Equational Characterizations of Cyclic Purity and Single Compactness

by

Balarabe Yusha'u

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICAL SCIENCES

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This thesis written by BALARABE YUSHA'U under the direction of his Thesis advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfilment of the requirements for the degree of MASTER OF SCIENCE in MATHEMATICAL SCIENCES.

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With genuine humility, I acknowledge your aid

YA ALLAH.

Without your guidance and help I cannot imagine: myself in this position nor the completion of this work. So, if this work is worth dedicating, it is dedicated to You

YA ALLAH.

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Thanks be to almighty Allah Who is out of His mercy good deeds are accomplished. May his blessings be upon his beloved messenger Muhammad SAW.

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Balarabe Yusha'u

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THESIS ABSTRACT

FULL NAME OF STUDENT: **Balarabe Yusha'u**

TITLE OF THESIS: **Equational Characterizations of Cyclic Purity and Single Compactness.**

MAJOR FIELD: **Mathematics.**

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The concepts cyclic purity and total purity for modules are defined by Simmons over commutative domains. In this work equational approaches are used to generalise these concepts and related results to modules over arbitrary rings. It is shown here that single splitness and single compactness as defined by Azumaya, coincide with these generalised versions of cyclic purity and total purity. Among other results, rings all of whose modules are totally pure are characterized, and coflatness is used to characterize rings over which every totally pure module is injective.

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خلاصة الرسالة

اسم الطالب الكامل : بالعربي يو شع
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عرف سمونز مبادىء النقاوة الدورية والنقاوة الكلية للمديولات (الفضاءات الحلقية) على المجالات التبديلية. في هذا العمل نستخدم المعادلات لتعميم هذه المبادىء والنتائج المتعلقة بها. لقد بينا هنا ان الفصل الاحادى والتراص الاحادي كما عرفها أزومايا تطابقان هذا التعميم للنقاوة الدورية والنقاوة الكلية. من بين النتائج في هذا العمل تم وصف الحلقات التي تكون كل مديولاتها نقية كليا. وتم استخدام تمام التسطح لوصف الحلقات التي تكون كل مديولاتها النقية متباينة.

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0.1 INTRODUCTION

The theory of purity for modules (in the sense of Cohn [6]) and the corresponding class of pure-injective modules were studied and developed by many authors (for example Cohn [6], Warfield [28], Stenström [26], Fuchs [12]). The existence and uniqueness (up to isomorphism) of pure-injective hulls was established by Maranda [19] for abelian groups, and by Fuchs [12] and Warfield [28] for modules. In [28], the following important result was proved using a topological argument.

Theorem 1 *An R-module M is pure-injective if, and only if, it is algebraically compact.*□

This result was first proved for general algebraic systems by Weglorz [29] using the equational characterization of purity, and was independently proved for modules over noetherian rings by Fuchs [12], and for modules over general rings by Stenström [26]. Shorter algebraic proofs were given subsequently by Azumaya [2] using homological algebra, and by Laradji [16] using equations. Weglorz [29] proved that pure-injective general algebras are precisely the equationally compact ones (for a definition of equational compactness of general algebras, see Mycielski's seminal paper [21]).

Cyclic purity for modules was studied by Simmons [25] over commutative domains. He defined a submodule N of an R-module M to be **cyclically pure**, if every coset $a + N \in M/N$ can be represented by an element whose annihilator is the same as that of the coset, that is, if for each $a + N \in M/N$ there exists $b \in N$ such that

$ann_R(a-b) = ann_R(a+N)$. Equationally, this is equivalent to saying that: any system of equations

$$r_j x = a_j \ (a_j \in N, \ r_j \in R, \ j \in J) \tag{0.1}$$

that is solvable in M is also solvable in N.

Simmons [25] discussed the basic properties of cyclic purity, its relationship to purity and to the weaker rd-purity, and also gave some homological aspects associated with it. He discovered that the **cyclically-pure-projective** modules (*i.e.*, modules that have the projective property relative to all cyclically-pure-exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules) are summands of direct sums of cyclics, but as he pointed out, he did not obtain any result for **cyclically-pure-injective** modules (*i.e.*, modules that have the injective property relative to all cyclically-pure-exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules). He established relationships between purity, rd-purity and cyclic purity over special classes of rings (for example, Prüfer domains, noetherian rings and Dedekind domains), and succeeded in characterizing those modules that are cyclically pure in any rd-pure extension (that is any module that contains them as rd-pure submodules). He called them **totally pure** modules.

A concept similar to cyclic purity was studied by Azumaya [3] over an arbitrary ring with identity, which he called **single splitness**. More precisely, he defined a submodule N of an R-module M to be **singly split** in M, if for every submodule M_o of M which is a single extension of N (*i.e.*, M_o/N is cyclic), N is a direct summand of

M_o. Various characterizations of these modules were given in [3]. Azumaya obtained many results on his **singl-pure-projective** modules (*i.e.*, modules C such that for every pure-exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules, A is singly split in B) and deduced some consequences from that. However, the only result he obtained for **singl-pure-injective** modules (*i.e.*, modules A such that for every pure-exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of R-modules, A is singly split in B) is the equivalence of the following statements.

(*a*) M is singly-pure-injective.

(*b*) M is singly compact.

(*c*) For any R-module B and a single-pure-extension A of B, every homomorphism $B \rightarrow M$ can be extended to a homomorphism $A \rightarrow M$.

This clearly implies that every singly split submodule of a singly-pure-injective module is single-pure-injective too.

As can be observed, not much has been done about singly-pure-injective modules as compared to their homological duals the singly-pure-projective ones. The question that naturally arises is whether it is possible to develop a theory of singly-pure-injective modules parallel to the theory of pure-injective modules.

We have seen that, all notions of purity discussed here can be characterized equationally. Although most proofs of results on the various types of purity for modules and related concepts that have appeared in the available literature involved homological and even topological arguments, we believe that by using equations, we can in many cases, get shorter and more direct proofs of the same results. As an illustration, let us state the following result, for which a short equational proof was given in [16].

Theorem 2 *Let M be an (R, S)-bimodule and let C be an algebraically compact right S-module. Then $Hom_s(M, C)$ is an algebraically compact right R-module.*□

It should be noted that other injective properties have also been characterized in terms of equations (see for example [15] for injectivity, [22] for rd-injectivity and cyclically-pure-injectivity, and [17] for quasi-injectivity).

It is in view of this, that it was decided to use equational approaches in this work, which is an attempt to discuss the following problems:

(1) Generalize as much as possible, the work of Simmons [25] on cyclic purity and total purity to modules over arbitrary rings.

(2) Compare cyclic purity and total purity as defined by Simmons [25] with Azumaya's single splitness and single pure-injectivity, respectively.

(3) Characterize some specific rings using total purity.

The work is organized as follows:

In Chapter 1, we give basic definitions and preliminary results that will be used in the subsequent chapters. This will include terms used but not defined in this introduction. In Chapter 2, we define compatibility and solvability of a system of equations. A relationship between injectivity and solvability of certain systems of equations is established. We conclude the chapter with equational proofs of some results on purity and absolute purity for modules.

In the third chapter, basic properties of cyclic purity are proved using equations, and some results in Simmons [25] are extended to arbitrary rings. Also rings all of whose modules are singly compact are characterized.

In Chapter 4, coflat modules are defined equationally, it is shown that this definition coincides with the one given by Damiano [9]. The results we established in the previous chapters are used to extend, and also to obtain equational proofs of some results. We end the chapter by characterizing rings in which every singly compact module is injective.

Chapter 1

PRELIMINARIES

In this chapter we give some basic definitions and preliminary results which are used in the subsequent chapters. For the basic ring and module theoretic notions (for example, modules, submodules, rings, subrings and ideals, rings and modules homomorphisms *etc.*) we refer to any standard rings or modules text (for example [1, 23, 24, 27]).

Throughout this work, unless stated otherwise, R is an associative ring with identity. An R-module, without further qualification, will always denote a left unital R-module. For any set I, M^I and $M^{(I)}$ denote respectively the direct product and direct sum of I copies of M, and their elements are represented by column vectors.

Definition 3 *By a* ***system of equations*** *over an R-module M, we mean a set of linear equations*

$$Ax = b \tag{1.1}$$

where $A = [\ r_{ij}]_{\substack{i\in I\\ j\in J}}$ *is a row-finite* $I \times J$ *matrix over R (I, J are index sets of arbitrary* *cardinalities), $b \in M^I$, and x is the column vector of unknowns indexed by J.*

Definition 4 *The system (1.1) is said to be* ***solvable*** *in M if there exists $m \in M^J$, such that $Am = b$. The system (1.1) is said to be* ***finitely solvable*** *in M if any finite subsystem of (1.1) is solvable in M.*

Definition 5 *Let M be an R-module. M is called* ***free*** *if M admits a basis.*

Proposition 6 *An R-module M is free if, and only if, M is isomorphic to $R^{(I)}$ for some I.*$\square$

Let $\{M_i\}_{i\in I}$ be a family of R-modules. The **direct product**, denoted by $\prod_{i\in I} M_i$, is the module whose underlying set is the cartesian product of the M_i, that is all vectors $m = (m_i)_{i\in I}$ where $m_i \in M_i$, and with module operations defined by $(m_i) + (n_i) = (m_i + n_i)$, and $r(m_i) = (rm_i)$ for all $r \in R$. The **direct sum** of the M_i denoted by $\bigoplus_{i\in I} M_i$, is the submodule of $\prod_{i\in I} M_i$ consisting of all (m_i) such that all but a finite number of m_i are zero.

For each $i \in I$, we can define canonical injections $\phi_i : M_i \longrightarrow \prod_{i\in I} M_i$, and canonical projections $\pi_i : \prod_{i\in I} M_i \longrightarrow M_i$. If $m_i \in M_i$, we set $\phi_i(m_i)$ to be the element of $\prod_{i\in I} M_i$ having m_i in the i^{th} coordinate and zero elsewhere, and $\pi_j((m_i)_{i\in I}) = m_j$ $(j \in I)$.

We note that

$$\pi_i\phi_j = \begin{cases} id_{M_i} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

These maps are also defined when the direct product is replaced by the direct sum, in this case $\sum_{i\in I} \phi_i\pi_i = id_{\bigoplus_{i\in I} M_i}$, this means that for each m in $\bigoplus_{i\in I} M_i$, almost all $\pi_i(m) = 0$

and $\sum_{i\in I}\phi_i\pi_i(m)=m$.

Let A, B, C be R-modules, and suppose we have homomorphisms as shown below

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1.2}$$

then (1.2) is said to be an **exact sequence** if $\operatorname{Im} f = \ker g$. It follows that the sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact if, and only if $\ker f = 0$ (that is if f is a monomorphism). Similarly, the sequence $B \xrightarrow{g} C \longrightarrow 0$ is exact if, and only if $\operatorname{Im} g = C$ (that is if g is an epimorphism). An exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called **short exact sequence.**

Definition 7 *Let M be an R-module. M is called **finitely generated** if there exists an exact sequence $R^n \longrightarrow M \longrightarrow 0$ for some positive integer n. In particular, M is **cyclic** if there exists sequence $R \longrightarrow M \longrightarrow 0$ that is exact. M is said to be **finitely presented** if there exists an exact sequence $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ for some positive integers n and m.*

Let V_1, V_2 and M be left R-modules, we denote the set of R-homomorphisms from V_1 to V_2 by $Hom_R(V_1,V_2)$ or, for short, $Hom(V_1,V_2)$. Let α be a homomorphism from V_1 to V_2. Consider the mapping α_* from $Hom(M,V_1)$ to $Hom(M,V_2)$ defined by setting $\alpha_*(f)=\alpha f$ for all elements f of $Hom(M,V_1)$. Similarly, we define α^* from $Hom(V_2,M)$ to $Hom(V_1,M)$ by $\alpha^*(g)=g\alpha$, for all elements g of $Hom(V_2,M)$. These mappings are $\mathbb{Z}$-homomorphisms, and are said to be **induced** by α. Now let

$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence of left R-modules, and let M be a left R-module. Then the sequences:

$$0 \longrightarrow Hom\,(M, A) \xrightarrow{f_*} Hom\,(M, B) \xrightarrow{g_*} Hom\,(M, C)$$

$$0 \longrightarrow Hom\,(C, M) \xrightarrow{f^*} Hom\,(B, M) \xrightarrow{g^*} Hom\,(A, M)$$

are exact (see [1, 5, 27]).

Definition 8 *An R-module M is said to be* ***projective*** *if for every short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R-modules and R-homomorphisms, the induced $\mathbb{Z}$-homomorphism $Hom\,(M, g) : Hom\,(M, B) \longrightarrow Hom\,(M, C)$ is surjective.*

Definition 9 *An R-module M is said to be* ***injective*** *if for every short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of R-modules and R-homomorphisms, the induced $\mathbb{Z}$-homomorphism $Hom\,(f, M) : Hom\,(B, M) \longrightarrow Hom\,(A, M)$ is surjective.*

Remark 1 *We shall discuss injective modules further in the next chapter.*

Now let U and V be left and right R-module, respectively, and let A be a free $\mathbb{Z}$-module generated by the set $U \times V$ (that is the set $\{(u, v) : u \in U,\ v \in V\}$). Consider the submodule B of A generated by all elements

$$(u + u', v) - (u, v) - (u', v)$$

$$(u, v + v') - (u, v) - (u, v')$$

$$(ur, v) - (u, rv)$$

where u, $u' \in U$, v, $v' \in V$, and $r \in R$.

There is a mapping $f : U \times V \longrightarrow T$ (where $T = A/B$ is a $\mathbb{Z}$-module) obtained by taking the inclusion mapping $g : U \times V \longrightarrow A$, followed by the natural homomorphism $h : A \longrightarrow T$. This mapping f is linear in both U and V, for the generating set of B is chosen to ensure this. Now let W be any $\mathbb{Z}$-module and let $\alpha : U \times V \longrightarrow W$ be any mapping that is linear in both U and V, then there corresponds a unique $\mathbb{Z}$-homomorphism $\beta : T \longrightarrow W$ such that $\alpha = \beta f$ (see [7, 23, 27]). The $\mathbb{Z}$-module T with these properties is called a **tensor product** of U and V and is denoted by $U \otimes_R V$ or, for short, $U \otimes V$. This module always exists, and is unique up to isomorphism (see [7, 23, 27]).

Let $f : M \longrightarrow M'$ and $g : N \longrightarrow N'$ be R-homomorphisms, where M and M' are right R-modules and N and N' are left R-modules. This induces a homomorphism $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ given by $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ (where $f \otimes g \in Hom_{\mathbb{Z}}(M \otimes N, M' \otimes N')$). (Note that $M \otimes N$ is generated by the elements $m \otimes n$, $m \in M$, $n \in N$). In particular, given a left R-module M and a submodule N of M, the inclusion $N \subseteq M$ induces a homomorphism $V \otimes N \longrightarrow V \otimes M$ for any right R-module V. This induced map in general need not be injective (see [7, 27]). However, if $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of left R-modules, and M is a right R-module, then this sequence induces an exact sequence

$$M \otimes A \xrightarrow{id_M \otimes f} M \otimes B \xrightarrow{id_M \otimes g} M \otimes C \longrightarrow 0$$

of $\mathbb{Z}$-modules and $\mathbb{Z}$-homomorphisms (see [27]).

In [6], Cohn defined an exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$ of left R-modules as **pure,** if the induced sequence $0 \longrightarrow A \otimes_R N \longrightarrow A \otimes_R M \longrightarrow A \otimes_R C \longrightarrow 0$ of $\mathbb{Z}$-modules is exact for every right R-module A (see also [27]). There is no loss of generality if N is considered to be a submodule of M and $N \longrightarrow M$ is the canonical injection in the above exact sequence, in this case, the submodule N is said to be **pure** in M. This was shown by Cohn [6] to be equivalent to the following equational characterization.

Theorem 10 *A submodule N of an R-module M is pure in M, if any finite set of equations*

$$\sum_{j=1}^{n} r_{ij}x_j = a_i \ \ (i = 1,\ 2,\ ...,\ m) \tag{1.3}$$

over N which is solvable in M is also solvable in N.□

A weaker form of purity is given by

Definition 11 *A submodule N of an R-module M is said to be* **rd-pure** *in M (rd here refers to relatively divisible), if* $rN = N \cap rM$ *for all* $r \in R$.

Equationally, this is equivalent to saying that the equation

$$rx = a \in N \tag{1.4}$$

is solvable in N whenever it is solvable in M.

Remark 2 *It can clearly be observed that, pure submodules are rd-pure.*

Let us also list some injective properties corresponding to the cited versions of purity. In the following, M always denotes an R-module.

Definition 12 *M is called* **rd-*injective*** *if it has the injective property with respect to all exact sequences $0 \longrightarrow B \longrightarrow A$, with B an rd-pure submodule of A.*

Remark 3 *The concept rd-injectivity was shown by Naudé, Naudé, and Pretorius to be equivalent to a certain equational property (see [22]).*

Definition 13 *A submodule N of an R-module M is called a* ***direct summand*** *of M if there exists a submodule N' of M such that $M = N \oplus N'$.*

Definition 14 *M is said to be* ***pure-injective*** *if for any R-module B and a pure-extension A of B (that is A contains B as a pure submodule), every homomorphism $B \to M$ can be extended to a homomorphism $A \to M$, or, equivalently, if M is a direct summand of any R-module in which it is pure.*

Remark 4 *It has been shown by Fuchs [12] and Warfield [28] that for any R-module M, there exists a pure-injective module $\hat{M}$ such that:*

(1) M is pure in $\hat{M}$, and

(2) for any pure-injective R-module N containing M as a pure submodule, if $\alpha : M \longrightarrow \hat{M}$ and $\beta : M \longrightarrow N$ are the inclusion maps, there exists a pure monomorphism $f : \hat{M} \longrightarrow N$ such that $\beta = f\alpha$.

Definition 15 *The R-module $\hat{M}$ in the above remark is called* ***pure-injective hull*** *or* ***pure-injective envelope*** *of M.*

The following definition has been introduced by Maddox [18].

Definition 16 *We say that M is* ***absolutely pure*** *if it is pure in every R-module containing it as a submodule.*

Pure-injective and singly pure-injective modules (see p. 3) are closely related to algebraically compact and singly compact modules as defined below:

Definition 17 *M is* ***algebraically compact*** *if any set of equations over M which is finitely solvable in M is solvable in M. M is called* ***singly compact*** *if any set of equations over M in one unknown which is finitely solvable in M is solvable in M.*

Before stating the next theorem, obtained by Warfield [28], we give the following definition:

Definition 18 *A left R-module M is* ***topologically compact*** *if there is a compact Hausdorff topology on M making it a topological group and such that the left multiplications by elements of R are continuous.*

Theorem 19 *(Warfield [28, Theorem 2]) The following conditions on a left R-module M are equivalent:*

(1) *M is pure-injective.*

(2) *M is a summand of a compact R-module.*

(3) *M is algebraically compact.*$\square$

As an illustration to our method, we give an equational proof of the following well-known result:

Proposition 20 *Let $\{M_i\}_{i\in I}$ be a family of left R-modules, the following are equivalent:*

(1) *The direct product $\prod_{i\in I} M_i$ is algebraically compact.*

(2) *Each M_i is algebraically compact.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of left R-modules, put $M = \prod_{i\in I} M_i$, and let the canonical injections and the canonical projections, associated with this direct product, be respectively denoted by $\phi_i : M_i \longrightarrow M$ and $\pi_i : M \longrightarrow M_i$ $(i \in I)$.

(1) $\Longrightarrow$ (2): Suppose that M is algebraically compact. We show that for each $i \in I$, M_i is algebraically compact. For a fixed $i \in I$, let

$$\sum_{j\in J} r_{kj}x_j = b_k \ \ (k \in K) \tag{1}$$

be any finitely solvable system of equations over M_i. Then the system

$$\sum_{j\in J} r_{kj}x_j = \phi_i(b_k) \ \ (k \in K) \tag{2}$$

is finitely solvable in M. Since M is algebraically compact, (2) is solvable in M by (say) $(m_j)_{j\in J}$. So, $\sum_{j\in J} r_{kj}m_j = \phi_i(b_k)$ $(k \in K)$. This implies that $\pi_i\left(\sum_{j\in J} r_{kj}m_j\right) = \sum_{j\in J} r_{kj}\pi_i(m_j) = \sum_{j\in J} r_{kj}m_{ji} = \pi_i\phi_i(b_k) = b_k$ $(k \in K)$ (where m_{ji} is the i^{th} component of m_j). Since $(m_{ji})_{j\in J}$ is in M_i^J, we have that (1) is solvable in M_i. Hence M_i is algebraically compact.

(2) $\Longrightarrow$ (1): Conversely, suppose that each M_i ($i \in I$) is algebraically compact. We show that the direct product M is also algebraically compact. For this, consider the system

$$\sum_{j\in J} r_{kj}x_j = b_k \ \ (k \in K) \tag{1}$$

over M. Suppose that (1) is finitely solvable in M. For each $i \in I$ this system gives rise to the system

$$\sum_{j\in J} r_{kj}x_j = \pi_i\,(b_k) \in M_i \ \ (k \in K) \tag{2}$$

which is finitely solvable. By hypothesis M_i is algebraically compact, it follows that (2) is solvable in M_i. So, there exists $m_{ij} \in M_i$ ($j \in J$) such that $\sum\limits_{j\in J} r_{kj}m_{ij} = \pi_i\,(b_k) = b_{ki}$ (for each $i \in I$, $k \in K$). Now let $m_j = (m_{ij})_{i\in I}$ be in M, then clearly $\sum\limits_{j\in J} r_{kj}m_j = \sum\limits_{j\in J} r_{kj}(m_{ij})_{i\in I} = \left(\sum\limits_{j\in J} r_{kj}m_{ij}\right)_{i\in I} = (\pi_i(b_k))_{i\in I} = (b_{ki})_{i\in I} = b_k$. Hence, the system is solvable in M and so, M is algebraically compact.□

Example 21 $\mathbb{Z}$ *is not algebraically compact. To show this, consider the following system of equations,* $2x_0+3x_1 = 1$, $2x_0+3^2x_2 = 1$, $2x_0+3^3x_3 = 1$, ..., $2x_0+3^nx_n = 1$, ... *over* $\mathbb{Z}$. *We claim that this system is finitely solvable, but not solvable. For an arbitrary* n *in* $\mathbb{N}$, *the set* $x_0 = \dfrac{1-3^n}{2}$, $x_1 = 3^{n-1}$, $x_2 = 3^{n-2}, \ldots, x_{n-1} = 3$, $x_n = 1$ *solves the subsystem consisting of the first* n *equations. However, suppose that the system is solvable in* $\mathbb{Z}$. *Then* $2x_0 + 3x_1 = 1$ *implies that* $3x_1 = 1 - 2x_0 = 3^2x_2 = 3^3x_3 = \ldots\ldots$ *Now suppose that* $a_0, a_1, a_2, a_3, \ldots..$ *is a solution of the system in* $\mathbb{Z}$, *then we have that* $3a_1 = 3^2a_2 = 3^3a_3 = \ldots.$, *so that* $3^n \mid a_1$ *for all* n *in* $\mathbb{N}$. *This implies that* $a_1 = 0$ *(otherwise* $|3^n| \le |a_1|$ *for all* $n \in \mathbb{N}$*), from which it would follow that* $a_0 = \dfrac{1}{2} \in \mathbb{Z}$, *a contradiction. Hence, the system is not solvable in* $\mathbb{Z}$.□

Remark 5 *We shall see later that $\mathbb{Z}$ is a singly compact abelian group.*

Now we define some specific rings that will be referred to later in this work.

Let R be a ring. An element $x \in R$ is called a **right zero divisor** if there exists a non-zero element $y \in R$ such that $yx = 0$. One can similarly define a **left zero divisor**. An element of R is called a **zero divisor** if it is both a right and a left zero divisor. An **integral domain** is a commutative ring with no non-zero zero divisor. A ring R is called **principal ideal domain** if it is an integral domain in which every ideal is generated by only one element in R, that is, for every ideal L of R, $L = Rr$ for some $r \in R$.

Definition 22 *A ring R is left (respectively right)* ***noetherian*** *if every left (respectively right) ideal of R is finitely generated.*

Remark 6 *It is a well-known fact that R is left noetherian if and only if every ascending chain of left ideals of R is stationary, if and only if every non-empty set of left ideals of R has a maximal element.*

Definition 23 *A ring R is* ***Von Neumann regular*** *if for each $a \in R$, there is an element $a' \in R$ with $aa'a = a$.*

Definition 24 *A ring R is* ***left coherent*** *if every finitely generated left ideal is finitely presented.*

Definition 25 *A ring R is left* ***semihereditary*** *if every finitely generated left ideal is projective. A semihereditary domain is called* ***Prüfer ring.***

Definition 26 *A ring R is left* ***hereditary*** *if every left ideal is projective. A hereditary domain is called* ***Dedekind ring****.*

Chapter 2

ON SYSTEMS OF EQUATIONS OVER MODULES

Here we give some basic definitions and preliminary results on systems of equations over modules and some related concepts. In Section 2.1, we define compatibility as a necessary condition for solvability. Some important characterizations of compatibility and solvability are also given. In Section 2.2, an equational characterization of injective modules is given. We also give equational proofs of some results related to injectivity. We end the section by defining, as well as equationally characterizing a weaker notion of injectivity, that is divisibility of modules. A relationship between injectivity and divisibility is also given. In Section 2.3, a result given by Fuchs in [13, p.115] on purity for abelian groups is extended to modules, and the result is proved using equations. We end the section with equational proofs of some results on absolutely pure modules, as well as stating without proof some results that characterize rings in terms of absolute purity.

2.1 COMPATIBILITY AND SOLVABILITY

Definition 27 *The system of equations*

$$Ax = b \tag{2.1}$$

over an R-module M (where $A = [\, r_{ij}]_{\substack{i\in I\\ j\in J}}$ *is a row-finite* $I \times J$ *matrix over R, I, J are index sets of arbitrary cardinalities,* $b = [b_i]_{i\in I}$ *is in* M^I*, and x is the column vector of unknowns indexed by J) is said to be* **compatible**, *if when a linear combination of left members of the equations vanishes, then it remains zero when the corresponding right members are substituted. More precisely if given any element* $\lambda \in R^{(I)}$*, the equation* $\lambda^T A = 0$ *implies that* $\lambda^T b = 0$.

Let us consider the following example. $0x = 1$ is a system of one equation over $\mathbb{Z}$. Clearly the system is not compatible and also not solvable in $\mathbb{Z}$.

The following result shows that finite solvability of any system of equations is stronger than compatibility of the system.

Proposition 28 *Let M be an R-module, then every finitely solvable system of equations over M is compatible.*

Proof. Suppose that the system (2.1) is finitely solvable. We show that it is compatible. For this, let $\alpha \in R^{(I)}$, and let I_o be the support of α (that is, the set $\{i \in I : \alpha_i \neq 0\}$). Suppose that $\alpha^T A = 0$. Since the system $\sum_{j\in J} r_{ij}x_j = b_i$ (where $i \in I_o$)

is finite, by hypothesis, the system is solvable by (say) $(m_j)_{j\in J}$, and so, $\sum_{j\in J} r_{ij}m_j = b_i$ (for all $i \in I_o$). This implies that $0 = \sum_{i\in I_o} \alpha_i \sum_{j\in J} r_{ij}m_j = \sum_{i\in I_o} \alpha_i b_i = \alpha^T b$ (since for all $i \in I \setminus I_o$, $\alpha_i = 0$). Hence, the system is compatible.□

Remark 7 *The converse of the above proposition is not always true. For example, consider the equation $2x = 1$ over $\mathbb{Z}$, then for any r in $\mathbb{Z}$, $r2 = 0$ implies that $r = 0$, and so $r1 = 0$, hence, the equation is compatible. However, the equation is clearly not solvable in $\mathbb{Z}$. In general, let R be any domain, and L be any non-zero proper ideal of R. Consider the system of equations $rx = r$ $(r \in L \setminus \{0\})$ indexed by L. Clearly, the system is compatible, but it is not solvable in L. This, together with Proposition 28, shows that compatibility is a necessary but not sufficient condition for the solvability of a system of equations.*

The concept of compatibility and solvability defined above can be looked at from another direction. Following Kertész [15], the left members of (2.1) can be thought of as elements of the free R-module X (X is isomorphic to $R^{(J)}$) over the set $(x_j)_{j\in J}$ of unknowns. Let Y be the submodule of X generated by all the left members $f_i = \sum_{j\in J} r_{ij}x_j$ of (2.1). The next two results are due to Kertész [15]. We now give their equational proofs.

Proposition 29 *Let X, Y be as above, then the correspondence $f_i \mapsto b_i$ $(i \in I)$ induces a homomorphism $\eta : Y \longrightarrow M$ if, and only if, the system (2.1) is compatible.*

Proof. Suppose that $f_i \mapsto b_i$ induces a homomorphism $\eta : Y \longrightarrow M$, we show that (2.1) is compatible. Let $\alpha \in R^{(I)}$, and suppose that $\sum_{i\in I} \alpha_i f_i = 0$ (almost all α_i are zero).

Since η is a homomorphism, we have $0 = \eta\left(\sum_{i\in I}\alpha_i f_i\right) = \sum_{i\in I}\alpha_i\eta(f_i)$. By hypothesis $\sum_{i\in I}\alpha_i\eta(f_i) = \sum_{i\in I}\alpha_i b_i$. This implies that $\sum_{i\in I}\alpha_i b_i = 0$. Hence the system is compatible. Conversely, suppose that (2.1) is compatible, since Y is generated by $f_i = \sum_{j\in J} r_{ij}x_j$, it follows that for all $y \in Y$, $y = \sum_i r_i f_i$ for some $r_i \in R$ (almost all r_i are zero). Now define a map $\eta : Y \longrightarrow M$, by $\eta\left(\sum_i r_i f_i\right) = \sum_i r_i b_i$, and let $\sum_i r_i f_i = \sum_i s_i f_i$, it follows that $\sum_i (r_i - s_i) f_i = 0$. Since the system is compatible, we have $\sum_i (r_i - s_i) b_i = 0$, and so, $\sum_i r_i b_i = \sum_i s_i b_i$. Hence, η is well-defined, and for any y_1, y_2 in Y, with $y_1 = \sum_i r_i f_i$, $y_2 = \sum_i s_i f_i$ for some r_i, $s_i \in R$ (almost all r_i, s_i are zero), $\eta(y_1 + y_2) = \eta\left(\sum_i r_i f_i + \sum_i s_i f_i\right) = \eta\left(\sum_i (r_i + s_i) f_i\right) = \sum_i (r_i + s_i) b_i = \sum_i r_i b_i + \sum_i s_i b_i = \eta(y_1) + \eta(y_2)$. Similarly, for all r in R, $\eta(ry_1) = \eta\left(r\sum_i r_i f_i\right) = \eta\left(\sum_i r r_i f_i\right) = \sum_i r r_i b_i = r\sum_i r_i b_i = r\eta(y_1)$. Hence η is a homomorphism.□

Proposition 30 *Let X, Y be as above, then $(g_j)_{j\in J}$ is a solution of* (2.1), *if, and only if, the correspondence $x_j \mapsto g_j$ $(j \in J)$ extends to a homomorphism $\theta : X \longrightarrow M$ whose restriction to Y is η.*

Proof. Suppose that $(g_j)_{j\in J}$ is a solution of (2.1), we show that the correspondence $x_j \mapsto g_j$ $(j \in J)$ extends to a homomorphism $\theta : X \longrightarrow M$ whose restriction to Y is η. Since X is freely generated by the set $(x_j)_{j\in J}$, for each $x \in X$, x can be written uniquely as $x = \sum_j s_j x_j$, for some $s_j \in R$ (almost all s_j are zero). Now define a map $\theta : X \longrightarrow M$ by $\theta\left(\sum_j s_j x_j\right) = \sum_j s_j g_j$, and let $\sum_j s_j x_j = \sum_j r_j x_j$, it follows that $\sum_j (r_j - s_j) x_j = 0$, and so, from the uniqueness above we have $r_j = s_j$ for all j. This means that $\sum_j (r_j - s_j) g_j = 0$, and so, $\sum_j r_j g_j = \sum_j s_j g_j$. Hence, θ is well-defined. For any x_1, $x_2 \in X$, $x_1 = \sum_j r_j x_j$, $x_2 = \sum_j s_j x_j$ for some s_j, $r_j \in R$ (almost all s_j, r_j are zero). $\theta(x_1 + x_2) = \theta\left(\sum_j r_j x_j + \sum_j s_j x_j\right) = \theta\left(\sum_j (r_j + s_j) x_j\right) = \sum_j (r_j + s_j) g_j = \sum_j r_j g_j + \sum_j s_j g_j = \theta(x_1) + \theta(x_2)$. Similarly, for all r in R, $\theta(rx_1) = \theta\left(r \sum_j r_j x_j\right) = \theta\left(\sum_j r r_j x_j\right) = \sum_j r r_j g_j = r \sum_j r_j g_j = r\theta(x_1)$. Hence, θ is a homomorphism. Now for all $y \in Y$, $y = \sum_i s_i \sum_{j \in J} r_{ij} x_j$ for some $(s_i)_{i \in I} \in R^{(I)}$, it follows that $\theta(y) = \theta\left(\sum_i s_i \sum_{j \in J} r_{ij} x_j\right) = \sum_i s_i \theta\left(\sum_{j \in J} r_{ij} x_j\right) = \sum_i s_i \sum_{j \in J} r_{ij} g_j = \sum_i s_i b_i$ (since $(g_j)_{j \in J}$ is a solution of (2.1)). This means that $\sum_i s_i b_i = \sum_i s_i \eta(f_i) = \eta\left(\sum_i s_i f_i\right) = \eta\left(\sum_i s_i \sum_{j \in J} r_{ij} x_j\right) = \eta(y)$. Hence, $\theta|_Y = \eta$.

Conversely, suppose that $x_j \mapsto g_j$ $(j \in J)$ extends to a homomorphism $\theta : X \longrightarrow M$ whose restriction to Y is η, we show that $(g_j)_{j \in J}$ is a solution of (2.1). For each i, we have $\sum_{j \in J} r_{ij} g_j = \sum_{j \in J} r_{ij} \theta(x_j) = \theta\left(\sum_{j \in J} r_{ij} x_j\right) = \theta(f_i) = \eta(f_i) = b_i$ (since θ is a homomorphism and $\theta|_Y = \eta$). Hence $(g_j)_{j \in J}$ is a solution of (2.1).□

Remark 8 *As observed above, compatibility is only a necessary but not sufficient condition for solvability of a system of equations. In the next section we shall see under what condition will compatibility imply solvability.*

2.2 EQUATIONAL CHARACTERIZATION OF INJECTIVITY

Injective modules have been defined in the previous chapter (see Definition 9) as modules that have the injective property relative to all short exact sequences. The next result gives various characterizations of injective modules.

Proposition 31 (*See for example* [24] .) *An R-module M is said to be injective if it satisfies any of the following equivalent conditions*

(1) *For any R-modules A and B with A containing B as submodule, if* $\alpha : B \longrightarrow A$ *is the canonical inclusion, then given any homomorphism* $f : B \longrightarrow M$*, there exists a homomorphism* $\beta : A \longrightarrow M$ *such that* $f = \beta\alpha$.

(2) *For any left ideal I of R, if* $\alpha : I \longrightarrow R$ *is the canonical inclusion, then given any homomorphism* $f : I \longrightarrow M$*, there exists a homomorphism* $\beta : R \longrightarrow M$ *such that* $f = \beta\alpha$.

(3) *Given any exact sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *of R-modules, the sequence* $0 \longrightarrow Hom_R(C, M) \longrightarrow Hom_R(B, M) \longrightarrow Hom_R(A, M) \longrightarrow 0$ *is exact.*

Remark 9 *The second condition of the above definition is known as Baer's criterion for injectivity.*

Theorem 32 *(See* [24] .) *Let* $\{M_i\}_{i \in I}$ *be a family of injective left R-modules. Then,* $\prod_{i \in I} M_i$ *is injective.*□

Theorem 33 *(See [24].) Let* $\{M_i\}_{i\in I}$ *be a family of left* R*-modules. If the direct sum* $\bigoplus_{i\in I} M_i$ *is injective, then each* M_i *is injective.*□

Remark 10 *The converse of the above theorem is true in general only if the index set* I *is finite, as the following result shows:*

Theorem 34 *The following statements are equivalent:*

(1) R *is a left noetherian ring.*

(2) *Every direct sum of injective* R*-modules is injective.*□

Remark 11 *The above theorem is known as Bass' theorem. We shall restate and prove this theorem in Chapter 4.*

Remark 12 *For any* R*-module* M*, there exists an injective* R*-module* $E(M)$ *such that:*

(1) M *is a submodule of* $E(M)$,

(2) *for any injective* R*-module* A *with* A *containing* M *as a submodule, if* $\alpha : M \longrightarrow E(M)$ *and* $\beta : M \longrightarrow A$ are the *inclusion maps, there exists a monomorphism* $f : E(M) \longrightarrow A$ *such that* $\beta = f\alpha$.

Definition 35 *The* R*-module* $E(M)$ *in the above remark is called* ***injective hull*** *or* ***injective envelope*** *of* M.

Remark 13 *For any* R*-module* M*, the injective hull* $E(M)$ *exists and is unique up to isomorphism (see for example* [24]*).*

We now prove the following equational characterization of injective modules.

Theorem 36 *For an R-module M, the following are equivalent:*

(1) *Every compatible system of equations over M is solvable.*

(2) *Every compatible system of equations over M in one unknown is solvable.*

(3) *M is injective.*

Proof. (1) $\Longrightarrow$ (2): If every compatible system of equations over M is solvable, then clearly every compatible system of equations over M in one unknown is solvable.

(2) $\Longrightarrow$ (3): Suppose that every compatible system of equations over M in one unknown is solvable, we show that M is injective. For this, let L be any left ideal of R, and let $\xi : L \longrightarrow M$ be an R-homomorphism. Consider the system $rx = \xi(r)$ $(r \in L)$. Let $\alpha_1, \alpha_2, ..., \alpha_n \in R$, and suppose that $\sum_{i=1}^{n} \alpha_i r_i = 0$ $(r_i \in L,\ i = 1,\ 2,\ ...,\ n)$, it follows that $0 = \xi\left(\sum_{i=1}^{n} \alpha_i r_i\right) = \sum_{i=1}^{n} \alpha_i \xi(r_i)$, which implies that the system is compatible. By hypothesis, the system is solvable by some $m \in M$. Now define a mapping $\beta : R \longrightarrow M$ by $\beta(r) = rm$, and suppose that $r = s$ $(r, s \in R)$. Then clearly $rm = sm$ for all m in M, and so, β is well-defined. For all r_1, r_2 in R, $\beta(r_1 + r_2) = (r_1 + r_2)m = r_1 m + r_2 m = \beta(r_1) + \beta(r_2)$. Similarly, for all $s \in R$, $\beta(sr) = srm = s\beta(r)$. So, β is an R-homomorphism. Now for all $r \in L$, $\beta(r) = rm = \xi(r)$, so β extends ξ. Hence by Baer's criterion, M is injective.

(3) $\Longrightarrow$ (1): Suppose that an R-module M is injective and let (2.1) be any compatible system of equations over M. Let X and Y be as in Proposition 29, since (2.1) is compatible, the correspondence $\sum_{j \in J} r_{ij} x_j \mapsto b_i$ induces a homomorphism $\eta : Y \longrightarrow M$. Since M is injective, η extends to a homomorphism $\theta : X \longrightarrow M$. Therefore, it follows

from Proposition 30 that the system is solvable.□

It is a well-known fact that an R-module M is injective if, and only if, it is a direct summand of every module containing it as a submodule (see for example [24]). Along this direction we prove the following.

Proposition 37 *A submodule N of an R-module M is a direct summand of M (that is, the canonical homomorphism $f : N \longrightarrow M$ splits) if, and only if, every compatible system of equations over N solvable in M is also solvable in N.*

Proof. Suppose that N is a direct summand of M, and let $\sum_{j\in J} r_{ij}x_j = b_i$ $(i \in I)$ be any compatible system of equations over N. Suppose that the system is solvable in M by (say) m_j $(j \in J)$. Then, $\sum_{j\in J} r_{ij}m_j = b_i \in N$. Now let $f : M \longrightarrow N$ be the projection map, then $b_i = f(b_i) = f\left(\sum_{j\in J} r_{ij}m_j\right) = \sum_{j\in J} r_{ij}f(m_j)$. Since $f(m_j) \in N$ (for all $j \in J$), the system is solvable in N.

Conversely, suppose that every compatible system of equations over N solvable in M is also solvable in N. We show that N is a direct summand of M. For this, consider the system of equations

$$\begin{aligned} r_1x_{m_1} + r_2x_{m_2} - x_{r_1m_1+r_2m_2} &= 0 \\ x_n &= n \end{aligned}$$

over N (for all $r_1, r_2 \in R$, $m_1, m_2 \in M$, $n \in N$), where the unknowns $(x_m)_{m\in M}$ are indexed by M. This system is clearly solvable in M by $x_m = m$ $(m \in M)$. By hypothesis, it has a solution n_m $(m \in M)$ in N. Consider the mapping $f : M \longrightarrow N$ given by $f(m) = n_m$. Clearly f is well-defined, and for any m_1, m_2 in M, $f(m_1 + m_2) = n_{m_1+m_2} = n_{m_1} + n_{m_2}$ (by taking $r_1 = r_2 = 1$ in the first equation, and using the fact that n_m is a solution of the system), and so we have $f(m_1 + m_2) = f(m_1) + f(m_2)$. Similarly, for all $r \in R$, $f(rm_1) = n_{rm_1} = rn_{m_1} = rf(m_1)$ (by taking $r_1 = r$ and $r_2 = 0$ in the first equation and using the fact that n_m is a solution of the system). Hence f is an R-homomorphism. Also for all $n' \in N$, $f(n') = n_{n'} = n'$ (by the second equation), which implies that f is a projection, and so, N is a direct summand of M.□

The next two results are modifications of Fuchs [13, p. 103, Exercises 1 and 2] from abelian groups to modules.

Proposition 38 *Let M be an injective R-module, then any system of equations over M is solvable in M whenever it is solvable in an R-module containing M.*

Proof. Suppose that the system (2.1) is solvable in N, where N is an R-module containing M, this implies that there exists $n \in N^J$ such that $An = b$. We show that the system is solvable in M. Since M is injective, by Theorem 36, it suffices to show that the system is compatible. For this, let $\alpha \in R^{(I)}$, and suppose that $\alpha^T A = 0$, then $0 = (\alpha^T A) n = \alpha^T (An)$. It follows from above that $\alpha^T b = 0$. Hence, the system is compatible. So, the system is solvable in M.□

Proposition 39 *A system of equations over an R-module M is compatible if, and only if, it is solvable in some R-module containing M.*

Proof. Suppose that the system (2.1) is compatible, and let $E(M)$ be the injective hull of M, then M is a submodule of $E(M)$ and the system is solvable in $E(M)$ by Theorem 36.

Conversely, if (2.1) is solvable in some R-module N containing M, then there exists $n \in N^J$ such that $An = b$. Now let $\alpha \in R^{(I)}$, and suppose that $\alpha^T A = 0$. Then we have that $(\alpha^T A)n = \alpha^T(An) = 0$. It follows from above that $\alpha^T b = 0$. Therefore, the system (2.1) is compatible.□

Let us now introduce a weaker form of injectivity.

Definition 40 *Let M be an R-module. An element $m \in M$ is said to be* ***divisible*** *if for every $r \in R$ with r not a right zero-divisor, there exists $m' \in M$ such that $rm' = m$. M is called* ***divisible*** *if every element $m \in M$ is divisible.*

One can easily show that M is divisible if each compatible equation of the form

$$rx = b \in M$$

is solvable in M.

Remark 14 *Since the system $rx = b$ (with r not a right zero-divisor) is compatible, it follows that every injective module is divisible. The two concepts, injectivity and divisibility, coincide if R is a principal ideal domain (see [24]).*

2.3 PURITY FOR MODULES

Different types of purity have been defined for modules. The definitions and a brief discussion about some of these types with their equational characterizations were given in the previous chapter. Although these are the forms of purity mostly referred to, there are various other types of purity for modules (see [3, 13, 22, 25 and 26]). In this section we prove more results in this subject using equations.

We first give the equational proof of the following important result. It is a generalization of Lemma 26.1 [13, p. 115] to modules.

Proposition 41 *Let A, B be submodules of an R-module M such that $B \subseteq A \subseteq M$. Then we have:*

(1) *If B is pure in A, and A is pure in M, then B is pure in M.*

(2) *If A is pure in M, then A/B is pure in M/B.*

(3) *If B is pure in M, and A/B is pure in M/B, then A is pure in M.*

Proof. Suppose that $\sum_{j=1}^{n} r_{ij}x_j = b_i$ $(i = 1,\ 2,\ \ldots,\ m)$ is any finite system of equations over B that is solvable in M, since B is a submodule of A, we have that $b_i \in A$ $(i = 1,\ 2,\ \ldots,\ m)$. By hypothesis, the system is solvable in A. Since B is pure in A, it follows that the system is solvable in B. Therefore, B is pure in M and so we have (1).

To show (2), suppose that the system $\sum_{j=1}^{n} r_{ij}x_j = a_i + B \in A/B$ $(i = 1,\ 2,\ \ldots,\ m)$ is solvable in M/B, then $\sum_{j=1}^{n} r_{ij}(m_j + B) = a_i + B$ (for some $m_j + B \in M/B$). This implies that $\sum_{j=1}^{n} r_{ij}m_j + B = a_i + B$, so that $\sum_{j=1}^{n} r_{ij}m_j - a_i = b_i$ (for some $b_i \in B$).

Hence, $\sum_{j=1}^{n} r_{ij}m_j = a_i + b_i \in A$. By hypothesis A is pure in M, so there exist $a'_j \in A$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} r_{ij}a'_j = a_i + b_i \in A$, which implies that $\sum_{j=1}^{n} r_{ij}a'_j - a_i = b_i$. So, $\sum_{j=1}^{n} r_{ij}a'_j + B = a_i + B \in A/B$, which shows that $\sum_{j=1}^{n} r_{ij}(a'_j + B) = a_i + B \in A/B$. Hence, A/B is pure in M/B.

For (3), let $\sum_{j=1}^{n} r_{ij}x_j = a_i \in A$ $(i = 1, 2, ..., m)$ be solvable in M, then there exist $m_j \in M$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} r_{ij}m_j = a_i \in A$, and so $\sum_{j=1}^{n} r_{ij}(m_j + B) = a_i + B \in A/B$. Since A/B is pure in M/B, there exist $a'_j \in A$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} r_{ij}\left(a'_j + B\right) = a_i + B$. This means there exist b_i in B $(1 \leq i \leq m)$ such that $\sum_{j=1}^{n} r_{ij}a'_j = a_i + b_i$, and it follows that $\sum_{j=1}^{n} r_{ij}\left(a'_j - m_j\right) = b_i \in B$. As B is pure in M and $a'_j - m_j \in M$, there exist $b'_j \in B$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} r_{ij}b'_j = b_i$. Therefore, we have $\sum_{j=1}^{n} r_{ij}\left(a'_j - b'_j\right) = a_i \in A$. Since $a'_j - b'_j \in A$ for $j = 1, 2, ..., n$, it follows that the system $\sum_{j=1}^{n} r_{ij}x_j = a_i$ is solvable in A. Hence A is pure in M.□

Remark 15 *It can be observed from* (2) *and* (3) *of the above proposition that the natural correspondence between submodules of M/B and submodules of M containing the pure submodule B preserves purity.*

The next result shows that every direct summand N of an R-module M is pure in M.

Proposition 42 *If N is a direct summand of an R-module M then N is pure in M.*

Proof. Suppose that N is a direct summand of an R-module M. We show that N is pure in M. For this, consider the system of equations

$$\sum_{j=1}^{n} r_{ij}x_j = a_i \ \ (i = 1,\ 2,\ ...,\ m) \tag{2.2}$$

over N, solvable in M by say m_j $(j = 1,\ 2,\ ...,\ n)$. It implies that $\sum_{j=1}^{n} r_{ij}m_j = a_i \in N$ $(i = 1,\ 2,\ ...,\ m)$. If $f : M \longrightarrow N$ is the projection map, it follows that $a_i = f(a_i) = f\left(\sum_{j=1}^{n} r_{ij}m_j\right) = \sum_{j=1}^{n} r_{ij}f(m_j)$. Since $f(m_j) \in N$ $(j = 1,\ 2,\ ...,\ n)$, we have that the system is solvable in N. So N is pure in M.□

Absolutely pure modules have been defined in the previous chapter as modules that are pure in every module containing them as submodules.

We now prove some results about absolute purity. First we establish a proof of the equational characterization of absolutely pure modules, stated (without proof) by Megibben [20, Theorem 1].

Theorem 43 *An R-module M is absolutely pure if and only if every compatible finite system of equations over M is solvable in M.*

Proof. Suppose that M is absolutely pure and let the system (2.2) be any compatible finite system of equations over M, by Proposition 39, the system is solvable in some R-module N containing M. Since M is absolutely pure, we have that M is pure in N, so the system is solvable in M.

Conversely, suppose that every compatible finite system of equations over M is solvable in M. We show that M is absolutely pure. For this, let N be any R-module containing M as a submodule, and suppose that (2.2) is any finite system of equations over M solvable in N. It follows from Proposition 39 that the system is compatible. By supposition, it is solvable in M, and so M is pure in N. Hence, M is absolutely pure.□

The following results are due to Maddox [18]. We now give their equational proofs.

Proposition 44 *Any pure submodule (and hence any direct summand) of an absolutely pure module is absolutely pure.*

Proof. Suppose that N is a pure submodule of an absolutely pure left R-module M. We show that N is absolutely pure. For this, consider the system of equations $\sum_{j=1}^{n} r_{ij}x_j = a_i$ $(i = 1, 2, ..., m)$ over N, suppose that the system is compatible. Since N is a submodule of M, it follows that the system is over M, and so, by Theorem 43 the system is solvable in M (M being absolutely pure). Since N is pure in M, we have that, the system is also solvable in N, and so, by Theorem 43, N is absolutely pure.□

Proposition 45 *An R-module M is absolutely pure if and only if M is pure in $E(M)$.*

Proof. If M is absolutely pure, then clearly M is pure in $E(M)$.

Conversely, let M be pure in $E(M)$, we show that M is absolutely pure. Let (2.2) be any compatible finite system of equations over M. We show that this system is solvable in M. Since the system is compatible, it follows from Theorem 36 that the system is solvable in $E(M)$. Also since the system is finite and M is pure in $E(M)$, we obtain that the system is solvable in M. So, by Theorem 43, M is absolutely pure.□

Remark 16 *In the above proposition, the injective hull can be replaced by any injective module containing M as a submodule.*

Remark 17 *It has been shown in* [18] *that any direct sum of absolutely pure modules is absolutely pure. Also it is a well-known fact (see Theorem* 34*) that a ring is left noetherian if, and only if, any direct sum of injective modules is injective. Hence, if the ring R is not left noetherian, then, there exist absolutely pure left R-modules which are not injective (for examples of absolutely pure modules that are not injective, see Maddox* [18]*). If the ring R is left noetherian, the two concepts injectivity and absolute purity coincide, as the following result shows:*

Theorem 46 *(Megibben* [20, *Theorem 3*]*) A ring R is left noetherian if, and only if, every absolutely pure R-module is injective.*□

Remark 18 *We shall give a proof of a generalized version of this result in Chapter 4.*

In the same order of ideas, we state without proof some results that characterize rings in terms of absolute purity.

Theorem 47 (*Maddox* [18, *Theorem 1*]) *If M is an R-module over a Dedekind domain R then the following statements are equivalent:*

(1) *M is absolutely pure.*

(2) *M is injective.*

(3) *M is divisible.*□

Theorem 48 (*Megibben* [20, *Theorem 5*]) *A ring R is regular if, and only if, every R-module is absolutely pure.*□

Theorem 49 *(Megibben* [20, *Theorem 2*]) *For a ring R the following conditions are equivalent:*

(1) *R is semihereditary.*

(2) *Each finitely generated submodule of a projective R-module is projective.*

(3) *The homomorphic image of an absolutely pure R-module is absolutely pure.*□

Theorem 50 *(Megibben* [20, *Theorem 6*]) *Let R be a commutative integral domain. Then R is a Prüfer ring if and only if every divisible R-module is absolutely pure.*□

Chapter 3

ON CYCLIC PURITY

We now concentrate more on the main theme of this thesis. In this chapter, we prove some of the basic properties of cyclic purity using equations. Some results in Simmons [25] are extended to arbitrary rings. This leads us to a new definition of total purity. It is shown that: cyclic purity as defined by Simmons [25] coincides with Azumaya's single splitness, and singly compact modules as defined by Azumaya [3] coincide with our newly defined totally pure modules. Also injective modules are characterized in terms of absolute purity and single compactness. We state without proof some results that show how the coincidence of different notions of purity characterizes some classes of rings. We end the chapter by characterizing rings all of whose modules are singly compact.

3.1 CYCLIC PURITY

We recall that a cyclically pure submodule N of an R-module M is defined by Simmons [25] as a submodule, in which every coset $a+N$ can be represented by an element whose annihilator is the same as that of the coset. That is, for each $a+N \in M/N$ there exists $b \in N$ such that $ann_R(a-b) = ann_R(a+N)$.

We now give the following alternative definition of cyclic purity.

Lemma 51 *Let N be a submodule of an R-module M. The following statements are equivalent:*

(1) *N is cyclically pure in M.*

(2) *For all $a \in M$, there exists $b \in N$ such that if $ra \in N$* (where $r \in R$)*, then $ra = rb$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that N is cyclically pure in M, and let $a+N \in M/N$ for some $a \in M$. By (1), there exists $b \in N$ such that $ann_R(a-b) = ann_R(a+N)$. Now suppose that $ra \in N$, it follows that $r(a+N) = N$. This implies that $r \in ann_R(a+N) = ann_R(a-b)$, and so $r(a-b) = 0$. Hence $ra = rb$.

(2) $\Longrightarrow$ (1): Suppose that $a+N \in M/N$ where $a \in M$, and let $r \in ann_R(a+N)$. It follows that $ra \in N$. So, by (2) there exists $b \in N$ such that $ra = rb$, this implies that $r(a-b) = 0$. Consequently, $r \in ann_R(a-b)$, and so $ann_R(a+N) \subseteq ann_R(a-b)$. Clearly for all $r \in ann_R(a-b)$ with b in N, $r \in ann_R(a+N)$. Hence $ann_R(a+N) = ann_R(a-b)$. So, N is cyclically pure in M.□

Now we prove the following important result that gives various useful characterizations of cyclically pure submodules. It is a generalization of Simmons [25, Proposition 1] to arbitrary rings.

Proposition 52 *For a submodule N of an R-module M, the following are equivalent:*

(1) N is cyclically pure in M.

(2) For any $a \in M$ and any left ideal L of R with $La \subseteq N$, there exists $b \in N$ such that $L(a-b) = 0$.

(3) Any system of equations

$$r_j x = b_j \in N \;\; (j \in J),$$

with $r_j \in R$ and one unknown x, is solvable in N whenever it is solvable in M.

(4) All cyclic modules possess the projective property relative to the exact sequence $0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$.

Proof. (1) $\Longrightarrow$ (2): Suppose that N is cyclically pure in M. Let $a \in M$, and let L be any left ideal of R such that $La \subseteq N$. Then $ra \in N$ for all $r \in L$. It follows that $r(a+N) = N$ for all $r \in L$. This implies that $r \in ann_R(a+N) = ann_R(a-b)$ for some $b \in N$ (since N is cyclically pure in M). Hence, $r(a-b) = 0$ for all $r \in L$, so that $L(a-b) = 0$.

(2) $\Longrightarrow$ (3): Assume (2) and consider the system $r_j x = b_j \in N \; (j \in J)$. Suppose that the system is solvable in M, then there exists $m \in M$ such that $r_j m = b_j \in N$ $(j \in J)$. Now let $L = \{r \in R : rm \in N\} = ann_R(m+N)$. Then L is a left ideal of R with $Lm \subseteq N$. So by (2), there exists $n \in N$ such that $L(m-n) = 0$. Since

$r_j m = b_j$ is in N for all $j \in J$, it follows that r_j is in L for all $j \in J$. This implies that $r_j m = r_j n = b_j \in N$ for all $j \in J$, and so, the system is solvable in N by n.

(3) $\Longrightarrow$ (4): Suppose that every system of equations in one unknown over N solvable in M is also solvable in N. Consider the exact sequence $0 \longrightarrow N \longrightarrow M \xrightarrow{f} M/N \longrightarrow 0$. Let $g : P \longrightarrow M/N$ be a homomorphism, where P is any cyclic R-module generated by (say) p. Since f is an epimorphism, there exists $m \in M$ such that $f(m) = g(p)$. Now let $L = \{r \in R : rm \in N\}$, and consider the system $rx = rm \in N$ $(\text{for all } r \in L)$. This system is clearly solvable in M by $m \in M$, so by hypothesis, there exists $n \in N$, such that $rm = rn$ for all $r \in L$. This implies that $r(m-n) = 0$ for all $r \in L$. Define a map $\varphi : P \longrightarrow M$ such that $\varphi(p) = m - n$, and let $rp = sp$ for some r, $s \in R$, then this implies that $(r-s)p = 0$, and so, $0 = g((r-s)p) = (r-s)g(p) = (r-s)(m+N)$. This shows that $(r-s)m \in N$, it follows that $r - s \in L$. Hence $(r-s)m =(r-s)n$, so that $r(m-n) = s(m-n)$. Hence φ is well-defined. Now for any p_1, p_2 in P, $p_1 = r_1 p$ and $p_2 = r_2 p$ for some r_1, r_2 in R. So $\varphi(p_1 + p_2) = \varphi(r_1 p + r_2 p) = \varphi((r_1 + r_2)p) = (r_1 + r_2)(m-n) = r_1(m-n) + r_2(m-n) = \varphi(p_1) + \varphi(p_2)$. Similarly for any $r \in R$, $\varphi(rp_1) = \varphi(rr_1 p) = rr_1(m-n) = r\varphi(p_1)$. Hence φ is a homomorphism. Since $\varphi(p) = m - n$, $f\varphi(p) = f(m-n) = f(m) - f(n) = f(m) = g(p)$. Since P is generated by p, by linearity we have $f\varphi = g$.

(4) $\Longrightarrow$ (1): Assume (4), then for any $a + N \in M/N$, let $L = \{r \in R : ra \in N\}$, which is a left ideal of R. Consider the cyclic module R/L and the exact sequence $0 \longrightarrow N \longrightarrow M \xrightarrow{f} M/N \longrightarrow 0$. Let $g : R/L \longrightarrow M/N$ be defined by $g(1 + L) = a + N$, suppose that $r_1 + L = r_2 + L$ (for some r_1, $r_2 \in R$). then $r_1 - r_2 \in L$. This

implies that $(r_1 - r_2)a \in N$, and so $r_1 a + N = r_2 a + N$. Hence g is well-defined. For any r_1, r_2 in R, $g((r_1 + L) + (r_2 + L)) = g((r_1 + r_2) + L) = (r_1 + r_2)a + N = r_1 a + r_2 a + N = (r_1 a + N) + (r_2 a + N) = g(r_1 + L) + g(r_2 + L)$. Similarly, for any $r \in R$, $g(r(r_1 + L)) = g(rr_1 + L) = rr_1 a + N = r(r_1 a + N) = rg(r_1 + L)$. Hence g is a homomorphism. So by hypothesis, there exists a homomorphism $\varphi : R/L \longrightarrow M$, such that $f\varphi = g$. Hence, $f(a) = a + N = g(1 + L) = f\varphi(1 + L) = f(\varphi(1 + L)) = f(a') = a' + N$ (where $a' = \varphi(1 + L)$). This implies that $a - a' \in N$. Now for all $r \in L$, $ra' = r\varphi(1 + L) = \varphi(r + L) = 0$. It follows that with $b = a - a'$, $ra = rb$, and so $r(a - b) = 0$ for all r in L. Hence, $ann_R(a + N) \subseteq ann_R(a - b)$. Similarly, if $r \in ann_R(a - b)$, then $ra = rb \in N$, and so $r \in ann_R(a + N)$. Hence, $ann_R(a - b) = ann_R(a + N)$. So N is cyclically pure in M.□

Remark 19 *The equivalence of* (1) *and* (3) *can be proved directly using equational arguments. Hence,* (3) *is the equational characterization of cyclic purity. It is now clear that cyclically pure submodules are necessarily rd-pure.*

The following result was given by Simmons [25, Remark C] for commutative domains. We prove it for arbitrary rings. We first say that an R-module M is **torsion-free** if, and only if, $M \neq 0$ and $ann_R(m) = 0$ for all $m \in M \setminus \{0\}$.

Corollary 53 *If N is a submodule of M with M/N torsion-free, then N is cyclically pure in M.*

Proof. Let M/N be torsion-free. This implies that $ann_R(a + N) = 0$ for all $a \in M$ with $a \notin N$. It follows that, for all $b \in N$, $ann_R(a - b) = 0$. Since N is a submodule.

$0 \in N$. Hence, for all $a + N \in M/N$, $ann_R(a - 0) = ann_R(a + N)$. This shows that N is cyclically pure in M.□

The next result shows that a direct summand is always cyclically pure.

Proposition 54 *If N is a direct summand of an R-module M then N is cyclically pure in M.*

Proof. Exactly the same as in Proposition 42 replacing the finite system (2.2) with an arbitrary system of equations in one unknown.□

Now we show that a cyclically pure submodule N of an R-module M is also cyclically pure in any submodule of M containing it.

Proposition 55 *Let N be cyclically pure in M, and let M' be any submodule of M containing N. Then N is cyclically pure in M'.*

Proof. Suppose that N is cyclically pure in M, and let M' be any submodule of M containing N. Consider the system of equations $r_j x = a_j \in N$ $(j \in J,\ r_j \in R)$. and suppose that it is solvable in M'. Since M' is a submodule of M, the system is solvable in M. By hypothesis N is cyclically pure in M, so the system is solvable in N. Hence N is cyclically pure in M'.□

Now we show that every cyclically pure submodule splits in all its single extensions, before then. let us recall the following definition.

Definition 56 (*Azumaya* [3]) *For any submodule N of an R-module M, M is said to be a* ***single extension*** *of N if the factor module M/N is cyclic, that is, if there is a cyclic submodule M_o of M such that $M = M_o + N$. If, in addition, N is pure in M, then M is said to be a* ***single pure extension*** *of N.*

Proposition 57 *If M is a single extension of N, and N is cyclically pure in M, then N is a direct summand of M.*

Proof. Suppose that M is a single extension of N, this implies that

$$M/N = (Rm_0 + N)/N, \text{ for some } m_0 \in M.$$

The system $rx = rm_0$ (for all $r \in R$ such that $rm_0 \in N$) is clearly solvable in M by m_0. Since N is cyclically pure in M, there exists $n_0 \in N$ such that $rm_0 = rn_0$ (for all $r \in R$, with $rm_0 \in N$). Now define $f : Rm_0 + N \longrightarrow N$ by $f(rm_0 + n) = rn_0 + n$ (where $r \in R$ and $n \in N$). Suppose that $rm_0 + n = sm_0 + n'$ (for some r, s in R and n, n' in N). This implies that $(r - s)m_0 = n' - n \in N$, and so $(r - s)m_0 \in N$. It follows that $(r - s)m_0 = (r - s)n_0$ (since N is cyclically pure in M). This shows that $rn_0 + n = sn_0 + n'$, and so f is well-defined. Now for all m_1, m_2 in $Rm_0 + N$, $m_1 = r_1 m_0 + n_1$, $m_2 = r_2 m_0 + n_2$ (for some r_1, $r_2 \in R$ and n_1, $n_2 \in N$). It follows that $f(m_1 + m_2) = f((r_1 m_0 + n_1) + (r_2 m_0 + n_2)) = f((r_1 + r_2) m_0 + (n_1 + n_2)) = (r_1 + r_2) n_0 + (n_1 + n_2) = (r_1 n_0 + n_1) + (r_2 n_0 + n_2) = f(r_1 m_0 + n_1) + f(r_2 m_0 + n_2)$. Similarly, for all $r' \in R$. $f[r'(rm_0 + n)] = f(r' r m_0 + r' n) = (r' r) n_0 + r' n = r'(rn_0 + n) = r' f(rm_0 + n)$. Hence, f is an R-homomorphism. Now for all $n \in N$,

$f(n) = f(0.m_0 + n) = 0.n_0 + n = n$. This shows that f is a projection map. So N is a direct summand of M.□

The following definitions are found in Azumaya [3], first suppose that A and C are left R-modules and $f : A \longrightarrow C$ is an epimorphism with kernel B, then:

Definition 58 *The epimorphism f is **pure** if $Hom(M, f) : Hom_R(M, A) \longrightarrow Hom_R(M, C)$ is an epimorphism for all finitely presented left R-modules M.*

Definition 59 *f is M-**pure** if $Hom(M, f) : Hom_R(M, A) \longrightarrow Hom_R(M, C)$ is an epimorphism for the left R-module M.*

Definition 60 *f is said to be **singly split** if f is M-pure for all cyclic left R-modules M.*

Remark 20 *With the same A, B, C as above, Azumaya [3] observed that by Warfield [28, Proposition 3] f is pure if and only if B is pure in A. This is equivalent to the equational characterization of purity we gave in Chapter 1.*

Definition 61 *If $\mu = [r_{ij}]_{\substack{i\in I \\ j\in J}}$ is any row-finite $I \times J$ matrix over R, then the submodule B of an R-module M is said to be μ-**pure** if a system of linear equations $\mu x = b$ (where $b \in M^I$ and x is the column vector of unknowns indexed by J) over B is solvable in B whenever it is solvable in M.*

Definition 62 *A submodule N of an R-module M is said to be **singly split** in M, if N is a direct summand of all its single extensions in M.*

Azumaya [3] gives the following characterizations of singly split submodules of an R-module.

Theorem 63 *Let M, N be left R-modules and $f : M \longrightarrow N$ be an epimorphism with kernel B. Then the following are equivalent:*

(1) *f is singly split.*

(2) *B is singly split in M.*

(3) *B is μ-pure in M for all matrices μ of single column over R.*

(4) *If M_o is any cyclic submodule of M then there is a homomorphism $M_o \longrightarrow B$ which fixes $M_o \bigcap B$ element-wise.*

We now show that cyclic purity of Simmons [25] and single splitness of Azumaya [3] coincide.

Proposition 64 *Let M be any R-module, and N be any submodule of M, the following are equivalent:*

(1) *N is cyclically pure in M.*

(2) *N is singly split in M.*

Proof. (1) $\Longrightarrow$ (2): If N is cyclically pure in M, then by Propositions 55 and 57, N is singly split in M.

(2) $\Longrightarrow$ (1): Suppose that N is singly split in M. Let the system $r_j x = b_j \in N$ $(j \in J)$ be solvable in M by (say) $m_0 \in M$, then the submodule $M' = Rm_0 + N$ of M is a single extension of N. Since N is singly split in M, there exists a homomorphism

$f : Rm_0 + N \longrightarrow N$ such that $f(n) = n$ (for all $n \in N$). By hypothesis $r_j m_0 \in N$ (for all $j \in J$), it follows that, $b_j = r_j m_0 = f(r_j m_0) = r_j f(m_0)$. Since $f(m_0) \in N$, the system is also solvable in N. This shows that N is cyclically pure in M.□

3.2 TOTAL PURITY

We recall that Simmons in [25] defined a submodule N of an R-module M (where R is commutative domain) to be totally pure if N is cyclically pure in any R-module containing it as an rd-pure submodule.

He characterizes the totally pure modules for Prüfer domains [25, Theorem 6] as follows:

Theorem 65 *For a Prüfer domain R, the following are equivalent:*

(1) *M is totally pure.*

(2) *M is cyclically pure in its pure injective hull $\hat{M}$.*

Using this result, Simmons [25] deduced the following result connecting divisibility and injectivity of modules over Prüfer domains.

Theorem 66 *Let R be a Prüfer domain and D a divisible R-module. The following are equivalent:*

(1) *D is totally pure.*

(2) *D is injective.*

On the other hand, let us recall the following definition.

Definition 67 *(Azumaya [3]) An R-module M is said to be* ***singly pure-injective,*** *if for every pure exact sequence* $0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$ *of R-modules, M is singly split in B.*

Azumaya [3] gives the following important result:

Theorem 68 *(Azumaya [3, Theorem 10]) The following conditions on a left R-module M are equivalent:*

(1) *M is singly pure-injective.*

(2) *M is a singly split submodule of a topologically compact R-module.*

(3) *M is singly compact.*

(4) *For any R-module B and a single pure extension A of B, every homomorphism* $B \longrightarrow M$ *can be extended to a homomorphism* $A \longrightarrow M$.

Remark 21 *Since in this work we are concerned with purely algebraic aspects, we are not going to refer to (2) in Theorem 68, we mention it only for completeness.*

In order to extend some of Simmons' results to arbitrary rings, we next give the following definitions.

Definition 69 *A submodule N of an R-module M will be called* ***singly pure*** *in M if every finite system of equations*

$$r_j x = b_j \;\; (b_j \in N, \;\; J \text{ finite})$$

in one unknown which is solvable in M is also solvable in N.

Definition 70 *An R-module M will be called* **totally pure**, *if M is cyclically pure in any R-module containing it as a singly pure submodule.*

Remark 22 *Since clearly every singly pure submodule is necessarily rd-pure, and since rd-purity coincides with purity over Prüfer domains [28, Corollary 5], our definition of total purity is weaker than Simmons' total purity but coincides with it for modules over Prüfer domains. Also it is clear that, each of purity and cyclic purity implies single purity.*

We give the following characterizations of total purity.

Theorem 71 *Let M be an R-module. The following are equivalent:*

(1) *M is totally pure.*

(2) *M is a direct summand of all its single extensions containing it as a singly pure submodule.*

(3) *M is cyclically pure in its pure injective hull $\hat{M}$.*

(4) *Every finitely solvable system of equations over M with one unknown is solvable in M.*

Proof. (1) $\Longrightarrow$ (2): Suppose that M is totally pure. Let N be a single extension of M containing M as a singly pure submodule, then M is cyclically pure in N (since M is totally pure). It follows from Proposition 57 that M is a direct summand of N.
(2) $\Longrightarrow$ (3): Suppose that M is a direct summand of all its single extensions containing it as a singly pure submodule. We show that M is cyclically pure in its pure injective hull $\hat{M}$. For this, let the system $r_j x = b_j \in M$ $(j \in J)$ be solvable in $\hat{M}$ by (say)

$m_0 \in \hat{M}$. Then the submodule $N = Rm_0 + M$ of $\hat{M}$ is a single extension of M, and the system is also solvable in N. Since M is pure in $\hat{M}$, then clearly, M is singly pure in N, and so, by hypothesis, M is a direct summand of N. So there exists a homomorphism $f : Rm_0 + M \longrightarrow M$ such that $f(m) = m$ (for all $m \in M$). By hypothesis $r_j m_0 \in M$, and so, $b_j = r_j m_0 = f(r_j m_0) = r_j f(m_0)$ for each $j \in J$. Since $f(m_0) \in M$, the system is also solvable in M. This shows that M is cyclically pure in $\hat{M}$.

(3) $\Longrightarrow$ (4): Let M be cyclically pure in $\hat{M}$. Suppose that $r_j x = b_j$ $(j \in J)$ is any finitely solvable system of equations over M. It follows from Theorem 19 that, the system is solvable in $\hat{M}$ ($\hat{M}$ being algebraically compact). Since M is cyclically pure in $\hat{M}$, we have that, the system is solvable in M. So, every finitely solvable system of equations over M is solvable in M.

(4) $\Longrightarrow$ (1): Suppose that every finitely solvable system of equations over M is solvable in M. Let N be any R-module containing M as a singly pure submodule. Let $r_j x = b_j$ $(j \in J)$ be any system of equations over M that is solvable in N. Then the system is finitely solvable in M, M being singly pure in N. By hypothesis the system is solvable in M, so that M is cyclically pure in N. It follows that M is totally pure.□

Remark 23 *The above is a generalization of* $(a) \Longleftrightarrow (b)$ *of Simmons* [25, *Theorem 6*]. *Also, as can be observed from* (4), *totally pure modules over arbitrary rings coincide with Azumaya's singly compact modules.*

The next result characterizes injective modules in terms of absolute purity and single compactness.

Proposition 72 *Let M be an R-module, the following are equivalent:*

(1) *M is injective.*

(2) *M is singly compact and absolutely pure.*

Proof. (1) $\Longrightarrow$ (2): Suppose that M is injective, and let $\sum_{j=1}^{n} r_{ij}x_j = a_i \in M$ $(i = 1, 2, ..., m)$ be any compatible finite system. Since M is injective, the system is solvable in M, by Theorem 43, M is absolutely pure. Also, if $s_j x = a_j \in M$ $(j \in J)$ is any finitely solvable system of equations over M. Then by Proposition 28, the system is compatible. Since M is injective, by Theorem 36, the system is solvable. Hence, M is singly compact.

(2) $\Longrightarrow$ (1): Suppose that M is singly compact and absolutely pure, we show that M is injective. For this, let $r_j x = b_j$ $(j \in J)$ be any compatible system of equations over M. Since M is absolutely pure, the system is finitely solvable. Also M is singly compact, it follows that, the system is solvable. By Theorem 36, M is injective.□

Remark 24 *We shall see in the next chapter that absolute purity can be replaced in the above proposition by a weaker property (coflatness).*

3.3 ACTION OF RINGS ON PURITIES

In general the different notions of purity for modules discussed earlier are different concepts, and sometimes incomparable, but a restriction on the class of rings will allow a comparison of these different concepts.

As has been observed by Warfield [28, Corollary 5], rd-purity and purity coincide for Prüfer domains. It follows that purity, rd-purity and single purity coincide for Prüfer domains.

We now state more results that show how the coincidence of different notions of purity characterizes some classes of rings. The next three results are found in Simmons [25].

Theorem 73 *(Simmons* [25, *Theorem 2*]*) A domain R is Prüfer if, and only if, cyclically pure submodules are pure.*

Theorem 74 (*Simmons* [25, *Theorem 3*]) *For a domain R, the following are equivalent:*

(1) *R is noetherian.*

(2) *Pure submodules are cyclically pure.*

Remark 25 *We shall give an equational proof of this result in the next chapter in a more general form.*

From the last two theorems, Simmons [25] deduced the following:

Theorem 75 (*Simmons* [25, *Corollary 4*]) *A domain R is Dedekind if, and only if, purity and cyclic purity coincide.*

Remark 26 *It follows from Theorem 75 that single purity, purity and cyclic purity coincide for Dedekind domains.*

We now give a characterization of rings all of whose modules are singly compact.

Theorem 76 *For a ring R, the following are equivalent:*

(1) *R is left noetherian.*

(2) *Every left R-module is singly compact.*

Proof. (1) $\Longrightarrow$ (2): Suppose that a ring R is left noetherian, and let M be any left R-module. Suppose that the system

$$r_j x = b_j \in M \ \ (j \in J) \tag{1}$$

is finitely solvable in M. For each finite subset L of J, let $I_L = \sum_{j \in L} Rr_j$, so that I_L is a left ideal of R. Since R is left noetherian, the family

$$\{I_L : L \text{ is a finite subset of } J\}$$

has a maximal member $I_{L_0} = \sum_{k=1}^{n} Rr_{j_k}$ ($L_0 = \{j_1, j_2, ..., j_n\}$ say) and for all j, $I_{L_0} + Rr_j = I_{L_0}$. So, for each j, $r_j = \sum_{k=1}^{n} \alpha_{jk} r_{j_k}$, for some $\alpha_{jk} \in R$. Since (1) is finitely solvable, we have that the system

$$r_j x = b_j \, (j \in L_0) \tag{2}$$

is solvable. Now, let m_0 be a solution of (2), and let $f : R \longrightarrow M$ be an R-homomorphism given by $f(1) = m_0$. Then for all $j \in L_0$, $f(r_j) = r_j f(1) = r_j m_0 = b_j$. Now for all $q \in J$, since $r_q = \sum_{k=1}^{n} \alpha_{qk} r_{j_k}$, for some $\alpha_{qk} \in R$, it follows that $f(r_q) =$

$f\left(\sum_{k=1}^{n}\alpha_{qk}r_{j_k}\right)=\sum_{k=1}^{n}\alpha_{qk}f\left(r_{j_k}\right)=\sum_{k=1}^{n}\alpha_{qk}b_{j_k}$ (since $j_k\in L_0$). Now let m_q be a solution of the system $r_jx=b_j\ (j\in L_0\cup\{q\})$. Then $b_q=r_qm_q=\sum_{k=1}^{n}\alpha_{qk}r_{j_k}m_q=\sum_{k=1}^{n}\alpha_{qk}b_{j_k}$. It follows that, $r_qm_0=r_qf(1)=f(r_q)=b_q$. This shows that, the system is solvable by m_0. So, M is singly compact.

(2) $\Longrightarrow$ (1): Suppose that every left R-module is singly compact, we prove that R is left noetherian. By Theorem 46, this is equivalent to showing that every absolutely pure left R-module is injective. To do this, let M be any absolutely pure module. By hypothesis, M is singly compact, it follows from Theorem 72 that M is injective. So R is left noetherian.□

Chapter 4

COFLATNESS AND SINGLE COMPACTNESS

In this chapter, coflat modules are defined equationally, it is shown that this definition coincides with the one given by Damiano [9] . The relationship between single purity and coflatness is established. Simmons [25, Theorem 3] , Simmons [25, Corollary 7] , Megibben [20, Theorem 3] are all generalized. Injective modules are characterized in terms of coflatness and single compactness. The notions of injectivity, absolute purity and coflatness are compared. Similarly, we compare injectivity, algebraic compactness and single compactness. We also show that singly compact modules share some properties with injective modules. An equational proof of Bass' Theorem is also given. Finally, we characterize rings in which every singly compact module is injective as precisely those that are Von Neumann regular.

Definition 77 *We say that an R-module M is* **coflat** *if every compatible finite system of equations*

$$r_j x = a_j \ \ (j = 1,\ 2,\ \dots,\ n)$$

over M in one unknown is solvable in M.

Definition 78 *(Damiano [9]) An R-module M is said to satisfy the* $\aleph$***-Baer criterion*** *if for every finitely generated left ideal I of R, every R-homomorphism* $f : I \longrightarrow M$ *extends to an R-homomorphism* $g : R \longrightarrow M$.

Damiano [9] defined an R-module M to be coflat if it satisfies a certain property, and he established that this property is equivalent to the satisfiability of the $\aleph$-Baer criterion.

Now we show that this definition coincides with our Definition 77.

Proposition 79 *For an R-module M, the following are equivalent:*

(1) *M is coflat.*

(2) *M satisfies the* $\aleph$*-Baer criterion.*

Proof. (1) $\Longrightarrow$ (2): Suppose that M is coflat. Let I be a left ideal of R finitely generated by $r_1,\ r_2,\ \dots,\ r_n$, and let $f : I \longrightarrow M$ be an R-homomorphism. Consider the system of equations $r_j x = f(r_j)$ $(1 \leq j \leq n)$ over M. Let $\alpha_1,\ \alpha_2,\ \dots,\ \alpha_n \in R$, and suppose that $\sum\limits_{j=1}^{n} \alpha_j r_j = 0$, then $0 = f\left(\sum\limits_{j=1}^{n} \alpha_j r_j\right) = \sum\limits_{j=1}^{n} \alpha_j f(r_j)$. Hence, the system is compatible. Since M is coflat, the system is solvable in M by (say) $m \in M$, and so, $r_j m = f(r_j)$ $(1 \leq j \leq n)$. Now define a mapping $\beta : R \longrightarrow M$ by $\beta(r) = rm$.

clearly β is well-defined. For all r_1, r_2 in R, $\beta(r_1+r_2) = (r_1+r_2)m = r_1m + r_2m = \beta(r_1) + \beta(r_2)$. Similarly, for all $s \in R$, $\beta(sr) = srm = s(rm) = s\beta(r)$. So, β is an R-homomorphism. Now for each $a \in I$, $a = \sum_{j=1}^{n} \lambda_j r_j$ (for some λ_j in R). It follows that $\beta(a) = am = \sum_{j=1}^{n} \lambda_j r_j m = \sum_{j=1}^{n} \lambda_j f(r_j) = f\left(\sum_{j=1}^{n} \lambda_j r_j\right) = f(a)$, so β extends f. Hence M satisfies the $\aleph$-Baer criterion.

(2) $\Longrightarrow$ (1): Suppose that M satisfies the $\aleph$-Baer criterion. Let $r_j x = a_j$ $(1 \leq j \leq n)$ be any compatible finite system of equations over M. We show that the system is solvable in M. For this let $I = \sum_{j=1}^{n} Rr_j$, then I is a finitely generated ideal of R. Consider a mapping $f : I \longrightarrow M$ given by $f\left(\sum_{j=1}^{n} \alpha_j r_j\right) = \sum_{j=1}^{n} \alpha_j a_j$, and suppose that $\sum_{j=1}^{n} \alpha_j r_j = \sum_{j=1}^{n} \lambda_j r_j$ (for some α_j, $\lambda_j \in R$, $1 \leq j \leq n$), this implies that $\sum_{j=1}^{n} (\alpha_j - \lambda_j) r_j = 0$, since the system is compatible, it follows that $\sum_{j=1}^{n} (\alpha_j - \lambda_j) a_j = 0$, and so, $\sum_{j=1}^{n} \alpha_j a_j = \sum_{j=1}^{n} \lambda_j a_j$, therefore, f is well-defined. For all $r_1, r_2 \in I$, with $r_1 = \sum_{j=1}^{n} \alpha_j r_j$, $r_2 = \sum_{j=1}^{n} \lambda_j r_j$ (for some α_j, $\lambda_j \in R$), it follows that $f(r_1 + r_2) = f\left(\sum_{j=1}^{n} \alpha_j r_j + \sum_{j=1}^{n} \lambda_j r_j\right) = f\left(\sum_{j=1}^{n} (\alpha_j + \lambda_j) r_j\right) = \sum_{j=1}^{n} (\alpha_j + \lambda_j) a_j = \sum_{j=1}^{n} \alpha_j a_j + \sum_{j=1}^{n} \lambda_j a_j = f(r_1) + f(r_2)$. Similarly, for all $r \in I$, $f(rr_1) = f\left(r \sum_{j=1}^{n} \alpha_j r_j\right) = f\left(\sum_{j=1}^{n} r\alpha_j r_j\right) = \sum_{j=1}^{n} r\alpha_j a_j = r\sum_{j=1}^{n} \alpha_j a_j = rf(r_1)$. So f is an R-homomorphism. By hypothesis, there exists an R-homomorphism $g : R \longrightarrow M$ that extends f, and so, $r_j g(1) = g(r_j) = f(r_j) = a_j$ $(1 \leq j \leq n)$. So, the system is solvable in M by $g(1)$. Hence, M is coflat.□

Remark 27 *It can easily be observed that coflat modules are exactly those that Eklof and Sabbagh* [10] *called* $\aleph_0$***-injective****, and also that Cohn* [7] *called* ***finitely divisible****, Gupta* [14] ***f-injective*** *and Colby* [8] $\aleph$***-injective****.*

The following result gives us a relationship between coflatness and single purity that we defined in Chapter 3.

Proposition 80 *For an R-module M, the following are equivalent:*

(1) *M is coflat.*

(2) *M is singly pure in every R-module containing it as a submodule.*

Proof. (1) $\Longrightarrow$ (2): Suppose that M is coflat, and let N be any R-module containing M as a submodule. Consider the system of equations $r_j x = b_j \in M$ $(j = 1,\ 2,\ \ldots,\ n)$ over M solvable in N. We show that the system is also solvable in M. By Proposition 39, the system is compatible. Since M is coflat, the system is solvable in M, and so, M is singly pure in N.

(2) $\Longrightarrow$ (1): Suppose that M is singly pure in every R-module containing it as submodule. We show that M is coflat. For this, let $r_j x = b_j \in M$ $(j = 1,\ 2,\ \ldots,\ n)$ be any compatible system of equations over M. Since the system is compatible, by Proposition 39, the system is solvable in some R-module N containing M as a submodule. Hence, by hypothesis M is singly pure in N, and so, the system is solvable in M. Therefore, M is coflat.□

The next result characterizes injective modules in terms of coflatness and single compactness.

Proposition 81 *For any R-module M. The following are equivalent:*

(1) *M is injective.*

(2) *M is both coflat and singly compact.*

Proof. Same as the proof of Proposition 72, restricting the first system of equations there to a system in one unknown.□

Remark 28 *This generalizes (a) $\Longleftrightarrow$ (c) of Simmons [25, Corollary 7]. Also, with appropriate modification of the proof of this result one can easily show that an R-module M is injective if, and only if, M is both absolutely pure and algebraically compact.*

To compare the notions of injectivity, absolute purity, and coflatness, we clearly have:

$$\text{injectivity} \Longrightarrow \text{absolute purity} \Longrightarrow \text{coflatness.}$$

The first arrow is not always reversible. For an example of a coflat module that is not injective see [9, Example 1.12]. As has been observed (see Theorem 46), injectivity and absolute purity coincide precisely if the ring R is left noetherian. If the ring R is left coherent, then we have that every coflat module is absolutely pure (Eklof and Sabbagh [10, Proposition 3.23]). It is not yet known for what largest class of rings is every coflat module absolutely pure.

Remark 29 *Since Von Neumann regular rings are coherent (see for example [27]), it follows that, a ring is Von Neumann regular if, and only if every R-module is coflat (Eklof and Sabbagh [10, Proposition 3.25]).*

Now we restate and prove the following result.

Proposition 82 *For a ring R, the following are equivalent:*

(1) *R is left noetherian.*

(2) *Every direct sum of injective left R-modules is injective.*

Proof. (1) $\Longrightarrow$ (2): Suppose that R is left noetherian. and let $\{M_i\}_{i\in I}$ be any family of injective left R-modules, we show that the direct sum $\bigoplus_{i\in I} M_i$ is injective. Since each M_i is absolutely pure, $\bigoplus_{i\in I} M_i$ is absolutely pure (see Remark 17). Also, by hypothesis, R is left noetherian, it follows from Theorem 76 that $\bigoplus_{i\in I} M_i$ is singly compact, and so by Proposition 72, $\bigoplus_{i\in I} M_i$ is injective.

(2) $\Longrightarrow$ (1): Conversely, suppose that every direct sum of injective left R-modules is injective. We show that R is left noetherian. For this, let $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ be an ascending chain of left ideals of R, put $E = \bigoplus_{n\in\mathbb{N}} E(R/I_n)$ (where $E(R/I_n)$ is the injective hull of R/I_n). For each $r \in L = \bigcup_{n\in\mathbb{N}} I_n$, there exists $k \in \mathbb{N}$ such that $r \in \bigcup_{n\geq k} I_n$, hence, the system

$$rx = (r + I_1,\ r + I_2,\ ...)\ \ (r \in L\ )$$

is clearly solvable in $\prod_{n\in\mathbb{N}} E(R/I_n)$ by $(1 + I_1,\ 1 + I_2,\ ...)$. By Proposition 39, the system is compatible. Since E is injective by hypothesis, there exists $a = (a_1 + I_1,\ a_2 + I_2,\ ..., a_t + I_t,\ 0, 0, ...) \in E$ such that $ra = (r + I_1,\ r + I_2,\ ...)$ for each $r \in L$. Hence for each $r \in L$, $r + I_{t+1} = 0$, that is $r \in I_{t+1}$. This proves that $L = I_{t+1}$, and so R is left noetherian.□

Remark 30 *The above is an equational proof of Bass' theorem (see Theorem* 34 *and Remark* 11).

The following two results are found in [9]. We now give their equational proof.

Proposition 83 *Let* $\{M_i\}_{i\in I}$ *be a family of left* R*-modules, the following are equivalent:*

(1) *The direct product* $\prod_{i\in I} M_i$ *is coflat.*

(2) *Each* M_i *is coflat.*

Proof. Let $\{M_i\}_{i\in I}$ be a family of left R-modules put $M = \prod_{i\in I} M_i$, and let the canonical injections and the canonical projections, associated with this direct product, be respectively denoted by $\phi_i : M_i \longrightarrow M$ and $\pi_i : M \longrightarrow M_i$ $(i \in I)$.

(1) $\Longrightarrow$ (2): Suppose that M is coflat. For a fixed $i \in I$, let

$$r_j x = a_j \ (j = 1,\ 2,\ \ldots,\ n) \tag{1}$$

be a compatible finite system of equations over M_i. Consider the system

$$r_j x = \phi_i(a_j) \ \ (j = 1,\ 2,\ \ldots,\ n) \tag{2}$$

over M, let $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$, and suppose that $\sum_{j=1}^{n} \alpha_j r_j = 0$. Since (1) is compatible, we have that $\sum_{j=1}^{n} \alpha_j a_j = 0$, which implies that $0 = \phi_i\left(\sum_{j=1}^{n} \alpha_j a_j\right) = \sum_{j=1}^{n} \alpha_j \phi_i(a_j)$, therefore, (2) is compatible too. Since M is coflat, (2) is solvable in M, by (say) $m \in M$, and so, $r_j m = \phi_i(a_j)$ $(j \in J)$. This implies that $\pi_i(r_j m) = r_j \pi_i(m) = r_j m_i = \pi_i \phi_i(a_j) = a_j$ $(j \in J)$ (where m_i is the i^{th} component of m). Since m_i is in M_i, we have that (1) is solvable in M_i, and hence M_i is coflat.

(2) $\Longrightarrow$ (1): Conversely, suppose that each M_i ($i \in I$) is coflat. We show that the direct product M is also coflat. For this, consider the system

$$r_j x = a_j \; (j = 1,\ 2,\ \ldots,\ n) \tag{1}$$

over M. Suppose that (1) is compatible. For each $i \in I$, this system gives rise to the system

$$r_j x = \pi_i (a_j) \in M_i \;\; (j = 1,\ 2,\ \ldots,\ n). \tag{2}$$

Now let $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n \in R$ and suppose that $\sum_{j=1}^{n} \alpha_j r_j = 0$. Since (1) is compatible, $\sum_{j=1}^{n} \alpha_j a_j = 0$. This implies that $0 = \pi_i \left(\sum_{j=1}^{n} \alpha_j a_j \right) = \sum_{j=1}^{n} \alpha_j \pi_i (a_j)$, and hence (2) is compatible. Since M_i is coflat, (2) is solvable in M_i by (say) $m_i \in M_i$. It follows that $r_j (m_i)_{i \in I} = (\pi_i(a_j))_{i \in I} = a_j$. So if we let $m = (m_i)_{i \in I} \in M$, then $r_j m = a_j$ ($j \in J$). Thus, the system is solvable in M, and so, M is coflat.□

Proposition 84 *For every family $\{M_i\}_{i \in I}$ of left R-modules, the following are equivalent:*

(1) *The direct sum $\bigoplus_{i \in I} M_i$ is coflat.*

(2) *Each M_i is coflat.*

Proof. Let $\{M_i\}_{i \in I}$ be a family of left R-modules, put $M = \bigoplus_{i \in I} M_i$, and let the canonical injections and the canonical projections, associated with this direct sum, be respectively denoted by $\varphi_i : M_i \longrightarrow M$ and $\pi_i : M \longrightarrow M_i$ ($i \in I$).

(1) $\Longrightarrow$ (2): This is the same as the proof of (1) $\Longrightarrow$ (2) of Proposition 83, replacing $M = \prod_{i \in I} M_i$ by $M = \bigoplus_{i \in I} M_i$.

(2) $\Longrightarrow$ (1): Conversely, suppose that each M_i ($i \in I$) is coflat. We show that M is coflat. For this let

$$r_j x = a_j \ (j = 1, 2, ..., n) \tag{1}$$

be any compatible finite system of equations over M. For each $i \in I$, consider the system

$$r_j x = \pi_i(a_j) \in M_i \ (j = 1, 2, ... n.) \tag{2}$$

Let $\alpha_1, \alpha_2, ..., \alpha_n \in R$ and suppose that $\sum_{j=1}^{n} \alpha_j r_j = 0$. Since (1) is compatible, $\sum_{j=1}^{n} \alpha_j a_j = 0$. This implies that $0 = \pi_i\left(\sum_{j=1}^{n} \alpha_j a_j\right) = \sum_{j=1}^{n} \alpha_j \pi_i(a_j)$, and hence (2) is compatible. Since M_i is coflat, (2) is solvable in M_i by (say) $m_i \in M_i$. It follows that $r_j m_i = \pi_i(a_j) = a_{ji}$. Now for each $j = 1, 2, ..., n$, let $s(a_j)$ be the support of a_j, and let $I_o = \bigcup_{j=1}^{n} s(a_j)$, since $a_j \in M$, we have that I_o is a finite subset of I. Now define n_i in M_i by

$$n_i = \begin{cases} m_i & if\ i \in I_o \\ 0 & if\ i \notin I_o \end{cases}$$

then $m = (n_i)_{i \in I}$ is in $\bigoplus_{i \in I} M_i$. For all $i \in I$,

$$\pi_i(a_j) = a_{ji} = r_j m_i = \begin{cases} r_j m_i & if\ i \in I_o \\ 0 & if\ i \notin I_o \end{cases},$$

that is, $a_{ji} = r_j n_i$. Hence, $r_j m = (\pi_i(a_j))_{i \in I} = (a_{ji})_{i \in I} = a_j$. It follows that the system (1) is solvable in M. Hence M is coflat.□

Remark 31 *With small appropriate modifications to the above proofs, one can easily show that an arbitrary direct sum or direct product of absolutely pure modules is absolutely pure, and thereby providing an equational proof of Maddox* [18. *Corollary*] .

We now prove the following result. It is a stronger version of Megibben [20, Theorem 3].

Theorem 85 *The following assertions are equivalent:*

(1) *Every coflat module is injective.*

(2) *Every coflat module is singly compact.*

(3) *Every absolutely pure module is injective.*

(4) *R is left noetherian.*

Proof. (1) $\Longrightarrow$ (2): Suppose that M is coflat, then by hypothesis M is injective, and so singly compact.

(2) $\Longrightarrow$ (3): Suppose that every coflat module is singly compact. Let M be any absolutely pure module, then M coflat. So by hypothesis M is singly compact. Hence by Proposition 81, M is injective.

(3) $\Longrightarrow$ (4): Suppose that every absolutely pure module is injective. We show that the ring R is left noetherian. For this, let $\{M_i\}_{i\in I}$ be any family of injective left R-modules. Since every injective module is absolutely pure, it follows that for each $i \in I$, M_i is absolutely pure, and so by Remark 17, $\bigoplus_{i\in I} M_i$ is absolutely pure. By hypothesis, $\bigoplus_{i\in I} M_i$ is injective. It follows from Proposition 82 that R is noetherian.

(4) $\Longrightarrow$ (1): Suppose that R is noetherian, and let M be coflat. Since R is noetherian, it follows from Theorem 76 that M is singly compact. So by Proposition 81, M is injective.□

Next we show that singly compact modules share some basic properties with the injective modules.

Proposition 86 *Let $\{M_i\}_{i\in I}$ be a family of left R-modules, the following are equivalent:*

(1) *The direct product $\prod_{i\in I} M_i$ is singly compact.*

(2) *Each M_i is singly compact.*

Proof. The proof is the same as the proof of Proposition 20 by replacing each of the arbitrary systems of equations there by a system of equations in one unknown.□

Proposition 87 *For a ring R, the following are equivalent:*

(1) *R is left noetherian.*

(2) *Every direct sum of singly compact R-modules is singly compact.*

Proof. (1) $\Longrightarrow$ (2): Suppose that R is left noetherian, and let $\{M_i\}_{i\in I}$ be any family of singly compact left R-modules. By Theorem 76, the left R-module $\bigoplus_{i\in I} M_i$ is singly compact.

(2) $\Longrightarrow$ (1): Conversely, suppose that, the direct sum of every family of singly compact left R-modules is singly compact. We show that R is left noetherian. For this, let $\{M_i\}_{i\in I}$ be any family of injective left R-modules. Since clearly, every injective module is coflat and singly compact, $\{M_i\}_{i\in I}$ is also a family of coflat, singly compact left R-modules. So by hypothesis, the direct sum $\bigoplus_{i\in I} M_i$ is singly compact. Also by Proposition 84, $\bigoplus_{i\in I} M_i$ is coflat. Hence, it follows from Proposition 81 that $\bigoplus_{i\in I} M_i$ is injective. By Proposition 82, R is noetherian.□

To compare the notions injectivity, algebraic compactness, and single compactness, we clearly have:

injectivity $\Longrightarrow$ algebraic compactness $\Longrightarrow$ single compactness

The arrows are not always reversible. For instance, $\mathbb{Z}$ is a noetherian ring (see for example [23]). If we consider $\mathbb{Z}$ as $\mathbb{Z}$-module, by Theorem 76, $\mathbb{Z}$ is singly compact, but we have shown in Chapter 1 that $\mathbb{Z}$ is not algebraically compact.

Similarly, consider $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}$-module. Since $\mathbb{Z}/2\mathbb{Z}$ is a finite $\mathbb{Z}$-module, then $\mathbb{Z}/2\mathbb{Z}$ is a compact $\mathbb{Z}$-module (in the discrete topology), and so, by Theorem 19 (Warfield [28, Theorem 2]), $\mathbb{Z}/2\mathbb{Z}$ is algebraically compact. However, consider the equation $2x = 1$ over $\mathbb{Z}/2\mathbb{Z}$. Clearly the system is compatible, but not solvable in $\mathbb{Z}/2\mathbb{Z}$. Hence, by Theorem 36, $\mathbb{Z}/2\mathbb{Z}$ is not injective.

We shall show later that if the ring R is Von Neumann regular, then the three concepts, injectivity, algebraic compactness and single compactness coincide. Before then, let us prove the following result:

Theorem 88 *For a ring R, the following conditions are equivalent:*

(1) *R is left noetherian.*

(2) *Every singly pure submodule is cyclically pure*

(3) *Every pure submodule is cyclically pure.*

(4) *For any family $\{M_i\}_{i\in I}$ of R-modules, $\bigoplus_{i\in I} M_i$ is cyclically pure in $\prod_{i\in I} M_i$.*

Proof. (1) $\Longrightarrow$ (2): Suppose that R is left noetherian, and let N be a singly pure submodule of an R-module M. By Theorem 76, M is singly compact. Hence, N is cyclically pure in M.

(2) $\Longrightarrow$ (3): Suppose that every singly pure submodule is cyclically pure, and let N be a pure submodule of an R-module M, then N is singly pure in M. Hence, by hypothesis N is cyclically pure in M. So every pure submodule is cyclically pure.

(3) $\Longrightarrow$ (4): Suppose that every pure submodule is cyclically pure. Let $\{M_i\}_{i\in I}$ be any family of R-modules. Let $\sum_{j=1}^{n} r_{kj}x_j = b_k$ $(k = 1, 2, ..., q)$ be any finite system of equations over $\bigoplus_{i\in I} M_i$ (where $b_k = (b_{ki})_{i\in I}$) Suppose that the system is solvable in $\prod_{i\in I} M_i$ by (say) $\mu_j = (m_{ij})_{i\in I}$ $(1 \leq j \leq n)$. Then it follows that $\sum_{j=1}^{n} r_{kj}m_{ij} = b_{ki}$ $(i \in I$, and $k = 1, 2, ..., q.)$ Now let $s(b_k)$ be the support of b_k $(k = 1, 2, ..., q,)$ and let $I_o = \bigcup_{k=1}^{q} s(b_k)$, since b_k is in $\bigoplus_{i\in I} M_i$, I_o is a finite subset of I. Now for each $1 \leq j \leq n$ define n_{ij} in $\prod_{i\in I} M_i$ by

$$n_{ij} = \begin{cases} m_{ij} & if\ i \in I_o \\ 0 & if\ i \notin I_o \end{cases}$$

and let $\lambda_j = (n_{ij})_{i\in I}$, then λ_j is in $\bigoplus_{i\in I} M_i$. For each $i \in I$ we have

$$b_{ki} = \sum_{j=1}^{n} r_{kj}m_{ij} = \begin{cases} \sum_{j=1}^{n} r_{kj}m_{ij} & if\ i \in I_o \\ 0 & if\ i \notin I_o \end{cases},$$

that is, $b_{ki} = \sum_{j=1}^{n} r_{kj}n_{ij}$. So, $\sum_{j=1}^{n} r_{kj}\lambda_j = b_k$ $(k = 1, 2, ..., q)$. Hence, the system is also solvable in $\bigoplus_{i\in I} M_i$, and consequently, $\bigoplus_{i\in I} M_i$ is pure in $\prod_{i\in I} M_i$. By hypothesis $\bigoplus_{i\in I} M_i$ is cyclically pure in $\prod_{i\in I} M_i$.

(4) $\Longrightarrow$ (1): Suppose that for every family $\{M_i\}_{i\in I}$ of R-modules, $\bigoplus_{i\in I} M_i$ is cyclically pure in $\prod_{i\in I} M_i$. We show that R is left noetherian. For this, let $\{M_i\}_{i\in I}$ be any family of injective left R-modules. Consider the system of equations $r_jx = a_j$ $(j \in J)$

over $\bigoplus_{i\in I} M_i$. Suppose that the system is compatible. Since the direct product $\prod_{i\in I} M_i$ is injective (see for example Theorem 32), it follows that the system is solvable in $\prod_{i\in I} M_i$. Since by hypothesis $\bigoplus_{i\in I} M_i$ is cyclically pure in $\prod_{i\in I} M_i$, we have that the system is also solvable in $\bigoplus_{i\in I} M_i$. It follows from Theorem 36 that $\bigoplus_{i\in I} M_i$ is injective, and so by Proposition 82, R is noetherian.□

Remark 32 *This is a generalization of* (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) *of Simmons [25. Theorem 3] (see Theorem* 74 *and Remark* 25*).*

Now we characterize rings in which every singly compact module is injective.

Theorem 89 *For a ring R, the following are equivalent:*

(1) *R is Von Neumann regular.*

(2) *Every singly compact module is coflat .*

(3) *Every singly compact R-module is injective.*

Proof. (1) $\Longrightarrow$ (2): Suppose that a ring R is Von Neumann regular, and let M be any singly compact R-module. Since R is Von Neumann regular, by Remark 29, M is coflat.

(2) $\Longrightarrow$ (3): Let M be any singly compact module, then by hypothesis M is coflat. It follows from Proposition 81 that M is injective.

(3) $\Longrightarrow$ (1): Suppose that every singly compact R-module is injective. We show that R is Von Neumann regular. By Remark 29, it suffices to show that every R-module is coflat. To show this, let M be any R-module, then M is pure in $\hat{M}$ (the pure

injective hull of M). Since $\hat{M}$ is singly compact, it follows from hypothesis that $\hat{M}$ is injective, and so, absolutely pure. Hence, M is a pure submodule of an absolutely pure module, it follows from Proposition 44 that M is absolutely pure, and so, coflat. Hence, R is Von Neumann regular.□

We now end this work with the following remarks:

Remark 33 *It follows from the above theorem that if the ring R is Von Neumann regular, then injectivity, algebraic compactness and single compactness coincide.*

Remark 34 *One can show that: Every singly pure submodule of a coflat module is coflat. The proof is the same as that of Proposition* 44 *except that we restrict the system of equations there to a system in one unknown. Similarly, one can show that: Every cyclically pure submodule of an injective (respectively singly compact) module is injective (respectively singly compact).*

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