Order Number 9413643

On the inversion of the Laplace transform

Al-Shuaibi, Abdulaziz Abdullah Othman, Ph.D.

King Fahd University of Petroleum and Minerals (Saudi Arabia), 1992

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

ON THE INVERSION OF THE LAPLACE TRANSFORM

BY

Abdulaziz Abdullah Othman Al-Shuaibi

A Dissertation Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

In

MATHEMATICAL SCIENCES

SEPTEMBER 1992

710

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261, SAUDI ARABIA

COLLEGE OF GRADUATE STUDIES

This dissertation, written by ABDULAZIZ ABDULLAH OTHMAN AL-SHUAIBI under the direction of his Dissertation Advisor and approved by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY IN MATHEMATICS.

Dissertation Committee

Chairman (Dr. Amin Boumenir)

Co-Chairman (Dr. Mohamed El-Gebeily)

Member (Dr. Mohammad Iqbal)

Member (Dr. Isik Tarman)

Member (Dr. Maamar Bettayeb)

Dr. Mohammad Al-Bar
Department Chairman

Dr. Ala H. Al-Rabeh
Dean, College of Graduate Studies

Date: September 12, 1992

Dedication

To the memory of my grandfather Othman Abdulrahman Mohammad Al-Shuaibi - May Allah have mercy on him!

Acknowledgements

The author wishes to express his deep gratitude to all who contributed to the completion of this work. In particular, he is indebted to:

Dr. Amin Boumenir, his advisor, for his ideas, guidance, stimulation and encouragement throughout this whole investigation; also, for his assistance in the final wording of the text.

Dr. Mohammad Al-Bar (Chairman, Department of Mathematical Sciences), for his kind support throughout this work.

Dr. Mohamed El-Gebeily, for his constructive comments and supportive ideas.

Dr. Mohammad Almulhim, for his assistance in the computational difficulties.

All members of the Committee for their useful comments.

Last, but not least, to My wife Mona Abdullah Mohammad Al-Shuaibi, for her patience throughout this research.

Table of Contents

Page

Dedication iii
Acknowledgement iv
List of Tables vii
List of Graphs viii
Dissertation Abstract ix

Chapter 1 Background

§1 The Laplace Transform 1

1.1 Preliminaries and Definition 1
1.2 Analyticity and Analytic Continuation of $\mathcal{L}$ 3
1.3 Asymptotics of f and $\mathcal{L}f$ 5
1.4 General Properties of $\mathcal{L}$ 7

§2 The Inverse Laplace Transform 9

2.1 Existence of the Inverse 9
2.2 Analytic Inversion of $\mathcal{L}f$ 10

§3 Numerical Inversion of $\mathcal{L}f$ 13

3.1 Ill-posed Problems 14
3.2 Regularization 14
3.3 Survey of Literature 15

Chapter II Numerical Inversion by the use of Legendre Polynomials

2.1 Introduction 20
2.2 Functions of the Type $\mathcal{W}_\beta$ 21
2.3 Functions of the type $\mathcal{D}_\alpha$ 22
2.4 Series and Asymptotics of Legendre Polynomials 26
2.5 The Inversion of the Laplace Transform 28
2.6 The Optimal Choice of β, λ 31

Page

2.7 Determination of c and α 32

2.8 Numerical Implementation 37

2.9 Concluding Remarks 44

Chapter III A Regularization Method

3.1 Introduction 46

3.2 Tikhonov Regularization 49

3.3 Regularization of the Laplace Transform 54

3.4 Approximation of the Regularized Solution F_α 57

3.5 Conclusion 67

Chapter IV A Spectral Representation Method For the Inversion of the Laplace Transform

4.1 Introduction 68

4.2 The Spectrum of $\mathcal{L}^2$ 70

4.3 The Spectral Analysis 77

4.4 The Inversion Formula 79

Chapter V Direct Spectral Analysis of the Laplace Transform

5.1 Introduction 83

5.2 The Spectral Representation of $\mathcal{L}^2$ 86

5.3 The Spectral Resolution of $\mathcal{L}^2$ 88

5.4 The Real Inversion Formula 91

5.5 Regularization by Truncation 93

5.6 Concluding Remarks 98

References 99

List of Tables

Page

1. Table of Example 1, $F(s) = \frac{1}{s}$ 39
2. Table of Example 2, $F(s) = 1/\left((s+1)^2+1\right)$ 40
3. Table of Example 3, $F(s) = 1/(s+2)^{3/2}(s+1)$ 41
4. Table of Example 4, $F(s) = \sqrt{s+2}/(s+1)^{3/2}$ 42
5. Table of Example 5, $F(s) = 1/\sqrt{s^2+1}$ 43

List of Graphs

Page

1. Graph of Example 1, $F(s) = \frac{1}{(s+1)^2}$ 64

2. Graph of Example 2, $F(s) = \frac{1}{(s+1)^2+1}$ 65

3. Graph of Example 3, $F(s) = \frac{1}{(s+\frac{1}{2})(s+\frac{3}{2})}$ 66

DISSERTATION ABSTRACT

NAME OF STUDENT	ABDULAZIZ ABDULLAH OTHMAN AL-SHUAIBI
TITLE OF STUDY	On the Inversion of the Laplace Transform
MAJOR FIELD	Mathematical Sciences
DATE OF DEGREE	September, 1992

The objective of this work is to develop new methods for the numerical inversion of the Laplace transform.

The 1st method is an approximation by Legendre polynomials and an estimation of the error bounds via some optimal parameters shall be derived.

The 2nd method overcomes the ill-posedness by using the Tikhonov regularization. The numerical treatment is based on convolution of Hermite polynomials approximations.

The spectral analysis provides a spectral representation leading to a real inversion formula, and by using pseudo-differential operators techniques, we shall obtain a direct approximation method.

DOCTOR OF PHILOSOPHY DEGREE

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
Dhahran, Saudi Arabia

September 1992

خلاصة رسالة

اسم الطالب : عبدالعزيز عبدالله عثمان الشعيبي .

عنوان الرسالة : التقريب العددي لمعكوس محولات لابلاس .

التخصص : العلوم الرياضية .

تاريخ الدرجة : سبتمبر ١٩٩٢ م .

منذ سنة ١٩٢٤ والباحثون مهتمون بموضوع التقريب العددي لمعكوس محولات لابلاس ، ولايزال البحث مستمراً منذ ذلك الوقت .

الهدف من الرساله هو تطوير طرق جديده للتقريب وايضاً ايجاد بعض الحدود للخطأ لإحدى الطرق المتداولة .

سنحاول ايجاد تمثيل طيفي للحصول على صيغه معكوس حقيقي تحليلي عن طريق مدخل التحليل الطيفي .

الطريقه الطيفيه ستعالج بطريقتين مختلفتين بينما الطريق العددية ستعالج عن طريق التصليح و طريقة العزوم .

دكتوراة الفلسفة

جامعة الملك فهد للبترول والمعادن

الظهران ، المملكة العربية السعودية

سبتمبر ١٩٩٢

CHAPTER 1

BACKGROUND

§1 The Laplace Transform

1.1 Preliminaries and Definitions

Let $\Omega = \{f | f$ is Lebsegue measurable for $t > 0, \quad f$ is real or complex valued $\}$ and let the parameter s be real or complex.

Definition 1.1.1 If f is in Ω then the **Laplace transform** of f denoted $\mathcal{L}f$ is defined by the function $F(s)$

$$F(s) := \mathcal{L}f = \int_0^\infty e^{-st} f(t) dt \tag{1.1}$$

for all s for which this improper integral converges.

Definition 1.1.2 The set of all complex numbers s such that the integral in (1.1) converges is called the **set of convergence** of the integral and denoted $C(f)$.

Remark 1.1.1 If the Laplace integral (1.1) of some function f converges at a point s_0 then it converges in the right-half plane $Re\ s > Re\ s_0$ [1]. Therefore the function $F(s)$ is well defined.

Clearly, there exist functions (such as $f(t) = e^{t^2}$) such that $C(f)$ is empty. A sufficient condition for $C(f)$ not to be empty can be easily seen to be the existence of constants μ and γ such that

$$|f(t)| \leq \mu e^{\gamma t} \quad \text{for all } t > 0. \tag{1.2}$$

Functions f with this property are said to be of **bounded exponential growth.**

Definition 1.1.3 The greatest lower bound of all γ such that μ exists and (1.2) holds is called the **growth indicator** of f and is denoted γ_f.

$$\gamma_f := \inf\{\gamma| \ |f(t)| \leq \mu e^{\gamma t}, \quad \text{for some } \mu \geq 0\}.$$

If no such γ exists, we set $\gamma_f = \infty$. It may happen that $\gamma_f = -\infty$, for instance, if $f(t) \equiv 0$ for all sufficiently large t.

If $\gamma_f < \infty$, then for $\gamma > \gamma_f$ there exists μ such that (1.2) holds. Consequently, if $s = \sigma + i\omega$ and $\sigma > \gamma$ then,

$$|e^{-st} f(t)| \leq \mu e^{-(\sigma-\gamma)t}$$

and hence the integral (1.1) exists. Thus, for functions in Ω such that $\gamma_f < \infty$, the domain of convergence $C(f)$ certainly contains the half plane $Re\, s > \gamma_f$. It may also contain some or all s such that $Re\, s = \gamma_f$ (for example, $f(t) = 1/(1+t^2), \gamma_f = 0$).

The following Lemma plays a basic role in the determination of the set $C(f)$.

Lemma 1.1.1 [30] *Let $f \in \Omega$, $0 \leq \beta < \frac{\pi}{2}$. If the point s_0 belongs to $C(f)$, then the Laplace integral (1.1) converges uniformly on the sector $S_\beta = \{s \,|\, Arg(s - s_0)| \leq \beta\}$.*

Theorem 1.1.1 [30] *The set of convergence of a Laplacian integral, if not empty, is either the full plane or a right half-plane, possibly including some or all of its boundary points.*

Lemma (1.1.1) implies, in particular, that if $s_0 \in C(f)$, then any s such that $Re\, s > Re\, s_0$ also belongs to $C(f)$, for any such s is contained in an angular domain $\text{Arg}(s - s_0) \leq \beta$ where $\beta < \frac{\pi}{2}$.

Definition 1.1.4 The infimum α_f of all real numbers α such that $C(f)$ contains an s with $Re\, s = \alpha$ is called the **abscissa of convergence** of the Laplace integral (1.1).

If $C(f)$ is empty, we set $\alpha_f = \infty$. if α_f is finite, then $C(f)$ contains all s such that $Re\, s > \alpha_f$, and no s such that $Re\, s < \alpha_f$. No general statement can be made concerning the convergence of the integral for $Re\, s = \alpha_f$.

1.2 Analyticity and Analytic Continuation of $\mathcal{L}f$.

One of the basic facts about the Laplace transformation is analyticity. It makes it possible to apply the powerful tools of complex analysis to the solution of real variable problems. See [30].

Theorem 1.2.1 *Let $f \in \Omega$, $\alpha_f < \infty$. Then $F = \mathcal{L}f$ is analytic in the interior of $C(f)$, that is, at all points s such that $Re\, s > \alpha_f$.*

The growth parameter measures the overall growth of a function of exponential type. We now introduce a measure for the growth in particular directions.

Definition 1.2.1 For real ϕ, let the function $\gamma(\phi)$ denote the infimum of all real numbers α such that for all sufficiently large $t > 0$,

$$\left|f\left(e^{i\phi}t\right)\right| < e^{\alpha t}.$$

The function thus defined is called the **indicator function of** f.

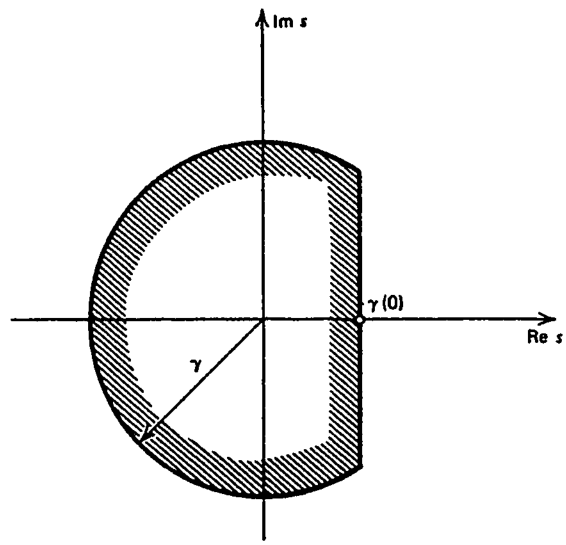
If the growth indicator of f is γ, then clearly $\gamma(\phi) \leq \gamma$ for all ϕ. It can happen that $\gamma(\phi) < \gamma$, and even that $\gamma(\phi) < 0$ for certain values of ϕ.

Example: $f(z) = e^z$.

$$\text{for } t \geq 0, \quad |f\left(e^{i\phi}t\right)| = e^{t\cos\phi}, \quad \text{thus} \quad \gamma(\phi) = \cos\phi.$$

The growth indicator of γ_f defined by (1.1.3) for arbitrary function $f \in \Omega$ of exponential growth clearly equals $\gamma(0)$.

Theorem 1.2.2 [30] *Let $f \in \Gamma_\lambda$ where Γ_λ is the totality of entire functions of exponential type with growth indicator $\gamma, \gamma \geq 0$, and let $\gamma(\phi)$ be the indicator function of f. Then $F = \mathcal{L}f$ can be extended to a function which is analytic in the union of the two region sets $|s| > \gamma$ (including $s = \infty$) and $Re\, s > \gamma(0)$.*



Theorem 1.2.3 *Under the hypotheses of theorem (1.2.2),* $F = \mathcal{L}f$ *can not be continued analytically into any half-plane* $Re\, s > \alpha_0$ *where* $\alpha_0 < \gamma(0)$*; that is,* F *has a singular point on the straight line segment* $Re\, s = \gamma(0), \quad |s| \leq \gamma$.

1.3 Asymptotics of f and $\mathcal{L}f$.

Let $f \in \Omega$ and let $\alpha_f < \infty$. It is clear that $F = \mathcal{L}f$ as an analytic function is continuous at all interior points of the set of convergence $C(f)$. This implies that

$$\lim_{s \to s_0} F(s) = F(s_0) \tag{1.3}$$

holds for all $s_0 \in C(f)$, and the following theorem illustrate the behavior of $\mathcal{L}f$ on the boundary of $C(f)$.

Theorem 1.3.1 [18] **(Abelian Theorem):** *Let $f \in \Omega$ and $F = \mathcal{L}f$. If $C(f)$ contains the boundary point s_0, then for every β such that $0 \leq \beta < \frac{\pi}{2}$,*

$$\lim_{s \to s_0} F(s) = F(s_0)$$

where the approach of s to s_0 is restricted to the angular domain $S_\beta = \{s \mid |\mathrm{Arg}(s - s_0)| \leq \beta\}$.

Now, for the behavior of $\mathcal{L}f$ near infinity we recall the following theorem

Theorem 1.3.2 [18] *Let $f \in \Omega$, $F = \mathcal{L}f$, and let s_0 be any complex number. Then for any β such that $0 \leq \beta < \frac{\pi}{2}$, $\lim_{s \to \infty} F(s) = 0$ provided that s tends to infinity in the angular domain $S_\beta = \{s \mid |\mathrm{Arg}(s - s_0)| \leq \beta\}$.*

The following theorem displays the relation of asymptotic behavior of both f and its image $\mathcal{L}f = F$.

Theorem 1.3.3 [18] *When the original function $f(t)$ has the asymptotic property:*

$$f(t) \sim Ae^{s_0 t} t^\lambda \text{ as } t \to \infty \ (t \to 0).$$

A and s_0 are complex, and $Re\,\lambda > -1$.

Then, $\mathcal{L}(f(t))(s) = F(s)$ exists for $Re\,s > Re\,s_0$ and it has for $A \neq 0$, a singular point at s_0, and $F(s)$ can be asymptotically represented by

$$F(s) \sim A\frac{\Gamma(\lambda + 1)}{(s - s_0)^{\lambda+1}} \text{ as } s \to s_0 \ (s \to \infty)$$

in the angular region $|\mathrm{Arg}(s - s_0)| < \dfrac{\pi}{2}$.

Theorem 1.3.4 [18] *Suppose that* $\mathcal{L}(f(t))(s) = F(s)$ *exists and that* $f(t)$ *has a limit* A *as* $t \to 0$, *then* $sF(s)$ *has the same limit* A as $s \to \infty$ *in the region* $|\mathrm{Arg}\, s| < \dfrac{\pi}{2}$,

$$f(0^+) = \lim_{s \to \infty} sF(s) \tag{1.4}$$

1.4 General Properties of $\mathcal{L}f$.

Let $f, g \in \Omega$ satisfy the following:

i) $\alpha_f < \infty, \quad \alpha_g < \infty$.

ii) $F(s) = \mathcal{L}(f(t))(s)$.

iii) $G(s) = \mathcal{L}(g(t))(s)$.

Then for Re $s > \max\{\alpha_f, \alpha_g\}$.

The following properties of the Laplace transform hold:

I. Linearity ;

for a, b constants

$$\mathcal{L}\left(af(t) + bg(t)\right)(s) = aF(s) + bG(s). \tag{1.5}$$

II. Laplace of the integral of a function ;

$$\mathcal{L}\left(\int_0^t f(t)dt\right)(s) = \frac{1}{s}F(s) \tag{1.6}$$

III. Laplace of the derivatives of a function;

$$\mathcal{L}\left(f^{(n)}(t)\right)(s) = s^n F(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \cdots - f^{(n+1)}(0^+). \tag{1.7}$$

IV. The similarity theorem ;

$$\mathcal{L}\left[f(at)\right](s) = \frac{1}{a}F(\frac{s}{a}), \quad a > 0 \tag{1.8}$$

V. Translation in the $t-$ plane;

$$\mathcal{L}[f(t-b)](s) = e^{-bs}F(s) \tag{1.9}$$

VI. The attenuation theorem

$$\mathcal{L}\left(e^{at}f(t)\right)(s) = F(s-a) \tag{1.10}$$

VII. The multiplication theorem

$$\mathcal{L}\left(t^n f(t)\right)(s) = (-1)^n F^{(n)}(s) \tag{1.11}$$

VIII. The division theorem

Suppose $\frac{f(t)}{t}$ has a finite abscissa of convergence α. Then

$$\mathcal{L}\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty F(u)du, \quad Re\, s > \alpha \tag{1.12}$$

IX. The convolution theorem

Suppose that $y = f * g$ and that $\Psi(s) = \mathcal{L}(y(t))(s)$, where the convolution of f with g is defined by

$$f * g = \int_0^t f(x)g(t-x)dx.$$

Then if $G = \mathcal{L}g$ and $F = \mathcal{L}f$ converge absolutely for $s = s_0$, then so is Ψ and we have

$$\Psi(s) = G(s)F(s) \tag{1.13}$$

for all s such that $Re\ s \geq Re\ s_0$.

§2 The Inverse Laplace Transform

2.1 Existence of the Inverse

Since we are in Ω, recall that $f_1 = f_2$ almost everywhere means that $m\{x | f_1(x) \neq f_2(x)\} = 0$, where m denotes the Lebesgue measure.

Theorem 2.1.1 [30] *Let $f_k \in \Omega$, $F_k = \mathcal{L}f_k (k = 1, 2)$. If $F_1(s) = F_2(s)$ for all s in some half-plane, then f_1 and f_2 are equal almost everywhere.*

Given a function F, we wish to know if it represents the Laplace transform of some function.

Theorem 2.1.2 [81] *Let $s = \sigma + i\tau$, and suppose that F satisfies the following:*

i) *$F(s)$ is analytic in the half-plane $\sigma > \alpha$.*

ii) *$F(\sigma_0 + iy)$ is Lebesgue integrable $(-\infty < y < \infty)$ for each $\sigma_0 > \alpha$.*

iii) $\lim_{|s|\to\infty} F(s) = 0$, *where* $|\text{Arg } s| < \frac{\pi}{2}$. *Then* $\exists\, f \in \Omega$ *such that*

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Moreover, the integral converges absolutely in $\alpha < \sigma < \infty$.

2.2 Analytical Inversion of $\mathcal{L}f$

The basic problem concerning the Laplace transform is the problem of inversion, i.e., given the value of $\mathcal{L}f = F$ as a function of s, how do we find the values of f as a function of t. Unfortunately, we can stress that there is no single method to answer this question. Instead, there are many particular methods geared to appropriate situations.

I. The simplest method is the availability of a table of Laplace transforms and inverses. But this is very limited and often suited for textbook exercises.

II. If, as happens sometimes, $F(s) = \mathcal{L}(f(t))(s)$ is known as a function of a complex variable over various regions of the s-plane, we can use a complex inversion formula

$$f(t) = \begin{cases} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds, & \text{for} \quad t \geq 0 \\ 0, & \text{for} \quad t < 0 \end{cases} \tag{1.14}$$

where c is chosen sufficiently large so as to have all singularities of F to the left of the vertical line $\text{Re } s = c$. [81] .

A particular version of (1.14) is for $t \geq 0$.

$$f(t) = \frac{1}{2\pi i} \int_C F(s)\, e^{st}\, ds \tag{1.15}$$

where C is a carefully chosen contour.

A number of analytic properties of $f(t)$ can be deduced from (1.14). Using the classical techniques of contour shifting and residue evaluation, we can obtain important information concerning the asymptotic behavior of $f(t)$ as $t \to \infty$. The drawback to this general approach is that the knowledge of $F(s)$ is required in the complex plane. In many of our applications and certainly in the most important cases, we **possess information concerning $F(s)$ only on the positive part of the real axis, or only at a discrete set of points i.e., the real inversion of the Laplace transform.** Moreover, the integrand e^{st} being too oscillatory for large values of $Im\ s$ on the line of integration may be untractable.

III. There is a number of real inversion formulae, of which the best known is the Post-Widder. Under appropriate assumptions one has [81]

$$f(t) = \lim_{k \to \infty} \left[\frac{(-1)^k}{k!} \left(\frac{k}{t} \right)^{k+1} F^{(k)} \left(\frac{k}{t} \right) \right]. \tag{1.16}$$

But, this has the disadvantage of the risky differentiation, when it comes to numerical approximations.

IV. Expanding the original function into power series

Theorem [42]. *If $F(s)$ is analytic at the point at infinity and in that neighborhood has a Laurent series expansion*

$$F(s) = \sum_{k=1}^{\infty} \frac{c_k}{s^k}, \tag{1.17}$$

Then, the original function $f(t)$ can be recovered by the formula

$$f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!} t^{k-1}. \tag{1.18}$$

V. Heaviside Expansion Theorem. Let $F(s)$ satisfy the following conditions

1. $F(s)$ is meromorphic

2. $F(s)$ is analytic in some half-plane $Re\ s > \alpha$.

3. there exists a sequence of circles

$$C_n = \{s \mid |s| = R_n\}, \quad R_1 < R_2 < \cdots (R_n \to \infty)$$

on which $F(s)$ converges uniformly to zero with respect to Arg s.

4. For any $c > \alpha$ the integral

$$\int_{c-i\infty}^{c+i\infty} F(s) ds$$

converges absolutely.

Then for the original function $f(t)$ we have

$$f(t) = \sum_{s_k} \operatorname*{res}_{s_k} F(s)\, e^{st} \tag{1.19}$$

where the residues are computed with respect to all poles of the function $F(s)$, and the summation is over group of poles lying in the annual regions between adjacent circles C_n. A special case of this is the case where $F(s) = \dfrac{A(s)}{B(s)}$, where the polynomial $A(s)$ has less degree than that of $B(s)$, i.e. $F(s)$ is a proper rational function. Then the original function $f(t)$ can be recovered by

$$f(t) = \sum_{k=1}^{n} \frac{1}{(n_k - 1)!} \lim_{s \to s_k} \frac{d^{n_k - 1}}{d\, s^{n_k - 1}} \left[F(s)(s - s_k)^{n_k} e^{st} \right] \tag{1.20}$$

where s_k's are the poles of $F(s)$, and the n_k's are their multiplicities, and the sum is taken over all distinct poles.

§3 Numerical Inversion of the L.T.

We have seen in the previous section that procedures for direct (analytical inversion) of the Laplace transform are deficient and disadvantageous; in fact, they can only be used in few special cases and with some computational difficulties encountered. It therefore seems inevitable to employ the powerful tools of numerical analysis. Unfortunately, the numerical inversion of the Laplace transform falls in the category of ill-posed problems.

3.1 Ill-posed Problems

Let A be an operator from a metric space X to a metric space Y. Consider the problem of determining an $f \in X$, such that the equation $Af = g$ is satisfied for some given $g \in Y$. This problem is called well-posed (in the Hadamard sense) if the following conditions are satisfied:

i) for each $g \in Y$ $\exists$ a solution $f \in X$

ii) this solution is unique

iii) this solution depends continuously on the data g.

If one or more of these conditions is not satisfied, the problem is called ill-posed (incorrectly, improperly posed).

In our problem, condition (iii) is not satisfied and hence ill-posedness arises. From numerical point of view, this is equivalent to the concept of instability which can be interpreted from Riemann Lebesgue lemma. A change $h(t) = A \sin \omega t$, in the original function $f(t)$ in (1.1) with a very large frequency ω produces a very small change in the data function $F(s)$. It is then clear that one has no choice but to go to the regularization methods proven to be effective ways of treating ill-posed problem [77].

3.2 Regularization

Regularization of ill-posed problems [2,53,77] is a phrase that is used for various approaches to overcome the lack of continuous dependence (as well as to bring about existence and uniqueness if necessary). Roughly speaking, this entails an analysis of an ill-posed problem via an analysis of an associated well-posed problem, or a family (usually a sequence, or a filter) of well-posed problems, provided this analysis yields meaningful solutions to the given problem.

We note that the various approaches to regularization involve one or more of the following intuitive ideas:

(a) a change of the concept of solutions

(b) a change of the space of solutions

(c) a change of the topologies of the spaces

(d) a change of the operator itself

(e) the concept of regularization operators.

Two of these concepts (b, e) will be employed in our work (see chapters II and III).

3.3 Survey of Literature

An extensive list of papers concerning the inversion of the Laplace transform, starting as early as 1934, has been collected and published by R. Piessens [57], and R. Piessens and N.D.P. Dang [59] covering the literature of this topic up to

1976. For a comprehensive overview of the theoretical and computational aspects of the subject one may consult the remarkable handbook of V.I. Krylov [42], and R.E. Bellman, R.E. Kalaba and J.A. Lockett [4], with the latter one geared toward physical and engineering applications of the problem.

More recently, Davis and Martin [16] have published a good paper of survey and comparison of methods with testing and evaluation of 16 different transforms.

We can categorize the numerical methods of inversion the Laplace transform as follows:

I. Methods which compute a sample.

The computing sample is of the form

$$I_n(t) = \int_0^\infty \delta_n(t,u)\, F(u) du$$

where the function $\delta_n(t,u)$ form a delta convergent sequence [26,81,71,74,83,14].

II. Methods which expand $f(t)$ in exponential functions.

That is by representing $f(t)$ with exponential functions by introducing e^{-rt} as a new independent variable in some set of orthonormal functions such as Lagurre, Legendre, Jacobi, Chebyshev, etc. [21,44,49,4,56,71,80,45,35,8,9,10,13,48,22,55]. Most of these methods are known sometimes as methods of moments or Galerkin's or collocation.

III. Gaussian Numerical Quadrature of the Bromwich Integral.

Gaussian quadrature is a well-known method for the approximation of integrals, based on ensuring that the rules are exact for polynomials. Such a rule has been developed for the inversion integral, designed to invert exactly Laplace transforms of the form $s^{-\alpha}\Phi(s^{-1})$ where $\Phi(s^{-1})$ is a polynomial in s^{-1}. [68,72,71,58]. More recently, [41] have developed an inversion technique by conformally mapping the Bromwich contour onto the unit circle. Talbot's method has also been investigated by [50] for the contour integration of the Bromwich integral. V.I. Ryabov [65] had studied the behavior of the coefficient of approximating this integral by Gauss quadrature methods of highest accuracy.

IV. Methods Based on Fourier Series.

In the cases when $f(t)$ is a real function the Bromwich integral yields

$$\begin{aligned}\mathrm{Re}[F(c+i\omega)] &= \int_0^\infty e^{-ct} f(t)\cos\omega t\, dt \\ \mathrm{Im}[F(c+i\omega)] &= -\int_0^\infty e^{-ct} f(t)\sin\omega t\, dt.\end{aligned}$$

The inversion theorems for Fourier sine and cosine transforms yields an alternative to the above equations

$$\begin{aligned}f(t) &= \frac{1}{2\pi} e^{ct} \int_0^\infty Re\ F(c+i\omega)]\cos\omega t\, dt \\ f(t) &= \frac{1}{2\pi} e^{ct} \int_0^\infty Im\ F(c+i\omega)]\cos\omega t\, dt\end{aligned}$$

in all these relations $c > \alpha_f$.

These transforms have been investigated by [20,19,11,15,46], and more recently [34,75,33].

V. Padé Approximations.

A considerable number of papers have been using Pade' approximations, where $F(s)$ is replaced by its Pade' approximation [44] and its references, [39,2]. Then the Heaviside theorem may be employed.

VI. Methods of real Inversion.

These methods utilize the classical real inversion formulae of the Laplace transform due to post and Widder [65]

$$f(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \, s^{n+1} \frac{d^n F(s)}{ds^n}\bigg|_{s=(n+1)/t}$$

These formulae have been receiving much attention in the past few years [82,23,40].

VII. Methods of Regularization.

The previous methods do not include regularization techniques which are known to be the best methods to invert the laplace transform in the presence of noisy data. Regularization methods have been considered by [22,1,79,47,38]

The concept of regularization and in particular of constructing a sequence of regularizing operators and the choice of an optimal Tikhonov regularization parameter involved has been a subject of active research, and was discussed thoroughly in Ph.D. dissertation of Bakker [2].

CHAPTER 2

NUMERICAL INVERSION BY THE USE OF LEGENDRE POLYNOMIALS

2.1 Introduction

A method for approximating the inverse Laplace transform numerically is presented, using a finite series of the classical Legendre polynomials. The method recovers a real-valued function $f(t)$ in a finite interval of the positive real axis when $f(t)$ belongs to a certain class of functions of the type $\mathcal{W}_\beta$ and requires the knowledge of its Laplace transform $F(s)$ only at a finite number of points on the real axis $s > 0$. The choice of these points is connected to some parameters α, β, γ, c to be introduced in the method. We shall test the method in some examples, with particular emphasis on the estimation of the error bounds and how much these bounds are affected by the total variations of the original function in question, as well as its least upper bound.

In the sequel, we intend to carry our work as follows:

(i) We start with mapping the interval $[0, \infty)$ to the interval $[-1, 1)$. this enables us to use the Legendre polynomials defined over the interval $[-1, 1)$.

(ii) We, then, introduce a class of functions of the type $\mathcal{W}_\beta$ in the interval $[-1, 1)$.

(iii) We show that the class $\mathcal{W}_\beta$ is large by showing that it includes a practical and more familiar class $\mathcal{D}_\alpha$.

(iv) We shall discuss some properties of the Legendre polynomials.

(v) We introduce a method of approximating inverse Laplace transforms when they belong to the class $\mathcal{W}_\beta$ using Legendre polynomials.

(vi) An inversion theorem together with error bounds will be proven, emphasizing how the parameter β, λ are important to minimize the number of polynomials needed when they are chosen optimally.

2.2 Functions of the Type $\mathcal{W}_\beta$

Let $f(t)$ be defined at each point of the positive real axis. Then for $\lambda > 0$, the following change of variable will map the interval $[0, \infty)$ to the interval $[-1, 1)$.

$$x : [0, \infty) \to [-1, 1)$$

$$x(t) = 1 - 2\, e^{-\lambda t}$$

$$t : [-1, 1) \to [0, \infty)$$

$$t(x) = -\frac{\ln\left(\frac{1-x}{2}\right)}{\lambda}$$

Given $f(t)$, define

$$h(t) = e^{-\beta t} f(t), \quad \text{for arbitrary} \quad \beta > \lambda > 0$$

$$g(x) = h\left(-\frac{\ln(\frac{1-x}{2})}{\lambda}\right)$$

$$g(x) = \left(\tfrac{1-x}{2}\right)^{\beta/\lambda} f\left(-\tfrac{1}{\lambda}\ln\left(\tfrac{1-x}{2}\right)\right)$$

Definition 2.2.1 A real valued function $f(t)$ is said to be of type $\mathcal{W}_\beta$ if $f(t) \equiv 0$ for $t < 0$, and if the corresponding $g(x)$ (as defined above) has a derivative of bounded variations in the interval $[-1, 1)$, i.e.

$$\text{total variation of } g' \text{ in } [-1,1) = V_{-1}^{1}\, g'(x) < \infty.$$

Remark 2.2.1 If $f(t) \in \mathcal{W}_\beta$, then the corresponding $g(x)$ is bounded and absolutely continuous and hence of bounded variation.

At the first glance, one may think that our hypothesis above is too restrictive and yields a relatively small class of functions for practical purposes. On the contrary, the following lemma will show that our class of functions of the type$\mathcal{W}_\beta$ is a large one indeed.

2.3 Functions of the Type $\mathcal{D}_\alpha$

Definition 2.3.1 We say a real valued function $f(t)$ is of type $\mathcal{D}_\alpha$ if:

(i) $f(t)$ is twice continuously differentiable in $[0, \infty)$.

(ii) For every $f(t)$, $\exists\ c \geq 0$ and $\alpha \in \mathcal{R}$ such that $|f''(t)| \leq ce^{\alpha t}$ for $t \geq 0$.

Lemma 2.3.1 *Let $\beta > \max\{\lambda, \alpha + \lambda\}$, where $\lambda > 0$, then any function of type $\mathcal{D}_\alpha$ is also of type $\mathcal{W}_\beta$.*

Proof: The differentiability of the associated $g(x)$ defined above follows immediately from part (i), and we only prove that $g'(x)$ is of bounded variation. For this, it suffices to show that $g''(x)$ is absolutely integrable in $[-1, 1)$. Now, changing variables again yields

$$\int_{-1}^{1} |g''(x)|dx = \frac{1}{2\lambda}\int_{0}^{\infty} |\{\beta(\beta-\lambda)f(t) + (\lambda-2\beta)f'(t)+f''(t)\}\, e^{(\lambda-\beta)t}|dt. \qquad (*)$$

This equation requires bounds for $|f'|$ and $|f|$, and to obtain such estimates, we observed that part (i) of definition (2.3.1) implies the existence of $f(0)$ and $f'(0)$.

Define

$$y(t) = f(t) - tf'(0) - f(0) \qquad (2.1)$$

Hence,

$$y''(t) = f''(t)$$

$$y'(t) = f'(t) - f'(0)$$

$$y'(0) = y(0) = 0.$$

Now, consider the following cases:

<u>Case (i)</u>: $\alpha = 0$, i.e., $|f''(t)| = |y''(t)| \le c$.

Integrating both sides of the inequality $-c \le y''(t) \le c$ yields,

$$\left.\begin{aligned} |y'(t)| &\le ct \\ |y(t)| &\le \frac{c}{2}t^2 \end{aligned}\right\} \qquad (2.2)$$

Using equation (*) together with (2.1) and (2.2) implies

$$\int_{-1}^{1} |g''(x)|dx \leq \frac{1}{2\lambda(\beta-\lambda)^2}[(\beta-\lambda)\{|(\lambda-2\beta)f'(0)+\beta(\beta-\lambda)f(0)|+c\}$$

$$+\quad c\beta + c(2\beta-\lambda)+\beta(\beta-\lambda)|f'(0)|]. \tag{2.3}$$

Case (ii): $|\alpha| > 0$, i.e., $|y''(t)| \leq ce^{\alpha t}$.

Again, integrating both sides of the inequality $-ce^{\alpha t} \leq y''(t) \leq ce^{\alpha t}$

$$\left.\begin{aligned} |y'(t)| &\leq \frac{c}{\alpha}\left[e^{\alpha t}-1\right] \\ |y(t)| &\leq \frac{c}{\alpha^2}\left[e^{\alpha t}-\alpha t-1\right] \end{aligned}\right\} \tag{2.4}$$

Putting (2.1) and (2.4) in equation (*) yields

$$\int_{-1}^{1} |g''(x)|dx \leq \frac{\alpha^2|\beta(\beta-\lambda)f(0)+(\lambda-2\beta)f'(0)|-c\beta(\beta-\lambda)-c\alpha(2\beta-\lambda)}{2\lambda\alpha^2(\beta-\lambda)}$$

$$+\quad \frac{\alpha\beta(\beta-\lambda)|f'(0)|-c\beta(\beta-\lambda)}{2\lambda\alpha(\beta-\lambda)^2}$$

$$+\quad \frac{c\beta(\beta-\lambda)+\alpha c(2\beta-\lambda)+\alpha^2 c}{2\lambda\alpha^2(\beta-\lambda-\alpha)} \tag{2.5}$$

Since we know that the total variation of $g'(x)$ in $[-1,1)$ cannot exceed $\int_{-1}^{1}|g''(x)|dx$, the lemma is proved.

Now, for later purposes we need to estimate least upper bounds for $|g'|$ in both cases $\alpha = 0$ and $|\alpha| > 0$. Using (2.1), (2.2), (2.4) and the definition of g', we have

Case (i): $\alpha = 0$

$$\begin{aligned} U_1 &= \sup_{-1\leq x<1} |g'(x)| \\ &= \sup_{0\leq t<\infty} \left|\frac{e^{-(\beta-\lambda)t}}{2\lambda}[f'(t)-\beta f(t)]\right| \end{aligned}$$

$$= \sup_{0\leq t<\infty} \left| \frac{e^{-(\beta-\lambda)t}}{2\lambda} [\{f'(0)+y'(t)\} \right.$$
$$\left. -\beta\{f(0)+tf'(0)+y(t)\}]\right|$$
$$\leq \sup_{0\leq t<\infty} \frac{e^{-(\beta-\lambda)t}}{2\lambda}\left\{|f'(0)- -\beta f(0)-t\beta f'(0)|+\frac{c\beta t^2}{2}+ct\right\} \quad (2.3')$$

Case (ii) $|\alpha|>0$:

$$U_2 = \sup_{1\leq x<1} |g'(x)|$$
$$\leq \sup_{0\leq t<\infty} \frac{e^{-(\beta-\lambda)t}}{2\lambda}\left\{|f'(0)-\beta f(0)-t\beta f'(0)|+\frac{c\beta}{\alpha^2}\left[e^{\alpha t}-\alpha t-1\right]+\right.$$
$$\left.+\frac{c}{\alpha}\left[e^{\alpha t}-1\right]\right\} \quad (2.5')$$

<u>**Notation**</u>: Let V_1, V_2 denote the right-hand side of inequalities (2.3), (2.5) respectively (standing for the total variation of g' in $[-1,1)$ when $\alpha=0$ and $|\alpha|>0$ respectively). Let U_1, U_2 denote the right-hand side of inequalities (2.3′) and (2.5′) respectively (standing for the least upper bound of $|g'|$ when $\alpha=0$ and $|\alpha|>0$ respectively).

Remark (2.3.1): The parameters β and λ ensure that $V_1^{-1}g'$ and $\sup|g'|$ are both finite. In fact, they can be minimized considerably if these parameters were chosen optimally.

Before we state our proposed method for inverting the Laplace transform of a given function of the type $\mathcal{W}_\beta$, we recall the main properties of Legendre polynomials.

2.4 Series and Asymptotics of Legendre Polynomials

For any $g(x) \in L^2[-1,1)$, we can have the following Legendre polynomials expansion

$$g(x) \cong \sum_{n\geq 0} a_n P_n(x) \qquad \text{for } \; x \in [-1,1) \tag{2.6}$$

where

$$\begin{aligned} a_n &= \frac{2n+1}{2}\int_{-1}^{1} g(x)P_n(x)dx \\ P_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k(n+k)!}{(n-k)!(k!)^2}\left(\frac{1-x}{2}\right)^k. \end{aligned} \tag{2.7}$$

The series $\sum_{n\geq 0} a_n P_n(x)$ is called the Legendre series of $g(x)$ and we can assert that if $g(x)$ is in $L^2[-1,1)$, then this series converges in the mean in $[-1,1)$ to $g(x)$ [69].

We can also affirm that if $g(x)$ is continuous in $[-1,1)$ and its Legendre series is uniformly convergent there, then

$$g(x) \equiv \sum_{n\geq 0} a_n P_n(x), \qquad \text{for } x \in [-1,1). \tag{2.8}$$

We list below an important approximation formula, to be used later for computational purposes, with the help of which we can significantly minimize the number of steps needed in our computation as well as the error involved.

Theorem 2.4.1 Jackson's Theorem [69] p. 205]: *Let $w(x)$ be of bounded variation in $[-1,1)$, and let $U = \sup_{-1\le x\le 1} |w(x)|$, and V denote the total variation of $w(x)$ in $[-1,1)$. Given the function*

$$g(x) = g(-1) + \int_{-1}^{x} w(x)dx, \tag{2.9}$$

then the coefficient a_n,

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} g(x)P_n(x)dx$$

of its Legendre series satisfy the inequality

$$|a_n| < \frac{4}{\sqrt{\pi}}(U+V)\frac{1}{n^{3/2}}, \qquad \text{for } n \ge 2. \tag{2.10}$$

Moreover, the Legendre series of $g(x)$ converges uniformly and absolutely to $g(x)$ in $[-1,1)$. The remainder of the series beginning with the $(n+1)$-st term satisfies the inequalities

$$|R_{n+1}(x)| < \frac{8}{\sqrt{\pi}}(U+V)\frac{1}{\sqrt{n}}, \qquad \text{for } |x| \le 1, \;\; n \ge 1 \tag{2.11}$$

$$|R_{n+1}(x)| < \frac{16\sqrt{2}}{\pi} \cdot \frac{U+V}{\sqrt[4]{1-\delta^2}} \cdot \frac{1}{n}, \qquad \text{for } |x| \le \delta < 1, \;\; n \ge 1. \tag{2.12}$$

The above discussions have furnished a fairly sufficient survey of the Legendre polynomials we intend to use as tools for our approximation as well as full description of functions of the type $\mathcal{W}_\beta$.

Now, we are in a position for departure for our main problem, namely, the construction of the technique to be used for inverting the Laplace transform.

2.5 The Approximate Inversion of the Laplace Transform:

The Inversion Theorem *Given the Laplace transform $F(s)$ for a real-valued function $f(t)$ of type $\mathcal{W}_\beta$, and given $\epsilon > 0$, there exists an integer N such that*

$$f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t), \qquad \text{for } 0 \leq t_0 \leq t \leq T < \infty$$

satisfies

$$\sup_{t_0 \leq t \leq T} |f(t) - f_a(t)| < \epsilon \qquad \text{where,}$$

$$\begin{aligned} a_n &= \lambda(2n+1)\sum_{k=0}^{n} \frac{(-1)^n \Gamma(1+n+k)}{(n-k)!(k!)^2} F(\beta + \lambda + \lambda k) \\ \tilde{P}_n(t) &= \sum_{k=0}^{n} \frac{(-1)^n \Gamma(1+n+k)}{(n-k)!(k!)^2} e^{-(k\lambda - \beta)t} \end{aligned}$$

and N can be chosen such that

$$N \geq \left[\frac{16 e^{\beta T}(U+V)}{\epsilon\sqrt{\pi}}\right]^2, \qquad \text{for } t_0 = 0 \tag{2.13}$$

or,

$$N \geq \frac{32\sqrt{2}e^{\beta T}(U+V)}{\epsilon\pi\sqrt[4]{1-\delta^2}}, \qquad \text{for } t_0 > 0 \tag{2.14}$$

The second estimate for N can be used if the function $f(t)$ is to be recovered in an interval interior to $[0, \infty)$, i.e., for $t \in [t_0, T]$ with $t_0 > 0$ and $T < \infty$ where $\delta = \max\left\{|1 - 2e^{-\lambda t_0}|,\ |1 - 2e^{-\lambda T}|\right\}$. U and V represent respectively the least upper bound of $|g'(x)|$ and total variation of $g'(x)$ in $[-1, 1)$.

Proof: We may assume without loss of generality that $F(s)$ is defined for Re $s > 0$; a simple translation in the imaginary axis can be done if this is not the case.

Now, let us follow the same notations and change of variables introduced earlier.

Put,

$$h(t) = e^{-\beta t} f(t), \qquad \text{for} \quad \beta > \lambda > 0$$

$$x = 1 - 2e^{-\lambda t}, \qquad \text{for} \quad t \geq 0$$

$$g(x) = h\left(\frac{-\ln(\frac{1-x}{2})}{\lambda}\right), \qquad \text{for} \quad x \in [-1, 1).$$

Since $f(t)$ is of type $\mathcal{W}_\beta$, remark (2.2.1) implies that $g(x)$ is the indefinite integral of its derivative

$$g(x) = g(-1) + \int_{-1}^{x} g'(x)dx.$$

Now, with U and V being the least upper bound of $|g'(x)|$ and the total variation of $g'(x)$ respectively, Jackson's theorem states that $g(x)$ can be approximated by the first N terms of its Legendre series in equation (2.8)

$$g(x) \cong \sum_{n=1}^{N} a_n P_n(x), \qquad \text{for } x \in [-1, 1)$$

where, by Rodrigues formula

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} \left(\frac{1-x}{2}\right)^k \\
a_n &= \frac{2n+1}{2} \int_{-1}^{1} g(x) P_n(x) dx \\
&= \frac{2n+1}{2} \int_{0}^{\infty} e^{-\beta t} f(t) P_n(1-2e^{-\lambda t})\,(2\lambda e^{-\lambda t})dt. \\
&= \lambda(2n+1) \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} \int_{0}^{\infty} f(t) e^{-[\beta+\lambda+\lambda k]t} dt \\
&= \lambda(2n+1) \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+n+k)}{(n-k)!(k!)^2} F(\beta+\lambda+\lambda k).
\end{aligned}$$

Now, to show that our uniform error in approximating the original function

$f(t)$ by the function $f_a(t)$ cannot exceed ϵ in magnitude, we consider the following two cases:

Case (i): $t_0 = 0$.

Then, inequalities (2.11) and (2.13) give the remainder

$$|R_{N+1}(x)| < e^{-\beta T} \cdot \frac{\epsilon}{2}, \qquad \text{for } x \in [-1,\ 1 - 2e^{-\lambda T}]$$

Hence,

$$g(x) = \sum_{n=1}^{N} a_n P_n(x) + R_{N+1}(x), \qquad \text{for } x \in [-1, 1).$$

Putting $x = 1 - 2e^{-\lambda t}$ we have

$$\begin{aligned} e^{-\beta t} f(t) &= \sum_{n=0}^{N} a_n P_n(1 - 2e^{-\lambda t}) + R_{N+1}(1 - 2e^{-\lambda t}) \\ f(t) &= \sum_{n=0}^{N} a_n \tilde{P}_n(t) + e^{\beta t} R_{N+1}(1 - 2e^{-\lambda t}) \end{aligned}$$

If we put $f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t)$. Then,

$$\max_{0 \le t \le T} |f_a(t) - f(t)| = \max_{0 \le t \le T} e^{\beta t} R_{N+1}(1 - 2e^{-\lambda t}) < \frac{\epsilon}{2}.$$

Case (ii) $t_0 > 0$:

Similarly inequalities (2.12) and (2.14) give the remainder

$$|R_{N+1}(x)| < e^{-\beta T} \cdot \frac{\epsilon}{2} \qquad \text{for } x \in [-\delta, \delta]$$

where, $\delta = \max\{|1 - 2e^{-\lambda t_0}|,\ 1 - 2e^{-\lambda T}|\}$.

Also, $g(x) = \sum_{n=0}^{N} a_n P_n(x) - R_{N+1}(x), \quad$ for $x \in [-\delta, \delta]$.

Putting $x = 1 - 2e^{-\lambda t}, \quad$ for $t \in [t_0, T]$, we get

$$\begin{aligned} e^{-\beta t} f(t) &= \sum_{n=0}^{N} a_n P_n(1 - 2e^{-\lambda t}) + R_{N+1}(1 - 2e^{-\lambda t}) \\ f(t) &= \sum_{n=0}^{N} a_n \tilde{P}_n(t) + e^{\beta t} R_{N+1}(1 - 2e^{-\lambda t}) \end{aligned}$$

If we put $f_a(t) = \sum_{n=0}^{N} a_n \tilde{P}_n(t)$, then

$$\max_{t_0 \le t \le T} |f(t) - f_a(t)| = \max_{t_0 \le t \le T} e^{\beta t} R_{N+1}(1 - 2e^{-\lambda t}) < \frac{\epsilon}{2}.$$

This completes the proof.

2.6 The Optimal Choice of β and λ:

In our numerical computations for functions of the type $\mathcal{D}_\alpha$, it is always desirable to minimize the time and effort needed in the computation to achieve the accuracy within the pre-assigned tolerance ϵ.

Thus, with the assumption that c and α are already known, we may take U and V in inequalities (2.13) and (2.14)

$$U = U_i(\beta, \lambda) \qquad i = 1 \text{ or } 2.$$

$$V = V_i(\beta, \lambda) \qquad i = 1 \text{ or } 2.$$

$i = 1$ or 2, depend on whether α in our class $\mathcal{D}_\alpha$ is zero or not respectively.

Now, we can pose the following optimization problem, responsible for minimizing our integer N that determines the number of polynomials needed to achieve the desired accuracy. The minimization is taken over β, λ, and we recommend to adopt the powerful computational algorithm SUMT [12]

Minimize

$$e^{\beta T}\left[U_i(\beta,\lambda) + V_i(\beta,\lambda)\right] \qquad i = 1 \text{ or } 2$$

subject to

$$\lambda > 0$$

$$\beta > \max\{\lambda,\ \lambda + \alpha\}.$$

When this minimum is achieved, say at $\lambda = \lambda_{\text{opt}}$ and $\beta = \beta_{\text{opt}}$, for a given c and α, then we may take $U = U_i(\beta_{\text{opt}}, \lambda_{\text{opt}})$ and $V = V_i(\beta_{\text{opt}}, \lambda_{\text{opt}})$ in our bounds (2.13) and (2.14). This gives an optimal choice of $N = N_{\text{opt}}$, with which we can advance in our calculations.

2.7 Determination of c and α $(\alpha > 0)$.

The problem arising in the optimization recommended above is the determination of the best constants c and α needed there, when we only have the given function $F(s)$ in hand. For this, we recall the following theorem.

Theorem 2.7.1 (Tauberian Theroem) [81] *If the function $f(t)$ satisfies the inequality $|f(t)| < Me^{\alpha t}$ for all $t > 0$, M being a positive constant, then*

$$\lim_{s\to\infty} sF(s) = f(0).$$

Clearly, our function $f(t)$ as it belongs to the class of functions of the type $\mathcal{D}_\alpha$ satisfies the hypothesis of Tauberian theorem. This can be shown with the help of the inequalities (2.2) and (2.4).

Therefore, given $F(s)$ we calculate the following limits involved in our bounds and known to exist by the hypothesis of our class

$$f(0) = \lim_{s\to\infty} sF(s)$$

$$f'(0) = \lim_{s\to\infty} s\,[sF(s) - f(0]$$

$$f''(0) = \lim_{s\to\infty} s\,[s^2F(s) - sf(0) - f'(0)]$$

Now, we can use these limits to estimate lower bounds for c and α.

Since, $|f''(t)| \leq ce^{\alpha t}$, it is immediate to see that a necessary condition is that

$$|f''(0)| \leq c$$

$$|s^2F(s) - sf(0) - f'(0)| \leq \frac{c}{s-\alpha}, \qquad \text{for } s > \alpha. \tag{2.15}$$

If the leftside of the above inequality, which is the Laplace transform of $f''(t)$ is different from zero, we have, for $s > \alpha$

$$\frac{s|s^2F(s) - sf(0) - f'(0)| - c}{|s^2F(s) - sf(0) - f'(0)|} \leq \alpha. \tag{2.16}$$

In most cases, inequality (2.16) will provide a good estimate for α directly, otherwise we need to estimate the maximum of the left side over all $s > 0$, and use it as a lower bound for α.

If the procedure of determining lower bounds for α and c is too difficult, depending on the nature of the function $F(s)$, then the following theorems [81] are recommended.

Definition 2.7.1: An operator $L_{k,t}[F(s)]$ is defined by the equation

$$L_{k,t}[F(s)] = \frac{(-1)^k}{k!} F^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}$$

for any real positive number t and any positive integer k.

<u>Condition A</u>: A function $F(s)$ satisfies condition A if it has derivatives of all orders in $(0 < s < \infty)$ and if there exists a constant M such that for $(0 < s < \infty)$

$$L_{k,t}[F(s)] < M \qquad (k = 1, 2, \ldots)$$

$$|sF(s)| < M.$$

Result: [81]

Condition A is necessary and sufficient that

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt,$$

where $f(t)$ is bounded in $(0 < t < \infty)$.

Result: [81]

If $F(s)$ is the Laplace transform of a function $f(t)$ with $f(t)$ bounded in $(0 < t < \infty)$, then

$$\lim_{k\to\infty}\left\{\sup_{0<t<\infty} |L_{k,t}[F(s)]|\right\} = \operatorname*{ess\,sup}_{0<t<\infty} |f(t)|.$$

Now suppose we are given $F(s)$ for a function $f(t)$ which is known to satisfy part (i) of the definition of $\mathcal{D}_\alpha$, and we want to know if condition (ii) is also satisfied, i.e., the existence of c and α such that $|f(t)| \leq ce^{\alpha t}$. Moreover, we want to estimate the least c and α. Then the results of Widder suggest the following.

$$\text{Let } G(s,\eta) = F(s-\eta)$$

where η is a real number to be fixed later. Then, the original function of $G(s,\eta)$ is simply $e^{\eta t}f(t)$. We try to find the least possible value η that makes the following limit exist

$$\lim_{k\to\infty}\left\{\sup_{0<t<\infty} |L_{k,t}[G(s,\eta)]|\right\} = \operatorname{ess}\sup_{0<t<\infty} \left|e^{\eta t}f(t)\right|.$$

If we succeed in doing so, then we can claim that

$$\left|e^{\eta t}f(t)\right| \leq c$$

and our c and α are on hand, with $\alpha = -\eta$.

Remark (2.7.1): What we actually need to estimate is c and α such that $|f''(t)| \leq ce^{\alpha t}$. For this, we can, without loss of generality, consider $\tilde{F}(s)$ instead of $F(s)$, where

$$\begin{aligned}\tilde{F}(s) &= \mathcal{L}(f''(t))(s) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Then, we simply carry the same analysis for $\tilde{F}$.

Example:

$$F(s) = \frac{1}{s - \frac{1}{2}}$$

$$f(t) = e^{\frac{1}{2}t}$$

$$G(s, \eta) = F(s - \eta) = \frac{1}{s - \eta - \frac{1}{2}}$$

$$g(t) = e^{(\frac{1}{2}+\eta)t}$$

$$L_{k,t}[G(s,\eta)] = \frac{(-1)^k}{k!} \frac{(-1)^k k!}{\left(\frac{k}{t} - \eta - \frac{1}{2}\right)^{k+1}} \left(\frac{k}{t}\right)^{k+1} = \left(\frac{k}{k - t(\eta + \frac{1}{2})}\right)^{k+1}$$

Clearly,

$$\lim_{k\to\infty} \left\{ \sup_{0<t<\infty} |L_{k,t}[G(s,\eta)]| \right\} = \begin{cases} 1 & \eta = -\frac{1}{2} \\ \infty & \eta \neq -\frac{1}{2} \end{cases}$$

hence, we choose $\eta = -\frac{1}{2}$ and the corresponding limit $c = 1$. Then, $|e^{\eta t} f(t)| \leq c = 1$, and $|f(t)| \leq ce^{-\eta t} = e^{\frac{1}{2}t}$. Giving $\alpha = \frac{1}{2}$ as expected.

2.8 Numerical Implementation

We use the software Mathematica to implement our technique. The following input statements provide the desirable results.

Algorithm

$K :=$ Machine precision (to be specified)

$L := N[\lambda, K] = \lambda_{\text{opt}}$ (to be specified)

$B := N[\beta, K] = \beta_{\text{opt}}$ (to be specified)

$m :=$ number of polynomials needed $= N_{\text{opt}}$ (to be specified)

$f[s\_] := f[s] = N[F(s), K] =$ The Laplace Transform (to be specified).

$a[n\_] := a[n] = N[(L+0.)(2n+1)\text{Sum}[(((-1)\hat{\ }i)((n+i)!)f[(B+L+0.)$
$+(L+0.)i])/((n-i)!((i!)\hat{\ }2)), \{i, 0, n\}], K] =$ (the coefficient a_n)

$f_a[x\_] := N[\text{Exp}[(B+0.)x]\text{Sum}[a[n]\text{Legendre P}[n, 1-2\text{Exp}[-x(L+0.)]],$
$\{n, 0, m\}], K] =$ (the approximation function).

$g[x\_] := N[f(x), K] =$ the exact function (to be specified).

$h[x\_] : N[\text{Abs}[f_a[x] - g[x]], K] =$ the error function

Table $[\{N[x,1], N[f_a[x],8], N[g[x],8], \text{Number Form}[h[x],2]\}, \{x, t_0, T, t_1\}]$, $[t_0, T]$ is the interval of approximation and t_1 is the increment size of the calculation (to be specified).

Examples:

In the following examples λ and β have were taken in an arbitrary trial and error manner, for the reason that U and V have singular and nondifferentiable terms. For this we shall only estimate the number N for the first example.

Example 1: For simple computations let us consider the function $f(x) = 1$, with the Laplace transform $F(s) = \frac{1}{s}$, and let us try to recover $f(x)$ for $0 < t_0 \leq x \leq 200 = T$, within an error $\epsilon = 10^{-1}$. Then the corresponding g, g' and g'' can be easily seen by definition (2.2.1)

$$
\begin{aligned}
g(x) &= \left(\frac{1-x}{2}\right)^{\frac{\beta}{\lambda}} \\
g'(x) &= -\frac{\beta}{2\lambda}\left(\frac{1-x}{2}\right)^{\frac{\beta}{\lambda}-1} \\
g''(x) &= -\frac{\beta(\beta-\lambda)}{4\lambda^2}\left(\frac{1-x}{2}\right)^{\frac{\beta}{\lambda}-2}
\end{aligned}
$$

Now, since $|f''(x)| = 0$, we may take $\alpha = c = 0$ in our definition (2.3.1).

Now,

$$
\begin{aligned}
U &= \sup_{-1\leq x\leq 1} |g'(x)| = \frac{\beta}{2\lambda} \\
V &\leq \int_{-1}^{\Gamma} |g''(x)| = \frac{\beta}{2\lambda}
\end{aligned}
$$

Now, if we choose β arbitrarily small and λ very close to β with $\lambda < \beta$. Then, by our inequality (2.14), we have

$$
\begin{aligned}
N &\geq \frac{32\sqrt{2}e^{200\beta}\left(\frac{\beta}{\lambda}\right)}{10^{-1}\pi\sqrt[4]{1-\delta^2}}, \quad \delta \simeq \frac{1}{2}. \\
&\simeq 150.
\end{aligned}
$$

$m = 2$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	1.0	1.0	0.0
20.0	1.0	1.0	0.0
40.0	1.0	1.0	0.0
60.0	1.0	1.-0	0.0
80.0	1.0	1.0	0.0
100.0	1.0	1.0	0.0
120.0	1.0	1.0	0.0
140.0	1.0	1.0	$3.1\ 10^{-14}$
160.0	1.0	1.0	$6.1\ 10^{-14}$
180.0	1.0	1.0	$2.0\ 10^{-14}$
200.0	1.0	1.0	$7.0\ 10^{-12}$

Example 2:

$$F(s) = \frac{1}{(s+1)^2+1}$$

$$f(t) = e^{-t}\sin t$$

(a)

$m = 10$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	-0.00000962189	0.0	$9.6\ 10^{-6}$
$\frac{\pi}{4}$	0.32239106	0.32239694	$5.9\ 10^{-6}$
$\frac{\pi}{2}$	0.20787082	0.20787958	$8.8\ 10^{-6}$
$\frac{3\pi}{4}$	0.067052245	0.06701974	$3.3\ 10^{-5}$
π	-0.00049518539	0.0	$5.0\ 10^{-5}$
$\frac{5\pi}{4}$	-0.01406353	-0.013932035	$1.3\ 10^{-4}$
$\frac{3\pi}{2}$	-0.0083793352	-0.008983291	$6.10\ 10^{-4}$
$\frac{7\pi}{4}$	-0.001941255	-0.0028961856	$9.5\ 10^{-4}$
2π	-0.0019625595	0.0	$2.0\ 10^{-3}$

(b)

$m = 20$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	2.5354036	0.0	$2.5\ 10^{-8}$
$\frac{\pi}{4}$	0.32239695	0.32239694	$1.1\ 10^{-8}$
$\frac{\pi}{2}$	0.20787955	0.20787958	$3.0\ 10^{-8}$
$\frac{3\pi}{4}$	0.067019656	0.06701974	$8.4\ 10^{-8}$
π	0.00000024	0.0	0.0
$\frac{5\pi}{4}$	-0.013931553	-0.013932035	$4.8\ 10^{-7}$
$\frac{3\pi}{2}$	-0.0089845545	-0.008983291	$1.3\ 10^{-6}$
$\frac{7\pi}{4}$	-0.002895472	-0.0028961856	$7.1\ 10^{-7}$
2π	0.000010587337	0.0	$1.1\ 10^{-5}$

Example 3:

$$F(s) = \frac{1}{(s+2)^{3/2}(s+1)}$$

$$f(t) = e^{-x} Erf(\sqrt{t}) - 2\sqrt{\frac{t}{\pi}}\, e^{-2t}$$

(a)

$m = 5$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	-0.0091284955	0.0	$9.1\ 10^{-3}$
0.2	0.051823691	0.048925307	$2.9\ 10^{-3}$
0.4	0.099266734	0.10090527	$1.6\ 10^{-3}$
0.6	0.13233794	0.13555411	$3.2\ 10^{-3}$
0.8	0.15181664	0.15304602	$1.2\ 10^{-3}$
1.0	0.1594026	0.15730278	$2.1\ 10^{-3}$
1.2	0.15723974	0.15251432	$4.7\ 10^{-3}$
1.4	0.14761049	0.14216317	$5.4\ 10^{-3}$
1.6	0.13274715	0.12884934	$3.9\ 10^{-3}$
1.8	0.11472268	0.11438317	$3.4\ 10^{-4}$
2.0	0.095393769	0.099949961	$4.6\ 10^{-3}$

(b)

$m = 25$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	-0.000046726633	0.0	$4.7\ 10^{-5}$
0.2	0.048926224	0.048925307	$9.2\ 10^{-7}$
0.4	0.10090542	0.10090527	$1.5\ 10^{-7}$
0.6	0.13555571	0.13555411	$1.6\ 10^{-6}$
0.8	0.15304517	0.15304602	$8.5\ 10^{-7}$
1.0	0.15730314	0.15730278	$3.6\ 10^{-7}$
1.2	0.15251373	0.15251432	$5.9\ 10^{-7}$
1.4	0.14216473	0.14216317	$1.6\ 10^{-6}$
1.6	0.12884655	0.12884934	$2.8\ 10^{-6}$
1.8	0.11438579	0.11438317	$2.6\ 10^{-6}$
2.0	0.09995069	0.099949961	$7.3\ 10^{-7}$

Example 4:

$$F(s) = \frac{\sqrt{s+2}}{(s+1)^{3/2}}$$

$$f(t) = te^{-\frac{3}{2}t} I_1(\frac{t}{2}) + (t+1) I_0(\frac{t}{2})$$

where I_1 and I_0 are the modified Bessel functions of degree 1 and zero respectively.

(a)

$m = 5$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	1.0000323	1.0	$3.2\ 10^{-5}$
0.2	0.8986087	0.89862316	$1.4\ 10^{-5}$
0.4	0.79809951	0.79810129	$1.8\ 10^{-6}$
0.6	0.70224679	0.70223497	$1.2\ 10^{-5}$
0.8	0.61322768	0.61321473	$1.3\ 10^{-5}$
1.0	0.5321391	0.53213443	$4.7\ 10^{-6}$
1.2	0.45935012	0.45935646	$6.3\ 10^{-6}$
1.4	0.39475522	0.39476999	$1.5\ 10^{-5}$
1.6	0.33795381	0.33797177	$1.8\ 10^{-5}$
1.8	0.28837582	0.28839136	$1.6\ 10^{-5}$
2.0	0.24536772	0.24537636	$8.6\ 10^{-6}$

(b)

$m = 10$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	0.99999984	1.0	$1.6\ 10^{-7}$
0.2	0.89862312	0.89862316	$3.7\ 10^{-8}$
0.4	0.79810128	0.79810129	$1.6\ 10^{-8}$
0.6	0.702235	0.70223497	$3.4\ 10^{-8}$
0.8	0.6132147	0.61321473	$3.6\ 10^{-8}$
1.0	0.53213446	0.53213443	$3.4\ 10^{-8}$
1.2	0.45935644	0.45935646	$1.7\ 10^{-8}$
1.4	0.39476997	0.39476999	$2.1\ 10^{-8}$
1.6	0.33797182	0.33797177	$4.7\ 10^{-8}$
1.8	0.28839135	0.28839136	$8.7\ 10^{-9}$
2.0	0.2453763	0.24537636	$5.2\ 10^{-8}$

Example 5:

$$
\begin{aligned}
F(s) &= \frac{1}{\sqrt{s^2+1}} \\
f(t) &= J_0(t) \\
&= \text{Bessel function of degree zero}
\end{aligned}
$$

(a)

$m = 10$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	1.0001498	1.0	$1.5\ 10^{-4}$
0.2	0.99008446	0.99002497	$5.9\ 10^{-5}$
0.4	0.96033677	0.96039823	$6.1\ 10^{-5}$
0.6	0.91201646	0.91200486	$1.2\ 10^{-5}$
0.8	0.84638125	0.84628735	$9.4\ 10^{-5}$
1.0	0.76516841	0.76519769	$2.9\ 10^{-5}$
1.2	0.6709762	0.67113274	$1.6\ 10^{-4}$
1.4	0.56677705	0.56685512	$7.8\ 10^{-5}$
1.6	0.4555604	0.45540217	$1.6\ 10^{-4}$
1.8	0.34031201	0.33998641	$3.3\ 10^{-4}$
2.0	0.2241224	0.22389078	$2.3\ 10^{-4}$

(b)

$m = 20$			
t	$f_a(t)$ Appr.	$f(t)$ Exact	Error
0.0	0.99999486	1.0	$5.1\ 10^{-6}$
0.2	0.99002349	0.99002497	$1.5\ 10^{-6}$
0.4	0.96039749	0.96039823	$7.4\ 10^{-7}$
0.6	0.91200681	0.91200486	$1.9\ 10^{-6}$
0.8	0.84628488	0.84628735	$2.5\ 10^{-6}$
1.0	0.76520046	0.76519769	$2.8\ 10^{-6}$
1.2	0.67113118	0.67113274	$1.6\ 10^{-6}$
1.4	0.56685251	0.56685512	$2.6\ 10^{-6}$
1.6	0.45540876	0.45540217	$6.6\ 10^{-6}$
1.8	0.3399851	0.33998641	$1.3\ 10^{-6}$
2.0	0.22388007	0.22389078	$1.1\ 10^{-5}$

2.9 Concluding Remarks

The problem of inverting the Laplace transform with the aid of Legendre polynomials has been proposed earlier by Erdelyi [21], Papoulis [55] and Lanczos [43]. More recently, Schoenberg [73] discussed and proposed solutions in the form of a minimization problem. Nashed [54] had investigated Schoenberg's work and derived some sharp exponentially decreasing error bounds, but with rough assumptions on the original function being under transformation.

In our work, we claim the following:

1. A method for approximately inverting the Laplace transform with the use of Legendre polynomial was derived and proven to be very successful, provided the approximated function belongs to the class $\mathcal{W}_\beta$, which was shown to be quite large as Lemma 2.3.1 suggests.

2. One of the major features of the method is the fact that there is no need to solve linear system of equations to obtain the coefficients of the approximating polynomial. In fact, these coefficients are calculated directly and accurately. This is not the case in previous methods adopting the same Legendre polynomials.

3. Error bounds in terms of the least upper bound and the total variation of the original function are obtained.

4. New parameters β, λ were introduced in the method that could minimize the

error bounds when chosen optimally.

5. The method was tested in few examples and displayed good results when the original function is sought in a finite interval of the positive real axis. Unfortunately, this is not the case if the original function is to be recovered for large values of $t > 0$.

6. One of the main difficulties arising in our method is the optimization problem. But a trial and error choice of β, λ with $\beta > \lambda > 0$ has proven to be adequate.

CHAPTER 3

A REGULARIZATION METHOD

3.1 Introduction

In the context of ill-posed problems, the Laplace transform inversion is highly instable and severely ill-posed. The lack of universal methods for inverting the Laplace transform stems from the fact that the space of functions f for which the Laplace transform exists is simply too big.

From an operator theoretic point of view, we may see the Laplace transform as an integral operator of the first kind.

$$\left.\begin{aligned} &\mathcal{L}f = g \\ &\mathcal{L}(f(t))(s) = \int_0^\infty f(t)e^{-st}dt \end{aligned}\right\} \tag{3.1}$$

Then, the ill-posedness of the Laplace inversion can be best explained by the open mapping theorem.

Theorem 3.1.1 (Open mapping, bounded inverse). *A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous.*

Corollary 3.1.1 *T^{-1} is bounded iff the range of T is closed.*

Example: This example illustrates that if the image of the element is not in the range of the operator, then the inverse does not exist.

Define the operator A,

$$Af = \int_a^b x^2 t\, f(t)dt.$$

Then the equation

$$Af = \sin x$$

can never have a solution. That is because the range of A is the span of x^2.

Returning to our operator $\mathcal{L}$, the following remark will illustrate the ill-posedness of the problem.

Let us define the usual p-norm for the integrable functions on an interval (a, b) by

$$\|g\|_p = \left(\int_a^b |g(x)|^p \, dx \right)^{1/p}, \qquad 1 \leq p < \infty.$$

Let us denote the intervals $[0, \infty)$, $(-\infty, \infty)$ by $\mathcal{R}_+$ and $\mathcal{R}$ respectively and let $L^p(a, b)$ denote the space of functions with finite p-norm.

Remark 3.1.1:

(i) $\mathcal{L}$ maps $L^2(\mathcal{R}_+)$ into itself

(ii) $\mathcal{L}^{-1}$ is unbounded on $L^2(\mathcal{R}_+)$.

Proof: Part (i) will be shown in chapter V, and for part (ii) we immediately see that for $f_\omega(t) = e^{-t}\sin\omega t$

$$\frac{\|\mathcal{L}f_\omega\|_2}{\|f_\omega\|_2} \longrightarrow 0, \qquad \text{as } \omega \longrightarrow \infty$$

hence $\mathcal{L}$ is not bounded away from zero and therefore $\mathcal{L}^{-1}$ is unbounded.

The unboundedness of $\mathcal{L}^{-1}$ suggests strongly the employment of Tikhonov regularization techniques, responsible of replacing our original model

$$f = \mathcal{L}^{-1}g \tag{3.2}$$

by a one parameter family of bounded regularized operators for which the problem becomes well-posed and a stable solution to the inverse is tractable.

Although most of the methods in the past, as discussed in Chapter I, had one way or another used the concept of regularization implicitly to tackle the inversion of the Laplace transform, they did not utilize regularization techniques in the sense of Tikhonov, i.e., approximating the operator $\mathcal{L}^{-1}$, which has the advantages of solving the problem in the presence of noise and for a larger class of transforms.

Regularization methods have been used by [79,47] and more recently, Essah and Delves [22], Brianzi and Frontini [7], Ang, Laund and Stenger [1], Bakusinskii [3] and Strahov [76].

In the sequel we shall treat the inverse problem by regularization in the following steps:

(i) We shall give a brief introduction of the concept of regularization.

(ii) We shall restrict the space and pose some conditions on f and g in (3.1).

(iii) We shall transform the integral equation unitarily to an integral equation of convolution type.

(i) We, then, employ the powerful techniques recommended by Tikhonov and Arsenin [77], together with Fourier transformations to obtain a regularized approximate Fourier transform.

(v) In the last step, we shall develop a numerical scheme to invert this Fourier transform, and therefore a numerical approximation of the approximate Fourier transform is obtained.

(vi) Error bounds will be derived throughout and we conclude our work with some numerical algorithms and examples.

3.2 Tikhonov Regularization

In the sequel, we restrict ourselves to Hilbert spaces X and Y and let K be a bounded linear operator.

$$\left.\begin{aligned} &K : X \to Y \\ &Kf = g \end{aligned}\right\}. \tag{3.3}$$

Let $\mathrm{Ker}(K)$ denote the null-space of K and let $\mathrm{Ker}(K)^{\perp}$ be its orthogonal complement in X.

Definition 3.2.1: A regularization scheme for equation (3.3), in general, consists of a family of bounded linear operators $\mathcal{R}(\alpha,\cdot)$

$$\mathcal{R}(\alpha,\cdot) : Y \to X, \quad \alpha > 0,$$

with the property of point-wise convergence

$$\lim_{\alpha - 0} \mathcal{R}(\alpha, g) = f$$

for all $f \in X$. The parameter α is called the regularization parameter.

If the solution f_0 of equation (3.3) belongs to $\mathrm{Ker}(K)^{\perp}$, then a regularizing operator for this equation is generated by [80, 81] the minimization problem

$$\min_{f \in X} \left\{ \|Kf - g\|_2^2 + \alpha \|f\|_2^2 \right\}, \quad \alpha > 0, \quad g \in Y. \tag{3.4}$$

An element $f \in X$ solves the problem (3.4) if and only if it satisfies the relation

$$(K^*K + \alpha I)f = K^*g$$

where K^* denotes the adjoint of K. Since $\|(K^* + \alpha I)f\|_2 \geq \alpha \|f\|_2$ for any $\alpha > 0$ and $f \in X$, it follows that $(K^*K + \alpha I)X$ is a closed subspace of X and consequently

$$[(K^*K + \alpha I)X]^{\perp} = \mathrm{Ker}(K^*K + \alpha I) = \{0\}$$

thus $(K^*K + \alpha I)^{-1}$ is defined on X, and the minimization problem (3.4) has a unique solution.

Now, it is clear that the operator

$$g \mapsto R(\alpha, g) := (K^*K + \alpha I)^{-1}\, K^*g \tag{3.5}$$

is well defined and continuous for each $\alpha > 0$.

Moreover, we get the approximation error

$$f_0 - R(\alpha, g) = \alpha(K^*K + \alpha I)^{-1} f_0$$

with the help of which [2], it may be shown that $R(\alpha, g) \to f_0$ as $\alpha \to 0$. Thus R is a regularizing operator for the equation (3.3). Ivanov [37] has shown some error bounds in such cases.

Remark 3.2.1 [2] If $\text{Ker}(K)^{\perp}$ is not finite dimensional, then in general, problem (3.4) cannot be solved exactly. In order to get a numerical approximation to $R(\alpha, g)$, we may replace (3.4) by

$$\min_{f \in X_n} \left\{ \|Kf - g\|_2^2 + \alpha \|f\|_2^2 \right\} \tag{3.6}$$

where X_n is an n-dimensional subspace of X. This minimization problem is solved by

$$R_n(\alpha, g) := \left[(KP_n)^* KP_n + \alpha I\right]^{-1} (KP_n)^* g \tag{3.7}$$

where P_n denotes the orthogonal projection from X onto X_n. The convergence of $R_n(\alpha, g)$ to $R(\alpha, g)$ as $n \to \infty$ is shown in [2].

In our work we shall be concerned with an integral operator of the convolution type. For this, we follow Tikhonov and Arsenin [77] in constructing a regularizing operator for equation (3.1) with the use of Fourier transform.

For definiteness, let us look at an equation of the form

$$Kf \equiv \int_{-\infty}^{\infty} H(s-t)f(t)dt = g(s) \tag{3.8}$$

and let us apply the Fourier transformation $\wedge$ and its inverse $\wedge^{-1}$ defined by

$$\left.\begin{aligned} \wedge: \quad & \hat{f}(\omega) \equiv \int_{-\infty}^{\infty} e^{ix\omega} f(x)dx \\ \wedge^{-1}: \quad & f(t) \equiv \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{-it\omega} d\omega \end{aligned}\right\} \tag{3.9}$$

Assumption 3.2.1

(i) $g(s) \in L^2(\mathcal{R})$.

(ii) $f(t) \in L^1(\mathcal{R})$.

(iii) $H(t) \in L^1(\mathcal{R})$.

By the convolution theorem we have for equation (3.8)

$$\hat{f}(\omega) = \frac{\hat{g}(\omega)}{\hat{H}(\omega)}. \tag{3.10}$$

The main difficulty in obtaining an inverse Fourier transform for $\hat{f}$ is that we need to suppress the influence of large values of ω as both $\hat{g}$ and $\hat{H}$ tend to zero. This immediately suggests the introduction of some stabilizing factor $S(\omega, \alpha)$, giving rise to the regularizing operator $R(\alpha, g)$.

$$f(t) \cong R_S(\alpha, g) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S(\omega,\alpha)\hat{g}(\omega)}{\hat{H}(\omega)} e^{-it\omega} d\omega \tag{3.11}$$

for details and properties of $S(\omega, \alpha)$ see [77], page 115.

Now suppose $M(\omega) > 0$ is an even function satisfying the following criterion:

Criterion 3.2.1

(i) It is piecewise-continuous on every finite interval

(ii) it is nonnegative and $M(\omega) > 0$ for $\omega \neq 0$

(iii) $M(\omega) \geq c > 0$ for sufficiently large $|\omega|$

(iv) for every $\alpha > 0$ the ratio $\hat{H}(-\omega)/[L(\omega) + \alpha M(\omega)]$ belongs to $L^2(\mathcal{R})$ where $L(\omega) = \hat{H}(\omega)\hat{H}(-\omega) = |\hat{H}(\omega)|^2$.

Then, if we set

$$S(\omega, \alpha) = \frac{L(\omega)}{L(\omega) + \alpha M(\omega)} \tag{3.12}$$

we obtain a class of regularizing operators for equation (3.8). Such a class is determined by the function $M(\omega)$. This yields the regularized solution

$$\hat{f}(\omega) = \frac{\hat{H}(\omega)\hat{g}(\omega)}{\alpha M(\omega) + |\hat{H}(\omega)|^2} \tag{3.13}$$

and its inverse

$$f_\alpha(t) = R_M(\alpha, g) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{H}(-\omega)\hat{g}(\omega)\exp[-i\omega t]}{L(\omega) + \alpha M(\omega)} d\omega$$

It was proven [77] that this regularized solution f_α minimizes the functional

$$U_\alpha(f, g) = \int_{-\infty}^{\infty} (Kf - g)^2 dt + \alpha\Omega[f]$$

with stabilizing factor of the form

$$\Omega[f] = \int_{-\infty}^{\infty} M(\omega)[f(\omega)]^2 \, d\omega.$$

Remark 3.2.1. Tikhonov and Arsenin have suggested to consider $M(\omega)$ of the type

$$M(\omega) = \sum_{k=0}^{n} q_k \omega^{2k}$$

where the q_k's are nonnegative constants and $q_n > 0$.

3.3 Regularization of the Laplace Transform

Let f and g satisfy (3.1) and suppose that the following assumption holds.

Assumption 3.3.1 Suppose that for some $\lambda > 0$ the function f in (3.1) satisfies

(i) $f(t) = O(t^{\lambda}), \qquad t \to 0$

(ii) $f(t) = O(t^{-\lambda}), \qquad t \to \infty.$

Change of Variables:

The following change of variables reduces our integral equation (3.1) into a convolution type integral:

$$\begin{aligned} t &= e^{-u} \\ s &= e^{v} \\ G(u) &= e^{u} g(e^{u}) \\ F(u) &= f(e^{-u}) \\ H(u) &= \exp[-e^{u}] \cdot e^{u} \end{aligned}$$

Equation (3.1), then, becomes

$$G(v) = \int_R H(v-u)F(u)\,du \tag{3.14}$$

Assumption 3.3.2 Let $g = \mathcal{L}f$ and suppose that f satisfies assumption (3.3.1). Furthermore, suppose

(i) $f(t) \in L^2_{\mathcal{R}_+}$

(ii) $\sqrt{t}\, g(t) \in L^2_{\mathcal{R}_+}$.

Remark 3.3.1 Assumptions (3.3.1) and (3.3.2) immediately imply that

(i) $G \in L^2(\mathcal{R})$

(ii) $F \in L^1(\mathcal{R}) \cap L^2(\mathcal{R})$.

Now, we have the hypothesis of assumption (3.2.1) satisfied, thus with the use of equation (3.12) applied to equation (3.13), we get the regularized solution

$$\hat{F}_\alpha(\omega) = \frac{\hat{H}(\omega)\hat{G}(\omega)}{\alpha M(\omega) + |\hat{H}(\omega)|^2} \tag{3.15}$$

where α is the regularizing parameters (to be chosen optimally) and $M(\omega)$ is a function satisfying criterion (3.2.1).

The following theorem will demonstrate the error in the regularizing method of approximating F by F_α.

Theorem 3.3.1 *Let $g = \mathcal{L}f$, and suppose that f satisfies assumption (3.3.2). Furthermore, suppose that there exists a constant E such that*

$$\int_{-\infty}^{\infty} e^{2\pi|x|} \left|\hat{G}(x)\right|^2 \, dx \le E^2 \qquad (*)$$

Then, with the choice $M(\omega) = |\omega|^2$ and $\alpha > 0$, the regularized solution F_α defined by (3.14) converges in $L^2(\mathcal{R})$ as $\alpha \to 0$ to the solution F of the equation

$$\hat{F}(\omega) = \frac{\hat{G}(\omega)}{\hat{H}(\omega)} \qquad (3.16)$$

The convergence is of order $O(\sqrt{\alpha})$.

Proof: The choice of $M(\omega) = |\omega|^2$ is seen immediately to satisfy criterion (3.2.1) and hence, the solution F_α defined by (3.15) is a regularized solution to equation (3.14).

Now, for the order of convergence, we first notice that

$$\begin{aligned} \hat{H}(\omega) &= \Gamma(1 + i\omega) \\ \hat{H}(-\omega) &= \bar{\hat{H}}(\omega) = \Gamma(1 - i\omega) \\ |\hat{H}(\omega)|^2 &= \frac{\pi\omega}{\sinh \pi\omega} \end{aligned}$$

The isometry of the Fourier transform, i.e., $\|\hat{F}\|_2 = \|F\|_2$, together with equations (3.14), (3.15) and the inequality $(*)$ immediately imply

$$\begin{aligned} \|F - F_\alpha\|_2 &= \|\hat{F} - \hat{F}_\alpha\|_2 \\ &= \left\| \frac{\hat{H}(-\omega)\hat{G}(\omega)}{\alpha|\omega|^2 + |\hat{H}(\omega)|^2} - \frac{\hat{G}(\omega)}{\hat{H}(\omega)} \right\|_2 \\ &= \left\| \frac{\alpha|\omega|^2 \hat{G}(\omega)}{\hat{H}(\omega)(\alpha|\omega|^2 + |\hat{H}(\omega)|^2)} \right\|_2 \\ &= \left[\int_{-\infty}^{\infty} \frac{\alpha^2|\omega|^4 \, |\hat{G}(\omega)|^2 \, d\omega}{|\hat{H}(\omega)|^2 \, (\alpha|\omega|^2 + |\hat{H}(\omega)|^2)^2} \right]^{1/2} \end{aligned}$$

$$
\begin{aligned}
&\leq \left[\int_{-\infty}^{\infty} \frac{\alpha^2|\omega|^4\,|\hat{G}(\omega)|^2\,d\omega}{2\alpha|\omega|^2\,|\hat{H}(\omega)|^4}\right]^{1/2}\\
&= \left[\int_{-\infty}^{\infty} \frac{\alpha\sinh^2\pi\omega\,|\hat{G}(\omega)|^2\,d\omega}{2\pi^2}\right]^{1/2}\\
&\leq \frac{E}{2\sqrt{2\pi}}\cdot\sqrt{\alpha}\\
&\to 0 \quad \text{as} \quad \alpha\to 0
\end{aligned}
$$

3.4 Approximation of the Regularized Solution F_α

Our main objective now is to invert the Fourier transform $\hat{F}_\alpha$ in order to obtain the desired F_α, but since the Fourier inverse of $\hat{F}_\alpha$ is not readily tabulated, neither easy to be found analytically by complex analysis procedures or otherwise, one can only attempt to solve this problem numerically.

Hermite Polynomials

It is quite natural to think of Hermite polynomials, when attempting to approximate functions in $L^2(\mathcal{R})$ being used in our Fourier transformations. For this, let us recall the following useful properties of the hermite polynomials [69,32]. The Hermite polynomials H_n $(n = 1, 2, \ldots)$ are defined by

$$
\begin{aligned}
&H_n(x) = e^{x^2}\tfrac{d^n e^{-x^2}}{dx^n} \qquad \text{for } x \in \mathcal{R}\\
&H_0(x) = 1\\
&H_n(x) = (-1)^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{n!(2x)^{n-2k}}{k!(n-2k)!}
\end{aligned}
\tag{3.17}
$$

Remark 3.4.1 Let $\varphi_n = d_n\, e^{-\frac{x^2}{2}}\, H_n(x)$, where $d_n = \dfrac{1}{\sqrt{2^n\, n! \sqrt{\pi}}}$. Then the set $\{\varphi_n\}$ constitutes a complete orthonormal set in $L^2(\mathcal{R})$. Furthermore, these functions φ_n are the eigenfunctions of the Fourier transform operator and we have

$$\hat{\varphi}_n = \sqrt{2\pi}\, i^n\, \varphi_n \tag{3.18}$$

For a detailed discussion of the proofs see [32]. Now, for the numerical approximation of $\hat{F}_\alpha$, set

$$\hat{F}_\alpha(\omega) = \Gamma(1 - i\omega) \cdot S_\alpha(\omega) \cdot \hat{G}(\omega), \tag{3.19}$$

where

$$S_\alpha(\omega) = \frac{1}{\alpha\omega^2 + \pi\omega/\sinh \pi w}.$$

Remark 3.4.2 The functions S_α, G, Γ are all $L^2(\mathcal{R})$ functions.

Now, consider the following approximation for the function S_α.

$$S_{\alpha,N}(\omega) = \sum_{n=0}^{N} a_n \varphi_n$$

where,

$$a_n = \int_{-\infty}^{\infty} S_\alpha(\lambda)\varphi_n(\lambda)d\lambda \tag{3.20}$$

Notice that $a_n = 0$ for n odd, hence

$$S_{\alpha,N}(\omega) = \sum_{n=0}^{[\frac{N}{2}]} a_{2n}\varphi_{2n}(\omega). \tag{3.21}$$

This in turn yields the approximation $\hat{F}_{\alpha,N}$ for the regularized solution $\hat{F}_\alpha$

$$\hat{F}_{\alpha,N}(\omega) = S_{\alpha,N}(\omega)\Gamma(1 - i\omega)\, \hat{G}(\omega). \tag{3.22}$$

Assumption 3.4.1. Let the hypothesis of theorem (3.3.1) be satisfied and assume that

$$\sup_{\omega} \left|\hat{G}(\omega)\right| < \infty.$$

Remark 3.4.3

(i) $\sup_{\omega} |\Gamma(1 - i\omega)| = 1.$

(ii) $\left(e^{-x^2/2}\right)^{\wedge}(\omega) = \sqrt{2\pi}\, e^{-\frac{\omega^2}{2}}.$

Now, the following theorem will imply the convergence of our scheme defined by (3.22)

Theorem 3.4.1. *Let the hypothesis of assumption (3.4.1) be satisfied. Then the approximation $F_{\alpha,N}$ defined by (3.22) will converge in the $L^2(\mathcal{R})$ sense to the regularized solution F_α defined by (3.19).*

Proof:

$$\begin{aligned}
\|F_{\alpha,N} - F_\alpha\|_2 &= \left\|\hat{F}_{\alpha,N} - \hat{F}_\alpha\right\|_2 \\
&= \left\|S_{\alpha,N}(\omega) \cdot \Gamma(1 - i\omega)\hat{G}(\omega) - S_\alpha(\omega) \cdot \Gamma(1 - i\omega)\hat{G}(\omega)\right\|_2 \\
&= \left\|(S_{\alpha,N}(\omega) - S_\alpha(\omega))\, \Gamma(1 - i\omega)\hat{G}(\omega)\right\|_2 \\
&\leq \|S_{\alpha,N} - S_\alpha\|_2 \sup_{\omega \in R} \left|\hat{G}(\omega)\right| \\
&\rightarrow 0 \quad \text{as} \quad N \rightarrow \infty.
\end{aligned}$$

Now, theorem (3.3.1) and (3.4.1) together with the triangle inequality easily imply the following

Corollary 3.4.1. $\|F - F_{\alpha,N}\|_2 \to 0$ *as* $\alpha \to 0, \quad N \to \infty.$

Now, for computational purposes recall the following:

Remark 3.4.4. Given $\hat{F}(\lambda) = K(\lambda)\,\hat{G}(\lambda)$, where K is an entire function, we can directly recover $F(x)$ via the inversion formula

$$F(x) = K(iD)G(x) \tag{3.23}$$

where $i = \sqrt{-1}$ and $D = \frac{d}{dx}$.

Theorem 3.4.2. *For the approximation scheme, defined by equation (3.22), we may recover* $F_{\alpha,N}$ *by the formula*

$$F_{\alpha,N}(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{[\frac{N}{2}]} a_{2n} H_{2n}(iD) \cdot [U * V * G](x)$$

where $U(x) = e^{-\frac{x^2}{2}}, \quad V(x) = e^{-x}\, e^{-e^{-x}}.$

Proof: Recall that $\Gamma(1 - i\omega) = \hat{V}(\omega)$ and

$$e^{-\frac{\omega^2}{2}} = \frac{1}{\sqrt{2\pi}}\hat{U}(\omega)$$

$$\begin{aligned} \hat{F}_{\alpha,N}(\omega) &= S_{\alpha,N}(\omega) \cdot \Gamma(1 - i\omega)\hat{G}(\omega) \\ &= \left(\sum_{n=0}^{[\frac{N}{2}]} a_{2n} \varphi_{2n}(\omega) \right) \cdot \Gamma(1 - i\omega) \cdot \hat{G}(\omega) \\ &= \left(\sum_{n=0}^{[\frac{N}{2}]} a_{2n} H_{2n}(\omega) \right) \cdot e^{-\frac{\omega^2}{2}} \cdot \Gamma(1 - i\omega) \cdot \hat{G}(\omega) \end{aligned}$$

$$= \left(\sum_{n=0}^{[\frac{N}{2}]} a_{2n} H_{2n}(\omega) \right) \cdot \frac{1}{\sqrt{2\pi}} \hat{U}(\omega) \cdot \hat{V}(\omega) \cdot \hat{G}(\omega)$$

Hence, the convolution theorem immediately implies

$$\hat{F}_{\alpha,N}(\omega) = \frac{1}{\sqrt{2\pi}} \left(\sum_{n=0}^{[\frac{N}{2}]} a_{2n} H_{2n}(\omega) \right) [U * V * G]^{\wedge}(\omega)$$

Now, remark (3.4.4) implies

$$F_{\alpha,N}(x) = \frac{1}{\sqrt{2\pi}} \left(\sum_{n=0}^{[\frac{N}{2}]} a_{2n} H_{2n}(iD) \right) (U * V * G)(x)$$

where,

$$H_{2n}(iD) = \sum_{k=0}^{n} (-1)^{n-2k} \frac{(2n)!(4)^{n-k} D^{2(n-k)}}{k!(2n-2k)!}. \tag{3.24}$$

This finally implies

$$F_{\alpha,N}(x) = \left[\frac{1}{\sqrt{2\pi}} \left\{ \sum_{n=0}^{[\frac{n}{2}]} a_{2n} \left(\sum_{k=0}^{n} \frac{(-1)^{n-2k} 2n!(4)^{n-k}}{k!(2n-2k)!} \frac{d^{2(n-k)} U(t)}{dt^{2(n-k)}} \right) \right\} * V(t) * G(t) \right] (x)$$

Remark 3.4.5. Recall that to obtain the approximation for the original function f we need to set

$$f_{\alpha,N}(t) = F_{\alpha,N}(-\log t).$$

<u>Algorithm:</u>

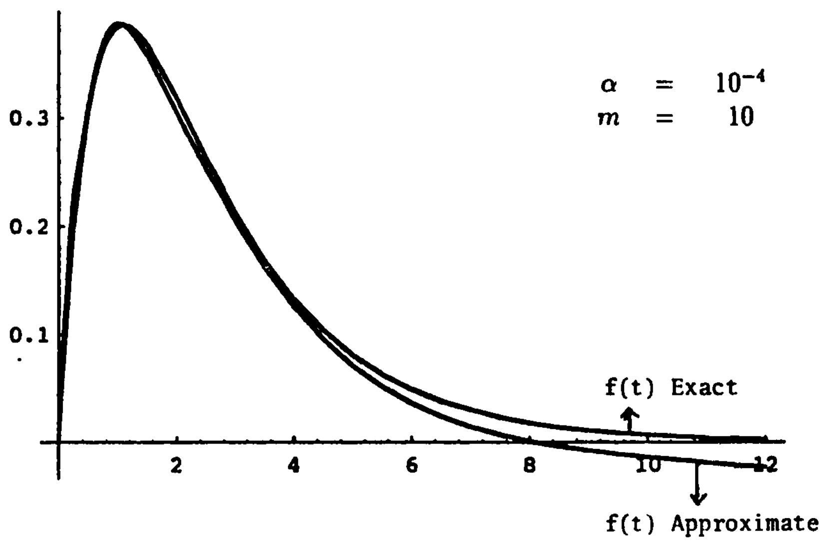
```
ClearAll[K,m,r,f,y,G,H,S,c,d,W,W1,A,AA,A1,GG,GG2,GG3,Y,HH1,P,PP,HH,HF,fa]
K:=10
m :=20
r :=N[.001,K]
f[x_] :=N[Exp[-x]  x,K]
y[s_] :=N[1./((s+1.)^2),K]
G[s_] :=N[Exp[s] y[Exp[s]],K]
H[u_] := H[u]=N[Exp[-u-Exp[-u]],K]
S[x_] :=N[Limit[N[1./(r (y^2) + (Pi y)/Sinh[Pi y]),K],y->x],K]
c[n_] :=N[Sqrt[1./(Sqrt[Pi](2^n)(n!))],K]
d[x_,n_] := d[x,n] = Dt[Exp[-(x^2)/4],{x,n}]
a[n_]:=a[n]=N[c[n] NIntegrate[Exp[-(x^2)/2] S[x]((-1)^n) HermiteH[n,x],{x,-10,-5,5,10}],K]
b[n_]:=b[n]=N[c[n] NIntegrate[Exp[-(x^2)/2] ((-1)^n)HermiteH[n,x] H[x],{x,-10,-5,5,10}],K]
g[n_]:=g[n]=N[c[n] NIntegrate[Exp[-(x^2)/2]((-1)^(n)) HermiteH[n,x] G[x],{x,-10,-5,5,10}],K]
Do[a[2n],{n,0,Floor[m/2]}]
Do[b[n],{n,0,m}]
Do[g[n],{n,0,m}]
Clear[p]
P[x_] :=N[(Sum[c[2 n]((-1)^(2 n)) a[2 n] HermiteH[2 n,I x],{n,0,Floor[m/2]}])(Sum[c[n]((-1)^n)
PP[x_] =N[Expand[P[x]],K]
A1 :=N[Limit[PP[x],x->0],K]
W1[n_]:=W1[n]=N[Coefficient[PP[x],x^n],K]
GG2[x_] :=Sum[W1[n]d[x,n],{n,1,2 m}]+A1 d[x,0]
GG3[x_] =Expand[Exp[(x^2)/4]GG2[x]]
HH1[x_] =N[Expand[Sum[c[n] g[n] ((-1)^n)HermiteH[n,x],{n,0,m}]],K]
Y[x_,u_] :=Expand[HH1[u-x] GG3[x]]
A[u_] =N[Limit[Y[x,u],x->0],K]
Clear[W]
W[n_]:=W[n]=N[Coefficient[Y[x,u],x^n],K]
```

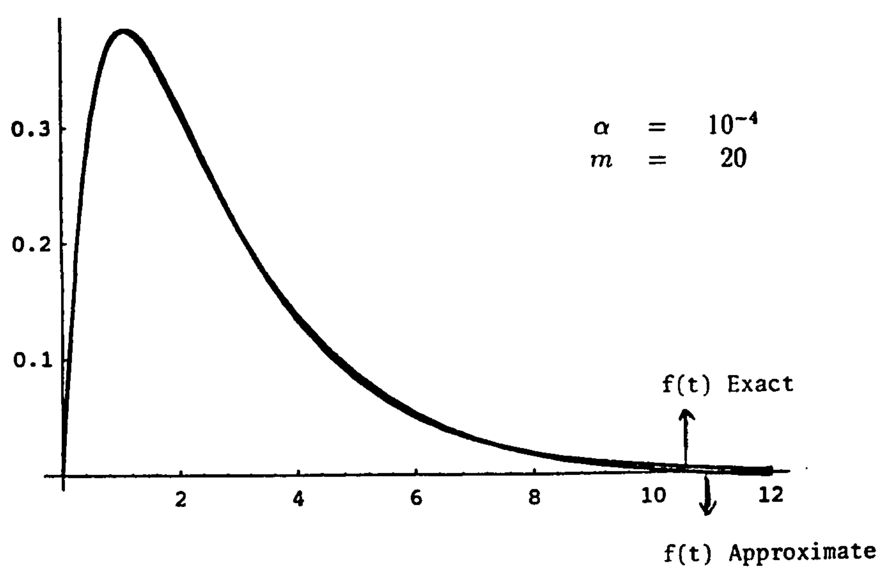
```
AA[u_,0_] :=N[Sqrt[2 Pi/3],K]
HH[u_,n_] :=N[(n!)(N[Sqrt[2 Pi/3],K])((2 u/3)^n)Sum[((3/(4 (u^2)))^k)/(((n-2 k)!)(k!)),{k,0,Fl
HF[u_] :=N[Sum[HH[u,n]W[n],{n,1,3 m}]+AA[u,0]A[u],K]
fa[u_] =N[Exp[-(u^2)/6]Expand[HF[u]],K]
Plot[{fa[-Log[u]],f[u]},{u,.01,10}]
```

Example 1:

$$F(s) = \frac{1}{(s+1)^2}$$

$$f(t) = te^{-t}$$

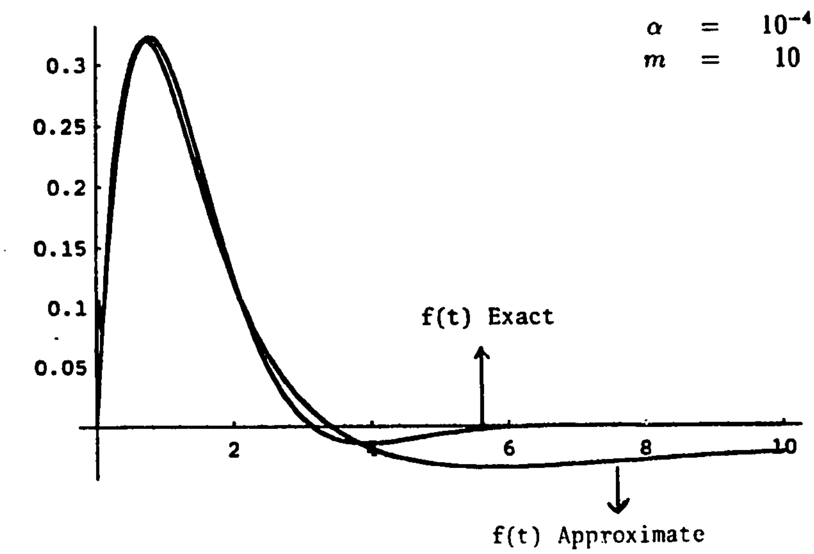


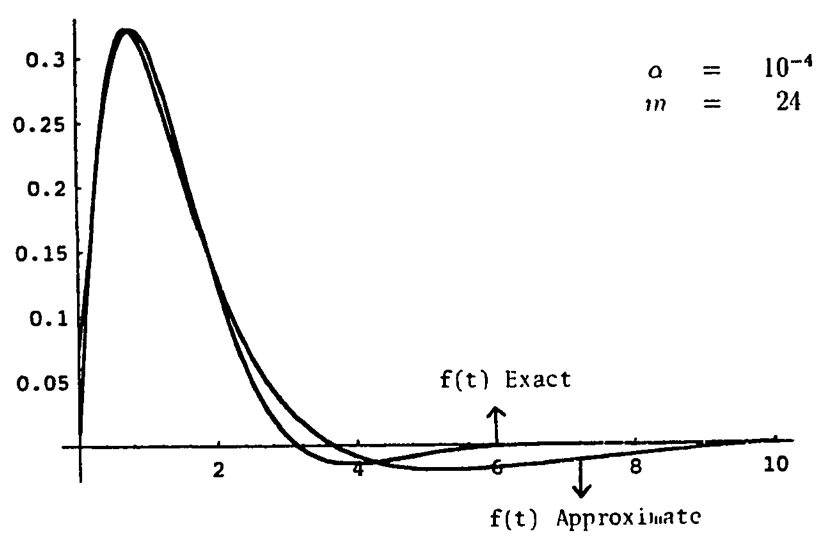


Example 2:

$$F(s) = \frac{1}{(s+1)^2+1}$$

$$f(t) = e^{-t}\sin t$$

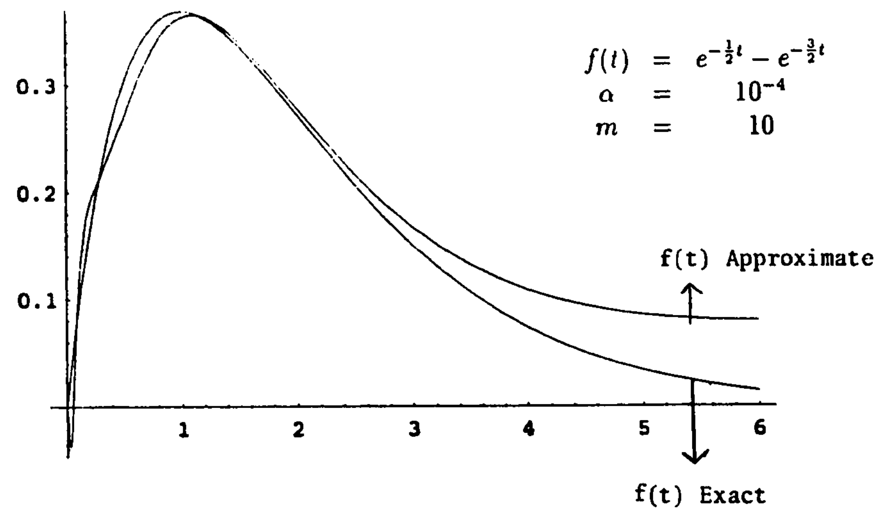


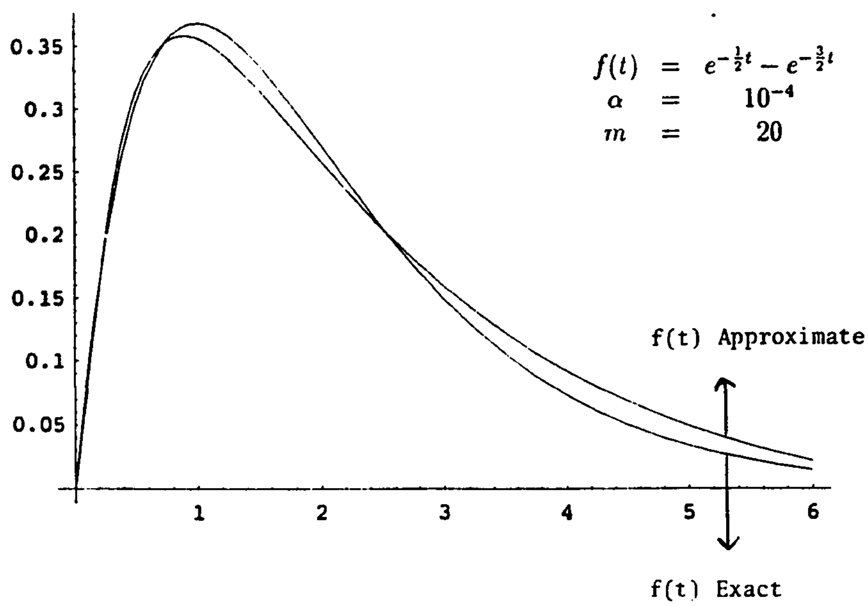


Example 3:

$$F(s) = \frac{1}{(s+\frac{1}{2})(s+\frac{1}{2})}$$

$$f(t) = e^{-t/2} - e^{-3t/2}$$





3.5 Conclusion

A new technique of numerically treating the Tikhonov regularization operators $\hat{F}_\alpha$ has displayed good results. But the slow convergence of the regularizing scheme was apparent. This suggests a more efficient programming algorithm, and/or finding an optimal α.

CHAPTER 4

A SPECTRAL REPRESENTATION METHOD FOR THE INVERSION OF THE LAPLACE TRANSFORM

4.1 Introduction

In this chapter, we shall study the spectral theory of the Laplace transform seen as an operator $\mathcal{L}$ acting on the Hilbert space $L^2[0,\infty)$ by finding its eigenvalues and their corresponding eigenfunctions, and taking advantage of the fact that its kernel is symmetric, and hence spectral analysis applies smoothly

$$\begin{aligned} \mathcal{L}(f(t))(s) &= \textstyle\int_0^\infty e^{-st} f(t)dt = g(s) \\ \mathcal{L}f &= g \end{aligned} \tag{4.1}$$

with its eigenvalue equation

$$\mathcal{L}f = \lambda f \tag{4.2}$$

Hildebrand [31] and Widder [81], have shown that equation (4.2) has solutions of the form

$$y(x,a) = \sqrt{\Gamma(1-a)}\, x^{a-1} \mp \sqrt{\Gamma(a)}\, x^{-a} \tag{4.3}$$

with corresponding eigenvalues $\lambda(a) = \mp\sqrt{\Gamma(a)\Gamma(1-a)}$ respectively for $0 < a < 1$.

It follows that all values of λ in the interval $(-\infty, -\sqrt{\pi}] \cup [\sqrt{\pi}, \infty)$ are spectral values. But if these are the only spectral values of the operator $\mathcal{L}$, then we immediately conclude that $\mathcal{L}^{-1}$ (which is known to exist) is bounded, contradicting the remark (3.1.1).

This suggests and motivates us to do more investigation on the nature of the spectrum and resolvent of the operator $\mathcal{L}$ and, therefore, will be our main concern in this chapter.

In the sequel, we intend to carry our work as follows:

1. For a more convenient and easier analysis we shall first switch our work into the Stieltjes transform $\mathcal{L}^2$ obtained by iterating the Laplace transform $\mathcal{L}$. Then we consider the eigenvalue equation

$$\mathcal{L}^2 f = \rho f \tag{4.4}$$

2. We use a result found in Hardy and Titchmarsh [78] to obtain all solutions of (4.4) in a direct and constructive way, where one does not need to guess solutions and verify them as done by Hildebrand and Widder.

3. We classify these solutions as they belong to different parts of the spectrum.

4. We show that the spectrum is simple, continuous and bounded.

5. As a consequence we shall derive the spectral representation associated with the Laplace transform, by computing its spectral measure explicitly.

6. This in turn gives a real inversion of the Laplace transform.

In order to study the real inversion of the Laplace transform, we shall restrict it to an operator mapping $L^2[0,\infty)$ into $L^2[0,\infty)$.

Then we shall try to find all eigenfunctions or solutions of

$$\mathcal{L}y \equiv \int_0^\infty e^{-st} y(t)dt = \lambda y(s). \tag{4.5}$$

As known in case the spectrum is continuous the solutions are not in the Hilbert space $L^2_{[0,\infty)}$. For this reason we shall not restrict the solutions to the Hilbert space $L^2_{[0,\infty)}$, only. The only property we can require is the solutions must be smooth, due to the smoothness of the kernel defining the operator $\mathcal{L}$, i.e. e^{-st}. Observe that if the equation $\mathcal{L}y = \lambda y$ holds then $\mathcal{L}^2 y = \lambda^2 y$ will also hold formally, where $\mathcal{L}^2$ is the Stieltjes transform defined by

$$\mathcal{L}^2(y(x))(s) = \int_0^\infty \frac{y(x)}{s+x} dx. \tag{4.6}$$

Equation (4.6) shall be the starting point of our discussion. We shall obtain all solutions of (4.6) and then by using a result on the spectrum of integral operators we shall discard all " bad eigenfunctions". By eigenfunctions we shall mean all solutions of $\mathcal{L}y = \lambda y$ such that $\lambda \in \sigma$, where σ is the spectrum of $\mathcal{L}$.

4.2 The Spectrum of $\mathcal{L}^2$

In order to solve equation (4.6) we shall make a change of variable so as to obtain an equation of convolution type. Setting

$$\phi(x) \equiv e^{\frac{x}{2}} y(e^x) \tag{4.7}$$

where $x \in (-\infty, +\infty)$, we have the operator A defined by

$$A : L^2(\mathcal{R} \to L^2(\mathcal{R})$$

$$A\phi = \int_{-\infty}^{\infty} \frac{\phi(x)}{2\cosh\left(\frac{x-s}{2}\right)}\, dx$$

To classify the spectrum of A, we must solve the following eigenvalue equation;

$$A\phi = \rho\phi$$

$$\phi(s) = \int_{-\infty}^{\infty} \frac{\phi(x)}{2\rho\cosh(\frac{x-s}{2})}\, dx \tag{4.8}$$

To solve this equation, we use a result proved by Titchmarsh [78]. Define the Fourier transform of k denoted $K(\omega)$

$$K(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k(t) e^{i\omega t} dt.$$

Result: Let $0 < c < c'$, and let $e^{c'|x|}k(x)$ belong to $L^1(-\infty, \infty)$ and $e^{-c|x|}\phi(x)$ belong to $L^2(-\infty, \infty)$. Then, if $\phi(x)$ satisfies

$$\phi(s) = \int_{-\infty}^{\infty} k(x-s)\phi(x) dx$$

it is of the form,

$$\phi(x) = \sum_{\nu} \sum_{p=1}^{q} C_{\nu,p} x^{p-1} e^{-iw_\nu x}$$

where w_ν runs through all zeros of $1 - \sqrt{2\pi}K(w)$ such that $|Im(w_\nu)| \leq c$. The $C_{\nu,p}$ are constants. q is the order of multiplicity of the zero w_ν and $K(w)$ is the Fourier transform of k.

It is not difficult to show that c' in the above theorem can not exceed $\frac{1}{2}$, for the function $e^{c'|x|}k(x)$ to be in $L^1(-\infty,\infty)$. Hence, we must have $0 < c < c' < \frac{1}{2}$. Our kernel $k(x) = \frac{1}{2\rho \cosh \frac{x}{2}}$ and its Fourier transform is $K(w) = \frac{\sqrt{\pi}}{\rho\sqrt{2}\cosh \pi w}$.

Now, set $1 - \sqrt{2\pi}K(w) = 0$. Then

$$w = \frac{\pm Ln[\frac{\pi}{\rho} + \sqrt{\frac{\pi^2}{\rho^2} - 1}]}{\pi}. \tag{4.9}$$

are its zeros, each with multiplicity 1. We require that $|\mathrm{Im}(w_\nu)| \leq c < \frac{1}{2}$ for the Theorem of Titchmarsh to apply.

Remark (4.1): That there is no other solution of any kind, i.e., for $c \geq \frac{1}{2}$, is shown by Hardy and Titchmarsh [78].

Remark (4.2): If we put $\rho = \pm i$ in (4.9), i.e., solving the equations $A\phi = \pm i\phi$, then we have $|Im(w_\nu)| = \frac{1}{2} > c$ and hence we have no solutions of any kind. This implies that the deficiency spaces $\mathcal{N}_i$ and $\mathcal{N}_{-i}$ defined by $L^2(-\infty,\infty) - R(A + iI)$ and $\mathcal{L}^2(-\infty,\infty) - R(A - iI)$ respectively, have zero dimensions. Hence the deficiency index of A is $(0,0)$. So the closure of A, denoted by $\overline{A}$ is a selfadjoint operator. This indeed gives a direct proof of the selfadjointness of $\overline{A}$.

Now that the operator $\overline{A}$ is selfadjoint, we need only to investigate the spectral values ρ on the real line, i.e., ρ is real.

Let $z = \frac{\pi}{\rho} + \sqrt{\frac{\pi^2}{\rho^2} - 1}$, and consider the following four cases:

i) $\rho = 0$.

The sequence of $L^2(-\infty,\infty)$ functions ψ_n

$$\psi_n(x) = 1 \quad n \le x \le n+1$$
$$= 0 \qquad \text{otherwise}$$

certainly have $\|\psi_n\| = 1$ and $\|A\psi_n\| \to 0$. Hence $\rho = 0$ is a point of the continuous spectrum.

ii) $-\infty < \rho < 0$.

Then, $z = -\frac{\pi}{|\rho|} + i\sqrt{1-\frac{\pi^2}{\rho^2}}$, $|\arg z| > \frac{\pi}{2}$ and $|\mathrm{Im}\, w| > \frac{1}{2}$. Hence we have no solutions.

iii) $0 < \rho \le \pi$.

Then, $z =$ positive real number, $\arg z = 0$ and $|\mathrm{Im}\, w| = 0$. Therefore, we have the solutions $\phi(s) = c_1 e^{i|w|s} + c_2 e^{-i|w|s}$, where c_1 and c_2 are arbitrary constants and $w = \dfrac{\pm Ln(\frac{\pi}{\rho} + \sqrt{\frac{\pi^2}{\rho^2} - 1})}{\pi}$. Also, $\rho = \frac{\pi}{\cosh \pi w}$.

iv) $\rho > \pi$.

Then,

$$z = \frac{\pi}{\rho} + i\sqrt{1 - \frac{\pi^2}{\rho^2}}, \quad |z| = 1, \ \arg z = \cos^{-1}\frac{\pi}{\rho} < \frac{\pi}{2}.$$

Hence, $w = \dfrac{\pm i \cos^{-1} \pi/\rho}{\pi}$ and $|\mathrm{Im}\, w| < \frac{1}{2}$. The solutions are $\phi(s) = c_3 e^{s|w|} + c_4 e^{-s|w|}$, with $\rho = \frac{\pi}{\cosh \pi w}$.

Notice that, the solutions in (iv) are those found in Hildebrand, as we shall see later.

The last case brings the question; which of these solutions are the eigenfunctionals? In other words it remains to classify these points as whether they belong to the spectrum or the resolvent of A. But before we do this, let us state the following results found in Pollard [61].

Let $k(x)$ be a function satisfying the following hypotheses:

1. $k(x)$ is real, measurable and even.

2. $k(x) \in L^1(-\infty, \infty)$.

3. $\int_{-\infty}^{\infty} |k(x)|\, dz \leq 1$.

4. $k(x) \in L^2(-\infty, \infty)$.

   Let $\tilde{K}(w) = \int_{-\infty}^{\infty} e^{-iwx} k(x) dx$.

Consider

$$Af(x) \equiv \int_R k(x-t) f(t) dt \tag{4.10}$$

Result 1: In order that λ belongs to the point spectrum of A it is necessary and sufficient that $\lambda - \tilde{K}(w)$ vanish on a set of positive measures.

Result 2: In order that λ belongs to the resolvent set of A it is necessary and sufficient that λ be different from zero and $\lambda - \tilde{K}(w)$ vanish nowhere.

Result 3: In order that λ belongs to the continuous spectrum of A it is necessary and sufficient that either (i) $\lambda = 0$ and $\tilde{K}(w) = 0$ on a set of at most measure zero, or (ii) $\lambda \neq 0$ and $\lambda = \tilde{K}(w)$ on a nonempty set of at most zero measure.

Now, we certainly have our kernel $k(x) = \frac{1}{2\cosh(\frac{x}{2})}$ of equation (4.9) satisfying the hypotheses mentioned above, with $\tilde{K}(w) = \int_{-\infty}^{\infty} \frac{e^{-iwx}}{2\cosh\frac{x}{2}}\, dx = \frac{\pi}{\cosh \pi w}$; hence, for $\rho > \pi$ we have $\rho - \check{K}(w) > 0$ and result 2 implies that ρ is in the resolvent set, where, for $0 < \rho \leq \pi$ we have by Result 3 that ρ belongs to the continuous spectrum.

As we are examining the spectrum of $\mathcal{L}$ we recall an interesting theorem by Gelfand and Shilov [72], which ensures our conclusion in the paragraph following result 3.

Theorem: [Shilov] *The generalized eigenfunctions $y(\lambda, x)$ of any self-adjoint operator A defined on the space $L^2(\mathcal{R})$ are derivatives (of the type $\partial^n/\partial x_1 \cdots \partial x_n$) of measurable functions which do not increase faster than $(1+|x|)^{3n/2+\epsilon}$, for arbitrary $\epsilon > 0$.*

Now, Hildebrand (as mentioned earlier) found that $y(x,a)$ defined by (4.3) solves the eigenvalue problem $\mathcal{L}y = \lambda y$, and hence solves $\mathcal{L}^2 y = \lambda y$. Then, with the same change of variables done in section (4.2), we can conclude that for $0 < a < 1$

$$\phi(x,a) = \Gamma(a)\, e^{(\frac{1}{2}-a)x} \mp \Gamma(1-a)\, e^{(a-\frac{1}{2})x}$$

are also solutions of equation (4.8)

$$A\phi = \rho\phi$$

But, Shilov theorem immediately excludes these solutions from our spectrum since they are derivatives of the functions $\tilde{\phi}$

$$\tilde{\phi}(x,a) = \frac{\Gamma(a)}{(\frac{1}{2}-a)}\, e^{(\frac{1}{2}-a)x} \pm \frac{\Gamma(1-a)}{(a-\frac{1}{2})}\, e^{(a-\frac{1}{2})x}$$

which obviously grows faster than $(1+|x|)^{\frac{3}{2}+\epsilon}$.

We then conclude that the spectrum of the operator A is continuous and it covers the interval $0 \leq \rho \leq \pi$.

Indeed, if we put $a = \frac{1}{2} + iw$ and notice that

$$\overline{\Gamma(a)} = \Gamma(\bar{a}) = \Gamma(1-a), \ |\Gamma(a)|^2 = \frac{\pi}{\cosh \pi w}.$$

Then the discussion in case (iii) implies that $\rho = |\Gamma(a)|^2$ is in the continuous spectrum of A with multiplicity 2, with the corresponding generalized eigenfunctions (eigenfunctionals) $\phi(s) = c_1(a)e^{-(1/2-a)s} \mp c_2(a)e^{(1/2-a)s}$, where c_1 and c_2 are arbitrary functions of a. By changing variables, i.e., put $s = Ln\,\eta$, $x = Ln\,\xi$, $y(s) = \frac{1}{\sqrt{\eta}}\phi(Ln\,\eta)$, then, we conclude that if ϕ_1 and ϕ_2 are solutions of (4.10). Then $y(s,\rho) = c_1(a)s^{-a} \mp c_2(a)s^{a-1}$ are solutions of

$$\rho y(s,\rho) = \int_0^\infty \frac{y(x,\rho)}{s+x} dx. \tag{4.11}$$

i.e., the operator $\mathcal{L}^2$ has a continuous spectrum covering $[0,\pi]$ and every spectral value $\rho(a) = |\Gamma(a)|^2 = \frac{\pi}{\cosh \pi w}$ has multiplicity 2, where $a = \frac{1}{2} + i\frac{\cosh^{-1}\frac{\pi}{\rho}}{\pi} = \frac{1}{2} + iw$.

Theorem: *The spectrum of $\mathcal{L}$ (acting on the space $L^2(\mathcal{R}_+)$) denoted σ is simple and continuous covering the interval $[-\sqrt{\pi}, \sqrt{\pi}]$. Moreover, the spectral values are $\lambda_1(a) = |\Gamma(a)| = \sqrt{\frac{\pi}{\cosh \pi w}}$ and $\lambda_2(a) = -|\Gamma(a)| = -\sqrt{\frac{\pi}{\cosh \pi w}}$ with the associated eigenfunctionals $y_1(x,\lambda) = \Gamma(a)x^{-a} + |\Gamma(a)|x^{a-1}$ and $y_2(x,\lambda) = \Gamma(a)x^{-a} - |\Gamma(a)|x^{a-1}$ respectively, where $a(\lambda) = \frac{\sin^{-1}\frac{\pi}{\lambda^2}}{\pi}$.*

Proof: For the equation $\mathcal{L}y = \lambda y$, one can directly verify that y_1 and y_2 are solutions. In fact these are the only solutions of $\mathcal{L}y = \lambda y$ for otherwise, $\mathcal{L}^2 y = \lambda y$ will have more than two independent solutions contradicting remark (4.1). Similarly if λ_1 or λ_2 has multiplicity greater than 1 then the eigenvalue ρ of $\mathcal{L}^2$ ($\rho = \lambda_1^2 = \lambda_2^2$) will have multiplicity greater than 2, a contradiction. Hence, our spectrum of $\mathcal{L}$ is continuous covering $[-\sqrt{\pi}, \sqrt{\pi}]$ and each eigenvalue is of multiplicity 1, hence the spectrum is simple.

We now have the eigenvalues and their corresponding generalized eigenfunctions for the operator $\mathcal{L}$ in hand, where $\mathcal{L}$ is bounded and selfadjoint. Hence we are in a position to start the construction of the spectral function.

4.3 The Spectral Analysis:

Since $\mathcal{L}$ is a self-adjoint operator with simple spectrum, there exists a nondecreasing function defined on the real line, μ say, and in this case the eigenfunctionals help define isometries

$$\left.\begin{aligned} &\wedge_1 : \mathcal{L}^2[0,\infty) \longrightarrow \mathcal{L}^2_{d\mu} \\ &\overset{\wedge_1}{f} = F(\lambda) = \textstyle\int_0^\infty f(x) y(x,\lambda)\,dx \\ &\overset{\wedge_1^{-1}}{F} = f(x) = \textstyle\int_{-\infty}^\infty F(\lambda)\,\overline{y(x,\lambda)}\,d\mu(\lambda). \end{aligned}\right\} \tag{4.12}$$

such that Parseval equality holds

$$(f,\psi) = \int_\sigma \widehat{f(\lambda)}^{1}\,\overline{\widehat{\psi(\lambda)}^{1}}\,d\mu(\lambda). \tag{4.13}$$

μ is called the spectral function associated with the operator $\mathcal{L}$. Recall that Parseval equality ensures that $\wedge_1$ is an isometry. This pair defines inverse isometric mapping of $L^2[0,\infty)$ onto $L^2_{d\mu}$ and of $L^2_{d\mu}$ onto the $L^2[0,\infty)$,

$$\mathcal{L}(f) = \int_{-\infty}^{\infty} \lambda F(\lambda)\, \overline{y(x,\lambda)}\, d\mu(\lambda) \Longleftrightarrow \lambda F(\lambda) = \int_0^{\infty} \mathcal{L}(f) y(x,\lambda) dx. \tag{4.14}$$

Formally we have for any $f \in C_0^\infty[0,\infty)$

$$\int_0^\infty e^{-xt} f(t)dt = \mathcal{L}(f) = \int_0^\infty f(t) \int_\sigma \lambda y(x,\lambda)\overline{y(t,\lambda)} d\mu(\lambda) dt,$$

So,

$$\text{the kernel of } \mathcal{L} = e^{-xt} = \int_\sigma \lambda y(x,\lambda)\overline{y(t,\lambda)} d\mu(\lambda).$$

We extensively rely on this formal equation together with the eigenfunctionals evaluated earlier to derive a formula for our spectral function μ.

$$\begin{aligned} e^{-xt} &= \int_{-\infty}^{\infty} \lambda y(x,\lambda)\overline{y(t,\lambda)} d\mu(\lambda) \\ &= \int_{-\sqrt{\pi}}^{\sqrt{\pi}} \lambda y(x,\lambda)\overline{y(t,\lambda)} d\mu(\lambda) \\ &= \int_0^{\sqrt{\pi}} \lambda \left[\Gamma(a)x^{-a} + |\Gamma(a)|x^{a-1}\right] \overline{[\Gamma(a)\, t^{-a} + |\Gamma(a)|t^{a-1}]} d\mu(\lambda) \\ &+ \int_{-\sqrt{\pi}}^{0} \lambda \left[\Gamma(a)x^{-a} - |\Gamma(a)|x^{a-1}\right] \overline{[\Gamma(a)t^{-a} - |\Gamma(a)|t^{a-1}]} d\mu(\lambda). \end{aligned}$$

Notice that $\lambda \equiv |\Gamma(a)| \equiv \left|\Gamma(\frac{1}{2} + iw)\right|$ and $a = \dfrac{1}{2} + i\dfrac{\sin^{-1}\frac{\pi}{\lambda^2}}{\pi}$ is an even function of λ. Changing λ to $-\lambda$ in the second integral of the above equation and assuming that μ is odd yields

$$e^{-xt} = \int_0^{\sqrt{\pi}} |\Gamma(a)|^2 \left[\Gamma(a)(xt)^{-a} + \Gamma(1-a)(xt)^{a-1}\right] \{\lambda d\mu(\lambda) + \lambda d\mu(\lambda)\}$$

$$e^{-xt} = -2\int_0^\infty |\Gamma(\tfrac{1}{2}+iw)|^2 \left[\Gamma(\tfrac{1}{2} + iw)(xt)^{-1/2-iw} + \Gamma(\tfrac{1}{2} - iw)(xt)^{-1/2+iw}\right] (d\mu\lambda(w)).$$

Putting $xt = e^s$ and changing w to $-w$ in the second piece of the integral yields,

$$e^{s/2}e^{-e^s} = -2\int_{-\infty}^{\infty} |\Gamma(\tfrac{1}{2} + i|w|)|^2 \Gamma(\tfrac{1}{2} + iw) \cdot e^{-iws} d\mu(\lambda(w)).$$

Using the inverse Fourier transform and the fact that

$$\int_{-\infty}^{\infty} e^{iws} e^{1/2s} e^{-e^s} ds = \Gamma(\tfrac{1}{2} + iw).$$

We finally get the formula for our spectral function μ,

$$\mu'(\lambda(w)).\lambda'(w) = \frac{1}{-4\pi|\Gamma(\frac{1}{2} + iw)|^2}.$$

Then,

$$\mu'(\lambda) = -\frac{1}{4\pi\lambda^2\lambda'}, \qquad \lambda > 0$$

Notice that,

$$\lambda' = \frac{d\lambda}{dw} = \{\sqrt{\frac{\pi}{\cosh(\pi w)}}\}',$$

hence,

$$\mu'(\lambda) = \frac{1}{2\pi\,\mathrm{sgn}(\lambda)\lambda^3\sqrt{\pi^2 - \lambda^4}} \qquad \text{for all} \quad \lambda \in \sigma.$$

4.4 The Inversion Formula:

Recall that $\mu(\lambda)$ is the spectral function for $\mathcal{L}$ where $\lambda(a) = \pm|\Gamma(a)| = \pm\sqrt{\frac{\pi}{\cosh \pi w}}$, $a = \frac{1}{2} + iw$ and w is real, $-\infty < w < \infty$. With these facts we can prove the following theorem.

Lemma (4.4.1) $\mathcal{L}^{-1} = \Lambda_1^{-1}\left(\frac{1}{\lambda}\right)\Lambda_1$

Proof: Using the definition of Λ_1 and Λ_1^{-1} in equations (4.12) and (4.13) together with the isometries defined by equation (4.14), we have

$$\mathcal{L}^{-1}(g(s))(x) \;=\; f(x)$$

$$= (F(\lambda))^{\wedge_1^{-1}}(x)$$
$$= \left(\frac{1}{\lambda}(\mathcal{L}f)^{\wedge_1}\right)^{\wedge_1^{-1}}$$

hence,

$$\mathcal{L}^{-1} = \wedge_1^{-1}\left(\frac{1}{\lambda}\right)\wedge_1.$$

The Theorem of Inversion: *Let $g(s)$ be a given $L^2[0,\infty)$ function representing the Laplace transform of an $L^2[0,\infty)$ function $f(x)$. Then, f can be recovered from g by the formula*

$$f(x) = \frac{1}{2\pi}\int_0^\infty g(s)\int_{-\infty}^\infty \frac{(xs)^{-a}}{\Gamma(1-a)}dw\,ds$$

where, $a = \frac{1}{2} + iw$.

Proof: We similarly manipulate the integrals as done in the earlier discussions, and we formally switch integrals (this will be justified in chapter V).

$$\begin{aligned}
f(x) &= \mathcal{L}^{-1}(g(s))(x) = \wedge_1^{-1}(\frac{1}{\lambda})\overset{\wedge_1}{g}(s) = \wedge_1^{-1}(\frac{1}{\lambda})\int_0^\infty g(s)y(s,\lambda)ds \\
&= \wedge_1^{-1}\int_{-\infty}^\infty \frac{1}{\lambda}g(s)y(s,\lambda)ds \\
&= \int_{-\infty}^\infty \overline{\int_0^\infty \frac{1}{\lambda}g(s)y(s,\lambda)ds}\,y(x,\lambda)d\mu(\lambda) \\
&= \int_0^\infty g(s)\int_{-\infty}^\infty \frac{1}{\lambda}\overline{y(s,\lambda)}y(x,\lambda)d\mu(\lambda)ds \\
&= \int_0^\infty g(s)\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \frac{1}{\lambda^2}\left[\lambda\,\overline{y(s,\lambda)}y(x,\lambda)\right]d\mu(\lambda)ds \\
&= \frac{1}{2\pi}\int_0^\infty g(s)\int_{-\infty}^\infty \frac{(xs)^{-a}}{\Gamma(1-a)}dw\,ds.
\end{aligned}$$

Corollary: *Let $g(s)$ belong to $L^2[0,\infty)$ and represent the Laplace transform of an $L^2(0,\infty)$ function. Let g_M be its Mellin transform. Then we can recover g from*

g_M by the Mellin inversion formula

$$g(s) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} g_M(a)s^{-a}da.$$

Proof: $a = \frac{1}{2} + iw, \quad -\infty < w < \infty.$

According to the previous theorem,

$$\begin{aligned}
\mathcal{L}(f(x))(\eta) &= g(\eta) = \frac{1}{2\pi}\int_0^{\infty} g(s)\int_{-\infty}^{\infty} s^{-a}\eta^{a-1}dw\,ds. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta^{a-1}\int_0^{\infty} g(s)s^{-a}ds\,dw. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta^{a-1}\int_0^{\infty} g(s)s^{-a}ds\,d\,[-i(a-\frac{1}{2})]. \\
&= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\eta^{-a}\int_0^{\infty} g(s)s^{a-1}ds\,da. \\
&= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\eta^{-a}g_M(a)da.
\end{aligned}$$

Example: Consider the $L^2[0,\infty)$ function $f(t) = e^{-t}$, its Laplace transform $g(s) = \frac{1}{s+1}$.

Now, let us use our formula to recover f.

$$f(x) = \frac{1}{2\pi}\int_0^{\infty}\frac{1}{s+1}\int_{-\infty}^{\infty}\frac{(xs)^{-a}}{\Gamma(1-a)}dw\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{x^{-a}}{\Gamma(1-a)}\int_0^{\infty}\frac{s^{-a}}{s+1}dsdw.$$

Put $s = e^{\xi}$ and $x = e^{\eta}$. Then,

$$\begin{aligned}
f(e^{\eta}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-(1/2+iw)\eta}}{\Gamma(\frac{1}{2}-iw)}\int_{-\infty}^{\infty}\frac{e^{(1/2-iw)\xi}}{1+e^{\xi}}d\xi\,dw. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-(1/2+iw)}\eta}{\Gamma(\frac{1}{2}-iw)}\cdot\frac{\pi}{\cosh\pi w}dw. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-(1/2+iw)\eta}}{\Gamma(\frac{1}{2}-iw)}\Gamma(\frac{1}{2}+iw)\Gamma(\frac{1}{2}-iw)\,dw. \\
&= \frac{e^{-\eta/2}}{2\pi}\int_{-\infty}^{\infty}\Gamma(\frac{1}{2}+iw)e^{-iw\eta}\,dw.
\end{aligned}$$

$$= \frac{e^{-\eta/2}}{2\pi}\left[2\pi e^{\eta/2} e^{-e^{\eta}}\right].$$
$$= e^{-e^{\eta}}$$

Hence, $f(t) = e^{-t}$.

CHAPTER 5

DIRECT SPECTRAL ANALYSIS OF THE LAPLACE TRANSFORM

5.1 Introduction

By comparing the Laplace transform with the differential operator, that is comparing two self-adjoint operators [6], we shall obtain $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$, where $\mathcal{L}$ is the Laplace transform and V is a change of variable. This will help us deduce an explicit spectral representation of the Laplace transform. Using this fact, we shall derive a simple real inversion method to compute the inverse Laplace transform.

It is known that the range of the Laplace transform is in the space of analytic functions. Therefore the domain of the inverse Laplace transform is contained in the set of entire functions on the right half plane. Recall that operators acting on the space of entire functions can be represented by differential operators of infinite order. One way of verifying this result for the inverse Laplace transform is to use methods of spectral theory. For this we shall restrict the Laplace transform to an operator mapping $L^2[0,\infty)$ into $L^2[0,\infty)$.

$$\begin{aligned} \mathcal{L} : L^2[0,\infty) &\longrightarrow L^2[0,\infty) \\ y(x) &\longrightarrow \mathcal{L}(y(x))(s) \equiv \int_0^\infty e^{-sx} y(x)dx. \end{aligned} \tag{5.1}$$

This is an integral operator with a symmetric kernel, and we shall try to show that it is a selfadjoint operator in $L^2[0,\infty)$. It is clear that

$$\mathcal{L}^2(y(x))(s) = \int_0^\infty \frac{y(x)}{s+x} dx. \tag{5.2}$$

Thus $\mathcal{L}^2$ can be seen as composition of projections and the Hilbert transform. We now shall use comparison techniques to find a spectral representation of the Laplace operator. To this end consider the following transformation V,

$$\begin{aligned} V : L^2[0,\infty) &\longrightarrow L^2(-\infty,\infty) \\ y &\longrightarrow Vy(x) = e^{x/2} y(e^x). \end{aligned} \tag{5.3}$$

It is easy to show that V is a unitary transformation, i.e.,

$$VV^* = I, \qquad \text{and} \quad V^{-1} = V^*$$

and the inverse operator is defined by

$$\begin{aligned} V^{-1} : L^2(-\infty,\infty) &\longrightarrow L[0,\infty) \\ g(x) &\longrightarrow V^{-1}g(x) = \frac{1}{\sqrt{x}} g(Lnx). \end{aligned}$$

The study of $\mathcal{L}^2$, involves the operator defined by $A \equiv V\mathcal{L}^2V^{-1}$, that is

$$\begin{aligned} A : \quad & L^2(-\infty,\infty) \longrightarrow L^2(-\infty,\infty) \\ & y(x) \rightarrow Ay(x) \equiv k * y = \int_{-\infty}^\infty k(x-\eta)y(\eta)d\eta \end{aligned} \tag{5.4}$$

where, $k(x) = \frac{1}{2\cosh\frac{x}{2}}$. A is obviously an integral operator of the convolution type.

We shall denote the Fourier transform $\wedge$ and its inverse by $\wedge^{-1}$

$$\wedge : f \longrightarrow \hat{f}(\lambda) \equiv \int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$$

$$\wedge^{-1} : \hat{f}(\lambda) \longrightarrow f(x) \equiv \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{-i\lambda x}d\frac{\lambda}{2\pi}.$$

Theorem 5.1.1: *The operator A defined on $L^2(-\infty,\infty)$ by (5.4) is a bounded selfadjoint operator with $\|A\| = \pi$.*

Proof: It is easy to show that the Fourier transform of k is

$$\hat{k}(\lambda) = \frac{\pi}{\cosh \pi\lambda}. \tag{5.5}$$

and so $\hat{k}(\lambda)$ is a bounded function. Using the Parseval relation $\|f\| = \|\hat{f}\|$ and the fact that $\sup_{\lambda\in\mathcal{R}} |\hat{k}(\lambda)| \leq \pi$ we have for any $f \in L^2(-\infty,\infty)$;

$$\|Af\| = \|k * f\| = \|\hat{k}\hat{f}\| \leq sup|\hat{k}(\lambda)|\|\hat{f}\| \leq \pi\,\|\hat{f}\| = \pi\,\|f\|.$$

Hence; $D_A = L^2(-\infty,\infty)$, and $\|A\| \leq \pi$. In fact the equality holds. This is shown by choosing

$$f_n \quad \text{such that} \quad \hat{f}_n(\lambda) = \begin{cases} \sqrt{n} & \frac{-1}{2n} < \lambda < \frac{1}{2n} \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$\int_{-\infty}^{\infty} \left[\hat{f}_n(\lambda)\right]^2 d\lambda = 1 \text{ and } \operatorname{supp} \hat{f}_n(\lambda) \to \{0\}.$$

Hence,

$$\lim_{n\to\infty} \|Af_n\| = \lim_{n\to\infty} \sqrt{\int [\hat{k}(\lambda)]^2[\hat{f}_n(\lambda)]^2 d\lambda} = \lim_{n\to\infty} \sqrt{[\hat{k}(0)]^2} \equiv |\hat{k}(0)| = \pi.$$

Since A is bounded and symmetric it is a selfadjoint operator in $L^2(-\infty,\infty)$.

Corollary 2. *The operator $\mathcal{L}^2$ is a bounded selfadjoint operator and,*

$$\|\mathcal{L}^2\| = \pi$$

Proof: Follows from the fact that A is a bounded selfadjoint operator, see Theorem (5.1.1) and $\mathcal{L}^2$ is unitarily equivalent to A.

5.2 The Spectral Representation of $\mathcal{L}^2$:

Let us find the spectral function of $\mathcal{L}^2$. In order to do so, we only need to find the spectral function of A. It is known that the spectrum is invariant under a unitary transformation of the operator.

By using Parseval relation for the Fourier transform

$$(Af,\psi) \equiv (k * f,\psi) \equiv (\widehat{k * f},\hat{\psi}) \equiv (\hat{k}(\lambda)\hat{f}(\lambda),\hat{\psi}(\lambda))$$

where, $\hat{f}(\lambda) \equiv \int_{-\infty}^{+\infty} f(x)\, e^{i\lambda x}dx$. From this we deduce,

$$(Af,\psi) = \int_{-\infty}^{\infty} \hat{k}(\lambda)\hat{f}(\lambda)\,\overline{\hat{\psi}(\lambda)}\, d\frac{\lambda}{2\pi}. \tag{5.6}$$

Let $a(\lambda)$ be the inverse of $\hat{k}(\lambda)$ defined by

$$a(\lambda) \equiv -\frac{1}{\pi} ln[\frac{\pi}{\lambda} + \sqrt{\frac{\pi^2}{\lambda^2} - 1}]$$

Clearly

$$\frac{\pi}{\cosh \pi a(\lambda)} = \lambda, \quad a(\lambda) < 0 \quad \lambda > 0.$$

We have from (5.6)

$$(Af,\psi) = \int_0^\pi \lambda\hat{f}(a(\lambda))\cdot\overline{\hat{\psi}(a(\lambda))}\,d\frac{a(\lambda)}{2\pi} + \int_\pi^0 \lambda\hat{f}(-a(\lambda))\overline{\hat{\psi}(-a(\lambda))}\,d\frac{-a(\lambda)}{2\pi}$$
$$(Af,\psi) = \int_0^\pi \lambda\left[\left\{\hat{f}(a(\lambda)\cdot\overline{\hat{\psi}(a(\lambda))}\right\} + \left\{\hat{f}(-a(\lambda))\cdot\overline{\hat{\psi}(-a(\lambda))}\right\}\right]d\frac{a(\lambda)}{2\pi}.$$

The above equation can be written in the matrix form as

$$(Af,\psi) = \int_0^\pi \lambda\left[\hat{f}(a(\lambda)),\ \hat{f}(-a(\lambda))\right]\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi}\end{bmatrix}\overline{\begin{bmatrix}\hat{\psi}(a(\lambda)) \\ \hat{\psi}(-a(\lambda))\end{bmatrix}}. \tag{5.7}$$

This only means that $\begin{bmatrix} e^{ixa(\lambda)} \\ e^{-ixa(\lambda)}\end{bmatrix}$ is the eigenfunctional of $\mathcal{L}^2$, so the multiplicity is two, and the associated spectral matrix is $\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi}\end{bmatrix}$, see [6].

Proposition 5.2.1: *The spectral function of A is* $\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0 \\ 0 & d\frac{a(\lambda)}{2\pi}\end{bmatrix}$ *and the multiplicity is two and* $\sigma = \operatorname{supp} da(\lambda) = [0,\pi]$.

Now we use our relation $A = V\mathcal{L}^2V^{-1}$ to deduce the spectral representation for $\mathcal{L}^2$

$$(\mathcal{L}^2 f,\psi) = (V^{-1}AVf,\ \psi) = (AVf,\ V\psi) = (\lambda\widehat{Vf},\widehat{V\psi}).$$

Define the transform $\hat{}^2$;

$$\overset{\wedge 2}{f}(\lambda) = \int_0^\infty f(x)x^{-\frac{1}{2}+ia(\lambda)}\,dx$$

Then, we obtain the following relation between $\hat{}^2$ and $\hat{}$ by using the isometry V

$$\hat{f}^2(\lambda) = \int_0^\infty f(x)x^{-\frac{1}{2}+ia(\lambda))}\,dx = \int_{-\infty}^\infty Vf(x)\,e^{ixa(\lambda)}\,dx = \hat{Vf}\,(a(\lambda)). \qquad (5.8)$$

Hence, equation (5.7) implies;

$$\begin{aligned}(\mathcal{L}^2 f,\psi) &= (AVf,V\psi)\\ &= \int_0^\pi \left[\hat{Vf}\,(a(\lambda)),\ \hat{Vf}\,(-a(\lambda))\right]\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0\\ 0 & d\frac{a(\lambda)}{2\pi}\end{bmatrix}\overline{\begin{bmatrix}\hat{V\psi}\,(a(\lambda))\\ \hat{V\psi}\,(-a(\lambda))\end{bmatrix}}\\ &= \int_0^\pi \left[\hat{f}^2(\lambda),\ \overline{\hat{f}^2}(\lambda)\right]\begin{bmatrix} d\frac{a(\lambda)}{2\pi} & 0\\ 0 & d\frac{a(\lambda)}{2\pi}\end{bmatrix}\begin{bmatrix}\overline{\hat{\psi}^2(\lambda)}\\ \hat{\psi}^2(\lambda)\end{bmatrix}\end{aligned}$$

This, simply means that our spectrum is continuous of multiplicity two covering $[0,\pi]$, which is the support of the spectral matrix function $\mu(\lambda) = \begin{bmatrix}\frac{a(\lambda)}{2\pi} & 0\\ 0 & \frac{a(\lambda)}{2\pi}\end{bmatrix}$, and the corresponding eigenfunctionals are $\begin{bmatrix} x^{-\frac{1}{2}+ia(\lambda)}\\ x^{-\frac{1}{2}-ia(\lambda)}\end{bmatrix}$.

5.3 The Spectral Resolution of $\mathcal{L}$:

In what follows we shall consider the square root $\mathcal{L}^2$. For simplicity set $w(\lambda) \equiv -\frac{1}{2} + ia(\lambda^2)$.

Now, with the help of the eigenfunctionals of $\mathcal{L}^2$ we shall construct the eigenfunctionals of $\mathcal{L}$ as follows: it is easy to see that the following combination

$$y(x,\lambda) = \Gamma(1+w)x^{\bar{w}} + s(\lambda)|\Gamma(1+w)|x^{w} \qquad (5.9)$$

satisfy $\mathcal{L}y = \lambda y$ where, $s(\lambda) = \begin{cases} +1 & \lambda > 0 \\ -1 & \lambda < 0 \end{cases}$ and $\lambda = s(\lambda)|\Gamma(1 + w(\lambda))|$.

Define the transform $\overset{\wedge_1}{} : \overset{\wedge_1}{f}(\lambda) = \int_0^\infty f(x)y(x,\lambda)dx$ and recall that $\hat{f}^2 = \int_0^\infty f(x)x^\omega dx$. Observe that

$$\overset{\wedge_1}{f}(\lambda) = \Gamma(1+w)\cdot\overline{\overset{\wedge_2}{f}}(\lambda^2) + s(\lambda)|\Gamma(1+w)|\overset{\wedge_2}{f}(\lambda^2). \tag{5.10}$$

It is clear that the multiplicity of the spectrum of $\mathcal{L}$ is either 1 or two. We claim that equation (5.10) defines a complete system of eigenfunctionals and so the multiplicity is only one. Recall that a system of eigenfunctionals is complete if

$$\hat{f}^1(\lambda) = 0 \ \text{ for } \lambda \in \sigma \quad \Longrightarrow \quad f = 0 \ \text{ in } L^2[0,\infty).$$

Since both $\lambda \in \sigma$ and $-\lambda \in \sigma$ we have from equation (5.11)) we obtain the following system

$$\overset{\wedge_1}{f}(\lambda) = 0 \ = \ \Gamma(1+w)\cdot\overline{\overset{\wedge_2}{f}}(\lambda^2) + |\Gamma(1+w)|\overset{\wedge_2}{f}(\lambda^2) \tag{5.11}$$

$$\overset{\wedge_1}{f}(-\lambda) = 0 \ = \ \Gamma(1+w)\cdot\overline{\overset{\wedge_2}{f}}(\lambda^2) - |\Gamma(1+w)|\overset{\wedge_2}{f}(\lambda^2) \tag{5.12}$$

The determinant of the system being no zero means that $\overline{\overset{\wedge_2}{f}}(\lambda^2) = 0$ and $\overset{\wedge_2}{f}(\lambda^2) = 0$. That is f= 0.

This transition formula (5.11), together with Parseval relation implies for all $f, \psi \in L^2[0,+\infty)$.

$$(\mathcal{L}f, \mathcal{L}\psi) = (\mathcal{L}^2 f, \psi).$$

For the left-hand-side use the Parseval equality associated with the operator $\mathcal{L}$ and for the right-hand-side use the parseval associated with $\mathcal{L}^2$ to obtain

$$\int_\sigma \lambda^2 \hat{f}_1(\lambda)\overline{\hat{\psi}_1}(\lambda)d\rho(\lambda) = \int_0^\pi \lambda\left[\hat{f}_2(\lambda)\overline{\hat{\psi}_2}(\lambda) + \overline{\hat{f}_2}(\lambda)\hat{\psi}_2(\lambda)\right] d\frac{a(\lambda)}{2\pi} \tag{5.13}$$

Now, we use this equation to evaluate our spectral function ρ of $\mathcal{L}$ and recall that

$$|\Gamma(1+w)|^2 = \lambda^2.$$

We now shall compute the left-hand-side of equation (5.14) using equation (5.11)

$$\begin{aligned}
\int_\sigma \lambda^2 \hat{f}_1(\lambda)\hat{\psi}_1(\lambda)d\rho(\lambda) &= \int_\sigma \lambda^2\left\{\Gamma(1+w)\overline{\hat{f}_2}(\lambda^2) + s(\lambda)|\Gamma(1+w)|\hat{f}_2(\lambda^2)\right\}\cdot \\
&\cdot \left\{\overline{\Gamma(1+w)}\hat{\psi}_2(\lambda^2) + s(\lambda)|\Gamma(1+w)|\overline{\hat{\psi}_2}(\lambda^2)\right\}d\rho(\lambda) \\
&= \int_\sigma \lambda^4\left\{\overline{\hat{f}_2}(\lambda^2)\hat{\psi}_2(\lambda^2) + \hat{f}_2(\lambda^2)\overline{\hat{\psi}_2}(\lambda^2)\right\}d\rho(\lambda) \\
&+ \int_\sigma S(\lambda)\left\{\overline{\Gamma(w+1)}|\Gamma(1+w)|\hat{f}_2(\lambda^2)\hat{\psi}_2(\lambda^2) + \right. \\
&+ \left. \Gamma(1+w)|\Gamma(1+w)|\overline{\hat{f}_2}(\lambda^2)\overline{\hat{\psi}_2}(\lambda^2)\right\}d\rho(\lambda)
\end{aligned}$$

Since the second integrand is an odd function, we only need to assume that the $\rho(\lambda)$ is an odd function for the above expression to reduce to

$$\int_\sigma \lambda^2 \hat{f}_1(\lambda)\hat{\psi}_1(\lambda)d\rho(\lambda) = \int_\sigma \lambda^4\left\{\overline{\hat{f}_2}(\lambda^2)\hat{\psi}_2(\lambda^2) + \hat{f}_2(\lambda^2)\overline{\hat{\psi}_2}(\lambda^2)\right\}d\rho(\lambda)$$

and use equation (5.14) to obtain

$$\begin{aligned}
\int_\sigma \lambda^4\{\overline{\hat{f}_2(\lambda^2)\hat{\psi}_2(\lambda^2)} \; + \; \hat{f}_2(\lambda^2)\overline{\hat{\psi}_2}(\lambda^2)\}d\rho(\lambda) = \\
\int_0^\pi \lambda\left[\overline{\hat{f}_2}(\lambda)\hat{\psi}_2(\lambda) + \hat{f}_2(\lambda)\overline{\hat{\psi}_2}(\lambda)\right]d\frac{a(\lambda)}{2\pi}
\end{aligned}$$

$$2\int_{\sigma+} \lambda^4 \{\overline{\hat{f}^2}(\lambda^2)\, \hat{\psi}^2(\lambda^2) + \hat{f}^2(\lambda^2)\, \overline{\hat{\psi}^2}(\lambda^2)\} d\rho(\lambda) =$$

$$\int_0^{\pi} \lambda \left[\overline{\hat{f}^2}(\lambda)\, \hat{\psi}^2(\lambda) + \hat{f}^2(\lambda)\, \overline{\hat{\psi}^2}(\lambda) \right] d\frac{a(\lambda)}{2\pi}$$

where $\sigma+ \equiv \sigma \cap [0, +\infty)$. Therefore we need to have

$$2\lambda^2 d\rho(\sqrt{\lambda}) = \lambda d\frac{a(\lambda)}{2\pi} \qquad \lambda \geq 0$$

From which we deduce that

$$\frac{\rho'(\sqrt{\lambda})\lambda}{\sqrt{\lambda}} = \frac{a'(\lambda)}{2\pi} = \frac{1}{2\pi\lambda\sqrt{\pi^2 - \lambda^2}}$$

from which we conclude that

$$\rho'(\lambda) = \frac{1}{2\pi\lambda^3\sqrt{\pi^2 - \lambda^4}} \qquad 0 \leq \lambda \leq \sqrt{\pi}$$

to recover all of $\rho(\lambda)$ on σ we need to recall that $\rho(\lambda)$ is an odd function

$$\rho'(\lambda) = \frac{1}{2\pi\lambda^3 s(\lambda)\sqrt{\pi^2 - \lambda^4}} \qquad -\sqrt{\pi} \leq \lambda \leq \sqrt{\pi}$$

where $s(\lambda) = \begin{cases} +1 & \lambda > 0 \\ -1 & \lambda < 0 \end{cases}$.

This formula coincides with the same one found earlier in Chapter IV.

5.4 Construction of a Real Inversion Formula by Using Pseudo-Differential Operators

Let f and its Laplace transform be in $L^2[0, \infty)$ and let V be the change of variables defined earlier; let $D = \frac{d}{dx}$. Then the following theorem furnishes a real inversion formula for $\mathcal{L}^{-1}$.

Theorem 5.4.1: $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$.

Proof: It is known that convolution operator can be represented as differential operators, indeed from (5.4) and (5.5))

$$Af(x) \equiv k * f(x)$$

So by taking the Fourier transform

$$\begin{aligned}
\widehat{Af}(\lambda) &= \hat{k}(\lambda)\hat{f}(\lambda) \\
&= \frac{\pi}{\cosh \pi\lambda}\hat{f}(\lambda) \\
\widehat{A^{-1}f}(\lambda) &= \frac{\cosh \pi\lambda}{\pi}\hat{f}(\lambda) \\
&= \frac{\widehat{\cosh(\pi\frac{-id}{dx})f}(\lambda)}{\pi} \\
A^{-1}f &= \frac{1}{\pi}\cos(\pi\frac{d}{dx})f(x)
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{L}^{-2} &= V^{-1}A^{-1}V = V^{-1}\frac{1}{\pi}\cos(\pi D)\ V \\
\mathcal{L}^{-1} &= \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V \qquad (5.14)
\end{aligned}$$

where $D \equiv \frac{d}{dx}$.

The analyticity of F helps a great deal in utilizing the formula of theorem 5.4.1. That is, one can always expand F in terms of its Fourier series in a small neighbourhood of the positive real axis, as we shall see by the following applications of the theorem.

Before we do so let us use the above theorem to justify our formal work in the inversion theorem of chapter IV, as promised.

Remark 5.4.1: $V^{-1}\frac{\cos(\pi D)}{\pi}Ve^{-st} = \int_{-\infty}^{\infty}\frac{(ts)^{-a}d\omega}{\Gamma(1-a)}$, where $a = \frac{1}{2} + i\omega$.

Proof: If we equate both formulae obtained for $\mathcal{L}^{-1}$ in chapter IV and V, we have

$$\frac{1}{\pi}V^{-1}\cos(\pi D)V\mathcal{L}F = \int_0^{\infty}F(s)\int_{-\infty}^{\infty}\frac{(st)^{-\frac{1}{2}-i\omega}}{\Gamma(\frac{1}{2}-i\omega)}d\omega$$

$$\frac{1}{\pi}V^{-1}\cos(\pi D)V\int_{-\infty}^{\infty}e^{-st}F(s)ds = \int_0^{\infty}F(s)\int_{-\infty}^{\infty}\frac{(st)^{-\frac{1}{2}-i\omega}}{\Gamma(\frac{1}{2}-i\omega)}dw$$

$$\int_0^{\infty}F(s)\left(\frac{1}{\pi}V^{-1}\cos(\pi D)Ve^{-st}\right)ds = \int_0^{\infty}F(s)\int_{-\infty}^{\infty}\frac{(st)^{-\frac{1}{2}-i\omega}}{\Gamma(\frac{1}{2}-i\omega)}dw$$

Therefore, it remains to show that

$$\frac{1}{\pi}V^{-1}\cos(\pi D)Ve^{-st} = \int_{-\infty}^{\infty}\frac{(st)^{-\frac{1}{2}-i\omega}}{\Gamma(\frac{1}{2}-i\omega)}dw.$$

But, using the definition of the Gamma function as will be shown later, we get

$$\begin{aligned}\frac{1}{\pi}V^{-1}\cos(\pi D)\left(e^{t/2}e^{-se^t}\right) &= \frac{1}{\pi}V^{-1}\cos(\pi D)\int_{-\infty}^{\infty}e^{it\omega}s^{-\frac{1}{2}+i\omega}\Gamma(\tfrac{1}{2}-i\omega)d\omega \\ &= V^{-1}\int_{-\infty}^{\infty}\frac{e^{it\omega}s^{-\frac{1}{2}+i\omega}}{\Gamma(\frac{1}{2}+i\omega)}d\omega \\ &= \int_{-\infty}^{\infty}\frac{(st)^{-\frac{1}{2}+i\omega}}{\Gamma(\frac{1}{2}+i\omega)}d\omega.\end{aligned}$$

5.5 Regularization Via Truncation

To apply theorem (5.4.1) one found it very difficult to use directly. For this we need to expand $F(s)$ in terms of its Fourier series, which in turn can only be done if $F(s)$ is periodical, otherwise we need to truncate F and expand safely in a finite interval $\ell \leq s \leq L$.

Now, let us discuss the error we commit by truncating F.

Assumption 5.5.1: Let $\tilde{\mathcal{R}}$ denote the interval $\mathcal{R} - [Ln\,\ell, Ln\,L]$ and define the error function E_L

$$E_{L,\ell}(\lambda) = \int_{\tilde{\mathcal{R}}} e^{i\lambda x}\, e^{x/2}\, F(e^x) dx.$$

Furthermore, suppose that for some $\alpha > 0$, $\beta > 0$ and $\epsilon > 0$

$$|E_{L,\ell}(\lambda)| \leq \ell^{\beta} \frac{e^{-\pi(1+\epsilon)|\lambda|}}{L^{\alpha}}.$$

Proposition 5.5.1: *Let $F = \mathcal{L}f$ and suppose we want to recover f from its Laplace transform using the formula $\mathcal{L}^{-1} = \frac{1}{\pi}\mathcal{L}V^{-1}\cos(\pi D)V$, with replacing F by its truncated approximation F_T,*

$$\begin{cases} F_T(s) = F(s) & s \in [\ell, L] \\ \quad\quad\;\; = 0 & \text{elsewhere}. \end{cases}$$

Furthermore, suppose that assumption 5.5.1 is satisfied. Then the error E_f we commit in recovering f by using F_T is of order $O(L^{-\alpha})\;O\big(\beta^{\ell}\big)$.

Proof: Let f_T and E_f denote the approximation of f using F_T and the error committed respectively. Then,

$$\begin{aligned}
\|E_f\|_2^2 &= \|f - f_T\|_2^2 = \left\|\frac{1}{\pi}\mathcal{L}V\cos(\pi D)\left(\widehat{VF} - \widehat{VF_T}\right)\right\|_2^2 \\
&\leq \frac{1}{\pi}\|\cosh\pi\lambda\,(VF - VF_T)\|_2^2 \\
&= \frac{1}{\pi}\left\|\cosh\pi\lambda\left(\int_{-\infty}^{\infty} e^{ix\lambda}e^{x/2}F(e^x) - \int_{Ln\,\ell}^{Ln\,L} e^{ix\lambda}e^{x/2}F(e^x)\right)dx\right\|_2^2 \\
&= \frac{1}{\pi}\|\cosh\pi\lambda E_{L,\ell}(\lambda)\|_2^2 \\
&\leq \frac{1}{\pi}\left\|\cosh\pi\lambda\,\ell^{\beta}\,e^{-\pi(1+\epsilon)|\lambda|}\right\|_2^2 \\
&= \frac{1}{4\pi}O\big(L^{-\alpha}\big)\;O\big(\ell^{\beta}\big) \\
&\rightarrow 0 \quad \text{as } \ell \rightarrow 0 \text{ and } L \rightarrow \infty.
\end{aligned}$$

Application 1: We shall work out an example on the computation of the inverse Laplace of a given function, by using equation (5.14). In case $e^{\frac{x}{2}}F(e^x)$ is continuous, it can also be approximated by polynomials over a finite interval. Let

$$F(x) \equiv \sum_{n\leq 0} a_n \frac{(Ln(x))^n}{\sqrt{x}} \quad \text{where} \quad n \leq 0 \quad \text{and} \quad 1 \leq x \leq L.$$

Clearly $\mathcal{L}^{-1}F(x) = \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)VF = \sum a_n \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)x^n$. We need to evaluate

$$\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}x^{n-2k}$$

Therefore

$$V^{-1}\frac{1}{\pi}\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}x^{-1/2}(Ln\,x)^{n-2k}$$

Thus

$$\mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)x^n = \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}\mathcal{L}\left[x^{-1/2}(Ln\,x)^{n-2k}\right]$$

and so

$$f(x) = \mathcal{L}^{-1}F(x) = \sum_{n\leq 0} a_n \sum_{0\leq 2k} \frac{1}{\pi}\frac{(-1)^k(\pi)^{2k}n!}{2k!(n-2k)!}\mathcal{L}\left[x^{-1/2}(Ln\,x)^{n-2k}\right]$$

Now, we need to evaluate $\mathcal{L}\left[x^{-1/2}\,(Ln(x))^n\right]$. For this, we change variables in the definition of the Gamma function by putting $t = sx$

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t}t^{z-1}dt \\ &= s^z\int_0^\infty e^{-sx}x^{z-1}dx\end{aligned}$$

Putting $x = e^y$, we have

$$s^{-z}\Gamma(z) = \int_{-\infty}^\infty e^{-se^y}\,e^{zy}dy$$

Now, similarly putting $x = e^y$

$$\begin{aligned} h(s) &= \mathcal{L}\left(x^{-1/2}(Ln\, x)^n\right) = \int_0^\infty e^{-sx} x^{-1/2} (Ln\,(x))^n dx \\ &= \int_{-\infty}^{\infty} e^{-se^y}\, e^{(\frac{1}{2}+n)y}\, dy \end{aligned}$$

hence,

$$h(s) = \frac{\Gamma(\frac{1}{2}+n)}{s^{\frac{1}{2}+n}}.$$

Finally, we get

$$f(x) = \mathcal{L}^{-1}F(x) = \sum_{n\leq 0} a_n \sum_{2k\geq 0} \frac{1}{\pi} \frac{(-1)^k(\pi)^{2k} n! \Gamma(\frac{1}{2}+n-2k)}{(2k)!(n-2k)! x^{\frac{1}{2}+n-2k}}$$

Example: Consider $F(s) = \frac{1}{\sqrt{s}}$, which is the Laplace transform of $f(t) = \frac{1}{\sqrt{\pi t}}$. Using application 1, we have

$$e^{x/2}F(e^x) = 1$$

Then,

$$\begin{aligned} f(t) &= \mathcal{L}V^{-1}\frac{\cos(\pi D)}{\pi}VF \\ &= \mathcal{L}V^{-1}\frac{\cos(\pi D)}{\pi}(1) \\ &= \mathcal{L}\left(\frac{1}{\pi\sqrt{x}}\right) \\ &= \frac{1}{\sqrt{\pi t}}. \end{aligned}$$

Application 2: Let $(\mathcal{L}f)(x) \equiv F(x)$ where $F(x)$ is defined by

$$F(x) = \frac{1}{\sqrt{x}} \sum_{n\geq 0} a_n \cos(n \ln x \frac{\pi}{L}) \quad 1 \leq x \leq e^L$$

where $\int_1^{e^L} |F(\eta)|^2 d\eta < \infty$. Clearly from (5.9)

$$\mathcal{L}^{-1}F(x) = \mathcal{L}V^{-1}\frac{1}{\pi}\cos(\pi D)VF$$

where

$$VF = e^{\frac{x}{2}}\,F(e^x) = \sum_{n\geq 0} a_n \cos(nx\frac{\pi}{L}).$$

we deduce that

$$\mathcal{L}^{-1}F(x) = \mathcal{L}\frac{1}{\pi}\,V^{-1}\,\cos(\pi D)\sum a_n \cos(nx\frac{\pi}{L}). \tag{5.15}$$

We now would like to evaluate the right hand side of equation (5.15). Recall that

$$\frac{1}{\pi}\cos(\pi D) \equiv \frac{1}{\pi}\sum_{k\geq 0}(-1)^k(\pi^{2k})\frac{D^{2k}}{2k!}$$

and $(-1)^k\pi^{2k}D^{2k}\cos(nx\frac{\pi}{L}) = (n\frac{\pi^2}{L})^{2k}\cos(nx\frac{\pi}{L})$. Hence

$$\begin{aligned}\frac{1}{\pi}\cos(\pi D)\cos(nx\frac{\pi}{L}) &= \frac{1}{\pi}\sum_{k\geq 0}\frac{1}{2k!}(n\frac{\pi^2}{L})^{2k}\cos(nx\frac{\pi}{L})\\ &= \frac{1}{\pi}\cosh(\frac{\pi^2 n}{L})\cos(nx\frac{\pi}{L}).\end{aligned}$$

and so

$$\frac{1}{\pi}V^{-1}\cos(\pi D)\cos(nx\frac{\pi}{L}) = \frac{1}{\pi}\frac{1}{\sqrt{x}}\cosh(\frac{\pi^2 n}{L})\cos(nlnx\frac{\pi}{L}).$$

The last remaining operation is the Laplace transform.

$$\mathcal{L}\frac{1}{\pi}\frac{1}{\sqrt{x}}\cosh(\frac{\pi^2 n}{L})\cos(nlnx\frac{\pi}{L}) = \frac{1}{\pi}\cosh(\frac{\pi^2 n}{L})\mathcal{L}\frac{1}{\sqrt{x}}\cos(nlnx\frac{\pi}{L}).$$

Therefore

$$f(x) = \mathcal{L}^{-1}(F(x)) = \sum_{n\geq 0} a_n \cosh(\frac{\pi^{2n}}{L})Re\{\frac{\Gamma(\frac{1}{2}+in\frac{\pi}{L})}{x^{\frac{1}{2}+in\frac{\pi}{L}}}\}.$$

Application 3: If we can also use a Fourier expansion

$$e^{\frac{x}{2}}F(e^x) \equiv \sum c_n e^{inx\frac{\pi}{L}} \quad \text{where} \quad |x| \leq L$$

and $\int_{e^{-L}}^{e^L} |F(\eta)|^2 d\eta < \infty$. Obviously $VF(x) \equiv \sum c_n e^{inx\frac{\pi}{L}}$ and

$$\frac{1}{\pi}\cos(\pi D)e^{inx\frac{\pi}{L}} = \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})e^{inx\frac{\pi}{L}}.$$

Thus $V^{-1}\frac{1}{\pi}\cos \pi D e^{inx\frac{\pi}{L}} = \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})x^{-\frac{1}{2}+in\frac{\pi}{L}}$. All we need now is to apply Laplace transform to obtain

$$\begin{aligned} f(x) \equiv \mathcal{L}^{-1}F(x) &= \sum c_n \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})\mathcal{L}x^{-\frac{1}{2}+in\frac{\pi}{L}} \\ &= \sum c_n \frac{1}{\pi}\cosh(n\frac{\pi^2}{L})\frac{\Gamma(\frac{1}{2}+in\frac{\pi}{L})}{x^{\frac{1}{2}+in\frac{\pi}{L}}}. \end{aligned}$$

5.6 Concluding Remarks

A spectral representation of the Laplace transform was obtained and used to obtain real analytical inversion formulae in both of chapters IV and V. Although these formulae are represented differently, they were proven to coincide by remark (5.4.1). They yield two different ways of interpreting the Laplace transform inverse. Moreover, regularization by truncation was suggested using the inversion formula in chapter V. The ill-posedness of $\mathcal{L}^{-1}$ resulting from the unboundedness was very clearly seen through the differential operator D involved in the formula $\mathcal{L}^{-1} = \mathcal{L}\frac{1}{\pi}V^{-1}\cos(\pi D)VF$.

References

[1] Ang, D.D., Lund, J. and Stenger, F. Complex variables and regularization methods of I.L.T., Mathematics of Computation. Vol (53) No. (188), (589-608), 1989.

[2] Bakker, B. Solution methods for operator equations of the first kind, Ph.D. Dissertation, 1982.

[3] Bakusinskii, A.B. Regularization algorithms for linear equations with unbounded operators. Soviet Math. Dokl. Vol. 9 (1968), No. 6, pp 1298-1300.

[4] Bellman, R.E., Kalaba, R.E. and J. Lockett, J. Numerical inversion of the Laplace transform, American Elsevier, New York, 1966.

[5] Berezanski, J.M. Expansions in eigenfunctions of selfadjoint operators, Amer. Math. Transl. Mono, vol 17, 1968.

[6] Boumenir, A. A comparison theorem for self-adjoint operators, Proc. Amer. Soc., vol 111, number 1, 1991, pop. 161-175.

[7] Brianzi, P. and Frontini, F.M. On the regularized inversion of the Laplace transform, Inverse Problem 7(1991), 355-368.

[8] Chang, R.Y., Chen, K.C., and Wang, M.L. 1985, *Int. J. Systems Sci.,* **16**, 1163.

[9] Chou, J.H., and Horng, I.R. 1986, *Int. J. Systems Sci.,* **17**, 735.

[10] Chung, H.Y., and Sun, Y.Y. 1976, *Electron. Lett.,* **22**, 1219; 1987 *Int. J. Systems Sci.*, **18**, 213.

[11] Cooley, J.W. and Tukey, J.W. *Math. Comp.* **19**(1965) 297-301.

[12] Cooper, L., Applied nonlinear programming for engineers and scientists. Aloray Publishers (1974).

[13] Cope, Davis K. Convergence of Piessen's method for NILT on the real line. SIAM Journal on Numerical Analysis. Vol(27), No(5), 1990, (1345-1354).

[14] Cost, T.L. Approximate Laplace transform inversions in viscoelastic stress analysis, *AIAA J.* **2** (1964) 2157-2166.

[15] Crump, K.S. *J. ACM* **23**(1976), 89-96.

[16] Davis, B and Martin, B. Numerical inversion of the Laplace transform: A survey and comparison of methods, J. Comput. Phys. 33(1979), 1-32.

[17] Dieudonne, J. Treatise in Analysis, vol. 2, Acad. Press, (1961).

[18] Doetsch, Gustav. Introduction to the theory and application of the Laplace transformation, Springer Verlag, Berlin-Heidelberg, 1974.

[19] Durbin, F. *Comput. J.* **17**(1974) 371-376.

[20] Dubner, H. and Abate, J. *J. ACM* **15**(1968) 115-123.

[21] Erdelyi, *Philos. Mag.* 34(1943), 533-537.

[22] Essah, W.A. and Delves, L.M. In the NILT, Inverse Problems, 4(1988), 705-724.

[23] Faraut, J. and Gindikia, S. Two inversion formulae for Laplace transforms in a cone. C. R. Acad. Sci Paris, t. 310, Sérié 1, pp.5-8, 1990.

[24] Feller, W. An Introduction to Probability Theory and its Applications, Vol II, Wiley, New York, 1971.

[25] Gave, Jr., D.P. Operational Res. **14**(1966) 444-459.

[26] Gelfand, I.M. and Shilov, G.E. "Generalized Functions," Academic Press, new York, 1964.

[27] Gilliam, D.S, Schulenberger, J.R. and Lund, J.R. "Spectral representation of the Laplace and Stieltjes transform," in *Computational and Combinatorial Methods in Systems Theory* (C.P. Byrne and A. Lindquist, eds.), Elseveir Science, North-Holland, 1986, pp. 69-74.

[28] Gravaes, R.R., – Marris and Hopkins, T.R., Reliable rational interpolation, Numer. Math., 36(111-128) 1981.

[29] Hardy, H. and Titchmarsh, E.C. Solution of an integral equation, Journal London Math. Soc. 4(1929), 360-4.

[30] Henrici, Peter. Applied and Computational Complex Analysis, Vol II, John Wiley & Sons, Inc., 1977.

[31] Hildebrand, F.B. Methods of Applied Mathematics, Second edition, Prentice-Hall, Inc., 1965.

[32] Hochstadt, H. Integral Equations, John Wiley and Sons, Inc., U.S.A., 1973.

[33] De Hoog, F.R., Knight, J.H. and Stokes, A.N. An improved method for numerical inversion of the Laplace transform. SIAM. J. Sci. Stat. Comput. Vol. 3. No. 3, 1982.

[34] Honig, G. and Hirdes, U. A method for the N.I.L.T., Journal of Computation and Applied Mathematics, 10 (1984), 113-132.

[35] Hwang, C. and Shih, Y.P. 1981, *Int. J. Control*, **34**, 577.

[36] Ikebe, Y. "Galerkin method for the numerical solution of Fredholm integral equations of the second kind," *SIAM Rev.* v. 14, 1972, pp. 465-491.

[37] Nvanov, V.V. The theory of approximate methods and their applications to the numerical solution of singular integral equations. Noordhoff, Leijden (1976).

[38] Iqbal, M. Numerical Laplace transform inversion by a regularization methods The Punjab University Journal of Mathematics, Vol. XX, pp. 71-96, 1987.

[39] Van, Iseghem. Jeannette. Laplace transformation inversion and Padé type approximants. Applied Numerical Mathematics, 3(1987), 529-538.

[40] Jagerman, D.L. An inversion technique for the Laplace transform, The Bell System Technical Journal, Vol. 61, No. 8, (pp. 1995-2002). 1982.

[41] Kitahara, N., Nagahara, D. and Yano, H. On the numerical inversion of the Laplace transform and its application, Journal of the Franklin Institute, Vol (325), No (2), 1988, 221-223.

[42] Krylov, V.I. and Skoblya, N.S. A hand book of methods of approximate Fourier transformation and inversion of the Laplace transform. Mir Publishers, Moscow (1974).

[43] Lanczos, C., "Applied Analysis", 624–635, Pitman, 1957.

[44] Longman, I.M. *J. Comp. Appl. Math* **1**(1975) 17-23.

[45] Lyness, J.N. and Giunta G 1986 A modification of the Weeks method for numerical inversion of the Laplace transform *Math. Comput.* **47** 313-22.

[46] MacDonald, J.R. *J. Appl. Phys.* **35**(1964) 3034-3041.

[47] Marti, J.T. "On a regularization method for Fredholm equations of the first kind using Sobolev spaces," in *Treatment of Integral Equations by Numerical Methods* (C. T. H. Baker and G.F. Miller, eds.), Academic Press, New York, 1982, pp. 59-66.

[48] Migneron, R. and Narayanan, K. S. S. Numerical inversion of Mellin moments and L. T., Computer Physics Communications 49 (1988), 457-463.

[49] Miller, M.K., and Guy, W.T., SIAM J. Numer. Anal. 3 (1966), 624–635.

[50] Murli, A. and Rizzardi, M. A logarithm 682. Talbot's method for the Laplace transform inversion problem, ACM Transactions on Mathematical Software. Vol (16), No (2), 1990, (158-168).

[51] Morozov, V.A. On the solution of functional equations by the method of regularization. Soviet Math. Dokl. 7(1966), 414-417.

[52] Naimark, M.A. Linear differential operators. Part II. Frederick Unger Publishing Co., Inc. 1968.

[53] Nashed, M.Z. Approximate Regularized Solutions to Improperly Posed Linear Integral and Operator Equations, Lecture Notes in Mathematics No. 430 (289-332). 1974.

[54] Nashed, M.Z., and Wahba, G. "Some exponentially decreasing error bounds for numerical inversion of the Laplace transform", J. Math. Anal. Appl., Vol. 52, 1975, pp. 265–283.

[55] Papoulis, *Quart. Appl. Math*, 14 (1956), 405–414.

[56] Piessens, R. and Branders, M. 1971 Numerical inversion of the Laplace transform using generalized Laguerre polynomials *Proc. IEE* **118** 1517-22.

[57] Piessens, R. *J. Comput. Appl. Math.* **1**(1975) 115-128.

[58] Piessens, R. *J. Engirg. Math.* **5**(1971) 1-9.

[59] Piessens, R. and Dang, N.D.P. *J. Comput. Appl. Math* **2**(1976) 225-228.

[60] Piessens, R. and Huysmany, R. (1984) Algorithm 619: automatic numerical inversion of the Laplace transform *ACM Trans. Math Soft.* **10** 348-53.

[61] Pollard, H. Integral Transforms. Duke Math. J. 13, 307–330. 1946.

[62] Popoulis, Quart. Appl. Math. 14(1956), 405-414.

[63] Prianzi, P. and Frontini M. On the regularized inversion of the Laplace transformation, Inverse Problems, 7 (pp. 355-368), 1991.

[64] Ryabov, V.M. L.T.I. formula based on Widder's theorem, Vestnik Leningradskogo Universiteta Mathematika, Vol. 22, No. 4, pp. 35-39, 1989.

[65] Ryabov, V. M. The behavior of quadrature formula coefficients for LTI with increasing number of nodes, Vestnik Leningradskogo Universiteta Mathematika, Vol (23), No,(3), (38-40), 1990.

[66] Salzer, H.E. *Math. Tables Other Aids Cmput*, **9**(1955) 164-177.

[67] Salzer, H.E. *J. Mathematical Phys.* **37**(1958) 89-109.

[68] Salzer, H.E. *J. Mathematical Phys.* **40**(1961) 72-86.

[69] Sansone, G. Orthogonal functions, Robert E. Krieger Publishing Co., Huntington, New York (1977).

[70] Schapery, R.A. *Proc. 4th U.S. Nat. Congr. Appl. Mech., ASME* **2**(1962) 1075-1085.

[71] Schmittroth, L.A. *Comm. ACM* **3**(1960) 171-173.

[72] Shirtliffe, C.J. and Stephenson D G *J. Mathematical Phys.* **40**(1961) 135-141.

[73] Schoenberg, I.J., Remarks concerning a numerical inversion of the Laplace transform due to Bellmana, Kalaba and Lockett, J. Math. Anal., 43(1973), 823–828.

[74] Stehfest, H. *Comm. ACM* **13**(1970) 47-49.

[75] Squirl. The numerical treatment of Laplace transforms, International Journal for Numerical Methods in Engineering, Vol 20, 1697-1702, (1984). and Applied Mathematics, 10 (1984), 113-132.

[76] Strahov, V.N. Numerical solution of incorrect problems representable by convolution integral equations. Soviet Math. Dokl. Vol. 9 (1968), No. 1, pp. 91-94.

[77] Tikhonov, A.N., Arsenin, V.Y. Solutions of ill-posed problems, V.H. Winston and Sons Publishers, Washington, D.C. (1977).

[78] Titchmarsh, E.C.; second edition. Introduction to the Theory of Fourier Integrals. Oxford at the Clarendon Press, 1948.

[79] Varah, J.M. 1983 Pitfalls in the numerical solution of linear ill-posed problems *SIAM J. Sci. Stat. Comput.* **4** No. 2, 164-176.

[80] Weeks, W.T. 1966 Numerical inversion of the Laplace transform using Laguerre functions *J. ACM* **13** 419-29.

[81] Widder, D.V. The Laplace transform, Princeton, Princeton University Press, (1972).

[82] Wilson, C. OBI. Error analysis of a Laplace transform inversion procedure. SIAM. J. Num. Anal., Vol. 27, No. 2, pp 457-489, 1990.

[83] Zakian, V. *Electron. Lett.* **6**(1970) 677-679