Stochastic Model for Love Wave Propagation in an Inhomogeneous Layer

by

Ramzy M. Al-Zayer

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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This thesis, written by RAMZY MOHAMMED AL-ZAYER under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of *MASTER OF SCIENCE* in *MATHEMATICS*.

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To my family

Mother Meriam

Wife Mariam

Kids Nezar, Fatema, Muntadhar

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All praise to "ALLAH", the Lord of the world, the Almighty, with Whose gracious help it was possible to accomplish this work. May peace and blessings be upon Mohammad the last of the Messengers!

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الخلاصة

العنوان :- نموذج تخميني لسريان موجات لوف خلال طبقة غير متناسقة الخواص .

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نستعرض في هذه الرسالة سريان موجات لوف خلال مجال تخميني . سوف نفترض أن الأرض مكونة من طبقة شبه نهائية ذات خواص متناسقة ومتوافقة في كل الاتجاهات مغطاة بطبقة ذات خواص تخمينية . هذا النموذج الرياضي يحل محل النموذج المعرف والغير تخميني الذي أشبع بحثاً . نتاج ذلك هو معادلة تفاضلية تخمينية مع شروط حدودية معروفة . نحلل هذه المعادلة باستخدام طريقتين مختلفتين للحصول على المعدل الوسطي الأول والثاني لمجال الإزاحة الموجة . ثم نعطي مثالاً رقمياً لهذه النتائج لتوضيح أعتمادية هذه المعدلات الوسطية على المكان وعلى مسافة التوافق . أخيراً نستخلص علاقة عامة لسرعة موجات لوف في الوسط التخميني .

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ABSTRACT

Title: **STOCHASTIC MODEL FOR LOVE WAVE PROPAGATION IN AN INHOMOGENEOUS LAYER**

Name: **Ramzy M. Al-Zayer**

Major Field: **Mathematics**

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In this thesis, I consider propagation of Love waves in a stochastic medium. I assume the Earth consists of a layer with randomly varying elastic properties overlying a homogeneous isotropic half-space. This is an alternative mathematical model to the deterministic model which has been used extensively. The model gives rise to a stochastic partial differential equation with deterministic boundary conditions. The first and the second moments of the displacement field are obtained using two different methods. A numerical example is given to show the dependence of these moments on *depth* and correlation distance. A general Love wave dispersion relation is also derived.

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Introduction

Love waves are very important seismic surface waves which propagate in the very top layer of the Earth's surface. In earthquake seismology, studying these waves gives us an insight into the nature of the source and the medium through which the waves propagate. In oil exploration, surface waves are considered to be noise. Since the pioneering work of Love [1911], these have been a focal topic of research.

Early classical layered model of the Earth consists of a homogeneous layer overlying a homogeneous half-space, Aki and Richard [1980]. But in reality the Earth in general and the near-surface layer in particular, have a very complex elastic parameter distribution. Also, the top layer interfaces are by no means perfectly horizontal as was proposed by Love. Furthermore, it is an approximation to represent the waves source by a deterministic function. Researchers realized the need to consider a more realistic model. The problem is how to model the variation in the above three elements of wave propagation; namely, the elastic parameter distribution, the boundary geometry, and the source terms.

Several surface shapes have been considered; for example, by Sato [1961], Paul [1965], and Wolf [1967]. Elastic parameters have also been described by different deterministic functions; for example, Ghosh [1970], Zaman, Asghar and Hanif [1991]. However, these deterministic models do not account for the point to point irregularities of the medium's properties, the irregular shape of surfaces separating the media, and the uncertainties of the source function. These are best described by stochastic modeling.

The irregularity and complexity of the properties of real media lead to a stochastic description of these media and there arise problems of wave propagation in stochastic media, Sobczyk [1985].

The use of stochastic models to describe the Earth's layer has gained popularity in the last few decades. One of the major projects which is being conducted in the Arctic Ocean by a group of research centers is the study of the thickness of the polar ice cap which gives an early indication about any climatic changes. However, because of the impurities and air bubbles in the ice, wave propagation in such a medium is not a simple problem. Stochastic modeling is employed in this case [Waves Young, 1997].

One of the earliest work in this direction is the book by Chernov [1970] who deals with stochastic wave propagation. Frisch [1968] has also discussed wave propagation in a random media. An excellent account of the methods employed in propagation and scattering of waves in a random media is given by Sobczyk [1985]. The background in stochastic differential equations can be found in Soong [1973] and Frisch [1968].

Scientists used some of the techniques developed in random wave propagation in some practical problems. As an example, Korvin [1977], [1983] used Keller's perturbation method to calculate the attenuation coefficient due to random inhomogeneities. Chu. Askar and Cakmak [1981] used the same method in their work to measure elastic properties in the laboratory. Li & Hudson [1995] used Born approximation to study elastic waves in a laterally heterogeneous layer.

This Thesis is an application to the first-order smoothing method in the problem of Love wave propagation in stochastic medium. We find the mean and variance of the mean field using two different methods. The first method is based upon the smoothing method on the layer. The second is to us the Green's function method. with the use of the smoothing method to find the first and second moments.

The first chapter is devoted for a quick review of some mathematical concepts we will be using in later chapters. In the second chapter, we explain why we need to replace the deterministic model by a stochastic one. The result of that is a stochastic differential equation which is the subject of Section 2.3. Love waves in a deterministic model of a layer over a half-space is analyzed in Chapter 3. The core of the thesis is in Chapters 4 and 5. In these chapters, the above two methods are used for Love wave propagation problem.

Numerical results are also given in some interesting cases. Matlab and Mathematica are two software which have been used in this thesis. To have a continuous analysis, some details were relegated to appendices at the end of the Thesis.

CHAPTER 1

MATHEMATICAL PRELIMINARIES

1.1 Stochastic Process and Field

A stochastic process can be characterized in two ways. The first is to describe a stochastic process $X(t), t \in T$ as a family of sample functions, *realizations*, of the variable t, for every fixed elementary event defined on the probability space $\{\Gamma, \mathcal{F}, P\}$. The set Γ is the space of elementary events, $\mathcal{F}$ is a σ-algebra of subsets of Γ and P is a probability measure defined on $\mathcal{F}$. This requires using measure theory to define probability on a functional space.

The sample theoretic approach is an alternative simpler approach to define a stochastic process. In this characterization, a stochastic process $\{X(t,\gamma), t \in T, \gamma \in \Gamma\}$ is regarded as a family of random variables say, $X(t_1), X(t_2), \ldots$. If the index set T is countable, $X(t)$ is said to be a discrete-parameter stochastic process, or it can be an interval, and the process is called a continuous-parameter. In the latter case, a random function is completely characterized by the n-point probability functions

$$F_n(t_1, t_2, \ldots, t_n; x_1, x_2, \ldots, x_n) = P\left[X(t_1) < x_1, \ldots, X(t_n) < x_n\right]. \qquad (1.1)$$

The family of all these joint probability functions for $n = 1, 2, \ldots$ and all possible values of the t_i, constitutes a finite characterization of the process $X(t)$.

For practical purposes, the stochastic process is characterized in terms of its moments; mean, mean square value, autocorrelation function and higher order moments;

$$M_X(t) = \langle X(t) \rangle; M_{X^2}(t) = \langle |X(t)|^2 \rangle; K_{XX}(t_1, t_2) = \langle X(t_1) X(t_2) \rangle; K_{XX}(t_1, \ldots, t_n)$$

$$= \langle X(t_1) \cdots X(t_n) \rangle, \qquad (1.2)$$

where angular brackets are expectations over all realizations of the process. The variance of the process $X(t)$ is defined by $\sigma_X^2(t) = \langle |X(t) - M_X(t)|^2 \rangle$. If $M_X(t) = 0$, then $\sigma_X^2(t) = M_{X^2}(t)$. If we have two or more stochastic processes, then their mutual dependence is conveniently characterized by the covariance matrix $R(t_1, t_2) = \{R_{X_i X_j}(t_1, t_2)\}, i, j = 1, 2, \ldots, n$, where

$$R_{X_i X_j}(t_1, t_2) = \langle [X_i(t) - M_{X_i}(t_1)][X_j(t_2) - M_{X_j}(t_2)] \rangle. \tag{1.3}$$

Depending on the properties of their distribution functions, many of the practical stochastic processes are statistically simple. These simplifications are based upon the memory of the process (e.g., the Markovian processes) or on the regularity over the index set T (e.g., the stationary processes).

The process $X(t), t \in T$ is *stationary* if its cumulative probability distributions stay invariant under an arbitrary translation of the time, or space in our case, parameter,

$$F_n(t_1, t_2, \ldots, t_n; x_1, x_2, \ldots, x_n) = F_n(t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau; x_1, x_2, \ldots, x_n);$$

$$t_j + \tau \in T, \quad j = 1, 2, \ldots, n. \tag{1.4}$$

A weakly stationary process is one that has the following properties:

$$M_X(t) = \text{ constant } < \infty$$

$$M_{X^2}(t) < \infty$$

$$K_{XX}(t_1, t_2) = K_{XX}(t_2 - t_1). \tag{1.5}$$

Ergodic theory introduces further simplifications. It relates time averages of the realizations to ensemble averages of the process. A stationary process is *ergodic* if for every function f we have $P\{\langle f \rangle = mf\} = 1$ where m is the time average equation. This means that the time-average taken for any realization agrees with the ensemble-average with probability one.

A stochastic process $X(t), t \in T$, is called *Gaussian* or *normal* if all its finite-dimensional distributions are Gaussian. In this case the process is completely specified by its mean $M_X(t)$ and covariance function $K_{XX}(t_1, t_2) = R_{X_1 X_1}(t_1, t_2)$.

The other way of simplifying a process is to assume it to be a *Markov diffusion* process; an example is the *Brownian* motion process (sometimes called Wiener process). A process is called *Markovian* when it is completely specified by the conditional distribution function

$$F(s, x : t, y) = P\{X(t) < y | X(s) = x\}. \tag{1.6}$$

The process is *diffusion* when the change within a small time interval is small. If all sample functions of a Wiener process are continuous and differentiable, then the model is called the Ornstein-Uhlenbeck process, Soong [1973].

The elastic parameters that describe the uppermost layer of the Earth are considered to be random functions. These processes are statistically simple, they are simplified further by assuming them to be ergodic and stationary. In addition, we will use the Ornstein-Uhlenbeck process assumption. This model has the following correlation function, Sobczyk [1985]:

$$K_{XX}(\tau) = \sigma^2 \exp(-b|\tau|), \tag{1.7}$$

where σ is the standard deviation of the process and $1/b$ is the correlation length (distance) which is defined as the value of τ for which the autocorrelation function $K_{XX}(\tau)$ decreases to $(1/e)$ times its value at $\tau = 0$.

Homogeneous Isotropic Random Fields

As a generalization of the one-dimensional stochastic process, a random field is used when the random function is a function of space $X(\vec{r})$, where $\vec{r} = (x, y, z)$ is a point in the Euclidean space $\mathbb{R}^3$. Similarly, the case of a stationary process in the one-dimensional case, homogeneity and isotropy of a random field play a key role in analysis. A random field is statistically homogeneous and isotropic if its statistical properties are invariant with respect to a shift of the system coordinates and rotations and reflections, respectively. In such a field the autocorrelation is a function of $|\vec{r}_1 - \vec{r}_2|$, i.e.,

$$K_{XX}(\vec{r}_1, \vec{r}_2) = K_{XX}(|\vec{r}_1 - \vec{r}_2|) = K_{XX}(r). \tag{1.8}$$

If we observe this field along a straight line, picked at random, of the space $\mathbb{R}^3$, we get a one-dimensional stationary function or a stochastic process.

1.2 The Deterministic Wave Equation

Assume an unbounded elastic solid, macroscopically homogeneous and in equilibrium with respect to all body and surface forces. If a disturbance (time-dependent stress) passes through the material, the displacement of the point $P(x, y, z)$ at any instant t is given by the vector $\vec{u}(x, y, z, t)$. The general equation of motion at point P is, Grant and West [1965]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial p_{ij}}{\partial x_k} \tag{1.9}$$

where u_i is the displacement in the i-th axis direction, p_{ik} is the stress tensor, ρ is the density of the material.

If the medium is homogeneous, isotropic, and perfectly elastic, we can use Hooke's law (written is tensor form)

$$p_{ik} = \lambda \theta \delta_{ik} + 2\mu e_{ik} \tag{1.10}$$

where δ_{ik} is the Kronecker delta, e_{ik} is the strain tensor

$$\text{and } \theta = \vec{\nabla} \cdot \vec{u}, \tag{1.11}$$

where λ, μ are Lame's constants.

Substituting (1.11) in (1.10), the resultant equation, in vector form, is

$$\rho\ddot{u} = (\lambda + \mu)\vec{\nabla}(\vec{\theta} \cdot \vec{u}) \ + \mu \nabla^2 \vec{u}. \tag{1.12}$$

Taking the divergence and the curl of both sides gives the standard wave equation of P-type and S-type waves, respectively, as follows:

$$\rho \frac{\partial^2 \theta}{\partial t^2} = (\lambda + 2\mu) \nabla^2 \theta \tag{1.13}$$

$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = \mu \nabla^2 \vec{\xi}, \tag{1.14}$$

where $\vec{\xi} = \vec{\nabla} \times \vec{u}$.

It is clear that both the principal stress (P) and the shearing stress (S) propagate with the respective velocities, $V_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$ and $V_s = \sqrt{\dfrac{\mu}{\rho}}$.

Equation (1.12) can be separated into two equations, one for the scalar potential ϕ, the other for the vector potential $\vec{\psi}$, each of which satisfies the wave equation. This is done by assuming

$$\vec{u} = \vec{\nabla}\phi - \vec{\nabla} \times \vec{\psi}, \tag{1.15}$$

where we impose the following condition:

$$\vec{\nabla} \cdot \vec{\psi} = 0. \tag{1.16}$$

If we introduce (1.15) into the equations of motion (1.12), we get

$$\rho\vec{\nabla}\ddot{\phi} - (\lambda + 2\mu)\vec{\nabla}(\nabla^2\phi) - \rho(\vec{\nabla} \times \ddot{\vec{\psi}}) + \mu \nabla^2 (\vec{\nabla} \times \vec{\psi}) = 0. \tag{1.17}$$

This complicated equation can be satisfied if we allow ϕ and $\vec{\psi}$ independently to satisfy the following two wave equations which represent P and S motions respectively.

$$\rho\ddot{\phi} = (\lambda + 2\mu) \nabla^2 \phi, \tag{1.18}$$

$$\rho\ddot{\vec{\psi}} = \mu \nabla^2 \vec{\psi}. \tag{1.19}$$

The use of the two potential functions ϕ and $\vec{\psi}$ makes it convenient to solve boundary value problems involving P and S waves and to make the continuity conditions at interfaces and the solution of boundary value problems easier to express.

In this Thesis we will study one component of the displacement field, namely, that in the y-direction. Therefore, we do not need to use these two potentials. Using equation (1.12), the wave equation in terms of the horizontal component of the displacement field is

$$\mu \nabla^2 v = \rho \frac{\partial^2 v}{\partial t^2}. \tag{1.20}$$

The solution of wave equation is greatly simplified if we assume a plane wave. This can be done far from the source where wavefronts lose their initial curvature. Also, as with electromagnetic waves, the S-waves are plane polarized. The particle motion is horizontal in the SH-waves and vertical in the SV-waves. Love waves, which we will study in this Thesis, are surface SH-waves. In case of plane wave, the solution can be found using Fourier transform and separation of variables.

In the above review, three assumptions were imposed upon the wave theory.

1. Perfect Elasticity.

The medium is assumed to be perfectly elastic. However, there is a broad variety of earth materials, including silts, clays, sand and shales, which do not behave as perfectly elastic under loading.

2. Isotropy

Properties of the medium at an arbitrary point are assumed to be the same in every direction. But the fact is that the earth consists of layers which were formed by subsequent stages of deposition, compaction and consolidation. Therefore, wave propagation characteristics differ in the depth direction from the horizontal direction. This model is called transversely-isotropic.

3. Homogeneity

Also, spatially, the medium's parameters are not always constant, but they slowly change. For example, there is a velocity gradient in the depth direction due to the increase of overburden pressure with depth. This property brings the most difficult mathematical problems to the seismic wave propagation theory.

Random Inhomogeneity

A second type of heterogeneity, very common in geological settings, is the uncorrelated or "random" type, in which the elastic properties change continuously but unsystematically from point to point A geological formation, which is hundreds of meters thick, can be considered statistically homogeneous. However, its texture usually varies considerably at the scale of a foot.

This is the kind of inhomogeneities we shall be concerned with. We will assume statistical homogeneity (assumption 3), but locally, the properties change randomly. Figure (1.1) shows an actual seismic velocity and density logs taken from a shallow borehole. It shows both the random as well as the systematic nature of the elastic parameters distribution. As we shall show, the random behavior of the velocity will scatter the seismic energy that propagates through it. To study the effect of scattering on waves, we introduce a mathematical model that represents the ground. We will assume that the elastic parameter, like the density, is a random field. One way to represent this field is to separate the parameter into a constant and a random function. For example,

$$\rho(x,y,z,\gamma) = \rho_0[1 + \epsilon\rho_1(x,y,z,\gamma)] \tag{1.21}$$

where $\rho_1(x,y,z,\gamma)$ is a random variable with zero mean and standard deviation 1, ϵ is chosen such that $\epsilon\rho_0$ is the rms value of the density variations. The mean density is everywhere the same ρ_0, but the local density varies randomly point-to-point.

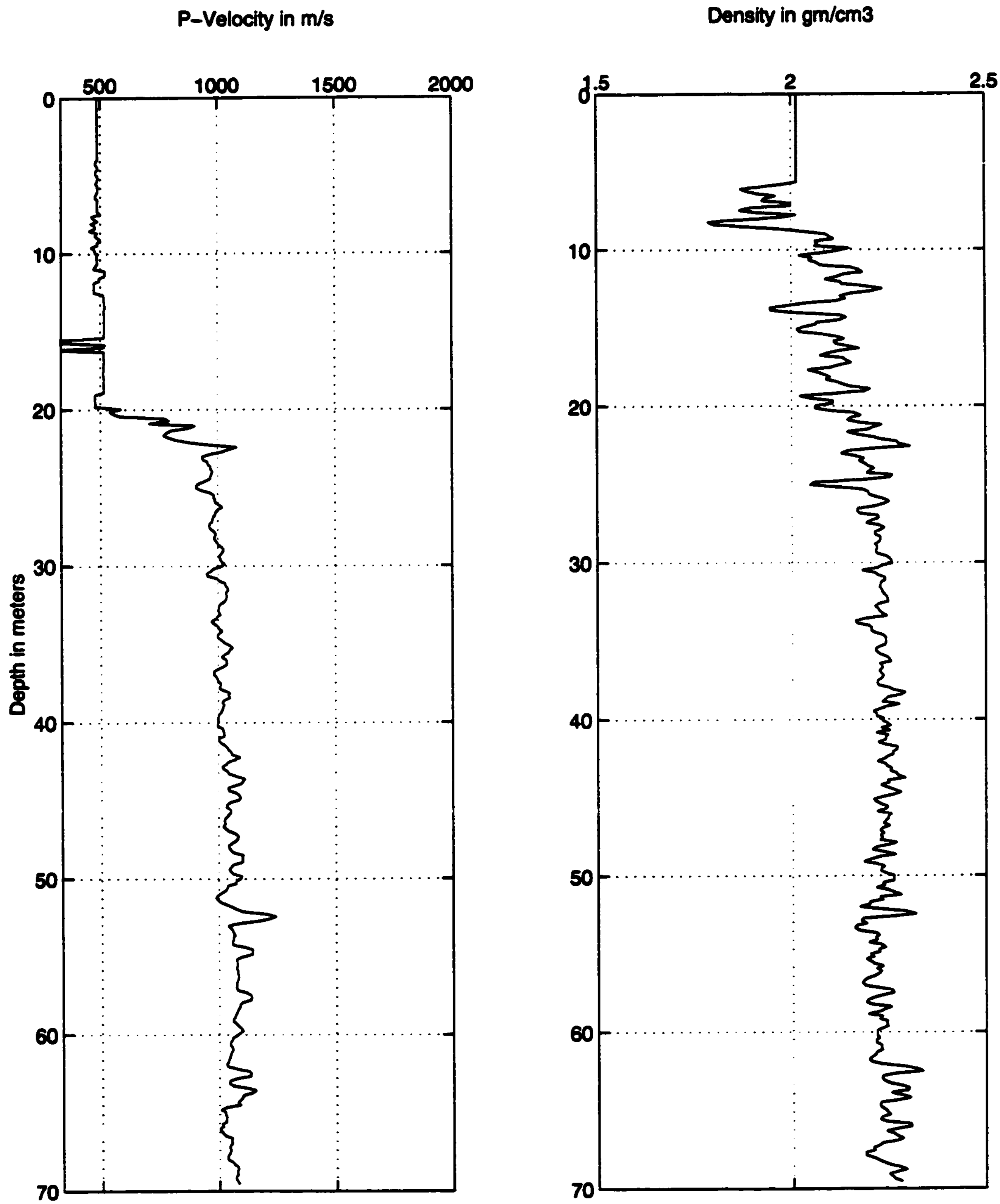

Figure 1.1: Well-logs showing p-wave velocity and medium's density at regular intervals (.5 ft).

1.3 Green's Function

Consider the differential equation

$$Lu = a_0(x)u'' + a_1(x)u' + a_2(x)u = f(x) \quad a < x < b \tag{1.22}$$

where a_0, a_1, a_2 are continuous and f is piecewise continuous in $a \leq x \leq b$. We require u to satisfy the two homogeneous boundary conditions

$$\left.\begin{aligned} B_1(u) = 0 &: \alpha_{11}u(a) + \alpha_{12}u(a) + \beta_{11}u(b) + \beta_{12}u'(b) = 0 \\ B_2(u) = 0 &: \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = 0 \end{aligned}\right\} \tag{1.23}$$

where the two vectors $(\alpha_{11}, \alpha_{12}, \beta_{11}, \beta_{12})$ and $(\alpha_{21}, \alpha_{22}, \beta_{21}, \beta_{22})$ are independent and real.

The Green's function is a particular fundamental solution satisfying two prescribed boundary conditions. So, the Green's function $G(x, x_0)$ is the solution of the auxiliary problem

$$LG = a_0(x)\frac{d^2G}{dx^2} + a_1(x)\frac{dG}{dx} + a_2(x)G = \delta(x - x_0), \quad a < x < b; \tag{1.24}$$

$$\text{with } B_1(G) = 0, \ B_2(G) = 0. \tag{1.25}$$

In other form, not involving the delta function, system (1.24) can be written as

$$\left.\begin{aligned} &LG = 0, \quad x \neq x_0; \\ &B_1(G) = 0, \quad B_2(G) = 0; \\ &G \ \text{ continuous at } \ x = x_0; \\ &\left.\frac{dG}{dx}\right|_{x=x_0^+} - \left.\frac{dG}{dx}\right|_{Px=x_0^-} = \frac{1}{a_0(x_0)}. \end{aligned}\right\} \tag{1.26}$$

Existence and Uniqueness

We state without proof the following theorem:

Theorem. *If the completely homogeneous system*

$$Lu = 0, \quad a < x < b; \quad B_1(u) = 0, \ B_2(u) = 0$$

has only the trivial solution, the Green's function exists and is unique.

Application

The Green's function is used to solve inhomogeneous systems and to translate the eigenvalue problem into an integral equation. To guarantee the existence of $G(x, x_0)$, system (1.24) is assumed to have the trivial solution only. To illustrate how Green's function is used, we will consider a completely inhomogeneous system. But, first, let us consider the following system

$$Lu = f, \ a < x < b; \quad B_1(u) = 0, \ B_2(u) = 0. \tag{1.27}$$

This system has the one and only solution

$$u(x) = \int_a^b G(x, x_0) f(x_0) dx_0. \tag{1.28}$$

This can be proved easily by direct substitution in (1.27).

One should note that the solution of (1.26) depends continuously on f. Let u_1 and u_2 be two solutions corresponding to f_1 and f_2, respectively. Then

$$|u_1 - u_2| \leq \int_a^b |G(x, x_0)| \, |f_1(x_0) - f_2(x_0)| dx_0 \leq M(b - a)\epsilon, \tag{1.29}$$

where $|G| \leq M$ (since it is continuous in the rectangle $a \leq x, x_0 \leq b$) and $|f_1(x) - f_2(x)| < \epsilon$. So the accuracy of the solution is proportional to the accuracy of f.

Before we go to the case of completely inhomogeneous system, we need the following preliminaries on linear operators. Given the operator L in (1.22), $u(x)$ and

$v(x)$ two arbitrary and twice-differentiable functions, integrate $\int_a^b vLudx$ by parts to get the following relation (known as Green's formula)

$$\int_a^b (vLu - uL^*v)dx = [J(u,v)]_a^b \tag{1.30}$$

where L^* is an operator (known as the formal adjoint of L) and J is known as the conjunct of u and v.

$$L^*v = a_0v'' + (2a_0' - a_1)v' + (a_0'' - a_1' + a_2) \tag{1.31}$$

$$J(u,v) = a_0(vu' - uv') + (a_1 - a_0')(uv). \tag{1.32}$$

The differential operator L is said to be formally self-adjoint if $L^* = L$. The necessary and sufficient condition for an operator to be formally self-adjoint is

$$a_0'(x) = a_1(x).$$

In such a case

$$L = \frac{d}{dx}\left[a_0(x)\frac{d}{dx}\right] + a_2(x). \tag{1.33}$$

Let D be the set of all twice-differentiable functions u which satisfy the two homogeneous boundary conditions given in (1.23). We say that v belongs to D^* if it is twice-differentiable and if

$$[J(u,v)]_a^b = 0 \quad \text{for every} \quad u \text{ in } D. \tag{1.34}$$

v will belong to D^* if it satisfies two conditions similar to (1.23) but with different coefficients known as the adjoint boundary conditions.

Definition. The boundary-value problem or system

$$Lu = f, \;\; a < x < b; \;\; B_1(u) = 0 \; B_2(u) = 0 \tag{1.35}$$

is said to be self-adjoint if $L = L^*$ and $D = D^*$.

Green's function is the solution of system (1.24). The adjoint Green's function $H(x, x_0)$ is the solution of

$$L^*H = \delta(x - x_0), \quad a < x, x_0 < b; \quad B_1^*(H) = 0,\ B_2^*(H) = 0 \tag{1.36}$$

which implies that $[J(G, H)]_a^b = 0$.

Multiplying (1.24) by $H(x, x_1)$ and (1.36) by $G(x, x_0)$ and integrate from $x = a$ to $x = b$ we get

$$H(x, x_0) = G(x_0, x). \tag{1.37}$$

If the system (12) is self-adjoint, then

$$G(x, x_0) = G(x_0, x) \tag{1.38}$$

This is the symmetry property of the Green's function.

Completely Inhomogeneous System

Consider the following system

$$Lu = f,\ a < x < b; \quad B_1(u) = \alpha,\ B_2(u) = \beta. \tag{1.39}$$

Multiply the differential equation in (1.39) by $G(x, x_0)$ and that in (1.24) by $u(x)$, subtract and integrate from $x = a$ to $x = b$, using Green's formula, we get

$$u(x_0) = \int_0^b G(x, x_0) f(x) dx - [J(u, G)]_a^b. \tag{1.40}$$

If $\alpha = \beta = 0$ in (1.24), then

$$u(x_0) = \int_a^b G(x_0, x) f(x) dx. \tag{1.41}$$

This is the same result we found in (1.15).

1.4 Perturbation Techniques

Given a linear stochastic operator $L(\vec{r},\gamma)$ and a non-random function $g(\vec{r})$, our problem is to solve for v in

$$L(\vec{r},\gamma)v(\vec{r}) = g(\vec{r}). \tag{1.42}$$

One of the most powerful approximation techniques is the perturbation approach. Perturbation methods are applicable to those cases where the random parametric variations are 'week' or 'small'. This class of problems is of great physical importance.

We consider a direct perturbation scheme for solving equation (1.42) when the differential operator $L(\vec{r},\gamma)$ can be expressed in the form

$$L(\vec{r},\gamma) = L_0(\vec{r}) + \epsilon L_1(\vec{r},\gamma) + \epsilon^2 L_2(\vec{r},\gamma) + \cdots \tag{1.43}$$

where ϵ is a small parameter and L_0 is a deterministic operator. The operators $L_1, L_2, \ldots$ are stochastic operators. In the direct application of the perturbation scheme, we seek a solution of equation (1.42) in the form

$$v(\vec{r}) = v_0(\vec{r}) + \epsilon v_1(\vec{r}) + \epsilon^2 v_2(\vec{r}) + \cdots \tag{1.44}$$

Upon substituting this solution representation and operator representation (1.43) into equation (1.42) and equating terms of the same order of ϵ, we get the following system:

$$\begin{aligned}
L_0(\vec{r})v_0(\vec{r}) &= g(\vec{r}) \\
L_0(\vec{r})v_1(\vec{r}) &= -L_1(\vec{r},\gamma)v_0(\vec{r}) \\
L_0(\vec{r})v_j(\vec{r}) &= -[L_1(\vec{r},\gamma)v_{j-1}(\vec{r}) + L_2(\vec{r},\gamma)v_{j-2}(\vec{r}) + \cdots \\
&\qquad + L_j(\vec{r},\gamma)v_0(\vec{r})], \quad j = 1, 2, \ldots
\end{aligned} \tag{1.45}$$

In the case where the differential operator $L_0(r)$ is linear with constant coefficients, its inverse operator, $L_0^{-1}(\vec{r})$ is well defined. It is an integral operator whose kernel is the weighting function associated with $L_0(\vec{r})$. The Green's function is an example of such weighting functions.

Considering only particular solutions, the solution of equations (1.45) in this case can be written in the form

$$\begin{aligned} v_0(\vec{r}) &= L_0^{-1}(\vec{r})g(\vec{r}) \\ v_j(\vec{r}) &= -L_0^{-1}(\vec{r})\sum_{i=1}^{j} L_i(\vec{r},\gamma)v_{i-j}(\vec{r}), \quad j = 1, 2, \ldots \end{aligned} \tag{1.46}$$

Hence, the solution of $v(\vec{r})$ can now be written as

$$v(\vec{r}) = [1 - \epsilon L_0^{-1}(\vec{r})L_1(\vec{r}) - \epsilon^2 L_0^{-1}(\vec{r})(-L_1(\vec{r},\gamma)L_0^{-1}(\vec{r})L_1(\vec{r},\gamma) + L_2(\vec{r},\gamma))] + \cdots L_0^{-1}(\vec{r})g(\vec{r}). \tag{1.47}$$

Equation (1.47) gives an explicit solution for $v(\vec{r})$. However, it is in general difficult to calculate anything beyond the simple moments of $v(\vec{r})$. This can be done only for the first few terms because the evaluation of higher-order terms becomes exceedingly complex, Soong [1973].

Most of the work on wave propagation in random media are based upon perturbation expansions. It has been shown that such expansions are generally either divergent or too slowly convergent for large values of $\vec{r}$, Frisch [1968]. The divergence problem is overcome by cutting off the series at the k-th term. In the second difficulty, terms proportional to a certain power of $|\vec{r}|$ arise in the series which makes it tend to infinity as $|\vec{r}| \rightarrow \infty$. Hence, the perturbation expansion is limited to a short distance from the source, Sobczyk [1985]. Therefore, formal perturbation techniques have been used in wave propagation analysis rather than the rigorous perturbation technique.

CHAPTER 2

WAVES IN RANDOM MEDIA

2.1 Stochastic Media

The theory of wave propagation in random media is concerned with motions in media whose properties have so complicated space and time dependence that they can only be described statistically.

The observation of structures of different materials and natural media leads to the conviction that classical models and theories are too idealized and do not adequately reflect the complexity and heterogeneity of numerous real media. One example of these theories is the classical elasticity theory. Treating the elastic continuum of the media as being homogeneous is an idealization which is valued only on a certain particular scale of magnitude. Long wavelength disturbance in metals, for example, can be considered as waves in homogeneous media.

However, a mixture of several discrete components (phases) make up a very complex heterogeneous material which cannot be idealized. These materials which are usually called composites, are of great importance in practice. Such real media as soils, rocks, concretes and ceramics provide further examples of materials with a very complicated structure. In order to embrace the heterogeneous, complicated and undetermined structure of real media, an alternative mathematical model of such a medium has to be introduced.

It is often assumed that the heterogeneity of a medium is regular; this means that properties of the medium are described by means of ordinary (non-random) functions of the spatial coordinates. Such media may be called regularly (or deterministically) heterogenous. An example of such medium is the one whose physical properties change linearly with depth. However, the inhomogeneity that we are concerned with is a random one.

Linear elasticity theory is based on Hooke's law and the equilibrium equation. If the model is homogeneous and isotropic, then the displacement field $\vec{u}$ is governed by the Lamé equation

$$(\lambda + \mu) \nabla (\vec{\nabla} \vec{u}) + \mu \nabla^2 \vec{u} - \rho \frac{\partial^2 \vec{u}}{\partial t^2} + \vec{F} = 0 \tag{2.1}$$

where λ and μ are the Lamé parameters, ρ is the density and $\vec{F}$ is the body force.

Some real media like sea water or the Earth's crust can be approximated by a stack of homogeneous and isotropic layers. Wave propagation in such media has been extensively studied.

One medium which displays significant microscopic heterogeneity of structure and plays an important role in practice is the porous medium. It is a continuous medium which consists of a solid matrix cemented together by clay and has small fluid-filled pores. To study wave propagation in such media, many mathematical models have been postulated, the best known being the poro-elasticity proposed by Biot, the porous continuum is regarded as uniform and its complicated geometry is left out of account, Korvin [1979].

A more satisfactory approach to the analysis of media with a heterogeneous and complicated structure should express the heterogeneity and randomness of the material in an explicit way. Such an approach is embodied in the stochastic modeling of complicated real media, Sobczyk [1985]. Many authors have used this type of model to study elastodynamic problems; Korvin [1977], Askar and Cokarak, [1998].

Stochastic or random medium is a mathematical model which accounts for the random heterogeneity of a real medium. Properties of the medium are described by means of random functions of space $X(\vec{r}, \gamma)$, where $\vec{r} = (x, y, z)$ and γ is an elementary event ($\gamma \in \Gamma$). $X(\vec{r}, \gamma)$ can be a scalar, vector or tensor. Also, the random field $X(\vec{r}, \gamma)$ may have continuously varying sample functions, or it may be just a discrete-valued function. $X(\vec{r}, \gamma)$ can be completely characterized by two ways; the n-point distribution functions or by the characteristic functional.

In any practical analysis the most we hope for is the lower order distribution functions of $X(\vec{r}, \gamma)$ or the low order moments like the mean function $M_X(\vec{r}) = \langle X(\vec{r}, \gamma) \rangle$ and the correlation function $K_{XX}(\vec{r}_1, \vec{r}_2)$. The correlation function plays a basic role in wave propagation in random media. In our analysis we will assume a statistically homogeneous and isotropic random field in which case $K_{XX}(\vec{r}_1, \vec{r}_2)$ is a function of the single scalar variable $q = |\vec{r}_2 - \vec{r}_1|$. The most common forms of such correlation function are

$$K_{XX}(q) = \sigma_X^2 e^{-q^2/a^2}, \tag{2.2}$$

$$K_{XX}(q) = \sigma_X^2 e^{-q/a}, \tag{2.3}$$

where σ_X^2 is the variance, $M_X(\vec{r}) = 0$, and a is a parameter called correlation radius or distance. More complicated forms of the correlation function are needed for certain media.

Continuous Stochastic Media

In such media, random functions are characterized by continuous and smooth realizations. A composite material with not too different constituents, whose properties as a whole change smoothly, is a good example.

In our analysis we will assume the layer through which the incident waves propagate, to be a continuous stochastic medium. The validity of such a model depends on the layer compaction and cementation. Because, if the layer is very porous and filled with air, this model is not adequate.

The problem of wave propagation analysis in continuous stochastic media leads to the stochastic differential equation

$$L(\vec{r}, \gamma)v(\vec{r}) = g(\vec{r}) \tag{2.4}$$

where L is a random differential operator with respect to space. We will assume L to be linear. This assumption enables us to use the superposition law and the Green's function.

We will restrict ourselves to the problem of scalar harmonic wave propagation in continuous stochastic medium (stochastic Helmholtz equation)

$$\nabla^2 v + k_0^2 n^2(\vec{r}, \gamma)v = g(\vec{r}). \tag{2.5}$$

Discrete Stochastic Medium

Mixing two materials with significantly different physical properties produces a real medium which can be modelled as a discrete stochastic medium. The properties of such media can be represented by random functions with discontinuous sample functions. Stress-free holes and pores filled with fluid are two examples of such media. Most of the literature considers the case of two-phase media in such manner.

2.2 Stochastic Models

Consider a medium where the local density ρ and rigidity μ show random inhomogeneities depending on the spatial coordinates (x, y, z). Each physical parameter can be expressed explicitly as a series, Korvin [1977]; e.g.,

$$\rho(\vec{r}, \epsilon) = \rho_0(\vec{r}) + \rho_1(\vec{r})\epsilon(\vec{r}) + \rho_2(\vec{r})\epsilon^2(\vec{r}) + O(|\epsilon|^3) \tag{2.6}$$

where ρ_0, ρ_1 and ρ_2 are given, non-random, functions and $\epsilon(\vec{r})$ is a homogeneous random field of zero mean. From the geophysical point of view, the following three cases seem to be the most important:

$$\rho(\vec{r}) = \rho_0(\vec{r}) + \epsilon(\vec{r}) \tag{2.7}$$

$$\rho(\vec{r}) = \rho_0(r)[1 + \epsilon(\vec{r})] \tag{2.8}$$

$$\rho(\vec{r}) = \frac{\rho_0(\vec{r})}{1+\epsilon(\vec{r})} = \rho_0(\vec{r}) - \rho_0(\vec{r})\epsilon(\vec{r}) + \rho_0(\vec{r})\epsilon^2(\vec{r}) + O(|\epsilon|^3) \tag{2.9}$$

Density distribution (2.9) is widely used in case of wave propagation in random media. This is due to the fact that in this case the wave number K assumes the form

$$K(\vec{r}) = K_0(\vec{r})[1 + \epsilon(\vec{r})] \tag{2.10}$$

which makes the perturbation methods easy to carry out, Keller [1964].

The second form (2.8) is used to describe waves in a turbulent atmosphere, Tataraski [1961]. The first type of distribution (2.7) seems to be the most appropriate in seismic practice, Korvin [1977].

Here, we shall limit ourselves to the functions $\rho_0(\vec{r}), \rho_1(\vec{r}), \rho_2(\vec{r}), \ldots$ which depend on the single coordinate z (i.e., the depth). Therefore, all these coefficients are constant along planes parallel to (x, y). In the region of study, if we pick random points p_1, p_2, then the totality of parameter variations (e.g., density $\rho(z)$ along a vertical line pointing downward in the positive z-axis) is just one realization of a stationary random process.

It must be mentioned here that such a random description is also valid for rigidity μ, and velocity β. To illustrate how density inhomogeneities are distributed in depth, real density measurements are shown in Fig (2.1) . If we assume a distribution of type (2.8), then the mean density for the top 30 m of the well is $\rho_0 = 2.1$ gm/cm^3 and the random parameter $\epsilon(\vec{r})$ has standard deviation $\epsilon = \pm.08$ gm/cm^2. Assuming autocorrelation function of the form

$$K_{\rho\rho}(z) = \epsilon^2 e^{-z/z_0}, \tag{2.11}$$

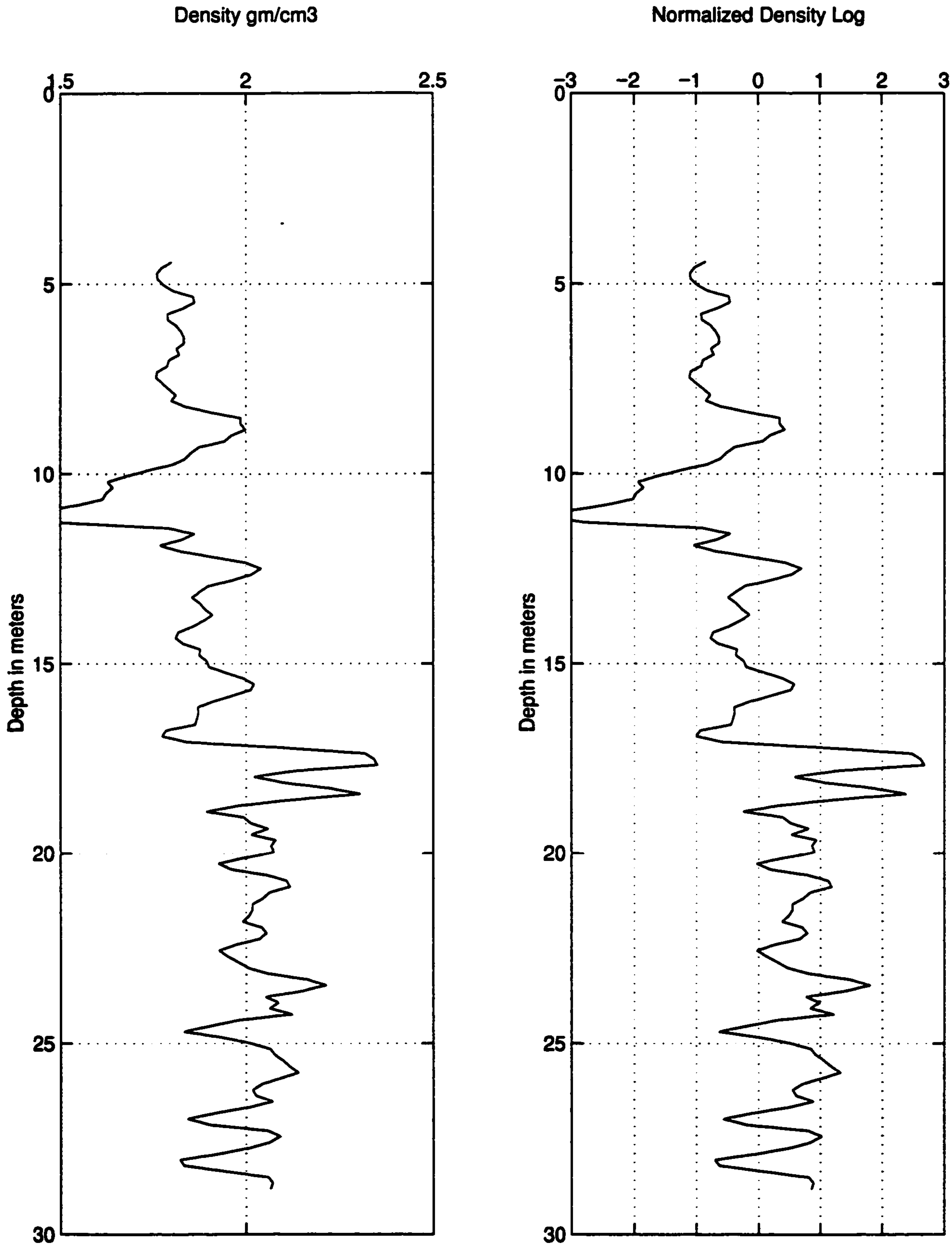

Figure 2.1: Density log and the (dimensionless) random part of it.

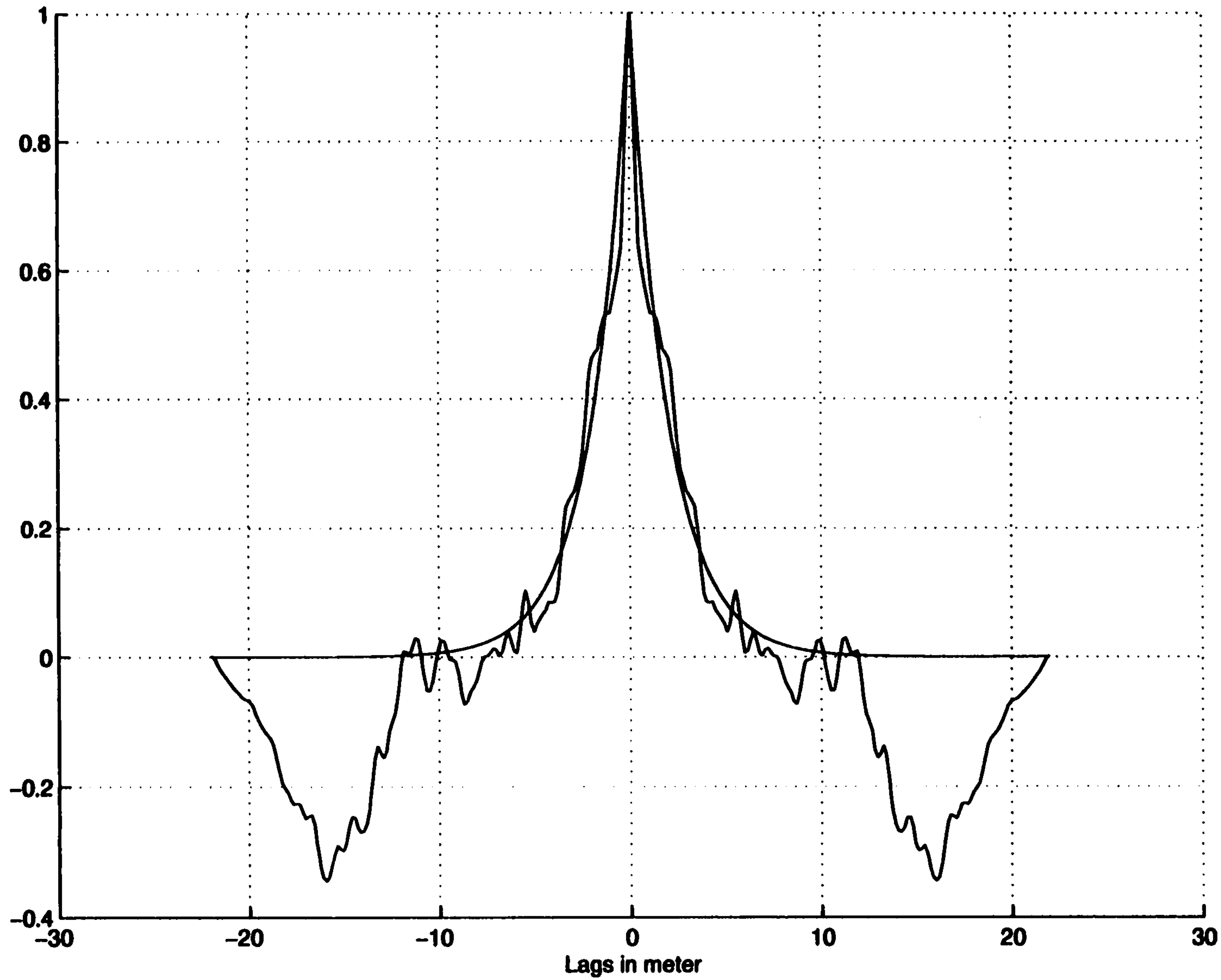

Figure 2.2: The autocorrelation function as computed from the random density fluctuation shown in Figure 2.1 fitted with the theoretical form e^{-2Z}.

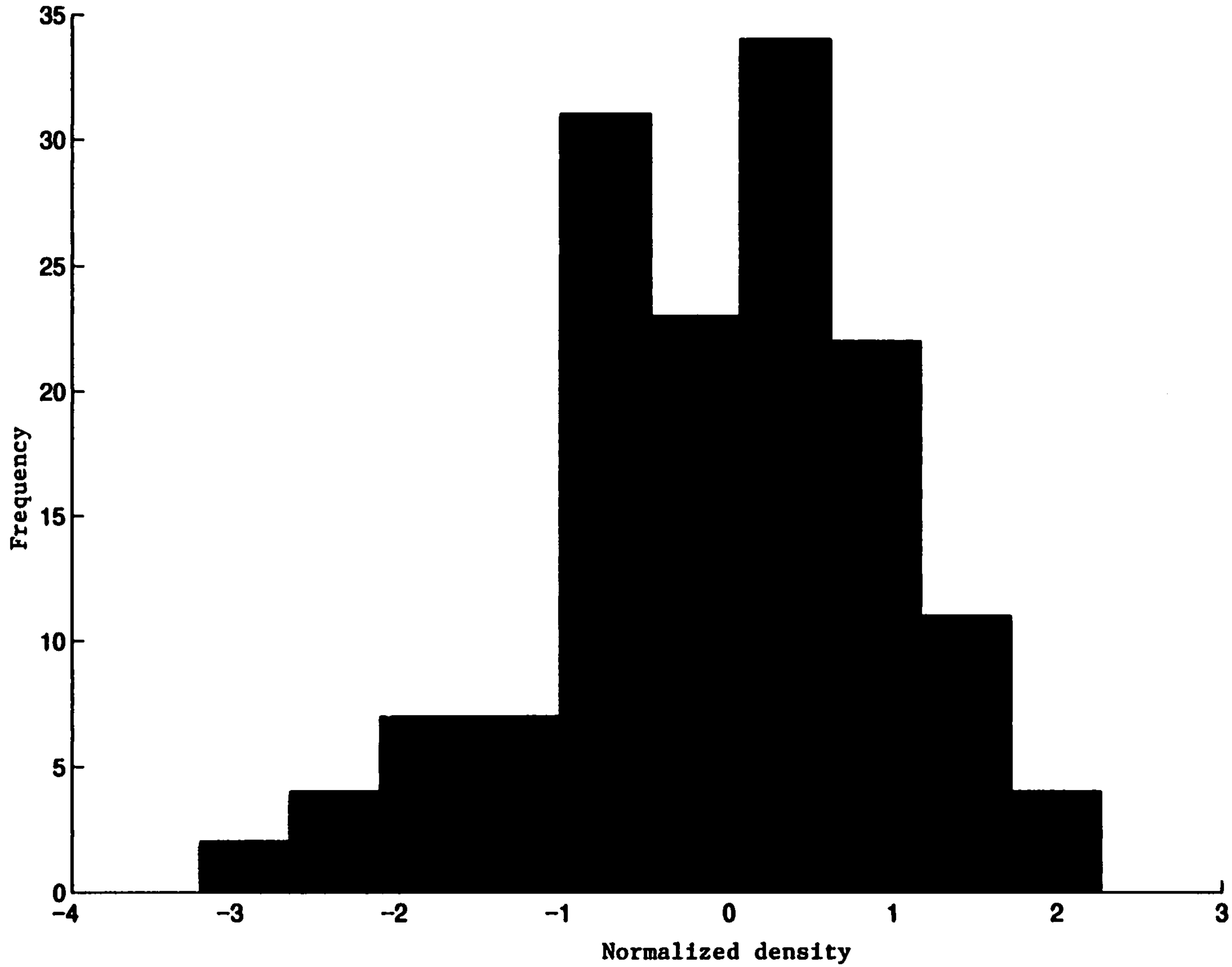

Figure 2.3: Histogram of the random density fluctuation in Figure 2.1.

we find that $z_0 \approx 2\,\mathrm{m}$ is the best correlation distance that fits the data.

In our study, we want ϵ to have a unit standard deviation; therefore we introduce a new parameter $\rho_1(z,\gamma)$, such that

$$\rho(z) = \rho_0(z)[1 + \epsilon\rho_1(z,\gamma)] \tag{2.12}$$

where the random parameter ρ_1 has zero mean and standard deviation of one. In our data, we found $\epsilon = .08$, $\rho_0 = 2.1$. The normalized random parameter $\rho_1(\vec{z},\gamma)$ is plotted in Fig. 2.1. Note that the difference between this and a computer generated random process is the zero correlation length in the latter one. Both autocorrelations, computed from the data, and the function (2.11) are shown in Fig. (2.2). Figure 2.3 is a histogram of $\rho_1(z,\gamma)$.

Each realization represents a density log which is acquired in situ by lowering measuring tools down the borehole and pulling it up slowly while it is recording. The mean value ρ_0 is taken with respect to all realizations. The autocorrelation function shown in Fig. (2.2) has been computed from a single realization, but if ergodicity holds, the function should characterize the whole area.

As you can seen from Fig. (1.1), the lower layer has more homogeneous parameters distribution. Therefore, we will assume that the earth model is composed of a top near-surface layer whose parameters are random stochastic processes. Below this layer is a semi-infinite half-space whose parameters are constants. In our study $\rho_0(\vec{r}), \rho_1(\vec{r}), \ldots,$ which appear in the general equation (2.6), are taken to be constant for simplification. Another more realistic model is to use simple linear functions for ρ_0 or ρ_1. Our model is a half-space covered by a layer. Equation (2.12) represents parameters distribution in the layer.

Korvin [1972] showed that there is a significant difference between the various forms of velocity distributions (2.7), (2.8), and (2.9) In particular, the models (2.7) and (2.8) slow the wave down.

2.3 Random Wave Propagation

There are two approaches to the problem of wave propagation in random media: the wave formalism and the ray formalism. The former is based upon linear partial differential wave equations. This formalism is used in sound wave scattering. The ray formalism is an asymptotic approximation, which applies to problems where the wavelength is much smaller than the correlation length of random homogeneities, such as the propagation of ultrasonic waves in the ocean.

In our analysis, the problem of wave propagation in random media leads to linear stochastic partial differential equations whose coefficients are random functions of space. These equations belong to the class of linear stochastic equations. There is no complete theory of linear differential equations with non-constant coefficients, the case gets worse when the coefficients are random functions. However, a lot of work has been done on this problem. Comprehensive survey articles are given by Frisch [1968] and by Barabanekov, Rytov and Tatarski [1961]. Chow [1975] gives a systematic presentation of a variety of approximation techniques for analyzing random wave motion when the random fluctuations in the elastic properties are weak. Sobczyk [1985] published a book "Stochastic Wave Propagation" which is being used as a main reference. The book gives a concise and unified exposition of the existing methods of analysis of linear stochastic waves with particular reference to the most recent results.

We will assume a continuous stochastic medium. Therefore, we will consider only those methods applicable to such a medium.

Wave Propagation in a Continuous Stochastic Medium

Waves can be defined as disturbances propagating in space with finite speed and carrying energy. Waves are described by the scalar displacement field $v(\vec{r}, t), \vec{r} = (x, y, z)$. $v(\vec{r}, t)$ depends on the source of disturbance and the properties of the medium in which the wave propagates. Similar wave equations can be written for P and SV waves by allowing v to be a vector. The simplest form of waves is the harmonic oscillation with a constant frequency ω. This form of waves will be considered throughout this Thesis. Harmonic waves are very important in applications because every wave can be represented as a superposition of plane harmonic waves with different frequencies and wave numbers, provided our wave equation is linear. Therefore, from now on, we will study stochastic linear partial differential equations which describe harmonic waves propagating through a random medium. These assumptions simplify our problem and enable us to use some very useful methods like Green's function method.

Waves in random media are subject to a phenomenon called scattering. Scattering causes attenuation of incident wave amplitude, retardation of its propagation velocity, depolarization and other phenomena.

The main objective of the analysis of wave propagation in continuous stochastic media is to provide quantitative and qualitative information concerning the scattering phenomena. The problem can be described in the form

$$L(\gamma)v = g$$

or

$$[L_0 + L_1(\gamma)]v = g$$

where v is the unknown wavefield, g a non-random source term, and both L_0 and L_1 are linear differential operators. But, unlike L_0 whose coefficients are constant, $L_1(\gamma)$'s coefficients are random functions with average values equal to zero. The stochastic Helmholtz scalar equation becomes

$$\nabla^2 v + k^2 n^2(\vec{r}, \gamma) v = g(\vec{r})$$

where k (the wave number) is a positive real number, $n(\vec{r}, \gamma)$ (characterizing physical parameters of the medium) is a given random function characterizing the inhomogeneity of the medium. The solution will be $v(\gamma, \vec{r})$ which is also a random field. In Section (1.2), we gave an outline of the wave equation derivation as well as Helmholtz equation.

Methods of Solution

There is no complete theory for random partial differential equations. Therefore, the analysis of wave propagation must be done using formal methods. Although no full mathematical justification exists, yet these methods have been very successful. The difficulties are of two kinds, physical and mathematical. The mathematical difficulties arise from the fact that, although the equations are linear, the solution depends nonlinearly upon the stochastic coefficients. Among these difficulties are the divergence of the standard perturbation expansions, and the impossibility to compute all the moments of the coefficients. The physical difficulties are due to the fact that some simplifications or assumptions do not have practical justifications. For example, the correlation function of the refractive index $n(\vec{r}, \gamma)$ is usually taken to be in the form $\exp\{-r/a\}$. This is not a good approximation for some models where the random changes are continuously differentiable sample functions.

In view of these difficulties, care is needed when using these methods. Existing formal methods are based on certain simplifying assumptions and physical hypotheses. Like in most of the related studies, we will study scalar equations only and assume our medium to be weakly inhomogeneous, i.e., that the fluctuations in its properties are small. This will enable us to use perturbation theory.

CHAPTER 3

LOVE WAVES

3.1 Basic Equations

Love wave is a type of surface wave which travels parallel to the surface of the Earth. It is horizontally polarized and is characterized by exponentially decaying amplitude with depth z.

Love wave recorded in a seismological station are interpreted to study the internal structure of the earth. However, they are considered as noise to be eliminated (in exploration geophysics). One of the early observations of seismology was the presence of large transverse (horizontal) components of displacement in the main tremor of an earthquake. However, such displacements do not occur for Rayleigh waves, which contain displacements only in the vertical plane.

Using the previously introduced Earth model of a homogeneous, isotropic half-space, such surface SH waves are not possible. Consider a half-space with density ρ, rigidity μ and velocity β, Fig. 3.1. Assume an incident wave which is harmonic in time and along the x-axis and decays exponentially with depth. Then the displacement in the y-direction is given by

$$v = Ae^{-bz}\exp[ik(x-ct)] \tag{3.1}$$

where $c = \omega/k$ is the wave propagation velocity and Real(b) must be positive. Substituting this solution to the governing scalar wave equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}, \tag{3.2}$$

after canceling the exponential terms and A we get

$$b = k\left[1-\left(\frac{c}{\beta}\right)^2\right]^{1/2}. \tag{3.3}$$

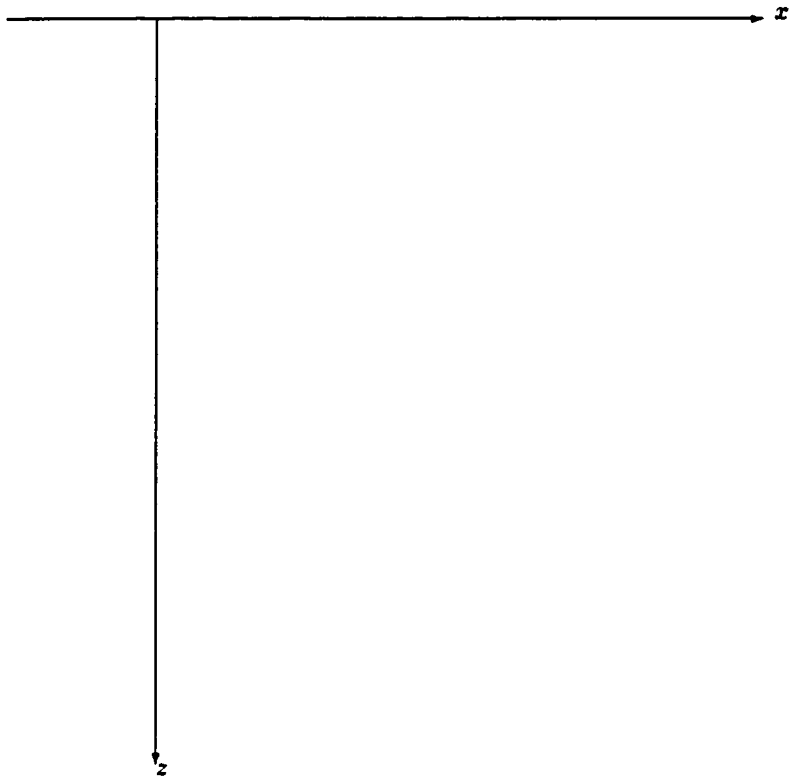

Figure 3.1: The Semi-infinite Half-space Model

Using the boundary condition at the free surface would mean $\frac{\partial v}{\partial z} = 0$ at $z = 0$. This implies $A = 0$ or $b = 0$. Neither case represents a surface wave. This discrepancy motivated Love [1911] to consider a more realistic model. He assumed that the Earth consists of an isotropic homogeneous half-space with an overlying horizontal layer. Although this model oversimplifies the actual Earth, Love was able to explain the resulting SH waves are surface waves trapped in the upper superficial layer. These waves are guided waves which propagate as multiple reflections within the layer. The main part of this analysis will be shown in the next section.

The actual surface layer of the Earth consists of an aggregate of different materials which are poorly consolidated. Early models have imposed ideal assumptions upon the elastic properties of the Earth's material, upon the nature and shape of the internal boundaries and upon the physical conditions at the seismic source.

Since 1911 considerable effort has been done to study Love wave propagation using more complicated Earth models. The complexities were introduced in the layers' interfaces, elastic parameter distributions and source nature. But, in all models up to now, this had a layer or more overlying a half-space.

Because the analysis of propagation in a layer with an arbitrary surface shape is quite formidable. specific surface shapes have been considered. For example. Sato [1961] studied a flat surface with a step, Takahashi [1964] assumed hyperbolic surface, and Paul [1965] a sinusoidal surface. Wolf [1967] solved the problem for a restricted class of shapes. Kelly [1983] studied Love wave propagation using numerical modeling. The models he used were of geological interest like channels. Lately, investigators have been treating the top or bottom surface of the layer as one having randomly varying shape which causes scattering of the incident wave field at the surface.

The other trend is to consider deterministic variation of the elastic properties. Various authors, for example, Ghosh [1970], Chattopadhya et al. [1981], have considered a model of a homogeneous half-space overlayed by an inhomogeneous layer.

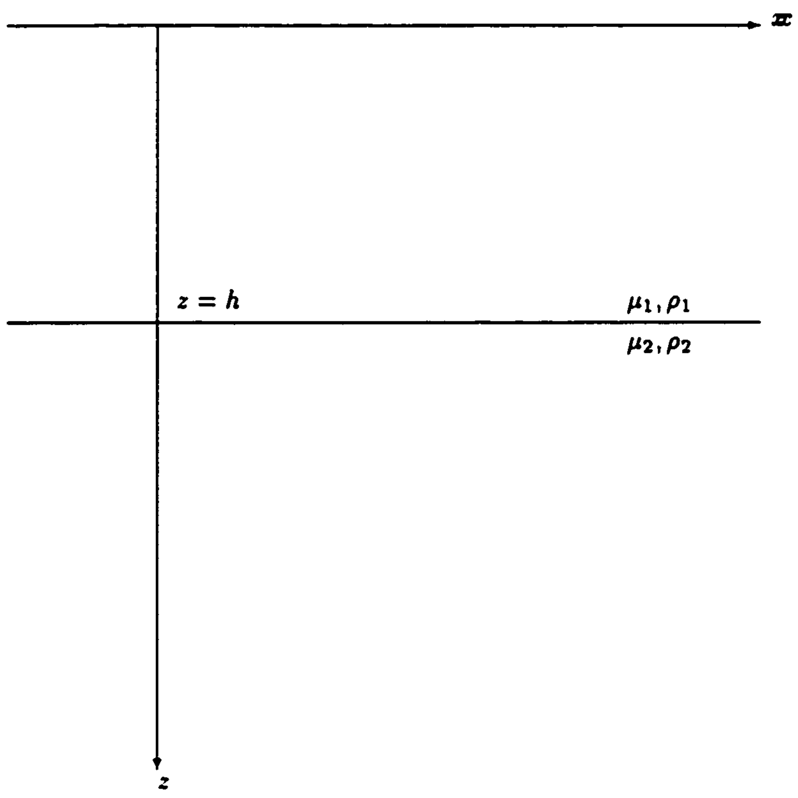

Figure 3.2: One Layer and a Half-space (2D) Model (Homogeneous Case)

Kazi and Abu-Safiya [1981-82] studied a two-layer model which was carried further by Zaman et al. [1989-90] who assumed the upper density in the layer to be variable.

The earth is generally assumed in both theory and practical application to be isotropic, or at most, to be composed of isotropic layers. Sufficiently detailed studies, however, often indicate the presence of an isotropy even in granular material. Anderson, [1961] studied the dispersion of Rayleigh waves in a layered anisotropic solid and Love waves in simple layered media. A more complete boundary value problem for SH waves in multi-layered material was given by the same author in 1962.

In this thesis we are considering the effect of the high frequency fluctuation of the density and the elastic parameters on Love wave propagation. These variables are modeled as random functions. As I was working on this problem Bhattacharyya [1998] used a similar mathematical model to solve for the dispersion equation for a combination of Love and Rayleigh type surface waves using different techniques.

3.2 Dispersion Relation

Consider a homogeneous elastic layer of thickness h having rigidity μ, and density ρ_1, overlying a homogeneous elastic half-space with rigidity μ_2 and density ρ_2. The horizontal plane $z = h$ is the interface between the two media, Figure 3.2. We denote by $V_i(x, z, t)$ the displacement component in the upper layer ($i = 1$) and in the half-space ($i = 2$) respectively. Throughout the subscript 1 refers to the layer and 2 refers to the half-space. Suppose that u, v and w are the displacement components in the three coordinate direction x, y and z, respectively.

$$\nabla^2 V_1 - \frac{1}{\beta_1^2}\frac{\partial^2 V_1(\vec{r})}{\partial t^2} = g(\vec{r}, t); \qquad \text{for the layer , } 0 \le z < h, \tag{3.4}$$

and

$$\nabla^2 V_2 - \frac{1}{\beta_2^2}\frac{\partial^2 V_2(\vec{r})}{\partial t^2} = 0; \qquad \text{for the half-space, } z > h \tag{3.5}$$

where $g(\vec{r}, t)$ is the source term.

The boundary conditions in this case are:

Free Surface Condition,

$$\frac{\partial V_1}{\partial z} = 0; \qquad \text{at } z = 0. \tag{3.6}$$

Continuity of stress at the interface of the two media

$$\mu_1 \frac{\partial V_1}{\partial z} = \mu_2 \frac{\partial V_2}{\partial z}; \qquad \text{at } z = h, \tag{3.7}$$

and continuity of displacement,

$$V_1 = V_2; \qquad \text{at } z = h. \tag{3.8}$$

Assume that V_1, V_2 and g have the following form

$$V_1(\vec{r}, t) = V_1(z)e^{i(kx-wt)}, \tag{3.9}$$

$$V_2(\vec{r}, t) = V_2(z)e^{i(kx-wt)}, \tag{3.10}$$

$$g(\vec{r}, t) = 4\pi\delta(z, h)e^{i(kx-wt)}. \tag{3.11}$$

Using conditions (3.9) to (3.11), motion equations (3.4) and (3.5) can be written as:

$$\frac{d^2V_1(z)}{dz^2} - \alpha_1^2 V_1(z) = \frac{4\pi}{\mu_1}\delta(z-h); \quad 0 \le z < h \tag{3.12}$$

$$\frac{d^2V_2(z)}{dz^2} - \alpha_2^2 V_2(z) = 0; \quad z > h \tag{3.13}$$

where

$$\alpha_1^2 = k^2 - k_1^2 \tag{3.14}$$

$$\alpha_2^2 = k^2 - k_2^2 \tag{3.15}$$

with

$$k_1^2 = \frac{w^2}{\beta_1^2} \text{ and } k_2^2 = \frac{w^2}{\beta_2^2}.$$

Also, boundary conditions (3.6) to (3.8) can be rewritten as:

$$\left.\frac{dV_1(z)}{dz}\right|_{z=0} = 0 \tag{3.16}$$

$$\mu_1 \left.\frac{dV_1(z)}{dz}\right|_{z=h} = \mu \left.\frac{dV_2(z)}{dz}\right|_{z=h} \tag{3.17}$$

$$V_1(z)|_{z=h} = V_2(z)|_{z=h}. \tag{3.18}$$

Now, suppose we have a plane wave, so $V_1(z)$ and $V_2(z)$ will have the following form:

$$V_1(z) = A_1e^{\alpha_1 z} + A_2e^{\alpha_1 z}, \tag{3.19}$$

$$V_2(z) = B_1e^{-\alpha_2 z}. \tag{3.20}$$

Note that the second term in $V_2(z)$ was eliminated to fulfill the condition that $V_2(z) \to 0$ as $z \to \infty$. Now, to derive the dispersion relation, we use the boundary conditions (3.16) to (3.18) to find A_1, A_2 and B_1 in (3.19) and (3.20). Condition (3.16) gives

$$A_1 - A_2 = 0. \tag{3.21}$$

Condition (3.17) gives

$$-\alpha_1\mu_1A_1e^{-\alpha_1 h} + \alpha_1\mu_1A_2e^{\alpha_1 h} = -\alpha_2\mu_2B_1e^{-\alpha_2 z}. \tag{3.22}$$

Condition (3.18) gives

$$A_1e^{-\alpha_1 h} + A_2e^{\alpha_1 h} = B_1e^{-\alpha_2 h}. \tag{3.23}$$

We can put equations (3.21) to (3.20) in the matrix form

$$\begin{pmatrix} -1 & 1 & 0 \\ \alpha_1\mu_1e^{-\alpha_1 h} & \alpha_1\mu_1e^{\alpha_1 h} & \alpha_2\mu_2e^{-\alpha_2 h} \\ e^{-\alpha_1 h} & e^{\alpha_1 h} & e^{-\alpha_2 h} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.24}$$

This system only has non-trivial solution if the determinant of the coefficient matrix is zero:

$$-\left(\alpha_1\mu_1e^{\alpha_1 h}e^{-\alpha_2 h} - \alpha_2\mu_2e^{\alpha_1 h}e^{-\alpha_2 h}\right) - \left(-\alpha_1\mu_1e^{-\alpha_1 h}e^{-\alpha_2 h} - \alpha_2\mu_2e^{-\alpha_1 h}e^{-\alpha_2 h}\right) = 0,$$

which leads to

$$\frac{e^{\alpha_1 h} - e^{-\alpha_1 h}}{e^{\alpha_1 h} + e^{-\alpha_1 h}} = \frac{\alpha_2\mu_2}{\alpha_1\mu_1}. \tag{3.25}$$

If we replace α_1 by $i\alpha$ we get the well-known dispersion relation for Love waves

$$\tan\alpha h - \frac{\alpha_2\mu_2}{\alpha\mu_1} = 0. \tag{3.26}$$

From equation (3.20), phase velocity $C < \beta_2$, otherwise $V_2 \not\to 0$ as $z \to \infty$. Also, $C > \beta_1$, otherwise (3.26) has no solution. So, Love wave propagation is velocity bounded by the layer and half-space S-wave velocities, i.e.

$$\beta_1 < C < \beta_2. \tag{3.27}$$

3.3 Green's Function Method

Let us assume again that we have a horizontal homogeneous isotropic elastic medium of thickness h overlying a semi-infinite half-space. Also, suppose we have a point source of disturbance. Figure (3.2) shows the geometry of the problem.

The equation of motion for SH waves in the presence of a source term is

$$\mu \nabla^2 v - \rho_0 \frac{\partial^2 v}{\partial t^2} = 4\pi\sigma_1(\vec{r}, t). \tag{3.28}$$

Suppose that both v and σ_1 are harmonic in time (i.e. $v(\vec{r}, t)$ is replaced by $v(\vec{r})\, e^{i\omega t}$ and $\sigma_1(\vec{r}, t)$ by $\sigma_1(\vec{r})e^{i\omega t}$). This will reduce equation (3.28) to

$$\nabla^2 v + k_1^2 v = \frac{4\pi}{\mu_1}\sigma_1(\vec{r}). \tag{3.29}$$

Now we will assume that v and σ_1 are both functions of x and z only. There are two ways to eliminate the dependence of v and σ_1 on the x-coordinate. We can either assume the functions to be harmonic in x or take the Fourier transform of (3.29) with respect to x. Throughout this Thesis, we will adopt the first approach.

For any point in the layer, substituting $v(x, z) = V_1(z)e^{ikx}, \sigma_1(x, z) = \delta(z - h)e^{ikx}$ in equation (3.29) we get

$$\frac{d^2 V_1(z)}{dz^2} - \alpha_1^2 V_1(z) = \frac{4\pi}{\mu_1}\delta(z - h). \tag{3.30}$$

Similar equations govern wave propagation in the half-space

$$\frac{d^2 V_2(z)}{dz^2} - \alpha_2^2 V_2(z) = 0. \tag{3.31}$$

Equations (3.30) and (3.31) are exactly the same equations we used in the previous section, equations (3.12) and (3.13). Boundary conditions (3.16) to (3.18) also hold in this case.

This is the boundary value problem we will try to solve. To solve this problem, we will use the Green's function method Stakgold [1978]. This method depends on the Green's function of the two operators in (3.30) and (3.31).

Now looking back at our problem, equations (3.30) and (3.31), we find that both operators are self-adjoint, since $a_1(x) = 0$ and $a_0(x) = 1$ and $a_1(x) = a_0'(x)$, where $a_0(x)$ and $a_1(x)$ are as in equation (1.22). Therefore, the symmetry property of the Green's function holds.

For the upper layer, the Green's function $G_1(x, x_0)$ satisfies the differential equation

$$\left.\begin{aligned} &\frac{d^2G_1(z,z_0)}{dz^2} - \alpha_1^2 G_1(z,z_0) = \delta(z-z_0) \\ &\frac{dG_1(z,z_0)}{dz} = 0 \quad \text{at } z=0 \text{ and } z=h. \end{aligned}\right\}. \tag{3.32}$$

For the lower half-space, the Green's function is the solution to

$$\left.\begin{aligned} &\frac{d^2G_2(z,z_0)}{dz^2} - \alpha_2^2 G_1(z,z_0) = \delta(z-z_0) \\ &\frac{dG_2(x,z_0)}{dz} = 0 \quad \text{at } z=h \text{ and as } z \text{ approaches } \infty. \end{aligned}\right\}. \tag{3.33}$$

The solutions for system (3.32) and (3.33) are given in Appendix (A).

Let us use these Green's functions to solve for the displacement field of the Love waves.

Multiplying equation (3.30) by $G_1(z, z_0)$ and (3.32) by $V_1(z)$, subtracting and integrating from $z = 0$ to $z = h$, we get

$$\int_0^h \left(G_1(z,z_0)\frac{d^2V_1}{dz^2} - V_1(z)\frac{d^2G_1(z,z_0)}{dz^2} \right) dz = \frac{4\pi}{\mu_1}\int_0^h \delta(z-h)G_1(z,z_0)dz$$
$$- \int_0^h V_1(z)\delta(z-z_0)dz. \tag{3.34}$$

Using boundary condition (3.16) and those at (3.32), we get

$$\begin{aligned} V_1(z_0) &= \frac{4\pi}{\mu_1}G_1(h,z_0) - \left[G_1(z,z_0)\frac{dV_1(z)}{dz} - \frac{dG_1(z,z_0)}{dz}V_1(z)\right]_0^h \\ &= \frac{4\pi}{\mu_1}G_1(h,z_0) - G_1(h,z_0)\left[\frac{dV_1(z)}{dz}\right]_{z=h}. \end{aligned} \tag{3.35}$$

In a similar way, multiplying (3.31) by $G_2(z,z_0)$ and equation (3.33) by $V_2(z)$, subtracting and integrating from $z = h$ to infinity,

$$\int_h^\infty \left(G_2(z,z_0)\frac{d^2V_2}{dz^2} - V_2(z)\frac{d^2G_2(z,z_0)}{dz^2}\right)dz = -\int_h^\infty V_2(z)\delta(z-z_0)dz. \tag{3.36}$$

Using the boundary conditions (3.33), and the fact that $\frac{dV_2}{dz} = 0$ as z approaches infinity, we get

$$\begin{aligned} V_2(z_0) &= -\left[G_2(z,z_0)\frac{dV_2(z)}{dz} - \frac{dG_2(z,z_0)}{dz}V_2(z)\right]_h^\infty \\ &= G_2(h,z_0)\left[\frac{dV_2(z)}{dz}\right]_{z=h}. \end{aligned} \tag{3.37}$$

Using boundary conditions (3.17) and (3.18)

$$\left[\frac{dV_1}{dz}\right]_{z=h} = \frac{\frac{4\pi}{\mu_1}G_1(h,h)}{G_1(h,h) + G_2(h,h)\frac{\mu_1}{\mu_2}} = \frac{\frac{4\pi\mu_2}{\mu_1}G_1(h,h)}{\mu_2 G_1(h,h) + \mu_1 G_2(h,h)}. \tag{3.38}$$

Substituting (3.38) in (3.35) and (3.37), and switching indexes gives

$$V_1(z) = \frac{4\pi}{\mu_1}\left(1 - \frac{\mu_2 G_1(h,h)}{(\mu_2 G_1(h,h) + \mu_1 G_2(h,h))}\right)G_1(z,h) \tag{3.39}$$

$$V_2(z) = \frac{4\pi G_1(h,h)}{(\mu_2 G_1(h,h) + \mu_1 G_2(h,h))}G_2(z,h). \tag{3.40}$$

Let

$$A = \mu_2 G_1(h,h) + \mu_1 G_2(h,h), \quad \text{then} \tag{3.41}$$

$$V_1(z) = \frac{4\pi G_2(h,h)}{A}G_1(z,h), \quad \text{and} \tag{3.42}$$

$$V_2(z) = \frac{4\pi G_1(h,h)}{A}G_2(z,h). \tag{3.43}$$

$G_1(z,z_0)$ and $G_2(z,z_0)$ have been derived in Appendix A. In this section we use $G_1(z,h)$ and $G_2(z,h)$; they are

$$G_1(z,h) = -\frac{1}{\alpha_1}\frac{e^{\alpha_1 z} + e^{-\alpha_1 z}}{e^{\alpha_1 h} - e^{-\alpha_1 h}} \tag{3.44}$$

$$G_2(z,h) = -\frac{1}{\alpha_2}\left[e^{-\alpha_2(z-h)}\right]. \tag{3.45}$$

If we equate the denominator in (3.42) to zero, we get the dispersion equation of Love waves

$$A(e^{\alpha_1 h} - e^{-\alpha_1 h}) = 0. \tag{3.46}$$

If $e^{\alpha_1 h} - e^{-\alpha_1 h} = 0$, then $\alpha_1 = 0$, which means that waves do not decay with depth. If $A = 0$, then

$$\frac{e^{\alpha_1 h} + e^{-\alpha_1 h}}{e^{\alpha_1 h} - e^{-\alpha_1 h}} = -\frac{\alpha_1 \mu_1}{\alpha_2 \mu_2}.$$

Replacing α_1 by $i\alpha$, where $\alpha = (k_1^2 - k^2)^{1/2}$, we get the well-known dispersion relation

$$\tan \alpha h = \frac{\alpha_2 \mu_2}{\alpha \mu_1}. \tag{3.47}$$

Numerical Example

To display how the mode functions $V_1(z)$ and $V_2(z)$ look like, the following numbers were plugged into Eqs. (3.42) and (3.43). We used $\overline{\rho}_1 = 2.0$ gm/cm^3, $\overline{\rho}_2 = 2.3$ gm/cm^3, $\beta_1 = 900$ m/s, $\beta_2 = 2000$ m/s and $h = 100\,\text{m}$. The plot is shown in Fig. 3.3. The dispersion relation is shown in Fig. 3.4.

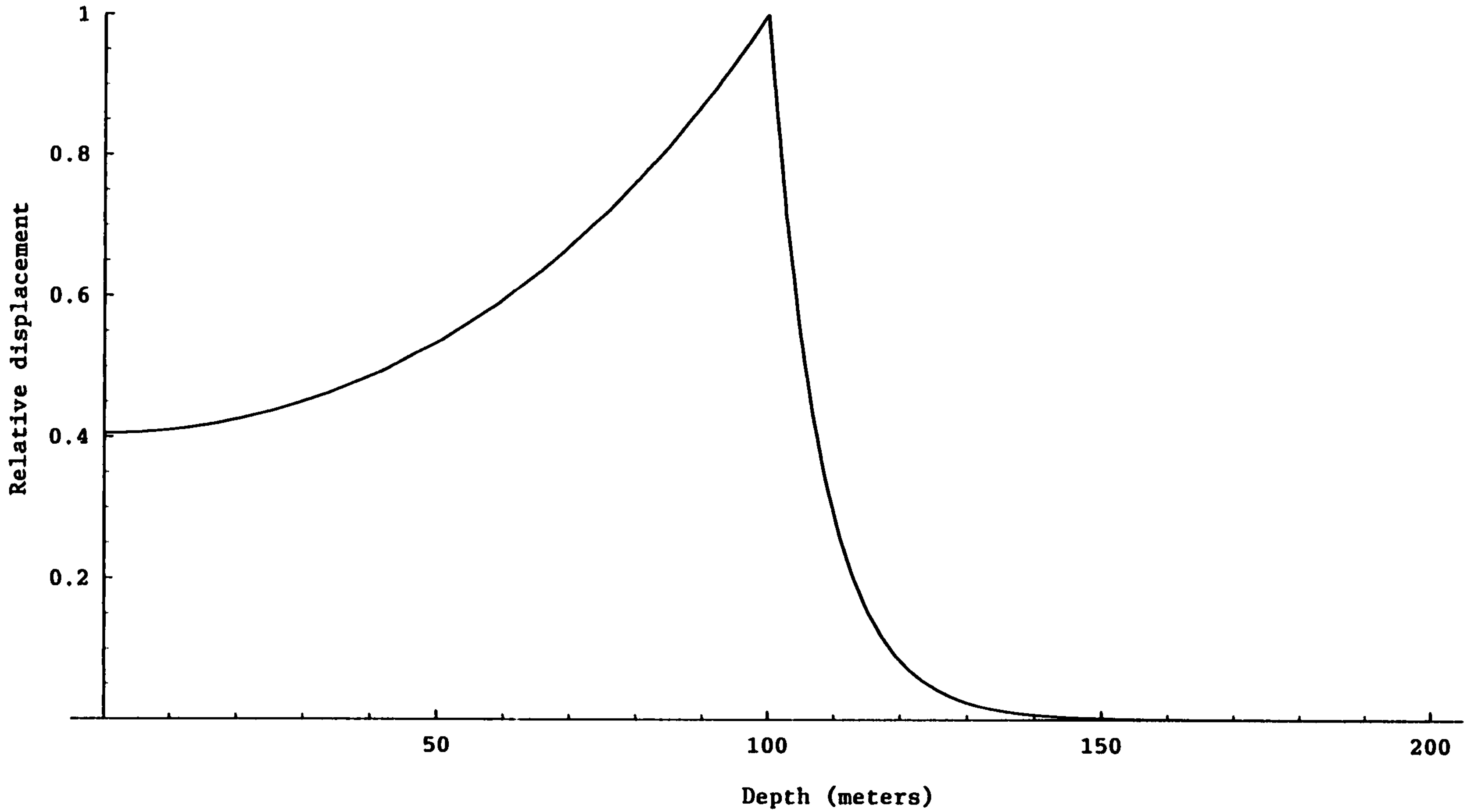

Figure 3.3: The mode functions $V_1(z)$ and $V_2(z)$, note how fast the displacement decays in the half-space.

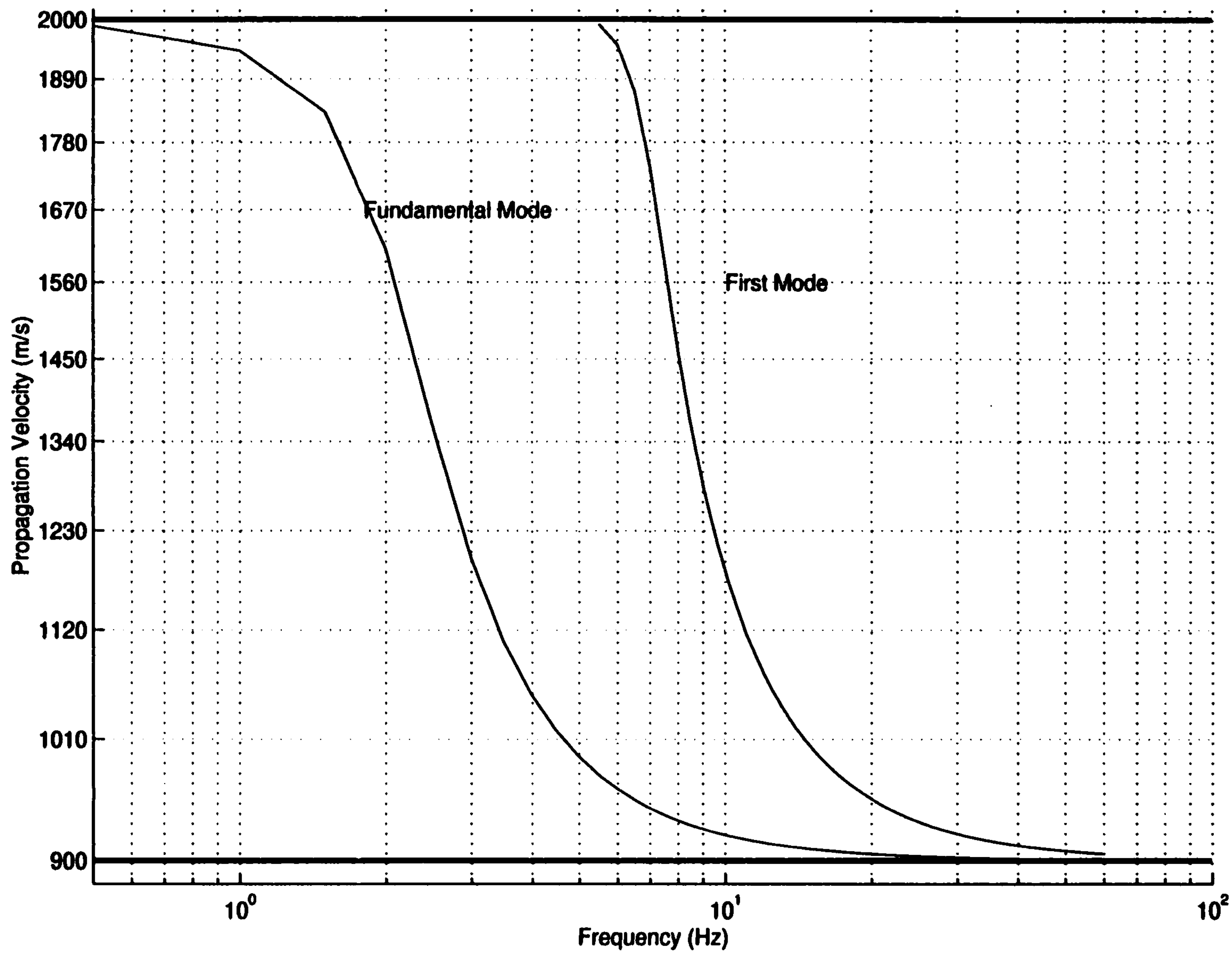

Figure 3.4: Dispersion curves for the first two modes of Love waves.

CHAPTER 4

LOVE WAVES IN RANDOM MEDIA: GREEN'S FUNCTION METHOD

Introduction.

The model that we shall study consists of a layer overlying a semi-infinite half-space. Geologically, it is known that the top layer of ~ 100m thickness has remarkably different internal structure than the deeper layers. The top layer usually consists of unconsolidated sediments of sand and gravel while the deeper layers are well-cemented and more homogeneous. Here, we shall study the effect of inhomogeneity of the top layer on Love wave propagation. For simplicity, we will assume the model to be infinite in the x and y directions; but having a finite thickness of h in the z direction. We shall confine ourselves to time-harmonic plane waves.

In the wave equation, there are three elastic parameters; these are density ρ and the two Lame's constants λ and μ. Since we are only interested in Love waves which are S-type waves, the λ parameter is of no concern to us.

In Chapter 3, we assumed the media to be perfectly homogeneous and isotropic. In such a case the two parameters ρ and μ were taken to be constant at every point of the two media. However, as we have discussed in Section 2.2, each parameter fluctuates in a random way. This fluctuation can be described mathematically by one of the equations (2.7–2.9). In this chapter, the half-space is taken to be completely homogeneous and isotropic. The overlying layer is assumed to be nonhomogeneous in a stochastic way. For simplicity of the mathematical model we assume that the rigidity is constant but the density changes with x and y. In the z-direction, $\rho(z, \gamma)$ is assumed to be a stochastic process (ergodic and statistically homogeneous).

This is basically the same model we used in Section 3.2 with the density ρ taken to be a one-dimensional random field. It should be mentioned here that rigidity μ increases when ρ increases. So the model we are considering has some limitations.

First Order Smoothing Perturbation Method

Many practical problems associated with continuous stochastic media lead to differential equations with random coefficients which can be written is

$$L(\gamma)u = g \tag{4.1}$$

where $L(\gamma)$ is a linear stochastic operator and g is a non-random function.

Classical perturbation expansion theory fails in the analysis of wave propagation in stochastic unbounded media. Therefore, different techniques have been proposed to solve for the mean and the higher moments of u. One approach is to find solutions for individual realizations of the random function u, then carry out the averaging. The second approach includes methods which seek the statistical moments of u rather than the solution itself.

Our aim is to find an appropriate equation for the mean $\langle u \rangle$ of the unknown random wave-field u. It has the form

$$\mathcal{L}\langle u \rangle = g \tag{4.2}$$

where $\mathcal{L}$ is a deterministic operator called an *effective operator* for L and $\langle \quad \rangle$ indicates statistical averaging over all realizations. Of course, the operator $\mathcal{L}$ needs to be invertible to be able to solve for the mean field $\langle u(z) \rangle$ and the correlation $\langle u(z)u(z') \rangle$. For example, assume that the operator $L(\gamma)$ is invertible (for almost all $\gamma \in \Gamma$); then

$$u = L^{-1}(\gamma)g.$$

Averaging gives

$$\langle u \rangle = \langle L^{-1}(\gamma) \rangle g,$$

or

$$\mathcal{L}\langle u \rangle = \langle L^{-1}(\gamma) \rangle^{-1} = g.$$

The only problem in this formal derivation is calculating the inverse operator $L^{-1}(\gamma)$. Let us represent the operator $L(\gamma)$ and $u(\gamma)$ as sums

$$L(\gamma) = \langle L \rangle + L_1, \quad \langle L_1 \rangle = 0, \tag{4.3}$$

$$u(\gamma) = \langle u \rangle + u_1, \quad \langle u_1 \rangle = 0, \tag{4.4}$$

where $L_0 = \langle L \rangle$ and $u_0 = \langle u \rangle$ are the mean deterministic operator and the mean solution, respectively. L_1 and u_1 represent the random fluctuating part of $L(\gamma)$ and $u(\gamma)$. Using (4.3), equation (4.1) becomes

$$(L_0 + L_1)u = g. \tag{4.5}$$

The following few steps are based on operator algebra. Assuming L_0 to be invertible

$$u = L_0^{-1}g - L_0^{-1}L_1 u. \tag{4.6}$$

Averaging gives

$$\langle u \rangle = L_0^{-1}g - L_0^{-1}\langle L_1 u \rangle. \tag{4.7}$$

From (4.6),

$$L_1 u = L_1(L_0^{-1}g) - L_1 L_0^{-1} L_1 u.$$

Averaging and using (4.3)

$$\langle L_1 u \rangle = -\langle L_1 L_0^{-1} L_1 u \rangle. \tag{4.8}$$

At this point, we have to introduce simplifying assumptions. One of these is a truncation or closure approximation. In the analysis of wave propagation, it is known as Bourret's local independence hypothesis, Sobczyk [1985].

$$\langle L_1 L_0^{-1} L_1 u\rangle \simeq \langle L_1 L_0^{-1} L_1\rangle\langle u\rangle, \tag{4.9}$$

which means that there is no correlation between the field fluctuation and the random coefficients. Various heuristic arguments have been made to justify (4.9).

If we grant the closure approximation, (4.7) becomes

$$L_0\langle u\rangle - \langle L_1 L_0^{-1} L_1\rangle\langle u\rangle = g. \tag{4.10}$$

Correlation Function of u

To obtain an expression for the correlation function for u, we need to iterate step (4.6) twice and use the closure approximation (4.9). The final result is

$$\langle u(\vec{r})u(\vec{r}_0)\rangle = \langle u(\vec{r})\rangle\langle u(\vec{r}_0)\rangle + L_0^{-1}\langle L_1 L_0^{-1} L_1\rangle\langle u(\vec{r})\rangle\langle u(\vec{r}_0)\rangle. \tag{4.11}$$

The variance is a special case of this equation when $\vec{r}_0$ is replaced by $\vec{r}$

$$\langle (u - \langle u\rangle)^2\rangle = \langle u^2\rangle - \langle u\rangle^2 = L_0^{-1}\langle L_1 L_0^{-1} L_1\rangle\langle u\rangle^2. \tag{4.12}$$

The variance can be interpreted as a measure of the scatter of the solution due to the stochasticity of the medium.

Dyson Equation

It should be pointed out that equation (4.10) is a special case of a more general solution for the mean field known as Dyson equation, McCoy [1983]. Equation (4.1), after using (4.3) and (4.4) and averaging, takes the form

$$L_0\langle u\rangle + \langle L_1 u_1\rangle = g. \tag{4.13}$$

Subtracting equation (4.13) from equation (4.1) gives

$$L_0 u_1 + (I - P)L_1 u_1 = -L_1\langle u\rangle, \tag{4.14}$$

where I denotes the identity operator and P is the operation of taking an ensemble average. Equation (4.14) is now treated as an equation for u_1 in which the rhs term plays the role of a known excitation. The solution of this equation, u_1, can be obtained by the classical Liouville-Neumann iteration series

$$u_1 = \sum_{n=0}^{\infty}(-1)^{n+1}\left[L_0^{-1}(I-P)L_1\right]^n L_0^{-1}L_1\langle u\rangle. \tag{4.15}$$

Applying L_1 to both sides and averaging gives

$$\langle L_1 u_1\rangle = -M\langle u\rangle, \tag{4.16}$$

where

$$M = \sum_{n=0}^{\infty}(-1)^n\langle L_1\left[L_0^{-1}(I-P)L_1\right]^n L_0^{-1}L_1\rangle, \tag{4.17}$$

is called the mass operator in quantum physics.

Substitution of (4.17) in (4.13) gives the Dyson equation

$$\mathcal{L}\langle u\rangle = (L_0 - M)\langle u\rangle = g. \tag{4.18}$$

Dyson equation is the most general result of the smoothing method. But, because of the infinite series, direct application of this equation to actual physical problems is difficult. Nevertheless, by approximating M, Dyson's equation plays an important role in investigating the mean field in stochastic media. For example, if we take $n=0$, equation (4.18) reduces to equation (4.12) which is the first-order smoothing approximation or, as it is often called, the Bourret approximation. As an application of this method, we will consider the Helmholtz equation.

Helmholtz Stochastic Equation

Consider the following stochastic Helmholtz equation:

$$L_0 u(\vec{r}) + X(\vec{r},\gamma)u(\vec{r}) = g(\vec{r}) \tag{4.19}$$

where L_0 is a deterministic Laplace differential operator with respect to $\vec{r}$, and $X(\vec{r},\gamma)$ is a given random field. Assuming L_0 to be invertible then it has a Green's function $G_0(\vec{r},\vec{r}_0)$ that satisfies

$$L_0^{-1}f(\vec{r}) = \int G_0(\vec{r},\vec{r}_0)f(\vec{r}_0)d\vec{r}_0. \tag{4.20}$$

Replace L_1 by $X(\vec{r},\gamma)$ in equation (4.10), and using (4.20), we get

$$L_0\langle u(\vec{r},\gamma)\rangle - \int G_0(\vec{r},\vec{r}_0)K_{XX}(\vec{r},\vec{r}_0)\langle u(\vec{r}_0)\rangle d\vec{r}_0 = g(\vec{r}), \tag{4.21}$$

where $K_{XX}(\vec{r}_1,\vec{r}_0)$ is the correlation function $\langle X(\vec{r}_1)X(\vec{r}_0)\rangle$.

So the equation is an integro-differential equation in which the solution function appears under the integral and under the differential operator.

Let us now find the variance of the stochastic field $u(\vec{r})$. From equation (4.12),

$$L_0V(\vec{r}) = L_0\langle(u-\langle u\rangle)^2\rangle = \langle L_1L_0^{-1}L_1\rangle\langle u\rangle^2. \tag{4.22}$$

Using the Green's function of L_0, (4.22) can be written as

$$L_0V(\vec{r}) - \int G_0(\vec{r},\vec{r}_0)K_{XX}(\vec{r},\vec{r}_0)\langle u(\vec{r}_0)\rangle^2 d\vec{r}_0. \tag{4.23}$$

This implies that

$$V(\vec{r}) = \int\int G_0(\vec{r},\vec{r}_2)G_0(\vec{r},\vec{r}_1)K_{XX}(\vec{r},\vec{r}_1)\langle u(\vec{r}_1)\rangle d\vec{r}_1 d\vec{r}_2. \tag{4.24}$$

Keller's Method

In 1964 J.B. Keller derived a more efficient equation of the mean field. Assume that the fluctuation in the random coefficients of $L(\gamma)$ are small, then we can express $L(\gamma)$ as a series consisting of operators and powers of ϵ, a small parameter which serves as a measure of the homogeneity of the medium. Equation (4.1) takes the form

$$\left[L_0 + \epsilon L_1(\gamma) + \epsilon^2 L_2(\gamma) + O(\epsilon^3)\right] u = g. \tag{4.25}$$

Setting $\epsilon = 0$, we get the first approximation equation which ensures existence of L_0^{-1}.

$$L_0 u_0 = g. \tag{4.26}$$

We can find u from equation (4.25). Using (4.26), we get

$$u = u_0 - L_0^{-1}(\epsilon L_1 + \epsilon^2 L_2)u + O(\epsilon^3). \tag{4.27}$$

The last equation can be solved using the method of successive approximation. Assuming $\langle L_1(\gamma)\rangle = 0$, the final equation is

$$\left\{L_0 + \epsilon^2\left[\langle L_2\rangle - \langle L_1 L_0^{-1} L_1\rangle\right]\right\}\langle u\rangle = g + O(\epsilon^3). \tag{4.28}$$

If we assume that $L_2 = L_3 = \cdots = 0$, and $\epsilon L_1 = V_1$, then we get a special case which corresponds to the Bourret approximation

$$\left[L_0 - \langle V_1 L_0^{-1} V_1\rangle\right]\langle u\rangle = g, \tag{4.29}$$

which is the same result we got in equation (4.10).

Formulation of the Problem

The above methodology is now applied to two-dimensional Love wave propagation in a layer overlying a semi-infinite half-space. This problem has been solved in Section 3.2 for perfectly homogeneous media. In this section we will introduce small inhomogeneities in some elastic parameters of the layer. The inhomogeneities will be modelled as random fields. As a consequence, the Love waves propagating in such a medium will also appear as a random displacement field.

Using the linearized theory of elasticity, the displacement equation of motion can be written as

$$\mu\,\nabla^2\,\vec{u} + (\lambda + \mu)\,\nabla\,\vec{\nabla}\cdot\vec{u} = \rho\frac{\partial^2\vec{u}}{\partial t^2}. \tag{4.30}$$

Using Helmholtz decomposition, $\vec{u} = \nabla\varphi + \nabla \times \vec{\psi}$, which implies that

$$\begin{aligned} \nabla^2\varphi &= \frac{1}{\alpha^2}\frac{\partial^2\varphi}{\partial t^2} \quad (4.31) \\ \nabla^2\vec{\psi} &= \frac{1}{\beta^2}\frac{\partial^2\vec{\psi}}{\partial t^2}. \quad (4.32) \end{aligned}$$

The displacement vector can be written as

$$\vec{u} = u\vec{\imath} + v\vec{\jmath} + w\hat{k}$$

where v is the displacement component in the y-direction.

$$v = \frac{\partial\varphi_y}{\partial y} - \frac{\partial\varphi_z}{\partial x} + \frac{\partial\varphi_x}{\partial z}. \quad (4.33)$$

If we consider the propagation of a plane wave characterized by the displacement potential $\varphi, \vec{\psi}$ and whose *normal* has no component in the y-direction, then the displacement component v satisfies

$$\nabla^2 v(r,t) - \frac{1}{\beta^2}\frac{\partial^2 v(r,t)}{\partial t^2} = \sigma(r,t) \quad (4.34)$$

where $\sigma(r,t)$ is the source function in the layer.

We will now study time harmonic waves and we will simplify the problem by a prior assumption that we know how v and σ depend on x. We will assume that they have the following simple separable harmonic form:

$$\begin{aligned} v(x,z,t) &= V(z)e^{i(kx-wt)}, \quad (4.35) \\ \sigma(x,z,t) &= S(z)e^{i(kx-wt)}, \quad (4.36) \end{aligned}$$

where k is the wave number. Another approach is to take the Fourier transform with respect to the spatial coordinate x. Using any of the above two approaches, we can reduce the PDE (4.34) to the following ODE

$$\frac{d^2V(z)}{dz^2} + \left(\frac{\omega^2\rho}{\mu} - k^2\right)V(z) = S(z). \quad (4.37)$$

Now, consider a layer of thickness h whose properties vary only in the z direction (the depth direction). Let the two parameters ρ and μ be two stochastic processes. So $\rho = \rho(z,\gamma), \mu = \mu(z,\gamma)$ where γ is a realization of the sample space. Because λ does not show up in our equation (4.37), density ρ and rigidity μ are the only two elastic parameters that concern us. Our aim is to solve for the mean displacement field $\langle v_1 \rangle$ – over all realizations of ρ and μ. Suppose that the density is

$$\rho_1 = \overline{\rho}_1[1 + \epsilon\rho(z,\gamma)]. \tag{4.38}$$

Here ϵ is a small parameter whose magnitude is a measure of homogeneity, $\overline{\rho}_1$ is the mean density and $\rho(\gamma, z)$ is a standardized stochastic process. It has a zero mean and a standard deviation of 1.

Rigidity of the layer μ_1 can be modelled the same way.

$$\mu_1 = \overline{\mu}_1[1 + \epsilon\mu(z,\gamma)]. \tag{4.39}$$

For the purpose of this thesis, $\mu(z,\gamma)$ will be taken to be zero. This might not be geophysically consistent with observed data. However, considerable simplification is gained using this assumption. Some authors combined the two parameters in one, $\beta^2 = \mu/\rho$, then considered β as a stochastic process, Korvin [1977-83], Chu et al. used both equations (4.38) and (4.39) in their 1981 paper.

Now, consider the boundary value problem involving a layer of uniform thickness whose properties are described above. It extends to infinity in both directions x and y and has a thickness of h. The layer is overlying a semi-infinite homogeneous half-space as shown in Figure (4.1). The half-space has a constant density ρ_2 and rigidity μ_2. Our aim is to find the mean field displacement of the Love wave in the upper layer. Substituting (4.38) in equation (4.37), we get the following two equations for

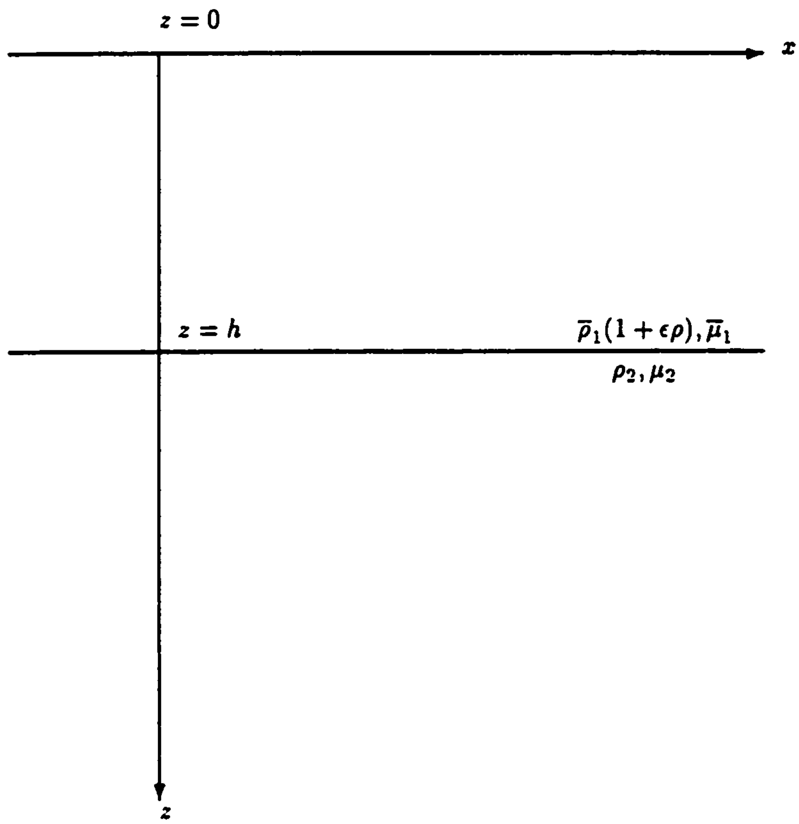

Figure 4.1: One Layer and a Half-space (2D) Model (Stochastic Case)

both the layer and the half-space

$$\left[\frac{d^2}{dz^2} + (\overline{k}_1^2 - k^2)\right] V_1(z,\gamma)$$

$$+ [\epsilon \overline{k}_1^2 \rho(z,\gamma)] V_1(z,\gamma) = S(z); \qquad \text{for the layer,} \tag{4.40}$$

$$\left[\frac{d^2}{dz^2} - (k^2 - \overline{k}_2^2)\right] V_2(z) = 0; \qquad \text{for the half-space,} \tag{4.41}$$

where $k_1^2 = \frac{w^2}{\overline{\mu}_1}\overline{\rho}_1[1+\epsilon\rho(z,\gamma)]$,

$$\overline{k}_1 = \frac{\omega^2\overline{\rho}_1}{\overline{\mu}_1}, \overline{k}_2 = \frac{\omega^2\overline{\rho}_2}{\overline{\mu}_2}, \ \ \alpha_1^2 = \overline{k}_1^2 - k^2, \ \alpha_2^2 = k^2 - \overline{k}_2^2. \tag{4.42}$$

Any solution to equation (4.41) and (4.42) must satisfy the following boundary conditions:

1. Normal stress at the free surface should vanish, i.e.,

$$\left[\frac{dV_1}{dz}\right]_{z=0} = 0. \tag{4.43}$$

2. Continuity of the stress and displacement at the boundary between the layer and the half-space gives

$$\mu_1 \left[\frac{dV_1}{dz}\right]_{z=h} = \mu_2 \left[\frac{dV_2}{dz}\right]_{z=h} \tag{4.44}$$

$$V_1(h) = V_2(h) = q. \tag{4.45}$$

Our goal now is to solve for the mean function $\langle V_1(z)\rangle$ and $V_2(z)$.

Method of Solution

The problem can be rewritten as

$$\frac{d^2V_1(z)}{dz^2} + \alpha_1^2 V_1(z) + \left[\epsilon \overline{k}_1^2 \rho(z,\gamma)\right] V_1(z) = S(z), \tag{4.46}$$

where $S(z)$ is the source term,

$$\frac{d^2V_2(z)}{dz^2} - \alpha_2^2 V_2(z) = 0, \tag{4.47}$$

$$\left|\frac{dV_1}{dz}\right|_{z=0} = 0, \tag{4.48}$$

$$\mu_1 \frac{dV_1}{dz}\bigg|_{z=h} = \mu_2 \frac{dV_2}{dz}\bigg|_{z=h}, \tag{4.49}$$

$$V_1(z)\,|_{z=h} = V_2(z)|_{z=h}\,. \tag{4.50}$$

To solve for $\langle V_1(z)\rangle$, we need first to solve the following auxiliary initial value problem (IVP) as defined by the differential equation (4.46) and the initial conditions

$$V_1(z)|_{z=0} = X_0, \tag{4.51}$$

$$\frac{dV_1(z)}{dz}\bigg|_{z=0} = Y_0. \tag{4.52}$$

Equation (4.46) is a Helmholtz equation; therefore, we can use the first-order smoothing approximation to convert the differential equation (4.46) into an integro-differential equation for the mean field similar to equation (4.21)

$$\left(\frac{d^2}{dz^2} + \alpha_1^2\right)\langle V_1(z)\rangle = \epsilon \bar{k}_1^2 \int_0^h G_1(z,z_0)K_{\rho\rho}(z,z_0)\langle V_1(z_0)\rangle dz_0 + S(z), \tag{4.53}$$

where $K_{\rho\rho}$ is the correlation function for the process $\rho(z,\gamma)$ and $G_1(z,z_0)$ is the Green's function.

The Green's function for the IVP is defined by the following system:

$$\left.\begin{aligned} &\frac{d^2G_1(z,z_0)}{dz^2} + \alpha_1^2 G_1(z,z_0) = \delta(z-z_0), \\ &\frac{dG_1(z,z_0)}{dz}\bigg|_{z=0} = 0,\ G_1|_{z=0} = 0. \end{aligned}\right\} \tag{4.54}$$

We solve this system in Appendix B and find $G_1(z,z_0)$ to be

$$G_1(z,z_0) = \begin{cases} \dfrac{1}{\alpha_1}\sin\alpha_1(z-z_0), & \text{where } z < z_0 \\ 0, & \text{otherwise} \end{cases} \tag{4.55}$$

A common choice for the correlation function is the Uhlenbeck-Ornstein process

$$K_{\rho\rho}(z - z_0) = \sigma^2 e^{-|z-z_0|b}. \tag{4.56}$$

where the standard deviation is $\sigma = 1$ and b is the inverse of the correlation length.

Case I. Let us first take $S(z)$ to be zero, i.e., assume that the wave is coming from a very far away source. Substitute (4.55) and (4.56) in equation (4.53), we get

$$\left(\frac{d^2}{dz^2} + \alpha_1^2\right) \langle V_1(z)\rangle = \frac{\epsilon \overline{k}_1^2}{\alpha_1} \int_0^h \sin \alpha_1 (z - z_0) e^{-|z-z_0|b} \langle V_1(z_0)\rangle dz_0. \tag{4.57}$$

Now, we have to solve the IVP which consists of equation (4.57) and the conditions (4.51) and (4.52). The integral in the rhs of equation (4.57) is of convolution type. It can be solved using Laplace transform.

We shall need the following Laplace transform properties:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt,$$
$$f(t) = \frac{i}{2\pi} \int_\Gamma e^{st} F(s)ds,$$

where the contour Γ is vertical and to the right of all singularities of $F(s)$ in the complex plane.

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g), \qquad \text{where } (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2} \tag{4.58}$$

Using these properties and the initial conditions (4.51) and (4.52), Eq. (4.57) is transformed into

$$F_1(s)(s^2 + \alpha_1^2) - sX_0 - Y_0 = \frac{\epsilon \overline{k}_1^2 F_1(s)}{(s+b)^2 + \alpha_1^2},$$

which gives

$$F_1(s) = X_0\frac{s[(s+b)^2+\alpha_1^2]}{D(s)} + Y_0\frac{[(s+b)^2+\alpha_1^2]}{D(s)}, \tag{4.59}$$

where

$$D(s) = [(s^2+\alpha_1^2)][(s+b)^2+\alpha_1^2]] - (\epsilon\overline{k}_1^2), \tag{4.60}$$

and $F_1(s)$ is the Laplace transform of $\langle V_1(z)\rangle$.

In order to take the inverse Laplace transform of equation (4.59), we shall need the following theorem from complex variable theory. If $f(z) = \frac{P(z)}{Q(z)}$, where both functions $P(z)$ and $Q(z)$ are analytic at z_0, and Q has a simple zero at z_0, while $P(z_0) \neq 0$, then

$$\text{Res }(f; z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

Then, by residue theorem,

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \text{ Res }(f, z_k)$$

Using the above theorem, $\langle V_1(z)\rangle$ can be found as

$$\langle V_1(z)\rangle = X_0 \sum_{k=1}^{4} A_k e^{s_k z} + Y_0 \sum_{k=1}^{4} B_k e^{s_k z}, \tag{4.61}$$

where

$$A_k = \left.\frac{s[(s+b)^2+\alpha_1^2]}{\frac{d}{ds}D(s)}\right|_{s=s_k}, \tag{4.62}$$

$$B_k = \left.\frac{[(s+b)^2+\alpha_1^2]}{\frac{d}{ds}D(s)}\right|_{s=s_k}, \tag{4.63}$$

where s_1, s_2, s_3 and s_4 are the zeros of $D(s)$. In Appendix C, we show how to find these zeros.

Referring to the boundary conditions (4.48) and (4.50) we eliminate the arbitrary values X_0 and Y_0.

$$\left[\frac{dV_1}{dz}\right]_{z=0} = 0 \Rightarrow Y_0 = 0, \tag{4.64}$$

$$V_2(h) = \langle V_1(h) \rangle \quad = \quad q \Rightarrow X_0 = \frac{q}{\sum_{k=1}^{4} A_k e^{s_k h}}, \tag{4.65}$$

where q is a constant.

Further from the half-space, differential equation (4.47), suppose $V_2(z)$ has the form

$$V_2(z) = q e^{-\alpha_2 z}, \tag{4.66}$$

Note that because $V_2(z)$ vanishes at infinity, the part of solution containing the term $e^{\alpha_2 z}$ has been left out.

Using the displacement continuity condition (4.50), we get

$$q e^{-\alpha_2 h} = \langle V_1(h) \rangle, \tag{4.67}$$

$$\text{which gives} \quad V_2(z) = q e^{\alpha_2 (h-z)}, \tag{4.68}$$

$$\text{so that} \quad \langle V_1(z) \rangle = \frac{q \sum_{k=1}^{4} A_k e^{s_k z}}{\sum_{k=1}^{4} A_k e^{s_k h}}. \tag{4.69}$$

Finally, imposing stress continuity (4.49) on equations (4.67) and (4.68), we get

$$\frac{\sum_{k=1}^{4} s_k A_k e^{s_k h}}{\sum_{k=1}^{4} A_k e^{s_k h}} = -\frac{\mu_2 \alpha_2}{\mu_1}. \tag{4.70}$$

This is the dispersion relation for Love waves in a stochastic medium.

Special Case.

For a homogeneous isotropic layer, we have $c = 0$ and $b = \frac{1}{a} = 0$, infinite correlation distance. This implies

$$x_1 = x_2 = -\alpha_1^2.$$

This gives

$$s_1 = s_2 = \alpha_1 i, \; s_3 = s_4 = -\alpha_1 i,$$

leading to

$$A_n = \frac{s(s^2+\alpha_1^2)}{4s(s^2+\alpha_1^2)} = \frac{1}{4},$$

so that

$$\sum_{k=1}^{4} A_k e^{s_k h} = \frac{1}{2}e^{\alpha_1 ih} + \frac{1}{2}e^{-\alpha_1 ih},$$

and

$$\sum_{k=1}^{4} A_k s_k e^{s_k h} = \frac{\alpha_1 i}{2}e^{\alpha_1 h} - \frac{\alpha_1 i}{2}e^{-\alpha_1 ih}.$$

Applying these to the general dispersion relation (4.70), we get

$$-\frac{\mu_2\alpha_2}{\mu_1} = \alpha_1 i\left(\frac{e^{\alpha_1 ih} - e^{-\alpha_1 ih}}{e^{\alpha_1 ih} + e^{-\alpha_1 ih}}\right) \Rightarrow \frac{\mu_2\alpha_2}{\mu_1\alpha_1} = \tan\alpha_1 h, \tag{4.71}$$

which reproduce the well known Love waves dispersion relation we found in Section 3.2 (Eq. (3.26).

In this case,

$$V_1(z) = q\left(\frac{e^{\alpha_1 iz} - e^{-\alpha_1 iz}}{e^{\alpha_1 ih} + e^{-\alpha_1 ih}}\right) \tag{4.72}$$

$$V_2(z) = qe^{\alpha_2(h-z)}. \tag{4.73}$$

Note that $V_2(z)$ remains unchanged in either case.

Case II. In this case we will assume that there is an energy source at the interface between the two media. We start again from the equation

$$\left(\frac{d^2}{dz^2} + \alpha_1^2\right)\langle V_1(z)\rangle = \frac{\epsilon\overline{k}_1^2}{\alpha_1}\int_0^z \sin\alpha_1(z-z_0)e^{-(z-z_0)b}\langle V_1(z_0)\rangle dz_0 + \frac{S(z)}{\mu_1}. \tag{4.74}$$

Let us take $S(z) = 4\pi\delta(z-h)$ a Dirac delta function which is a line source at $z = h$. The Laplace transform gives $\mathcal{L}\{\delta(z-h)\} = e^{-sh}$. Laplace transforming equation (4.74), and using initial conditions (4.51) and (4.52), we get

$$F_1(s)(s^2+\alpha_1^2) - sX_0 - Y_0 = \frac{\epsilon\overline{k}_1^2 F_1(s)}{(s+b)^2+\alpha_1^2} + e^{-sh}, \tag{4.75}$$

so that

$$F_1(s) = X_0 \frac{s[(s+b)^2+\alpha_1^2]}{D(s)} + Y_0 \frac{[(s+b)^2+\alpha_1^2]}{D(s)} + \frac{e^{-sh}[(s+b)^2+\alpha_1^2]}{D(s)}, \qquad (4.76)$$

where

$$D(s) = (s^2+\alpha_1^2)[(s+b)^2+\alpha_1^2] - \epsilon \overline{k}_1^2. \qquad (4.77)$$

Zeros of $D(s)$ are given in Appendix C. We use residue theorem again to find the inverse Laplace transform of (4.77) as

$$\langle V_1(z) \rangle = X_0 \sum_{k-1}^{4} A_k e^{s_k z} + Y_0 \sum_{k=1}^{4} B_k e^{s_k z} + \sum_{k=1}^{4} C_k e^{s_k z}, \qquad (4.78)$$

where A_k and B_k are the same as in (4.62) and (4.63).

$$C_k = \left. \frac{e^{-sh}[(s+b)^2+\alpha_1^2]}{\frac{d}{ds}D(s)} \right|_{s=s_k} \qquad (4.79)$$

and s_1, s_2, s_3 and s_4 are zeros of $D(s)$.

The boundary condition (4.48) implies that $Y_0 = 0$. Using displacement and stress continuity conditions (4.49) and (4.50), we get the following system:

$$\begin{pmatrix} \sum_{k=1}^{4} S_k A_k e^{S_k h} \frac{\mu_2}{\mu_1} \alpha_2 e^{-\alpha_2 h} \\ \sum_{k=1}^{4} A_k e^{S_k h} - e^{\alpha_2 h} \end{pmatrix} \begin{pmatrix} X_0 \\ q \end{pmatrix} = \begin{pmatrix} -\sum_{k=1}^{4} S_k C_k e^{S_k h} \\ -\sum_{k=1}^{4} C_k e^{S_k h} \end{pmatrix}.$$

For this system to have a non-trivial solution, assuming no source present case, the determinant of the coefficients matrix must be zero. This condition gives rise to the dispersion relation (4.70). So, the solution of X_0 is in terms of q.

$$X_0 = \frac{q e^{-\alpha_2 h} - \sum_{k=1}^{4} C_k e^{s_k h}}{\sum_{1}^{4} A_k e^{s_k h}}. \qquad (4.80)$$

For the half-space $V_2(z)$ is as in equation (4.68), i.e. it has the form

$$V_2(z) = q e^{\alpha_2 (h-z)}. \qquad (4.81)$$

For $\langle V_1(z)\rangle$, we substitute (4.80) in (4.78) to get

$$\langle V_1(z)\rangle = \frac{qe^{-\alpha_2 h} - \sum_{=1}^{4} C_k e^{s_k h}}{\sum_{k=1}^{4} A_k e^{s_k h}} \sum_{k=1}^{4} A_k e^{s_k z} + \sum_{k=1}^{4} C_k e^{s_k z}. \tag{4.82}$$

Numerical Example

We will use the same parameters we used in Section 3.3 example. In addition, will take $\epsilon = .1$ and $b = 2$. The dispersion curve for this case is very similar to the homogeneous case, Fig. 3.4, with slightly greater propagation phase velocities. This seems to be contrary to the common belief that stochasticity slows waves down, Korvin [1977]. However, the model we used perturb ρ only and assume constant μ. This does not represent rocks behavior, where density and rigidity are proportional to each other. Figure 4.2 shows that the difference increases with frequency and it is less at the fundamental mode. The same information can be gained from Table 4.1.

The displacement mode function for this case is shown in Figure 4.3, red curve, compared with that of the homogeneous case, blue curve. This plot confirms an established fact that randomness causes attenuation of propagated waves. The attenuation gets less as we approach the surface, Figure 4.4. Note that there is a 4% decrease of the surface and 0% in the half-space. The relation between the correlation length inverse b and the attenuation percentage is shown in Figures 4.5A and 4.5B. The difference between the homogeneous case and the stochastic case goes to zero very rapidly after $b = .5$. Although this seems to be unexpected, it agrees with the findings of other researchers, Korvin [1979]. Finally, as it is expected the difference increases with increasing the homogeneity measure ϵ, Figure 4.6.

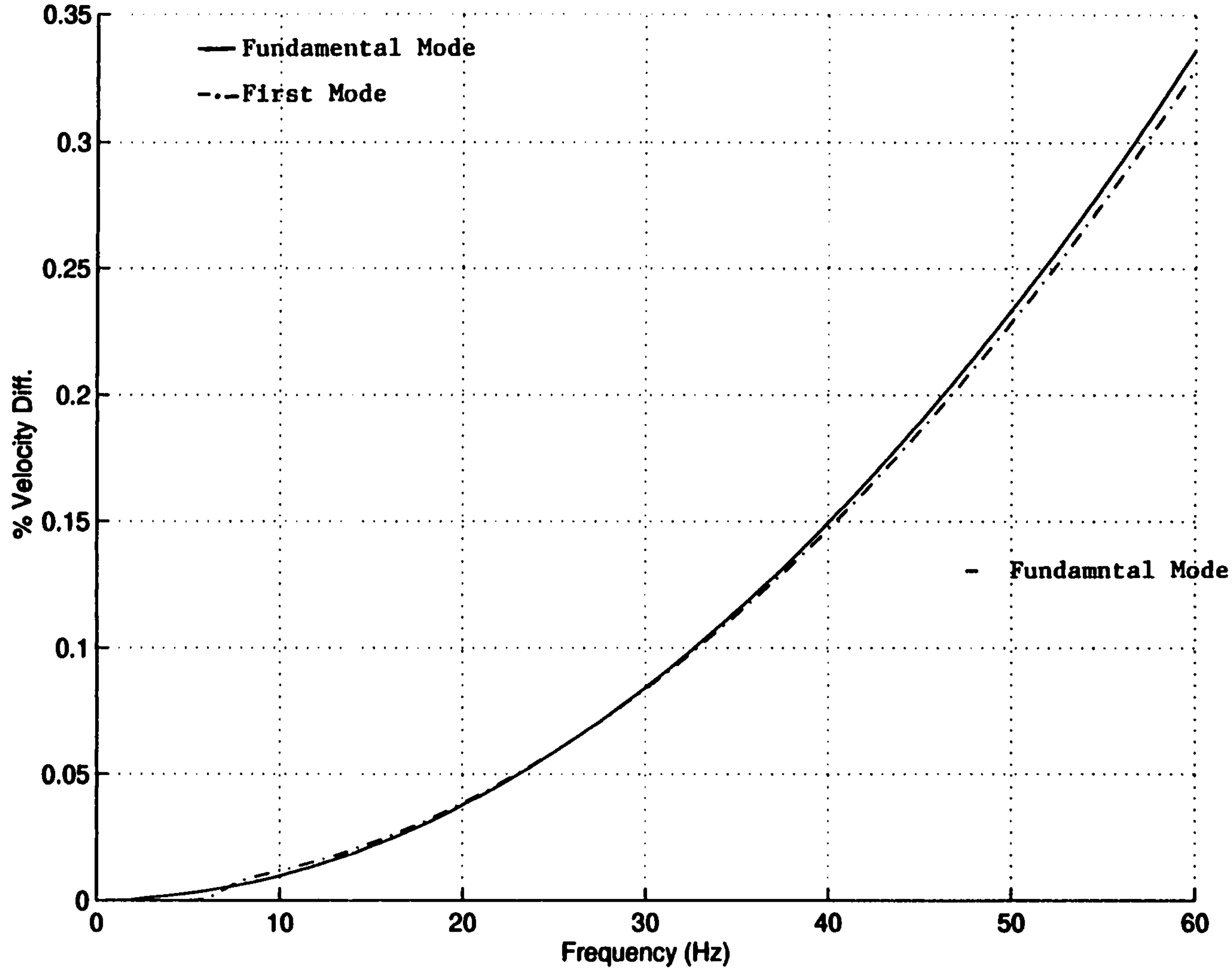

Figure 4.2: The difference in program phase velocity, between the homogeneous case and the stochastic case, as a function of frequency

Table 4.1: Wave number for the fundamental mode (K_0) and the first mode (K_1) for the two cases and the difference percentage.

Freq.	K0 (homog.)	K0 (stoch.)	%Diff.	K1 (stoch.)	K1 (homog.)	%Diff.
0.50	0.00158066	0.00158066	0.00013	0.00000000	0.00000000	0.00000
1.00	0.00323586	0.00323586	0.00008	0.00000000	0.00000000	0.00000
1.50	0.00514847	0.00514846	0.00017	0.00000000	0.00000000	0.00000
2.00	0.00781259	0.00781255	0.00045	0.00000000	0.00000000	0.00000
2.50	0.01157284	0.01157273	0.00093	0.00000000	0.00000000	0.00000
3.00	0.01575031	0.01575011	0.00130	0.00000000	0.00000000	0.00000
3.50	0.01987881	0.01987849	0.00161	0.00000000	0.00000000	0.00000
4.00	0.02388925	0.02388878	0.00195	0.00000000	0.00000000	0.00000
4.50	0.02779866	0.02779800	0.00237	0.00000000	0.00000000	0.00000
5.00	0.03163065	0.03162977	0.00280	0.00000000	0.00000000	0.00000
5.50	0.03540394	0.03540277	0.00329	0.01736745	0.01736737	0.00021
6.00	0.03913217	0.03913067	0.00383	0.01932078	0.01932046	0.00081
6.50	0.04282523	0.04282335	0.00440	0.02184211	0.02184119	0.00215
7.00	0.04649038	0.04648804	0.00503	0.02530444	0.02530240	0.00439
7.50	0.05013302	0.05013016	0.00570	0.02967525	0.02967197	0.00655
8.00	0.05375726	0.05375380	0.00643	0.03446013	0.03445582	0.00802
8.50	0.05736625	0.05736212	0.00719	0.03929017	0.03928496	0.00909
9.00	0.06096249	0.06095760	0.00802	0.04402771	0.04402162	0.00998
9.50	0.06454793	0.06454220	0.00888	0.04864116	0.04863416	0.01085
10.00	0.06812416	0.06811749	0.00979	0.05313406	0.05312607	0.01172
10.50	0.07169246	0.07168476	0.01074	0.05751950	0.05751045	0.01262
11.00	0.07525390	0.07524506	0.01175	0.06181182	0.06180161	0.01357
11.50	0.07880934	0.07879926	0.01279	0.06602405	0.06601257	0.01456
12.00	0.08235953	0.08234809	0.01389	0.07016738	0.07015453	0.01560
12.50	0.08590508	0.08589216	0.01504	0.07425122	0.07423689	0.01668
13.00	0.08944651	0.08943200	0.01622	0.07828343	0.07826749	0.01782
13.50	0.09298428	0.09296804	0.01747	0.08227058	0.08225291	0.01900
14.00	0.09651876	0.09650067	0.01875	0.08621817	0.08619865	0.02022
14.50	0.10005029	0.10003020	0.02008	0.09013087	0.09010936	0.02150
15.00	0.10357916	0.10355694	0.02145	0.09401261	0.09398897	0.02282
15.50	0.10710562	0.10708112	0.02287	0.09786677	0.09784086	0.02419
16.00	0.11062990	0.11060297	0.02435	0.10169623	0.10166790	0.02561
16.50	0.11415218	0.11412266	0.02586	0.10550349	0.10547259	0.02707
17.00	0.11767264	0.11764037	0.02742	0.10929072	0.10925709	0.02858
17.50	0.12119144	0.12115626	0.02903	0.11305980	0.11302328	0.03013
18.00	0.12470871	0.12467044	0.03069	0.11681238	0.11677281	0.03173
18.50	0.12822456	0.12818303	0.03239	0.12054993	0.12050714	0.03337
19.00	0.13173912	0.13169415	0.03414	0.12427373	0.12422753	0.03507
19.50	0.13525248	0.13520388	0.03593	0.12798492	0.12793514	0.03680
20.00	0.13876473	0.13871231	0,03778	0.13168452	0.13163097	0.03859
20.50	0.14227594	0.14221950	0.03967	0.13537343	0.13531593	0.04042
21.00	0.14578620	0.14572554	0.04161	0.13905248	0.13899082	0.04230
21.50	0.14929555	0.14923048	0.04358	0.14272238	0.14265637	0.04421
22.00	0.15280408	0.15273438	0.04562	0.14638382	0.14631325	0.04618
22.50	0.15631183	0.15623728	0.04769	0.15003737	0.14996204	0.04819
23.00	0.15981884	0.15973923	0.04981	0.15368359	0.15360327	0.05025
23.50	0.16332517	0.16324027	0.05198	0.15732296	0.15723745	0.05236
24.00	0.16683087	0.16674044	0.05420	0.16095593	0.16086500	0.05450
24.50	0.17033595	0.17023978	0.05646	0.16458293	0.16448635	0.05670
25.00	0.17384047	0.17373831	0.05877	0.16820431	0.16810185	0.05894
25.50	0.17734446	0.17723606	0.06113	0.17182042	0.17171184	0.06122
26.00	0.18084794	0.18073305	0.06353	0.17543159	0.17531665	0.06356
26.50	0.18435095	0.18422932	0.06598	0.17903809	0.17891655	0.06593
27.00	0.18785350	0.18772487	0.06847	0.18264021	0.18251181	0.06835
27.50	0.19135563	0.19121974	0.07101	0.18623819	0.18610266	0.07082
28.00	0.19485736	0.19471394	0.07360	0.18983225	0.18968935	0.07334
28.50	0.19835871	0.19820747	0.07624	0.19342261	0.19327207	0.07589
29.00	0.20185969	0.20170037	0.07892	0.19700947	0.19685101	0.07850
29.50	0.20536033	0.20519264	0.08166	0.20059301	0.20042636	0.08115
30.00	0.20886064	0.20868430	0.08443	0.20417340	0.20399829	0.08384
30.50	0.21236063	0.21217535	0.08725	0.20775080	0.20756693	0.08658
31.00	0.21586033	0.21566580	0.09012	0.21132536	0.21113245	0.08937
31.50	0.21935975	0.21915567	0.09304	0.21489722	0.21469497	0.09220
32.00	0.22285890	0.22264496	0.09600	0.21846650	0.21825461	0.09508
32.50	0.22635779	0.22613368	0.09901	0.22203333	0.22181151	0.09800
33.00	0.22985643	0.22962184	0.10206	0.22559783	0.22536575	0.10097
33.50	0.23335484	0.23310943	0.10516	0.22916009	0.22891745	0.10398
34.00	0.23685303	0.23659648	0.10832	0.23272023	0.23246671	0.10704
34.50	0.24035100	0.24008298	0.11151	0.23627833	0.23601360	0.11014
35.00	0.24384876	0.24356894	0.11475	0.23983448	0.23955822	0.11329
35.50	0.24734633	0.24705435	0.11804	0.24338077	0.24310065	0.11649
36.00	0.25084370	0.25053923	0.12138	0.24694127	0.24664095	0.11972
36.50	0.25434090	0.25402358	0.12476	0.25049207	0.25017920	0.12301
37.00	0.25783792	0.25750740	0.12819	0.25404123	0.25371547	0.12634
37.50	0.26133477	0.26099069	0.13166	0.25758881	0.25724981	0.12972
38.00	0.26483146	0.26447345	0.13519	0.26113488	0.26078228	0.13314
38.50	0.26832800	0.26795568	0.13876	0.26467951	0.26431295	0.13661
39.00	0.27182439	0.27143739	0.14237	0.26822274	0.26784185	0.14012
39.50	0.27532063	0.27491857	0.14603	0.27176463	0.27136905	0.14368
40.00	0.27881674	0.27839923	0.14974	0.27530524	0.27489458	0.14729
40.50	0.28231271	0.28187937	0.15350	0.27884460	0.27841849	0.15094
41.00	0.28580856	0.28535898	0.15730	0.28238277	0.28194082	0.15463
41.50	0.28930428	0.28883807	0.16115	0.28591978	0.28546161	0.15837
42.00	0.29279988	0.29231664	0.16504	0.28945569	0.28898089	0.16216
42.50	0.29629537	0.29579468	0.16898	0.29299053	0.29249871	0.16599
43.00	0.29979075	0.29927219	0.17297	0.29652433	0.29601509	0.16987
43.50	0.30328602	0.30274918	0.17701	0.30005714	0.29953006	0.17379
44.00	0.30678118	0.30622564	0.18109	0.30358899	0.30304366	0.17776
44.50	0.31027625	0.30970157	0.18522	0.30711991	0.30655591	0.18177
45.00	0.31377122	0.31317696	0.18939	0.31064993	0.31006684	0.18583
45.50	0.31726609	0.31665183	0.19361	0.31417908	0.31357648	0.18994
46.00	0.32076088	0.32012616	0.19788	0.31770740	0.31708484	0.19409
46.50	0.32425557	0.32359995	0.20219	0.32123490	0.32059195	0.19828
47.00	0.32775019	0.32707321	0.20656	0.32476161	0.32409783	0.20253
47.50	0.33124471	0.33054592	0.21096	0.32828757	0.32760251	0.20682
48.00	0.33473916	0.33401809	0.21541	0.33181278	0.33110599	0.21115
48.50	0.33823354	0.33748971	0.21992	0.33533729	0.33460830	0.21553
49.00	0.34172783	0.34096079	0.22446	0.33886110	0.33810946	0.21995
49.50	0.34522206	0.34443132	0.22905	0.34238424	0.34160949	0.22442
50.00	0.34871621	0.34790129	0.23369	0.34590672	0.34510838	0.22894
50.50	0.35221030	0.35137070	0.23838	0.34942858	0.34860618	0.23350
51.00	0.35570431	0.35483956	0.24311	0.35294983	0.35210287	0.23811
51.50	0.35919827	0.35830786	0.24789	0.35647048	0.35559849	0.24276
52.00	0.36269216	0.36177559	0.25271	0.35999055	0.35909304	0.24746
52.50	0.36618599	0.36524275	0.25758	0.36351006	0.36258653	0.25220
53.00	0.36967976	0.36870935	0.26250	0.36702903	0.36607898	0.25699
53.50	0.37317347	0.37217537	0.26746	0.37054746	0.36957039	0.26183
54.00	0.37666713	0.37564081	0.27247	0.37406538	0.37306078	0.26671
54.50	0.38016073	0.37910567	0.27753	0.37758281	0.37655015	0.27164
55.00	0.38365428	0.38256995	0.28263	0.38109974	0.38003851	0.27661
55.50	0.38714778	0.38603364	0.28778	0.38461620	0.38352588	0.28163
56.00	0.39064123	0.38949674	0.29298	0.38813220	0.38701226	0.28669
56.50	0.39413463	0.39295925	0.29822	0.39164775	0.39049765	0.29180
57.00	0.39762798	0.39642116	0.30350	0.39516287	0.39398207	0.29696
57.50	0.40112128	0.39988247	0.30884	0.39867755	0.39746551	0.30216
58.00	0.40461454	0.40334318	0.31422	0.40219183	0.40094800	0.30741
58.50	0.40810776	0.40680328	0.31964	0.40570570	0.40442952	0.31271
59.00	0.41160093	0.41026276	0.32511	0.40921918	0.40791010	0.31805
59.50	0.41509406	0.41372164	0.33063	0.41273227	0.41138973	0.32343
60.00	0.41858715	0.41717989	0.33619	0.41624499	0.41486841	0.32886

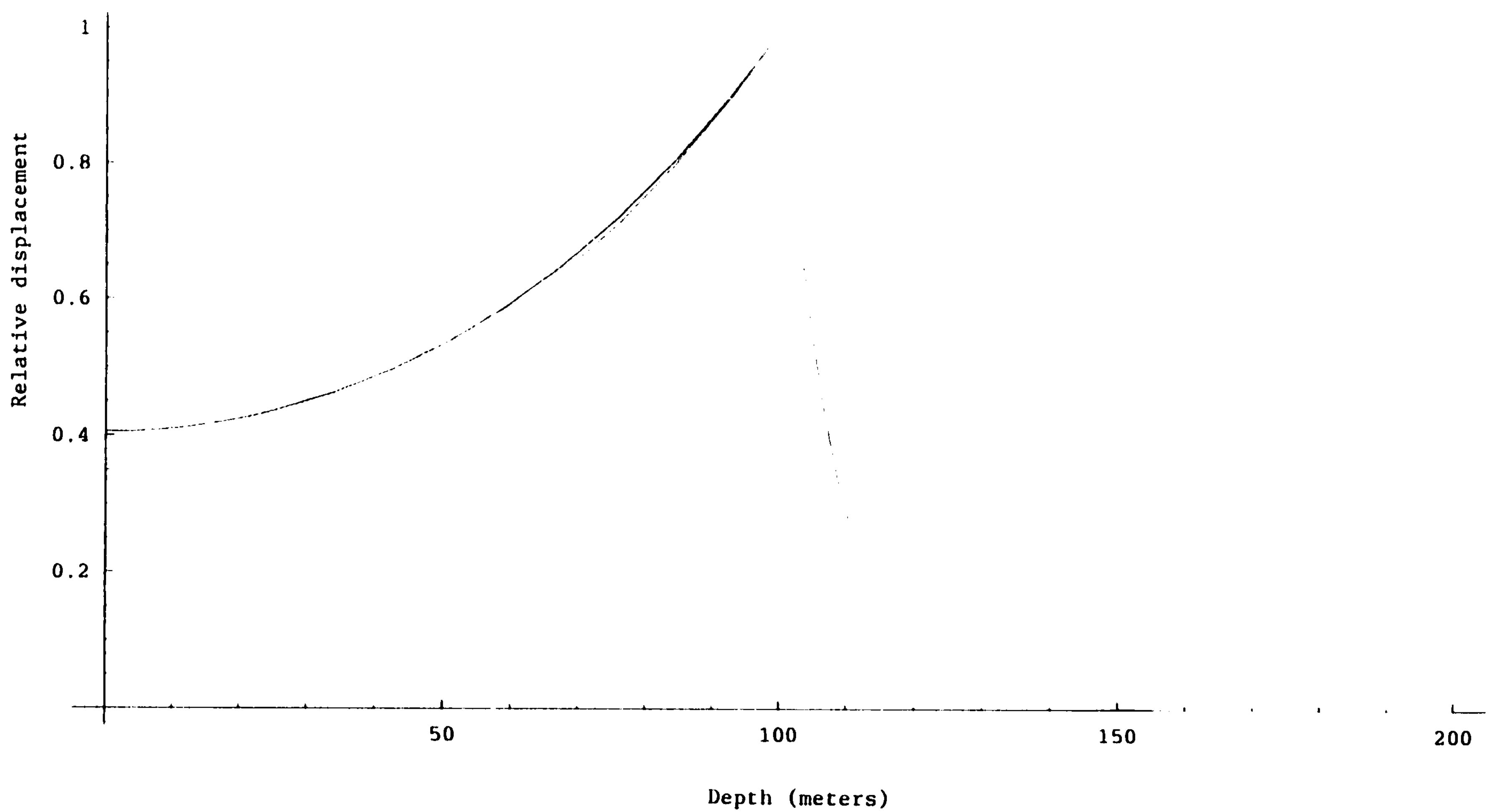

Figure 4.3: Displacement function as a function of depth z, homogeneous medium case (blue curve) compared to stochastic medium case (red curve), $b = 0.5$, $\varepsilon = 0.1$.

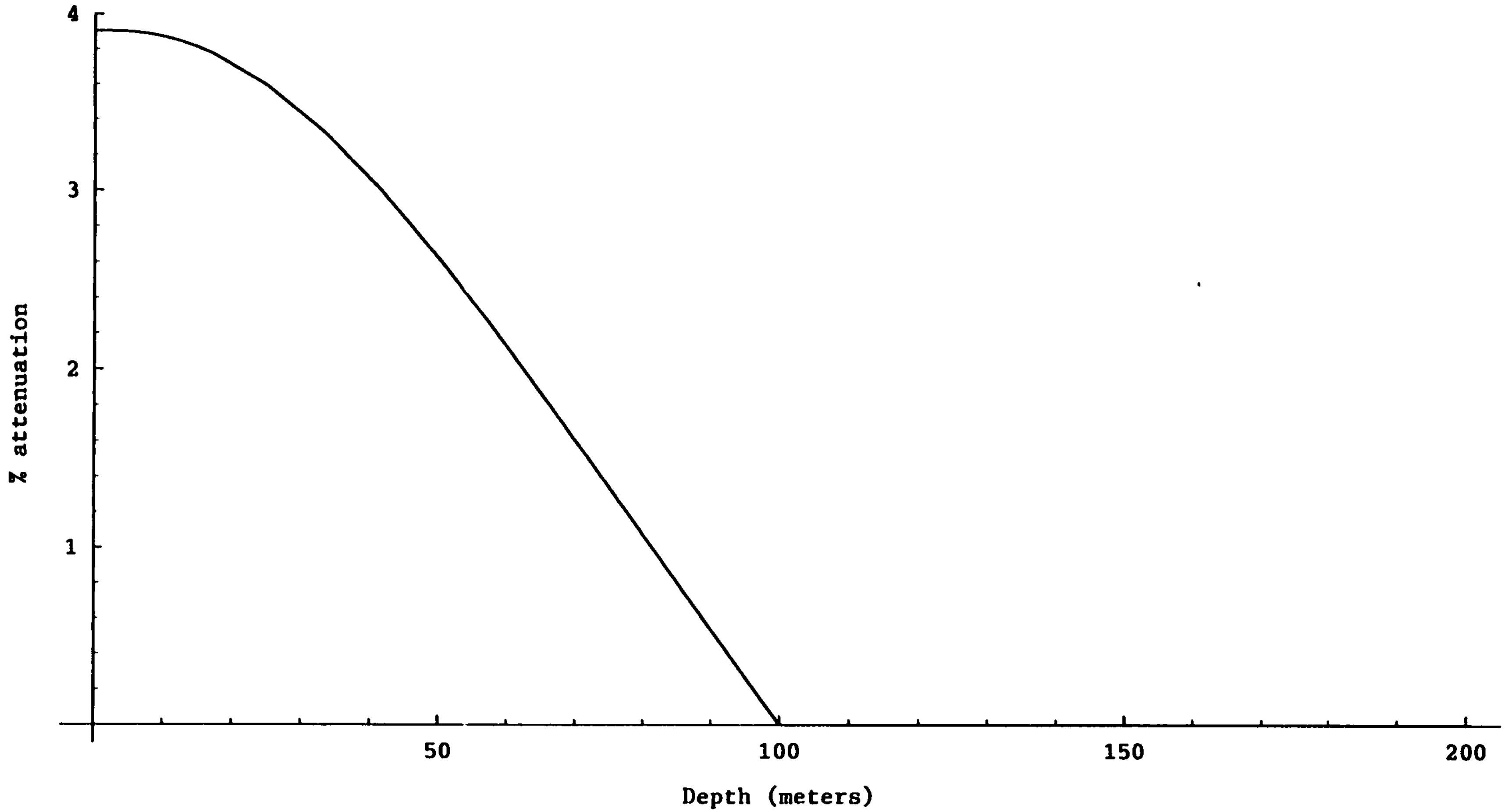

Figure 4.4: The percentage difference between the two curves is shown in Figure 4.3.

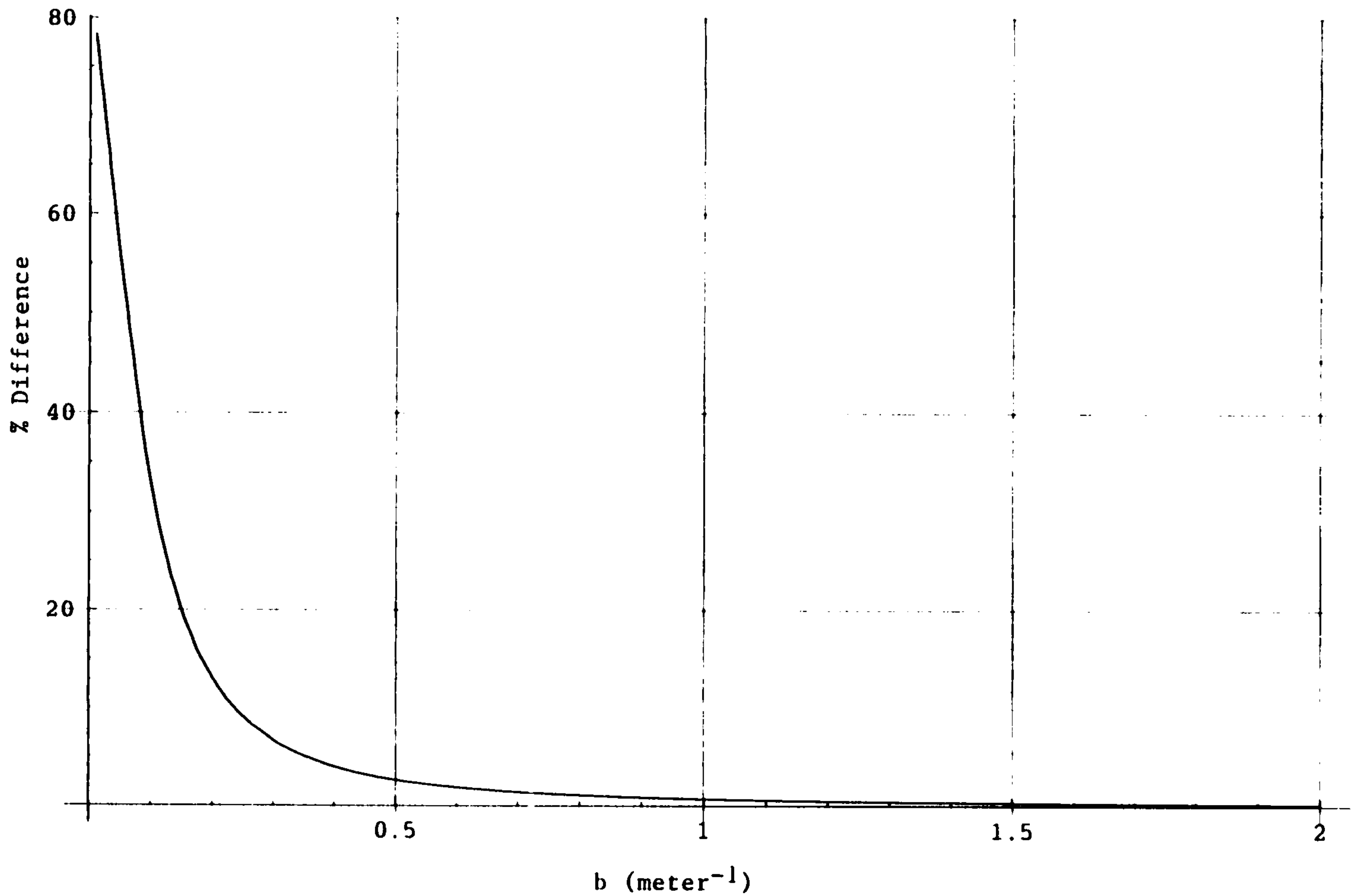

Figure 4.5A: The relation between correlation length inverse b and displacement difference of the two cases at z = /2 and $\varepsilon = 0.1$

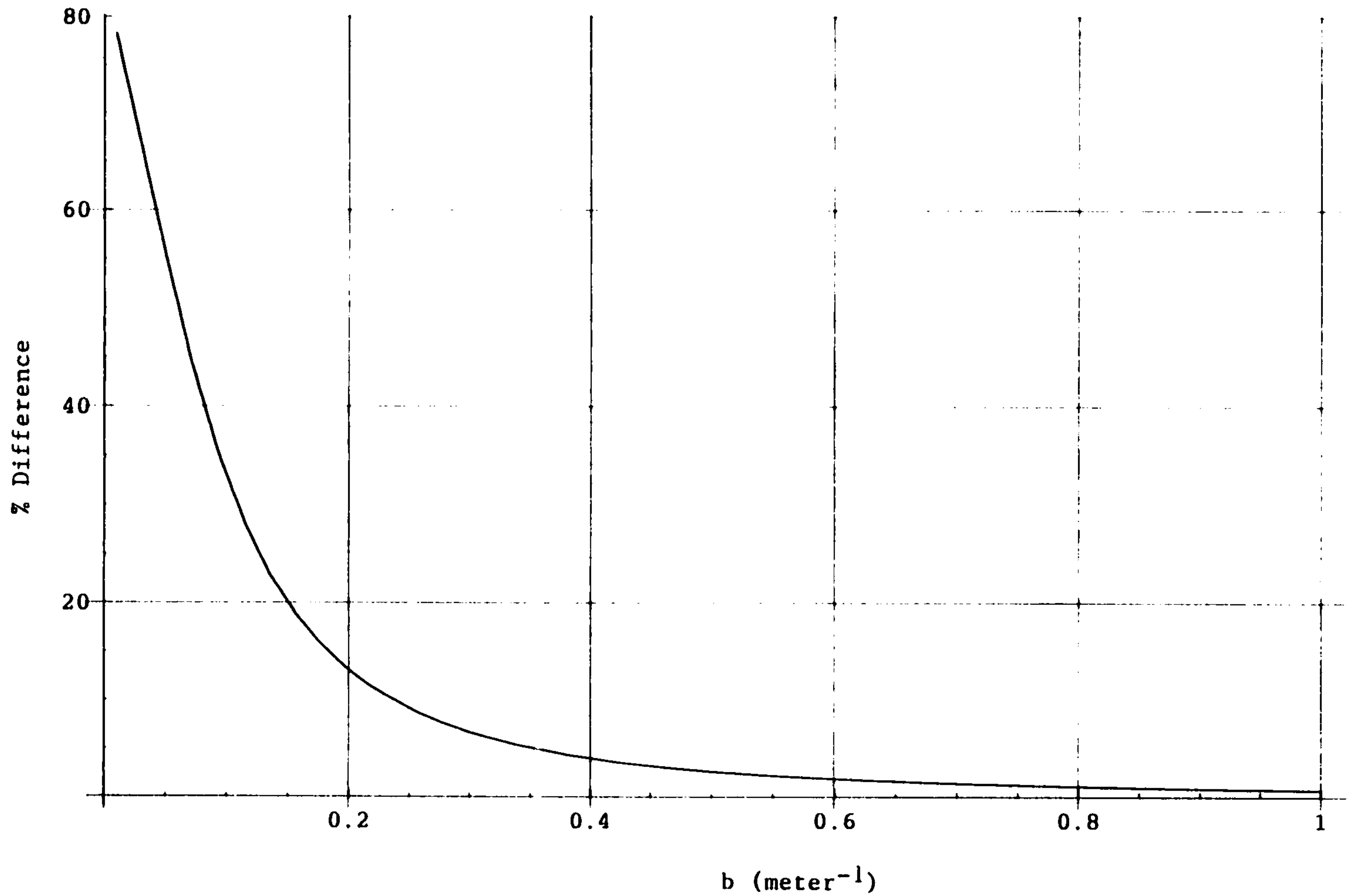

Figure 4.5B: The relation between correlation length inverse b and displacement difference of the two cases at $z = h/2$ and $\varepsilon = 0.1$.

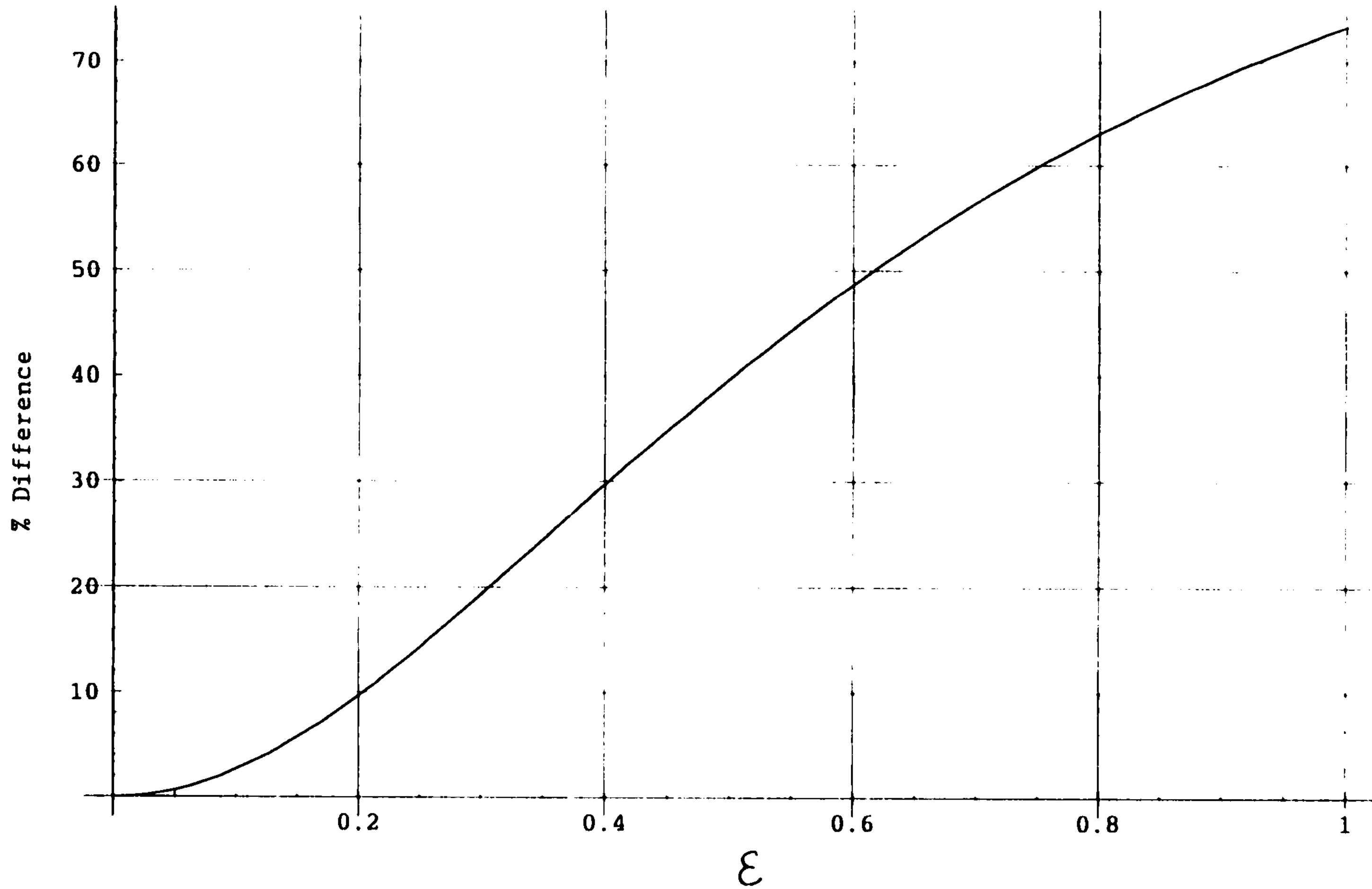

Figure 4.6: The percentage of the displacement difference between the two cases increases with homogeneity ε , $z = h/2$, $b = 0.5$.

CHAPTER 5

LOVE WAVE IN RANDOM MEDIA; PERTURBATION EXPANSION METHOD

Introduction

In Chapter 3 we have seen how efficient it is to solve our problem using Green's function method. The model we used there was an ideal model, i.e., a homogeneous and isotropic layer overlying a semi-infinite homogeneous and isotropic half-space. Also, we assumed the interface between the two media to be perfectly horizontal.

In Chapter 2 we have concluded that an alternative model is needed to represent the uncertainties in the properties of the layer. In Chapter 4, we assumed the layer to have random properties. This stochastic model required an approximation method be used to get an integro-differential equation for the mean field. Using Laplace transform, we were able to find the mean displacement field.

This Chapter combines the above two methods, the Green's function method and the smoothing approximation method. We will use the same model as in Chapter 4 and the same approximation technique; however, rather than using Laplace transform for the layer, the Green's function method will be used, followed by the approximation method which is employed to obtain the variance of the field. But before that we will use normal perturbation technique to derive a formula for the variance, assuming that the mean field is the field which is corresponding to homogeneous media described by the mean parameters $\overline{\rho}_1$ and $\overline{\mu}_1$.

Problem Formulation

We begin by the Love wave equation in the top layer and the half-space, respectively

$$\mu_1 \nabla^2 v_1 - \rho_1 \frac{\partial^2 v_1}{\partial t^2} = 4\pi\sigma_1(r,t) \tag{5.1}$$

$$\mu_2 \nabla^2 v_2 - \rho_1 \frac{\partial^2 v_2}{\partial t^2} = 0. \tag{5.2}$$

Let

$$\left.\begin{aligned} v_1(r) &= V_1(z)e^{i(kx-wt)} \\ v_2(r) &= V_2(z)e^{i(kx-wt)} \\ \sigma_1(r) &= \delta(z-h)e^{i(kx-wt)} \\ \rho_1 &= \overline{\rho}_1[1+\epsilon\rho(z,\gamma)] \\ \mu_1 &= \overline{\mu}_1[1+\epsilon\mu(z,\gamma)] \end{aligned}\right\}. \tag{5.3}$$

For now, we will assume the rigidity to be constant $\mu_1 = \overline{\mu}_1$. $\rho(z,\gamma)$ is a homogeneous and ergodic one-dimensional field with a mean of zero and standard deviation one, ϵ is a small parameter which acts as a measure of the inhomogeneity of the medium.

Substituting equations (5.3) in (5.1) and (5.2), we get

$$\frac{d^2V_1(z)}{dz^2} - \alpha_1^2 V_1(z) = \frac{4\pi\delta(z-h)}{\mu_1} - \overline{k}_1^2\epsilon\rho(z,\gamma)V_1(z)$$

$$\frac{d^2V_2(z)}{dz^2} - \alpha_2^2 V_2(z) = 0$$

where

$$\alpha_1^2 = k^2 - \overline{k}_1^2 ; \overline{k}_1^2 = \frac{\omega^2}{\overline{\beta}_1^2} = \frac{\omega^2\overline{\rho}_1}{\overline{\mu}_1}$$

where $\overline{\beta}_1, \overline{\rho}, \overline{\mu}$ are the mean velocity, density and rigidity of the layer, respectively. Also, $\alpha_2^2 = k^2 - k_1^2 = k^2 - \frac{\omega^2}{\overline{\beta}_2^2}$. Therefore, we have the following system to solve

$$\frac{d^2V_1(z)}{dz^2} - \alpha_1^2 V_1(z) = g(z) \tag{5.4}$$

$$\text{where} \quad g(z) = \left(\frac{(4\delta(z-h)}{\mu_1} - \overline{k}_1^2\epsilon\rho(z,\gamma)V_1(z)\right) \tag{5.5}$$

$$\frac{d^2V_2(z)}{dz^2} - \alpha_2^2 V_2(z) = 0. \tag{5.6}$$

These two ODE's satisfy the following boundary conditions

$$\frac{dV_1(z)}{dz} = 0 \quad \text{at} \quad z = 0 \tag{5.7}$$

$$V_1(z) = V_2(z) \quad \text{at} \quad z = h \tag{5.8}$$

$$\mu_1 \frac{dV_1}{dz} = \mu_2 \frac{dV_2}{dz} \quad \text{at} \quad z = h \tag{5.9}$$

Method of Solution

We will first use the Green's function Stakgold [1978], method to solve for $V_1(z)$ and $V_2(z)$. Let $G_1(z, z_0)$ be Green's function of the layer, then

$$\frac{d^2G_1(z,z_0)}{dz^2} - \alpha_1^2 G_1(z,z_0) = \delta(z - z_0), \quad 0 \le z, z_0 \le h \tag{5.10}$$

$G_1(z, z_0)$ satisfies

$$\frac{dG_1}{dz} = 0 \quad \text{at} \quad z = 0,\ z = h. \tag{5.11}$$

Multiplying ODE (5.4) by $G_1(z, z_0)$ and ODE (5.10) by $V_1(z)$; then subtracting and integrating from $z = 0$ to $z = h$ gives

$$\int_0^h \left(G_1(z,z_0)\frac{d^2V_1(z)}{dz^2} - V_1(z)\frac{d^2G_1(z,z_0)}{dz^2} \right) dz = \int_0^h \frac{4\pi}{\mu_1} G_1(z,z_0)\delta(z-h)dz$$
$$- \bar{k}_1^2 \epsilon \int_0^h \rho(z,\gamma)G_1(z,z_0)V_1(z)dz - \int_0^h V_1(z)\delta(z-z_0)dz$$

which gives

$$\left[G_1(z-z_0)\frac{dV_1(z)}{dz} - V_1(z)\frac{dG_1(z,z_0)}{dz} \right]_0^h = \frac{4\pi}{\mu_1}G_1(h,z_0) - V_1(z_0) - \bar{k}_1^2\epsilon\xi(z_0,\gamma). \tag{5.12}$$

Using boundary conditions (5.7) and (5.11), after switching index, gives

$$V_1(z) = \frac{4\pi}{\mu_1}G_1(z,h) - G_1(z,h)\left[\frac{dV_1(z)}{dz}\right]_{z=h} - \bar{k}_1^2\epsilon\xi(z,\gamma) \tag{5.13}$$

$$\text{where} \quad \xi(z,\gamma) = \int_0^h \rho(z,\gamma)V_1(z_0)G_1(z,z_0)dz_0, \tag{5.14}$$

z is any point in the layer $0 < z < h$.

Similarly $G_2(z, z_0)$ is the Green's function of the lower half-space. It is the solution of the problem:

$$\left.\begin{aligned}
&\frac{d^2G_2(z,z_0)}{dz^2} - \alpha_1^2 G_2(z,z_0) = \delta(z-z_0)\\
&\frac{dG_2}{dz} = 0 \quad \text{at} \quad z = h\\
&\frac{dG_2}{dz} = 0, \;\; G_2(z,z_0) = 0 \quad \text{as} \quad z \to \infty.
\end{aligned}\right\} \tag{5.15}$$

Multiplying ODE (5.6) by $G_2(z,z_0)$ and ODE (5.15) by $V_2(z)$, subtracting and integrating from $z = h$ to $z = \infty$ gives

$$\left.\begin{aligned}
&\int_h^\infty \left(G_2(z,z_0)\frac{d^2V_2(z)}{dz^2} - V_2(z)\frac{d^2G_2}{dz^2}\right) = -\int_h^\infty \delta(z-z_0)V_2(z)dz\\
&\Rightarrow V_2(z_0) = \left[G_2(z,z_0)\frac{dV_2(z)}{dz} - V_2(z)\frac{dG_2(z,z_0)}{dz}\right]_\infty^h.
\end{aligned}\right\}. \tag{5.16}$$

Honoring the boundary conditions (5.15) and switching index, we get

$$V_2(z) = G_2(z,h)\left[\frac{dV_2(z)}{dz}\right]_{z=h}. \tag{5.17}$$

So, given the two equations (5.13) and (5.17) and using boundary conditions (5.8) and (5.9), we seek an explicit solution for $V_1(z)$ and $V_2(z)$

$$\Rightarrow V_2(z) = G_2(z,h)\frac{\mu_1}{\mu_2}\left[\frac{dV_1(z)}{dz}\right]_{z=h} \tag{5.18}$$

$$\begin{aligned}
\Rightarrow \frac{\mu_1}{\mu_2}G_2(h,h)\left[\frac{dV_1(z)}{dz}\right]_{z=h} &= \frac{4\pi}{\mu_1}G_1(h,h)\\
&\quad - G_1(h,h)\left[\frac{dV_1(z)}{dz}\right]_{z=h} - \overline{k}_1^2\epsilon\xi(h,\gamma)\\
\Rightarrow \left[\frac{dV_1(z)}{dz}\right]_{z=h} &= \frac{\frac{4\pi\mu_2}{\mu_1}G_1(h,h) - \epsilon\overline{k}_1^2\mu_2\xi(h,\gamma)}{\mu_2G_1(h,h) + \mu_1G_2(h,h)}\\
&= \frac{1}{A}\left(\frac{4\pi\mu_2}{\mu_1}G_1(h,h) - \overline{k}_1^2\epsilon\mu_2\xi(h,\gamma)\right)
\end{aligned} \tag{5.19}$$

where

$$A = \mu_2G_1(h,h) + \mu_1G_2(h,h),$$

and

$$\xi(h,\gamma) = \int_0^h G_1(z,h)\rho(z,\gamma)V_1(z)dz.$$

We substitute (5.14) in (5.13) to get

$$V_1(z) = \frac{4\pi}{\mu_1} G_1(z,h)\left(1 - \frac{\mu_2}{A} G_1(h,h)\right) + \epsilon \overline{k}_1^2 \left(\frac{\mu_2}{A} G_1(z,h)\xi(h,\gamma) - \xi(z,\gamma)\right). \quad (5.20)$$

Similarly, if we substitute (5.19) in (5.18), we get

$$V_2(z) = \frac{G_2(z,h)}{A}\left(4\pi G_1(h,h) - \overline{k}_1^2 \epsilon \mu_1 \xi(h,\gamma)\right). \quad (5.21)$$

It is important to note that if we set $\epsilon = 0$, then equations (5.20) and (5.21) reduce to the solution of the homogeneous case we found previously, in section 3.3 equations (3.39) and (3.40).

Suppose that we can separate the solution $V_1(z)$ in (5.20) into two parts. The first one, $V_1^{(0)}(z)$, is the solution for a perfectly homogeneous medium and the other part $V_1^{(1)}(z)$, is the scatter field induced by the random inhomogeneity

$$V_1(z) = V_1^{(0)}(z) + \epsilon V_1^{(1)}(z), \quad (5.22)$$

where

$$\begin{aligned} V_1^{(0)}(z) &= \frac{4\pi}{\mu_1} G_1(z,h)\left(1 - \frac{\mu_2}{A} G_1(h,h)\right) && (5.23)\\ V_1^{(1)}(z) &= \overline{k}_1^2 \left(\frac{\mu_2}{A} G_1(z,h)\xi(h,\gamma) - \xi(z,\gamma)\right). && (5.24) \end{aligned}$$

Similarly

$$V_2(z) = V_2^{(0)}(z) - \epsilon V_2^{(1)}(z) \quad (5.25)$$

where z is any point in the half-space; $h < z < \infty$, then,

$$\begin{aligned} V_2^{(0)}(z) &= \frac{4\pi}{A} G_1(h,h) G_2(z,h) && (5.26)\\ V_2^{(1)}(z) &= \frac{\overline{k}_1^2 \mu_1}{A} \xi(h,\gamma) G_2(z,h). && (5.27) \end{aligned}$$

Since $V_1^{(1)}(z)$ and $V_2^{(1)}(z)$ are random solutions, the natural question is to find the mean scattered field and its variance. Also, we would like to know the variance of this solution. But, as you notice, $V_1^{(1)}(z)$ and $V_2^{(1)}(z)$ involve integral expressions which depend on $V_1(z)$. Zero order approximation, we will replace $V_1(z)$ by $V_1^{(0)}(z)$ in the integrand. This results

$$\begin{aligned} V_1^{(1)}(z) &= \overline{k}_1^2 \left(\frac{\mu_2}{A} G_1(z,h) \int_0^h G_1(h,z_0)\rho(z_0,\gamma)V_1^{(0)}(z_0)dz_0 \right. \\ &\qquad \left. - \int_0^h G_1(z,z_0)\rho(z_0,\gamma)V_1^{(0)}(z_0)dz_0 \right) \\ &= \overline{k}_1^2 \int_0^h \left(\frac{\mu^2}{A} G_1(z,h)G_1(h,z_0) - G_1(z,z_0) \right) \rho(z_0,\gamma)V_1^{(0)}(z_0)dz_0 \end{aligned} \tag{5.28}$$

Let

$$J_1(z,z_0) = \overline{k}_1^2 \left(\frac{\mu_2}{A} G_1(z,h)G_1(h,z_0) - G_1(z,z_0) \right) V_1^{(0)}(z_0). \tag{5.29}$$

Then

$$V_1^{(1)}(z) = \int_0^h J_1(z,z_0)\rho(z_0,\gamma)dz_0. \tag{5.30}$$

Similarly,

$$V_2^{(1)}(z) = \frac{\overline{k}_1^2\mu_1}{A} G_2(z,h) \int_0^h G_1(h,z_0)\rho(z_0,\gamma)V_1^{(0)}(z_0)dz_0. \tag{5.31}$$

Let

$$J_2(z,z_0) = \frac{\overline{k}_1^2\mu_1}{A} G_2(z,h)G_1(h,z_0)V_1^{(0)}(z_0) \tag{5.32}$$

Then

$$V_2^{(1)}(z) = \int_0^h J_2(z,z_0)\rho(z_0,\gamma)dz_0. \tag{5.33}$$

Now $J_1(z,z_0)$ and $J_2(z,z_0)$ are piecewise continuous deterministic functions whose first and second derivatives are continuous since $G_1(z,z_0)$ has this property. $\rho(z,\gamma)$ is a stochastic process of zero mean and unit standard deviation.

$$\langle \rho(z_0,\gamma) = 0$$

$$\langle \rho^2(z_0,\gamma) \rangle = 1.$$

Using these properties of the stochastic process $\rho(z,\gamma)$ we get

$$\langle V_1^{(1)}(z) \rangle = \int_0^h J_1(z,z_0)\langle \rho(z_0,\gamma) \rangle dz = 0$$

$$\langle V_2^{(1)}(z) \rangle = \int_0^h J_2(z,z_0)\langle \rho(z_0,\gamma) \rangle dz_0 = 0.$$

Let us find the variance of the solution which we expect to be a function of z given by

$$\left\langle V_1^{(1)^2}(z)\right\rangle = \int_0^h\int_0^h J_1(z,z_1)J_1(z,z_2)\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle dz_1dz_2 \quad (5.34)$$

$$\left\langle V_2^{(1)^2}(z)\right\rangle = \int_0^h\int_0^h J_2(z,z_1)J_2(z,z_2)\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle dz_1dz_2 \quad (5.35)$$

where $\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle$ is the second order correlation function associated with ρ. For substantial simplification, it is customary to assume $\rho(z,\gamma)$ to be a white noise or a weakly correlated process, Frisch [1968].

However, in our case we will assume that $\rho(z,\gamma)$ is an Uhlenbeck-Ornstein process that is a centered, stationary, Gaussian and Markovian random process. Having the autocorrelation function

$$\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle = \sigma^2\exp(-|z_1-z_2|b). \quad (5.36)$$

Here σ is the standard deviation of the process and b is the inverse of the correlation length.

Using $\sigma = 1$ and substituting (5.36) in (5.34) and (5.35) yields the expressions

$$\left\langle V_1^{(1)^2}(z)\right\rangle = \int_0^h J_1(z,z_2)\int_0^h J_1(z,z_1)e^{-|z_1-z_2|b}dz_1dz_2 \quad (5.37)$$

$$\left\langle V_2^{(1)^2}(z)\right\rangle = \int_0^h J_2(z,z_2)\int_0^h J_2(z,z_1)e^{-|z_1-z_2|b}dz_1,dz_2. \quad (5.38)$$

Evaluating (2.37) and (2.38) is a long and tedious task. However, we found that the computer software "Mathematica" is a good tool to be used for this purpose. Appendix (D) has some details as well as printout of the result.

Properties of such solutions will be examined next using a numerical example. We shall study the dependence of the variance on depth (independent variable z) and also on the correlation length b^{-1} and ϵ.

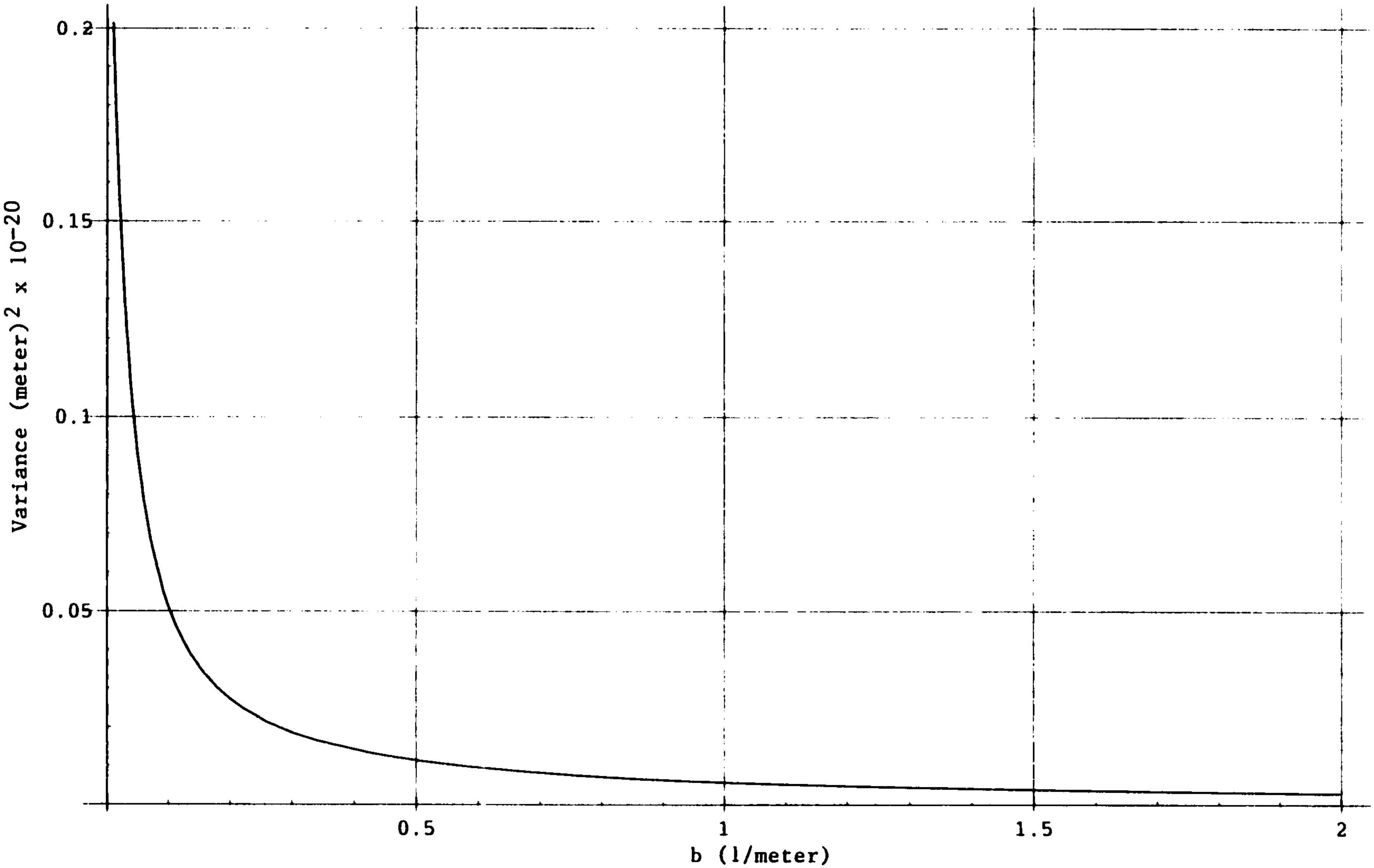

Figure 5.1A: The variance decreases very fast as b increases, this is similar to Figure 4.5, $z = h/2$, $\varepsilon = 0.1$.

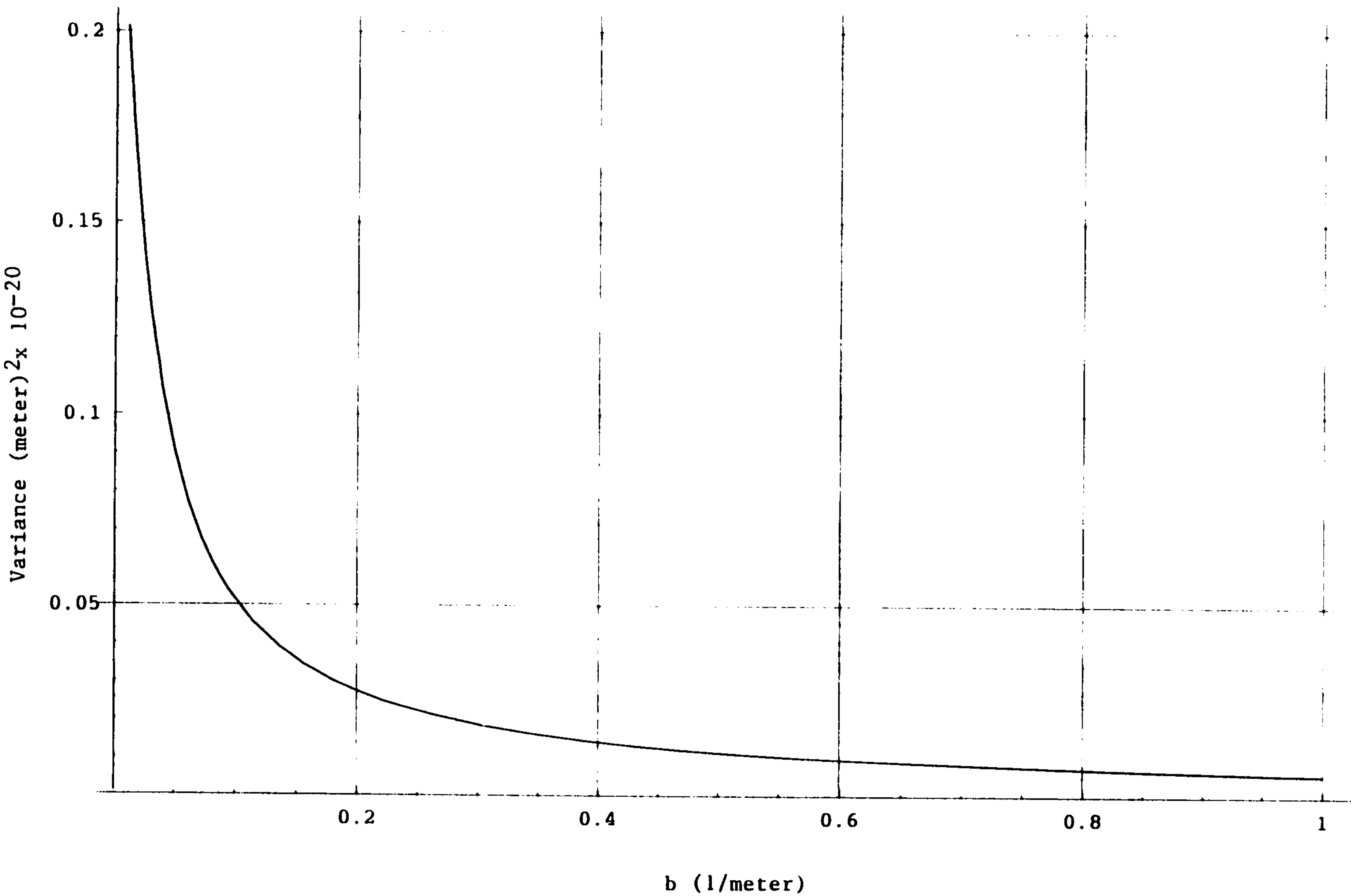

Figure 5.1B: The variance decreases very fast as b increases, this is similar to Figure 4.5, $z = h/2$, $\varepsilon = 0.1$.

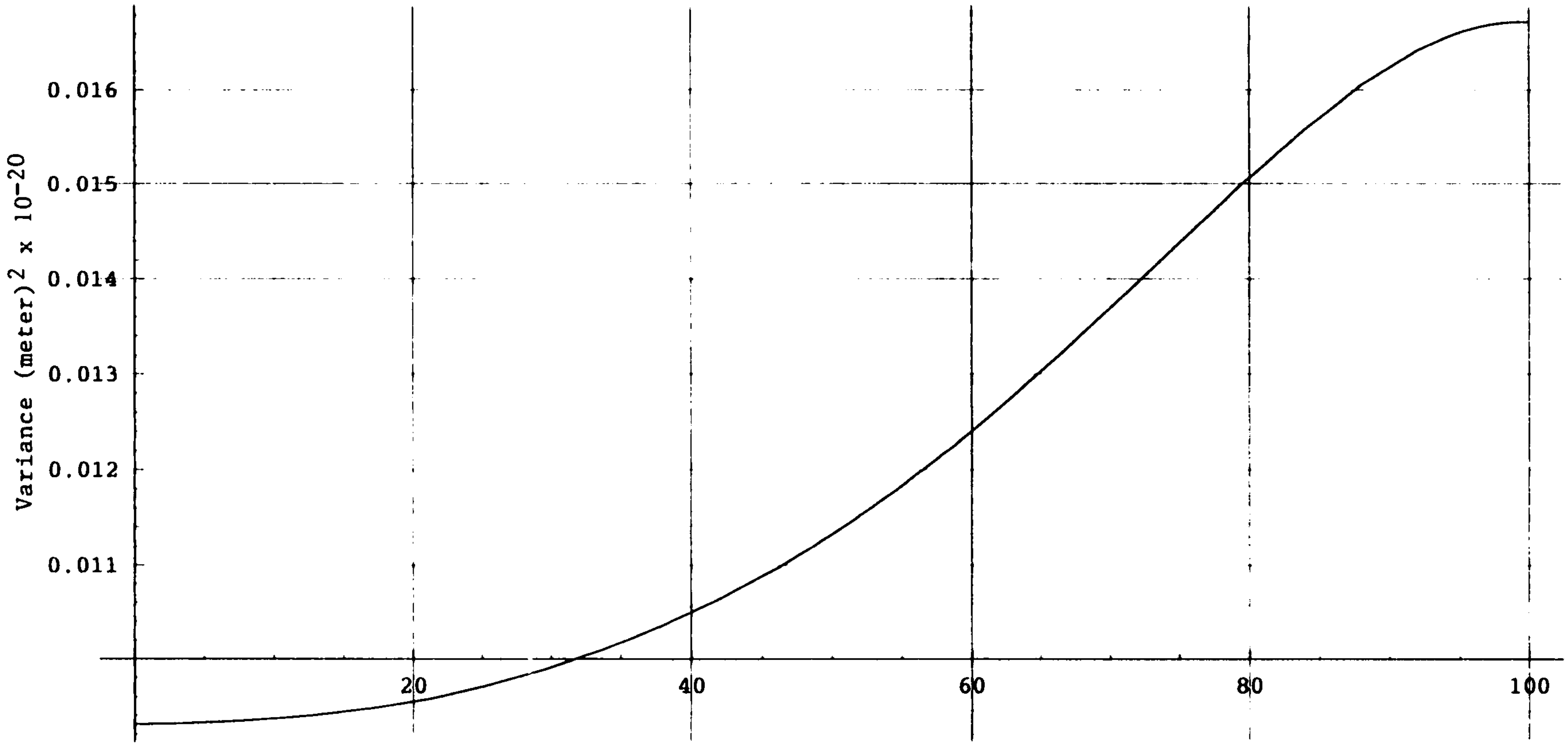

Figure 5.2: The variance as a function of depth z,
b = 0.5, ε = 0.1 .

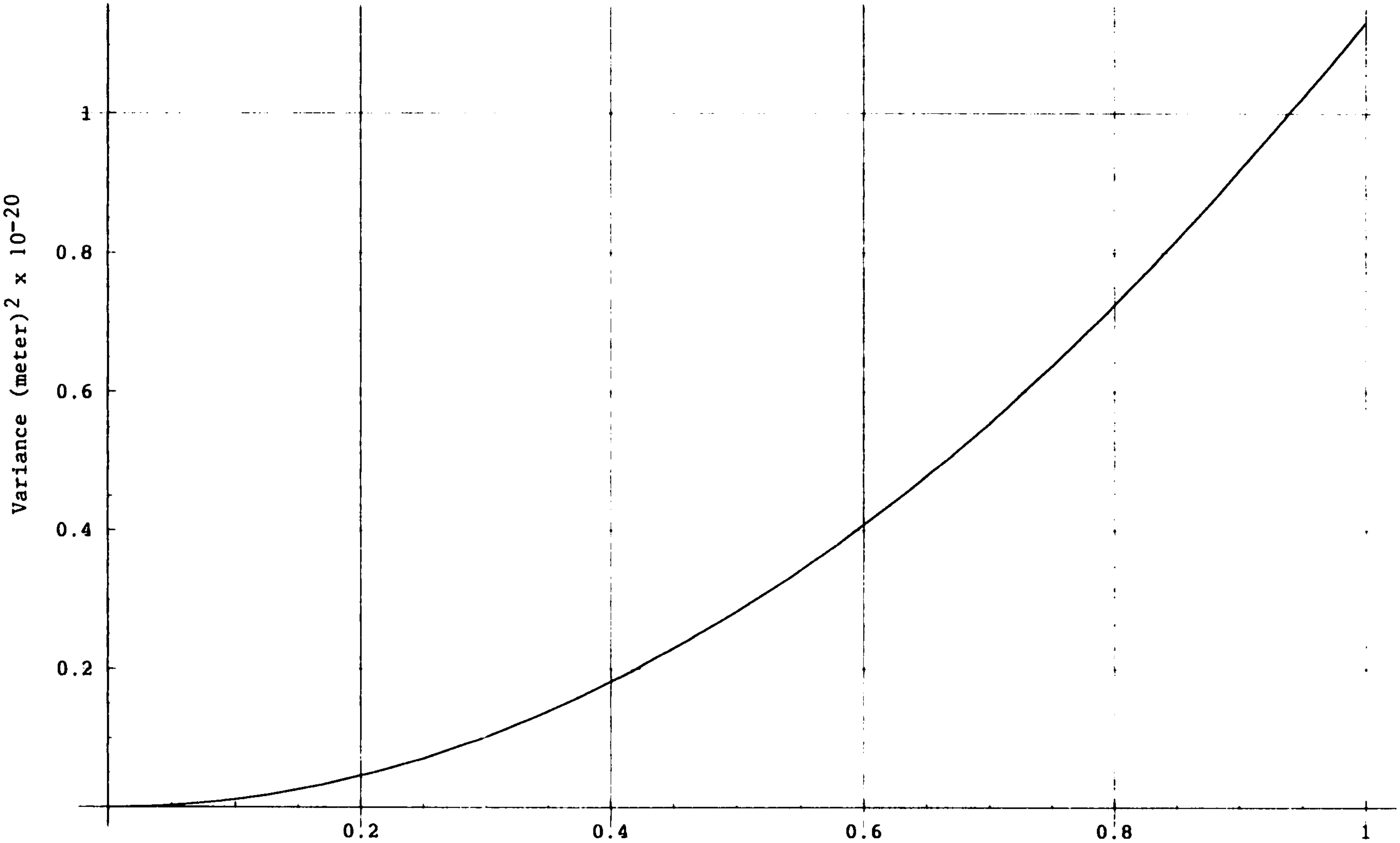

Figure 5.3: The deviation of the solution from the mean solution increases as the inhomogeneity ε increases, b = 0.5, z = h/2 .

Numerical Example

We will use the same example used in Chapter 4. The evaluated integral Eq. (5.37) has been plotted against correlation distance inverse $1/a$ (Figure 5.1), depth z (Figure 5.2) and ϵ (inhomogeneity measure). The variance vanishes rapidly as b increases. Korvin [1977] also found that the variance of the random wave field in an infinite medium is increasing with increasing correlation distance, i.e., decreases with increasing b. Since the source is put at $z = h$, maximum variance is observed at that point. Finally, the variance increases as the inhomogeneity of the medium measured by ϵ increases.

Formal Perturbation Method

In the last section we have seen that rigorous perturbation method gave us a solution (Eq. 5.20) which is not easy to compute. Also, if we try to find the mean solution, we get a solution that corresponds to a layer whose properties are the mean properties of our problem. This method does not take into account stochasticity of the medium. To overcome this problem, we first find the integro-differential equation which governs the mean solution in the layer. Then, we apply the boundary conditions on the mean solution. We start from the boundary problem (Eqs. 5.4 – 5.9). Equation 5.4 can be written as

$$[L_0 + L_1(\gamma)]V_1(z) = p(z) \tag{5.40}$$

where

$$L_0 = \left(\frac{d^2}{dz^2} - \alpha_1^2\right), \; L_1 = \overline{k}_1^2 \epsilon \rho(z,\gamma), \; p(z) = \frac{4\pi\delta(z-h)}{\mu_1}. \tag{5.41}$$

This is a Helmholtz stochastic equation, similar to equation 4.19 whose mean equation is (4.21):

$$\frac{d^2\langle V_1(z)\rangle}{dz^2} - \alpha_1^2\langle V_1(z)\rangle = \epsilon_1\overline{k}_1^2\int_0^h G_1(z,z_0)K_{\rho\rho}(z,z_0)\langle V_1(z_0)\rangle dz_0 + p(z) \tag{5.42}$$

which can be written as

Now we repeat the same steps we did in the last section. We get

$$\langle V_1(z)\rangle = \frac{4\pi}{\mu_1}G_1(z,h) - G_1(z,h)\left[\frac{d\langle V_1(z)\rangle}{dz}\right]_{z=h} + \epsilon\overline{k}^2\ell(z) \tag{5.43}$$

where

$$\ell(z) = \int_0^h\int_0^h G_1(z_1,z)G_1(z_1,z_2)K_{\rho\rho}(z_1,z_2)\langle V_1(z_2)\rangle dz_1dz_2 \tag{5.44}$$

$$V_2(z) = G_2(z,h)\left[\frac{dV_2(z)}{dz}\right[_{z=h}. \tag{5.45}$$

Using boundary conditions 5.7 to 5.9, we get solutions similar to 5.20 and 5.21

$$\langle V_1(z)\rangle = \frac{4\pi}{\mu_1}G_1(z,h)\left(1-\frac{\mu_2}{A}G_1(h,h)\right) - \epsilon\overline{k}_1^2\left(\frac{\mu_2}{A}G_1(z,h)\ell(h) - \ell(z)\right) \tag{5.46}$$

$$V_2(z) = \frac{G_2(z,h)}{A}\left(4\pi G_1(h,h) + \epsilon\overline{k}_1^2\mu_1\ell(h)\right). \tag{5.47}$$

Notice that the contribution of the randomness of the medium is proportional to ϵ. Note that (5.46) and (5.47) are similar to equations (5.20) and (5.21) except for the integral ξ.

CONCLUSION

In this thesis we solved a boundary value problem. One of the differential equations has stochastic coefficients. One of the obvious attempts was to apply a rigorous perturbation technique, assuming small deviations from the mean. Although calculating the random field is very difficult in this case, the deviation of the solution from the mean has been studied in Chapter 5. Instead, the integro-differential equations have been found first, then the mean Love wave displacement field has been found using two different methods; Laplace transform (Chapter 4) and Green's function method, Chapter 5. We notice that the effect of the inhomogeneities gets stronger as the frequencies increase. Also, the deviation of the mean is larger for fundamental mode then the higher ones. Both methods confirm that the as the correlation length decreases the deviation from the mean, for both the displacement field and phase propagation velocity, approach zero very fast. Also, more amplitude attenuation of the displacement field occurs in case of stochastic medium. This attenuation increases as we move away from the interface toward the surface.

In this thesis only density was taken to be random, for simplification purpose. However, more general analysis should take into account the stochasticities in the rigidity μ_1. This requires two other correlation functions $\langle\mu(z)\mu(z_0)\rangle$ and $\langle\mu(z)\rho(z)\rangle$ be used. Chu, Askar and Cakmak [1981]. Future work should also consider the nonlocal inhomogeneity, e.g., linear increase of ρ and μ in depth. Comparison can also be made when β_1 is taken to be random rather than ρ_1 and/or μ_1.

APPENDIX A

Deriving the Green's Functions

The Green's function for the layer is the solution of the following problem

$$\left.\begin{array}{l} \dfrac{d^2G_1}{dz^2} - \alpha_1^2 G_1 = \delta(z - z_0); \quad 0 < z < h \\ \dfrac{dG_1}{dz} = 0 \quad \text{at} \quad z = 0 \text{ and } z = h. \end{array}\right\}. \tag{C.1}$$

This is equivalent to the following problem:

$$\frac{d^2G_1}{dz^2} - \alpha_1^2 G_1 = 0 \quad 0 \le x < x_0 \quad \text{and} \quad x_0 < x \le h; \tag{C.2}$$

$$\frac{dG_1}{dz} = 0 \quad \text{at both} \quad z = 0 \text{ and } z = h; \tag{C.3}$$

$$G_1 \quad \text{is continuous at} \quad x = x_0. \tag{C.4}$$

$$\left.\frac{dG_1}{dz}\right|_{z=z_0^+} - \left.\frac{dG_1}{dz}\right|_{z=z_0} = 1. \tag{C.5}$$

To solve this system, first we find the Green's function of an infinite medium. Here, $G(z)$ satisfies

$$\frac{d^2G}{dz^2} - \alpha^2 G = 0 \quad \text{and vanishes at } z = -\infty \text{ and } z = \infty.$$

$$G(z) = \begin{cases} Ae^{\alpha z}, & -\infty < z < z_0 \\ Be^{-\alpha z}, & z_0 < z < \infty \end{cases}. \tag{C.6}$$

G is continuous at $z = z_0$; which implies that

$$G(z, z_0) = \begin{cases} Ae^{\alpha(z-z_0)}, & -\infty < z < z_0 \\ Ae^{\alpha(z_0-z)}, & z_0 < z < \infty \end{cases} \tag{C.7}$$

$G(z, z_0)$ satisfies the jump condition. This gives

$$[G_1'(z, z_0)]_{z=z_0^+} - [G'(z, z_0)]_{z=z_0^-} = -A\alpha e^{\alpha(z_0-z_0)} - A\alpha e^{\alpha(z_0-z_0)} = 1 \Rightarrow A = -\frac{1}{2\alpha}. \tag{C.8}$$

Substituting this result in (A.7) we get

$$G(z, z_0) = -\frac{e^{-\alpha|z-z_0|}}{2\alpha}. \tag{C.9}$$

This is the infinite medium's Green's function.

Now, we impose boundaries in the infinite medium. First, we consider the layer which is bounded by $z = 0$ and $z = h$. Assume it has the Green's function $G_1(z, z_0)$ whose form is

$$G_1(z, z_0) = -\frac{e^{-\alpha|z-z_0|}}{2\alpha_1} + Ae^{\alpha_1 z} + Be^{-\alpha_1 z}. \tag{C.10}$$

Using boundary conditions (A.3), we get

$$G_1(z, z_0) = -\left(\frac{1}{2\alpha_1}\right)\left[e^{-\alpha|z-z_0|} + \frac{e^{\alpha z}\left(e^{-\alpha(h+z_0)} - e^{-\alpha(h-z_0)}\right)}{e^{\alpha h} - e^{-\alpha h}} + \frac{e^{-\alpha z}\left(e^{\alpha(h-z_0)} + e^{-\alpha(h-z_0)}\right)}{e^{\alpha h} - e^{-\alpha h}}\right].$$

Now, for the half-space, let the Green's function be $G_2(z, z_0)$ which satisfies

$$\frac{d^2G_2}{dz^2} - \alpha_2^2 G_2 = 0; \quad h \leq x < x_0 \quad \text{and} \quad x_0 < x < \infty; \tag{C.11}$$

$$\frac{dG_2}{dz} = 0 \quad \text{at} \quad z = h \ \text{and} \ z = \infty \tag{C.12}$$

$$G_2 \quad \text{is continuous at} \quad x = x_0 \tag{C.13}$$

$$\left.\frac{dG_2}{dz}\right|_{z=z_0^+} - \left.\frac{dG_2}{dz}\right|_{z=z_0^-} = 1. \tag{C.14}$$

Again, we will assume that the Green's function $G_2(z, z_0)$ is the sum of infinite medium's Green's function plus two terms which can be found from the boundary conditions. So, $G_2(z, z_0)$ has the form:

$$G_2(z, z_0) = -\frac{e^{-\alpha_2|z-z_0|}}{2\alpha_2} + Ce^{\alpha_2 z} + De^{-\alpha_2 z}. \tag{C.15}$$

Using the boundary conditions (A.12), we get

$$G_2(z, z_0) = -\left(\frac{1}{2\alpha_2}\right)\left[e^{-\alpha_2|z-z_0|} + e^{-\alpha_2(z+z_0-2h)}\right]. \tag{C.16}$$

APPENDIX B

Deriving the Green's Function

We calculate Green's function from the following homogeneous differential equation

$$\frac{d^2G}{dz^2} + q^2 G = 0. \tag{B.1}$$

So

$$G(z) = \begin{cases} A\cos qx + B\sin qz & \text{if } z < z_0 \\ C\cos qz + D\sin qz & \text{if } z > z_0 \end{cases} \tag{B.2}$$

$$G'(z) = \begin{cases} -Aq\sin qz + Bq\cos qz & \text{if } z < z_0 \\ -Cq\sin qz + Dq\cos qz & \text{if } z > z_0 \end{cases} \tag{B.3}$$

The homogeneous boundary conditions are:

$$\left.\frac{dG}{dz}\right|_{z=0} = 0, \quad G|_{z=0} = 0. \tag{B.4}$$

This implies that

$$A = 0, \qquad \text{and} \quad B = 0. \tag{B.5}$$

Using (B.5) implies that

$$G(z, z_0) = \begin{cases} 0 & \text{if } z < z_0, \\ C\cos qz + D\sin qz & \text{if } z > z_0. \end{cases} \tag{B.6}$$

Then continuity condition gives,

$$C\cos qz_0 + D\sin qz_0 = 0 \quad xq\sin qz_0 \tag{B.7}$$

Jump condition gives

$$-qC\sin qz_0 + Dq\cos qz_0 = +1 \quad \times \cos qz_0 \tag{B.8}$$

Multiplying (B.7) by $q \sin qz_0$, (B.8) by $\cos qz_0$, then subtracting, we get

$$Dq\sin^2 qz_0 + Dq\cos^2 qz_0 = +\cos qz_0 \qquad \Rightarrow C = -\frac{\sin qz_0}{q}, D = +\frac{\cos qz_0}{q} \quad \text{(B.9)}$$

Substituting C and D in (B.6) gives

$$G(z, z_0) = \left\{ \begin{array}{ll} \dfrac{-\sin qz_0 \cos qz}{q} + \dfrac{\sin qz \cos qz_0}{q} & \text{if } z > z_0 \\ 0 & \text{otherwise} \end{array} \right\}. \quad \text{(B.10)}$$

This can be written as

$$G(z, z_0) = \frac{-1}{q} \sin q(z - z_0) \qquad \text{where } z > z_0. \quad \text{(B.11)}$$

Identically G can be written as

$$G(z, z_0) = \frac{1}{q} \sin q(z - z_0), \qquad \text{where } z < z_0. \quad \text{(B.12)}$$

APPRNFIC C

Finding the Roots

$$D(s) = (s^2 + \alpha_1^2)[(s+b)^2 + \alpha_1^2] - c^2, \tag{C.1}$$

where $c^2 = \epsilon^2 \overline{k}^4$.

Let

$$x = \left(s + \frac{b}{2}\right)^2; \text{ or } s = \pm\sqrt{x} - \frac{b}{2}. \tag{C.2}$$

First for $s = \sqrt{x} - \frac{b}{2}$, then

$$\left.\begin{aligned}
D(s) &= \left(\left(\sqrt{x} - \frac{b}{2}\right)^2 + \alpha_1^2\right)\left[\left(\sqrt{x} + \frac{b}{2}\right)^2 + \alpha_1^2\right] - c^2 \\
&= \left(\sqrt{x} - \frac{b}{2}\right)^2\left(\sqrt{x} + \frac{b}{2}\right)^2 + \alpha_1^2\left(\sqrt{x} - \frac{b}{2}\right)^2 + \alpha_1^2\left(\sqrt{x} + \frac{b}{2}\right)^2 + \alpha_1^4 - c^2 \\
&= \left(x - \frac{b^2}{4}\right)^2 + \alpha_1^2\left(2x + 2\frac{b^2}{4}\right) + \alpha_1^4 - c^2 \\
&= x^2 - \left(\frac{b^2}{2} - 2\alpha_1^2\right)x + \left(\frac{b^4}{16} + \alpha_1^2 2\frac{b^2}{4} + \alpha_1^4 - c^2\right) \\
&= x^2 + \left(2\alpha_1^2 - \frac{b^2}{2}\right)x + \left(\alpha_1^4 + \frac{b^2}{2}\alpha_1^2 + \frac{b^4}{16} - c^2\right) \\
&= x^2 + jx + k,
\end{aligned}\right\} \tag{C.3}$$

where

$$j = \left(2\alpha_1^2 - \frac{b^2}{2}\right) \tag{C.4}$$

and

$$k = \left(\alpha_1^4 + \frac{b^2}{2}\alpha_1^2 + \frac{b^4}{16} - c^2\right). \tag{C.5}$$

Solving this equation

$$\Rightarrow x = \frac{-j \pm \sqrt{j^2 - 4k}}{2} \tag{C.6}$$

$$= \frac{-\left(2\alpha_1^2 - \frac{b^2}{2}\right) \pm \sqrt{\left(2\alpha_1^2 - \frac{b^2}{2}\right)^2 - 4\left(\alpha_1^4 + \frac{b^2}{2}\alpha_1^2 + \frac{b^4}{16} - c^2\right)}}{2}$$

$$= \frac{b^2}{4} - \alpha_1^2 \pm \sqrt{-\alpha_1^2 b^2 + c^2} \tag{C.7}$$

So, we have the following two roots for (C.6)

$$\left.\begin{aligned} x_1 &= \left(\frac{b^2}{4} - \alpha_1^2\right) + \sqrt{c^2 - \alpha_1^2 b_1^2}, \\ x_2 &= \left(\frac{b^2}{4} - \alpha_1^2\right) - \sqrt{c^2 - \alpha_1^2 b^2}. \end{aligned}\right\} \tag{C.8}$$

Similarly if we substitute $s = -\sqrt{x} - \frac{b}{2}$ in (C.1), we get the same roots x_1 and x_2.

Substituting back in equation (C.2), we get the following four roots:

$$s_1 = \sqrt{x_1} - \frac{b}{2},\ s_2 = \sqrt{x_2} - \frac{b}{2},\ s_3 = -\left(\sqrt{x_1} + \frac{b}{2}\right),\ s_4 = -\left(\sqrt{x_2} + \frac{b}{2}\right). \tag{C.9}$$

APPENDIX D

In this appendix we give some details of computing the variance integral

$$\langle f^{(2)^2}\rangle = \int_0^h \int_0^h J(z,z_1)J(z,z_2)\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle dz_1 dz_2 \tag{D.1}$$

where

$$J(z,z_0) = \overline{k}_1^2 \int_1^{(1)}(z_0)\left[\frac{\mu_2}{A}G_1(h,z_0) - G_1(z,z_0)\right]. \tag{D.2}$$

and

$$\langle\rho(z_1,\gamma)\rho(z_2,\gamma)\rangle = \sigma^{-2}e^{-|z_1-z_2|b}. \tag{D.3}$$

We found Green's function

$$G(z,z_0) = -\frac{1}{2\alpha_1}\left[I_1(z,z_0) + e^{-\alpha_1|z-z_0|}\right] \tag{D.4}$$

where

$$I_1(z,z_0) = \frac{e^{-\alpha_1 z}\left(e^{-\alpha_1(h+z_0)} + e^{-\alpha_1(h-z_0)}\right)}{m} + \frac{e^{-\alpha_1 z}\left(e^{\alpha_1(h-z_0)} + e^{-\alpha_1(h-z_0)}\right)}{m}$$

where

$$m = e^{\alpha_1 h} - e^{-\alpha_1 h}.$$

Then

$$G_1(z_0,h) = -\frac{1}{m\alpha_1}\left[e^{\alpha_1 z_0} + e^{-\alpha_1 z_0}\right].$$

We also found

$$f_1^{(1)}(z_0) = \frac{4\pi}{\mu_1}G_1(h,z_0)\left(1 - \frac{\mu_2}{A}G_1(h,h)\right). \tag{D.5}$$

Using (D.4) and (D.5), (D.2) becomes

$$\begin{aligned} J(z,z_0) \quad = \quad & \frac{\overline{k}_1^2\mu_2}{\mu_1 A}\left(1 - \frac{\mu_2}{A}G_1(h,h)\right)\left[\frac{\mu_2}{G}G_1(h,z_0)G_1(h,z_0)G_1(h,z_0)\right. \\ & \left. + \frac{1}{2\alpha_1}G_1(h,z_0)I(z,z_0) + \frac{1}{2\alpha_1}G_1(h,z_0)e^{-\alpha_1|z-z_0|}\right] \end{aligned} \tag{D.6}$$

Let

$$\begin{aligned} CN1 &= \frac{4\pi \overline{k}_1^2 \mu_2}{\mu_1 A}\left(1 - \frac{\mu_2}{A} G_1(h,h)\right) \\ CN2 &= \frac{2\pi \overline{k}_1^2}{\alpha_1 \mu_1}\left(1 - \frac{\mu_2}{A} G_1(h,h)\right). \end{aligned}$$

Then

$$J(z,z_0) = \begin{cases} J_1(z,z_0) = CN1 G_1(h,z_0) G_1(h,z_0) + CN2 G_1(h,z_0) I(z,z_0) \\ \quad + CN2 G_1(h,z_0) e^{-\alpha_1(z-z_0)} \quad \text{for } z > z_0 \\ J_2(z,z_0) = CN1 G_1(h,z_0) G_1(h,z_0) + CN2 G_1(h,z_0) I(z,z_0) \\ \quad + CN2 G_1(h,z_0) e^{-\alpha_1(z_0-z)} \quad \text{for } z < z_0 \end{cases} \tag{D.7}$$

$$\begin{aligned} \langle f_1^{(2)^2}(z) &= \int_0^h J(z,z_2) \int_0^h J(z,z_1) e^{-|z_1-z_2|b} dz_1 dz_2 \\ &= \int_0^z J_1(z,z_2) \left[\int_0^{z_2} J_1(z,z_1) e^{-(z_2-z_1)b} dz_1 \right. \\ &\quad \left. + \int_{z_2}^z J_1(z,z_1) e^{-z(z_1-z_0)b} + \int_z^h J_2(z,z_1) e^{-(z_1-z_2)b} dz_1 \right] dz_2 \\ &\quad + \int_z^h J_2(z,z_2) \left[\int_0^z J_1(z,z_1) e^{-(z_2-z_1)b} dz_1 \right. \\ &\quad \left. + \int_z^{z_2} J_2(z,z_1) e^{-(z_2-z_1)b} dz + \int_{z_2}^h J_2(z,z_1) e^{-(z_1-z_2)b} dz_1 \right] dz_2. \end{aligned} \tag{D.8}$$

Now we use Mathematica Computer Software to evaluate this tedious and long integral. The attached sheets are the output of the program.

grating the first term

```
intg1[z_,b_] = (-5*E^(4*a1*h - 2*a1*(h - z))*(a1*cn2 - cn1*E^(a1*(h - z)) + a1*cn2*E^(2*a1*(h - z)))^2)/
  (4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(4*a1*h - 2*a1*(h - z) + 2*a1*z)*
   (a1*cn2 - cn1*E^(a1*(h - z)) + a1*cn2*E^(2*a1*(h - z)))^2)/
  (a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(4*a1*h - 2*a1*(h - z) + 4*a1*z)*
   (a1*cn2 - cn1*E^(a1*(h - z)) + a1*cn2*E^(2*a1*(h - z)))^2)/
  (4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (5*(cn1^2*E^(4*a1*h) + 2*a1^2*cn2^2*E^(4*a1*h) + a1^2*cn2^2*E^(2*a1*(3*h - z)) -
     2*a1*cn1*cn2*E^(a1*(5*h - z)) + a1^2*cn2^2*E^(2*a1*(h + z)) -
     2*a1*cn1*cn2*E^(a1*(3*h + z))))/(4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h - 3*a1*z)*(cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) -
     a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^5*b*(-1 + E^(2*a1*h))^4) - (E^(a1*h - a1*z)*
   (cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) - a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/(a1^5*b*(-1 + E^(2*a1*h))^4)
 - (4*(2*a1^2 - b^2)*E^(a1*h - a1*z)*
   (cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) - a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
     a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^5*b*(-1 + E^(2*a1*h))^4) - (4*(2*a1^2 - b^2)*E^(a1*h - a1*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) -
 (8*(2*a1^2 - b^2)*E^(a1*h - a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
     a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h + a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
     a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
  (a1^5*b*(-1 + E^(2*a1*h))^4) + ((E^(b*z)*
     (cn1^2*E^(4*a1*h) + 2*a1^2*cn2^2*E^(4*a1*h) + a1^2*cn2^2*E^(2*a1*(3*h - z)) -
       2*a1*cn1*cn2*E^(a1*(5*h - z)) + a1^2*cn2^2*E^(2*a1*(h + z)) -
       2*a1*cn1*cn2*E^(a1*(3*h + z))))/(4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4) +
   (E^((2*a1 + b)*z)*(cn1^2*E^(4*a1*h) + 2*a1^2*cn2^2*E^(4*a1*h) +
       a1^2*cn2^2*E^(2*a1*(3*h - z)) - 2*a1*cn1*cn2*E^(a1*(5*h - z)) +
```



```
       a1^2*cn2^2*E^(2*a1*(h + z)) - 2*a1*cn1*cn2*E^(a1*(3*h + z))))/
    (a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4))/E^((4*a1 + b)*z) +
 ((4*(2*a1^2 - b^2)*E^(a1*h + 3*a1*z)*
     (cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) - a1*cn2*E^(a1*(h + 3*z)))*
     (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
    (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
   (4*(2*a1^2 - b^2)*E^(a1*h - a1*z)*
     (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
     (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
    (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) +
   (8*(2*a1^2 - b^2)*E^(a1*h + a1*z)*
     (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
     (a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z)))/
    (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4))/E^((2*a1 + b)*z) +
 (4*E^(a1*h - a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
     a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h - 2*a1*z) - cn1*E^(a1*h - a1*z))*z
  )/(a1^4*b*(-1 + E^(2*a1*h))^4) +
 ((2*a1^3*cn2^2*E^(4*a1*h) - a1^2*b*cn2^2*E^(4*a1*h) -
     4*a1^2*cn1*cn2*E^(4*a1*h + a1*(h - z)) + 2*a1*b*cn1*cn2*E^(4*a1*h + a1*(h - z)) -
     2*b*cn1^2*E^(4*a1*h + 2*a1*(h - z)) - 4*a1^2*b*cn2^2*E^(4*a1*h + 2*a1*(h - z)) -
     4*a1^2*cn1*cn2*E^(4*a1*h + 3*a1*(h - z)) + 2*a1*b*cn1*cn2*E^(4*a1*h + 3*a1*(h - z)) +
     2*a1^3*cn2^2*E^(4*a1*h + 4*a1*(h - z)) - a1^2*b*cn2^2*E^(4*a1*h + 4*a1*(h - z)) -
     2*a1^3*cn2^2*E^(2*a1*(h - z) + 2*a1*(3*h - z)) -
     a1^2*b*cn2^2*E^(2*a1*(h - z) + 2*a1*(3*h - z)) +
     4*a1^2*cn1*cn2*E^(2*a1*(h - z) + a1*(5*h - z)) +
     2*a1*b*cn1*cn2*E^(2*a1*(h - z) + a1*(5*h - z)) -
     2*a1^3*cn2^2*E^(2*a1*(h - z) + 2*a1*(h + z)) -
     a1^2*b*cn2^2*E^(2*a1*(h - z) + 2*a1*(h + z)) +
     4*a1^2*cn1*cn2*E^(2*a1*(h - z) + a1*(3*h + z)) +
     2*a1*b*cn1*cn2*E^(2*a1*(h - z) + a1*(3*h + z)))*z)/
  (a1^4*(2*a1 - b)*(2*a1 + b)*E^(2*a1*(h - z))*(-1 + E^(2*a1*h))^4)
```

grating the second term

```
intg2[z_,b_] := (-2*(a1 - b)*E^(a1*h - a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
    (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
    (a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z)))/
```



```
  (a1^5*(2*a1 - b)*b*(-1 + E^(2*a1*h))^4) +
 (2*(a1 - b)*E^(a1*h + a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z)))/
  (a1^5*(2*a1 - b)*b*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z - (4*a1 + b)*z)*(-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) +
     2*a1*b*E^(4*a1*z) + b^2*E^(4*a1*z))*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z)))/
  (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) -
 (2*E^(a1*h - a1*z - (4*a1 + b)*z)*(-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) +
     2*a1*b*E^(4*a1*z) + b^2*E^(4*a1*z))*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z)))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h - 5*a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (2*(a1 + b)*E^(a1*h - 3*a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (a1^5*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (2*(a1 + b)*E^(a1*h - a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
```



```
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (a1^5*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h + 3*a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z - (4*a1 + b)*z)*(-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) +
     2*a1*b*E^(4*a1*z) + b^2*E^(4*a1*z))*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z)))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 E^(b*z)*((2*E^(a1*h - a1*z - (4*a1 + b)*z)*
      (-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) + 2*a1*b*E^(4*a1*z) +
        b^2*E^(4*a1*z))*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
        a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) -
        cn1*E^(a1*h + a1*z)))/(a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
    (E^(a1*h + a1*z - (4*a1 + b)*z)*
      (-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) + 2*a1*b*E^(4*a1*z) +
        b^2*E^(4*a1*z))*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
        a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) -
        cn1*E^(a1*h + a1*z)))/(a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) +
    (E^(a1*h - 3*a1*z - (4*a1 + b)*z)*
      (-2*a1*b + b^2 - 8*a1^2*E^(2*a1*z) + 2*b^2*E^(2*a1*z) + 2*a1*b*E^(4*a1*z) +
        b^2*E^(4*a1*z))*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
        a1*cn2*E^(a1*(h + 3*z)))*(-(a1*cn2*E^(2*a1*h)) - a1*cn2*E^(2*a1*z) +
        cn1*E^(a1*h + a1*z)))/(a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4)) +
 (2*(8*a1^2 - 3*b^2)*E^(a1*h - a1*z + (2*a1 + b)*z - (4*a1 + b)*z)*
   (-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) + a1*cn2*E^(a1*(h + 3*z)))*
   (a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*z)/
  (a1^4*(2*a1 - b)*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4)
```

rating the third term

```
Intg3[z_,b_] := -((E^(a1*h - a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
    (-2*a1*b*cn1*E^(4*a1*h - b*h) - b^2*cn1*E^(4*a1*h - b*h) -
      4*a1^3*cn2*E^(a1*h - b*h - a1*z) - 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) +
      2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) - 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) +
      2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) + a1*b^2*cn2*E^(a1*h - b*h + a1*z) -
      4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) + a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) +
      2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) - a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) +
      4*a1^3*cn2*E^(a1*h - a1*z - b*z) - a1*b^2*cn2*E^(a1*h - a1*z - b*z) +
      4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) -
      2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) -
      a1*b^2*cn2*E^(a1*h + a1*z - b*z) + 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) -
      a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) + 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) +
      b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) +
      2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
      2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
      b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
      8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
      2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
      4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
      a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
      2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) -
      a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z))))/
```



```
   (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4)) -
  (2*E^(a1*h - a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
   (-2*a1*b*cn1*E^(4*a1*h - b*h) - b^2*cn1*E^(4*a1*h - b*h) -
     4*a1^3*cn2*E^(a1*h - b*h - a1*z) - 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) +
     2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) - 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) +
     2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) -
     2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) + a1*b^2*cn2*E^(a1*h - b*h + a1*z) -
     4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) + a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) +
     2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) - a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) +
     4*a1^3*cn2*E^(a1*h - a1*z - b*z) - a1*b^2*cn2*E^(a1*h - a1*z - b*z) +
     4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) -
     2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) -
     a1*b^2*cn2*E^(a1*h + a1*z - b*z) + 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) -
     a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) + 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) +
     b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) + 2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
     b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
     8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
     2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
     2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
     b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
     8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
     2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
     4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
     a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
     4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
     a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
     2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
     a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
     2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) -
     a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (E^(a1*h - a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
   (2*a1*b*cn1*E^(4*a1*h - b*h) + b^2*cn1*E^(4*a1*h - b*h) +
```



```
     4*a1^3*cn2*E^(a1*h - b*h - a1*z) + 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) -
     2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) + 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) -
     2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) - 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) +
     2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) - a1*b^2*cn2*E^(a1*h - b*h + a1*z) +
     4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) - a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) -
     2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) + a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) -
     4*a1^3*cn2*E^(a1*h - a1*z - b*z) + a1*b^2*cn2*E^(a1*h - a1*z - b*z) -
     4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) +
     2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) +
     a1*b^2*cn2*E^(a1*h + a1*z - b*z) - 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) +
     a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) - 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) -
     b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) - 2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
     b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
     8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) +
     2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) +
     2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
     b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
     8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
     2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
     4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) -
     a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) -
     4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
     a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
     2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
     a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
     2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) +
     a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 E^(b*z)*((2*E^(a1*h - a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
     (-2*a1*b*cn1*E^(4*a1*h - b*h) - b^2*cn1*E^(4*a1*h - b*h) -
       4*a1^3*cn2*E^(a1*h - b*h - a1*z) - 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) +
       2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) - 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) +
       2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) -
```



```
       2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) + a1*b^2*cn2*E^(a1*h - b*h + a1*z) -
       4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) + a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) +
       2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) - a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) +
       4*a1^3*cn2*E^(a1*h - a1*z - b*z) - a1*b^2*cn2*E^(a1*h - a1*z - b*z) +
       4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) -
       2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) -
       a1*b^2*cn2*E^(a1*h + a1*z - b*z) + 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) -
       a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) + 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) +
       b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) +
       2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
       b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
       8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
       2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
       2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
       b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
       8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
       2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
       4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
       a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
       4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
       a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
       2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
       a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
       2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) -
       a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z))))/
    (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
   (E^(a1*h + a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
     (-2*a1*b*cn1*E^(4*a1*h - b*h) - b^2*cn1*E^(4*a1*h - b*h) -
       4*a1^3*cn2*E^(a1*h - b*h - a1*z) - 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) +
       2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) - 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) +
       2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) -
       2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) + a1*b^2*cn2*E^(a1*h - b*h + a1*z) -
       4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) + a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) +
```

```
2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) - a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) +
4*a1^3*cn2*E^(a1*h - a1*z - b*z) - a1*b^2*cn2*E^(a1*h - a1*z - b*z) +
4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) -
2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) -
a1*b^2*cn2*E^(a1*h + a1*z - b*z) + 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) -
a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) + 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) +
b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) +
2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) -
2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) +
4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) -
a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)))/
(a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) +
(E^(a1*h - 3*a1*z)*(a1*cn2*E^(2*a1*h) + a1*cn2*E^(2*a1*z) - cn1*E^(a1*h + a1*z))*
 (2*a1*b*cn1*E^(4*a1*h - b*h) + b^2*cn1*E^(4*a1*h - b*h) +
  4*a1^3*cn2*E^(a1*h - b*h - a1*z) + 2*a1^2*b*cn2*E^(a1*h - b*h - a1*z) -
  2*a1*b^2*cn2*E^(a1*h - b*h - a1*z) + 4*a1^3*cn2*E^(3*a1*h - b*h - a1*z) -
  2*a1^2*b*cn2*E^(3*a1*h - b*h - a1*z) - 2*a1*b^2*cn2*E^(3*a1*h - b*h - a1*z) +
  2*a1^2*b*cn2*E^(a1*h - b*h + a1*z) - a1*b^2*cn2*E^(a1*h - b*h + a1*z) +
  4*a1^3*cn2*E^(3*a1*h - b*h + a1*z) - a1*b^2*cn2*E^(3*a1*h - b*h + a1*z) -
  2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z - b*z) + a1*b^2*cn2*E^(3*a1*h - 3*a1*z - b*z) -
  4*a1^3*cn2*E^(a1*h - a1*z - b*z) + a1*b^2*cn2*E^(a1*h - a1*z - b*z) -
```



```
  4*a1^3*cn2*E^(3*a1*h - a1*z - b*z) - 2*a1^2*b*cn2*E^(3*a1*h - a1*z - b*z) +
  2*a1*b^2*cn2*E^(3*a1*h - a1*z - b*z) + 2*a1^2*b*cn2*E^(a1*h + a1*z - b*z) +
  a1*b^2*cn2*E^(a1*h + a1*z - b*z) - 4*a1^3*cn2*E^(3*a1*h + a1*z - b*z) +
  a1*b^2*cn2*E^(3*a1*h + a1*z - b*z) - 2*a1*b*cn1*E^(2*a1*h + 2*a1*z - b*z) -
  b^2*cn1*E^(2*a1*h + 2*a1*z - b*z) -
  2*a1*b*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) +
  b^2*cn1*E^(a1*h - b*h - a1*z + a1*(-h + z)) -
  8*a1^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) +
  2*b^2*cn1*E^(3*a1*h - b*h - a1*z + a1*(-h + z)) +
  2*a1*b*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) -
  b^2*cn1*E^(3*a1*h - 3*a1*z - b*z + a1*(-h + z)) +
  8*a1^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) -
  2*b^2*cn1*E^(3*a1*h - a1*z - b*z + a1*(-h + z)) +
  4*a1^3*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) -
  a1*b^2*cn2*E^(3*a1*h - b*h - a1*z + 2*a1*(-h + z)) -
  4*a1^3*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) +
  a1*b^2*cn2*E^(3*a1*h - a1*z - b*z + 2*a1*(-h + z)) -
  2*a1^2*b*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) -
  a1*b^2*cn2*E^(4*a1*h - b*h - a1*z + a1*(-h + 2*z)) +
  2*a1^2*b*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z)) +
  a1*b^2*cn2*E^(2*a1*h + a1*z - b*z + a1*(-h + 2*z))))/
 (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4))
```

Integrating the fourth term

```
Intg4[z_,b_] := (((cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) - a1*cn2*E^(a1*(h + 3*z)))*
    (a1*cn2*E^(a1*h) - cn1/E^(a1*z) + a1*cn2*E^(a1*h - 2*a1*z))*
    (-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) - b^2*E^(-2*a1*z + b*z) +
     8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) - 2*b^2*E^(-2*a1*z + (2*a1 + b)*z) +
     2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) - b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
   (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*E^(a1*z)*(-1 + E^(2*a1*h))^4) +
  (E^(a1*h - a1*z)*(cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) -
```



```
    a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h) + (a1*cn2)/E^(2*a1*z) +
    a1*cn2*E^(2*a1*h - 2*a1*z) - 2*cn1*E^(a1*h - a1*z))*
   (-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) - b^2*E^(-2*a1*z + b*z) +
    8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) - 2*b^2*E^(-2*a1*z + (2*a1 + b)*z) +
    2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) - b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(3*a1*h - a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
    a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + (a1*cn2)/E^(2*a1*z) - cn1*E^(a1*h - a1*z))*
   (-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) - b^2*E^(-2*a1*z + b*z) +
    8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) - 2*b^2*E^(-2*a1*z + (2*a1 + b)*z) +
    2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) - b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4)/E^(b*h) -
((E^(2*a1*h - 3*a1*z)*(cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) -
    a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2*E^(a1*h) - cn1/E^(a1*z) +
    a1*cn2*E^(a1*h - 2*a1*z))*(-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) -
    b^2*E^(-2*a1*z + b*z) + 8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) -
    2*b^2*E^(-2*a1*z + (2*a1 + b)*z) + 2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) -
    b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
  (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h - a1*z)*(cn1*E^(2*a1*(h + z)) - a1*cn2*E^(a1*(3*h + z)) -
    a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + a1*cn2*E^(2*a1*h) + (a1*cn2)/E^(2*a1*z) +
    a1*cn2*E^(2*a1*h - 2*a1*z) - 2*cn1*E^(a1*h - a1*z))*
   (-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) - b^2*E^(-2*a1*z + b*z) +
    8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) - 2*b^2*E^(-2*a1*z + (2*a1 + b)*z) +
    2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) - b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h + a1*z)*(-(cn1*E^(2*a1*(h + z))) + a1*cn2*E^(a1*(3*h + z)) +
    a1*cn2*E^(a1*(h + 3*z)))*(a1*cn2 + (a1*cn2)/E^(2*a1*z) - cn1*E^(a1*h - a1*z))*
   (-8*a1^2 + 4*b^2 - 2*a1*b*E^(-2*a1*z + b*z) - b^2*E^(-2*a1*z + b*z) +
    8*a1^2*E^(-2*a1*z + (2*a1 + b)*z) - 2*b^2*E^(-2*a1*z + (2*a1 + b)*z) +
    2*a1*b*E^(-2*a1*z + (4*a1 + b)*z) - b^2*E^(-2*a1*z + (4*a1 + b)*z)))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4))/E^(b*z)
```


Integrating the fifth term

```
Intg5[z_, b_] :=
 -((E^(6*a1*h - 5*a1*z - a1*(h + z))*
      (a1*cn2 + a1*cn2*E^(2*a1*z) - cn1*E^(a1*(-h + z)))*
      (a1*cn2*E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(a1*h + 2*a1*z)))/
     (4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4)) +
  (E^(2*a1*h - a1*z - a1*(h + z))*
     (a1*cn2 + a1*cn2*E^(2*a1*z) - cn1*E^(a1*(-h + z)))*
     (a1*cn2*E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(a1*h + 2*a1*z)))/
    (4*a1^5*(2*a1 - b)*(-1 + E^(2*a1*h))^4) -
  (E^(4*a1*h - 3*a1*z - a1*(h + z))*
     (-2*a1^3*cn2^2*E^(a1*h) + 2*a1^2*b*cn2^2*E^(a1*h) -
      2*a1^3*cn2^2*E^(3*a1*h) + 2*a1^2*b*cn2^2*E^(3*a1*h) +
      2*a1^2*cn1*cn2*E^(a1*z) - a1*b*cn1*cn2*E^(a1*z) +
      2*a1^2*cn1*cn2*E^(2*a1*h + a1*z) -
      3*a1*b*cn1*cn2*E^(2*a1*h + a1*z) -
      2*a1^3*cn2^2*E^(a1*h + 2*a1*z) +
      3*a1^2*b*cn2^2*E^(a1*h + 2*a1*z) -
      4*a1^3*cn2^2*E^(3*a1*h + 2*a1*z) +
      4*a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z) +
      2*a1^2*cn1*cn2*E^(2*a1*h + 3*a1*z) -
      3*a1*b*cn1*cn2*E^(2*a1*h + 3*a1*z) +
      a1^2*b*cn2^2*E^(a1*h + 4*a1*z) -
      2*a1^3*cn2^2*E^(3*a1*h + 4*a1*z) +
      2*a1^2*b*cn2^2*E^(3*a1*h + 4*a1*z) -
      a1*b*cn1*cn2*E^(a1*h + a1*(-h + z)) +
      4*a1^2*cn1*cn2*E^(3*a1*h + a1*(-h + z)) -
      3*a1*b*cn1*cn2*E^(3*a1*h + a1*(-h + z)) -
      4*a1*cn1^2*E^(2*a1*h + a1*z + a1*(-h + z)) +
      4*b*cn1^2*E^(2*a1*h + a1*z + a1*(-h + z)) -
      [illegible]
```

```
    4*a1^2*cn1*cn2*E^(3*a1*h + 2*a1*z + a1*(-h + z)) -
    3*a1*b*cn1*cn2*E^(3*a1*h + 2*a1*z + a1*(-h + z)) -
    2*a1^3*cn2^2*E^(3*a1*h + 2*a1*(-h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*(-h + z)) +
    2*a1^2*cn1*cn2*E^(2*a1*h + a1*z + 2*a1*(-h + z)) -
    a1*b*cn1*cn2*E^(2*a1*h + a1*z + 2*a1*(-h + z)) -
    2*a1^3*cn2^2*E^(3*a1*h + 2*a1*z + 2*a1*(-h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z + 2*a1*(-h + z))))/
  (2*a1^5*(2*a1 - b)*b*(-1 + E^(2*a1*h))^4) +
 (E^(2*a1*h - a1*z - a1*(h + z))*
   (-2*a1^3*cn2^2*E^(a1*h) + 2*a1^2*b*cn2^2*E^(a1*h) -
    2*a1^3*cn2^2*E^(3*a1*h) + 2*a1^2*b*cn2^2*E^(3*a1*h) +
    2*a1^2*cn1*cn2*E^(a1*z) - a1*b*cn1*cn2*E^(a1*z) +
    2*a1^2*cn1*cn2*E^(2*a1*h + a1*z) -
    3*a1*b*cn1*cn2*E^(2*a1*h + a1*z) -
    2*a1^3*cn2^2*E^(a1*h + 2*a1*z) +
    3*a1^2*b*cn2^2*E^(a1*h + 2*a1*z) -
    4*a1^3*cn2^2*E^(3*a1*h + 2*a1*z) +
    4*a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z) +
    2*a1^2*cn1*cn2*E^(2*a1*h + 3*a1*z) -
    3*a1*b*cn1*cn2*E^(2*a1*h + 3*a1*z) +
    a1^2*b*cn2^2*E^(a1*h + 4*a1*z) -
    2*a1^3*cn2^2*E^(3*a1*h + 4*a1*z) +
    2*a1^2*b*cn2^2*E^(3*a1*h + 4*a1*z) -
    a1*b*cn1*cn2*E^(a1*h + a1*(-h + z)) +
    4*a1^2*cn1*cn2*E^(3*a1*h + a1*(-h + z)) -
    3*a1*b*cn1*cn2*E^(3*a1*h + a1*(-h + z)) -
    4*a1*cn1^2*E^(2*a1*h + a1*z + a1*(-h + z)) +
    4*b*cn1^2*E^(2*a1*h + a1*z + a1*(-h + z)) -
    a1*b*cn1*cn2*E^(a1*h + 2*a1*z + a1*(-h + z)) +
    4*a1^2*cn1*cn2*E^(3*a1*h + 2*a1*z + a1*(-h + z)) -
    3*a1*b*cn1*cn2*E^(3*a1*h + 2*a1*z + a1*(-h + z)) -
    2*a1^3*cn2^2*E^(3*a1*h + 2*a1*(-h + z)) +
```



```
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*(-h + z)) +
    2*a1^2*cn1*cn2*E^(2*a1*h + a1*z + 2*a1*(-h + z)) -
    a1*b*cn1*cn2*E^(2*a1*h + a1*z + 2*a1*(-h + z)) -
    2*a1^3*cn2^2*E^(3*a1*h + 2*a1*z + 2*a1*(-h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z + 2*a1*(-h + z))))/
  (2*a1^5*(2*a1 - b)*b*(-1 + E^(2*a1*h))^4) +
 (E^(8*a1*h - a1*z - a1*(2*h + z))*
   (-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z))*
   (-(a1*cn2) + cn1*E^(a1*h + a1*z) - a1*cn2*E^(a1*h + a1*(-h + 2*z))))
 /(4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
  (E^(4*a1*h + 3*a1*z - a1*(2*h + z))*
   (-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(a1*h + a1*z))*
   (-(a1*cn2) + cn1*E^(a1*h + a1*z) - a1*cn2*E^(a1*h + a1*(-h + 2*z))))
 /(4*a1^5*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
  (E^(5*a1*h - a1*z)*((2*a1^3*cn2^2)/E^(a1*(h + z)) +
    (a1^2*b*cn2^2)/E^(a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
    a1*b*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) -
    a1*b*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*z - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*z - a1*(h + z)) +
    4*a1^3*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) +
    2*a1^2*b*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) -
    a1*b*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) -
    2*a1*b*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
    4*a1*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) +
```



```
    2*b*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
    2*a1*b*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) -
    a1*b*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(a1*h - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
    3*a1*b*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(a1*h + 2*a1*z - a1*(2*h + z)) +
    2*b*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(2*a1*h + 3*a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) +
    a1^2*b*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
    2*a1*b*cn1*cn2*E^
     (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
    a1^2*b*cn2^2*E^
     (2*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
    a1^2*b*cn2^2*E^(4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z))))/
  (2*a1^5*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
 (E^(3*a1*h + a1*z)*((2*a1^3*cn2^2)/E^(a1*(h + z)) +
    (a1^2*b*cn2^2)/E^(a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
    a1*b*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) -
```



```
    a1*b*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*z - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*z - a1*(h + z)) +
    4*a1^3*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) +
    2*a1^2*b*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) -
    a1*b*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) -
    2*a1*b*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
    4*a1*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) +
    2*b*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
    2*a1*b*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) -
    a1*b*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    a1^2*b*cn2^2*E^(a1*h - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
    3*a1*b*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(a1*h + 2*a1*z - a1*(2*h + z)) +
    2*b*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(2*a1*h + 3*a1*z - a1*(2*h + z)) -
    a1*b*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
    a1^2*b*cn2^2*E^(2*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) +
    a1^2*b*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
    2*a1*b*cn1*cn2*E^
```

```
    (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
   a1^2*b*cn2^2*E^
    (2*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
   a1^2*b*cn2^2*E^(4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z))))/
 (2*a1^5*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
((E^(3*a1*h - a1*z)*(-(a1*cn2) - a1*cn2*E^(2*a1*z) +
     cn1*E^(a1*h + a1*z))*
    (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
     a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
     4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
     a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
     4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
     2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
     2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
     2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
     a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
     4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
     a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
     2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
     b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
     2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
     b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
     8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
     2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
     4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
     a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
     2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
     a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 ((a1*cn2*E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(a1*h + 2*a1*z))*
   (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
    4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
```



```
    a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
    2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
    2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
    b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
    8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
    a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
    2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
    a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
 (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*E^(a1*z)*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h - a1*z)*(a1*cn2 + a1*cn2*E^(2*a1*h) +
    a1*cn2*E^(2*a1*z) - 2*cn1*E^(a1*h + a1*z) +
    a1*cn2*E^(2*a1*h + 2*a1*z))*
  (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
    4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
    2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
```



```
    2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
    b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
    8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
    a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
    2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
    a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4))/E^(b*h) -
((E^(a1*h + a1*z)*(-(a1*cn2) - a1*cn2*E^(2*a1*z) +
     cn1*E^(a1*h + a1*z))*
   (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
    4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
    2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
    2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
    b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
    8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
    a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
    2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
```



```
    a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) +
 (E^(2*a1*h - 3*a1*z)*(a1*cn2*E^(a1*h) - cn1*E^(a1*z) +
    a1*cn2*E^(a1*h + 2*a1*z))*
  (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
    4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
    2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
    2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
    4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
    2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
    2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
    b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
    8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
    4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
    a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
    2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
    a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
  (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(2*a1*h))^4) +
 (E^(a1*h - a1*z)*(a1*cn2 + a1*cn2*E^(2*a1*h) +
    a1*cn2*E^(2*a1*z) - 2*cn1*E^(a1*h + a1*z) +
    a1*cn2*E^(2*a1*h + 2*a1*z))*
  (-2*a1^2*b*cn2*E^(3*a1*h - 3*a1*z + b*z) -
    a1*b^2*cn2*E^(3*a1*h - 3*a1*z + b*z) +
    4*a1^3*cn2*E^(a1*h - a1*z + b*z) -
    a1*b^2*cn2*E^(a1*h - a1*z + b*z) +
```

```
4*a1^3*cn2*E^(3*a1*h - a1*z + b*z) -
2*a1^2*b*cn2*E^(3*a1*h - a1*z + b*z) -
2*a1*b^2*cn2*E^(3*a1*h - a1*z + b*z) +
2*a1^2*b*cn2*E^(a1*h + a1*z + b*z) -
a1*b^2*cn2*E^(a1*h + a1*z + b*z) +
4*a1^3*cn2*E^(3*a1*h + a1*z + b*z) -
a1*b^2*cn2*E^(3*a1*h + a1*z + b*z) -
2*a1*b*cn1*E^(2*a1*h + 2*a1*z + b*z) +
b^2*cn1*E^(2*a1*h + 2*a1*z + b*z) +
2*a1*b*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) +
b^2*cn1*E^(3*a1*h - 3*a1*z + b*z + a1*(-h + z)) -
8*a1^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
2*b^2*cn1*E^(3*a1*h - a1*z + b*z + a1*(-h + z)) +
4*a1^3*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) -
a1*b^2*cn2*E^(3*a1*h - a1*z + b*z + 2*a1*(-h + z)) +
2*a1^2*b*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z)) -
a1*b^2*cn2*E^(2*a1*h + a1*z + b*z + a1*(-h + 2*z))))/
(a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(2*a1*h))^4))/E^(b*z) +
(E^(3*a1*h - a1*z)*((4*a1^4*cn2^2)/E^(a1*(h + z)) -
(a1^2*b^2*cn2^2)/E^(a1*(h + z)) +
8*a1^4*cn2^2*E^(2*a1*h - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(2*a1*h - a1*(h + z)) -
3*a1^2*b^2*cn2^2*E^(2*a1*h - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*z - a1*(h + z)) +
12*a1^4*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
```



```
4*a1^3*b*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
5*a1^2*b^2*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) +
8*a1^4*cn2^2*E^(4*a1*h + 2*a1*z - a1*(h + z)) -
2*a1^2*b^2*cn2^2*E^(4*a1*h + 2*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
2*a1^2*b^2*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 4*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h + 4*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(4*a1*h + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(4*a1*h + a1*(-h + z) - a1*(h + z)) +
16*a1^2*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
2*a1*b*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
5*b^2*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^
  (2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^
  (2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(4*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^
  (4*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
```



```
2*a1*b^2*cn1*cn2*E^
  (3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^
  (2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^
  (4*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
2*a1^3*b*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
2*a1*b*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
b^2*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) +
2*a1^3*b*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
2*a1^3*b*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
a1^2*b^2*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
2*a1^2*b*cn1*cn2*E^
  (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
a1*b^2*cn1*cn2*E^
  (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
2*a1^3*b*cn2^2*E^
  (4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
a1^2*b^2*cn2^2*E^
  (4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)))*h)/
(a1^4*(2*a1 - b)*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4) -
(E^(3*a1*h - a1*z)*((4*a1^4*cn2^2)/E^(a1*(h + z)) -
  (a1^2*b^2*cn2^2)/E^(a1*(h + z)) +
```



```
8*a1^4*cn2^2*E^(2*a1*h - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(2*a1*h - a1*(h + z)) -
3*a1^2*b^2*cn2^2*E^(2*a1*h - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(a1*h + a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(3*a1*h + a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*z - a1*(h + z)) +
12*a1^4*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
4*a1^3*b*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) -
5*a1^2*b^2*cn2^2*E^(2*a1*h + 2*a1*z - a1*(h + z)) +
8*a1^4*cn2^2*E^(4*a1*h + 2*a1*z - a1*(h + z)) -
2*a1^2*b^2*cn2^2*E^(4*a1*h + 2*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(3*a1*h + 3*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) -
2*a1^2*b^2*cn2^2*E^(2*a1*h + 4*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 4*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h + 4*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(2*a1*h + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(4*a1*h + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(4*a1*h + a1*(-h + z) - a1*(h + z)) +
16*a1^2*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
2*a1*b*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
5*b^2*cn1^2*E^(3*a1*h + a1*z + a1*(-h + z) - a1*(h + z)) -
```

```
8*a1^3*cn1*cn2*E^(2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1^2*b*cn1*cn2*E^
 (2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^
 (2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(4*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^
 (4*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*h + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(4*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^
 (3*a1*h + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^
 (2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^
 (4*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
2*a1^3*b*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(3*a1*h - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(2*a1*h + a1*z - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(4*a1*h + a1*z - a1*(2*h + z)) +
2*a1*b*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
b^2*cn1^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) +
2*a1^3*b*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(3*a1*h + 2*a1*z - a1*(2*h + z)) -
2*a1^2*b*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(4*a1*h + 3*a1*z - a1*(2*h + z)) +
```



```
  2*a1^3*b*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
  a1^2*b^2*cn2^2*E^(4*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) -
  2*a1^2*b*cn1*cn2*E^
   (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
  a1*b^2*cn1*cn2*E^
   (3*a1*h + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
  2*a1^3*b*cn2^2*E^
   (4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
  a1^2*b^2*cn2^2*E^(4*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z))
)*z)/(a1^4*(2*a1 - b)*b*(2*a1 + b)*(-1 + E^(2*a1*h))^4)
```

rating the sixth term

```
Intg6[z_,b_] := -(E^(-(a1*h) - 5*a1*z - a1*(h + z))*(a1*cn2 + a1*cn2*E^(2*a1*z) - cn1*E^(a1*(-h + z)))
   (-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(-(a1*h) + a1*z)))/
  (4*a1^5*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
 (E^(-5*a1*h - a1*z - a1*(h + z))*(a1*cn2 + a1*cn2*E^(2*a1*z) - cn1*E^(a1*(-h + z)))*
   (-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(-(a1*h) + a1*z)))/
  (4*a1^5*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
 (E^(2*a1*h - a1*z - a1*(2*h + z))*((a1*cn2)/E^(a1*h) - cn1*E^(a1*z) +
    a1*cn2*E^(a1*(-h + 2*z)))*(-((a1*cn2)/E^(a1*h)) + cn1*E^(a1*z) -
    a1*cn2*E^(-(a1*h) + 2*a1*z)))/(4*a1^5*(2*a1 - b)*(-1 + E^(-2*a1*h))^4) -
 (E^(-2*a1*h + 3*a1*z - a1*(2*h + z))*
   ((a1*cn2)/E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(a1*(-h + 2*z)))*
   (-((a1*cn2)/E^(a1*h)) + cn1*E^(a1*z) - a1*cn2*E^(-(a1*h) + 2*a1*z)))/
  (4*a1^5*(2*a1 - b)*(-1 + E^(-2*a1*h))^4) -
 (E^(-(a1*h) - 3*a1*z - a1*(h + z))*
   (-2*a1^3*cn2^2 - 2*a1^2*b*cn2^2 - (2*a1^3*cn2^2)/E^(2*a1*h) -
    (2*a1^2*b*cn2^2)/E^(2*a1*h) - 4*a1^3*cn2^2*E^(2*a1*z) - 4*a1^2*b*cn2^2*E^(2*a1*z) -
    2*a1^3*cn2^2*E^(4*a1*z) - 2*a1^2*b*cn2^2*E^(4*a1*z) +
    4*a1^2*cn1*cn2*E^(a1*(-h + z)) + 3*a1*b*cn1*cn2*E^(a1*(-h + z)) -
    2*a1^3*cn2^2*E^(2*a1*(-h + z)) - a1^2*b*cn2^2*E^(2*a1*(-h + z)) +
    2*a1^2*cn1*cn2*E^(-3*a1*h + a1*z) + a1*b*cn1*cn2*E^(-3*a1*h + a1*z) +
```



```
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z) + 3*a1*b*cn1*cn2*E^(-(a1*h) + a1*z) -
   2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z) - 3*a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z) +
   2*a1^2*cn1*cn2*E^(-(a1*h) + 3*a1*z) + 3*a1*b*cn1*cn2*E^(-(a1*h) + 3*a1*z) -
   a1^2*b*cn2^2*E^(-2*a1*h + 4*a1*z) + a1*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z)) -
   4*a1*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z)) -
   4*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z)) +
   4*a1^2*cn1*cn2*E^(2*a1*z + a1*(-h + z)) + 3*a1*b*cn1*cn2*E^(2*a1*z + a1*(-h + z)) +
   a1*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z)) +
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z)) +
   a1*b*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z)) -
   2*a1^3*cn2^2*E^(2*a1*z + 2*a1*(-h + z)) - a1^2*b*cn2^2*E^(2*a1*z + 2*a1*(-h + z))))/
  (2*a1^5*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
 (E^(-3*a1*h - a1*z - a1*(h + z))*(-2*a1^3*cn2^2 - 2*a1^2*b*cn2^2 -
   (2*a1^3*cn2^2)/E^(2*a1*h) - (2*a1^2*b*cn2^2)/E^(2*a1*h) - 4*a1^3*cn2^2*E^(2*a1*z) -
   4*a1^2*b*cn2^2*E^(2*a1*z) - 2*a1^3*cn2^2*E^(4*a1*z) - 2*a1^2*b*cn2^2*E^(4*a1*z) +
   4*a1^2*cn1*cn2*E^(a1*(-h + z)) + 3*a1*b*cn1*cn2*E^(a1*(-h + z)) -
   2*a1^3*cn2^2*E^(2*a1*(-h + z)) - a1^2*b*cn2^2*E^(2*a1*(-h + z)) +
   2*a1^2*cn1*cn2*E^(-3*a1*h + a1*z) + a1*b*cn1*cn2*E^(-3*a1*h + a1*z) +
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z) + 3*a1*b*cn1*cn2*E^(-(a1*h) + a1*z) -
   2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z) - 3*a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z) +
   2*a1^2*cn1*cn2*E^(-(a1*h) + 3*a1*z) + 3*a1*b*cn1*cn2*E^(-(a1*h) + 3*a1*z) -
   a1^2*b*cn2^2*E^(-2*a1*h + 4*a1*z) + a1*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z)) -
   4*a1*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z)) -
   4*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z)) +
   4*a1^2*cn1*cn2*E^(2*a1*z + a1*(-h + z)) + 3*a1*b*cn1*cn2*E^(2*a1*z + a1*(-h + z)) +
   a1*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z)) +
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z)) +
   a1*b*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z)) -
   2*a1^3*cn2^2*E^(2*a1*z + 2*a1*(-h + z)) - a1^2*b*cn2^2*E^(2*a1*z + 2*a1*(-h + z))))/
  (2*a1^5*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
 (E^(a1*h - a1*z)*(2*a1^3*cn2^2*E^(-4*a1*h - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-4*a1*h - a1*(h + z)) + 2*a1^3*cn2^2*E^(-2*a1*h - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-2*a1*h - a1*(h + z)) -
```



```
   2*a1^2*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) +
   a1*b*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) -
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
   a1*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
   2*a1^3*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) +
   4*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
   2*a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
   2*a1^2*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
   a1*b*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
   2*a1^3*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
   4*a1^2*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
   2*a1*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
   4*a1*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
   2*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
   4*a1^2*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
   2*a1*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
   2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
   2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
   a1*b*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
   2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
   a1^2*b*cn2^2*E^(-3*a1*h - a1*(2*h + z)) - a1^2*b*cn2^2*E^(-(a1*h) - a1*(2*h + z)) +
   a1*b*cn1*cn2*E^(a1*z - a1*(2*h + z)) +
   3*a1*b*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) -
   a1^2*b*cn2^2*E^(-3*a1*h + 2*a1*z - a1*(2*h + z)) -
   2*b*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
   a1^2*b*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) +
   a1*b*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) +
   a1*b*cn1*cn2*E^(-2*a1*h + 3*a1*z - a1*(2*h + z)) -
   a1^2*b*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) -
```

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    a1^2*b*cn2^2*E^(-2*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) +
    2*a1*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
    a1^2*b*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
    a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z))))/
  (2*a1^5*(2*a1 - b)*b*(-1 + E^(-2*a1*h))^4) -
  (E^(-(a1*h) + a1*z)*(2*a1^3*cn2^2*E^(-4*a1*h - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-4*a1*h - a1*(h + z)) + 2*a1^3*cn2^2*E^(-2*a1*h - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-2*a1*h - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) +
    a1*b*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
    a1*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) +
    4*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
    2*a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
    a1*b*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
    2*a1^3*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
    2*a1*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
    4*a1*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
    2*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
    4*a1^2*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
    2*a1*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
    2*a1^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    a1*b*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
    2*a1^3*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
    a1^2*b*cn2^2*E^(-3*a1*h - a1*(2*h + z)) - a1^2*b*cn2^2*E^(-(a1*h) - a1*(2*h + z)) +
```



```
    a1*b*cn1*cn2*E^(a1*z - a1*(2*h + z)) +
    3*a1*b*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) -
    a1^2*b*cn2^2*E^(-3*a1*h + 2*a1*z - a1*(2*h + z)) -
    2*b*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
    a1^2*b*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) +
    a1*b*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) +
    a1*b*cn1*cn2*E^(-2*a1*h + 3*a1*z - a1*(2*h + z)) -
    a1^2*b*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) -
    a1^2*b*cn2^2*E^(-2*a1*h - a1*(2*h + z) + a1*(-h + 2*z)) +
    2*a1*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
    a1^2*b*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
    a1^2*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(2*h + z) + a1*(-h + 2*z))))/
  (2*a1^5*(2*a1 - b)*b*(-1 + E^(-2*a1*h))^4) +
 E^(b*h)*((E^(-3*a1*h - a1*z)*(-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(-(a1*h) + a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
      2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
      a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
      a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
      2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z))))/
   (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
  (E^(-(a1*h) - a1*z)*(a1*cn2 + (a1*cn2)/E^(2*a1*h) + a1*cn2*E^(2*a1*z) -
     2*cn1*E^(-(a1*h) + a1*z) + a1*cn2*E^(-2*a1*h + 2*a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
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      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
      2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
      a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
      a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
      2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z))))/
   (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
  (((a1*cn2)/E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(-(a1*h) + 2*a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
      2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
      a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
      a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
      2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z))))/
   (a1^4*(2*a1 - b)*b*(2*a1 + b)^2*E^(a1*z)*(-1 + E^(-2*a1*h))^4)) -
 E^(b*z)*((E^(-(a1*h) - 3*a1*z)*(-(a1*cn2) - a1*cn2*E^(2*a1*z) + cn1*E^(-(a1*h) + a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
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      2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
      a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
      a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
      2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z))))/
   (a1^4*(2*a1 - b)^2*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
  (E^(-(a1*h) - a1*z)*(a1*cn2 + (a1*cn2)/E^(2*a1*h) + a1*cn2*E^(2*a1*z) -
     2*cn1*E^(-(a1*h) + a1*z) + a1*cn2*E^(-2*a1*h + 2*a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
      2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
      a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
      a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
      2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
      b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
      8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
      2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
      a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z))))/
   (a1^4*(2*a1 - b)*b^2*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) +
  (E^(-2*a1*h + a1*z)*((a1*cn2)/E^(a1*h) - cn1*E^(a1*z) + a1*cn2*E^(-(a1*h) + 2*a1*z))*
    ((-2*a1*b*cn1)/E^(b*h) - (b^2*cn1)/E^(b*h) - 4*a1^3*cn2*E^(-3*a1*h - b*h - a1*z) -
      2*a1^2*b*cn2*E^(-3*a1*h - b*h - a1*z) + 2*a1*b^2*cn2*E^(-3*a1*h - b*h - a1*z) -
      4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z) + 2*a1^2*b*cn2*E^(-(a1*h) - b*h - a1*z) +
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2*a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z) - 2*a1^2*b*cn2*E^(-3*a1*h - b*h + a1*z) +
a1*b^2*cn2*E^(-3*a1*h - b*h + a1*z) - 4*a1^3*cn2*E^(-(a1*h) - b*h + a1*z) +
a1*b^2*cn2*E^(-(a1*h) - b*h + a1*z) +
2*a1*b*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) -
b^2*cn1*E^(-3*a1*h - b*h - a1*z + a1*(-h + z)) +
8*a1^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
2*b^2*cn1*E^(-(a1*h) - b*h - a1*z + a1*(-h + z)) -
4*a1^3*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
a1*b^2*cn2*E^(-(a1*h) - b*h - a1*z + 2*a1*(-h + z)) +
2*a1^2*b*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)) +
a1*b^2*cn2*E^(-(b*h) - a1*z + a1*(-h + 2*z)))/
(a1^4*(2*a1 - b)*b*(2*a1 + b)^2*(-1 + E^(-2*a1*h))^4)) +
(E^(-(a1*h) - a1*z)*((4*a1^4*cn2^2)/E^(a1*(h + z)) - (a1^2*b^2*cn2^2)/E^(a1*(h + z)) +
4*a1^4*cn2^2*E^(-4*a1*h - a1*(h + z)) - a1^2*b^2*cn2^2*E^(-4*a1*h - a1*(h + z)) +
8*a1^4*cn2^2*E^(-2*a1*h - a1*(h + z)) + 2*a1^3*b*cn2^2*E^(-2*a1*h - a1*(h + z)) -
3*a1^2*b^2*cn2^2*E^(-2*a1*h - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
8*a1^4*cn2^2*E^(2*a1*z - a1*(h + z)) - 2*a1^2*b^2*cn2^2*E^(2*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) +
12*a1^4*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) +
4*a1^3*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
5*a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*z - a1*(h + z)) - a1^2*b^2*cn2^2*E^(4*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) +
2*a1^3*b*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
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2*a1^2*b^2*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
16*a1^2*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
5*b^2*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(2*a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*h) - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(-(a1*h) - a1*(2*h + z)) +
2*a1^2*b*cn1*cn2*E^(a1*z - a1*(2*h + z)) + a1*b^2*cn1*cn2*E^(a1*z - a1*(2*h + z)) +
2*a1^2*b*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) -
2*a1*b*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
b^2*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) +
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2*a1^2*b*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) -
a1^2*b^2*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) +
2*a1^2*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
2*a1^3*b*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
a1^2*b^2*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)))*h)/
(a1^4*(2*a1 - b)*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4) -
(E^(-(a1*h) - a1*z)*((4*a1^4*cn2^2)/E^(a1*(h + z)) - (a1^2*b^2*cn2^2)/E^(a1*(h + z)) +
4*a1^4*cn2^2*E^(-4*a1*h - a1*(h + z)) - a1^2*b^2*cn2^2*E^(-4*a1*h - a1*(h + z)) +
8*a1^4*cn2^2*E^(-2*a1*h - a1*(h + z)) + 2*a1^3*b*cn2^2*E^(-2*a1*h - a1*(h + z)) -
3*a1^2*b^2*cn2^2*E^(-2*a1*h - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(-3*a1*h + a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(h + z)) +
8*a1^4*cn2^2*E^(2*a1*z - a1*(h + z)) - 2*a1^2*b^2*cn2^2*E^(2*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-4*a1*h + 2*a1*z - a1*(h + z)) +
12*a1^4*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) +
4*a1^3*b*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
5*a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-(a1*h) + 3*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(4*a1*z - a1*(h + z)) - a1^2*b^2*cn2^2*E^(4*a1*z - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) +
2*a1^3*b*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
2*a1^2*b^2*cn2^2*E^(-2*a1*h + 4*a1*z - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(a1*(-h + z) - a1*(h + z)) -
```



```
8*a1^3*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-2*a1*h + a1*(-h + z) - a1*(h + z)) +
16*a1^2*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
5*b^2*cn1^2*E^(-(a1*h) + a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(2*a1*z + a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(2*a1*z + a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) -
2*a1^2*b*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
3*a1*b^2*cn1*cn2*E^(-2*a1*h + 2*a1*z + a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*(-h + z) - a1*(h + z)) -
8*a1^3*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
2*a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(2*a1*z + 2*a1*(-h + z) - a1*(h + z)) +
4*a1^4*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
a1^2*b^2*cn2^2*E^(-2*a1*h + 2*a1*z + 2*a1*(-h + z) - a1*(h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*h) - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(-(a1*h) - a1*(2*h + z)) +
2*a1^2*b*cn1*cn2*E^(a1*z - a1*(2*h + z)) + a1*b^2*cn1*cn2*E^(a1*z - a1*(2*h + z)) +
2*a1^2*b*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(-2*a1*h + a1*z - a1*(2*h + z)) -
2*a1*b*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
b^2*cn1^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) -
a1^2*b^2*cn2^2*E^(-(a1*h) + 2*a1*z - a1*(2*h + z)) +
2*a1^2*b*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) +
a1*b^2*cn1*cn2*E^(3*a1*z - a1*(2*h + z)) -
2*a1^3*b*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) -
```

```
a1^2*b^2*cn2^2*E^(-(a1*(2*h + z)) + a1*(-h + 2*z)) +
2*a1^2*b*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) +
a1*b^2*cn1*cn2*E^(-(a1*h) + a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
2*a1^3*b*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)) -
a1^2*b^2*cn2^2*E^(2*a1*z - a1*(2*h + z) + a1*(-h + 2*z)))*z)/
(a1^4*(2*a1 - b)*b*(2*a1 + b)*(-1 + E^(-2*a1*h))^4)
```

ming up the six terms into one function

```
Intg[z_,b_] := Intg1[z,b]+Intg2[z,b]+Intg3[z,b]+Intg4[z,b]+Intg5[z,b
```


ming up the six terms into one function

```
ntg6[z,b]
```

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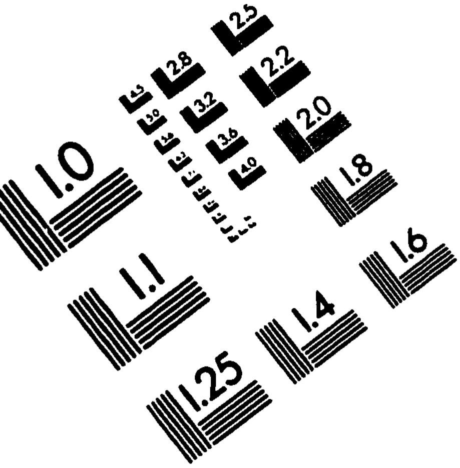
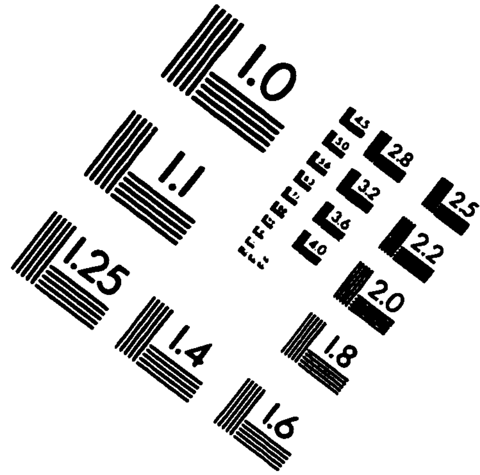
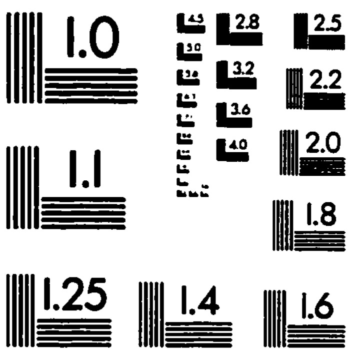
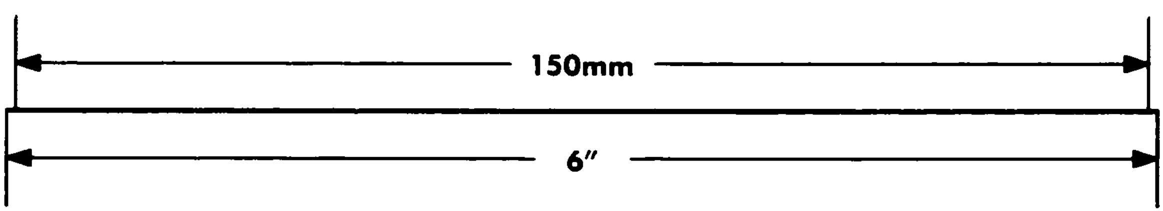
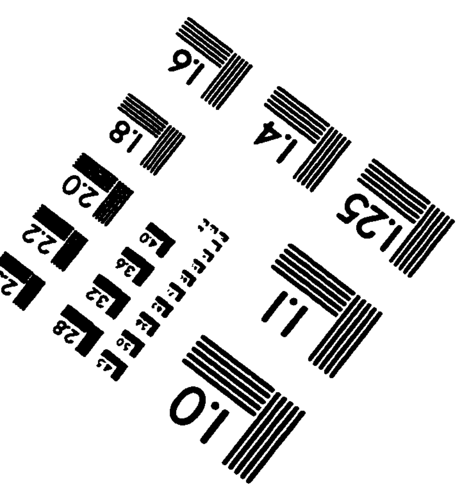
IMAGE EVALUATION TEST TARGET (QA-3)

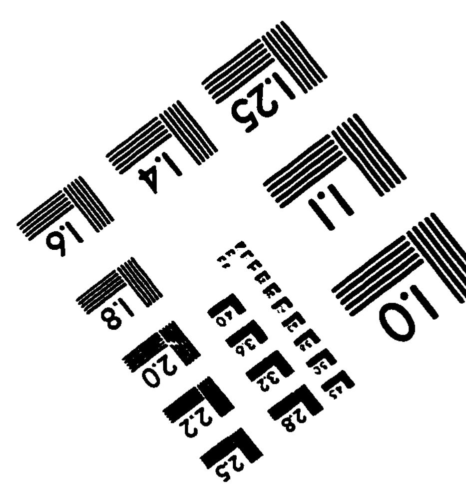












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