = qb$, for some $b \in B$. We show that $fa = gb$. Suppose not; then

$$\begin{aligned} d(fa, gb) &\leq F(\max\{d(pa, qb), d(pa, gb), d(qb, gb)\}) \\ &\leq F(d(fa, gb)) \\ &< d(fa, gb) \end{aligned}$$

a contradiction. Thus $fa = pa = gb = qb$.

(b) If $fa \in B$ and f and p are weakly compatible, then

$$ffa = fpa = pfa = ppa.$$

We show that fa is a common fixed point of f and p. If not; then

$$\begin{aligned} d(ffa, fa) &= d(ffa, gb) \\ &\leq F(\max\{d(pfa, qb), d(pfa, gb), d(qb, gb)\}) \\ &\leq F(d(ffa, fa)) \\ &< d(ffa, fa) \end{aligned}$$

sets a contradiction. Thus

$$fa = ffa = pfa.$$

(c) As in (b), we can prove that gb is a common fixed point of g and q.

(d) Since $fa = gb$, therefore fa is a common fixed point of f, g, p and q. The proof is similar if qB, fB or gB is complete. Finally, if $u \neq v$ are two common fixed points of f, g, p and q, then

$$\begin{aligned} d(u,v) &= d(fu,gv) \\ &\leq F(\max\{d(pu,qv), d(pu,gv), d(qv,gv)\}) \\ &\leq F(d(u,v)) \\ &< d(u,v) \end{aligned}$$

gives a contradiction. Thus $u = v$ proves the uniqueness of the common fixed point. ■

2.3 Invariant Approximation

In this section, we obtain common fixed points of best approximation on the basis of results obtained in the previous article. Our work provides analogues of most of the well-known results for the class of weakly compatible mappings on a metric space.

Recently, Hussain and Khan [51] have obtained in Theorem 3.1, a generalization of Theorem 3 by Sahab et al. [100] for a class of noncommuting selfmappings on a Hausdorff locally convex space. An analogue of Theorem 3.1 in [51] is given below in the setup of an arbitrary metric space.

Theorem 2.3.1 Let M be a subset of a metric space X and f and g be selfmappings of X. Assume that u is a common fixed point of f and g, and

$D = P_M(u)$ is nonempty. Suppose that:

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$ on D,

(ii) $gD = D$, $f(\partial M) \subseteq M$ (∂M denotes the boundary of M), and fD or D is complete,

(iii) f is g-nonexpansive on $D \cup \{u\}$,

(iv) for all $x \neq y$ in D, (2.2.1) holds.

Then f and g have a unique common fixed point in $P_M(u)$.

Proof. Let $y \in D$. Then $gy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and since $f(\partial M) \subseteq M$, it follows that $fy \in M$. As f is g-nonexpansive on $D \cup \{u\}$, so

$$d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Now, $fy \in M$ and $gy \in D$ imply that $fy \in D$; consequently, f and g are selfmappings of D. By Theorem 2.2.1, there exists a unique $b \in D$ such that $b = fb = gb$. ■

The following example illustrates our theorem.

Example 2.3.2 Let $X = \mathbb{R}$ and $M = [1, 4]$. Define

$$f(x) = \frac{1}{3}(x + 2) \quad \text{and} \quad g(x) = \frac{1}{2}(x + 1).$$

The mappings f and g being commuting are weakly compatible and satisfy the property $(E \cdot A)$ for the sequence $\left\{1 + \frac{1}{n}\right\}, \quad n = 1, 2, \ldots$. Also,

$$|fx - fy| < |gx - gy|, \quad \text{for all} \quad x \neq y \quad \text{in} \quad M.$$

All the conditions of Theorem 2.3.1 are satisfied. Clearly, $P_M(0) = \{1\}$ and 1 is the common fixed point of f and g.

The existence of a unique common fixed point from the set of best approximations for four weakly compatible mappings is established in the next result. It is remarked that the study of best approximations in the context of four mappings is a new one in the literature.

Theorem 2.3.3 Let f, g, p and q be selfmappings of a metric space X and M be a subset of X. Assume that u is a common fixed point of f, g, p and q, and $D = P_M(u)$ is nonempty. Suppose that:

(i) the pairs (f, p) and (g, q) are weakly compatible, and the pair (f, p) or (g, q) satisfies the property $(E \cdot A)$ on D,

(ii) $pD = D,\ qD = D,\ f(\partial M) \subseteq M,\ g(\partial M) \subseteq M$, and D, fD, or gD is complete,

(iii) f is p-nonexpansive and g is q-nonexpansive on $D \cup \{u\}$,

(iv) for all $x, y \in D$, (2.2.2) holds.

Then f, g, p and q have a unique common fixed point in $P_M(u)$.

Proof. As in the proof of Theorem 2.3.1, we can prove that $fy \in D$ and $gy \in D$. Thus f, g, p and q are selfmappings of D. Therefore, by Theorem 2.2.4, there exists a unique $b \in D$ such that b is a common fixed point of f, g, p and q. ■

We establish the analogues of Theorem 3.2 by Al-Thagafi [3] and Theorem 3.3 due to Hussain and Khan [51] in the following result.

Theorem 2.3.4 Let f and g be selfmappings of a metric space X and M be a subset of X. Assume that u is a common fixed point of f and g, and $D^* = D_M^g(u)$ is nonempty. Suppose that:

(i) f and g are weakly compatible and satisfy the property $(E \cdot A)$ on D^*,

(ii) g is nonexpansive on $P_M(u) \cup \{u\}$ and f is g-nonexpansive on $D^* \cup \{u\}$,

(iii) $gD^* = D^*$, $f(\partial M) \subseteq M$, and fD^* or D^* is complete,

(iv) for all $x \neq y$ in D^*, (2.2.1) holds.

Then f and g have a unique common fixed point in D^*.

Proof. Let $y \in D^*$. Then $gy \in D^*$. By the definition of D^*, $y \in \partial M$ and so $fy \in M$. As f is g-nonexpansive on $D^* \cup \{u\}$,

$$d(fy, u) = d(fy, fu) \leq d(gy, u).$$

Therefore, $fy \in P_M(u)$. Since g is nonexpansive on $P_M(u) \cup \{u\}$, therefore

$$d(gfy, u) = d(gfy, gu) \leq d(fy, u) = d(fy, fu) \leq d(gy, gu) = d(gy, u).$$

Thus, $gfy \in P_M(u)$ and so $fy \in C_M^g(u)$. Therefore, $fy \in D^*$. Hence f and g are selfmappings of D^*. Thus, by Theorem 2.2.1, there exists a unique $b \in D^* \subset P_M(u)$ such that $b = fb = gb$. ■

2.4 Eigenvalue Problems

The aim of this section is to seek solutions of certain nonlinear eigenvalue problems for operators defined on a normed space and closed balls of a reflexive Banach space.

We now apply Theorem 2.2.1 to solve an eigenvalue problem as follows:

Theorem 2.4.1 Let X be a normed space and f be a selfmapping of X with $f(0) \neq 0$. Suppose that:

(i) there exists a sequence $\{x_m\}$ such that

$$\lim_{m\to\infty} f_n x_m = \lim_{m\to\infty} x_m = t, \quad \text{for some} \ \ t \in X,$$

where $f_n = \left(1 - \dfrac{1}{n}\right) f, \ \ n = 2, 3, 4, \ldots,$

(ii) X or fX is complete,

(iii) for all $x \neq y$ in X, the following condition holds:

$$\begin{aligned} \|fx - fy\| \quad \leq \quad & \max\{\|x - y\|, r\|f_n x - x\| + \alpha\|f_n y - y\|, \\ & \frac{1}{2}(\|f_n y - x\| + \|f_n x - y\|)\Big\} \end{aligned} \tag{2.4.1}$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then $M_n = 1/\left(1 - \frac{1}{n}\right)$ is an eigenvalue of f.

Proof. Clearly, f_n and I (the identity mapping on X) are commuting and satisfy the property $(E \cdot A)$. Note that

$$\|f_n x - f_n y\| < \|fx - fy\|, \quad \text{for each} \quad n > 1.$$

By this and (iii), for all $x \neq y$ in X and each $n > 1$, (2.2.1) is satisfied for the mappings f_n and I. By Theorem 2.2.1, there exists $x_n \in X$ such that $x_n = f_n x_n$ for each $n > 1$; that is, $fx_n = M_n x_n$ for each $n > 1$. This and $f(0) \neq 0$ imply that $x_n \neq 0$ for each $n > 1$. Thus, for each $n > 1$, x_n is an eigenvector and M_n is an eigenvalue for f. ■

Example 2.4.2 Let $X = \mathbb{R}^2$ and f be defined by

$$f(x, y) = (x - 1, y + 1).$$

Clearly, $f(0, 0) \neq (0, 0)$ and (2.4.1) holds in view of

$$|f(x_1, y_1) - f(x_2, y_2)| = |(x_1, y_1) - (x_2, y_2)|.$$

Now, for the sequence $(x_n, y_n) = \left(\frac{1}{n} - 1, \frac{1}{n} + 1\right), \quad n = 1, 2, \ldots,$

$$\lim_{n\to\infty} f_2(x_n, y_n) = \lim_{n\to\infty} \frac{1}{2} f(x_n, y_n) = (-1, 1) = \lim_{n\to\infty} (x_n, y_n).$$

Thus, by Theorem 2.4.1, $M_2 = 2$ is an eigenvalue of f. The corresponding eigenvector is $(-1, 1)$.

In the sequel, B denotes the closed ball $B = \{x \in X : \|x\| \leq r\}$.

Theorem 2.4.3 Let X be a reflexive Banach space and $f : B \to X$ be weakly continuous with $f(0) \neq 0$. Suppose that for each $x \in \partial B$ and for $k \in (0, 1]$, one of the following conditions holds:

(i) $\|fx\| \leq \max\{\|kfx - x\|, \|x\|\}$,

(ii) there exists $p > 1$ such that $\|fx\|^p \leq \|kfx - x\|^p + \|x\|^p$.

Then $M = \dfrac{1}{k}$ is an eigenvalue for f.

Proof. Let $M = \dfrac{1}{k}, \quad k \in (0, 1]$. Define, $f_k = kf$. Assume that (i) or (ii) is satisfied; then we get one of the following:

(a) $\|f_k x\| \leq \|fx\| \leq \max\{\|f_k x - x\|, \|x\|\}$,

(b) $\|f_k x\|^p \leq \|fx\|^p \leq \{\|f_k x - x\|^p, \|x\|^p\}$.

By Theorem 1.2.5, there exists $u \in B$ such that $f_k u = u$. So $fu = Mu$. This and $f(0) \neq 0$ imply that $u \neq 0$. Thus u is an eigenvector for f and so M is an eigenvalue for f as desired. ∎

As an application of Theorem 2.4.3, we obtain the following analogue of Theorem 3.2 in [7].

Theorem 2.4.4 Let C be a closed and bounded subset of $\mathbb{R}^n$ and $T : L^p(C) \to L^p(C)$. Suppose that:

(i) $H = H(t,s) : C \times \mathbb{R} \to \mathbb{R}$ is weakly continuous with respect to s uniformly in t,

(ii) $x(t) \in L^p(C)$ implies $H(t, Tx(t)) \in L^p(C)$,

(iii) for $x(t) \in L^p(C)$ with $\|x(t)\|_p = 1$,

$$\|H(t,Tx(t))\|_p \leq \max\{1, \|kH(t,Tx(t)) - x(t)\|_p\},$$

where $k \in (0,1]$,

(iv) $H(t, T(0)) \neq 0$.

Then the operator equation

$$H(t,Tx(t)) = ux(t) \tag{2.4.2}$$

has a solution in B_1, the closed unit ball of $L^p(C)$, for each $u = \dfrac{1}{k}, \quad k \in (0,1]$.

Proof. We know that $L^p \quad (1 < p < \infty)$ is a reflexive Banach space. The operator S defined by

$$Sx(t) = H(t,Tx(t))$$

maps $L^p(C)$ into itself by (ii). In view of (iii), for the operator $S : B_1 \to L^p(C)$,

$$\|Sx(t)\|_p \leq \max\{\|x(t)\|_p, \|kSx(t) - x(t)\|_p\}$$

for each $x(t) \in \partial B_1$ and $k \in (0, 1]$. By (iv), we have $S(0) \neq 0$. Now, let $\{x_n(t)\}$ converges weakly to $x(t)$. This implies, by (i), that $\{H(t, x_n(t)\}$ converges weakly to $Sx(t)$. Thus S is weakly continuous. Now all the conditions of Theorem 2.4.3 are satisfied and hence for each $u = \frac{1}{k}, \quad k \in (0, 1]$, we get, $Sx(t) = ux(t)$; that is, the operator equation (2.4.2) has a solution in B_1 for each $u = \frac{1}{k}, \quad k \in (0, 1]$. ■

The following example supports the above theorem.

Example 2.4.5 The eigenvalue problem

$$e^t - \|x(t)\| = ux(t)$$

has a nontrivial solution in the closed unit ball B_1 of $L^2([0, 1])$.

Solution Set $H(t, s) = e^t - s$, $Tx(t) = \|x(t)\|$, $C = [0, 1]$ and $p = 2$ in Theorem 2.4.4.

(i) Assume that $\{s_n\}$ converges weakly to s. Then, for any continuous linear functional f, we have

$$|f(H(t, s_n)) - f(H(t, s))| = |f(s_n - s)| = |f(s) - f(s_n)|.$$

Thus $H(t, s_n)$ converges weakly to $H(t, s)$ and so H is weakly continuous with respect to s uniformly on t.

(ii) If $x(t) \in L^2$, then clearly

$$H(t, Tx(t)) = e^t - \|x(t)\| \in L^2([0,1]).$$

(iii) For $x(t) \in L^2([0,1])$ with $\|x(t)\|_2 = 1,$ we get $H(t, Tx(t)) = e^t - 1.$

So,

$$\|H(t, Tx(t))\|_2 = \left(\int_0^1 |e^t - 1|^2 dt \right)^{1/2} < 1.$$

(iv) Since

$$\|H(t, T(0))\|_2 = \|e^t\|_2 = \left(\int_0^1 e^{2t} dt \right)^{1/2} > 1.$$

So, H does not map B_1 into itself and $H(t, T(0)) \neq 0$.

Now, the conclusion follows from Theorem 2.4.4.

CHAPTER 3

ITERATIVE ALGORITHMS FOR A FINITE FAMILY OF MAPPINGS

3.1 Introduction

Finding common fixed points of a finite family $\{T_i : i = 1, 2, \ldots, k\}$ of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings. Approximating common fixed points of a finite family of nonexpansive mappings by iteration has been studied by many authors (see, for example, Kuhfittig [77], Rhoades [99] and Takahashi and Shimoji [123]). Ghosh and Debnath [39] proved some convergence results for common fixed points of finite families of quasi-nonexpansive mappings in a uniformly convex Banach space (see also, Ahmed and Zeyada [2], Dotson, Jr. [34], Ghosh and Debnath [38, 40], Maiti and Ghosh [82] and Petryshyn and Williamson, Jr. [94]).

Goebel and Kirk [42], in 1972, introduced the notion of an asymptotically nonexpansive mapping and established that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space X and T is an asymptotically nonexpansive selfmapping of C, then T has a fixed point. Bose [20] initiated in 1978, the study of iterative construction for fixed points of asymptotically nonexpansive mappings. Xu and Ori [138], in 2001, introduced an implicit iteration process for a finite family of nonexpansive mappings. Sun [117], in 2003, modified this implicit iteration process for a finite family of asymptotically quasi-nonexpansvie mappings. Khan and Takahashi [73] have approximated common fixed points of two asymptotically nonexpansive mappings by the modified Ishikawa iteration. Recently, Shahzad and Udomene [110] established convergence theorems for the modified Ishikawa iteration of two asymptotically quasi-nonexpansive mappings to a common fixed point of the mappings.

For a finite family of mappings, it is desirable to devise a general iteration scheme which extends the modified Mann iteration (1.3.5), the modified Ishikawa iteration (1.3.6), Khan and Takahashi scheme [73] and the three-step iteration (1.3.7) by Xu and Noor [133], simultaneously. Thereby, to achieve this goal,we introduce a new iteration process for a finite family $\{T_i : i = 1, 2, \ldots, k\}$ of asymptotically quasi-nonexpansive mappings as follows:

Let C be a convex subset of a Banach space X and $x_1 \in C$. Suppose that $\alpha_{in} \in [0, 1]$, $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$. Let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of selfmappings of C. The iteration scheme is defined as follows:

$$\begin{aligned}
x_{n+1} &= (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n\, y_{(k-1)n}, \\
y_{(k-1)n} &= (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n\, y_{(k-2)n}, \\
y_{(k-2)n} &= (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n\, y_{(k-3)n}, \\
\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n\, y_{1n}, \\
y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n\, y_{0n}, \qquad (3.1.1)
\end{aligned}$$

where $y_{0n} = x_n$ for all n.

Clearly, the iteration process (3.1.1) generalizes the modified Mann iteration (1.3.5), the modified Ishikawa iteration (1.3.6) and the three-step iteration scheme (1.3.7) from one mapping to the finite family of mappings $\{T_i : i = 1, 2, \ldots, k\}$.

Throughout this chapter, we assume that $F = \bigcap_{i=1}^{k} F(T_i)$.

The main purpose of this chapter is to:

(i) establish a necessary and sufficient condition for convergence of the iteration scheme (3.1.1) to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space;

(ii) prove weak and strong convergence results of the iteration scheme (3.1.1) to a common fixed point of a finite family of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive mappings in a uniformly convex Banach space;

(iii) obtain weak and strong convergence results about common fixed points of a finite family of quasi-nonexpansive mappings in a Banach space by a generalized Ishikawa iterative scheme.

Our work is a significant generalization of the corresponding results of Khan and Takahashi [73], Kuhfittig [77], Petryshyn and Williamson [94], Qihou [96], Schu [103], Shahzad and Udomene [110], Tan and Xu [126] and Xu and Noor [133]. Moreover, our results provide analogue of the results of Sun [117], for the iteration scheme (3.1.1) instead of the implicit iteration.

3.2 Convergence of Iterative Algorithms

The aim of this section is to prove some results for the iterative process (3.1.1) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space. We begin with the following:

Lemma 3.2.1 Let C be a nonempty closed convex subset of a Banach space, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$ where $\{u_{in}\}$ are sequences in $[0, +\infty)$ with $\lim_{n\to\infty} u_{in} = 0$ for each i. Assume that $F \neq \phi$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$ for each i. Define the sequence $\{x_n\}$ as in (3.1.1). Then

(a) there exists a sequence $\{\nu_n\}$ in $[0, +\infty)$ such that $\sum_{n=1}^{\infty} \nu_n < +\infty$ and

$$\|x_{n+1} - p\| \leq (1 + \nu_n)^k \|x_n - p\|,$$

for all $p \in F$ and all n;

(b) there exists a constant $M > 0$ such that

$$\|x_{n+m} - p\| \leq M\|x_n - p\|,$$

for all $p \in F$ and $n, m = 1, 2, 3, \ldots$

Proof. (a) Let $p \in F$ and

$$\nu_n = \max_{1\leq i\leq k} u_{in}, \quad \text{for all} \quad n = 1, 2, 3, \ldots.$$

Since $\sum_{n=1}^{\infty} u_{in} < +\infty$ for each i, therefore $\sum_{n=1}^{\infty} \nu_n < +\infty$. Now we have

$$\begin{aligned}
\|y_{1n} - p\| &\leq (1-\alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| \\
&\leq (1-\alpha_{1n})\|x_n - p\| + \alpha_{1n}(1+u_{1n})\|x_n - p\| \\
&= (1+\alpha_{1n}u_{1n})\|x_n - p\| \\
&\leq (1+\nu_n)\|x_n - p\|.
\end{aligned}$$

Assume that

$$\|y_{jn} - p\| \leq (1+\nu_n)^j \|x_n - p\|$$

holds for some $1 \leq j \leq k-2$. Then

$$\begin{aligned}
\|y_{(j+1)n} - p\| &\leq (1-\alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}\|T_{j+1}^n\, y_{jn} - p\| \\
&\leq (1-\alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1+u_{(j+1)n})\|y_{jn} - p\| \\
&\leq (1-\alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1+u_{(j+1)n})(1+\nu_n)^j\|x_n - p\| \\
&\leq (1-\alpha_{(j+1)n})\|x_n - p\| + \alpha_{(j+1)n}(1+\nu_n)^{j+1}\|x_n - p\| \\
&= (1-\alpha_{(j+1)n})\|x_n - p\| \\
&\quad +\alpha_{(j+1)n}\left(1+\sum_{r=1}^{j+1}\frac{(j+1)j\cdots(j+2-r)}{r!}\nu_n^r\right)\|x_n - p\| \\
&= \left[1+\alpha_{(j+1)n}\sum_{r=1}^{j+1}\frac{(j+1)j\cdots(j+2-r)}{r!}\nu_n^r\right]\|x_n - p\| \\
&\leq (1+\nu_n)^{j+1}\|x_n - p\|.
\end{aligned}$$

Thus, by induction, we have

$$\|y_{in} - p\| \leq (1+\nu_n)^i\|x_n - p\|, \text{ for all } i = 1, 2, \ldots, k-1. \tag{3.2.1}$$

Now, by (3.2.1), we obtain

$$\begin{aligned}
\|x_{n+1}-p\| &\le (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}\|T_k^n y_{(k-1)n}-p\| \\
&\le (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+u_{kn})\|y_{(k-1)n}-p\| \\
&\le (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+u_{kn})(1+\nu_n)^{k-1}\|x_n-p\| \\
&\le (1-\alpha_{kn})\|x_n-p\|+\alpha_{kn}(1+\nu_n)^{k}\|x_n-p\| \\
&= \left[1-\alpha_{kn}+\alpha_{kn}\left(1+\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right)\right]\|x_n-p\| \\
&= \left[1+\alpha_{kn}\sum_{r=1}^{k}\frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right]\|x_n-p\| \\
&\le (1+\nu_n)^k\|x_n-p\|.
\end{aligned}$$

This completes the proof of (a).

(b) If $t \ge 0$, then $1+t \le e^t$ and so,

$$(1+t)^k \le e^{kt}, \quad k=1,2,\ldots.$$

Thus, from part (a), we get

$$\begin{aligned}
\|x_{n+m}-p\| &\le (1+\nu_{n+m-1})^k\|x_{n+m-1}-p\| \\
&\le \exp\{k\nu_{n+m-1}\}\|x_{n+m-1}-p\| \le \cdots \\
&\le \exp\left\{k\sum_{i=1}^{n+m-1}\nu_i\right\}\|x_n-p\| \\
&\le \exp\left\{k\sum_{i=1}^{\infty}\nu_i\right\}\|x_n-p\|.
\end{aligned}$$

Setting $M=\exp\left\{k\sum_{i=1}^{\infty}\nu_i\right\}$, completes the proof. $\blacksquare$

The above lemma generalizes Theorem 3.1 for two asymptotically quasi-nonexpansive mappings by Shahzad and Udomene [110] to the case of any finite family of such mappings.

The next result deals with a necessary and sufficient condition for the convergence of $\{x_n\}$ generated by the iteration process (3.1.1) to a point of F; for this we follow the arguments of Qihou ([96, Theorem 1).

Theorem 3.2.2 Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$. Suppose that $F \neq \phi$, $x_1 \in C$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$ for all i. Then the iterative sequence $\{x_n\}$, defined by (3.1.1), converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. We will only prove the sufficiency; the necessity is obvious. ¿From Lemma 3.2.1(a), we have

$$\|x_{n+1} - p\| \leq (1 + \nu_n)^k \|x_n - p\|,$$

for all $p \in F$ and all n. Therefore,

$$\begin{aligned} d(x_{n+1}, F) &\leq (1 + \nu_n)^k d(x_n, F) \\ &= \left(1 + \sum_{r=1}^{k} \frac{k(k-1)\cdots(k-r+1)}{r!}\nu_n^r\right) d(x_n, F). \end{aligned}$$

As $\sum_{n=1}^{\infty} \nu_n < +\infty$, so

$$\sum_{n=1}^{\infty} \sum_{r=1}^{k} \frac{k(k-1)\cdots(k-r+1)}{r!} \nu_n^r < +\infty.$$

By Lemma 1.3.5 and $\liminf_{n\to\infty} d(x_n, F) = 0$, we get that

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence. From Lemma 3.2.1(b), we have

$$\|x_{n+m} - p\| \leq M\|x_n - p\|, \text{ for all } p \in F \text{ and all } n, m =, 1, 2, 3, \ldots \quad (3.2.2)$$

Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore for each $\epsilon > 0$, there exists a natural number n_1 such that

$$d(x_n, F) \leq \frac{\epsilon}{3M}, \text{ for all } n \geq n_1.$$

Hence, there exists $z_1 \in F$ such that

$$\|x_{n_1} - z_1\| \leq \frac{\epsilon}{2M}. \quad (3.2.3)$$

¿From (3.2.2) and (3.2.3), for all $n \geq n_1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - z_1\| + \|x_n - z_1\| \\ &\leq M\|x_{n_1} - z_1\| + M\|x_{n_1} - z_1\| \\ &\leq M\left(\frac{\epsilon}{2M}\right) + M\left(\frac{\epsilon}{2M}\right) \\ &= \epsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence and so converges to $q \in X$. Finally, we show that $q \in F$. For any $\overline{\epsilon} > 0$, there exists a natural number n_2 such that

$$\|x_n - q\| \leq \frac{\overline{\epsilon}}{2(2+\nu_1)}\ , \text{ for all } n \geq n_2. \tag{3.2.4}$$

Again, $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists a natural number $n_3 \geq n_2$ such that

$$d(x_n, F) \leq \frac{\overline{\epsilon}}{3(4+3\nu_1)}\ , \text{ for all } n \geq n_3.$$

Thus, there exists $z_2 \in F$ such that

$$\|x_{n_3} - z_2\| \leq \frac{\overline{\epsilon}}{2(4+3\nu_1)}. \tag{3.2.5}$$

From (3.2.4) and (3.2.5), for any $T_i, \quad i = 1, 2, \ldots, k$, we get

$$\begin{aligned}
\|T_i q - q\| &= \|T_i q - z_2 + z_2 - T_i x_{n_3} + T_i x_{n_3} - z_2 + z_2 - x_{n_3} + x_{n_3} - q\| \\
&\leq \|T_i q - z_2\| + 2\|T_i x_{n_3} - z_2\| + \|x_{n_3} - z_2\| + \|x_{n_3} - q\| \\
&\leq (1+\nu_1)\|q - z_2\| + 2(1+\nu_1)\|x_{n_3} - z_2\| \\
&\quad + \|x_{n_3} - z_2\| + \|x_{n_3} - q\| \\
&\leq (1+\nu_1)\|x_{n_3} - q\| + (1+\nu_1)\|x_{n_3} - z_1\| + 2(1+\nu_1)\|x_{n_3} - z_2\| \\
&\quad + \|x_{n_3} - z_2\| + \|x_{n_3} - q\| \\
&= (2+\nu_1)\|x_{n_3} - q\| + (4+3\nu_1)\|x_{n_3} - z_2\| \\
&\leq (2+\nu_1)\frac{\overline{\epsilon}}{2(2+\nu_1)} + (4+3\nu_1)\frac{\overline{\epsilon}}{2(4+3\nu_1)} \\
&= \overline{\epsilon}.
\end{aligned}$$

Since $\overline{\epsilon}$ is arbitrary, therefore

$$\|T_i q - q\| = 0, \quad \text{for all} \quad i,$$

i.e., $T_i q = q, i = 1, 2, \ldots, k$. Thus $q \in F$. ■

Remark 3.2.3 Theorem 3.2.2 contains as special cases, Theorem 3.2 of Shahzad and Udomene [110] and Theorem 1 by Qihou [96] together with its Corollaries 1 and 2, which are themselves extensions of the results of Ghosh and Debnath [40] and Petryshyn and Williamson [94].

An asymptotically nonexpansvie mapping is asymptotically quasi-nonexpansive, so we have:

Corollary 3.2.4 Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymsptotically nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \leq (1 + u_{in})\|x - y\|$, for all $x, y \in C$ and $i = 1, 2, \ldots, k$. Suppose that $F \neq \phi$, $x_1 \in C$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$, for all i. Then the iterative sequence $\{x_n\}$, defined by (3.1.1), converges strongly to a point $p \in F$ if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$.

Corollary 3.2.5 Let C, $\{T_i : i = 1, 2, \ldots, k\}$, F and u_{in} be as in Theorem 3.2.2. Then the iterative sequence $\{x_n\}$, defined by (3.1.1), converges strongly to a point $p \in F$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.

Corollary 3.2.6 Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically nonexpansive selfmappings of C. Suppose that $F \neq \phi$, $x_1 \in C$ and $\sum_{n=1}^{\infty} u_{in} < +\infty$ for all i. Let $\{x_n\}$ be the sequence defined by (3.1.1). If $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $i = 1, 2, \ldots, k$ and one of the mappings is semi-compact, then $\{x_n\}$ converges strongly to $p \in F$.

Proof. Let T_ℓ be semi-compact for some $1 \leq \ell \leq k$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$. Hence

$$\|p - T_i p\| = \lim_{n_j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, \text{ for all } i = 1, 2, \ldots, k.$$

Thus, $p \in F$ and by Corollary 3.2.5, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. ■

Theorem 3.2.7 Let C, $\{T_i : i = 1, 2, \ldots, k\}$, F and u_{in} be as in Theorem 3.2.2. Suppose that there exists a mapping T_j which satisfies the following conditions:

(i) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$;

(ii) there exists a constant M such that

$$\|x_n - T_j x_n\| \geq M d(x_n, F), \quad \text{for all} \quad n.$$

Then the sequence $\{x_n\}$, defined by (3.1.1), converges strongly to a point $p \in F$.

Proof. From (i) and (ii), it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. By Theorem 3.2.2, $\{x_n\}$ converges strongly to a common fixed point of the family of mappings. ■

3.3 Results in Uniformly Convex Banach Spaces

In this section, we establish some weak and strong convergence results for the iterative scheme (3.1.1) by removing the condition $\liminf_{n\to+\infty} d(x_n, F) = 0$ from the results obtained in Section 3.2; for this we have to consider the class of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive mappings on a uniformly convex Banach space.

Lemma 3.3.1 Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, 3, \ldots, k\}$ a family of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i)$, where $\{u_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$, for each $i \in \{1, 2, 3, \ldots, k\}$. Assume that $F \neq \phi$ and the sequence $\{x_n\}$ is as in (3.1.1) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. Then

(i) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$;

(ii) $\lim_{n\to\infty} \|x_n - T_j^n y_{(j-1)n}\| = 0$, for each $j = 1, 2, , \ldots, k$;

(iii) $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$, for each $j = 1, 2, \ldots, k$.

Proof. Let $p \in F$ and $\nu_n = \max_{1\le i\le k} u_{in}$, for all n.

(i) By Lemma 1.3.5(i) and Lemma 3.2.1(a), it follows that

$$\lim_{n\to\infty} \|x_n - p\| \ \text{ exists for all } p \in F.$$

Assume that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{3.3.1}$$

(ii) The inequality (3.2.1) and (3.3.1) give that

$$\limsup_{n\to\infty} \|y_{jn} - p\| \le c, \text{ for } 1 \le j \le k-1. \tag{3.3.2}$$

We also note that:

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_{kn})(x_n - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\
&\le (1-\alpha_{kn})\|x_n - p\| + \alpha_{kn}(1+v_n)\|y_{(k-1)n} - p\| \\
&\quad \cdots\cdots\cdots\cdots \\
&\le (1-\alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n})(1+v_n)^{k-j}\|x_n - p\| \\
&\quad +\alpha_{kn}\alpha_{(k-1)n}\cdots\alpha_{(j+1)n}(1+v_n)^{k-j}\|y_{jn} - p\|.
\end{aligned}$$

Therefore,

$$\|x_n - p\| \leq \frac{\|x_n - p\|}{\delta^{k-j}} - \frac{\|x_{n+1} - p\|}{\delta^{k-j}(1+v_n)^{k-j}} + \|y_{jn} - p\|$$

and hence

$$c \leq \liminf_{n\to\infty} \|y_{in} - p\|, \quad \text{for } 1 \leq j \leq k-1. \tag{3.3.3}$$

¿From (3.3.2) and (3.3.3), we have

$$\lim_{n\to\infty} \|y_{jn} - p\| = c, \text{ for each } j = 1, 2, 3, \ldots, k-1.$$

That is,

$$\lim_{n\to\infty} \|(1-\alpha_{jn})(x_n - p) + \alpha_{jn}(T_j^n y_{(j-1)n} - p)\| = c,$$

for each $j = 1, 2, 3, \ldots, k-1$.

Also, from (3.3.2), we obtain

$$\limsup_{n\to\infty} \|T_j^n y_{(j-1)n} - p\| \leq c, \text{ for each } j = 1, 2, 3, \ldots, k-1.$$

By Lemma 1.3.6, we get

$$\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0, \text{ for each } j = 1, 2, 3, \ldots, k-1. \tag{3.3.4}$$

For the case $j = k$, by (3.2.1), we have

$$\begin{aligned}
\|T_k^n y_{(k-1)n} - p\| &\leq (1+u_{kn})\|y_{(k-1)n} - p\| \\
&\leq (1+u_{kn})(1+\nu_n)^{k-1}\|x_n - p\|.
\end{aligned}$$

But $\lim_{n\to\infty} \|x_n - p\| = c$, by part (i). So,

$$\limsup_{n\to\infty} \|T_k^n y_{(k-1)n} - p\| \leq c.$$

Moreover,

$$\lim_{n\to\infty} \|(1-\alpha_{kn})(x_n - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\| = c.$$

Again by Lemma 1.3.6, we get

$$\lim_{n\to\infty} \|x_n - T_k^n y_{(k-1)n}\| = 0. \tag{3.3.5}$$

Thus, (3.3.4) and (3.3.5) imply that

$$\lim_{n\to\infty} \|T_j^n y_{(j-1)n} - x_n\| = 0, \text{ for each } j = 1, 2, 3, \ldots, k. \tag{3.3.6}$$

(iii) For $j = 1$, from part (ii), we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0.$$

If $j = 2, 3, 4, \ldots, k$, then we have

$$\begin{aligned}
\|T_j^n x_n - x_n\| &= \|(T_j^n x_n - T_j^n y_{(j-1)n}) + (T_j^n y_{(j-1)n} - x_n)\| \\
&\leq L\|x_n - y_{(j-1)n}\|^\gamma + \|T_j^n y_{(j-1)n} - x_n\| \\
&= L(\alpha_{(j-1)n}\|x_n - T_{j-1}^n y_{(j-2)n}\|)^\gamma + \|T_j^n y_{(j-1)n} - x_n\| \to 0.
\end{aligned}$$

Hence,

$$||T_j^n x_n - x_n\| \to 0 \text{ as } n \to \infty, \text{ for } 1 \leq j \leq k. \tag{3.3.7}$$

Let us observe that:

$$\begin{aligned}\|x_n - T_j x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\ &\quad + \|T_j^{n+1} x_{n+1} - T_j^{n+1} x_n\| + \|T_j^{n+1} x_n - T_j x_n\| \\ &\leq \alpha_{kn}\|x_n - T_k^n y_{(k-1)n}\| + \|x_{n+1} - T_j^{n+1} x_{n+1}\| \\ &\quad + L(\alpha_{kn}\|x_n - T_k^n y_{(k-1)n}\|)^\gamma + L\|T_j^n x_n - x_n\|^\gamma.\end{aligned}$$

Using (3.3.6) and (3.3.7), we get

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \text{ for } 1 \leq j \leq k.$$

■

Theorem 3.3.2 Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying the Opial property and let $\{T_i : i = 1, 2, 3, \ldots, k\}$ be a family of $(L-\gamma)$ uniform Lipschitz and asymsptoticaly quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + u_{in})\|x - p_i\|$ for all $x \in C$ and $p_i \in F(T_i)$, $\;i = 1, 2, 3, \ldots, k$ where $\{u_{in}\}$ are sequences in $[0, \infty)$ with $\sum_{n=1}^{\infty} u_{in} < \infty$ for each $i = 1, 2, 3 \ldots, k$. Let the sequence $\{x_n\}$ be as in (3.1.1) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. If $F \neq \phi$ and each $I - T_i, i = 1, 2, 3, \ldots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \ldots, k\}$.

Proof. Let $p \in F$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 3.3.1(i) and hence $\{x_n\}$ is bounded. Since a uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $z_1 \in C$. By Lemma 3.3.1,

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \text{for} \;\; i = 1, 2, \ldots, k.$$

Since $I - T_i$ is demiclosed at 0 for $i = 1, 2, 3, \ldots, k$, so we obtain $T_i z_1 = z_1$. That is, $z_1 \in F$. In order to show that $\{x_n\}$ converges weakly to z_1, take another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $z_2 \in C$. Again, as above, we can prove that $z_2 \in F$. Next, we show that $z_1 = z_2$. Assume $z_1 \neq z_2$. Then by the Opial property

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\
&= \lim_{n\to\infty} \|x_n - z_2\| \\
&= \lim_{n_k\to\infty} \|x_{n_k} - z_2\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - z_1\| \\
&= \lim_{n\to\infty} \|x_n - z_1\|.
\end{aligned}$$

This contradiction proves that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, 3, \ldots, k\}$. ■

Theorem 3.3.3 Under the hypotheses of Lemma 3.3.1, assume that, for some $1 \leq i \leq k$, T_i^m is semi-compact for some positive integer m. Then $\{x_n\}$ converges strongly to some common fixed point of the family $\{T_j : j = 1, 2, 3, \ldots, k\}$.

Proof. By Lemma 3.3.1(iii), we have

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \text{ for } 1 \leq j \leq k. \tag{3.3.8}$$

Fix $i \in \{1, 2, 3, \ldots, k\}$ and suppose T_i^m to be semi-compact for some $m \geq 1$. ¿From (3.3.8), we obtain

$$\begin{aligned} \|T_i^m x_n - x_n\| &\leq \|T_i^m x_n - T_i^{m-1} x_n\| + \|T_i^{m-1} x_n - T_i^{m-2} x_n\| \\ &\quad + \cdots + \|T_i^2 x_n - T_i x_n\| + \|T_i x_n - x_n\| \\ &\leq \|T_i x_n - x_n\| + (m-1)L\|T_i x_n - x_n\|^\gamma \to 0. \end{aligned}$$

Since $\{x_n\}$ is bounded and T_i^m is semi-compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to q \in C$. Hence, from (3.3.8), we have

$$\|q - T_i q\| = \lim_{n\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0, \text{ for all } i = 1, 2, 3, \ldots, k.$$

Thus $q \in F$ and by Corollary 3.2.5, $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, 3, \ldots, k\}$. ■

Our results in Chapter 2 can be used to guarantee the existence of a unique common fixed point of families of two or four mappings. We apply Theorem 2.2.1 and Theorem 2.2.4 to obtain the following two results, respectively.

Theorem 3.3.4 Let C be a nonempty closed convex subset of a uniformly convex Banach space X satisfying Opial's property, and S and T be $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive selfmappings of C with $\sum_{n=1}^{\infty} u_n < +\infty$ and $\sum_{n=1}^{\infty} \nu_n < +\infty$ ($\{u_n\}$ and $\{\nu_n\}$ are the corresponding sequences for S and T, respectively). Suppose that S and T are weakly compatible, and $I-S$ and $I-T$ are demiclosed at 0. If the conditions (i) - (iii) in Theorem 2.2.1 are satisfied (where $f = S$ and $g = T$), then the modified Ishikawa iteration scheme (1.3.6), with $\alpha_n, \beta_n \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, converges weakly to a unique common fixed point of S and T.

Proof. By Theorem 2.2.1, there exists $p \in C$ such that p is a unique common fixed point of S and T. Now, by Theorem 3.3.2, $\{x_n\}$ converges weakly to p.

Theorem 3.3.5 Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, 3, 4\}$ be a family of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive selfmappings of C with $\sum_{n=1}^{\infty} u_{in} < +\infty, i = 1, 2, 3, 4$ ($\{u_{in}\}$ is the corresponding sequence for T_i). Suppose that $I - T_i, i = 1, 2, 3, 4$, are demiclosed at 0, and the pairs (T_1, T_3) and (T_2, T_4) are weakly compatible. If the conditions (i)-(iii) in Theorem 2.2.4 are satisfied (where $f = T_1, g = T_2, p = T_3, q = T_4$), then the seuence $\{x_n\}$, defined by (3.1.1) where $k = 4$ and $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, converges weakly to a unique common fixed point of T_1, T_2, T_3 and T_4.

Proof. By Theorem 2.2.4, the mappings T_1, T_2, T_3 and T_4 have a unique common fixed point $p \in C$. Now, Theorem 3.3.2 implies that $\{x_n\}$ converges weakly to p.

3.4 Asymptotically Nonexpansive Mappings

The family of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive mappings in Lemma 3.3.1 can be replaced by a family of asymptotically nonexpansive mappings. We state this result as follows; the proof is similar to that of Lemma 3.3.1.

Lemma 3.4.1 Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically nonexpansive selfmappings of C. Assume that $F \neq \phi$ and $\sum_{i=1}^{\infty} u_{in} < \infty$ for each $i = 1, 2, \ldots, k$. Let $\{x_n\}$ be as in (3.1.1) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in \left(0, \frac{1}{2}\right)$. Then (i), (ii) and (iii) of Lemma 3.3.1 hold.

Remark 3.4.2 Note that

(a) Lemma 3.3.1(i) extends Lemma 2.1 of Tan and Xu [126]. Lemma 3.3.1(ii) extends Theorem 3.3 of Shahzad and Udomene [110] for two uniformly continuous asymptotically quasi-nonexpansive mappings to any finite family of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive mappings.

(b) Lemma 3.4.1(ii) and Lemma 3.4.1(iii) contain as special cases, Lemma 2.2 of Xu and Noor [133] and Lemma 1.5 of Schu [103], respectively.

On the lines of the proof of Theorem 3.3.2 and using Lemma 1.3.10 and Lemma 3.4.1, the following result can be easily proved.

Theorem 3.4.3 Under the hypotheses of Lemma 3.4.1, assume that the space X satisfies the Opial property. Then the sequence $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

The special cases of Theorem 3.4.3 are Theorems 3.1-3.2 of Tan and Xu [126] and Theorem 2.1 due to Schu [103].

Following the arguments of the proof of Theorem 3.3.3, we can prove:

Theorem 3.4.4 Under the assumptions of Lemma 3.4.1, suppose that, for some $1 \leq i \leq k$, and a positive integer m, T_i^m is semi-compact. Then $\{x_n\}$ converges strongly to some common fixed point of the family of mappings.

The above theorem contains as a special case, Theorem 2.2 of Schu [103].

Remark 3.4.5 To guarantee the existence of a common fixed point of a finite family of asymptotically nonexpansive mappings, one can use, for example, Theorem 1.3.12 for a commutative semigroup of such mappings.

Remark 3.4.6 Theorem 3.2.2, Corollary 3.2.5, Theorem 3.3.3 and Theorem 3.4.4 about the iteration scheme (3.1.1) are analogues of Theorem 3.1, Corollary 3.2, Theorem 3.3 and Theorem 3.4, in the context of implcit iteration process, by Sun [117], respectively.

3.5 Quasi-Nonexpansive Mappings

We introduce a generalization of the Ishikawa iterative scheme by improving the Kuhfittig iteration scheme (1.3.3) as follows:

Definition 3.5.1 Let C be a convex subset of a Banach space X, $x_1 \in C$, $U_0 = I$ (the identity mapping on C), $\alpha_n, \beta_{jn} \in (0,1]$, for all $n = 1,2,3,\ldots,$ and $j = 1,2,\ldots,k$, and $\{Ti : i = 1,2,\ldots,k\}$ be a family of selfmappings of C. We define a generalization of Ishikawa iterative scheme as:

$$\begin{aligned} U_1 &= (1-\beta_{1n})I + \beta_{1n}T_1U_0, \\ U_2 &= (1-\beta_{2n})I + \beta_{2n}T_2U_1, \\ \cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ U_k &= (1-\beta_{kn})I + \beta_{kn}T_kU_{k-1}, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_k U_{k-1}\, x_n\,. \end{aligned} \tag{3.5.1}$$

Indeed, if $k = 2$ and $T_1 = T_2 = T$ in (3.5.1), then we get the Ishikawa iteration (1.3.2).

We present strong and weak convergence results of the generalized Ishikawa successive approximations (3.5.1) to a common fixed point of a family of quasi-nonexpansive mappings $\{T_i : i = 1, 2, \ldots, k\}$ in the context of a Banach space.

Theorem 3.5.2 Let C be a nonempty closed convex subset of a strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of quasi-nonexpansive selfmappings of C with $F \neq \phi$. Then the sequence $\{x_n\}$, defined by (3.5.1), converges strongly to a common fixed point of the family if and only if
$\lim_{n\to\infty} \inf d(x_n, F) = 0$.

Proof. The necessity is obvious. Thus we will only prove the sufficiency. It is easy to see that the families $\{U_1, \ldots, U_k\}$ and $\{T_1, \ldots, T_k\}$ have the same set of common fixed points. We prove that U_j and T_jU_{j-1}, $j = 1, 2, \ldots, k$ are quasi-nonexpansive selfmappings of C. Let $p \in F$. Clearly, T_1U_0 is quasi-nonexpansive. Now,

$$\begin{aligned}
\|U_1x - p\| &= \|(1-\beta_{1n})x + \beta_{1n}T_1x - (1-\beta_{1n})p - \beta_{1n}\,p\| \\
&\leq (1-\beta_{1n})\|x-p\| + \beta_{1n}\|T_1x - p\| \\
&\leq (1-\beta_{1n})\|x-p\| + \beta_{1n}\|x-p\| \\
&= \|x-p\|.
\end{aligned}$$

So, U_1 is quasi-nonexpansive. Subsequently,

$$\|T_2U_1x - p\| \leq \|U_1x - p\| \leq \|x - p\|,$$

and hence, T_2U_1 is quasi-nonexpansive.

Similarly, we can prove that U_2 is quasi-nonexpansive.

Repeating this procedure, we prove that U_j and $T_jU_{j-1}, \quad j = 1, 2, \ldots, k$ are quasi-nonexpansive.

If $p \in F$, then in view of the fact that $\{T_1, \ldots, T_k\}$ and $\{U_1, \ldots, U_k\}$ have the same common fixed points, $p \in \bigcap_{i=1}^{k} F(U_i)$; therefore, $p \in F(T_kU_{k-1})$ and so $F \subseteq F(T_kU_{k-1})$. Thus,

$$\liminf_{n\to\infty} \|x_n - F(T_kU_{k-1})\| = 0$$

because $\liminf_{n\to\infty} \|x_n - F\| = 0$.

By Theorem 1.3.8, the sequence $\{x_n\}$ defined by (3.5.1) converges strongly to a fixed point y of T_kU_{k-1}.

Next we show that y is a common fixed point of T_k and U_{k-1} $(k \geq 2)$. For this, we first show that $T_{k-1}U_{k-2}\, y = y$. Suppose not; then the closed line segment $[y, T_{k-1}U_{k-2}\, y]$ has positive length. Let

$$z = U_{k-1}\, y = (1 - \beta_{(k-1)n})\, y + \beta_{(k-1)n}T_{k-1}U_{k-2}\, y.$$

Since $F \neq \phi$ and $\{T_1, \ldots, T_k\}$ and $\{U_1, \ldots, U_k\}$ have the same common fixed points, therefore

$$T_{k-1}U_{k-2}\, p = p, \quad \text{for} \quad p \in F.$$

By the quasi-nonexpansiveness of T_kU_{k-2} and T_k, we get

$$\|T_{k-1}U_{k-2}\,y - p\| \leq \|y - p\| \tag{3.5.2}$$

and

$$\|T_k z - p\| \leq \|z - p\|.$$

In view of $T_k z = T_k U_{k-1}\,y = y$, it follows that

$$\|y - p\| \leq \|z - p\|.$$

As X is strictly convex, for noncollinear vectors a and b in X, we have $\|a + b\| < \|a\| + \|b\|$ (see [130, Definition 4.1.1]). This implies that

$$\begin{aligned}
\|y - p\| &\leq \|z - p\| \\
&= \|(1 - \beta_{(k-1)n})\,y + \beta_{(k-1)n}T_{k-1}U_{k-2}\,y \\
&\quad -(1 - \beta_{(k-1)n})\,p - \beta_{(k-1)n}\,p\| \\
&< (1 - \beta_{(k-1)n})\|y - p\| + \beta_{(k-1)n}\|T_{k-1}U_{k-2}\,y - p\|.
\end{aligned}$$

So, we get

$$\|y - p\| < \|T_{k-1}U_{k-2}\,y - p\|$$

which contradicts (3.5.2). Hence,

$$T_{k-1}U_{k-2}\,y = y.$$

Subsequently,

$$U_{k-1}\,y = (1 - \beta_{(k-1)n})y + \beta_{(k-1)n}T_{k-1}U_{k-2}\,y = y$$

and

$$y = T_k U_{k-1}\, y = T_k\, y.$$

Thus, y is a common fixed point of T_k and U_{k-1}.

Since $T_{k-1}U_{k-2}\, y = y$, we may repeat the above procedure to show that

$$T_{k-2}U_{k-3}\, y = y$$

and thereby y must be a common fixed point of T_{k-1} and U_{k-2}. Continuing in this manner, we conclude that $T_1U_0\, y = y$ and y is a common fixed point of T_2 and U_1. Consequently, y is a common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$.

■

Corollary 3.5.3 Let C be a nonempty weakly compact convex subset of a strictly convex Banach space, and $\{T_i : i = 1, 2, \ldots, k\}$ a commutative family of quasi-nonexpansive selfmappings of C. Then the sequence $\{x_n\}$, defined by (3.5.1), converges strongly to a common fixed point of the family if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. By Theorem 1.3.11, it follows that $F \neq \phi$. Since C is weakly compact, therefore it is closed strongly. The result follows from Theorem 3.5.2. ■

Theorem 3.5.4 Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of quasi-nonexpansive selfmappings of C with $F \neq \phi$. Let $\{x_n\}$ be defined by (3.5.1) with $0 < a \leq \alpha_n \leq b < 1$ and $0 < \beta_{jn} \leq \ \beta < 1$. If the map $T_k U_{k-1}$ satisfies the Condition B (see Section 1.3), then $\{x_n\}$ converges strongly to a common fixed point of the family.

proof. A uniformly convex space is strictly convex, so one can use the arguments of the proof of Theorem 3.5.2 with the exception that we will employ Theorem 1.3.9 in lieu of Theorem 1.3.8. ■

Remark 3.5.5 Note that Theorem 3.5.4 is an extension of Theorem 1 of Maiti and Ghosh [82], and Theorems 1 and 2 of Senter and Dotson, Jr. [104].

Theorem 3.5.6 Let C be a nonempty closed convex subset of a Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of quasi-nonexpansive self-mappings of C with $F \neq \phi$. Then the sequence $\{x_n\}$, defined by (3.5.1), converges strongly to a common fixed point of the family if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Similar to that of Theorem 3.2.2 and hence is omitted. ■

In the sequel, we obtain some results for a family of nonexpansive mappings $\{T_i : i = 1, 2, \ldots, k\}$ without the condition $\liminf_{n\to\infty} d(x_n, F) = 0$.

Theorem 3.5.7 Let C be a compact convex subset of a strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive selfmappings of C with $F \neq \phi$. Then the sequence $\{x_n\}$, defined by (3.5.1) with $\{\alpha_n\}$ satisfying Condition A (see Section 1.3) and $\beta_{jn} = \beta_j$ for all n and $j = 1, 2, \ldots, k$, converges strongly to a common fixed point of the family.

Proof. It is easy to show that U_j and $T_j U_{j-1}$, $j = 1, 2, \ldots, k$ are nonexpansive selfmappings of C, and the families $\{T_1, \ldots, T_k\}$ and $\{U_1, \ldots, U_k\}$ have the same set of common fixed points.

By Theorem 1.3.7, the sequence $\{x_n\}$ defined by (3.5.1) converges strongly to a fixed point y of $T_k U_{k-1}$. The rest of the proof is similar to that of Theorem 3.5.2 and is omitted. ■

Corollary 3.5.8 [77, Theorem 1]. Let C be a compact convex subset of a strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive selfmappings of C with $F \neq \phi$. Then the iterative sequence $\{x_n\}$ in (1.3.3), converges strongly to a common fixed point of the family.

The following result is an improvement of the theorem of Rhoades [99] in the sense that we use (3.5.1) instead of (1.3.3); the same proof carries over for the modified scheme.

Theorem 3.5.9 Let C be a closed convex subset of a uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive selfmappings of C with $F \neq \phi$. Then the sequence $\{x_n\}$, defined by (3.5.1) with $\{\alpha_n\}$ satisfying Condition A and $\beta_{jn} = \beta_j$ for all n and $j = 1, 2, \ldots, k$, converges weakly to a common fixed point of the family.

Remark 3.5.10 (i) Theorem 3.5.6 is an extension of Corollary 1 of Qihou [96] for a family of quasi-nonexpansive mappings; this corollary of Qihou itself improves Theorem 1.1 and 1.1$'$ of Petryshyn and Williamson [94] and Theorem 3.1 of Ghosh and Debnath [40].

(ii) Theorem 3.5.6 generalizes Theorem 3.5.9 to an arbitrary Banach space setting where the iteration scheme (3.5.1) converges strongly to a common fixed point of a finite family of quasi-nonexpansive mappings.

CHAPTER 4

COINCIDENCES OF LIPSCHITZ TYPE HYBRID MAPPINGS

4.1 Introduction

Pant [88] generalized weak commutativity of mappings to R-weak commutativity and then, in [89], he introduced the notion of pointwise R-weak commuting mappings which extends R-weak commutativity and compatibility, simultaneously. Pant and Pant [91], obtained a unique common fixed point of two noncompatible pointwise R-weakly commuting selfmappings under a strict contractive condition on a metric space.

Singh and Mishra [112] considered the notion of (IT)-commutativity for a hybrid pair of a single-valued and a multivalued mapping and proved that R-weakly commuting hybrid pairs need not be weakly compatible (see [112, Example 1]). Recently, Kamran [64] introduced that the concept "f is T-weakly commuting" for hybrid mappings f and T to generalize (IT)-commuting mappings (see [64, Example 3.8]); then he extended Theorem 1 of Aamri and El Moutawakil [1] to hybrid selfmappings f and T where f is T-weakly commuting. Singh and Hashim [111] generalized the results in [1] for a hybrid pair of (IT)-commuting nonself mappings under strict contractive conditions (see also, Chang [24] and Sastry et al. [102]).

In this chapter, we establish new coincidence and common fixed point results for hybrid mappings (not necessarily continuous) satisfying Lipschitz type conditions on a metric space. Our results extend the results of Kamran [64] and Singh and Hashim [111]. As applications, we demonstrate the existence of common fixed points from the set of best approximations in metric spaces. Further, we provide a solution of an eigenvalue problem for a multivalued mapping on a normed space.

4.2 Coincidences of Hybrid Mappings

We obtain some coincidence and common fixed point theorems for a hybrid pair of mappings (not necessarily continuous) satisfying the property (E.A) and Lipschitz type conditions on a metric space X. We begin with a generalization of Theorems 3.4 and 3.10 of Kamran [64] and Theorem 3.1 due to Singh and Hashim [111] (see also Theorem 2.1 of Pant and Pant [91]); the Lipschitz type condition we use, on the one hand, is simpler than their contractive conditions and on the other hand, contains as a special case the condition due to Pant [90].

Theorem 4.2.1 Let $Y \subseteq X$, $S : Y \to C(X)$ and $f : Y \to X$ be such that:

(i) f and S satisfy the property (E.A); i.e., there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Sx_n = A \in C(X)$ and $\lim_{n\to\infty} fx_n = t \in A$;

(ii) fY is a complete subspace or SY is a complete subspace with $SY \subseteq fY$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in Y$ with $fa = t$, the following Lipschitz type condition holds:

$$\begin{aligned} H(Sx_n, Sa) \quad \leq \quad & (1 + u_n) \max \{r_n d(fx_n, fa), \\ & r_n d(Sx_n, fx_n) + \alpha_n d(Sa, fa), \\ & r_n d(Sx_n, fa) + \alpha_n d(Sa, fx_n)\} \qquad (4.2.1) \end{aligned}$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are sequences in $[0, +\infty)$ with $\lim_{n\to\infty} u_n = 0$, $\lim_{n\to\infty} r_n = r$ and $\lim_{n\to\infty} \alpha_n = \alpha$, for some $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then a is a coincidence point of S and f. Moreover, if $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$, then S and f have a common fixed point.

Proof. If fY is complete, then $\lim_{n\to\infty} fx_n = t = fa$ for some $a \in Y$. We show that $fa \in Sa$ Suppose not; taking the limit as $n \to \infty$ in (4.2.1), we get

$$\begin{aligned} H(A, Sa) &\leq \max\{rd(fa, fa), r\,d(A, fa) + \alpha d(Sa, fa), \\ &\quad r\,d(A, fa) + \alpha d(Sa, fa)\} \\ &= \alpha\, d(Sa, fa). \end{aligned}$$

Since $fa = t \in A$, it follows from the definition of the Hausdorff metric H that

$$d(fa, Sa) \leq H(A, Sa) \leq \alpha d(Sa, fa).$$

Since $0 \leq \alpha < 1$, we get a contradiction. Thus $fa \in Sa$.

Now assume that $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$. Thus $fa = ffa \in Sfa$ and so fa is a common fixed point of S and f. Similarly the case SY is complete and $SY \subseteq fY$ can be verified. ■

The following example shows that our Theorem extends substantially Theorems 3.4 and 3.10 of Kamran [64] and Theorem 3.1 of Singh and Hashim [111].

Example 4.2.2 Let X be the space of usual reals. Define $fx = x^2$ and

$$Sx = \begin{cases} [0, x^3] & \text{if} \quad x \geq 0 \\ [x^3, 0] & \text{if} \quad x < 0 \end{cases}$$

Note that the contractive condition of Theorem 3.1 in [111] is not satisfied; in particular, the contractive condition of Theorems 3.4 and 3.10 in [64] does not hold (take $x = 2$ and $y = 0$). Hence those theorems are not applicable here. Now, f and S satisfy the property (E.A) for the sequences $\left\{\frac{1}{n}\right\}$ and $\left\{1 - \frac{1}{n}\right\}$; in case of $\left\{\frac{1}{n}\right\}$; $t = 0$, $a = 0$ and (4.2.1) is satisfied because

$$H\left(S\left(\frac{1}{n}\right), S(0)\right) = \frac{1}{n^3} \leq \frac{1}{n^2} = d\left(f\left(\frac{1}{n}\right), f(0)\right).$$

Same concerns the case of the sequence $\left\{1 - \frac{1}{n}\right\}$. All the conditions of Theorem 4.2.1 are satisfied and S and f have common fixed points 0 and 1.

Corollary 4.2.3 Let $Y \subseteq X$ and $f, g : Y \to X$ be such that:

(i) f and g satisfy the property (E.A); i.e., there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$;

(ii) fY is a complete subspace or gY is a complete subspace with $gY \subseteq fY$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in Y$ with $fa = t$, the following condition holds:

$$
\begin{aligned}
d(gx_n, ga) \;\; \leq \;\; & (1+u_n)\max\{r_n d(fx_n, fa), \\
& r_n d(gx_n, fx_n) + \alpha_n d(ga, fa), \\
& r_n d(gx_n, fa) + \alpha_n d(ga, fx_n)\}
\end{aligned}
$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are as in the statement of Theorem 4.2.1.

Then a is a coincidence point of f and g. Further, if $fa \in Y$, f and g are weakly compatible and $ffa = fa$, then f and g have a common fixed point.

Proof. By Theorem 4.2.1, a is a coincidence point of f and g. Since f and g are weakly compatible, it follows that

$$ffa = fga = gfa = gga.$$

Thus ga is a common fixed point of f and g. ■

Remark 4.2.4 If the condition (iii) in Corollary 4.2.3 is replaced by the following condition: for all $x, y \in Y$ with $x \neq y$,

$$
\begin{aligned}
d(gx, gy) \;\; < \;\; & \max\{d(fx, fy), r\, d(gx, fx) + \alpha\, d(gy, fy), \\
& \frac{1}{2}[d(gx, fy) + d(gy, fx)]\}
\end{aligned}
$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$, then f and g have a unique common fixed point. Moreover, if $r = \alpha = \frac{1}{2}$, then we obtain Corollary 3.6 of Singh and Hashim [111] which itself is an extension of Theorem 1 of Aamri and El Moutawakil [1].

The following result extends Theorem 4 of Sastry and Murthy [101] which itself is a generalization of the Theorem of Pant [90].

Theorem 4.2.5 Let $Y \subseteq X$ and $f, g : Y \to X$ be such that:

(i) f and g satisfy the property (E.A);

(ii) fY is complete or gY is complete with $gY \subseteq fY$;

(iii) g is f-continuous; i.e., if $fx_n \to fx$, then $gx_n \to gx$ whenever $\{x_n\}$ is a sequence in Y and $x \in Y$.

Then f and g have a coincidence point. Further, if a is a coincidence point of f and g such that $fa \in Y$, f and g are weakly compatible, and

$$d(fa, ffa) \neq \max\{d(fa, gfa), d(ffa, gfa)\}$$

whenever the right hand side is nonzero, then f and g have a common fixed point.

Proof. By (i), there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \text{ for some } t \in X.$$

If fY is complete, then

$$\lim_{n\to\infty} fx_n = fa, \text{ for some } a \in Y.$$

By (iii), $\lim_{n\to\infty} gx_n = ga$. Thus $fa = ga$. Weak compatibility of f and g implies that $fga = gfa$ and so

$$ffa = fga = gfa = gga.$$

Suppose $ffa \neq fa$, then

$$d(fa, ffa) \neq \max\{d(fa, gfa), d(gfa, ffa)\} = d(fa, ffa)$$

a contradiction. Thus $fa = ffa = gfa$; i.e., fa is a common fixed point of f and g. ■

The following theorem concerning four mappings improves upon Theorem 3.2 in [111] (compare the result with [1, Theorem 2] and [91, Theorem 2.3]).

Theorem 4.2.6 Let $Y \subseteq X$, $S, T : Y \to C(X)$ and $f, g : Y \to X$ be such that:

(i) there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} Tx_n = A \in C(X) \text{ and } \lim_{n\to\infty} gx_n = t \in A;$$

(ii) fY or TY is a complete subspace, gY or SY is a complete subspace, $SY \subseteq gY$ and $TY \subseteq fY$;

(iii) for any sequence $\{y_n\}$ in Y with $\lim\limits_{n\to\infty} gy_n = t$ and each $x \in Y$ with $y_n \neq x$, the following condition holds:

$$\begin{aligned} H(Sx, Ty_n) \leq\ & (1+u_n)\max\{r_n d(fx, gy_n), \\ & \alpha_n[d(gy_n, Ty_n) + d(fx, Sx)], \\ & \alpha_n[d(fx, Ty_n) + d(gy_n, Sx)]\} \end{aligned} \tag{4.2.2}$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are sequences in $[0, +\infty)$ with $\lim\limits_{n\to\infty} u_n = 0$, $\lim\limits_{n\to\infty} r_n = r$, $\lim\limits_{n\to\infty} \alpha_n = \alpha$, for some $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then:

(a) f and S have a coincidence point, and g and T have a coincidence point;

(b) if a is a coincidence point of f and S with $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point;

(c) if b is a coincidence point of g and T with $gb \in Y$, g is T-weakly commuting at b and $ggb = gb$, then g and T have a common fixed point;

(d) S, T, f and g have a common fixed point provided that (b) and (c) hold.

Proof. (a) By (i) and $TY \subseteq fY$, there exists a sequence $\{y_n\}$ in Y such that $fy_n \in Tx_n$, for each n, and

$$\lim_{n\to\infty} fy_n = t \in A = \lim_{n\to\infty} Tx_n.$$

We show that $\lim_{n\to\infty} Sy_n = A$. If not,then there exists a subsequence $\{Sy_k\}$ of $\{Sy_n\}$, a positive integer n and a real number $\epsilon > 0$ such that for $k \geq n$, we have $H(Sy_k, A) \geq \epsilon$. From (iii), we get

$$\begin{aligned} H(Sy_k, Tx_k) &\leq (1+u_k)\max\{r_k d(fy_k, gx_k), \\ &\quad \alpha_k\left[d(gx_k, Tx_k) + d(fy_k, Sy_k)\right], \\ &\quad \alpha_k\left[d(fy_k, Tx_k) + d(gx_k, Sy_k)\right]\} \\ &\leq (1+u_k)\max\{r_k d(fy_k, gx_k), \\ &\quad \alpha_k[d(gx_k, Tx_k) + d(fy_k, A) + H(A, Sy_k)], \\ &\quad \alpha_k[d(fy_k, Tx_k) + d(gx_k, A) + H(A, Sy_k)]\}. \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} H(Sy_k, A) \leq \alpha \lim_{k\to\infty} H(A, Sy_k).$$

Since $0 \leq \alpha < 1$, we get a contradiction. Thus

$$\lim_{n\to\infty} Sy_n = A.$$

Consequently, f and S satisfy the property (E.A) for the sequence $\{y_n\}$. If fY or TY is complete, then there exists a point $a \in Y$ such that $\lim_{n\to\infty} fy_n = t = fa$. We show that $fa \in Sa$. If not, then

$$H(Sa, Tx_n) \leq (1 + u_n) \max\{r_n d(fa, gx_n), \alpha_n[d(gx_n, Tx_n) + d(fa, Sa)],$$

$$\alpha_n[d(fa, Tx_n) + d(gx_n, Sa)]\}.$$

Taking the limit as $n \to \infty$, we have

$$H(Sa, A) \leq \alpha\, d(fa, Sa).$$

Thus,

$$d(Sa, fa) \leq H(Sa, A) \leq \alpha\, d(fa, Sa)$$

a contradiction by virtue of $fa = t \in A$. Thus $fa \in Sa$; i.e., a is a coincidence point of f and S. Since $SY \subseteq gY$, therefore there exists a sequence $\{z_n\}$ in Y such that $gz_n \in Sy_n$, for each n, and

$$\lim_{n\to\infty} gz_n = t \in A = \lim_{n\to\infty} Sy_n.$$

As above, we can show that $\lim_{n\to\infty} Tz_n = A$. If gY or SY is complete, then there exists a point $b \in Y$ such that $\lim_{n\to\infty} gz_n = t = gb$. Take the sequence $b_n = b$, for all n, so, $\lim_{n\to\infty} gb_n = t$. Suppose $gb \notin Tb$. Using (iii) and taking the limit as $n \to \infty$, we obtain

$$H(Sa, Tb) \leq \alpha\, d(gb, Tb).$$

Hence

$$d(gb, Tb) = d(fa, Tb) \leq H(Sa, Tb) \leq \alpha\, d(gb, Tb);$$

a contradiction. Thus $gb \in Tb$; i.e., b is a coincidence point of g and T.

(b) Now, if $fa \in Y$, f is S-weakly commuting at a and $ffa = fa$, then

$$fa = ffa \in Sfa$$

and so fa is a common fixed point of f and S.

(c) Similar to case (b).

(d) Immediate, in view of $fa = gb = t$. ■

4.3 Approximation Results

As an application of Theorems 4.2.1, we obtain common fixed points from the set of best approximations in a metric space in the following theorem which extends Theorem 3.14 of Kamran [64].

Theorem 4.3.1 Let $M \subset X$, $u \in X$ and $D = P_M(u)$ be nonempty. Suppose that $f : X \to X$ and $S : X \to C(X)$ satisfy:

(i) there exists a sequence $\{x_n\}$ in D such that

$$\lim_{n\to\infty} Sx_n = A \in C(D) \text{ and } \lim_{n\to\infty} fx_n = t \in A;$$

(ii) fD is complete or SD is complete with $SD \subseteq fD$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in D$ with $fa = t$, (4.2.1) holds.

If D is S-invariant and $fD = D$, then a is a coincidence point of f and S. Further, if f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point in $P_M(u)$.

Proof. Since $SD \subseteq D$, it follows that S maps D into $C(D)$. The result follows from Theorem 4.2.1. ■

The existence of common fixed points from the set of best approximations for four mappings is established in the next result which can be easily verified on the basis of Theorem 4.2.6.

Theorem 4.3.2 Let $M \subset X$, $u \in X$ and $D = P_M(u)$ be nonempty and complete. Assume that $f, g : X \to X$ and $S, T : X \to C(X)$ satisfy:

(i) there exists a sequence $\{x_n\}$ in D such that

$$\lim_{n\to\infty} Tx_n = A \in C(D) \text{ and } \lim_{n\to\infty} gx_n = t \in A;$$

(ii) for any sequence $\{y_n\}$ in D with $\lim_{n\to\infty} gy_n = t$ and each $x \in D$, (4.2.2) holds.

(iii) D is S and T-invariant, $fD = D$ and $gD = D$.

Then:

(a) f and S have a coincidence point $a \in D$, and g and T have a coincidence point $b \in D$;

(b) if f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point from D;

(c) if g is T-weakly commuting at b and $ggb = gb$, then g and T have a common fixed point from D;

(d) S, T, f and g have a common fixed point from D provided that (b) and (c) hold.

Recently, Hussain and Khan [51] obtained in Theorem 3.1, a generalization of Theorem 3 by Sahab et al. [100] for a class of noncommuting single-valued selfmappings of a Hausdorff locally convex space. An improvement of Theorem 3.1 in [51] is given below for hybrid mappings in the setup of a metric space.

Theorem 4.3.3 Let $M \subset X$ and $D = P_M(u)$ be nonempty where u is a common fixed point of the mappings $f, g : X \to X$. Suppose that:

(i) f and g satisfy the property (E.A) on D;

(ii) fD is complete or gD is complete with $gD \subseteq fD$;

(iii) $fD = D$ and $g(\partial M) \subseteq M$ (here ∂M denotes the boundary of M);

(iv) g is f-nonexpansive on $D \bigcup \{u\}$; i.e., $d(gx, gy) \leq d(fx, fy)$, for all $x, y \in D \bigcup \{u\}$ with $x \neq y$.

Then f and g have a coincidence point a in D. Further, if f and g are weakly compatible and $ffa = fa$, then f and g have a unique common fixed point in D.

Proof. Let $y \in D$. Then $fy \in D$. By the definition of $P_M(u)$, $y \in \partial M$ and so $gy \in M$. By (iv), we have

$$d(gy, u) = d(gy, gu) \leq d(fy, fu) = d(fy, u).$$

Now, $gy \in M$ and $fy \in D$ imply that $gy \in D$; consequently, f and g are selfmappings of D. Now, the result follows from Corollary 4.2.3 and Remark 4.2.4. ■

We establish an analogue of Theorem 3.2 in [3] and Theorem 3.3 [51] for hybrid mappings (which need not be continuous) on a metric space.

Theorem 4.3.4 Let $M \subset X$ and $D^* = D_M^f(u)$ be nonempty where u is a common fixed point of the mappings $f : X \to X$ and $S : X \to C(X)$. Suppose that

(i) there exists a sequence $\{x_n\}$ in D^* such that

$$\lim_{n\to\infty} Sx_n = A \in C(D^*) \text{ and } \lim_{n\to\infty} fx_n = t \in A;$$

(ii) fD^* is complete or SD^* is complete with $SD^* \subseteq fD^*$;

(iii) for all $x_n, n = 1, 2, 3, \ldots$ and $a \in D^*$ with $fa = t$, (4.2.1) holds.

If D^* is S-invariant and $fD^* = D^*$, then a is a coincidence point of f and S. Further, if f is S-weakly commuting at a and $ffa = fa$, then f and S have a common fixed point in D^*.

Proof. Since $SD^* \subseteq D^*$, therefore S maps D^* into $C(D^*)$. The result follows from Theorem 4.2.1. ■

For yet another application of Theorem 4.2.1, we solve an eigenvalue problem.

Theorem 4.3.5 Let E be a normed space, $Y \subseteq E$ and $S : Y \to C(E)$ satisfy:

(i) there exists a sequence $\{x_n\}$ in Y such that

$$\lim_{n\to\infty} Sx_n = A \in C(E) \text{ and } \lim_{n\to\infty} \lambda x_n = t \in A$$

where λ is a real number;

(ii) λY is a complete subspace or SY is a complete subspace with $SY \subseteq \lambda Y$;

(iii) for all $x_n, n = 1, 2, 3, \dots$ and $a \in Y$ where $a = t/\lambda$, the following condition holds:

$$H(Sx_n, Sa) \leq (1 + u_n) \max\{r_n ||x_n - a||, r_n \, d(Sx_n, \lambda x_n) + \alpha_n \, d(Sa, \lambda a),$$
$$r_n \, d(Sx_n, \lambda a) + \alpha_n \, d(Sa, \lambda x_n)\}$$

where $\{u_n\}, \{r_n\}$ and $\{\alpha_n\}$ are as in Theorem 4.2.1.

Then S has an eigenvalue.

Proof. Let $f : Y \to E$ be defined by $fx = \lambda x$. Then, by Theorem 4.2.1, $fa \in Sa$; i.e., $\lambda a \in Sa$. Thus λ is an eigenvalue of S and a is the corresponding eigenvector. ■

CHAPTER 5

RANDOM SOLUTIONS

5.1 Introduction

The main purpose of this chapter is to establish random fixed point theorems. Here we provide random analogues of some results from Chapters 2-4. In Section 5.2, we introduce random iteration algorithms which are random versions of the iteration procedures (1.3.3), (3.1.1) and (3.5.1). We study the convergence of these random iterations to a common fixed point of different classes of random operators. Section 5.3 deals with random fixed points of multivalued inward random operators on a separable Banach space with $\epsilon_\alpha(X) < 1$. Finally, in Section 5.4, we obtain random common fixed points theorems in Banach spaces and metric spaces; these results are stochastic versions of Theorems 2.2.1, 2.2.4 and 4.2.1.

5.2 Random Iterative Algorithms

Approximation of random fixed points by iterative processes has been studied by several authors (see, Beg [9], Choudary [26-27], Duan and Li [36]). In 2006, Beg and Abbas [11] studied convergence of different random iterative algorithms for weakly contractive and asymptotically nonexpansive random operators in the setting of a Banach space.

In this section, we extend the iterative procedures (1.3.3), (3.1.1) and (3.5.1) to the random case. It is shown that the random schemes converge to a random common fixed point of the families of quasi-nonexpansive (asymptotically quasi-nonexpansive) random operators in Banach spaces. It is remarked that our random schemes contain as special cases the random Mann iteration (1.5.1), the random Ishikawa iteration (1.5.2) and the three-step random iteration from [11].

Let C be a nonempty closed convex subset of a separable Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of random operators from $\Omega \times C$ to C. Let $\xi_n : \Omega \to C$ be a sequence of mappings where ξ_1 is assumed to be measurable.

We begin with the random version of Kuhfitting iterative scheme (1.3.3):

Definition 5.2.1 Let $0 < \alpha < 1$. For each $\omega \in \Omega$, define

$$\xi_{n+1}(\omega) = (1-\alpha)\xi_n(\omega) + \alpha T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))), \tag{5.2.1}$$

where $U_i : \Omega \times C \to C, i = 1, 2, \ldots, k$, are random operators given by

$$\begin{aligned}
U_0(\omega, \xi_n(\omega)) &= \xi_n(\omega), \\
U_1(\omega, \xi_n(\omega)) &= (1-\alpha)\xi_n(\omega) + \alpha\, T_1(\omega, U_0(\omega, \xi_n(\omega))), \\
U_2(\omega, \xi_n(\omega)) &= (1-\alpha)\xi_n(\omega) + \alpha\, T_2(\omega, U_1(\omega, \xi_n(\omega))), \\
\cdots\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
U_k(\omega, \xi_n(\omega)) &= (1-\alpha)\xi_n(\omega) + \alpha\, T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))),
\end{aligned}$$

for each $\omega \in \Omega$.

We introduce a generalization of the random Ishikawa iteration to the case of a finite family of mappings as follows (note that this scheme is the random case of the iteration scheme (3.5.1)):

Definition 5.2.2 Let $0 < \alpha_n, \beta_{jn} \leq 1$, for all $n = 1, 2, 3, \ldots$ and $j = 1, 2, \ldots, k$. Then for each $\omega \in \Omega$, define

$$\xi_{n+1}(\omega) = (1-\alpha_n)\xi_n(\omega) + \alpha_n T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))), \tag{5.2.2}$$

where $U_i : \Omega \times C \to C, i = 1, 2, \ldots, k$, are random operators given by

$$\begin{aligned}
U_0(\omega, \xi_n(\omega)) &= \xi_n(\omega), \\
U_1(\omega, \xi_n(\omega)) &= (1-\beta_{1n})\xi_n(\omega) + \beta_{1n}T_1(\omega, U_0(\omega, \xi_n(\omega))), \\
U_2(\omega, \xi_n(\omega)) &= (1-\beta_{2n})\xi_n(\omega) + \beta_{2n}\, T_2(\omega, U_1(\omega, \xi_n(\omega))), \\
\cdots\cdots\cdots\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
U_k(\omega, \xi_n(\omega)) &= (1-\beta_{kn})\xi_n(\omega) + \beta_{kn}\, T_k(\omega, U_{k-1}(\omega, \xi_n(\omega))),
\end{aligned}$$

for each $\omega \in \Omega$.

Remark 5.2.3 If we take $k = 2$ and $T_1 = T_2 = T$ in (5.2.2), then we get the random Ishikawa iterative scheme (1.5.2).

The random version of the iterative scheme (3.1.1) is given in the following:

Definition 5.2.4 Let $0 \le \alpha_{in} \le 1$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$. Then, for each $\omega \in \Omega$, define

$$\xi_{n+1}(\omega) = (1-\alpha_{kn})\xi_n(\omega) + \alpha_{kn}T_k^n(\omega, U_{k-1}(\omega, \xi_n(\omega))), \tag{5.2.3}$$

where $U_i : \Omega \times C \to C, i = 1, 2, \ldots, k$, are random operators given by

$$\begin{array}{rcl}
U_0(\omega, \xi_n(\omega)) & = & \xi_n(\omega), \\
U_1(\omega, \xi_n(\omega)) & = & (1-\alpha_{1n})\xi_n(\omega) + \alpha_{1n} T_1^n(\omega, U_0(\omega, \xi_n(\omega))), \\
U_2(\omega, \xi_n(\omega)) & = & (1-\alpha_{2n})\xi_n(\omega) + \alpha_{2n} T_2^n(\omega, U_1, (\omega, \xi_n(\omega))), \\
\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots\ldots \\
U_k(\omega, \xi_n(\omega)) & = & (1-\alpha_{k_n})\xi_n(\omega) + \alpha_{k_n} T_k^n(\omega, U_{k-1}(\omega, \xi_n(\omega))),
\end{array}$$

for each $\omega \in \Omega$.

First we present an analogue of Theorem 1 in [53] for random operators in the following:

Lemma 5.2.5 Let C be a nonempty compact convex subset of a separable Banach space X, and $T : \Omega \times C \to C$ a nonexpansive random operator. Then T has a random fixed point and $\{\xi_n\}$, defined by (1.5.3), converges strongly to a fixed point of T.

Proof. The compactness of C implies that $\{\xi_n\}$ has a convergent subsequence $\{\xi_{n_k}\}$. Assume that

$$\xi_{n_k}(\omega) \to \zeta(\omega), \text{ for each}\, \omega \in \Omega. \tag{5.2.4}$$

By Proposition 1.5.7, we have

$$\lim_{k\to\infty} \|\xi_{n_k}(\omega) - T(\omega, \xi_{n_k}(\omega))\| = 0, \text{ for each}\, \omega \in \Omega.$$

We utilize nonexpansiveness of T to obtain, for each $\omega \in \Omega$,

$$\begin{aligned}
\|T(\omega,\zeta(\omega))-\zeta(\omega)\| &\leq \|T(\omega,\zeta(\omega))-T(\omega,\xi_{n_k}(\omega))\| \\
&\quad +\|T(\omega,\xi_{n_k}(\omega))-\xi_{n_k}(\omega)\|+\|\xi_{n_k}(\omega)-\zeta(\omega)\| \\
&\leq \|\zeta(\omega)-\xi_{n_k}(\omega)\|+\|T(\omega,\xi_{n_k}(\omega))-\xi_{n_k}(\omega)\| \\
&\quad +\|\xi_{n_k}(\omega)-\zeta(\omega)\| \\
&= 2\|\zeta(\omega)-\xi_{n_k}(\omega)\|+\|T(\omega,\xi_{n_k}(\omega))-\xi_{n_k}(\omega)\|.
\end{aligned}$$

Taking the limit as $k \to \infty$, we get

$$T(\omega,\zeta(\omega))=\zeta(\omega), \text{ for each } \omega \in \Omega.$$

Moreover,

$$\begin{aligned}
\|\xi_{n+1}(\omega)-\zeta(\omega)\| &= \|(1-\alpha_n)\xi_n(\omega)+\alpha_n T(\omega,\xi_n(\omega))-\zeta(\omega)\| \\
&\leq (1-\alpha_n)\|\xi_n(\omega)-\zeta(\omega)\| \\
&\quad +\alpha_n\|T(\omega,\xi_n(\omega))-T(\omega,\zeta(\omega))\| \\
&\leq (1-\alpha_n)\|\xi_n(\omega)-\zeta(\omega)\|+\alpha_n\|\xi_n(\omega)-\zeta(\omega)\| \\
&= \|\xi_n(\omega)-\zeta(\omega)\|, \qquad (5.2.5)
\end{aligned}$$

for each $\omega \in \Omega$ and any positive integer n. From (5.2.4), it follows that for any $\epsilon > 0$, there exists an integer n_0 such that

$$\|\xi_{n_0}(\omega)-\zeta(\omega)\| < \epsilon,$$

for each $\omega \in \Omega$. Therefore, by (5.2.5), we get

$$\|\xi_n(\omega) - \zeta(\omega)\| < \epsilon,$$

for any integer $n \geq n_0$ and each $\omega \in \Omega$. Since ϵ is arbitrary, therefore

$$\xi_n(\omega) \to \zeta(\omega),$$

for each $\omega \in \Omega$. The mapping $\zeta : \Omega \to C$, being the limit of a sequence of measurable mappings, is also measurable. Thus ζ is a random fixed point of T. ■

The following result generalizes Theorem 1 of Khufittig [77] for random operators.

Theorem 5.2.6 Let C be a nonempty compact convex subset of a separable strictly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpansive random operators from $\Omega \times C$ to C with $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$. Then $\{\xi_n\}$, defined by (5.2.1), converges strongly to a random common fixed point of the family.

Proof. It is easy to see that $\xi : \Omega \to C$ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$ if and only if ξ is a random common fixed point of $\{U_i : i = 1, 2, \ldots, k\}$, for each $\omega \in \Omega$.

Define $S_i : \Omega \times C \to C$ by

$$S_i(\omega, x) = T_i(\omega, U_{i-1}(\omega, x)), i = 1, 2, 3, \ldots, k.$$

Obviously, U_i and $S_i, i = 1, 2, \ldots, k$, are nonexpansive.

By Lemma 5.2.5, $\{\xi_n\}$ defined by (5.2.1), converges strongly to a random fixed point $\zeta : \Omega \to C$ of S_k. Next we show that ζ is a random common fixed point of T_k and $U_{k-1}(k \geq 2)$. For this, we first show that ζ is a random fixed point of S_{k-1}. Suppose not; then the closed line segment $[\zeta(\omega), S_{k-1}(\omega, \zeta(\omega))]$ has positive length for some $\omega \in \Omega$. Assume that

$$\begin{aligned} \psi(\omega_1) &= U_{k-1}(\omega_1, \zeta(\omega_1)) \\ &= (1-\alpha)\zeta(\omega_1) + \alpha T_{k-1}(\omega_1, U_{k-2}(\omega_1, \zeta(\omega_1))), \end{aligned}$$

for some $\omega_1 \in \Omega$. Since $\{T_i : i = 1, 2, \ldots, k\}$ and $\{U_i : i = 1, 2, \ldots, k\}$ have the same random common fixed points and $D \neq \phi$, therefore

$$S_{k-1}(\omega_1, \theta(\omega_1)) = T_{k-1}(\omega_1, U_{k-2}(\omega_1, \theta(\omega_1))) = \theta(\omega_1),$$

where $\theta \in D$. By the nonexpansiveness of S_{k-1} and T_k, we have

$$\begin{aligned} \|S_{k-1}(\omega_1, \zeta(\omega_1)) - \theta(\omega_1)\| &= \|S_{k-1}(\omega_1, \zeta(\omega_1)) - S_{k-1}(\omega_1, \theta(\omega_1))\| \\ &\leq \|\zeta(\omega_1) - \theta(\omega_1)\| \end{aligned} \tag{5.2.6}$$

and

$$\begin{aligned} \|T_k(\omega, \psi(\omega_1)) - \theta(\omega_1)\| &= \|T_k(\omega_1, \psi(\omega_1)) - T_k(\omega_1, \theta(\omega_1))\| \\ &\leq \|\psi(\omega_1) - \theta(\omega_1)\| \end{aligned}$$

(5.2.7)

In view of

$$T_k(\omega_1, \psi(\omega_1)) = T_k(\omega_1, U_{k-1}(\omega_1, \zeta(\omega_1))) = S_k(\omega_1, \zeta(\omega_1)) = \zeta(\omega_1),$$

it follows, by (5.2.7), that

$$\|\zeta(\omega_1) - \theta(\omega_1)\| \leq \|\psi(\omega_1) - \theta(\omega_1)\|.$$

As X is strictly convex, we obtain

$$\begin{aligned}
\|\zeta(\omega_1) - \theta(\omega_1)\| &\leq \|\psi(\omega_1) - \theta(\omega_1)\| \\
&= \|(1-\alpha)\zeta(\omega_1) + \alpha T_{k-1}(\omega_1, U_{k-2}(\omega_1, \zeta(\omega_1))) \\
&\quad -(1-\alpha)\theta(\omega_1) - \alpha\theta(\omega_1)\| \\
&< (1-\alpha)\|\zeta(\omega_1) - \theta(\omega_1)\| \\
&\quad +\alpha\|T_{k-1}(\omega_1, U_{k-2}(\omega_1, \zeta(\omega_1))) - \theta(\omega_1)\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\zeta(\omega_1) - \theta(\omega_1)\| &< \|T_{k-1}(\omega_1, U_{k-2}(\omega_1, \zeta(\omega_1))) - \theta(\omega_1)\| \\
&= \|S_{k-1}(\omega_1, \zeta(\omega_1)) - \theta(\omega_1)\|
\end{aligned}$$

which contradicts (5.2.6).

Hence, ζ is a random fixed point of S_{k-1}. Subsequently;

$$U_{k-1}(\omega, \zeta(\omega)) = (1-\alpha)\zeta(\omega) + \alpha T_{k-1}(\omega, U_{k-2}(\omega, \zeta(\omega))) = \zeta(\omega),$$

for each $\omega \in \Omega$, and so

$$\zeta(\omega) = S_k(\omega, \zeta(\omega)) = T_k(\omega, U_{k-1}(\omega, \zeta(\omega))) = T_k(\omega, \zeta(\omega)),$$

for each $\omega \in \Omega$. Thus ζ is a random common fixed point of T_k and U_{k-1}.

Since $T_{k-1}(\omega, U_{k-2}(\omega, \zeta(\omega))) = \zeta(\omega)$, for each $\omega \in \Omega$, we may repeat the above procedure to show that $T_{k-2}(\omega, U_{k-3}(\omega, \zeta(\omega))) = \zeta(\omega)$, for each $\omega \in \Omega$, and thereby ζ must be a random common fixed point of T_{k-1} and U_{k-2}. Continuing in this manner, we conclude that $T_1(\omega, U_0(\omega, \zeta(\omega))) = \zeta(\omega)$, for each $\omega \in \Omega$ and ζ is a random common fixed point of T_2 and U_1. Consequently, ζ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. ■

The weak convergence result of the Theorem of Rhoades [99], is established for random operators in the following result.

Theorem 5.2.7 Let C be a nonempty bounded closed convex subset of a separable uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of nonexpnsive random operators from $\Omega \times C$ to C with $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$. Then $\{\xi_n\}$, defined by (5.2.1), converges weakly to a random common fixed point of the family.

Proof. Suppose that the mappings $S_i, i = 1, 2, \ldots, k$, are defined as in the proof of Theorem 5.2.6. Since X is uniformly convex, therefore it is reflexive and so, $\{\xi_n\}$ has a subsequence $\{\xi_{n_j}\}$ converging weakly to $\zeta : \Omega \to C$.

Now by Proposition 1.5.7,

$$\lim_{j\to\infty} \|\xi_{n_j}(\omega) - S_k(\omega, \xi_{n_j}(\omega))\| = 0, \quad \text{for each} \quad \omega \in \Omega.$$

Hence by Lemma 1.2.8, we get

$$S_k(\omega, \zeta(\omega)) = \zeta(\omega), \quad \text{for each} \quad \omega \in \Omega.$$

That is, ζ is a random fixed point of S_k.

A uniformly convex space is strictly convex, so one can use the corresponding arguments of the proof of Theorem 5.2.6 to show that ζ is a random common fixed point of $\{T_i : i = 1, 2, \ldots, k\}$. ■

In what follows, we provide a random form of Lemma 3.2.1.

Lemma 5.2.8 Let C be a nonempty closed convex subset of a separable Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of asymptotically quasi-nonexpansive continuous random operators from $\Omega \times C$ to C, i.e.,

$$\|T_i^n(\omega, \eta(\omega)) - \xi_i(\omega)\| \le (1 + u_{in}(\omega))\|\eta(\omega) - \xi_i(\omega)\|,$$

for each $\omega \in \Omega$, where $\xi_i : \Omega \to C$ is a random fixed point of T_i for each $i, \eta : \Omega \to C$ is any measurable mapping, and $u_{in} : \Omega \to [0, \infty)$ a sequence of measurable mappings with $\lim_{n\to\infty} u_{in}(\omega) = 0$, for each $\omega \in \Omega$ and $i = 1, 2, \ldots, k$. Assume that $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$ and $\sum_{n=1}^{\infty} u_{in}(\omega) < +\infty$, for each $\omega \in \Omega$ and $i = 1, 2, \ldots, k$. Let $\{\xi_n\}$ be as in (5.2.3) and $\zeta \in D$. Then

(i) there exists a sequence of measurable mappings $\nu_n : \Omega \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \nu_n(\omega) < +\infty$, for each $\omega \in \Omega$, and

$$\|\xi_{n+1}(\omega) - \zeta(\omega)\| \leq (1 + \nu_n(\omega))^k \|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $n = 1, 2, 3, \ldots$;

(ii) there exists $M > 0$ (depending on ω) such that

$$\|\xi_{n+m}(\omega) - \zeta(\omega)\| \leq M \|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $n, m = 1, 2, 3, \ldots$.

Proof. (i) Let $\nu_n : \Omega \to [0, \infty)$ be defined by

$$\nu_n(\omega) = \max_{1 \leq i \leq k} u_{in}(\omega), \text{ for each}\, \omega \in \Omega \,\text{and}\, n = 1, 2, 3, \ldots$$

Clearly, $\{\nu_n\}$ is a sequence of measurable mappings. Since $\sum_{n=1}^{\infty} u_{in}(\omega) < +\infty$, for each i and $\omega \in \Omega$, therefore

$$\sum_{n=1}^{\infty} \nu_n(\omega) < +\infty, \text{ for each}\, \omega \in \Omega.$$

As in the proof of Lemma 3.2.1, we can prove, by induction, that

$$\|U_{in}(\omega, \xi_n(\omega)) - \zeta(\omega)\| \leq (1 + \nu_n(\omega))^i \|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $1 \leq i \leq k-1$. Thus

$$\begin{aligned}
\|\xi_{n+1}(\omega) - \zeta(\omega)\| &\leq (1-\alpha_{kn})\|\xi_n(\omega) - \zeta(\omega)\| \\
&\quad +\alpha_{kn}\|T_k^n(\omega, U_{k-1}(\omega, \xi_n(\omega))) - \zeta(\omega)\| \\
&\leq (1-\alpha_{kn})\|\xi_n(\omega) - \zeta(\omega)\| \\
&\quad +\alpha_{kn}(1 + u_{kn}(\omega))(1+\nu_n(\omega))^{k-1}\|\xi_n(\omega) - \zeta(\omega)\| \\
&\leq (1+\alpha_{kn})||\xi_n(\omega) - \zeta(\omega)\| \\
&\quad +\alpha_{kn}(1+\nu_n(\omega))^k\|\xi_n(\omega) - \zeta(w)\| \\
&\leq (1+\nu_n(\omega))^k\|\xi_n(\omega) - \zeta(\omega)\|,
\end{aligned}$$

for each $\omega \in \Omega$ and $n = 1, 2, 3, \ldots$.

(ii) If $t \geq 0$, then $1 + t \leq e^t$ and so, $(1+t)^k \leq e^{kt}, k = 1, 2, \ldots$. Thus from part (i), for each $\omega \in \Omega$, we get

$$\begin{aligned}
\|\xi_{n+m}(\omega) - \zeta(\omega)\| &\leq (1+\nu_{n+m-1}(\omega))^k\|\xi_{n+m-1}(\omega) - \zeta(\omega)\| \\
&\leq \exp\{k\nu_{n+m-1}(\omega)\}\|\xi_{n+m-1}(\omega) - \zeta(\omega)\| \leq \ldots. \\
&\leq \exp\left\{k\sum_{i=n}^{n+m-1}\nu_i(\omega)\right\}\|\xi_n(\omega) - \zeta(\omega)\| \\
&\leq \exp\left\{k\sum_{i=1}^{\infty}\nu_i(\omega)\right\}\|\xi_n(\omega) - \zeta(\omega)\|.
\end{aligned}$$

Let $\exp\left\{k\sum_{i=1}^{\infty}\nu_i(\omega)\right\} = M$ (depending on ω). Thus

$$\|\xi_{n+m}(\omega) - \zeta(\omega)\| \leq M\|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $n, m = 1, 2, 3, \ldots.$ ■

As an application of Lemma 5.2.8, we obtain a strong convergence result for a family of random operators.

Theorem 5.2.9 Under the assumptions of Lemma 5.2.8, the random iteration scheme $\{\xi_n\}$, defined by (5.2.3), converges strongly to a random common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$ if and only if $\lim_{n\to\infty} \inf\ d(\xi_n(\omega), D) = 0$.

Proof. We will only prove the sufficiency, the necessity is obvious. From Lemma 5.2.8, we have

$$\|\xi_{n+1}(\omega) - \zeta(\omega)\| \leq (1 + \nu_n(\omega))^k\|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $n = 1, 2, 3, \ldots$, where $\zeta : \Omega \to C$ is a random common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$. Therefore

$$\begin{aligned} d(\xi_{n+1}(\omega), D) &\leq (1 + \nu_n(\omega))^k d(\xi_n(\omega), D) \\ &= \left(1 + \sum_{r=1}^{k} \frac{k(k-1)\ldots(k-r+1)}{r!}\nu_n^r(\omega)\right) d(\xi_n(\omega), D). \end{aligned}$$

¿From $\sum_{n=1}^{\infty} \nu_n < +\infty$, for each $\omega \in \Omega$, we have

$$\sum_{n=1}^{\infty} \sum_{r=1}^{k} \frac{k(k-1)\dots(k-t+1)}{r!} \nu_n^r(\omega) < +\infty, \quad \text{for each} \quad \omega \in \Omega.$$

Thus, by Lemma 1.3.5, we have

$$\lim_{n\to\infty} d(\xi_n(\omega), D) = 0.$$

As in the proof of Theorem 3.2.2, we can prove that $\{\xi_n\}$ is a Cauchy sequence and hence it is convergent. So there exists a measurable mapping $\psi : \Omega \to C$ such that $\lim_{n\to\infty} \xi_n(\omega) = \psi(\omega)$, for each $\omega \in \Omega$. Now we can easily prove that $\psi \in D$ by using the arguments used in the proof of Theorem 3.2.2. ■

An analogue of this theorem for a finite family of quasi-nonexpansive random operators may similarly be verified. We include its statement for completeness sake.

Theorem 5.2.10. Let C be a nonempty closed convex subset of a separable Banach space X, and $\{T_i : i = 1, 2, \dots, k\}$ a family of quasi-nonexpansive continuous random operators from $\Omega \times C$ to C with $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$. Then the generalized random Ishikawa iteration (5.2.2), converges strongly to a random common fixed point of the family if and only if $\lim_{n\to\infty} \inf\ d(\xi_n(\omega), D) = 0$.

Corollary 5.2.11 Under the assumptions of Lemma 5.2.8, $\{\xi_n\}$ defined by

(5.2.3), converges strongly to $\zeta \in D$ if and only if there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ which converges to ζ.

A stochastic version of Lemma 3.3.1 is obtained in the result to follow.

Lemma 5.2.12 Let C be a nonempty closed convex subset of a separable uniformly convex Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ a family of $(L - \gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive random operators from $\Omega \times C \to C$ with the sequence of measurable mappings $u_{in} : \Omega \to [0, \infty)$ satisfying $\sum_{n=0}^{\infty} u_{in}(\omega) < +\infty$, for each $\omega \in \Omega, i = 1, 2, \ldots, k$. Suppose that $D = \bigcap_{i=1}^{k} RF(T_i) \neq \phi$ and the sequence $\{\xi_n\}$ is as in (5.2.3), where $\alpha_{in} \in [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$. Then for each $\omega \in \Omega$,

(i) $\lim_{n\to\infty} \|\xi_n(\omega) - \zeta(\omega)\|$ exists for each $\zeta \in D$;

(ii) $\lim_{n\to\infty} \|\xi_n(\omega) - T_j^n(\omega, U_{j-1}(\omega, \xi_n(\omega)))\| = 0, j = 1, 2, \ldots, k$;

(iii) $\lim_{n\to\infty} \|\xi_n(\omega) - T_j(\omega, \xi_n(\omega))\| = 0, j = 1, 2, \ldots, k$.

Proof. Let $\zeta \in D$ and $\nu_n : \Omega \to [0, \infty)$ be defined by

$$\nu_n(\omega) = \max_{i \leq i \leq k} u_{in}(\omega), \quad \text{for each} \quad \omega \in \Omega \quad \text{and} \quad n = 1, 2, 3, \ldots.$$

(i) By Lemma 5.2.8 (i), we have

$$\|\xi_{n+1}(\omega) - \zeta(w)\| \leq \left(1 + \sum_{r=1}^{k} \frac{k(k-1)\ldots(k-r+1)}{r!}\nu_n^r(\omega)\right) \|\xi_n(\omega) - \zeta(\omega)\|,$$

for each $\omega \in \Omega$ and $n = 1, 2, 3, \ldots$. Since $\sum_{n=1}^{\infty} \nu_n(\omega) < +\infty$, for each $\omega \in \Omega$, therefore

$$\sum_{n=1}^{\infty} \sum_{r=1}^{k} \frac{k(k-1)\ldots(k-r+1)}{r!} \nu_n^r(\omega) < +\infty, \quad \text{for each}\, \omega \in \Omega.$$

Thus, from Lemma 1.3.5, it follows that $\lim_{n\to\infty} \|\xi_n(\omega) - \zeta(\omega)\|$ exists, for each $\omega \in \Omega$ and all $\zeta \in D$.

The proofs of (ii) and (iii) are similar to their counter parts in Lemma 3.3.1 and so are omitted. ■

We now establish weak and strong convergence of the random iteration (5.2.3) to a common fixed point of the family.

Theorem 5.2.13 Under the hypotheses of Lemma 5.2.12, assume that X satisfies Opial's property and $I - T_i$ is a demiclosed random operator at 0 for each $i = 1, 2, \ldots, k$. Then the sequence $\{\xi_n\}$, defined by (5.2.3), converges weakly to a random common fixed point of the family.

Proof. Let $\zeta \in D$. Then, by Lemma 5.2.12(i), $\lim_{n\to\infty} \|\xi_n(\omega) - \zeta(\omega)\|$ exists for each $\omega \in \Omega$, and hence $\{\xi_n\}$ is bounded. Since X is reflexive, there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ converging weakly to a measurable mapping $\eta_1 : \Omega \to C$, for each $\omega \in \Omega$. By Lemma 5.2.12(iii) and the demiclosedness of

$I - T_i$, we obtain

$$T_i(\omega, \eta_1(\omega)) = \eta_1(\omega), \text{ for each}\, \omega \in \Omega.$$

Thus, $\eta_1 \in D$. In order to show that $\{\xi_n\}$ converges weakly to η_1, take another subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ converging weakly to a measurable mapping $\eta_2 : \Omega \to C$, for each $\omega \in \Omega$. Again, as above, $\eta_2 \in D$. Assume that $\eta_1(\omega) \neq \eta_2(\omega)$ for some $\omega \in \Omega$. Then, by Opial's property, we have

$$\begin{aligned}
\lim_{n\to\infty} \|\xi_n(\omega) - \eta_1(\omega)\| &= \lim_{n_j\to\infty} \|\xi_{n_j}(\omega) - \eta_1(\omega)\| \\
&< \lim_{n_j\to\infty} \|\xi_{n_j}(\omega) - \eta_2(\omega)\| \\
&= \lim_{n\to\infty} \|\xi_n(\omega) - \eta_2(\omega)\| \\
&= \lim_{n_k\to\infty} \|\xi_{n_k}(\omega) - \eta_2(\omega)\| \\
&< \lim_{n_k\to\infty} \|\xi_{n_k}(\omega) - \eta_1(\omega)\| \\
&= \lim_{n\to\infty} \|\xi_n(\omega) - \eta_1(\omega)\|.
\end{aligned}$$

This contradiction proves that $\eta_1(\omega) = \eta_2(\omega)$, for each $\omega \in \Omega$. Thus $\{\xi_n\}$ converges weakly to a random common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$. ■

Theorem 5.2.14. Under the hypotheses of Lemma 5.2.12, assume that, for some $1 \leq j \leq k$ and a positive integer m, T_j^m is semi-compact. Then $\{\xi_n\}$, defined by (5.2.3), converges strongly to a random common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$.

Proof. By Lemma 5.2.12 (iii), we obtain

$$\begin{aligned}
\|T_j^m(\omega,\xi_n(\omega))-\xi_n(\omega)\| &\leq \|T_j^m(\omega,\xi_n(\omega))-T_j^{m-1}(\omega,\xi_n(\omega))\| \\
&\quad +\|T_j^{m-1}(\omega,\xi_n(\omega))-T_j^{m-2}(\omega,\xi_n(\omega))\| \\
&\quad +\ldots+\|T_j^2(\omega,\xi_n(\omega))-T_j(\omega,\xi_n(\omega))\| \\
&\quad +\|T_j(\omega,\xi_n(\omega))-\xi_n(\omega)\| \\
&\leq (m-1)L\|T_j(\omega,\xi_n(\omega))-\xi_n(\omega)\|^\gamma \\
&\quad +\|T_j(\omega,\xi_n(\omega))-\xi_n(\omega)\| \to 0.
\end{aligned}$$

Since $\{\xi_n\}$ is bounded and T_j^m is semi-compact, $\{\xi_n\}$ has a convergent subsequence $\{\xi_{n_k}\}$ converging to a measurable mapping $\eta:\Omega\to C$. Hence, again by Lemma 5.2.12 (iii), we have

$$\|\eta(\omega)-T_i(\omega,\eta(\omega))\|=\lim_{n\to\infty}\|\xi_{n_k}(\omega)-T_i(\omega,\xi_{n_k}(\omega))\|=0,$$

for each $\omega\in\Omega$ and $i=1,2,\ldots,k$. Thus $\eta\in D$ and so by Corollary 5.2.11, $\{\xi_n\}$ converges to η. ■

Remark 5.2.15 On the lines of the proofs of Lemma 5.2.12, Theorem 5.2.13 and Theorem 5.2.14, we can easily prove analogues of these results for a family of asymptotically nonexpansive random operators instead of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive random operators.

5.3 Multivalued Inward Random Operators

One of the most general fixed point theorems for multivalued nonexpansive selfmappings has been obtained by Kirk and Massa [75], in 1990, proving the existence of fixed points in Banach spaces for which the asymptotic center of a bounded sequence in a closed bounded convex subset is nonempty and compact. This occurs, for instance, if X is a uniformly convex space, but it is known that (see [76]) when X is nearly uniformly convex, the asymptotic center of a bounded sequence can be a noncompact set. This fact forced Benavides and Ramirez [16] to generalize Kirk-Massa theorem to a class of Banach spaces where the asymptotic center of a sequence is not necessarily a compact set. Specifically, they gave a fixed point theorem for a multivalued nonexpansive and $1-\chi$-contractive compact convex valued selfmapping of a Banach space whose characteristic of noncompact convexity associated to the separation meausre of noncompactness is less than 1. In 2004, Benavides and Ramirez [17] obtained results similar to those in [16] for nonself mappings satisfying the inwardness condition. In particular, they proved the following result.

Theorem 5.3.1 Let X be a Banach space such that $\epsilon_\alpha(X) < 1$ and C be a closed bounded convex subset of X. If $T : C \to KC(X)$ is a nonexpansive and $1-\chi$-contractive mapping such that $T(C)$ is a bounded set, and which satisfies $Tx \subset I_C(x)$, for all $x \in C$, then T has a fixed point.

Shahzad and Lone [109] extended the above theorem for SL mappings (see Definition 1.4.18), and Plubtieng and Kumam [95] have obtained a random analogue of Theorem 5.3.1.

In this section, we randomize the results of Shahzad and Lone [109]; incidently our results extend the work of Plubtieng and Kumam [95].

Lemma 5.3.2. Let C be a nonempty closed bounded convex subset of a Banach space X. Suppose that $T : C \to KC(X)$ is an SL mapping such that

$$Tx \subset I_C(x), \text{ for all } x \in C.$$

If $\{x_n\}$ is a sequence in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then there exists an ultranet $\{x_{n_\alpha}\}$ of $\{x_n\}$ such that

$$Tx \bigcap I_A(x) \neq \phi, \text{ for all } x \in A,$$

where $A = A\left(C, \{x_{n_\alpha}\}\right)$.

Proof. Let $\{n_\alpha\}$ be an ultranet of the positive integers $\{n\}$. The compactness of Tx_{n_α} implies that for each n_α, we can take $y_{n_\alpha} \in Tx_{n_\alpha}$ such that

$$\|x_{n_\alpha} - y_{n_\alpha}\| = d\left(x_{n_\alpha}, Tx_{n_\alpha}\right).$$

By the compactness of Tx, for each $x \in A$, there exists a sequence $\{z_{n_\alpha}\}$ in Tx such that

$$\lim_{\alpha \to \infty} z_{n_\alpha} = z \in Tx$$

and

$$\begin{aligned} \|y_{n_\alpha} - z_{n_\alpha}\| &= d\left(y_{n_\alpha}, Tx\right) \\ &\leq H\left(Tx_{n_\alpha}, Tx\right). \end{aligned}$$

We show that $z \in I_A(x)$. As T is an SL mapping and $\{x_{n_\alpha}\}$ is asymptotically T-regular, so we have

$$\lim_{\alpha \to \infty} \sup H\left(Tx_{n_\alpha}, Tx\right) \leq \lim_{\alpha \to \infty} \sup \|x_{n_\alpha} - x\|$$

for all $x \in A$. Thus, we obtain

$$\begin{aligned} \lim_{\alpha \to \infty} \|x_{n_\alpha} - z\| &= \lim_{\alpha \to \infty} \|y_{n_\alpha} - z_{n_\alpha}\| \\ &\leq \lim_{\alpha \to \infty} \sup H\left(Tx_{n_\alpha}, Tx\right) \\ &\leq \lim_{\alpha \to \infty} \sup \|x_{n_\alpha} - x\| \\ &= r \end{aligned}$$

where $r = r\left(C, \{x_{n_\alpha}\}\right)$. As $z \in Tx \subseteq I_C(x)$, so there exist $\lambda \geq 0$ and $\nu \in C$ such that

$$z = x + \lambda(\nu - x).$$

If $\lambda \leq 1$, then by the convexity of $C, z \in C$ and hence, by the definition of $A, z \in A \subseteq I_A(x)$. Now, assume that $\lambda > 1$, then we can write

$$\nu = \frac{1}{\lambda} z + \left(1 - \frac{1}{\lambda}\right) x.$$

Thus

$$\lim_{\alpha \to \infty} \|x_{n_\alpha} - \nu\| \leq \frac{1}{\lambda} \lim_{\alpha \to \infty} \|x_{n_\alpha} - z\| + \left(1 - \frac{1}{\lambda}\right) \lim_{\alpha \to \infty} \|x_{n_\alpha} - x\| \leq r.$$

This implies that $\nu \in A$ and so $z \in I_A(x)$. ■

The following result, on the one hand, is a stochastic version of Theorem 3.3 of Shahzad and Lone [109] (which itself extends Theorem 3.4 of Benavides and Ramirez [17]), and on the other hand, improves Theorem 3.3 of Plubtieng and Kumam [95].

Theorem 5.3.3. Let C be a nonempty closed bounded convex separable subset of a Banach space X with $\epsilon_\alpha(X) < 1$ and $T : \Omega \times C \to KC(X)$, a continuous $1-\chi-$contractive and SL random operator. If T satisfies the inwardness condition

$$T(\omega, x) \subset I_C(x), \text{ for all}\, x \in C \,\text{and}\, \omega \in \Omega,$$

then T has a random fixed point.

Proof. Let $x_0 \in C$ be fixed and consider the measurable mapping $x_0(\omega) =$

x_0, for all $\omega \in \Omega$. For each $n \geq 1$, define $T_n(\omega, .) : C \to KC(X)$ by

$$T_n(\omega, x) = \frac{1}{n}x_0(\omega) + \left(1 - \frac{1}{n}\right) T(\omega, x), \text{ for all}\, x \in C.$$

Then T_n is contractive and $T_n(\omega, x) \subset I_C(x)$, for all $x \in C$. Hence, by Lemma 1.4.11, each T_n has a deterministic fixed point $z_n(\omega) \in C$. So,

$$d\left(z_n(\omega), T(\omega, z_n(\omega))\right) \leq \frac{1}{n} \text{ diam } C \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$F_n(\omega) = \left\{ x \in C : d(x, T(\omega, x)) \leq \frac{1}{n} \operatorname{diam} C \right\}$$

is nonempty closed and convex. By Lemma 1.5.12, each F_n is measurable. Then, by Lemma 1.5.9, F_n admits a measurable selector $x_n(\omega)$ such that

$$d(x_n(\omega), T(\omega, x_n(\omega))) \leq \frac{1}{n} \operatorname{diam} C \to 0 \quad \text{as} \quad n \to \infty.$$

Define $\quad f : \Omega \times C \to [0, \infty)$ by

$$f(\omega, x) = \lim_{n\to\infty} \sup \|x_n(\omega) - x\|, x \in C.$$

It is easy to see that $f(\omega, .)$ is continuous and $f(., x)$ is measurable, so by Lemma 1.5.11, $f(., x)$ is measurable. Obviously, $f(\omega, .)$ is convex. Therefore, by Lemma 1.4.5, it is weakly lower semicontinuous. Note that, $\epsilon_\alpha(X) < 1$, so X is reflexive. Therefore, C is weakly compact (see[129]). Hence, by Lemma 1.5.13, the marginal function

$$r(\omega) = \inf_{x \in C} f(\omega, x)$$

and the marginal mapping

$$A(\omega) = \{x \in C : f(\omega, x) = r(\omega)\}$$

are measurable. By Remark 1.4.13, $A(\omega)$ is a weakly compact convex subset of C. For any $\omega \in \Omega$, we may assume that the sequence $\{x_{n_\alpha}(\omega)\}$ is an ultranet in C. Note that $A(\omega) = A\left(C, \{x_{n_\alpha}(\omega)\}\right)$ and $r(\omega) = r\left(C, \{x_{n_\alpha}(\omega)\}\right)$. We can apply Lemma 1.4.17 to obtain

$$r_C(A(\omega)) \leq \lambda r\left(C, \{x_{n_\alpha}(\omega)\}\right),$$

where $\lambda = 1 - \triangle_{X,\alpha}(1^-) < 1$.

For each $\omega \in \Omega$ and $n \geq 1$, we define the multivalued contraction $T_n^1(\omega, .) : A(\omega) \to KC(X)$ by

$$T_n^1(\omega, x) = \frac{1}{n} x_1(\omega) + \left(1 - \frac{1}{n}\right) T(\omega, x), \text{ for each } x \in C. \tag{5.3.1}$$

By Lemma 5.3.2, we have

$$T(\omega, x) \bigcap I_{A(\omega)}(x) \neq \phi, \text{ for all } x \in A(\omega). \tag{5.3.2}$$

Since $I_{A(\omega)}(x)$ is convex, it follows, from (5.3.1) and (5.3.2), that

$$T_n^1(\omega, x) \bigcap I_{A(\omega)}(x) \neq \phi, \text{ for all } x \in A(\omega).$$

Let B be a bounded subset of C. Since T is $1-\chi$-contractive and $T_n^1(\omega, \beta) =$

$\frac{1}{n}x_1(\omega) + \left(1 - \frac{1}{n}\right) T(\omega, \beta)$, so

$$\begin{aligned}
\chi(T_n^1(\omega, \beta)) &= \chi\left(\frac{1}{n}x_1(\omega) + (1 - \frac{1}{n})T(\omega, B)\right) \\
&= \chi\left((1 - \frac{1}{n})T(\omega, B)\right) \\
&= \left(1 - \frac{1}{n}\right)\chi(T(\omega, B)) \\
&\leq \left(1 - \frac{1}{n}\right)\chi(B) \\
&< \chi(B).
\end{aligned}$$

Thus $T_n^1(\omega, .)$ is χ-condensing. Hence, by Lemma 1.4.10, $T_n^1(\omega, .)$ has a fixed point $z_n^1(\omega) \in A(\omega)$; i.e., $F(\omega) \bigcap A(\omega) \neq \phi$.

Clearly,

$$d\left(z_n^1(\omega), T(\omega, z_n^1(\omega))\right) \leq \frac{1}{n} \operatorname{diam} C \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$F_n^1(\omega) = \{x \in A(\omega) : d(x, T(\omega, x)) \leq \frac{1}{n} \operatorname{diam} C\}$$

is nonempty closed and convex for each $n \geq 1$. By Lemma 1.5.12, each F_n^1 is measurable. So, by Lemma 1.5.9 we can choose x_n^1 a measurable selector of F_n^1. Thus we have $x_n^1 \in A(\omega)$ and $d\left(x_n^1(\omega), T(\omega, x_n^1(\omega))\right) \to 0$ as $n \to \infty$. Let $f_2 : \Omega \times C \to [0, \infty)$ be defined by

$$f_2(\omega, x) = \lim_{n \to \infty} \sup \|x_n^1(\omega) - x\|, \text{ for all}\, \omega \in \Omega.$$

As above, f_2 is measurable and weakly lower semicontinuous. Also, the marginal function

$$r_2(\omega) = \inf_{x \in A(\omega)} f_2(\omega, x)$$

and the marginal mapping

$$A^1(\omega) = \{x \in A(\omega) : f_2(\omega, x) = r_2(\omega)\}$$

are measurable. Since $A^1(\omega) = A\left(A(\omega), \{x_n^1(\omega)\}\right)$, it follows that $A^1(\omega)$ is weakly compact and convex. We also note that $r_2(\omega) = r\left(A(\omega), \{x_n^1(\omega)\}\right)$. Again, for any $\omega \in \Omega$, we can assume that the sequence $\left\{x_{n_\alpha}^1(\omega)\right\}_\alpha$ is an ultranet in $A^1(\omega)$. As above, by Lemma 5.3.2 and Lemma 1.4.17, we obtain

$$T(\omega, x(\omega)) \bigcap I_{A^1(\omega)}(x(\omega)) \neq \phi, \text{ for all } x(\omega) \in A^1(\omega)$$

where $A^1(\omega) = A\left(A(\omega), \left\{x_{n_\alpha}^1(\omega)\right\}\right)$ and

$$r_C(A^1(\omega)) \leq \lambda r\left(A(\omega), \left\{x_n^1(\omega)\right\}\right) \leq \lambda r_C(A(\omega)) \tag{5.3.3}$$

By induction, for each $m \geq 1$, we take a sequence $\{x_n^m(\omega)\}_n \subset A^{m-1}$ such that

$$\lim_{n \to \infty} d\left(x_n^m(\omega), T(\omega, x_n^m(\omega))\right) = 0, \tag{5.3.4}$$

for each $\omega \in \Omega$. By means of the ultranet $\left\{x_{n_\alpha}^m(\omega)\right\}_\alpha$, we construct the set $A^m = A\left(C, \left\{x_{n_\alpha}^m(\omega)\right\}\right)$ such that

$$r_C(A^m) \leq \lambda^m r_C(A).$$

Since $\text{diam} A^m(\omega) \leq 2r_C(A^m)$ and $\lambda < 1$, it follows that $\lim\limits_{m\to\infty} \text{diam}\, A^m(\omega) = 0$. Note that $\{A^m(\omega)\}$ is a descending sequence of weakly compact subsets of C for each $\omega \in \Omega$. Thus, by Cantor's intersection theorem, we have $\bigcap\limits_m A^m(\omega) = \{z(\omega)\}$ for some $z(\omega) \in C$. Furthermore, we see that

$$H\left(A^m(\omega), \{z(\omega)\}\right) \leq \text{ diam}\, A^m(\omega) \to 0 \text{ as } m \to \infty.$$

Therefore, by Lemma 1.5.10, $z(\omega)$ is measurable.

Finally, we show that $z(\omega)$ is a random fixed point of T. For each $m \geq 1$, we have

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega))) &\leq \|z(\omega) - x_n^m(\omega)\| + d\left(x_n^m(\omega), T\left(\omega, x_n^m(\omega)\right)\right) \\ &+ H\left(T(\omega, x_n^m(\omega)), T(\omega, z(\omega))\right) \end{aligned} \tag{5.3.5}$$

Since T is an SL mapping, for $m \geq 1$ and $\{x_n^m(\omega)\}$ asymptotically T-regular, so we have

$$\lim_{n\to\infty} \sup H\left(T\left(\omega, x_n^m(\omega), T(\omega, z(\omega)\right)\right) \leq \lim_{n\to\infty} \sup \|x_n^m(\omega) - z(\omega)\| \tag{5.3.6}$$

Thus, by (5.3.4) - (5.3.6), we obtain

$$\begin{aligned} d(z(\omega), T(\omega, z(\omega)) &\leq 2 \lim_{n\to\infty} \sup \|z(\omega) - x_n^m(\omega)\| \\ &\leq 2\, diam\, A^m(\omega). \end{aligned}$$

Taking the limit as $m \to \infty$, we have $z(\omega) \in T(\omega, z(\omega))$. ■

We remark that if C is a weakly compact subset of a reflexive Banach space satisfying the nonstrict Opial's property, then we can follow the proof of Theorem 4.5 in [16] to deduce that a nonexpansive mapping $T : C \to K(X)$, with bounded range is $1 - \chi$-contractive. Then, in view of Theorem 5.3.3, we can state the following corollary.

Corollary 5.3.4. Let X be a Banach space satisfying the nonstrict Opial's property and $\epsilon_\alpha(X) < 1$. Suppose that C is a nonempty closed bounded convex separable subset of X and $T : \Omega \times C \to KC(X)$ is a nonexpansive random operator such that $T(C)$ is a bounded set and

$$T(\omega, x) \subset I_C(x), \text{ for all}\, x \in C \text{ and } \omega \in \Omega.$$

Then T has a random fixed point.

Remark 5.3.5. The ultranet in Lemma 5.3.2 can be replaced by a sequence which is asymptotically uniform with respect to C (see [41,78]). This allows us to rewrite the proof of Theorem 5.3.3 to β and χ moduli of noncompact convexity.

The following two theorems follow from the above remark and Lemma 1.4.16.

Theorem 5.3.6. Let C be a nonempty closed bounded convex separable subset of a Banach space X with $\epsilon_\beta(X) < 1$ and $T : \Omega \times C \to KC(X)$, continuous $1-\chi$-contrctive and SL random operator. If T satisfies the inwardness condition, then T has a random fixed point.

Theorem 5.3.7. Let X be a Banach space satisfying the nonstrict Opial's property and $\epsilon_\chi(X) < 1$. Suppose that C is a nonempty closed bounded convex separable subset of X and $T : \Omega \times C \to KC(X)$ is a nonexpansive random operator such that $T(C)$ is a bounded set and T satisfies the inwardness condition. Then T has a random fixed point.

5.4 Random Common Fixed Points

In this section we establish results about random coincidence and random common fixed points of nonself mappings in separable metric and Banach spaces; in particular, we provide random versions of Theorems 2.2.1 and 2.2.4. Further, we apply Theorem 4.2.1 to obtain random common fixed points of hybrid random operators.

We shall need the following result.

Theorem 5.4.1 [107, Theorem 3.1] Let X and Y be separable Banach spaces. Let C be a nonempty weakly compact subset of X, and $f, T : \Omega \times C \to Y$ be continuous random operators such that, for each $\omega \in \Omega, T(\omega, C)$ is bounded and $(f-T)(\omega, .)$ is demiclosed at 0. If the set $\{x \in C : f(\omega, x) - T(\omega, x) = 0\}$ is nonempty for each $\omega \in \Omega$, then there exists a measurable mapping $\xi : \Omega \to$

C such that $f(\omega, \xi(\omega)) - T(\omega, \xi(\omega)) = 0$, for each $\omega \in \Omega$.

We begin with a random version of Theorem 2.2.1

Theorem 5.4.2 Let C be a subset of a separable metric space X, and $S, T : \Omega \times C \to X$ be random operators such that

(i) S and T satisfy the random property (E.A);

(ii) $T(\omega, C)$ is a complete subspace of X, or $S(\omega, C)$ is a complete subspace with $S(\omega, C) \subseteq T(\omega, C)$, for every $\omega \in \Omega$;

(iii) for all $x \neq y$ in C and every $\omega \in \Omega$, the following condition holds:

$$\begin{aligned} d(S(\omega, x), S(\omega, y)) < \max\{ & d(T(\omega, x), T(\omega, y)), \\ & rd(S(\omega, x), T(\omega, x)) + \alpha d(S(\omega, y), T(\omega, y)), \\ & \tfrac{1}{2}[d(S(\omega, x), T(\omega, y)) + d(S(\omega, y), T(\omega, x))]\} \end{aligned} \tag{5.4.1}$$

where $r \in [0, +\infty)$ and $\alpha \in [0, 1)$.

Then S and T have a random coincidence point. Further, if $\zeta : \Omega \to C$ is a random coincidence point of S and T such that $S(\omega, \zeta(\omega)) \in C$ for each $\omega \in \Omega$, and S and T are weakly compatible, then S and T have a unique random common fixed point.

Proof. By (i) there exists a sequence of measurable mappings $\xi_n : \Omega \to C$

such that

$$\lim_{n\to\infty} S(\omega, \xi_n(\omega)) = \lim_{n\to\infty} T(\omega, \xi_n(\omega)) = \xi(\omega),$$

for each $\omega \in \Omega$, where $\xi : \Omega \to X$ is a measurable mapping. Suppose that $T(\omega, C)$ is complete for every $\omega \in \Omega$, then there exists a measurable mapping $\zeta : \Omega \to C$ such that

$$T(\omega, \zeta(\omega)) = \xi(\omega), \text{ for each } \omega \in \Omega.$$

We show that $S(\omega, \zeta(\omega)) = T(\omega, \zeta(\omega))$, for every $\omega \in \Omega$. By (iii), we have

$$\begin{aligned} d(S(\omega, \xi_n(\omega)), S(\omega, \zeta(\omega))) \quad < \quad & \max\{d(T(\omega, \xi_n(\omega), T(\omega, \zeta(\omega))), \\ & rd(S(\omega, \xi_n(\omega)), T(\omega, \xi_n(\omega))) \\ & +\alpha d(S(\omega, \zeta(\omega)), T(\omega, \zeta(\omega))), \\ & \frac{1}{2}[d(S(\omega, \xi_n(\omega)), T(\omega, \zeta(\omega))) \\ & +d(S(\omega, \zeta(\omega)), T(\omega, \xi_n(\omega)))]\}, \end{aligned}$$

for every $\omega \in \Omega$. Taking the limit as $n \to \infty$, we get

$$\begin{aligned} d(T(\omega, \zeta(\omega)), S(\omega, \zeta(\omega))) \quad \leq \quad & \max\{\alpha d(S(\omega, \zeta(\omega)), T(\omega, \zeta(\omega))), \\ & \frac{1}{2}d(S(\omega, \zeta(\omega)), T(\omega, \zeta(\omega)))\}, \end{aligned}$$

for every $\omega \in \Omega$. This is possible only if

$$d(T(\omega, \zeta(\omega)), S(\omega, \zeta(\omega))) = 0, \quad \text{for every} \quad \omega \in \Omega,$$

that is,

$$T(\omega, \zeta(\omega)) = S(\omega, \zeta(\omega)), \quad \text{for every} \quad \omega \in \Omega.$$

If $S(\omega, \zeta(\omega)) \in C$ for each $\omega \in \Omega$, and S and T are weakly compatible, then

$$\begin{aligned} T(\omega, T(\omega, \zeta(\omega))) &= T(\omega, S(\omega, \zeta(\omega))) \\ &= S(\omega, T(\omega, \zeta(\omega))) \\ &= S(\omega, S(\omega, \zeta(\omega))), \end{aligned}$$

for every $\omega \in \Omega$. We show that

$$S(\omega, S(\omega, \zeta(\omega))) = S(\omega, \zeta(\omega)), \text{ for every } \omega \in \Omega.$$

Suppose not, then for some $\omega \in \Omega$, we get

$$\begin{aligned} d(S(\omega, S(\omega, \zeta(\omega))), S(\omega, \zeta(\omega))) &< \max\{d(T(\omega, S(\omega, \zeta(\omega))), T(\omega, \zeta(\omega))), \\ &\quad rd(S(\omega, S(\omega, \zeta(\omega))), T(\omega, S(\omega, \zeta))), \\ &\quad +\alpha d(S(\omega, \zeta(\omega)), T(\omega, \zeta(\omega))), \\ &\quad \frac{1}{2}[d(S(\omega, S(\omega, \zeta(\omega))), T(\omega, \zeta(\omega))) \\ &\quad +d(S(\omega, \zeta(\omega)), T(\omega, S(\omega, \zeta(\omega))))]\} \\ &= d(S(\omega, S(\omega, \zeta(\omega))), S(\omega, \zeta(\omega))) \end{aligned}$$

a contradiction. Thus

$$T(\omega, S(\omega, \zeta(\omega))) = S(\omega, S(\omega, \zeta(\omega)) = S(\omega, \zeta(\omega)),$$

for every $\omega \in \Omega$. Similarly, we can prove the case, $S(\omega, C)$ is complete with $S(\omega, C) \subseteq T(\omega, C)$, for every $\omega \in \Omega$. The uniqueness can be easily verified by using (iii).

In the context of Banach spaces we can replace the random property (E.A) by the deterministic one as follows:

Theorem 5.4.3 Let C be a nonempty weakly compact subset of separable Banach space X and $S, T : \Omega \times C \to X$ be continuous random operators such that

(i) the mappings $S(\omega, .)$ and $T(\omega, .)$ satisfy the property (E.A), for each $\omega \in \Omega$;

(ii) $T(\omega, C)$ is complete or $S(\omega, C)$ is complete with $S(\omega, C) \subseteq T(\omega, C)$, for every $\omega \in \Omega$;

(iii) for all $x \neq y$ in C and every $\omega \in \Omega$, (5.4.1) holds;

(iv) $(S-T)(\omega, .)$ is demiclosed at 0 and $T(\omega, C)$ is bounded, for each $\omega \in \Omega$.

Then S and T have a random coincidence point. Further, if $\zeta : \Omega \to C$ is a random coincidence point of S and T such that $S(\omega, \zeta(\omega)) \in C$ for each $\omega \in \Omega$, and S and T are weakly compatible, then S and T have a unique random common fixed point.

Proof. By Theorem 2.2.1, the mappings $S(\omega, .)$ and $T(\omega, .)$ have a deterministic coincidence point; that is, the set $\{x \in C : S(\omega, x) = T(\omega, x)\}$ is nonempty, for each $\omega \in \Omega$. Thus, by Theorem 5.4.1, there exists a measurable mapping $\zeta : \Omega \to C$ such that

$$S(\omega, \zeta(\omega)) = T(\omega, \zeta(\omega)), \text{ for each } \omega \in \Omega;$$

that is, ζ is a random coincidence point of S and T. The rest of the proof is similar to that of Theorem 5.4.2 and is omitted.

The next result is about four mappings which can be proved on the lines of the proof of Theorem 2.2.4.

Theorem 5.4.4 Let C be a subset of a separable metric space X, and $F, G, S, T : \Omega \times C \to X$ be random operators such that:

(i) the pair (F, S) or (G, T) satisfies the random property (E.A);

(ii) $F(\omega, C), G(\omega, C), S(\omega, C)$ or $T(\omega, C)$ is complete, for every $\omega \in \Omega$;

(iii) $F(\omega, C) \subseteq T(\omega, C)$ and $G(\omega, C) \subseteq S(\omega, C)$, for every $\omega \in \Omega$;

(iv) for all x, y in C and every $\omega \in \Omega$, the following condition holds:

$$\begin{aligned} d(F(\omega, x), G(\omega, y)) \quad \leq \quad & \phi(\max\{d(S(\omega, x), T(\omega, y)), \\ & d(S(\omega, x), G(\omega, y)), d(T(\omega, y), G(\omega, y))\}), \end{aligned}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and $0 < \phi(t) < t$, for all $t \in (0, +\infty)$.

(v) $(F - S)(\omega, .)$ and $(G - T)(\omega, .)$ are demiclosed at 0, and $S(\omega, C)$ and $T(\omega, C)$ are bounded, for each $\omega \in \Omega$.

Then:

(a) F and S have a random coincidence point, and G and T have a random coincidence point,

(b) If $\zeta : \Omega \to C$ is a random coincidence point of F and S such that $S(\omega, \zeta(\omega)) \in C$ for each $\omega \in \Omega$, and S and F are weakly compatible, then F and S have a random common fixed point,

(c) If $\eta : \Omega \to C$ is a random coincidence point of G and T such that $T(\omega, \eta(\omega)) \in C$ for each $\omega \in \Omega$, and G and T are weakly compatible, then G and T have a random common fixed point,

(d) F, G, S and T have a unique random common fixed point provided (b) and (c) hold.

Theorem 5.4.5 Let C be a nonempty weakly compact subset of a separable Banach space X, and $F, G, S, T : \Omega \times C \to X$ be continuous random operators satisfying conditions (ii) - (iv) in Theorem 5.4.4. If the pair $(F(\omega, .), S(\omega, .))$

or $(G(\omega, .), T(\omega, .))$ satisfies the property (E.A), for each $\omega \in \Omega$, then (a) - (d) in Theorem 5.4.4 hold.

Proof. By Theorem 2.2.4, F and S have a deterministic coincidence point, and G and T have a deterministic coincidence point. Now, Theorem 5.4.1 implies that F and S have a random coincidence point, and G and T have a random coincidence point; this completes the proof of (a). Using the arguments of the proof of Theorem 5.4.4, we can easily prove (b) - (d).

For random common fixed points of hybrid mappings, we need the following useful results.

Theorem 5.4.6 [108, Theorem 3.1]. Let M be a nonempty separable weakly compact subset of a Banach space X, and $f : \Omega \times M \to M$ a random operator which is both continuous and weakly continuous. Assume that $T : \Omega \times M \to CB(M)$ is a continuous random operator such that $(f - T)(\omega, .)$ is demiclosed at 0 for each $\omega \in \Omega$. If f and T have a deterministic coincidence point, then f and T have a random coincidence point.

Theorem 5.4.7 [108, Theorem 3.12]. Let M be a nonempty separable complete subset of a metric space X, and let $T : \Omega \times M \to C(X)$ and $f : \Omega \times M \to X$ be continuous random operators satisfying condition (A°); that is, for any sequence $\{x_n\}$ in $X, D \in C(X)$ such that $d(x_n, D) \to 0$

and $d(fx_n, Tx_n) \to 0$ as $n \to \infty$, there exists $y \in D$ with $fy \in Ty$. If f and T have a deterministic coincidence point, then f and T have a random coincidence point.

We apply Theorem 4.2.1 to prove the following:

Theorem 5.4.8 Let Y be a nonempty separable weakly compact subset of a Banach space X, and $f : \Omega \times Y \to Y$ a random operator. Assume that $S : \Omega \times Y \to CB(Y)$ is continuous and $f(\omega, .)$ and $S(\omega, .)$ satisfy conditions (i) - (iii) in Theorem 4.2.1, for all $\omega \in \Omega$. Suppose that f is both continuous and weakly continuous and $(f - S)(\omega, .)$ is demiclosed at 0 for each $\omega \in \Omega$. Then f and S have a random coincidence point. Moreover, if for each $\omega \in \Omega$ and any $x \in M, f(\omega, x) \in S(\omega, x)$ implies $f(\omega, f(\omega, x)) = f(\omega, x)$, and f is S-weakly commuting random operator, then f and S have a random common fixed point.

Proof. By Theorem 4.2.1 f and S have a deterministic coincidence point. By Theorem 5.4.6, f and S have a random coincidence point $\psi : \Omega \to Y$; i.e. $f(\omega, \psi(\omega)) \in S(\omega, \psi(\omega))$ for each $\omega \in \Omega$. Let $\xi(\omega) = f(\omega, \psi(\omega))$ for each $\omega \in \Omega$. Then $\xi : \Omega \to Y$ is measurable. Thus, for each $\omega \in \Omega$, we have

$$\xi(\omega) = f(\omega, \psi(\omega)) = f(\omega, f(\omega, \psi(\omega))) = f(\omega, \xi(\omega)).$$

Since f is S-weakly commuting, we get

$$\xi(\omega) = f(\omega, \xi(\omega)) \in S(\omega, f(\omega, \psi(\omega))) = S(\omega, \xi(\omega))$$

for each $\omega \in \Omega$. Hence ξ is a random common fixed point of f and S.

Similarly, a combination of Theorem 4.2.1 and Theorem 5.4.7, gives the following result.

Theorem 5.4.9 Let Y be a nonempty separable complete subset of a metric space X, and $f : \Omega \times Y \to X$ be a continuous random operator. Assume that the random operator $T : \Omega \times Y \to C(X)$ is continuous and $f(\omega, .)$ and $S(\omega, .)$ satisfy the conditions (i) - (iii) in Theorem 4.2.1, for all $\omega \in \Omega$. If $f(\omega, .)$ and $S(\omega, .)$ satisfy the condition (A°) for each $\omega \in \Omega$, then f and T have a random coincidence point. Moreover, if for each $\omega \in \Omega$ and any $x \in M, f(\omega, x) \in S(\omega, x)$ implies $f(\omega, f(\omega, x)) = f(\omega, x)$, and f is S-weakly commuting random operator, then f and S have a random common fixed point.

We apply Theorem 5.4.3 to prove weak and strong convergence of the random modified Ishikawa iteration ((5.2.3) with $k = 2, T_1 = S$ and $T_2 = T$) to a unique random common fixed point of two random operators S and T.

Theorem 5.4.10 Let C be a nonempty weakly compact convex subset of a separable uniformly convex Banach space X, and S and T be two $(L - \gamma)$

uniform Lipschitz and asymptotically quasi-nonexpansive continuous random operators from $\Omega \times C \to C$. Suppose that S and T are weakly compatible. If the conditions (i) – (iv) of Theorem 5.4.3 are satisfied, and $(I - S)(\omega, \cdot)$ and $(I - T)(\omega, \cdot)$ are demiclosed at 0, then the random modified Ishikawa iteration converges weakly to a unique random common fixed point of S and T.

Proof. By Theorem 5.4.3, S and T have a unique random common fixed point (say ζ). Then, by Lemma 5.2.12 (i), $\lim_{n\to\infty} \|\xi_n(\omega) - \zeta(\omega))\|$ exists for each $\omega \in \Omega$, and hence $\{\xi_n\}$ is bounded. Since X is reflexive, there exists a subsequence $\{\xi_{n_j}\}$ of $\{\xi_n\}$ converging weakly to a measurable map $\eta : \Omega \to C$, for each $\omega \in \Omega$. By Lemma 5.2.12 (iii) and the demiclosedness of $I - S$ and $I - T$, $S(\omega, \eta(\omega)) = T(\omega, \eta(\omega)) = \eta(\omega)$, for each $\omega \in \Omega$. Thus, $\eta(\omega) = \zeta(\omega)$, for each $\omega \in \Omega$. In order to show that $\{\xi_n\}$ converges weakly to ζ, take another subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ converging weakly to a measurable map $\psi : \Omega \to C$, for each $\omega \in \Omega$. Again, as above, $\psi(\omega) = \zeta(\omega)$, for each $\omega \in \Omega$, so $\{\xi_n\}$ converges weakly to ζ. ■

On the lines of the proof of the above theorem, we can prove the following:

Corollary 5.4.11 Let C and X be as in Theorem 5.4.10, and S and T be two asymptotically nonexpansive random operators from $\Omega \times C \to C$. Suppose that S and T are weakly compatible and the conditions (i)-(iv) of Theorem

5.4.3 are satisfied. Then the random modified Ishikawa iteration converges weakly to a unique random common fixed point of S and T.

Theorem 5.4.12 Let C be a nonempty weakly compact convex subset of a separable uniformly convex Banach space X, and S and T be as in Lemma 5.2.12. Suppose that S and T are weakly compatible, the conditions (i)-(iv) of Theorem 5.4.3 hold and for some integer m, T^m or S^m is semi-compact. Then the random modified Ishikawa iteration converges strongly to a unique random common fixed point of S and T.

Proof. By Theorem 5.4.3, S and T have a unique random common fixed point. Let T^m be semi-compact (the proof is similar if S^m is semi-compact). By Lemma 5.2.12, we obtain

$$\begin{aligned}
\|T^m(\omega,\xi_n(\omega)) - \xi_n(\omega)\| &\leq \|T^m(\omega,\xi_n(\omega)) - T^{m-1}(\omega,\xi_n(\omega))\| \\
&\quad + \|T^{m-1}(\omega,\xi_n(\omega)) - T^{m-2}(\omega,\xi_n(\omega))\| \\
&\quad + \ldots + \|T^2(\omega,\xi_n(\omega)) - T(\omega,\xi_n(\omega))\| \\
&\quad + \|T(\omega,\xi_n(\omega)) - \xi_n(\omega)\| \\
&\leq (m-1)L\|T(\omega,\xi_n(\omega)) - \xi_n(\omega)\|^{\gamma} \\
&\quad + \|T(\omega,\xi_n(\omega)) - \xi_n(\omega)\| \to 0, \quad \text{as} \;\; n \to \infty.
\end{aligned}$$

Since $\{\xi_n\}$ is bounded and T^m is semi-compact, $\{\xi_n\}$ has a convergent subsequence $\{\xi_{n_k}\}$ converging to a measurable map $\eta : \Omega \to C$. Hence, again by Lemma 5.2.12 (iii), we have

$$\|\zeta(\omega)-S(\omega,\zeta(\omega))\| = \|\zeta(\omega)-T(\omega,\zeta(\omega))\| = \lim_{n\to\infty} \|\xi_{n_k}(\omega)-T(\omega,\xi_{n_k}(\omega))\| = 0,$$

for each $\omega \in \Omega$. Thus $RF(S,T) = \{\zeta\}$. As $\lim_{n\to\infty} \|\xi_n(\omega) - \zeta(\omega)\|$ exists, so $\{\xi_n\}$ converges strongly to ζ. ■

Remark 5.4.13 (i) Following the arguments of the proof Theorem 5.4.12, we can prove analogue of this result for two asymptotically nonexpansive random operators instead of $(L-\gamma)$ uniform Lipschitz and asymptotically quasi-nonexpansive random operators.

(ii) Corollary 5.4.11 extends Theorems 3.1-3.2 of Tan and Xu [126] to the case of two asymptotically nonexpansive random operators.

(iii) Theorem 2.1 [27] and Theorems 3.1-3.3 [36] deal with one continuous random operator whereas Theorem 5.4.12 gets hold of two continuous random operators.

(iv) Theorem 5.4.12 is a random version of Theorem 3.4 of Shahzad and Udomene [110] without the assumption that the set of common fixed point is nonempty.

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