

# A Generalized Theory for Bending of Thick Isotropic Rectangular Plates

by

Ammar Khalil Hafedh Mohammed

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES  
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DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the  
Requirements for the Degree of

**DOCTOR OF PHILOSOPHY**

In

**CIVIL ENGINEERING**

June, 1989

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**AMMAR KHALIL HAFEDH MOHAMMED**

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COLLEGE OF GRADUATE STUDIES

This dissertation, written by AMMAR KHALIL HAFEDH MOHAMMAD under the direction of his Dissertation Advisor and approved by the Dissertation Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

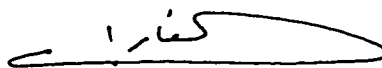
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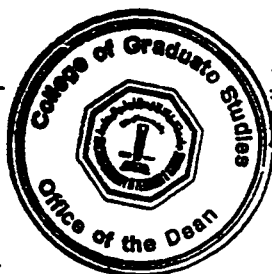
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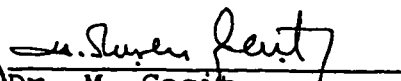


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بسم الله الرحمن الرحيم

أهدي رسالة الدكتوراة هذه إلى :

والدي العزيزين ،

وإلى

زوجتي ( أم ياسر ) العزیزة ،

وإلى

ابني المحبوبين : ياسر ومهنا

THIS PH.D DISSERTATION IS DEDICATED TO :

MY DEAR PARENTS

MY DEAR WIFE ( UM YASER )

AND

MY BELOVED SONS : YASER AND MOHANNED

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## خلاصة الرسالة

اسم الطالب : عمار خليل حافظ محمد

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لقد تطورت في السنوات الاخيرة نظريات منقحة للصفايح . حاولت كل هذه النظريات اعتبار تأثير الجهود العرضية والجهود العمودية والتمدد العمودي للصفايح والتي تزداد أهميتها بازياد نسبة سمك الصفيحة الى طولها . تنتمي النظرية المشتقة في هذه الرسالة الى هذا القسم من النظريات الا أنها تختلف عن باقي النظريات بأنها تفترض توزيعا عاما للجهود بالبداية والتي تقود الى اشتقاق نظرية لانحناء الصفايح أكثر دقة من سابقتها .

عند مقارنة النتائج من هذه بالنظرية الأكيدة - نظرية المرونة - وغيرها من النظريات المنقحة ، تبين أن هذه النظرية تعطي نتائج أقرب ماتكسون من نتائج النظرية الأكيدة وذلك بالنسبة - لانحراف الصفيحة والجهود المستوية الى تتعرض لها حتى عندما تصل نسبة سمك الصفيحة الى طولها الى " ٣ " ، بالنسبة لانحناء الاسطوانى ، و الى نسبة من سمك الصفيحة الى طولها الى " ١ " بالنسبة للصفايح المستطيلة .

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## Chapter 1

### INTRODUCTION

The behavior of a plate is affected greatly by its thickness. For this reason, plates can be divided into three categories [1]:

- (1) thin plates with small deflections
- (2) thin plates with large deflections
- (3) thick plates.

In order to simplify the theory of plates, many assumptions have been made when developing a theory for thin plates with small deflections. These assumptions can be summarized as [1]:

- (1) No stretching of the middle plane of the plate. This plane remains neutral during bending.
- (2) Points of the plate lying initially on a normal-to-the middle plane of the plate remain on the normal to the middle surface of the plate after bending.
- (3) The normal stresses in the direction transverse to the plate can be disregarded.

As a result to the above assumptions, many limitations are imposed on the classical theory of plates. As the thickness of the plate increases, the effect of transverse stresses and strains on the deflection of the plate and on the inplane stresses can not be neglected. Also, the resulting governing equation for deflection of the middle surface is of the fourth order which implies that two boundary conditions on each edge are needed for solution. This contradicts the requirement of satisfying three boundary conditions on each edge as elasticity theory states.

In order to overcome some of the limitations of thin or classical plate theory, researchers have developed a number of refined theories. Reissner [2] was the first to provide a refined theory that takes into account shear deformation. He did not include the effect of transverse normal strain. A special variational theorem was used by Reissner to develop his theory. As a result of his work, only mid-plane displacement  $w_0$  and bending moments and shear forces were modified. Stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  were not modified in Reissner's theory.

Some other theories [3,4,5,6] were developed to include the effects of transverse shear, transverse normal stress, and transverse normal strain. However as in all previous refined theories only the displacement "w" was corrected and the inplane stresses:  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  were left as for the Kirchoff thin plate theory.

Another refined theory was developed by Kromm [7,8]. Kromm introduced more general stress distributions across the thickness of the plate. But Kromm neglected the effects of the transverse normal stress,  $\sigma_z$  and normal strain,  $\epsilon_z$ .

Panc [9] had modified Kromm's work by deriving the governing equation for the function  $f_1(z)$ , used by Kromm, in a different way. Panc called this refined theory a "Generalized Theory".

In the present work, a new refined theory will be developed making use of Panc's generalized theory and a refined theory presented by Baluch et al. [10]. Figure 1.1 summarizes the state of the art and highlights characteristics of present formulation.

The effect of the transverse shear stresses, the transverse normal stress, and the transverse normal strain on the deflection "w" and on inplane stresses:  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  will be considered. Also, a general stress distribution across the thickness of the plate will be assumed. Solution of problems of bending for isotropic thick rectangular plates with different boundary conditions (i.e.: simply supported, free or clamped at  $y = \pm b/2$ ) will be considered. Also the applied load will be of general form (i.e.: concentrated, uniformly distributed or other continuous distribution).

In this present work, the importance of developing a refined theory that takes into account the effects of normal stress  $\sigma_z$ , and

shearing stresses  $\tau_{xz}$ ,  $\tau_{yz}$  on inplane stresses and on deflection will be illustrated explicitly. The normal stress  $\sigma_z$ , for example, will be shown to have values of the same order as the inplane stresses  $\sigma_x$ ,  $\sigma_y$ , and  $\tau_{xy}$  for plates of appreciable thickness.

A Levy type semi-inverse method will be followed to obtain the solution for bending of isotropic rectangular plates. In order to test the present theory, some problems of thick isotropic rectangular plates will be considered and compared to already existing theories and to exact solution, whenever it may exist.



**FIG. 1.1 : STATE OF ART + PRESENT THEORY**

1. NEGLECTS INFLUENCE OF :

$\tau_{xz}$  ,  $\tau_{yz}$

ON DEFLECTION .

2. NEGLECTS INFLUENCE OF :

$\sigma_z$  ,  $\epsilon_z$

ON PLATE RESPONSE .

**CLASSICAL**

1.  $\sigma_z$  ,  $\epsilon_z$

MISSING .

2.  $\sigma_x$  ,  $\sigma_y$  ,  $\tau_{xy}$  ,  $\tau_{xz}$  ,

$\tau_{yz}$

NOT CORRECTED .

**REISSNER**

FIG. 1.1 (CONTINUED)

1. ILL CONDITIONING .
  2. STRESSES NOT FOUND .
- (IN-PLANE PROBLEM NOT SOLVED)

*BALUCH, VOYIADJIS, and  
AZAD*

1. INCLUDES EFFECTS OF :  
 $\tau_{xz}$  ,  $\tau_{yz}$  ,  
 $\sigma_z$  , and  $\epsilon_z$   
ON PLATE RESPONSE .
2. IN-PLANE PROBLEM SOLVED .
3. STRESSES FOUND
4. ILL CONDITIONING REMOVED.

*PRESENT*

---

## Chapter 2

### THEORETICAL BACKGROUND

In this chapter, basic relations in the classical theory of isotropic elastic plates will be shown. Particular simplifications are introduced into the governing equations of the mathematical theory of elasticity. These simplifications give results which do not differ significantly from those obtained from the exact equations for the range of definition of the problem.

The simplifying assumptions used in various plates theories come from using the definition of a plate as a body which has one dimension which is small and also from results of elementary beam theory.

The stress-strain relations for an isotropic body are given by [9]:

$$\epsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)] \quad (2.1)$$

$$\epsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_x + \sigma_z)] \quad (2.2)$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x + \sigma_y)] \quad (2.3)$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \quad (2.4)$$

$$\gamma_{xz} = \frac{1}{G} \tau_{xz} \quad (2.5)$$

$$\gamma_{yz} = \frac{1}{G} \tau_{yz} \quad (2.6)$$

In the classical theory of plates, the following assumptions are adopted:

$$\sigma_z = 0 \quad (2.7.1)$$

$$\varepsilon_z = 0 \quad (2.7.2)$$

$$\gamma_{xz} = 0 \quad (2.7.3)$$

$$\gamma_{yz} = 0 \quad (2.7.4)$$

For small deflections, compared with the plate thickness  $h$ , the strain-displacement relations in rectangular coordinates are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (2.8.1)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (2.8.2)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad (2.8.3)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (2.8.4)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (2.8.5)$$

Because of the assumption in equation (2.7.2) the deflection function depends on the variables  $x$  and  $y$ , thus:

$$w = w(x, y) \quad (2.8.6)$$

Introducing equation (2.8.6) and (2.7.3), (2.7.4) into (2.8.4) and (2.8.5) yields for the displacements  $u$  and  $v$  after performing integration with respect to  $z$ :

$$u = -z \frac{\partial w}{\partial x} + u_0(x, y) \quad (2.8.7)$$

$$v = -z \frac{\partial w}{\partial y} + v_0(x, y) \quad (2.8.8)$$

where:  $u_0$ ,  $v_0$  are functions of integration. These functions define a state of plane strain of the plate (i.e. deformations independent of  $z$ ). They correspond to forces acting in the middle plane of the plate or to a uniform heating of the plate. These functions can be neglected during bending, if the only load acting on the plate is normal to its surface, and if the edges of the plate are free to move in the plane of the plate.

Introducing the simplifications (or assumptions) in (2.7.1-4), the stress-strain relations become:

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y) \quad (2.9.1)$$

$$\varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x) \quad (2.9.2)$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \quad (2.9.3)$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \quad (2.9.4)$$

The above set of equations represent the elasticity relations used in the classical theory of isotropic plates.

Consider an element of volume  $dx dy dz$  (Fig. 2.1). Then the stress components acting on this element must satisfy three conditions of equilibrium which are expressed in the absence of body forces by the equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (2.10)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \quad (2.11)$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \quad (2.12)$$

The shearing stresses satisfy conditions of symmetry which result from equations of moment equilibrium

$$\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned} \quad (2.13)$$

The equilibrium equations in 2.10, 2.11, and 2.12 are also known as the Cauchy equations. In the solution of plate problems, the stress components are usually replaced by the corresponding resultants per unit length. These resultants are denoted by bending moments, twisting moments, and shearing forces. They are defined by:

$$M_x = \int_{-h/2}^{+h/2} \sigma_x z dz \quad (2.14.1)$$

$$M_y = \int_{-h/2}^{+h/2} \sigma_y z dz \quad (2.14.2)$$

$$M_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} z dz \quad (2.14.3)$$

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz} dz \quad (2.14.4)$$

$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz} dz \quad (2.14.5)$$

Neglecting body forces, the equilibrium equations in terms of the internal forces as defined by equations (2.14) and the lateral load  $p(x,y)$  acting on an element  $hdx dy$  of a plate (Fig. 2.2) take the form:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x \quad (2.15)$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \quad (2.16)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad (2.17)$$

The relations given above represent the basis of the classical theory of elastic isotropic plates.

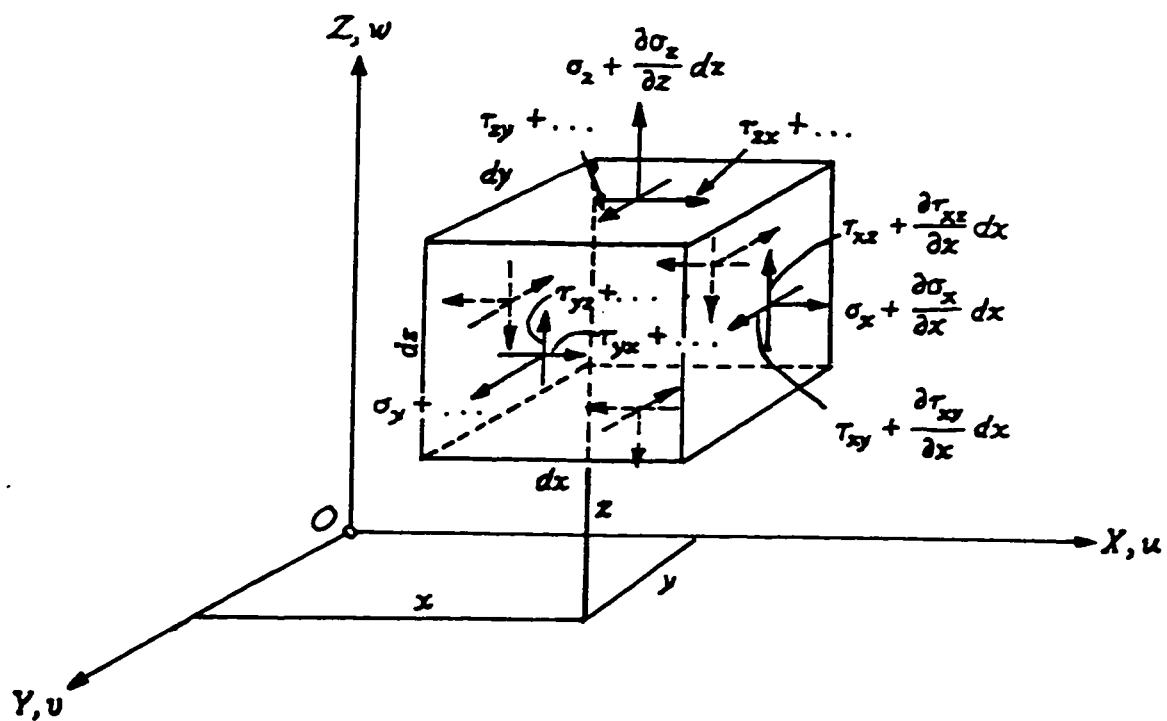


Figure 2.1 : Three-Dimensional Element (Note : +... = increment)



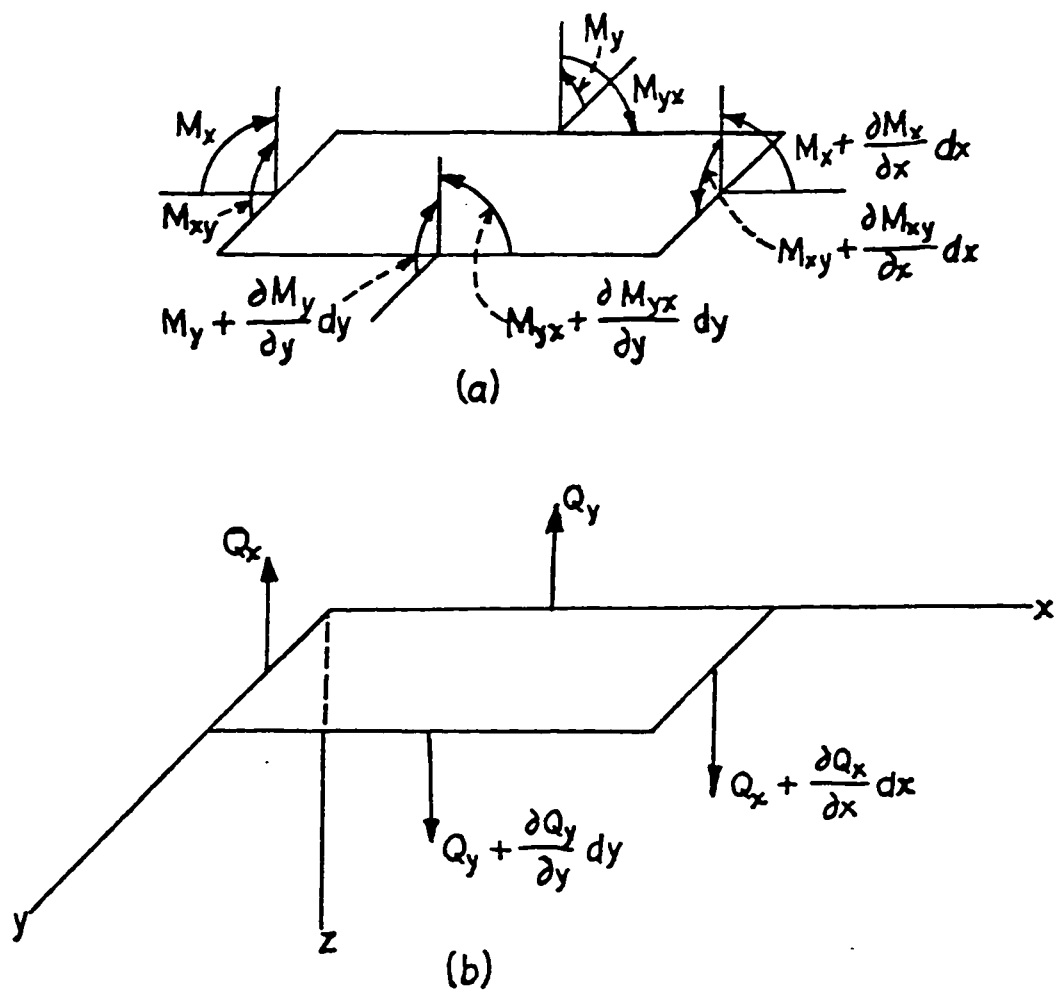


Figure 2.2 : a) Resultant Moments.  
b) Resultant Shear Forces.

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## Chapter 3

### FORMULATION

#### 3.1 Governing Equations for the Bending Problem

The following generalized assumption has been introduced by Kromm [7,8] to approximate the variation of the transverse normal stress <sup>(1)</sup>

$$\sigma_z = p(x,y) f_1(z) \quad (3.1)$$

If the load  $p(x,y)$  acts only at the upper surface  $z = -h/2$  of the plate, the function  $f_1(z)$  must satisfy the boundary conditions:

$$f_1(-h/2) = -1, f_1(+h/2) = 0 \quad (3.2)$$

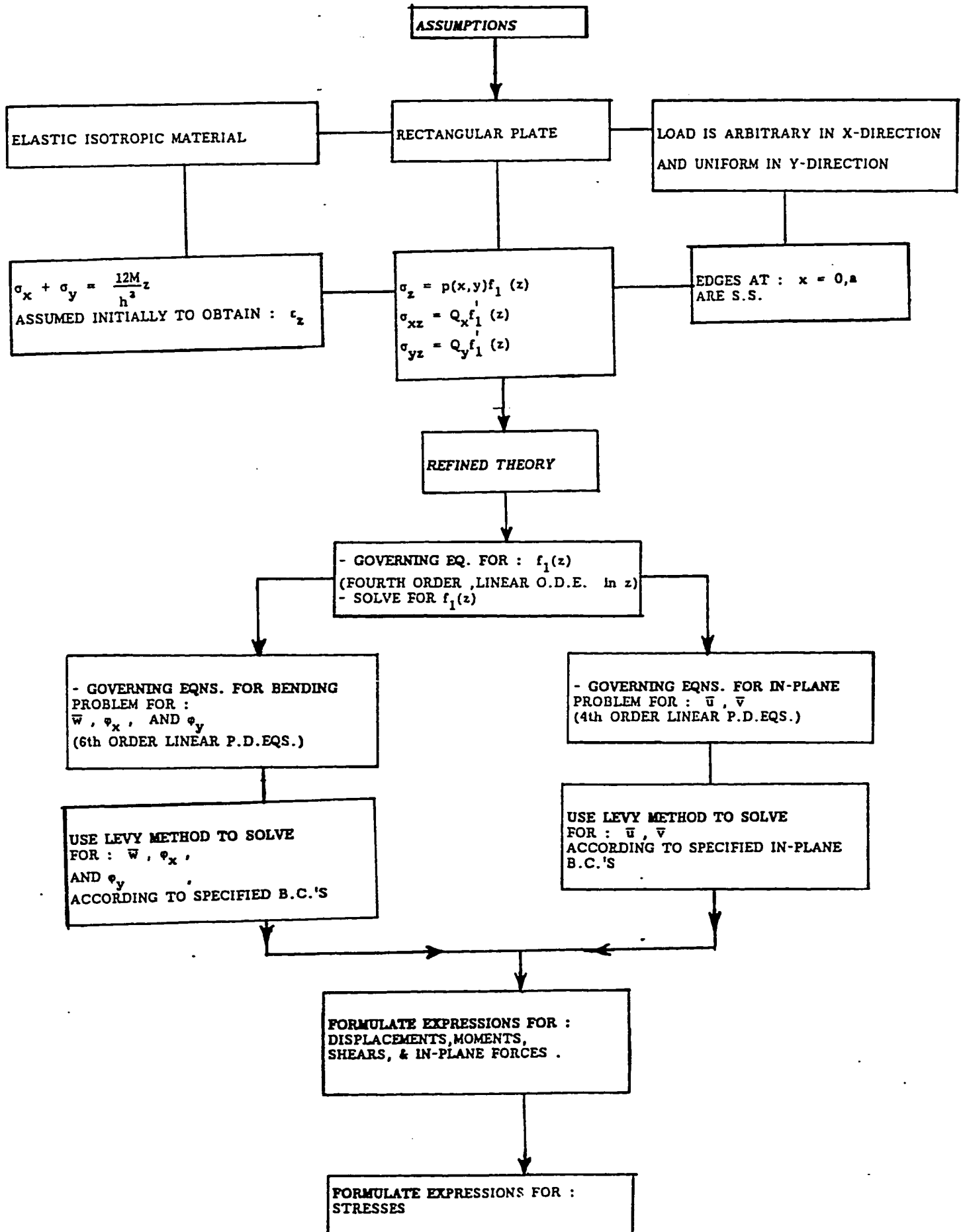
The distribution of transverse shears is assumed in the form:

$$\begin{aligned} \tau_{xz} &= Q_x(x,y) \bar{f}_2(z) \\ \tau_{yz} &= Q_y(x,y) \bar{f}_2(z) \end{aligned} \quad (3.3)$$

---

(1) See Figure 3.1 for a flowchart presentation of the theory developed.

Figure 3.1 : Flowchart For Present Theory.



where  $\bar{f}_2(z)$  must satisfy the stress boundary conditions at the surface of the plate i.e.

$$\bar{f}_2(\pm h/2) = 0 \quad (3.4)$$

On substituting equations (3.1) and (3.3) into the stress differential equation of equilibrium

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (3.5)$$

one obtains

$$\left[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right] \bar{f}_2(z) + p(x,y) \frac{df_1(z)}{dz} = 0 \quad (3.6)$$

However

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad (3.7)$$

Thus for identical satisfaction of equation (3.6) one should have

$$\bar{f}_2(z) = \frac{df_1(z)}{dz} = f_1'(z) \quad (3.8)$$

Thus  $\tau_{xz}, \tau_{yz}$  can be written as:

$$\tau_{xz} = Q_x f_1'(z) \quad (3.9)$$

$$\tau_{yz} = Q_y f_1'(z)$$

and conditions given in equation (3.4) can be written as

$$f_1'(\pm h/2) = 0 \quad (3.10)$$

The transverse normal strain  $\epsilon_z$  is given by:

$$\epsilon_z = \frac{1}{E} [\sigma_z - \mu(\sigma_x + \sigma_y)] \quad (3.11)$$

Using equation (3.1) in (3.11)

$$\epsilon_z = \frac{\partial w}{\partial z} = \frac{1}{E} (p(x,y)f_1(z)) - \frac{\mu}{E} \frac{(12M)z}{h^3} \quad (3.12)$$

where

$$M = M_x + M_y \quad (3.13)$$

and  $\sigma_x + \sigma_y$  has been assumed to be of the form

$$\sigma_x + \sigma_y = \frac{12M}{h^3} z \quad (3.14)$$

The above linear distribution for the stresses  $\sigma_x$  and  $\sigma_y$  was used as an input stress to enable us to get an expression for  $\epsilon_z$ , which on integration, yields a rational assumed form for the transverse displacement  $w$ .

Integrating (3.12) with respect to  $z$  yields the rational form for  $w$  as:

$$w(x,y,z) = \frac{1}{E} p(x,y) f_2(z) - \frac{6\mu M}{Eh^3} z^2 + w_0(x,y) \quad (3.15)$$

where

$$f_2(z) = \int f_1(z) dz \quad (3.16)$$

$$w_0(x,y) = \text{transverse displacement of the surface } z = 0. \quad (3.17)$$

The displacements  $u(x,y,z)$  and  $v(x,y,z)$  are obtained by making use of the strain-displacement relations:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{xz} = \frac{\tau_{xz}}{G} \quad (3.18.1)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{yz} = \frac{\tau_{yz}}{G} \quad (3.18.2)$$

Using equations (3.3) and (3.15) in (3.18.1) and integrating with respect to  $z$  gives for  $u$

$$u = -z \frac{\partial w_0}{\partial x} + \frac{Q_x}{G} f_1(z) - \frac{1}{E} \frac{\partial p}{\partial x} f_3(z) + \frac{2\mu}{Eh^3} \frac{\partial M}{\partial x} z^3 + u_0(x,y) \quad (3.19)$$

where

$$f_3(z) = \int f_2(z) dz \quad (3.19.1)$$

$$u_0(x,y) = \text{u-displacement of the mid surface} \quad (3.19.2)$$

Proceeding similarly, one may obtain an expression for the displacement  $v$  in the form

$$\begin{aligned} v = & -z \frac{\partial w_0}{\partial y} + \frac{Q_y}{G} f_1(z) - \frac{1}{E} \frac{\partial p}{\partial y} f_3(z) \\ & + \frac{2\mu}{Eh^3} \frac{\partial M}{\partial y} z^3 + v_0(x,y) \end{aligned} \quad (3.20)$$

where

$$v_0(x,y) = \text{v-displacement of the mid surface} \quad (3.20.1)$$

In refined theories taking into account influence of transverse shear only,  $u_0$  and  $v_0$  are taken to be identically zero.

The remaining stress-strain relations are

$$\sigma_x = \frac{E}{(1-\mu^2)} [\epsilon_x + \mu\epsilon_y] + \frac{\mu}{(1-\mu)} \sigma_z \quad (3.21.1)$$

$$\sigma_y = \frac{E}{(1-\mu^2)} [\epsilon_y + \mu\epsilon_x] + \frac{\mu}{(1-\mu)} \sigma_z \quad (3.21.2)$$

$$\tau_{xy} = G\gamma_{xy} \quad (3.21.3)$$

The strain-displacement relations are given by

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (3.21.4)$$

Substituting equations (3.1), (3.19), (3.20) and (3.21.4) into the set (3.21.1), (3.21.2) and (3.21.3) yields

$$\begin{aligned} \sigma_x = & \frac{E}{(1-\mu^2)} \left[ -z \frac{\partial^2 w_0}{\partial x^2} + \frac{f_1(z)}{G} \frac{\partial Q_x}{\partial x} - \frac{f_3(z)}{E} \frac{\partial^2 p}{\partial x^2} + \frac{2\mu}{Eh^3} \frac{\partial^2 M}{\partial x^2} z^3 \right. \\ & \left. + \mu \left\{ -z \frac{\partial^2 w_0}{\partial y^2} + \frac{f_1(z)}{G} \frac{\partial Q_y}{\partial y} - \frac{f_3(z)}{E} \frac{\partial^2 p}{\partial y^2} + \frac{2\mu}{Eh^3} \frac{\partial^2 M}{\partial y^2} z^3 \right\} \right] \\ & + \frac{E}{(1-\mu^2)} \left( \frac{\partial u_0}{\partial x} + \frac{\mu \partial v_0}{\partial y} \right) + \frac{\mu p}{(1-\mu)} f_1(z) \end{aligned} \quad (3.22)$$

$$\begin{aligned} \sigma_y = & \frac{E}{(1-\mu^2)} \left[ -z \frac{\partial^2 w_0}{\partial y^2} + \frac{f_1(z)}{G} \frac{\partial Q_y}{\partial y} - \frac{f_3(z)}{E} \frac{\partial^2 p}{\partial y^2} + \frac{2\mu}{Eh^3} \frac{\partial^2 M}{\partial y^2} z^3 \right. \\ & \left. + \mu \left\{ -z \frac{\partial^2 w_0}{\partial x^2} + \frac{f_1(z)}{G} \frac{\partial Q_x}{\partial x} - \frac{f_3(z)}{E} \frac{\partial^2 p}{\partial x^2} + \frac{2\mu}{Eh^3} \frac{\partial^2 M}{\partial x^2} z^3 \right\} \right] \\ & + \frac{E}{(1-\mu^2)} \left( \frac{\partial v_0}{\partial y} + \frac{\mu \partial u_0}{\partial x} \right) + \frac{\mu p}{(1-\mu)} f_1(z) \end{aligned} \quad (3.23)$$

$$\begin{aligned} \tau_{xy} = & \frac{E}{2(1+\mu)} \left\{ -2z \frac{\partial^2 w_0}{\partial x \partial y} + \frac{f_1(z)}{G} \frac{\partial Q_x}{\partial y} + \frac{f_1(z)}{G} \frac{\partial Q_y}{\partial x} \right. \\ & \left. - \frac{2f_3(z)}{E} \frac{\partial^2 p}{\partial x \partial y} + \frac{4\mu}{Eh^3} \frac{\partial^2 M}{\partial x \partial y} z^3 \right\} \end{aligned}$$



$$+ \frac{E}{2(1+\mu)} \left[ \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \right] \quad (3.24)$$

Using the definitions for the moment stress resultants

$$\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x z \, dz \quad ; \quad M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz \\ M_{xy} &= - \int_{-h/2}^{h/2} \tau_{xy} z \, dz \end{aligned} \quad (3.25)$$

one obtains

$$\begin{aligned} M_x &= \frac{E}{(1-\mu^2)} \left[ \frac{-h^3}{12} \frac{\partial^2 w_o}{\partial x^2} + \frac{h^3}{12G} F_1 \frac{\partial Q_x}{\partial x} - \frac{h^3}{12E} F_3 \frac{\partial^2 p}{\partial x^2} \right. \\ &\quad + \frac{\mu h^5}{40Eh^3} \frac{\partial^2 M}{\partial x^2} + \mu \left\{ - \frac{h^3}{12} \frac{\partial^2 w_o}{\partial y^2} + \frac{h^3}{12G} F_1 \frac{\partial Q_y}{\partial y} \right. \\ &\quad \left. \left. - \frac{h^3}{12E} F_3 \frac{\partial^2 p}{\partial y^2} + \frac{\mu h^5}{40Eh^3} \frac{\partial^2 M}{\partial y^2} \right\} \right] \\ &\quad + \frac{\mu h^3 p}{12(1-\mu)} F_1 \end{aligned} \quad (3.26.1)$$

where :

$$F_1 = \frac{12}{h^3} \int_{-h/2}^{h/2} z f_1(z) dz \quad (3.26.2)$$

$$F_3 = \frac{12}{h^3} \int_{-h/2}^{h/2} z f_3(z) dz \quad (3.26.3)$$

or

$$M_x = D \left[ \frac{\partial \phi_x}{\partial x} + \mu \frac{\partial \phi_y}{\partial y} + \frac{\mu(1+\mu)}{E} p F_1 \right] \quad (3.27.1)$$

$$M_y = D \left[ \frac{\partial \phi_y}{\partial y} + \mu \frac{\partial \phi_x}{\partial x} + \frac{\mu(1+\mu)}{E} p F_1 \right] \quad (3.27.2)$$

$$M_{xy} = -\frac{D(1-\mu)}{2} \left[ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right] \quad (3.27.3)$$

where <sup>(1)</sup> :

$$\begin{aligned} \phi_x &= -\frac{\partial w_o}{\partial x} + \frac{F_1}{G} Q_x - \frac{F_3}{E} \frac{\partial p}{\partial x} + \frac{3\mu}{10Eh} \frac{\partial M}{\partial x} \\ &= -\frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N} \frac{\partial p}{\partial x} + \frac{1}{R} \frac{\partial M}{\partial x} \end{aligned} \quad (3.27.4)$$

$$\phi_y = -\frac{\partial w_o}{\partial y} + \frac{Q_y}{S} - \frac{1}{N} \frac{\partial p}{\partial y} + \frac{1}{R} \frac{\partial M}{\partial y} \quad (3.27.5)$$

in which

$$S = \frac{G}{F_1} \quad (3.27.6)$$

$$N = \frac{E}{F_3} \quad (3.27.7)$$

$$R = \frac{10Eh}{3\mu} \quad (3.27.8)$$

In order to obtain the governing differential equation for  $w_o$ ,

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(1) See Appendix (A-4) for physical interpretation of  $\phi_x$  and  $\phi_y$

one first eliminates  $\varphi_x$  and  $\varphi_y$  by using equations (3.27.4) and (3.27.5) in equation (3.27.1) resulting in

$$\begin{aligned} M_x = & -D \left[ \frac{\partial^2 w_o}{\partial x^2} + \mu \frac{\partial^2 w_o}{\partial y^2} \right] + \frac{h^3}{6} F_1 \frac{\partial Q_x}{\partial x} - \frac{\mu h^3 p}{12(1-\mu)} F_1 \\ & - \frac{D}{N} \left[ \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial^2 p}{\partial y^2} \right] + \frac{D}{R} \left[ \frac{\partial^2 M}{\partial x^2} + \mu \frac{\partial^2 M}{\partial y^2} \right] \end{aligned} \quad (3.28)$$

Similarly, one obtains for the moments  $M_y$  and  $M_{xy}$  the expressions

$$\begin{aligned} M_y = & -D \left[ \frac{\partial^2 w_o}{\partial y^2} + \mu \frac{\partial^2 w_o}{\partial x^2} \right] + \frac{h^3}{6} F_1 \frac{\partial Q_y}{\partial y} - \frac{\mu h^3 p}{12(1-\mu)} F_1 \\ & - \frac{D}{N} \left[ \frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \right] + \frac{D}{R} \left[ \frac{\partial^2 M}{\partial y^2} + \mu \frac{\partial^2 M}{\partial x^2} \right] \end{aligned} \quad (3.29)$$

$$\begin{aligned} M_{xy} = & D(1-\mu) \frac{\partial^2 w_o}{\partial x \partial y} - \frac{h^3}{12} F_1 \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) \\ & + \frac{D(1-\mu)}{N} \frac{\partial^2 p}{\partial x \partial y} - \frac{D(1-\mu)}{R} \frac{\partial^2 M}{\partial x \partial y} \end{aligned} \quad (3.30)$$

The remaining two equations of equilibrium are

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x \quad (3.31)$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \quad (3.32)$$

By substituting equations (3.28) and (3.30) in equation (3.31), one

obtains

$$Q_x - \frac{h^3 F_1}{12} \Delta Q_x = -D \frac{\partial}{\partial x} \Delta w_o - \frac{h^3 F_1}{12(1-\mu)} \frac{\partial p}{\partial x} - \frac{D}{N} \frac{\partial}{\partial x} \Delta p + \frac{D}{R} \frac{\partial}{\partial x} \Delta M \quad (3.33)$$

Similarly, substitution of equations (3.29) and (3.30) into equation (3.32) yields

$$Q_y - \frac{h^3 F_1}{12} \Delta Q_y = -D \frac{\partial}{\partial y} \Delta w_o - \frac{h^3 F_1}{12(1-\mu)} \frac{\partial p}{\partial y} - \frac{D}{N} \frac{\partial}{\partial y} \Delta p + \frac{D}{R} \frac{\partial}{\partial y} \Delta M \quad (3.34)$$

Finally, on substituting equations (3.33) and (3.34) in equation (3.7) yields the plate differential equation in terms of displacement  $w_o$

$$D \Delta^2 w_o = p - \frac{h^3 F_1}{6(1-\mu)} \Delta p + \frac{\mu h^3 F_1}{12(1-\mu)} \Delta p - \frac{D}{N} \Delta^2 p + \frac{D}{R} \Delta^2 M \quad (3.35)$$

### 3.2 Governing Equations for the Inplane Problem

On substituting for  $\sigma_x, \sigma_y$  and  $\tau_{xy}$  from equations (3.22), (3.23) and (3.24) into

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz ; N_y = \int_{-h/2}^{h/2} \sigma_y dz ; N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz \quad (3.36)$$

and further making use of the inplane equilibrium equation

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (3.37)$$

results in the following differential equation in terms of displacement  $u_o$  and  $v_o$

$$\begin{aligned} \frac{\partial^2 u_o}{\partial x^2} + \frac{(1-\mu)}{2} \frac{\partial^2 u_o}{\partial y^2} + \frac{(1+\mu)}{2} \frac{\partial^2 v_o}{\partial x \partial y} &= \frac{(1+\mu)}{Eh} F_2 \frac{\partial p}{\partial x} \\ &+ \frac{F_4}{Eh} \frac{\partial}{\partial x} \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) - \frac{(1-\mu^2)}{Eh} F_2 \left( \frac{\partial^2 Q_x}{\partial x^2} + \frac{\partial^2 Q_x}{\partial y^2} \right) \end{aligned} \quad (3.38)$$

where:

$$F_2 = \int_{-h/2}^{h/2} f_1(z) dz, \quad F_4 = \int_{-h/2}^{h/2} f_3(z) dz \quad (3.38.1)$$

Similarly, operating on the other inplane equilibrium equation

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (3.39)$$

yields

$$\begin{aligned} \frac{\partial^2 v_o}{\partial y^2} + \frac{(1-\mu)}{2} \frac{\partial^2 v_o}{\partial x^2} + \frac{(1+\mu)}{2} \frac{\partial^2 u_o}{\partial x \partial y} \\ = \frac{(1+\mu)}{Eh} F_2 \frac{\partial p}{\partial y} + \frac{F_4}{Eh} \frac{\partial}{\partial y} \left( \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right) \end{aligned}$$

$$- \frac{(1-\mu^2)}{Eh} F_2 \left[ \frac{\partial^2 Q_y}{\partial x^2} + \frac{\partial^2 Q_y}{\partial y^2} \right] \quad (3.40)$$

### 3.3 Boundary Conditions

Physical interpretation for the terms  $\phi_x, \phi_y$  follows the same reasoning previously used in [10]. Thus  $\phi_x$  is the rotation of a vertical element  $x = \text{constant}$  of the plate and  $\phi_y$  is the rotation of a vertical element  $y = \text{constant}$  of the plate. Also, average displacement functions  $\bar{u}$ ,  $\bar{v}$  and  $\bar{w}$  are used here in all boundary conditions where [11] <sup>(1)</sup>

$$\bar{w} = w_o + \frac{P}{N} - \frac{M}{R} \quad (3.41)$$

Since the order of equations in bending is six and in inplane problem is four, three boundary conditions are needed to be specified for bending and two boundary conditions for the inplane problem at each end.

#### Bending Problem

##### 1. Simply Supported Edge ( $x = 0$ )

(1) See Appendix (A-4) for physical interpretation of

$\phi_x, \phi_y, \bar{u}, \bar{v},$  and  $\bar{w}$ .

$$\bar{w}(0,y) = 0, \phi_y(0,y) = 0, M_x(0,y) = 0 \quad (3.42)$$

2. Clamped Edge ( $x = 0$ )

$$\bar{w}(0,y) = 0, \phi_y(0,y) = 0, \phi_x(0,y) = 0 \quad (3.43)$$

3. Free Edge ( $x = 0$ )

$$M_x(0,y) = 0, Q_x(0,y) = 0, M_{xy}(0,y) = 0 \quad (3.44)$$

Inplane Problem

1. Edge Clamped Against Stretching ( $x = 0$ )

$$\bar{u}(0,y) = 0, \bar{v}(0,y) = 0 \quad (3.45)$$

2. Edge Free to Stretch ( $x = 0$ )

$$N_x(0,y) = 0, \bar{v}(0,y) = 0 \quad (3.46)$$

**3.4 Derivation of the Function  $f_1(z)$**

In order to derive the exact form of  $f_1(z)$  that satisfies the four boundary conditions given by equations (3.2) and (3.10), one starts with the stress differential equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \quad (3.47.1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \quad (3.47.2)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \quad (3.47.3)$$

Solving for  $\frac{\partial \tau_{xz}}{\partial z}$  and  $\frac{\partial \tau_{yz}}{\partial z}$  from equations (3.47.1) and (3.47.2) by using expressions for  $\sigma_x$ ,  $\sigma_y$ ,  $\tau_{xy}$  from equations (3.22), (3.23) and (3.24) and then substituting the result in the derivative of equation (47.3) with respect to  $z$  yields

$$\begin{aligned} \frac{E}{(1-\mu^2)} \left[ z \Delta^2 w_o + \left( \frac{2-\mu}{2G} \right) f_1(z) \Delta p + \frac{f_3(z)}{E} \Delta^2 p \right. \\ \left. - \frac{2\mu}{Eh^3} z^3 \Delta^2 M - \frac{\partial}{\partial x} \Delta u_o - \frac{\partial}{\partial y} \Delta v_o \right] \\ + p f_1''(z) = 0 \end{aligned} \quad (3.48)$$

Differentiating equation (3.48) twice with respect to  $z$  and using the relation  $f_3''(z) = f_1(z)$  yields the following fourth order differential equation in  $f_1(z)$

$$p f_1^{(iv)}(z) + \frac{(2-\mu)}{(1-\mu)} f_1''(z) \Delta p + \frac{f_1(z)}{(1-\mu^2)} \Delta^2 p = \frac{12\mu}{Eh^3} z \Delta^2 M \quad (3.49)$$

Expand the loading function  $p(x,y)$  in double Fourier series

$$p(x,y) = \sum_m \sum_n p_{mn} \sin \alpha_m x \sin \beta_n y \quad (3.50.1)$$



The solution for  $M$  can be shown to be :

$$M = M_h + M_p$$

where  $M_h$  is the homogeneous part of the solution (i.e when  $p = 0$ ) and  $M_p$  the particular solution.

Substituting for  $M$  in equation (3.49) above, one obtains for the homogeneous part of the solution corresponding to  $p = 0$  the relationship that

$$\Delta^2 M_h = 0 \quad (3.50.1a)$$

Relation (3.50.1a) indicates that it is the particular solution of  $M$  that plays a role in determination of the function  $f_1(z)$ .

The particular solution for  $M(x,y)$  corresponding to the loading  $p(x,y)$  given by (3.50.1) may be taken to be of the form

$$M_p(x,y) = \sum_m \sum_n M_{mn} \sin \alpha_m x \sin \beta_n y \quad (3.50.2)$$

Substituting the expansions given by equations (3.50.1) and (3.50.2) into equation (3.49) and dividing by  $p_{mn}$  yields

$$f_1^{(iv)}(z) - \bar{A}f_1''(z) + \bar{B}f_1(z) = \bar{C}z \quad (3.50.3)$$

where

$$\bar{A} = \frac{(2-\mu)}{(1-\mu)} (\alpha_m^2 + \beta_n^2) \quad (3.50.4)$$

$$\bar{B} = \frac{(\alpha_m^2 + \beta_n^2)^2}{(1-\mu^2)} \quad (3.50.5)$$

$$\bar{C} = \frac{12\mu M_{mn}}{h^3(1-\mu^2)P_{mn}} (\alpha_m^2 + \beta_n^2) \quad (3.50.6)$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b} \quad (3.50.7)$$

Equation (3.50.3) is a fourth order non-homogeneous differential equation in  $f_1(z)$  whose solution is given by

$$\begin{aligned} f_1(z) = f_{1p}(z) + f_{1h}(z) = & A_0 + A_1 z + A_2 \cosh \bar{a}z \\ & + A_3 \sinh \bar{a}z + A_4 \cosh \bar{b}z + A_5 \sinh \bar{b}z \end{aligned} \quad (3.51)$$

where

$$\bar{a} = \frac{\sqrt{(\bar{A} + \sqrt{\bar{A}^2 - 4\bar{B}})/2}}{2} \quad (3.51.1)$$

$$\bar{b} = \frac{\sqrt{(\bar{A} - \sqrt{\bar{A}^2 - 4\bar{B}})/2}}{2} \quad (3.51.2)$$

and  $f_{1p}(z)$  is the particular solution as given by  $A_0 + A_1 z$ , and  $f_{1h}(z)$  being the homogeneous solution. Coefficients in the particular solution are readily found to be

$$A_0 = 0 \quad (3.52)$$

$$A_1 = \frac{12\mu M_{mn}}{h^3 p_{mn}} \quad (3.53)$$

and the constants  $A_2$  through  $A_5$  involved in the homogeneous solution are found by using the four conditions given by equations (3.2) and (3.10).

Subsequent to obtaining  $f_1(z)$ , all other functions dependent on  $f_1(z)$  are readily obtained and given by:

$$\begin{aligned} f_2(z) = & \frac{A_1}{2} z^2 + \frac{A_2}{a} \sinh \bar{a}z + \frac{A_3}{a} \cosh \bar{a}z \\ & + \frac{A_4}{b} \sinh \bar{b}z + \frac{A_5}{b} \cosh \bar{b}z + C_1 \end{aligned} \quad (3.54)$$

$$\begin{aligned} f_3(z) = & \frac{A_1}{6} z^3 + \frac{A_2}{a^2} \cosh \bar{a}z + \frac{A_3}{a^2} \sinh \bar{a}z \\ & + \frac{A_4}{b^2} \cosh \bar{b}z + \frac{A_5}{b^2} \sinh \bar{b}z + C_1 z + C_2 \end{aligned} \quad (3.55)$$

$$\begin{aligned} F_1 = & A_1 + \frac{12}{h^3} \left[ \frac{h}{a} \cosh \frac{\bar{a}h}{2} - \frac{2}{a^2} \sinh \frac{ah}{2} \right] A_3 \\ & + \frac{12}{h^3} \left[ \frac{h}{b} \cosh \frac{bh}{2} - \frac{2}{b^2} \sinh \frac{\bar{b}h}{2} \right] A_5 \end{aligned} \quad (3.56)$$

$$\begin{aligned}
F_3 = A_1 + \frac{12}{h^3} \left[ \frac{h}{\bar{a}} \cosh \frac{\bar{a}h}{2} - \frac{2}{\bar{a}^2} \sinh \frac{\bar{a}h}{2} \right] A_3 \\
+ \frac{12}{h^3} \left[ \frac{h}{\bar{b}} \cosh \frac{\bar{b}h}{2} - \frac{2}{\bar{b}^2} \sinh \frac{\bar{b}h}{2} \right] A_5 \quad (3.57)
\end{aligned}$$

$$F_2 = \left[ \frac{2}{\bar{a}} \sinh \frac{\bar{a}h}{2} \right] A_2 + \left[ \frac{2}{\bar{b}} \sinh \frac{\bar{b}h}{2} \right] A_4 \quad (3.58)$$

$$F_4 = C_2 h + \left[ \frac{2}{\bar{a}^3} \sinh \frac{\bar{a}h}{2} \right] A_2 + \left[ \frac{2}{\bar{b}^3} \sinh \frac{\bar{b}h}{2} \right] A_4 \quad (3.59)$$

The constant  $C_1$  appearing in  $f_2(z)$  is found by imposing the condition (with no loss in generality) that

$$w(x, y, 0) = w_0(x, y) \quad (3.60)$$

in equation (3.15) resulting in

$$C_1 = - \left[ \frac{A_3}{\bar{a}^3} + \frac{A_5}{\bar{b}^3} \right] \quad (3.61)$$

Similarly the constant  $C_2$  appearing in  $f_3(z)$  is found by imposing the condition that

$$u(x, y, 0) = u_0(x, y) \quad (3.62)$$

in equation (19) resulting in

$$C_2 = \frac{2(1+\mu)}{\alpha_m^2} (A_2 + A_4) - \left( \frac{A_2}{\bar{a}^2} + \frac{A_4}{\bar{b}^2} \right) \quad (3.63)$$

As an additional check on the particular solution for plate deflection  $w_o$ , one may differentiate equation (3.48) with respect to  $z$  and then set  $z = 0$  in the resulting expression which yields

$$w_{oo} = \frac{h^3}{12} \left[ \frac{P_{mn}}{(\alpha_m^2 + \beta_n^2)^2} \right] \left[ AA_1 - BC_1 - A_3 \left( \frac{\bar{a}^3}{\bar{a}} - \bar{a}A + \frac{B}{\bar{a}} \right) \right. \\ \left. - A_5 \left( \frac{\bar{b}^3}{\bar{b}} - \bar{b}A + \frac{B}{\bar{b}} \right) \right] \quad (3.64)$$

where:

$$w_o = \sum_m \sum_n w_{oo} \sin \alpha_m x \sin \beta_n y \quad (3.65)$$

and  $p(x,y)$  is as given by equation (3.50.1).

It should be noticed that equation (3.64) give particular solution for  $w_o$  which should coincide with the particular solution obtained from the differential equation derived for the plate deflection  $w_o$  i.e. equation (3.35).

## Chapter 4

### SOLUTION OF PROBLEM BY SEMI-INVERSE LEVY TYPE METHOD

#### 4.1 Solution of the Bending Problem

##### 4.1.1 Derivation of the Governing Equations

From work in Chap. 3, one has the following:

$$M_x = D \left[ \frac{\partial \varphi_x}{\partial x} + \mu \frac{\partial \varphi_y}{\partial y} + \kappa p \right] \quad (4.1)$$

$$M_y = D \left[ \frac{\partial \varphi_y}{\partial y} + \mu \frac{\partial \varphi_x}{\partial x} + \kappa p \right] \quad (4.2)$$

$$M_{xy} = \frac{-D(1-\mu)}{2} \left[ \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right] \quad (4.3)$$

where:

$$\varphi_x = - \frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N} \frac{\partial p}{\partial x} + \frac{1}{R} \frac{\partial M}{\partial x} \quad (4.4)$$

$$\phi_y = -\frac{\partial w_o}{\partial y} + \frac{Q_x}{S} - \frac{1}{N} \frac{\partial p}{\partial y} + \frac{1}{R} \frac{\partial M}{\partial y} \quad (4.5)$$

$$S = \frac{G}{F_1} \quad (4.6)$$

$$N = \frac{E}{F_3} \quad (4.7)$$

$$R = \frac{10Eh}{3\mu} \quad (4.8)$$

$$\kappa = \frac{\mu(1+\mu)F_1}{E} \quad (4.9)$$

$$F_1 = \frac{12}{h^3} \int_{-h/2}^{+h/2} z f_1(z) dz \quad (4.10)$$

$$F_3 = \frac{12}{h^3} \int_{-h/2}^{+h/2} z f_3(z) dz \quad (4.11)$$

Using equations (4.4) and (4.5) and the following equation:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \quad (4.12)$$

one obtains alternate forms for  $M_x$ ,  $M_y$  and  $M_{xy}$  as:

$$M_x = -D \left[ \frac{\partial^2 w_o}{\partial x^2} + \mu \frac{\partial^2 w_o}{\partial y^2} \right] + \frac{h^3}{6} F_1 \frac{\partial Q_x}{\partial x} - \frac{\mu h^3 F_1}{12(1-\mu)} p$$

$$\frac{-D}{N} \left( \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial^2 p}{\partial y^2} \right) + \frac{D}{R} \left( \frac{\partial^2 M}{\partial x^2} + \mu \frac{\partial^2 M}{\partial y^2} \right) \quad (4.13)$$

$$M_y = -D \left[ \frac{\partial^2 w_o}{\partial y^2} + \mu \frac{\partial^2 w_o}{\partial x^2} \right] + \frac{h^3}{6} F_1 \frac{\partial Q_y}{\partial y} - \frac{\mu h^3 F_1}{12(1-\mu)} p$$

$$\frac{-D}{N} \left( \frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \right) + \frac{D}{R} \left( \frac{\partial^2 M}{\partial y^2} + \mu \frac{\partial^2 M}{\partial x^2} \right) \quad (4.14)$$

$$M_{xy} = D(1-\mu) \frac{\partial^2 w_o}{\partial x \partial y} - \frac{h^3}{12} F_1 \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right)$$

$$+ \frac{D(1-\mu)}{N} \frac{\partial^2 p}{\partial x \partial y} - \frac{D(1-\mu)}{R} \frac{\partial^2 M}{\partial x \partial y} \quad (4.15)$$

where:

$$M = M_x + M_y$$

Defining average transverse displacement  $\bar{w}$  and average rotations  $\phi_x, \phi_y$  as (Appendix A-4)

$$\bar{w} = w_o + \frac{p}{N} - \frac{M}{R} \quad (4.16)$$

$$\phi_x = - \frac{\partial \bar{w}}{\partial x} + \frac{Q_x}{S} \quad (4.17)$$

$$\phi_y = - \frac{\partial \bar{w}}{\partial y} + \frac{Q_y}{S} \quad (4.18)$$

the set of equations (4.4), (4.5), (4.13), (4.14) and (4.15) are



rewritten in the form

$$M_x = -D \left( \frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2} \right) + \frac{h^3}{6} F_1 \frac{\partial Q_x}{\partial x} - \frac{\mu h^3 F_1}{12(1-\mu)} P \quad (4.19)$$

$$M_y = -D \left( \frac{\partial^2 \bar{w}}{\partial y^2} + \mu \frac{\partial^2 \bar{w}}{\partial x^2} \right) + \frac{h^3}{6} F_1 \frac{\partial Q_y}{\partial y} - \frac{\mu h^3 F_1}{12(1-\mu)} P \quad (4.20)$$

$$M_{xy} = D(1-\mu) \frac{\partial^2 \bar{w}}{\partial x \partial y} - \frac{h^3}{12} F_1 \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) \quad (4.21)$$

Eliminating shears from equations (4.19), (4.20), (4.21) by using equations (4.17) and (4.18), one obtains:

$$M_x = \left[ -D + \frac{h^3 F_1}{6} S \right] \frac{\partial^2 \bar{w}}{\partial x^2} - D\mu \frac{\partial^2 \bar{w}}{\partial y^2} + \frac{h^3 F_1}{6} S \frac{\partial \varphi_x}{\partial x} - \frac{h^3 \mu F_1}{12(1-\mu)} P \quad (4.22)$$

$$M_y = \left[ -D + \frac{h^3 F_1}{6} S \right] \frac{\partial^2 \bar{w}}{\partial y^2} - D\mu \frac{\partial^2 \bar{w}}{\partial x^2} + \frac{h^3 F_1}{6} S \frac{\partial \varphi_y}{\partial y} - \frac{h^3 \mu F_1}{12(1-\mu)} P \quad (4.23)$$

$$M_{xy} = \left[ D(1-\mu) - \frac{h^3 F_1}{6} S \right] \frac{\partial^2 \bar{w}}{\partial x \partial y} - \frac{h^3 F_1}{12} S \left( \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \right) \quad (4.24)$$

Using equations (4.17), (4.18) to eliminate shears in the equilibrium equations (3.31), (3.32), one obtains:

$$\left[ \left( -D + \frac{h^3 F_1}{6} S \right) \frac{\partial^3}{\partial x^3} - \left( D - \frac{h^3 F_1}{6} S \right) \frac{\partial^3}{\partial x \partial y^2} - S \frac{\partial}{\partial x} \right] \bar{w} \\ + \left[ \frac{h^3 F_1}{6} S \frac{\partial^2}{\partial x^2} + \frac{h^3 F_1}{12} S \frac{\partial^2}{\partial y^2} - S \right] \varphi_x$$

$$+ \left[ \frac{h^3 F_1 S}{12} \frac{\partial^2}{\partial x \partial y} \right] \varphi_y = \frac{\mu h^3 F_1}{12(1-\mu)} \frac{\partial p}{\partial x} \quad (4.25)$$

$$\begin{aligned} & \left[ \left( -D + \frac{h^3 F_1 S}{6} \right) \frac{\partial^3}{\partial y^3} - \left( D - \frac{h^3 F_1 S}{6} \right) \frac{\partial^3}{\partial x^2 \partial y} - S \frac{\partial}{\partial y} \right] \bar{w} \\ & + \left[ \frac{h^3 F_1 S}{12} \frac{\partial^2}{\partial x \partial y} \right] \varphi_x + \left[ \frac{h^3 F_1 S}{6} \frac{\partial^2}{\partial y^2} + \frac{h^3 F_1 S}{12} \frac{\partial^2}{\partial x^2} - S \right] \varphi_y \\ & = \frac{\mu h^3 F_1}{12(1-\mu)} \frac{\partial p}{\partial y} \end{aligned} \quad (4.26)$$

The third equation involving  $\bar{w}$ ,  $\varphi_x$ , and  $\varphi_y$  is obtained by substituting equations (4.17) and (4.18) into equation (4.12):

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \bar{w} + \left[ \frac{\partial}{\partial x} \right] \varphi_x + \left[ \frac{\partial}{\partial y} \right] \varphi_y = \frac{-p}{S} \quad (4.27)$$

The set of equations (4.25) through (4.27) represents a sixth order bending problem.

By using the set of equations (4.25), (4.26), and (4.27), the governing plate differential equation in terms of the average transverse displacement  $\bar{w}$  can be obtained as<sup>(1)</sup>:

$$M'(\Delta^3 \bar{w}) + N'(\Delta^2 \bar{w}) = A\Delta^2 p + B\Delta p + Cp \quad (4.28)$$

(1) See Appendix (A-1) for derivation of this equation.

where:

$$M' = \frac{h^3 F_1 SD}{12} \quad (4.29.1)$$

$$N' = - SD \quad (4.29.2)$$

$$A = - \frac{(2-\mu)}{(1-\mu)} \left( \frac{h^3 F_1}{12} \right)^2 S \quad (4.29.3)$$

$$B = + \frac{(3-2\mu)}{12(1-\mu)} h^3 F_1 S \quad (4.29.4)$$

$$C = - S \quad (4.29.5)$$

$$\Delta = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad (4.29.6)$$

### 4.1.2 Solution of Bending Problem by Semi-Inverse Levy type Method

For plates with the pair of edges at  $x=0$ ,  $x=a$  being simply supported, see figure 4.1 , the solution to equation (4.28) may be expressed in the Levy form as:

$$\bar{w}(x,y) = \bar{w}_1(x) + \bar{w}_2(x,y) \quad (4.30)$$

in which the governing equations to be satisfied by  $\bar{w}_1$  and  $\bar{w}_2$  are given by [with Load  $p = p(x)$ ]:

$$M' \frac{d^6 \bar{w}_1}{dx^6} + N' \frac{d^4 \bar{w}_1}{dx^4} = A \frac{d^4 p}{dx^4} + B \frac{d^2 p}{dx^2} + Cp \quad (4.31)$$

and:

$$M' \Delta^3 \bar{w}_2 + N' \Delta^2 \bar{w}_2 = 0 \quad (4.32)$$

Expanding the load in a half range sine series

$$p = \sum_{m=1}^{\infty} p_m \sin \alpha_m x \quad (4.33)$$

with:  $\alpha_m = \frac{m\pi}{a}$  (4.33.1)

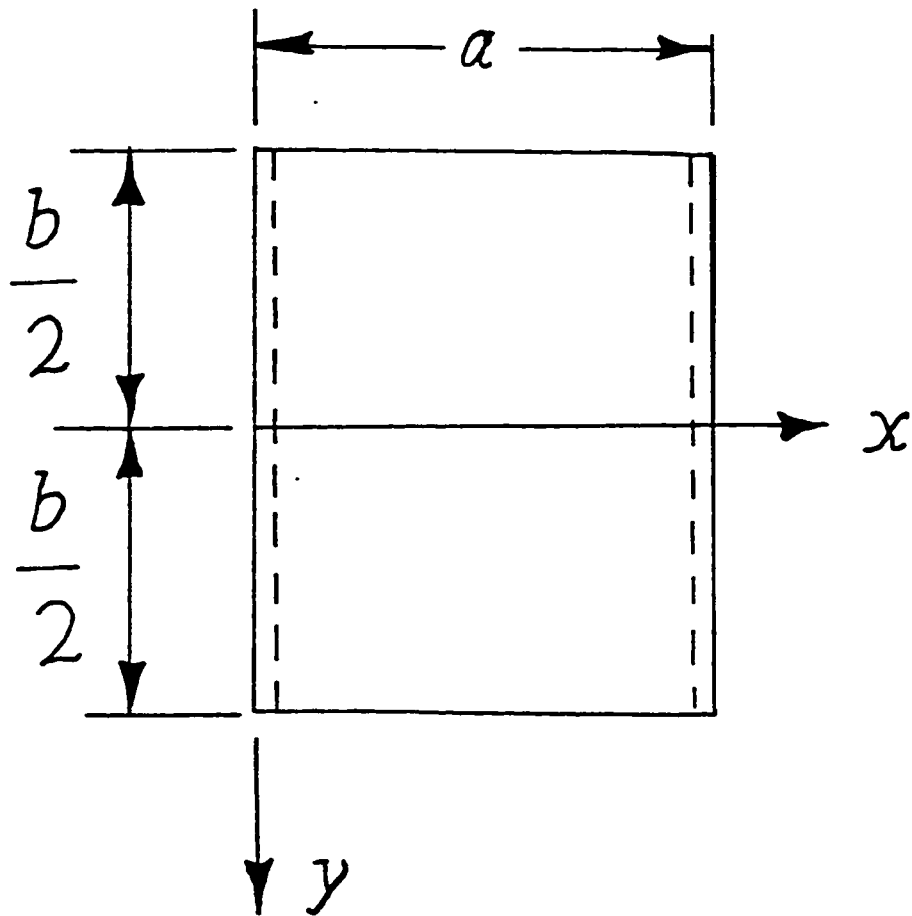


Figure 4.1 : Coordinate Axis For The Plate.

and the function  $\bar{w}_1$  expressed in the form:

$$\bar{w}_1 = \sum_{m=1}^{\infty} \beta_m \sin \alpha_m x \quad (4.34)$$

The Fourier coefficients  $\beta_m$  are then determined from equation (4.31)

to be:

$$\beta_m = \left\{ \frac{A\alpha_m^4 - B\alpha_m^2 + C}{-M'\alpha_m^6 + N'\alpha_m^4} \right\} P_m \quad (4.35)$$

The solution for  $\bar{w}_2$  may be taken in the following form:

$$\bar{w}_2 = \sum_{m=1}^{\infty} Y_m(y) \sin \alpha_m x \quad (4.36)$$

in which  $Y_m$  is obtained by substituting appropriate expressions for  $\bar{w}_2$  and its derivatives in equation (4.32). The function  $Y_m(y)$  can be shown to be:

$$Y_m(y) = A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + C_m \sinh \alpha_m y \\ + D_m \alpha_m y \cosh \alpha_m y + E_m \cosh \gamma_m y + F_m \sinh \gamma_m y \quad (4.37)$$

where:

$$\gamma_m^2 = \alpha_m^2 - \frac{N'}{M'} \quad (4.37-1)$$

Restricting the development to plates with loading and boundary conditions that are symmetrical with respect to the x-axis necessitates

$$C_m = D_m = F_m = 0$$

The complete solution for  $\bar{w}$  becomes:

$$\begin{aligned} \bar{w} &= \sum_{m=1}^{\infty} \bar{w}_m(y) \sin \alpha_m x \\ &= \sum_{m=1}^{\infty} (A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y \\ &\quad + E_m \cosh \gamma_m y + \beta_m) \sin \alpha_m x \end{aligned} \quad (4.38)$$

where:

$$\begin{aligned} \bar{w}_m(y) &= A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y \\ &\quad + E_m \cosh \gamma_m y + \beta_m \end{aligned} \quad (4.39)$$

In a similar way the same set of equations (4.25), (4.26), (4.27) can be used to obtain the governing equations for the average rotations  $\phi_x$  and  $\phi_y$ .

For the symmetric problem considered, the solutions are of the form:

$$\phi_x = \sum_{m=1}^{\infty} \phi_{xm}(y) \cos \alpha_m x \quad (4.40)$$

$$\phi_y = \sum_{m=1}^{\infty} \phi_{ym}(y) \sin \alpha_m x \quad (4.41)$$

where:

$$\begin{aligned} \varphi_{xm}(y) = & A_m' \cosh \alpha_m y + B_m' \alpha_m y \sinh \alpha_m y \\ & + E_m' \cosh \gamma_m y + \beta_m' \end{aligned} \quad (4.42)$$

$$\begin{aligned} \varphi_{ym}(y) = & C_m'' \sinh \alpha_m y + D_m'' \alpha_m y \cosh \alpha_m y \\ & + F_m'' \sinh \gamma_m y + \beta_m'' \end{aligned} \quad (4.43)$$

It should be noticed that due to symmetrical loading and boundary conditions with respect to the x-axis,  $\varphi_{xm}$  is even in "y" while  $\varphi_{ym}$  is odd in "y".

*Relations between the constants in  $\bar{w}$ ,  $\varphi_x$  and  $\varphi_y$  :*

In view of the order of the plate problem, there exists a linear dependence among the nine constants  $A_m$  through  $F_m''$ . One way of arriving at these relationships, together with the particular solutions  $\beta_m'$  and  $\beta_m''$ , is by the following procedure:

Substituting equations (4.1), (4.2), and (4.3) into equations (3.31) and (3.32) and using equations (4.17) and (4.18) to eliminate the transverse shears yields

$$\varphi_x + \frac{\partial \bar{w}}{\partial x} = \frac{D}{S} \left[ \frac{\partial^2 \varphi_x}{\partial x^2} + \frac{(1-\mu)}{2} \frac{\partial^2 \varphi_x}{\partial y^2} + \frac{(1+\mu)}{2} \frac{\partial^2 \varphi_y}{\partial x \partial y} + \kappa \frac{\partial p}{\partial x} \right] \quad (4.44)$$



$$\varphi_y + \frac{\partial \bar{w}}{\partial y} = \frac{D}{S} \left[ \frac{\partial^2 \varphi_y}{\partial y^2} + \frac{(1-\mu)}{2} \frac{\partial^2 \varphi_y}{\partial x^2} + \frac{(1+\mu)}{2} \frac{\partial^2 \varphi_x}{\partial x \partial y} + \kappa \frac{\partial p}{\partial y} \right] \quad (4.45)$$

Substituting for  $\varphi_x$ ,  $\varphi_y$ ,  $\bar{w}$  and  $p$  from equations (4.38), (4.40), (4.41), and (4.33), respectively, into equations (4.44) and (4.45), the following coupled ordinary differential equations in  $\varphi_{xm}(y)$  and  $\varphi_{ym}(y)$  are obtained:

$$\begin{aligned} & \left[ \frac{D(1-\mu)}{S} \frac{d^2}{dy^2} - \alpha_m^2 \frac{D}{S} - 1 \right] \varphi_{xm} + \left[ \frac{D(1+\mu)}{S} \alpha_m \frac{d}{dy} \right] \varphi_{ym} \\ & = \alpha_m \bar{w}_m - \kappa \frac{D}{S} \alpha_m p_m \end{aligned} \quad (4.46)$$

and:

$$\begin{aligned} & - \left[ \frac{D(1+\mu)}{S} \alpha_m \frac{d}{dy} \right] \varphi_{xm} + \left[ \frac{D}{S} \frac{d^2}{dy^2} - \frac{D(1-\mu)}{S} \alpha_m^2 - 1 \right] \varphi_{ym} \\ & = \frac{d\bar{w}_m}{dy} \end{aligned} \quad (4.47)$$

Uncoupling equations (4.46) and (4.47) for  $\varphi_{xm}$  and  $\varphi_{ym}$  results in:

$$\begin{aligned} & \left\{ \left[ \left( \frac{D}{S} \right)^2 \left( \frac{1-\mu}{2} \right) \right] \frac{d^4}{dy^4} + \left[ -(1-\mu) \left( \frac{D}{S} \right)^2 \alpha_m^2 - \frac{(3-\mu)D}{2S} \right] \frac{d^2}{dy^2} \right. \\ & \left. + \left[ \left( \frac{D}{S} \right)^2 \alpha_m^4 \frac{(1-\mu)}{2} + \frac{D}{S} \left( \frac{3-\mu}{2} \right) \alpha_m^2 + 1 \right] \right\} \{ \varphi_{xm} \} \end{aligned}$$

$$= \alpha_m \left[ \frac{D(1-\mu)}{S} \frac{d^2}{2} \frac{d^2}{dy^2} - \frac{D(1-\mu)}{S} \frac{\alpha_m^2}{2} - 1 \right] (\bar{w}_m - \kappa \frac{D}{S} P_m) \quad (4.48)$$

Similarly one obtains for  $\varphi_{ym}$ :

$$\begin{aligned} & \left\{ \left[ \left( \frac{D}{S} \right)^2 \left( \frac{1-\mu}{2} \right) \right] \frac{d^4}{dy^4} + \left[ -(1-\mu) \left( \frac{D}{S} \right)^2 \alpha_m^2 - \frac{(3-\mu)D}{2} \frac{D}{S} \right] \frac{d^2}{dy^2} \right. \\ & \left. + \left[ \left( \frac{D}{S} \right)^2 \frac{\alpha_m^4 (1-\mu)}{2} + \frac{D}{S} \left( \frac{3-\mu}{2} \right) \alpha_m^2 + 1 \right] \right\} \{\varphi_{ym}\} \\ & = \left[ \frac{D(1-\mu)}{S} \frac{d^3}{2} \frac{d^3}{dy^3} - \frac{D(1-\mu)}{S} \frac{\alpha_m^2}{2} \frac{d}{dy} - \frac{d}{dy} \right] (\bar{w}_m - \kappa P_m \frac{D}{S}) \quad (4.49) \end{aligned}$$

The required relationships among the constants, together with solutions for  $\beta_m^i$  and  $\beta_m^n$  are established by substituting relations in equations (4.33), (4.39), (4.42), and (4.43) into equations (4.48) and (4.49).

Then these relationships are given by:

$$A_m^i = -\alpha_m A_m - \frac{2D}{S} \alpha_m^3 B_m \quad (4.50.1)$$

$$B_m^i = -\alpha_m B_m \quad (4.50.2)$$

$$E_m^i = \frac{\alpha_m}{\left[ \frac{D}{S} (\gamma_m^2 - \alpha_m^2) - 1 \right]} E_m \quad (4.50.3)$$

$$\beta_m^i = \frac{-\alpha_m (\beta_m - \kappa \frac{D}{S} P_m)}{\left( \frac{D}{S} \alpha_m^2 + 1 \right)} \quad (4.50.4)$$

$$C_m^n = -\alpha_m A_m - \left( \frac{2D}{S} \alpha_m^3 + \alpha_m \right) B_m \quad (4.50.5)$$

$$D_m^n = -\alpha_m B_m \quad (4.50.6)$$

$$F_m^n = \frac{\gamma_m}{\left[ \frac{D}{S} (\gamma_m^2 - \alpha_m^2) - 1 \right]} E_m \quad (4.50.7)$$

$$\beta_m^n = 0 \quad (4.50.8)$$

**4.1.3 Derivation of the Non-Dimensional Form of  $f_1(z)$  and Related Constants:**

Consider the governing differential equation for  $f_1(z)$  (equation 3-49):

$$f_1^{(iv)}(z) - \bar{A} f_1''(z) + \bar{B} f_1(z) = \bar{C}z \quad (3-49)$$

where:

$$\bar{A} = \left[ \frac{2 - \mu}{1 - \mu} \right] \alpha_m^2 \quad (4-51.1)$$

$$\bar{B} = \left[ \frac{\alpha_m^4}{1 - \mu^2} \right] \quad (4-51.2)$$

$$\bar{C} = \frac{12\mu\alpha_m^4}{h^3(1 - \mu^2)} \frac{M_m}{P_m} \quad (4-51.3)$$

It can be shown that:

$$\bar{C} = \frac{\alpha_m^4}{(1 - \mu^2)} A_1 \quad (4-51.4)$$

Therefore, from equations (4-51.3), and (4-51.4), we get:

$$M_m = \frac{h^3 P_m}{12\mu} A_1 \quad (4-51.5)$$

where:

The particular solution for  $M = M_x + M_y$  can be written as:

$$M_p(x) = \sum_{m=1}^{\infty} M_m \sin \alpha_m x \quad (4-51.6)$$

It can be shown that  $M_m$  will be given by:<sup>(1)</sup>

$$M_m = P_m \left[ \frac{1 + \mu}{\alpha_m^2} + \frac{\mu h^3 F_1}{12} \right] \quad (4-52)$$

Substituting for  $F_1$  from equation (3-5b) and for  $M_m$  from equation (4-51.5) into the above equation results in:

$$\begin{aligned} & \left[ (1 - \mu^2) \right] A_1' + \frac{12\mu^2}{h^3} \left[ \frac{2}{\bar{a}^2} \sinh \frac{\bar{a}h}{2} - \frac{h}{\bar{a}} \cosh \frac{\bar{a}h}{2} \right] A_3 \\ & + \frac{12\mu^2}{h^3} \left[ \frac{2}{\bar{b}^2} \sinh \frac{\bar{b}h}{2} - \frac{h}{\bar{b}} \cosh \frac{\bar{b}h}{2} \right] A_5 = 0 \end{aligned} \quad (4-53)$$

Equation (4-53) together with equations (3-2) and (3-10) represent the boundary conditions that the function  $f_1(z)$  must satisfy.

From equation (3-5):

$$\begin{aligned} f_1(z) = & A_1' z + A_2 \cosh \bar{a}z + A_3 \sinh \bar{a}z + A_4 \cosh \bar{b}z \\ & + A_5 \sinh \bar{b}z \end{aligned}$$

-----  
(1) See Appendix (A-3) for derivation of this equation.

let :

$$A_1' = \frac{A_1}{h} \quad (4-54)$$

Then  $f_1(z)$  can be rewritten as:

$$\begin{aligned} f_1(z) = A_1 \left( \frac{z}{h} \right) + A_2 \cosh \bar{a}z + A_3 \sinh \bar{a}z \\ + A_4 \cosh \bar{b}z + A_5 \sinh \bar{b}z \end{aligned} \quad (4-55)$$

From the boundary condition on  $f_1(z)$ :  $f_1(-h/2) = -1$

one obtains

$$\begin{aligned} -\frac{1}{2}A_1 + \cosh \frac{\bar{a}h}{2} A_2 - \sinh \frac{\bar{a}h}{2} A_3 \\ + \cosh \frac{\bar{b}h}{2} A_4 - \sinh \frac{\bar{b}h}{2} A_5 = -1 \end{aligned} \quad (4-56)$$

and the boundary condition  $f_1(+h/2) = 0$  results in

$$\begin{aligned} \frac{1}{2} A_1 + \cosh \frac{\bar{a}h}{2} A_2 + \sinh \frac{\bar{a}h}{2} A_3 + \cosh \frac{\bar{b}h}{2} A_4 \\ + \sinh \frac{\bar{b}h}{2} A_5 = 0 \end{aligned} \quad (4-57)$$

the boundary condition  $f_1'(-h/2) = 0$  yields

$$\begin{aligned}
A_1 - \bar{a}h \sinh \frac{\bar{a}h}{2} A_2 + \bar{a}h \cosh \frac{\bar{a}h}{2} A_3 - \bar{b}h \sinh \frac{\bar{b}h}{2} A_4 \\
+ \bar{b}h \cosh \frac{\bar{b}h}{2} A_5 = 0
\end{aligned} \tag{4-58}$$

And the boundary condition  $f_1'(+h/2) = 0$  results in

$$\begin{aligned}
A_1 + \bar{a}h \sinh \frac{\bar{a}h}{2} A_2 + \bar{a}h \cosh \frac{\bar{a}h}{2} A_3 + \bar{b}h \sinh \frac{\bar{b}h}{2} A_4 \\
+ \bar{b}h \cosh \frac{\bar{b}h}{2} A_5 = 0
\end{aligned} \tag{4-59}$$

Thus equations (4-53), (4-56), (4-57), (4-58), and (4-59) can be solved for the constants  $A_1$  through  $A_5$ .

(Note that  $A_1'$  in equation (4-53) has to be replaced by  $A_1$  given by equation (4-54) ).

Therefore the function of  $f_1(z)$  given by equation (4-55) is now completely known.

*Solution for other functions and constants related to  $f_1(z)$ :*

The expression  $F_1$  will be rewritten in the following form:

$$F_1 = \frac{1}{h} \bar{F}_1 \quad (4-59.1)$$

where:

$$\begin{aligned} \bar{F}_1 = A_1 + 12 \left[ \frac{1}{\bar{a}h} \cosh \frac{\bar{a}h}{2} - \frac{2}{(\bar{a}h)^2} \sinh \frac{\bar{a}h}{2} \right] A_3 \\ + 12 \left[ \frac{1}{\bar{b}h} \cosh \frac{\bar{b}h}{2} - \frac{2}{(\bar{b}h)^2} \sinh \frac{\bar{b}h}{2} \right] A_5 \end{aligned} \quad (4-59.2)$$

Similarly  $F_3$  is rewritten as:

$$F_3 = h\bar{F}_3 \quad (4-59.3)$$

where:

$$\begin{aligned} \bar{F}_3 = \frac{1}{40} A_1 + \frac{12}{(\bar{a}h)^3} \left[ \cosh \frac{\bar{a}h}{2} - \frac{2}{(\bar{a}h)} \sinh \frac{\bar{a}h}{2} \right] A_3 \\ + \frac{12}{(\bar{b}h)^3} \left[ \cosh \frac{\bar{b}h}{2} - \frac{2}{(\bar{b}h)} \sinh \frac{\bar{b}h}{2} \right] A_5 + \bar{C}_1 \end{aligned} \quad (4-59.4)$$

in which

$$C_1 = h\bar{C}_1 \quad (5-59.5)$$



and:

$$\bar{C}_1 = \left[ \frac{1}{\bar{a}h} A_3 + \frac{1}{\bar{b}h} A_5 \right] \quad (4-59.6)$$

$F_2$  is rewritten as:

$$F_2 = h\bar{F}_2 \quad (4-59.7)$$

where:

$$\bar{F}_2 = \left[ \frac{2}{\bar{a}h} \sinh \frac{\bar{a}h}{2} \right] A_2 + \left[ \frac{2}{\bar{b}h} \sinh \frac{\bar{b}h}{2} \right] A_4 \quad (4-59.8)$$

$F_4$  is rewritten as:

$$F_4 = h^3\bar{F}_4 \quad (4-59.9)$$

where:

$$\begin{aligned} \bar{F}_4 = & \left[ \frac{2}{(\bar{a}h)^3} \sinh \frac{\bar{a}h}{2} \right] A_2 + \left[ \frac{2}{(\bar{b}h)^3} \sinh \frac{\bar{b}h}{2} \right] A_4 \\ & + \bar{C}_2 \end{aligned} \quad (4-59.10)$$

in which

$$C_2 = h^2\bar{C}_2 \quad (4-59.11)$$

and:

$$\begin{aligned}\bar{C}_2 &= \left[ \frac{2(1 + \mu)}{\alpha_m^2 h^2} - \frac{1}{(\bar{a}h)^2} \right] A_2 \\ &+ \left[ \frac{2(1 + \mu)}{\alpha_m^2 h^2} - \frac{1}{(\bar{a}h)^2} \right] A_4\end{aligned}\quad (4-59.12)$$

The function  $f_2(z)$  is rewritten as:

$$f_2(z) = h\bar{f}_2(z) \quad (4-59.13)$$

where:

$$\begin{aligned}\bar{f}_2(z) &= \left[ \frac{1}{2} \left( \frac{z}{h} \right)^2 \right] A_1 + \left[ \frac{1}{(\bar{a}h)} \sinh \bar{a}z \right] A_2 \\ &+ \left[ \frac{1}{(\bar{a}h)} \cosh \bar{a}z \right] A_3 + \left[ \frac{1}{(\bar{b}h)} \sinh \bar{b}z \right] A_4 \\ &+ \left[ \frac{1}{(\bar{b}h)} \cosh \bar{b}z \right] A_5 + \bar{C}_1\end{aligned}\quad (4-59.14)$$

And the function  $f_3(z)$  is rewritten as:

$$f_3(z) = h^2 \bar{f}_3(z) \quad (4-59.15)$$

where:

$$\bar{f}_3(z) = \left[ \frac{1}{6} \left( \frac{z}{h} \right)^3 \right] A_1 + \left[ \frac{1}{(\bar{a}h)^2} \cosh \bar{a}z \right] A_2$$

$$\begin{aligned}
& + \left[ \frac{1}{(\bar{a}h)^2} \sinh \bar{a}z \right] A_3 + \left[ \frac{1}{(\bar{b}h)^2} \cosh \bar{b}z \right] A_4 \\
& + \left[ \frac{1}{(\bar{b}h)^2} \sinh \bar{b}z \right] A_5 + \bar{C}_1 \left( \frac{z}{h} \right) + \bar{C}_2 \quad (4-59.16)
\end{aligned}$$

Having all the functions and constants related to  $f_1(z)$  written in a non-dimensional form, one proceeds now to write the other expressions in a non-dimensional form as follows:

The constant  $\beta_m$  appearing in equation (4-38) is rewritten as follows:

$$\beta_m = k_1 \frac{p_o a^4}{Eh^3} \quad (4-59.17)$$

where:

$$\begin{aligned}
k_1 = 48 & \left[ (2 - \mu)(1 + \mu) \frac{\bar{F}_1^2}{144} (m\pi)^4 (h/a)^4 \right. \\
& \left. + (3 - 2\mu)(1 + \mu) \frac{\bar{F}_1}{12} (m\pi)^2 (h/a)^2 + 1 - \mu^2 \right] \\
& / \left\{ (m\pi)^5 \left[ \frac{\bar{F}_1}{12} (m\pi)^2 (h/a)^2 + 1 \right] \right\} \quad (4-59.18)
\end{aligned}$$

The parameter  $\gamma_m$  appearing in equation (4-43.1) is rewritten as:

$$\begin{aligned}\gamma_m &= \frac{1}{h} \sqrt{(m\pi)^2 (h/a)^2 + \frac{12}{\bar{F}_1}} \\ &= \frac{1}{a} \bar{\gamma}_m = \frac{1}{a} \left[ \frac{a}{h} \sqrt{(m\pi)^2 (h/a)^2 + \frac{12}{\bar{F}_1}} \right]\end{aligned}\quad (4-59.19)$$

and  $\gamma_m \cdot \frac{b}{2}$  (a term that will be needed later) can be written as:

$$\frac{\gamma_m \cdot b}{2} = \frac{1}{2} \left( \frac{b}{a} \right) \left( \frac{a}{h} \right) \sqrt{(m\pi)^2 (h/a)^2 + \frac{12}{\bar{F}_1}} \quad (4-59.20)$$

One also has the terms:

$$\frac{D}{S} \alpha_m^2 + 1 = \frac{\bar{F}_1}{6(1-\mu)} (m\pi)^2 (h/a)^2 + 1 = \frac{1}{k_{11}} \quad (4-59.21)$$

And:

$$\frac{D}{S} (\gamma_m^2 - \alpha_m^2) - 1 = \frac{(1+\mu)}{(1-\mu)} = \frac{1}{k_{22}} \quad (4-59.22)$$

#### 4.1.4 Expressions For Moments and Shear Forces in the Plate

Making use of the relations in equations (4-50.1) to (4-50.8), one can write:

$$\varphi_{xm} = A_m (-\alpha_m \cosh \alpha_m y)$$

$$\begin{aligned}
& + B_m \left( -\frac{2D}{S} \alpha_m^3 \cosh \alpha_m y - \alpha_m^2 y \sinh \alpha_m y \right) \\
& + (k_{22} \alpha_m \cosh \gamma_m y) E_m + \beta_m' \quad (4-60)
\end{aligned}$$

$$\begin{aligned}
\phi_{ym} & = (-\alpha_m \sinh \alpha_m y) A_m \\
& + \left[ -\left( \frac{2D}{S} \alpha_m^3 + \alpha_m \right) \sinh \alpha_m y - \alpha_m^2 y \cosh \alpha_m y \right] B_m \\
& + (k_{22} \gamma_m \sinh \gamma_m y) E_m \quad (4-61)
\end{aligned}$$

Substituting appropriate expressions using equations (4-60), (4-61) and (4-33) into equations (4-1), (4-2) and (4-3), results in expressions for the bending and twisting moments as:

$$\begin{aligned}
M_x & = \left\{ \left[ (1 - \mu)(m\pi)^2 \cosh \alpha_m y \right] A_m \right. \\
& + \left[ \frac{2\bar{F}_1 m\pi^4 (h/a)^2}{6} \cosh \alpha_m y - 2\mu(m\pi)^2 \cosh \alpha_m y \right. \\
& \left. + (1 - \mu)(\alpha_m y)(m\pi)^2 \sinh \alpha_m y \right] B_m \\
& - \left[ K_{22} \left[ (m\pi)^2 - \frac{\mu}{(h/a)^2} \left[ (m\pi)^2 (h/a)^2 + \frac{12}{\bar{F}_1} \right] \right] \cosh \gamma_m y \right] E_m \\
& \left. + \left[ \beta_m' + \overline{kp}_m \right] \left[ \frac{p_o a^2}{12(1 - \mu^2)} \right] \sin \alpha_m x \right\} \quad (4-62)
\end{aligned}$$

where:

$$\bar{\beta}_m = \frac{-(m\pi)^2(k_1 - k_2)k_{11}}{12(1 - \mu^2)} \quad (4-62.1)$$

where:

$$kp_m \frac{D}{S} = k_2 \left[ \frac{P_o a^4}{Eh^3} \right] \quad (4-62.2)$$

And:

$$k_2 = \frac{2\mu(1 + \mu) \bar{F}_1^2}{3(1 - \mu)(m\pi)} (h/a)^4 \quad (4-62.3)$$

And:

$$kp_m = \overline{kp_m} \left[ \frac{P_o a^2}{12(1 - \mu^2)} \right] \quad (4-62.4)$$

$$\overline{kp_m} = \frac{4\mu(1 + \mu) \bar{F}_1}{(m\pi)} (h/a)^2 \quad (4-62.5)$$

Similarly  $M_y$  can be written as:

$$M_y = \frac{P_o a^2}{12(1 - \mu)^2} \left\{ -(1 - \mu)(m\pi)^2 \cosh \alpha_m y A_m \right.$$

$$\begin{aligned}
& + \left[ \frac{-2\bar{F}_1 (m\pi)^4 (h/a)^2}{6} \cosh \alpha_m y \right. \\
& \quad \left. - 2(m\pi)^2 \cosh \alpha_m y - \alpha_m y (m\pi)^2 (1 - \mu) \sinh \alpha_m y \right] B_m \\
& + k_{22} \left[ \frac{1}{(h/a)^2} \left[ (m\pi)^2 (h/a)^2 + \frac{12}{\bar{F}_1} \right] - \mu (m\pi)^2 \right] E_m \cosh \gamma_m y \\
& \quad \left. - \mu \bar{\beta}_m + \overline{kp}_m \right\} \sin \alpha_m x \tag{4-63}
\end{aligned}$$

Similarly for  $M_{xy}$ :

$$\begin{aligned}
M_{xy} &= \frac{p_o a^2}{24(1 + \mu)} \left\{ 2(m\pi)^2 \sinh \alpha_m y A_m \right. \\
& \quad + \left[ \frac{4\bar{F}_1 (h/a)^2 (m\pi)^4}{6(1 - \mu)} \sinh \alpha_m y \right. \\
& \quad \left. + 2(m\pi)^2 \sinh \alpha_m y + 2(m\pi)^2 \alpha_m y \cosh \alpha_m y \right] B_m \\
& \quad \left. + \left[ (-2k_{22} (m\pi) \bar{\gamma}_m \sinh \gamma_m y) E_m \right] \right\} \cos \alpha_m x \tag{4-64}
\end{aligned}$$

Similarly the shear force  $Q_x$  can be written as:

$$Q_x = \frac{p_o a}{12(1 - \mu^2)} \left\{ - 2(m\pi)^3 \cosh \alpha_m y B_m \right.$$

$$\begin{aligned}
& + \left[ \frac{12k_{22}(m\pi)}{(h/a)^2 \bar{F}_1} \cosh \gamma_m y \right] E_m \\
& + \frac{6(1-\mu)(m\pi)}{\bar{F}_1 (h/a)^2} \left\{ \bar{\beta}_m + \bar{\beta}'_m \right\} \cos \alpha_m x \quad (4-65)
\end{aligned}$$

where:

$$\begin{aligned}
\beta_m D &= k_1 \left[ \frac{p_o a^4}{Eh^3} \right] D \\
&= \bar{\beta}_m \left[ \frac{p_o a}{12(1-\mu^2)} \right]
\end{aligned}$$

or:

$$\bar{\beta}_m = k_1 \quad (4-65.1)$$

The expression for  $Q_y$  can be written as:

$$\begin{aligned}
Q_y &= \left[ \frac{p_o a}{12(1-\mu^2)} \right] \left\{ (-2(m\pi)^3 \sinh \alpha_m y) B_m \right. \\
& \quad \left. + \left[ \frac{12k_{22}(\bar{\gamma}_m)}{\bar{F}_1 (h/a)^2} \sinh \gamma_m y \right] E_m \right\} \sin \alpha_m x \quad (4-66)
\end{aligned}$$



## 4.2 Solution of the Inplane Problem

### 4.2.1 Formulation in Terms of Average Inplane Displacements $\bar{u}$ and $\bar{v}$

To start with, expressions for average inplane displacements  $\bar{u}$  and  $\bar{v}$  are derived as follows:

Define:

$$\bar{u} = \frac{1}{h} \int_{-h/2}^{+h/2} u dz \quad (4-67)$$

and:

$$\bar{v} = \frac{1}{h} \int_{-h/2}^{+h/2} v dz \quad (4-68)$$

Substituting for  $u$  from equation (3-19) into equation (4-67), one obtains

$$\bar{u} = u_o + \frac{F_2}{Gh} Q_x - \frac{F_4}{Eh} \frac{\partial p}{\partial x} \quad (4-69)$$

Similarly substituting for  $v$  from equation (3-20) into equation (4-68), yields

$$\bar{v} = v_o + \frac{F_2}{Gh} Q_x - \frac{F_4}{Eh} \frac{\partial p}{\partial y} \quad (4-70)$$

Noting that:

$$M = M_x + M_y$$

Then from equations (4-1), (4-2) and the above equation, one obtains:

$$M = D \left[ (1 + \mu) \left( \frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} \right) + 2Kp \right] \quad (4-71.1)$$

Thus:

$$\frac{\partial^2 M}{\partial x^2} = D \left[ (1 + \mu) \left( \frac{\partial^3 \phi_x}{\partial x^3} + \frac{\partial^3 \phi_y}{\partial x^2 \partial y} \right) + 2K \frac{\partial^2 p}{\partial x^2} \right] \quad (4-71.2)$$

$$\frac{\partial^2 M}{\partial y^2} = D \left[ (1 + \mu) \left( \frac{\partial^3 \phi_x}{\partial x \partial y^2} + \frac{\partial^3 \phi_y}{\partial y^3} \right) + 2K \frac{\partial^2 p}{\partial y^2} \right] \quad (4-71.3)$$

Similarly using equations (4-69), (4-70), one has:

$$\begin{aligned} \frac{\partial u_o}{\partial x} + \mu \frac{\partial v_o}{\partial y} &= \left[ \frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \right] \\ &+ \frac{F_4}{Eh} \left( \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial^2 p}{\partial y^2} \right) \\ &- \frac{F_2}{Gh} \left( \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right). \end{aligned} \quad (4-71.4)$$

Also:

$$\frac{1}{G} \left[ \frac{\partial Q_x}{\partial x} + \mu \frac{\partial Q_y}{\partial y} \right] = \frac{1}{F_1} \left[ \left[ \frac{\partial \varphi_x}{\partial x} + \mu \frac{\partial \varphi_y}{\partial y} \right] + \left[ \frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2} \right] \right] \quad (4-71.5)$$

Using the previous expressions and equation (3-22), the stress  $\sigma_x$  can be written in terms of average displacements  $\bar{w}$ ,  $\bar{u}$ ,  $\bar{v}$  and average rotations  $\varphi_x$  and  $\varphi_y$  as follows:

$$\begin{aligned} \sigma_x = & \frac{E}{(1 - \mu^2)} \left[ \left( \frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2} \right) \left[ -z + \left( f_1(z) - \frac{F_2}{h} \right) \frac{1}{F_1} \right] \right. \\ & + \left. \left( \frac{\partial^3 \varphi_x}{\partial x^3} + \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \mu \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \mu \frac{\partial^3 \varphi_y}{\partial y^3} \right) \right. \\ & \left. \left[ D(1 + \mu) \left( -\frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right) \right] \right. \\ & + \left. \left[ \frac{\partial \varphi_x}{\partial x} + \mu \frac{\partial \varphi_y}{\partial y} \right] \left[ \left( f_1(z) - \frac{F_2}{h} \right) \frac{1}{F_1} \right] \right. \\ & + \left. \left( \frac{\partial^2 \bar{p}}{\partial x^2} + \mu \frac{\partial^2 \bar{p}}{\partial y^2} \right) \left[ \frac{z}{N} - \frac{f_3(z)}{E} + \frac{F_4}{Eh} \right] \right. \\ & + \left. 2KD \left[ -\frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right] \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \right] \\
& + \frac{\mu p}{(1 - \mu)} f_1(z) \tag{4-72}
\end{aligned}$$

$$\begin{aligned}
\sigma_y = & \frac{E}{(1 - \mu^2)} \left[ \left( \frac{\partial^2 \bar{w}}{\partial y^2} + \mu \frac{\partial^2 \bar{w}}{\partial x^2} \right) \left[ -z + \frac{1}{F_1} \left( f_1(z) - \frac{F_2}{h} \right) \right] \right. \\
& + \left[ \frac{\partial^3 \varphi_y}{\partial y^3} + \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \mu \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \mu \frac{\partial^3 \varphi_x}{\partial x^3} \right] \\
& \left. \left[ D(1 + \mu) \left( -\frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right) \right] \right. \\
& + \left[ \frac{\partial \varphi_y}{\partial y} + \mu \frac{\partial \varphi_x}{\partial x} \right] \left[ \frac{1}{F_1} \left( f_1(z) - \frac{F_2}{h} \right) \right] \\
& + \left[ \frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \right] \left[ \frac{z}{N} - \frac{f_3(z)}{E} + \frac{F_4}{Eh} \right. \\
& + 2KD \left. \left[ -\frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right] \right] \\
& + \left[ \frac{\partial \bar{v}}{\partial y} + \mu \frac{\partial \bar{u}}{\partial x} \right] \\
& + \frac{\mu p}{(1 - \mu)} f_1(z) \tag{4-73}
\end{aligned}$$

Noting that:

$$\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} = \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + \frac{2F_1}{Eh} \frac{\partial^2 p}{\partial x \partial y} - \frac{F_2}{Gh} \left( \frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} \right) \quad (4-73.1)$$

$$\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} = \frac{G}{F_1} \left( \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2 \frac{\partial^2 \bar{w}}{\partial x \partial y} \right) \quad (4-73.2)$$

$$\frac{\partial^2 M}{\partial x \partial y} = D \left[ (1 + \mu) \left( \frac{\partial^3 \phi_x}{\partial x^2 \partial y} + \frac{\partial^3 \phi_y}{\partial x \partial y^2} \right) + 2K \frac{\partial^2 p}{\partial x \partial y} \right] \quad (4-73.3)$$

and

$$\frac{\partial^2 w_o}{\partial x \partial y} = \frac{\partial^2 \bar{w}}{\partial x \partial y} - \frac{1}{N} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{R} \frac{\partial^2 M}{\partial x \partial y}$$

Using the above relations into equation (3-24), we get for  $\tau_{xy}$  :

$$\begin{aligned} \tau_{xy} = G & \left[ \frac{\partial^2 \bar{w}}{\partial x \partial y} \left( -2z + \frac{2}{F_1} \left( f_1(z) - \frac{F_2}{h} \right) \right) \right. \\ & + \left. \left[ \frac{\partial^3 \phi_x}{\partial x^2 \partial y} + \frac{\partial^3 \phi_y}{\partial x \partial y^2} \right] \left[ (1 + \mu) D \left( \frac{-2z}{R} + \frac{4\mu z^3}{Eh^3} \right) \right] \right. \\ & + \left. \left[ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \right] \left[ \frac{1}{F_1} \left( f_1(z) - \frac{F_2}{h} \right) \right] \right] \end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{\partial^2 p}{\partial x \partial y} \right] \left[ \frac{2z}{N} - \frac{2f_3(z)}{E} + 2 \frac{F_4}{Eh} \right. \\
& \quad \left. + 2KD \left[ \frac{-2z}{R} + \frac{4\mu z^3}{Eh^3} \right] \right] \\
& + \left[ \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right] \quad (4-74)
\end{aligned}$$

Using equations (3-36) yields expressions for inplane stress resultant

$N_x$ :

$$N_x = \frac{E}{(1 - \mu^2)} \left[ h \left[ \frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \right] \right] + \frac{\mu p}{(1 - \mu)} F_2 \quad (4-75)$$

Similarly using equation (4-73) into second of equations (3-3b) yields:

$$N_y = \frac{E}{(1 - \mu^2)} \left[ h \left[ \mu \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right] \right] + \frac{\mu p}{(1 - \mu)} F_2 \quad (4-76)$$

The expression for  $N_{xy}$  is given by

$$N_{xy} = Gh \left[ \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right] \quad (4-77)$$

Using equations (4-75), (4-76), and (4-77) into the inplane equilibrium equations (3-37) and (3-39), yields the inplane governing equations in terms of average inplane displacements  $\bar{u}$ ,  $\bar{v}$ :

$$\begin{aligned} \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{(1 + \mu)}{2} \frac{\partial^2 \bar{v}}{\partial x \partial y} + \frac{(1 - \mu)}{2} \frac{\partial^2 \bar{u}}{\partial y^2} \\ = \frac{-\mu(1 + \mu) F_2}{Eh} \frac{\partial p}{\partial x} \end{aligned} \quad (4-78)$$

And:

$$\begin{aligned} \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{(1 + \mu)}{2} \frac{\partial^2 \bar{u}}{\partial x \partial y} + \frac{(1 - \mu)}{2} \frac{\partial^2 \bar{v}}{\partial x^2} \\ = \frac{-\mu(1 + \mu) F_2}{Eh} \frac{\partial p}{\partial y} \end{aligned} \quad (4-79)$$

#### 4.2.2 Solution for $\bar{u}$ and $\bar{v}$ :

It can be shown that the inplane governing equations (4-78) and (4-79) can be uncoupled for  $\bar{u}$  and  $\bar{v}$  to give:

$$\Delta^2 \{\bar{u}\} = k_3 \frac{\partial}{\partial x} \{\Delta p\} \quad (4-80)$$

And:

$$\Delta^2 \{\bar{v}\} = k_3 \frac{\partial}{\partial y} \{\Delta p\} \quad (4-81)$$

where:

$$k_3 = \frac{-\mu(1 + \mu)F_2}{Eh} \quad (4-81.1)$$

Since from equation (4-33):

$$p = \sum p_m \sin \alpha_m x$$

thus

$$\frac{\partial}{\partial x} \Delta p = \sum -\alpha_m^3 p_m \cos \alpha_m x \quad (4-81-2)$$

Assume that  $\bar{u}$  will be of the following form:

$$\bar{u} = \sum \bar{u}_m(y) \cos \alpha_m x \quad (4-82)$$

Then:  $\Delta^2 \bar{u}$  from equation (4-82) is:

$$\begin{aligned} \Delta^2 \bar{u} &= \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right]^2 \bar{u} \\ &= \sum \left[ \alpha_m^4 \bar{u}_m - 2\alpha_m^2 \frac{d^2 \bar{u}_m}{dy^2} + \frac{d^4 \bar{u}_m}{dy^4} + \right] \cos \alpha_m x \quad (4-82.1) \end{aligned}$$

Substituting equations (4-82.1), (4-81.2) into equation (4-80) yields the governing equation for  $\bar{u}$  as

$$\frac{d^4 \bar{u}_m}{dy^4} - 2\alpha_m^2 \frac{d^2 \bar{u}_m}{dy^2} + \alpha_m^4 \bar{u}_m = -\alpha_m^3 p_m k_3 \quad (4-82.2)$$

The solution of the above linear differential equation is given by



$$\bar{u}_m = \bar{u}_p + \bar{u}_h \quad (4-82.3)$$

From equation (4-82.2) the particular solution for  $\bar{u}$  may be shown to be

$$\bar{u}_p = -k_3 \frac{p_m}{\alpha_m} \quad (4-82.4)$$

It can be shown that  $\bar{u}_h$  will be of the following form:

$$\begin{aligned} \bar{u}_h = & C_1 \cosh \alpha_m y + C_2 \alpha_m y \sinh \alpha_m y + C_3 \alpha_m y \cosh \alpha_m y \\ & + C_4 \sinh \alpha_m y \end{aligned} \quad (4-82.5)$$

Assuming that  $\bar{u}$  will be symmetric with respect to the x-axis, then:

$$C_3 = C_4 = 0$$

and equation (4-82.3) yields for  $\bar{u}_m$  the expression

$$\bar{u}_m = \left[ C_1 \cosh \alpha_m y + C_2 \alpha_m y \sinh \alpha_m y + \bar{u}_p \right] \quad (4-83)$$

Similarly:

$$\bar{v}_m = \left[ C'_3 \sinh \alpha_m y + C'_4 \alpha_m y \cosh \alpha_m y \right] \quad (4-84)$$

Note that  $\bar{v}$  is antisymmetric with respect to the x-axis.

To find relations between the constants in  $\bar{u}$  and those in  $\bar{v}$ , appropriate expressions using equations (4-83) and (4-84) are substituted into equation (4-78). This results in:

$$C'_3 = C_1 - k_4 C_2 \quad (4-84-1)$$

$$C'_4 = C_2$$

where:

$$k_4 = \frac{1 + k_1}{k_2} \quad (4-84-2)$$

and:

$$k_1 = \frac{1 - \mu}{2} \quad (4-84-3)$$

$$k_2 = \frac{1 + \mu}{2} \quad (4-84-4)$$

Thus equation (4-84) can be rewritten for  $\bar{v}_m$  as:

$$\bar{v}_m = \left[ C_1 (\sinh \alpha_m y) + C_2 ( \alpha_m y \cosh \alpha_m y - k_4 \sinh \alpha_m y ) \right] \quad (4-85)$$

### 4.3 Boundary Conditions for the Bending Problem

The plate will be always simply supported along the edges at  $x = 0$  and  $x = a$ . The edges at  $y = \pm b/2$  can be simply supported, clamped, or free.

**Case I: A Plate Uniformly Loaded and Simply Supported at  $y = \pm b/2$ .**

For a simply supported edge at  $y = \pm b/2$ , the boundary conditions that need to be satisfied are:

$$\bar{w}(x, \pm b/2) = 0 \quad (4-86.1)$$

$$\phi_x(x, \pm b/2) = 0 \quad (4-86.2)$$

$$M_y(x, \pm b/2) = 0 \quad (4-86.3)$$

Using equation (4-38) for  $\bar{w}$ , boundary condition in equation (4-86.1) gives:

$$\begin{aligned} A_m \left[ \cosh \frac{\alpha_m b}{2} \right] + B_m \left[ \frac{\alpha_m b}{2} \sinh \frac{\alpha_m b}{2} \right] \\ + E_m \left[ \cosh \frac{\gamma_m b}{2} \right] = -\beta_m \end{aligned} \quad (4-87)$$

From equation (4-17), one has:

$$\phi_x = -\frac{\partial \bar{w}}{\partial x} + \frac{Q_x}{S}$$

but :

$$\left. \frac{\partial \bar{w}}{\partial x} \right|_{y = \pm b/2} = 0 \quad (\text{since } \bar{w}(x, \pm b/2) = 0)$$

Thus  $\varphi_x(x, \pm b/2) = 0$  implies that  $\frac{Q_x}{S}(x, \pm b/2) = 0$  \*

From equation (4-65), one has:

$$\begin{aligned} & \left( 2(m\pi)^3 \cosh \frac{\alpha_m b}{2} \right) B_m - \left( \frac{12k_{22}(m\pi)}{(h/a)^2 \bar{F}_1} \cosh \frac{\gamma_m b}{2} \right) E_m \\ & = \frac{6(1-\mu)m\pi}{\bar{F}_1 (h/a)^2} (\bar{\beta}_m + \bar{\beta}'_m) \end{aligned} \quad (4-88)$$

For the boundary condition in equation (4-86.3), one gets from equation (4-1):

$$M_y \Big|_{y = \pm b/2} = D \left[ \frac{\partial \varphi_y}{\partial y} + Kp_m \right] \quad (4-88.1)$$

The term  $\frac{\partial \varphi_x}{\partial x} \Big|_{y = \pm b/2} = 0$  is missing in equation (4-88.1) \*

since  $\varphi_x(x, \pm b/2) = 0$  which implies that

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\* Such modifications in boundary conditions are necessary to avoid effects of ill conditioning

$$\frac{\partial \varphi_x}{\partial x} \Big|_{y = \pm b/2} = 0$$

Thus expansion of (4-88.1) results in

$$\begin{aligned} & \left[ -(m\pi)^2 \cosh \frac{\alpha_m b}{2} \right] A_m \\ & - \left[ 2(m\pi)^2 \cosh \frac{\alpha_m b}{2} + \frac{\alpha_m b}{2} (m\pi)^2 \sinh \frac{\alpha_m b}{2} \right] B_m \\ & - \left[ a^2 \gamma_m^2 \cosh \frac{\gamma_m b}{2} \right] E_m + \left( 1 - \frac{2}{\mu} \right) \overline{Kp}_m = 0 \end{aligned} \quad (4-89)$$

*Case II: Plate Uniformity Loaded and Clamped at  $y = \pm b/2$*

$$\overline{w}(x, \pm b/2) = 0 \quad (4-87)$$

$$\varphi_x(x, \pm b/2) = 0 \quad (4-88)$$

$$\varphi_y(x, \pm b/2) = 0 \quad (4-89.1)$$

from equation (4-61) and boundary condition in equation (4-89.1), one has:

$$\begin{aligned} & \left[ \alpha_m h \sinh \frac{\alpha_m b}{2} \right] A_m + \left[ \left[ \frac{2(m\pi)^3 (h/a)^3 F_1}{6(1-\mu)} \right. \right. \\ & \left. \left. + (m\pi)(h/a) \sinh \frac{\alpha_m b}{2} + \frac{\alpha_m b}{2} (m\pi)(h/a) \cosh \frac{\alpha_m b}{2} \right] B_m \end{aligned}$$

$$- \left[ k_{22}(\gamma_m h) \sinh \frac{\gamma_m b}{s} \right] E_m = 0 \quad (4-90)$$

**Case III: Plate Uniformly Loaded and Free at  $y = \pm b/2$**

Boundary conditions for this case are:

$$M_y(x, \pm b/2) = 0 \quad (4-91.1)$$

$$Q_y(x, \pm b/2) = 0 \quad (4-91.2)$$

$$M_{xy}(x, \pm b/2) = 0 \quad (4-91.3)$$

Once again, ill conditioning of the non-modified system led to numerical problems. The following equivalent set of equations were used instead:

$$M_y(x, \pm b/2) = 0 \quad (4-91.4)$$

$$Q_y - \frac{\partial M_{xy}}{\partial x} = 0 \quad (4-91.5)$$

$$Q_y = 0 \quad (4-91.6)$$

Note: If  $Q_y(x, \pm b/2) = 0$  in equation (4-91.6) then equation (4-91.5)

implies that:

$$\frac{\partial M_{xy}}{\partial x} \Big|_{y = \pm b/2} = 0 \quad \text{or} \quad M_{xy}(x, \pm b/2) = 0 \quad (\text{which is equa-})$$

tion 4-91.3)

Also from equation (4-65) for  $Q_x$ :

$$\begin{aligned} \frac{\partial Q_x}{\partial y} &= \frac{p_o a}{12(1 - \mu^2)} \left\{ \left[ - 2(m\pi)^3 \alpha_m \sinh \alpha_m y \right] B_m \right. \\ &\quad \left. + \left[ \frac{12k_{22}(m\pi) \gamma_m}{\bar{F}_1(h/a)^2} \sinh \gamma_m y \right] E_m \right\} \cos \alpha_m x \\ &= \alpha_m Y(y) \cos \alpha_m x \end{aligned} \quad (4-91.7)$$

where

$$\begin{aligned} Y(y) &= \frac{p_o a}{12(1 - \mu^2)} \left\{ \left[ - 2(m\pi)^3 \sinh \alpha_m y \right] B_m \right. \\ &\quad \left. + \left[ \frac{12k_{22} \bar{\gamma}_m}{\bar{F}_1(h/a)^2} \sinh \gamma_m y \right] E_m \right\} \end{aligned} \quad (4-91.8)$$

Also from previous work

$$Q_y = Y(y) \sin \alpha_m x \quad (4-66)$$

Thus the boundary condition that  $Q_y(x, \pm b/2) = 0$  implies that

$$Y(\pm b/2) = 0. \text{ (from equation (4-66))}$$

Thus equation (4-91.7) yields that

$$\frac{\partial Q_x}{\partial y} \Big|_{y = \pm b/2} = 0$$

From the above ( and from equation (4.21) for  $M_{xy}$ ) it is seen that:

$$\frac{\partial M_{xy}}{\partial x} \Big|_{y = \pm b/2} = D(1 - \mu) \frac{\partial^3 \bar{w}}{\partial x^2 \partial y} \Big|_{y = \pm b/2}$$

$$\begin{aligned} \frac{\partial M_{xy}}{\partial x} &= \frac{p_0 a^4}{12(1 - \mu^2)} \left[ - \left( (1 - \mu) \alpha_m^3 \sinh \alpha_m y \right) A_m \right. \\ &\quad \left. - B_m \left[ (1 - \mu) \alpha_m^3 \sinh \alpha_m y + (1 - \mu) \alpha_m^4 y \cosh \alpha_m y \right] \right. \\ &\quad \left. - \left[ (1 - \mu) \alpha_m^2 \gamma_m \sinh \gamma_m y \right] E_m \right] \sin \alpha_m x \quad (4-91.9) \end{aligned}$$

Substituting for  $Q_y$  from equation (4-66) and for  $\frac{\partial M_{xy}}{\partial x}$  from equation (4-91.9) into boundary condition in equation (4-91.5) yields

$$\begin{aligned} &\left[ (1 - \mu)(m\pi)^3 \sinh \frac{\alpha_m b}{2} \right] A_m + B_m \left[ - (1 + \mu)(m\pi)^3 \sinh \frac{\alpha_m b}{2} \right. \\ &\quad \left. + (1 - \mu) \frac{\alpha_m b}{2} (m\pi)^3 \cosh \frac{\alpha_m b}{2} \right] \\ &+ E_m \left[ 12 \frac{k_{22}}{\bar{F}_1 (h/a)^2} + (1 - \mu) (m\pi)^2 \right] a \gamma_m \sinh \frac{\gamma_m b}{2} = 0 \quad (4-92) \end{aligned}$$

Consider boundary condition as given by equation (4-91.6):



$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = 0 \quad (4-93.1)$$

Since  $\frac{\partial^2 \bar{w}}{\partial x \partial y} \Big|_{y = \pm b/2} = 0$ , it can be shown that

$$\frac{\partial M_y}{\partial y} = -D \left( \frac{\partial^3 \bar{w}}{\partial y^3} \right) + \frac{h^2 \bar{F}_1}{6} \frac{\partial^2 Q_y}{\partial y^2} \quad (4-93.2)$$

Also it can be shown that:

$$\begin{aligned} \frac{\partial^3 \bar{w}}{\partial y^3} = & \left[ (2\alpha_m^3 \sinh \alpha_m y) B_m \right. \\ & \left. + \left( \frac{12}{\bar{F}_1 (h/a)^2} \gamma_m \sinh \gamma_m y \right) E_m \right] \sin \alpha_m x \end{aligned} \quad (4-93.3)$$

and:

$$\frac{\partial^2 Q_y}{\partial y^2} = \frac{p_o a^4}{12(1 - \mu^2)} \left\{ \left( \frac{144 k_{22}}{\bar{F}_2^2 (h/a)^4} \gamma_m \sinh \gamma_m y \right) E_m \right\} \sin \alpha_m x \quad (4-93.4)$$

Substituting from equations (4-93.3), (4-93.4) into (4-93.2) and then into (4-93.1), we get:

$$Q_y \Big|_{y = \pm b/2} = 0$$

$$\left( (1 - \mu)(m\pi)^3 \sinh \frac{\alpha_m b}{2} \right) A_m + B_m \left( - (1 + \mu)(m\pi)^3 \sinh \frac{\alpha_m b}{2} \right)$$

$$\begin{aligned}
& + (1 - \mu) \frac{\alpha_m b}{2} (m\pi)^3 \cosh \frac{\alpha_m b}{2} \left. \right) \\
& + E_m \left[ \frac{-12(1 - 2k_{22})}{\bar{F}_1 (h/a)^2} + (1 - \mu) (m\pi)^2 \right] a \gamma_m \sinh \frac{\gamma_m b}{2} = 0 \quad (4-94)
\end{aligned}$$

#### 4.4 Boundary Conditions for the Inplane Problem

One notes that due to the form of  $\bar{v}$  which is due to the method of obtaining solution by Levy method that:

$$\bar{v}(0, y) = \bar{v}(a, y) = 0 \quad (4-95)$$

So due to the use of the Levy method for solution, the edges at  $x = 0$  and at  $x = a$  are always free to stretch in the  $x$ -direction. Thus  $N_x$  will vanish at the edges at  $x = 0$  and at  $x = a$ . For this reason boundary conditions on inplane displacements can be specified on the edges at  $y = \pm b/2$ . We have two cases:

**Case I- Edges at  $y = \pm b/2$  clamped against stretching:**

In this case the following boundary conditions apply:

$$\bar{u}(x, \pm b/2) = 0 \quad (4-96.1)$$

$$\bar{v}(x, \pm b/2) = 0 \quad (4-96.2)$$

Substituting from equations (4-83) and (4-85) into the above boundary conditions yields

$$\left[ \cosh \frac{\gamma_m b}{2} \right] C_1 + \left[ \frac{\alpha_m b}{2} \sinh \frac{\gamma_m b}{2} \right] C_2 = - \bar{u}_p \quad (4-96.3)$$

and

$$\left[ \sinh \frac{\gamma_m b}{2} \right] C_1 + \left[ \frac{\alpha_m b}{2} \cosh \frac{\alpha_m b}{2} - k_4 \sinh \frac{\gamma_m b}{2} \right] C_2 = 0 \quad (4-96.4)$$

**Case II- Edges at  $y = \pm b/2$  are free to stretch in the  $y$ -direction only:**

In this case the following boundary conditions apply:

$$N_y = (x, \pm b/2) = 0 \quad (4-96.5)$$

$$\bar{u}(x, \pm b/2) = 0 \quad (4-96.6)$$

From boundary condition as given by equation (4-96.5), and making use of equation (4-76) yields

$$C_1 (1 - \mu) \alpha_m \cosh \alpha_m y + C_2 (1 - k_4) \alpha_m \cosh \alpha_m y + (1 - \mu) \alpha_m^2 y \sinh \alpha_m y = - \frac{k_6}{k_5} p_m + \mu \alpha_m \bar{u}_p \quad (4-96.7)$$

where:

$$k_5 = \frac{Eh}{(1 - \mu^2)} \quad (4-96.8)$$

and:

$$k_{\sigma} = \frac{\mu F_2}{(1 - \mu)} \quad (4-96.9)$$

#### 4.5 Expressions for Stresses in a Non-dimensional Form

The stress  $\sigma_x$  can be written as:

$$\sigma_x = \bar{\sigma}_x \left( \frac{P_0}{(h/a)^2} \right) \quad (4-97.1)$$

Similarly other stresses can be written as:

$$\sigma_y = \bar{\sigma}_y \left( \frac{P_0}{(h/a)^2} \right) \quad (4-97.2)$$

$$\tau_{xy} = \bar{\tau}_{xy} \left( \frac{P_0}{(h/a)^2} \right) \quad (4-97.3)$$

$$\tau_{xz} = \bar{\tau}_{xz} \left( \frac{P_0}{(h/a)} \right) \quad (4-97.4)$$

$$\tau_{yz} = \bar{\tau}_{yz} \left( \frac{P_0}{(h/a)} \right) \quad (4-97.5)$$

where:

$$\bar{\sigma}_x = \left\{ \frac{1}{(1 - \mu^2)} \left[ \bar{I}_1(y) \bar{g}_1(z) + \frac{\mu}{12(1 - \mu)} \bar{I}_2(y) \bar{g}_2(z) \right] \right.$$

$$\begin{aligned}
& + \bar{I}_3(y)\bar{g}_3(z) + \bar{I}_4(y)\bar{g}_4(z) + \bar{I}_7(y)\bar{g}_2(z) \\
& + \bar{I}_5(y) \left] + \bar{I}_6 f_1(z) \right\} \sin \alpha_m x
\end{aligned} \tag{4-98}$$

$$\begin{aligned}
\bar{\sigma}_y = & \left\{ \frac{1}{(1-\mu^2)} \left[ \bar{J}_1(y)\bar{g}_1(z) + \frac{\mu}{12(1-\mu)} \bar{J}_2(y)\bar{g}_2(z) \right. \right. \\
& + \bar{J}_3(y)\bar{g}_3(z) + \bar{J}_4(y)\bar{g}_4(z) + \bar{J}_7(y)\bar{g}_2(z) \\
& \left. \left. + \bar{J}_5(y) \right] + \bar{J}_6 f_1(z) \right\} \sin \alpha_m x
\end{aligned} \tag{4-99}$$

$$\begin{aligned}
\bar{\tau}_{xy} = & \frac{1}{(1+\mu)} \left[ \bar{L}_1(y)\bar{g}_1(z) + \bar{L}_2(y)\bar{g}_2(z) \frac{\mu}{12(1-\mu)} \right. \\
& + \frac{1}{2} \bar{L}_3(y)\bar{g}_3(z) + \bar{L}_4(y)\bar{g}_4(z) \\
& \left. + \bar{L}_6(y)\bar{g}_2(z) + \bar{L}_5(y) \right] \cos \alpha_m x
\end{aligned} \tag{4-100}$$

And:

$$\bar{I}_1(y) = \frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2} \tag{4-101.01}$$

$$\bar{I}_2(y) = \frac{\partial^3 \phi_x}{\partial x^3} + \frac{\partial^3 \phi_y}{\partial x^2 \partial y} + \mu \frac{\partial^3 \phi_x}{\partial x \partial y^2} + \frac{\partial^3 \phi_y}{\partial y^3} \tag{4-101.02}$$

$$\bar{I}_3(y) = \frac{\partial \phi_x}{\partial x} + \mu \frac{\partial \phi_y}{\partial y} \tag{4-101.03}$$

$$I_4(y) = \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial^2 p}{\partial y^2} \quad (4-101.04)$$

$$I_5(y) = \frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \quad (4-101.05)$$

$$I_6(y) = \frac{4\mu(h/a)^2}{(1 - \mu)(m\pi)} \quad (4-101.06)$$

$$I_7(y) = \frac{-4\mu^2(m\pi) \bar{F}_1}{6(1 - \mu)} (h/a)^4 \quad (4-101.07)$$

$$\bar{I}_1(y) = a^2 I_1(y) \quad (4-101.08)$$

$$\bar{I}_2(y) = a^2 h^2 I_2(y) \quad (4-101.09)$$

$$\bar{I}_3(y) = a^2 I_3(y) \quad (4-101.10)$$

$$\bar{I}_4(y) = h^2 I_4(y) \quad (4-101.11)$$

$$\bar{I}_5(y) = A \left[ \frac{h^2}{a^2} \right] I_5(y) \quad (4-101.12)$$

And:

$$\bar{J}_1(y) = a^2 J_1(y) \quad (4-101.13)$$

$$\bar{J}_2(y) = a^2 h^2 J_2(y) \quad (4-101.14)$$

$$\bar{J}_3(\mathbf{y}) = a^2 J_3(\mathbf{y}) \quad (4-101.15)$$

$$\bar{J}_4(\mathbf{y}) = h^2 J_4(\mathbf{y}) \quad (4-101.16)$$

$$\bar{J}_5(\mathbf{y}) = h \left( \frac{h^2}{a^2} \right) J_5(\mathbf{y}) \quad (4-101.17)$$

$$\bar{J}_6(\mathbf{y}) = \bar{I}_6(\mathbf{y}) \quad (4-101.18)$$

$$\bar{J}_7(\mathbf{y}) = \bar{I}_7(\mathbf{y}) \quad (4-101.19)$$

where:

$$J_1(\mathbf{y}) = \frac{\partial^2 \bar{w}}{\partial y^2} + \mu \frac{\partial^2 \bar{w}}{\partial x^2} \quad (4-101.20)$$

$$J_2(\mathbf{y}) = \frac{\partial^3 \phi_y}{\partial y^3} + \frac{\partial^3 \phi_x}{\partial x \partial y^2} + \mu \frac{\partial^3 \phi_y}{\partial x^2 \partial y} + \frac{\partial^3 \phi_x}{\partial x^3} \quad (4-101.21)$$

$$J_3(\mathbf{y}) = \frac{\partial \phi_y}{\partial y} + \mu \frac{\partial \phi_x}{\partial x} \quad (4-101.22)$$

$$J_4(\mathbf{y}) = \frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \quad (4-101.23)$$

$$J_5(\mathbf{y}) = \frac{\partial \bar{v}}{\partial y} + \mu \frac{\partial \bar{u}}{\partial x} \quad (4-101.24)$$

Also:

$$L_1(y) = \frac{\partial^2 \bar{w}}{\partial x \partial y} \quad (4-102.01)$$

$$L_2(y) = \frac{\partial^3 \phi_x}{\partial x^2 \partial y} + \frac{\partial^3 \phi_y}{\partial x \partial y^2} \quad (4-102.02)$$

$$L_3(y) = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \quad (4-102.03)$$

$$L_4(y) = \frac{\partial^2 p}{\partial x \partial y} \quad (4-102.04)$$

$$L_5(y) = \frac{1}{2} \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \quad (4-102.05)$$

$$L_6(y) = L_4(y) \quad (4-102.06)$$

And:

$$\bar{L}_1(y) = a^2 L_1(y) \quad (4-102.07)$$

$$\bar{L}_2(y) = a^2 h^2 L_2(y) \quad (4-102.08)$$

$$\bar{L}_3(y) = a^2 L_3(y)$$

$$\bar{L}_4(y) = 0 \quad (\text{since } L_4(y) = 0) \quad (4-102.09)$$

$$\bar{L}_5(y) = a \left( \frac{h^2}{a^2} \right) L_5(y) \quad (4-102.10)$$



$$\bar{L}_6(y) = \bar{L}_4(y) = 0 \quad (4-102.11)$$

Also;

$$g_1(z) = hg_1(z) \quad (4-103.1)$$

$$\bar{g}_1(z) = \left[ \frac{1}{\bar{F}_1} (f_1(z) - \bar{F}_2) - (z/h) \right] \quad (4-103.2)$$

$$\bar{g}_2(z) = \frac{\mu}{E} \left[ 2\left(\frac{z}{h}\right)^3 - \frac{3}{10}\left(\frac{z}{h}\right) \right] \quad (4-103.3)$$

$$g_3(z) = \frac{1}{F_1} \left[ f_1(z) - \frac{F_2}{h} \right] \quad (4-103.4)$$

$$\bar{g}_3(z) = \frac{1}{\bar{F}_1} \left[ f_1(z) - \bar{F}_2 \right] \quad (4-103.5)$$

$$g_4(z) = \frac{h^2}{E} \bar{g}_4(z) \quad (4-103.6)$$

$$\bar{g}_4(z) = \left[ -\bar{f}_3(z) + \bar{F}_3\left(\frac{z}{h}\right) + \bar{F}_4 \right] \quad (4-103.7)$$

---

## Chapter 5

### APPLICATIONS

#### 5.1 Cylindrical Bending

Two problems are considered to test the validity of the present formulation.

##### *Example 5.1.1*

An infinite plate strip of thickness "h" subjected to the stress field:

$$\sigma_z(x, y, -h/2) = -q_0 \sin \frac{\pi x}{L} \quad (5.1)$$

is considered first.

An exact elasticity solution exists for this problem [15]. Also this case was used in [10] to evaluate a higher order plate theory.

The dependent variables may be assumed to be in the form:

$$w_0 = w_{00} \sin \frac{\pi x}{L}$$

$$u_0 = u_{00} \cos \frac{\pi x}{L}$$

$$\begin{aligned}
v_o &= v_{oo} \sin \frac{\pi x}{L} \\
Q_x &= Q_{ox} \cos \frac{\pi x}{L} \\
Q_y &= Q_{oy} \cos \frac{\pi x}{L} \\
\phi_x &= \phi_{ox} \cos \frac{\pi x}{L} \\
\phi_y &= \phi_{oy} \cos \frac{\pi x}{L} \\
M_x &= M_{ox} \sin \frac{\pi x}{L} \\
M_y &= M_{oy} \sin \frac{\pi x}{L} \\
M_{xy} &= M_{oxy} \sin \frac{\pi x}{L}
\end{aligned} \tag{5.2}$$

The boundary conditions are as given by equations (3.42) and (3.46).

Substituting equations (3.66) and (3.67) into equations (3.7), (3.31), (3.32), (3.27.4), (3.27.5), (3.28), (3.29) and (3.30), one may solve for the unknown coefficients in the set of equations (3.67).

The solution for the transverse deflection  $w_o$  is given by:

$$w_o = \frac{P_m}{\alpha_m^4 D} \left[ 1 + \frac{(2-\mu)h^3}{12(1-\mu)} \alpha_m^2 F_1 - \alpha_m^4 D/N \right]$$

$$\begin{aligned}
& + \frac{\mu h^2 \alpha_m^2}{40(1-\mu)} - \frac{\mu^2 h^5 \alpha_m^4 F_1}{480(1-\mu)^2} \\
& + \frac{\mu^2 h^5 \alpha_m^4}{240(1-\mu)^2(1+\mu)} F_1 ] \sin \alpha_m x \quad (5.3)
\end{aligned}$$

where

$$\alpha_m = \alpha_1 = \frac{\pi}{L}, \quad p_m = p_1 = q_0 \quad (\text{for } m = 1) \quad (5.3.1)$$

Solving for the stress  $\sigma_x$ , we get:

$$\begin{aligned}
\sigma_x = & \left\{ E \alpha_m^2 w_{00} \frac{z}{(1-\mu^2)} - \frac{(2-\mu)}{(1-\mu)} p_m f_1(z) + \frac{p_m \alpha_m^2}{(1-\mu^2)} f_3(z) \right. \\
& \left. - \frac{2\mu \alpha_m^2 M_0}{h^3(1-\mu^2)} z^3 - \frac{p_m}{h(1-\mu^2)} [(\mu^2 - \mu - 2)F_2 + \alpha_m^2 F_4] \right\} \sin \alpha_m x \quad (5.4)
\end{aligned}$$

where

$$M_0 = M_{0x} + M_{0y} \quad (5.4.1)$$

If one solves the same problem using the shear deformation generalized theory of Panc [9], the expression for  $\sigma_x$  may be shown to be given by

$$\sigma_x = \frac{E \alpha_m^2}{(1-\mu^2)} w_{m0} - \frac{2p_m}{(1-\mu)} \left[ f_{1m}(z) + \frac{1}{2} \right] \quad (5.5)$$

where

$$w_{mo} = \frac{P_m}{k_m \alpha_m^4}$$

$$k_m = \frac{2E}{(1-\mu^2)\lambda_m^3} \left( \frac{\lambda_m h}{2} - \tanh \frac{\lambda_m h}{2} \right) \quad (5.6)$$

$$\lambda_m^2 = \frac{2}{(1-\mu)} \alpha_m^2$$

$$f_{1m}(z) = -\frac{1}{2} \left[ 1 - \frac{\lambda_m z \operatorname{ch}(\lambda_m h/2) - \operatorname{sh}(\lambda_m z)}{(\lambda_m h/2) \operatorname{ch}(\lambda_m h/2) - \operatorname{sh}(\lambda_m h/2)} \right]$$

Figure 5.1 shows results for  $w_0$  and Figures 5.2 to 5.9 show results for  $\sigma_x$ , as given by the exact solution [15], Panc [9], Baluch [10], and the present work.

The effect of normal strain on  $w_0$  becomes very clear for  $h/L > 1.0$  as shown in Figure 5.1. As  $h/L$  increases, the present work gives results which are closest to the exact solution.

The present work, as shown in Figures 5.2 to 5.9, gives the best results for stress  $\sigma_x$  as compared to the exact solution. For  $h/L > 1.0$ , previous work by Baluch [10] and Panc [9] failed to give good results for stresses. The present work yields almost exact results even up to  $h/L = 3.0$ , which is representative of an extremely thick plate. Figs. 5.4 through 5.9 show that  $\sigma_x$  from the present theory is almost superposed on the exact solution for  $h/L$  upto 3.0,

whereas the other refined theories yield diverging solutions and which are thus not plotted.

**Example 5.1.2**

An infinite plate strip of thickness "h" subjected to a uniformly distributed load "p" at  $z = -h/2$ . For this case, the previous expressions derived for  $w_0$  and  $\sigma_x$  in example (5.1.1) are still valid except that for this case:

$$\alpha_m = \frac{m\pi}{L}, P_m = \frac{4p}{m\pi} \quad m = 1, 3, 5, 7, \dots, \quad (5.7)$$

Figure 5.10 shows results for  $w_0$  and Figures 5.11 to 5.18 show results for  $\sigma_x$ , as given by the exact solution [15], Panc [9], and the present work.

The effect of normal strain on  $w_0$  is again apparent for  $h/L > 1.0$  as shown in Figure 5.10. The present work yields  $w_0$  which is close to the exact solution as  $h/L$  is increased.

The  $\sigma_x$  stresses from the present theory yield results initially indistinguishable from the exact theory for  $h/L$  upto as high as 3.0 (Figs.: 5.11 to 5.18).

Figures 5.19 to 5.21 depict the variation of the transverse normal stress  $\sigma_z$  with the ratio  $h/L$ . As with the case of  $\sigma_x$  stresses,

the present formulation yields results for  $\sigma_z$  almost identical to the exact solution. It is also of interest to note that as the plate becomes thicker, the maximum magnitude of the bending stress  $\sigma_x$  becomes of the same order as that of the transverse normal stress  $\sigma_z$ .

## 5.2 Examples for Rectangular Plates

A rectangular plate of sides  $a$  (along  $x$ -axis) and  $b$  (along  $y$ -axis) loaded uniformly and with the edges at  $x=0$ ,  $x=a$  being simply supported was considered. The following cases were chosen to give examples for such isotropic rectangular plates (in all cases considered, Poisson's ratio  $\mu$  was taken to be 0.3).

### NOTE :

In the figures that follow the notation

**BC.h/a-I(OR II)**

is used to indicate :

**BC** : Indicates the type of boundary condition

**SS** : indicates a simply supported edge.

**SC** : indicates a clamped edge.

**SF** : indicates a Free edge.

**h/a** : is the value of (thickness to span) ratio.

**I OR II** : indicates whether the edges at

$$y = \pm b/2$$

are not allowed to stretch in the  $y$ -direction

(I)

OR are allowed to do so (II).

### 5.2.1 A Square Plate Uniformly Loaded with All Edges Simply Supported (SS) :

The boundary conditions that need to be satisfied for the bending problem for this case are given by equations (4.87), (4.88),



and (4.89).

The boundary conditions that need to be satisfied for the inplane problem are given by equations (4.96.3), (4.96.4) for edges at  $y = \pm b/2$  not allowed to stretch in the  $y$ -direction (Case I) and by equations (4.96.3) and (4.96.7) for edges at  $y = \pm b/2$  allowed to stretch in the  $y$ -direction only (Case II). Table 5.1 shows the results for deflection  $\bar{w}$  obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR) [12], refined theory in reference [11] RTB, and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.2 for  $M_x$  and Table 5.3 for  $M_y$ ).

Also the stress  $\sigma_x$  is obtained and results are compared with results from other theories for Case I in Figures 5.22 to 5.30 and results are shown in Figures 5.31 to 5.43 for Case II.

The variation of the transverse shear stress  $\tau_{xz}$  is shown in Figures 5.44 to 5.47. The results are in qualitative agreement with the elasticity solution for bending of thick curved bar by force at end [14].

The results shown demonstrate clearly the effect of including the influence of transverse stresses and strains and normal stress and

strain on the deflection and on the resultant moments. This effect becomes very clear as  $h/a$  for the plate increases up to as high as  $h/a = 1.0$ .

The graphs for the stresses show the non-linearity in the stresses as  $h/a$  ratio increases. Also it is shown clearly in the graphs that the neutral plane is shifted and it does not coincide any more with the mid-plane as CPT and RTR predicts. The magnitude of the inplane stresses  $\sigma_x$ ,  $\sigma_y$ ,  $\sigma_{xy}$  decreases, as the ratio  $h/a$  of the plate increases, to an order of magnitude similar to that of the normal stress  $\sigma_z$  and thus  $\sigma_z$  cannot be neglected for thick plates.

### 5.2.2 A Square Plate Uniformly Loaded with Clamped Edges at $y = \pm b/2$ (SC) :

Table 5.4 shows the results for deflection  $\bar{w}$  obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR), refined theory in reference [11] RTB , and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.5 for  $M_x$  and Table 5.6 for  $M_y$ ).

Also the stress  $\sigma_x$  is obtained and results are compared with results from other theories for Case I in Figures 5.48 to 5.53 and results are shown in Figures 5.54 to 5.59 for Case II.

Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made based on the above results for this case (i.e : simple/clamped plate).

### 5.2.3 A Square Plate Uniformly Loaded with Free Edges at $y = \pm b/2$ (SF) :

Table 5.7 shows the results for deflection  $\bar{w}$  obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR) [12], refined theory in reference [11] RTB , and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.8 for  $M_x$  and Table 5.9 for  $M_y$ ).

Also the stress  $\sigma_x$  is obtained and results are compared with results from other theories for Case I in Figures 5.60 to 5.66 and results are shown in Figures 5.67 to 5.72 for Case II.

Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made based on the above results for this case (i.e : simple/free plate).

### 5.2.4 A Square Plate Simply Supported All Around and Loaded With A Line Load At $x = a/2$ ( See Figure 5-A ) :

Assuming that the plate (simply supported all around) is subjected to a line load at :  $x = x_1$  , in this case  $p_m$  can be shown to

be given by :

$$p_m = \frac{2p_o}{a} \sin \frac{m\pi x}{a} \quad (5.2.4-1)$$

Table 5.10 shows the results of deflection at center of the plate for this case of loading.

Table 5.11 shows the results of the resultant moment  $M_x$  at the center of the plate.

Table 5.12 shows the results of the resultant moment  $M_y$  at the center of the plate. The results were compared with results from CPT. Results from both RTR and RTB were not available. The importance of using a refined theory such as the one presented here is clear from the results shown in these tables. For a ratio of  $h/a$  as high as 1.0 , the deflection obtained from this theory is almost 7 times the one obtained by CPT.

Stresses are not shown for this case since the load does not converge when expanded in single Fourier series but rather it's integral converges.

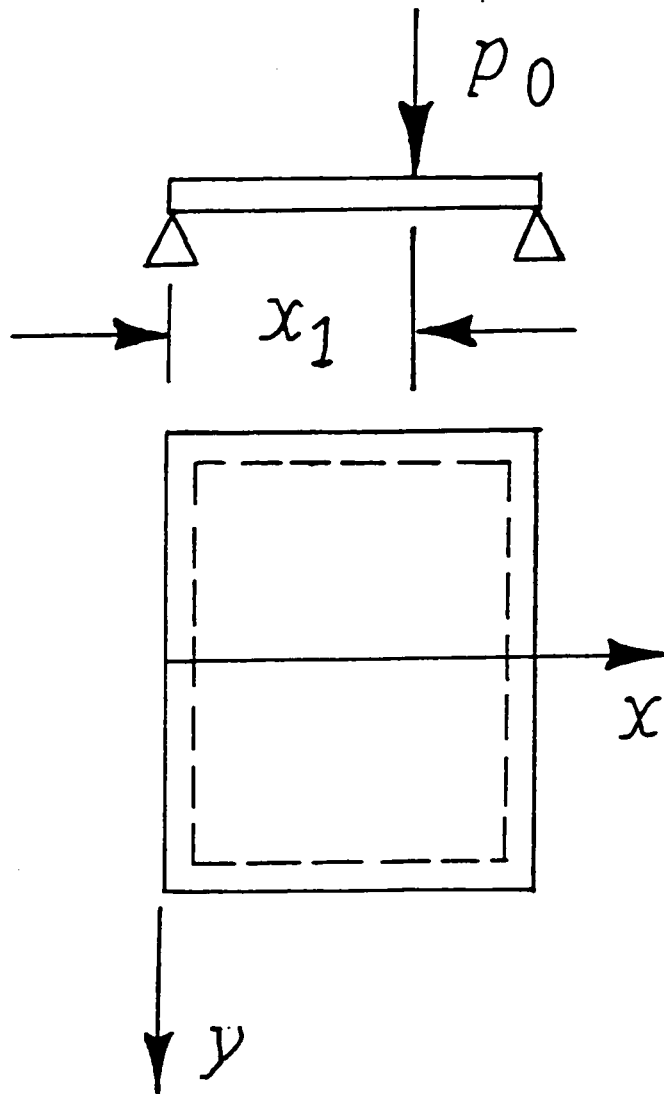


Figure 5-A : Line Load  $P_0$  At  $x = x_1$

### 5.2.5 A Square Plate Simply Supported All Around and Loaded With A Strip Load :

Assuming that the plate (simply supported all around) is subjected to a strip load of width =  $u$  and centered at  $x = \xi$ , in this case  $p_m$  can be shown to be given by :

$$p_m = \frac{4p_o}{m\pi} \sin \frac{m\pi\xi}{a} \sin \frac{m\pi u}{a} \quad (5.2.5-1)$$

Table 5.13 shows the results of deflection at center of the plate for this case of loading.

Table 5.14 shows the results of the resultant moment  $M_x$  at the center of the plate.

Table 5.15 shows the results of the resultant moment  $M_y$  at the center of the plate. The results were compared with results from CPT. Results from both RTR and RTB were not available. The importance of using a refined theory such as the one presented here is clear from the results shown in these tables. For a ratio of  $h/a$  as high as 1.0, the deflection obtained from this theory is almost 7 times the one obtained by CPT.

Also the stress  $\sigma_x$  is obtained and results are compared with results from other theories for Case II in Figures 5.73 to 5.77 .

Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made

based on the above results for this case.

Also it may be noted that this case of loading represents a general case of strip loading since the width and center of the strip load can be varied to obtain any case of strip loading including the case of uniformly loaded plate.

For the case of distributed loading on both the top and bottom surfaces of the plate, the problem can be solved by superposition. The problem will be divided into two problems. The first will be a plate loaded at top; and this will be solved as shown in the previous sections on the type of loading (i.e. : a line load, a strip load, or a uniform load). The second problem will be for a plate loaded at the bottom only; and this can be solved by reversing the z-axis (i.e. positive z-axis will be upward). Thus this second problem will be equivalent to the first problem with the z-axis being reversed. The solution for the whole problem will be obtained by superposing solutions from the first and second problems.

#### 5.2.6 A Plot Of $w(x,y,z)$ Across The Plate :

Substituting for  $w_0(x,y)$  from equation (3-41) in equation (3-15), the expression for  $w(x,y,z)$  can be rewritten as follows:

$$w(x,y,z) = \frac{p(x)}{E} f_2(z) - \frac{6\mu M(x,y)z^2}{Eh^3} + \bar{w}(x,y) - \frac{p(x)}{N} + \frac{M(x,y)}{R} \quad (5.2.6-1)$$

Substituting for  $N$  and  $R$  from equations (4.7) and (4.8), respectively, in equation (5.2.5-1) and rearranging results in

$$w(x, y, z) = \frac{P(x)}{E} \{f_2(z) - F_3\} + \frac{M(x, y)}{E} \left\{ \frac{3\mu}{10h} - \frac{6\mu z^2}{h^3} \right\} + \bar{w}(x, y)$$

Noting that

$$F_3 = h\bar{F}_3 \quad (4-59.3)$$

and

$$f_2(z) = h\bar{f}_2(z) \quad (4-59.13)$$

the expression for  $w(x, y, z)$  can be rewritten as follows:

$$w(x, y, z) = \frac{1}{E} \left\{ p(x) \left[ h\bar{f}_2(z) - h\bar{F}_3 \right] + \frac{3\mu M(x, y)}{h} \left[ \frac{1}{10} - 2\left(\frac{z}{h}\right)^2 \right] \right\} + \bar{w}(x, y)$$

Making use of equation (4.33) for  $p(x)$  and noting that

$$\bar{w}(x, y) = \sum_{m=1}^{\infty} \bar{w}_m(y) \sin \alpha_m x \quad (4.38)$$

$$M_m(y) = M_{xm}(y) + M_{ym}(y) \quad (3.13)$$

and

$$M_x(x, y) = \sum_{m=1}^{\infty} M_{xm}(y) \sin \alpha_m x \quad (4-62)$$



$$M_y(x,y) = \sum_{m=1}^{\infty} M_{ym}(y) \sin \alpha_m y \quad (4-63)$$

the expression for deflection  $w(x,y,z)$  can be rewritten in the following form :

$$\begin{aligned} w(x,y,z) = & \sum_{m=1}^{\infty} \frac{P_o a^4}{Eh^3} \left\{ p_m \left(\frac{h}{a}\right)^4 \left[ \bar{f}_2(z) - \bar{F}_3 \right] \right. \\ & + 3 \mu \left(\frac{h}{a}\right)^2 M_m(y) \left[ \frac{1}{10} - 2\left(\frac{z}{h}\right)^2 \right] \\ & \left. + \bar{W}_m(y) \right\} \sin \alpha_m x \end{aligned} \quad (5.2.6-2)$$

where

$$\bar{W}_m(y) = \frac{Eh^3}{p_o a^4} \bar{w}_m(y) \quad (5.2.6-3)$$

Figures 5.78,79,80 show deflection of TOP surface of the plate given by RTR and RTP for  $h/a = 0.1, 0.5$  and  $1.0$ , respectively .

Figures 5.81,82,83 show deflection of middle surface of the plate given by RTR and RTP for  $h/a = 0.1, 0.5$  and  $1.0$ , respectively .

Figures 5.84,82,83 show deflection of bottom surface of the plate given by RTR and RTP for  $h/a = 0.1, 0.5$  and  $1.0$ , respectively .

Figures 5.87,88,89 show deflection of top, middle, and bottom surfaces of the plate given by RTR and RTP for  $h/a = 0.1, 0.5$  and  $1.0$ , respectively .

From the graphs the effect of including the normal strain on deflection is very clear. Also, the present work can give the deflec-

tion as a function of  $z$  whereas RTR is giving " average deflection " across the depth of the plate. The present theory is predicting deflection at top to be much more than deflection at bottom of the plate as the ratio  $h/a$  of the plate increases. This result is expected; since as the plate thickness increases the load will be taken mostly by the top layers and the bottom layers will hardly feel the load.

### 5.2.7 Verifying Equilibrium Of The Plate In The Vertical Direction :

Edge reactions at edges of the plate should balance the applied load:

$$I = \int_0^a [ Q_y (x, +b/2) - Q_y (x, -b/2) ] dx + \int_{-b/2}^{b/2} [ Q_x (a, y) - Q_x (0, y) ] dy \quad (5.2.7-1)$$

After performing the integrations in the above equation, it can be shown that :

$$I = \frac{p_o ab}{12(1-\mu^2)} \left\{ \frac{24k_{22}(\cos(m\pi) - 1)}{F_1} \left(\frac{h}{a}\right)^2 \left\{ \frac{m\pi}{\gamma_m b} - \frac{\gamma_m}{a_m b} \right\} \sinh\left(\frac{\gamma_m b}{2}\right) E_m + \frac{6(1-\mu)(m\pi)}{F_1 \left(\frac{h}{a}\right)^2} \left[ \beta_m + \bar{\beta}_m \left| \cos(m\pi) - 1 \right| \right] \right\} \quad (5.2.7-2)$$

Table 5.16 shows that total reaction of the edges of the plate is equal to the uniformly applied loads for different types of support at  $y = \pm b/2$ . The results are satisfactory compared with classical theory since the latter gives unbalanced concentrated reaction of about 26 % whereas there is no evidence of such unbalanced reaction in this work.

#### **5.2.8 Effect of inplane stretching on inplane stresses :**

To study the effect of inplane stretching on inplane stresses,  $\sigma_y$  was evaluated at the center of a simply supported plate for the two cases :

when edges at  $y = \pm b/2$  are allowed to stretch in the y-direction (case-I)

and when edges at  $y = \pm b/2$  are not allowed to stretch in the y-direction (case-II).

The results are shown in Figures 5.90 to 5.92 .

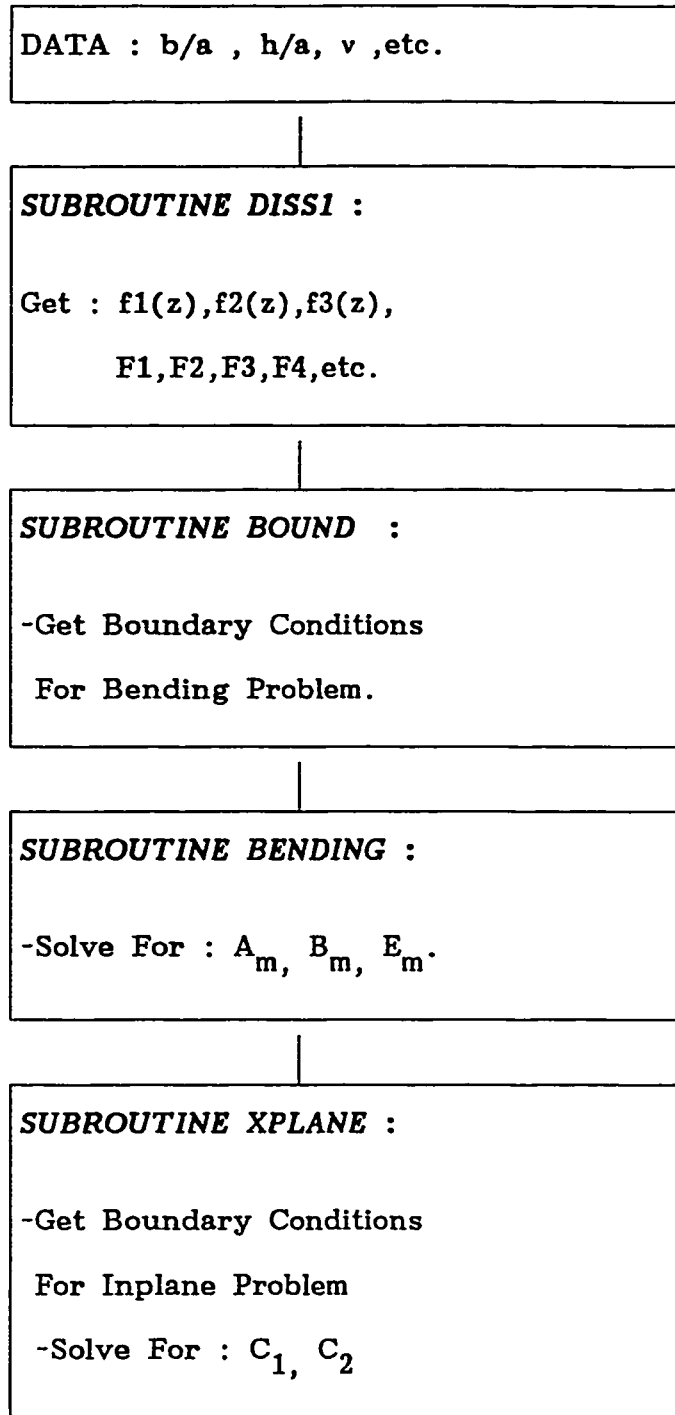
From the results it is noticed that the in-plane compressive stresses increase by 10-15 % for case-I over those for case-II. Also it is noticed that the in-plane tensile stresses decrease by 10-15 % for case-I over those for case-II. For thin plates the in-plane stresses were the same for both cases since the effect of the in-plane forces for thin plates is extremely small.

### **5.3 Computer Program**

A computer program (DISS2) is developed to get the solution for any rectangular plate that is simply supported at  $x=0,a$  and can have any boundary condition on edges at  $y = \pm b/2$ . A flowchart is given in Fig. 5-B to show the structure of this program. A program listing is included in the Appendix A-5-1.

It should be noted that this program can handle solutions according to RTB or RTP by the use of the parameter IBALCH. (See program listing for more details).

A similar program DISS4 is developed for the case of plate strips (i.e for the case of Cylindrical Bending). The plate strip can have any boundary condition at  $x=0,x=l$  (i.e at edges of the plate strip). A program listing for DISS4 is included in the Appendix A-5-2.

**FIG. 5-B : Flowchart For The Computer Program DISS2**

**Figure 5-B (Continued) : Flowchart For The Computer Program****DISS2****SUBROUTINE FORCE :**

-Get Resultant Moments :

$$M_x, M_y, M_{xy}$$

-Get Resultant Inplane Forces :

$$N_x, N_y, N_{xy}$$

-Get Shear Forces :

$$Q_x, Q_y$$

**SUBROUTINE STRES :**

-Get Stresses :

$$\sigma_x, \sigma_y, \sigma_z,$$

$$\tau_{xy}, \tau_{xz}, \tau_{yz}$$

**PRINT RESULTS For :**

DEFLECTION, MOMENTS,  
SHEARS AND STRESSES

#### 5.4 Conclusions

1. It may be concluded that the use of generalized distribution of transverse normal and shear stresses (as originally presented by Kromm [7,8] in the development of a new refined thick plate theory (along the lines of earlier presentation [10,11] yields a formulation that captures all essential characteristics of the exact three dimensional elasticity problem. This is reflected in that results for stresses obtained from the present formulation are almost identical to the exact solution up to ratios of  $h/a = 3.0$  (for the case of cylindrical bending). This ratio characterizes a significantly thick plate, and all previously known refined theories breakdown at this level of plate thickness.

For the case of rectangular plates , the results are satisfactory up to  $h/a = 1.0$

2. Based on comparison of resultant moments and forces :  $M_x$  ,  $M_y$  ,  $M_{xy}$  ,  $Q_x$  ,  $Q_y$  from classical thin plate theory and refined theories , a plate is considered to be thick for a ratio of  $h/a \geq 0.1$ . Thus for plates for which  $h/a \geq 0.1$  a refined theory - such as the one presented in this work - should be used to analyze the behavior of such plates completely.

3. It is shown in the results that as  $h/a$  increases (from 0.1 and above) , inplane bending and twisting shear stresses decrease to a level where they are of equivalent order as  $\sigma_z$  and therefore  $\sigma_z$  cannot be neglected.
4. This theory allows for in-plane movement of the plate, yielding new type of boundary conditions in the form of loosely or rigidly supported simple or clamped edges.  
The case of rigidly supported edges yields in-plane compression forces not present in any of the previous refined theories .  
The effect of these forces is accentuated as  $h/a$  increases. In-plane compressive normal stress  $\sigma_y$  increases by 10-15 % if the edges at  $y = \pm b/2$  are not allowed to stretch.
5.  $f_1(z)$  is the function that is responsible for yielding 3-Dimensional type behavior ( in terms of stresses ) from an essentially 2-Dimensional analysis for stress resultants and displacements .



6. Present theory (RTP) corrects stresses as  $h/a$  becomes large whereas Reissner's theory (RTR) predicts always linear distribution for the stresses :  $\sigma_x$  ,  $\sigma_y$  ,  $\sigma_{xy}$  , and parabolic distribution for the stresses :  $\tau_{xz}$  ,  $\tau_{yz}$  , and assumes that :  $\sigma_z = 0$  .

Present theory gives non-linear distribution similar to exact solution from theory of elasticity for deep beam type members.( For all stresses:  $\sigma_x$  ,  $\sigma_y$  ,  $\sigma_z$  ,  $\sigma_{xy}$  ,  $\tau_{xz}$  ,and  $\tau_{yz}$  )

7. Present theory captures ' transition from " beam bending problem" to " column type problem " as plate gets thicker ' better than Reissner's theory.
8. Present work solves the numerical problem of ill-conditioning which occurs in the previous companion refined theory [10,11]. The ill-conditioning in the previous formulation was a serious shortcoming as some of the results presented in References [10,11] are in discrepancy with those presented by the most well known of refined theories i.e. Reissner theory [12].

9. The variation of the transverse shear stress  $\tau_{xz}$  agrees qualitatively with the elasticity solution for bending of thick curved bar by force at end.
10. The results for vertical equilibrium of the plate are satisfactory compared with the classical theory of plates since the latter gives unbalanced concentrated reaction of about 26% whereas there is no evidence of such unbalanced reaction in this work.
11. The transverse normal stress  $\sigma_z$  of previous theory [11] (RTB) is not a function of thickness of the plate, whereas present one is a function of thickness. This reflects clearly the role of  $f_1(z)$  on plate behavior.

FIG.5.1 : DEFLECTION COEFFICIENT K VS H/L  
 { K = MAX. W0 (REFINED THEORY) / W0 (CLASSICAL THEORY) }  
 { P0 = SIN(PI\*X/L) }

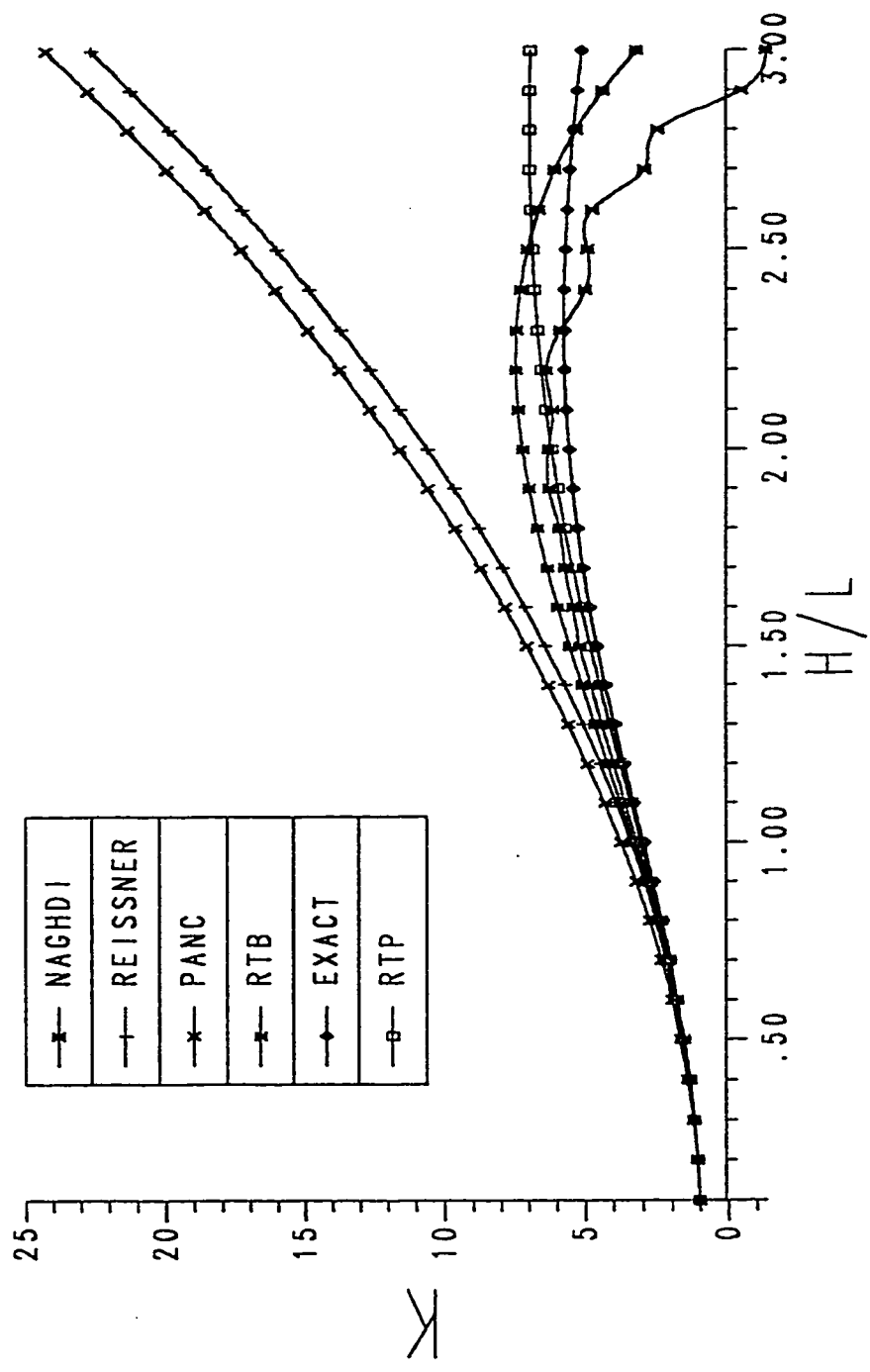


FIG.5.2 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.1, P=P0\*SIN(PI\*X/L) )

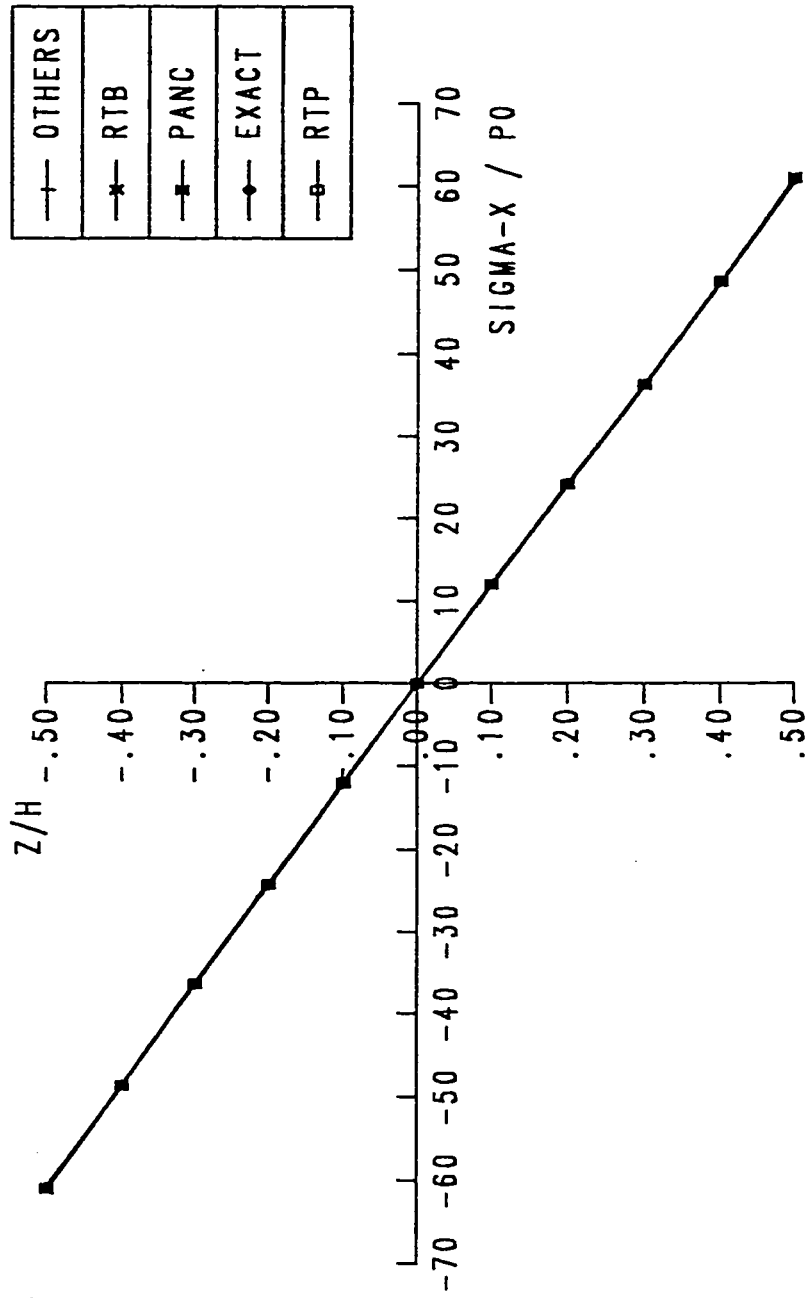


FIG.5.3 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.3, P=P0\*SIN(PI\*X/L))

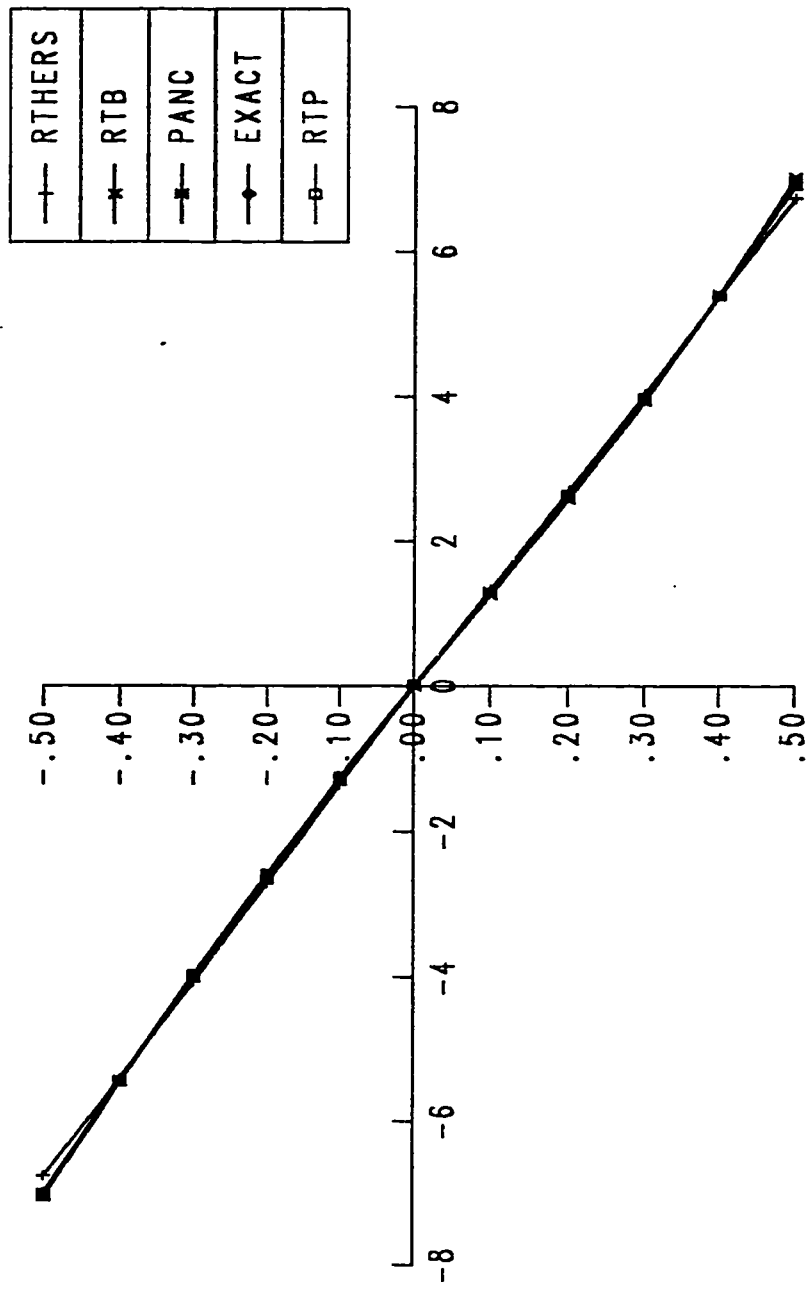


FIG.5.4 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.5, P=P0\*SIN(PI\*X/L) )

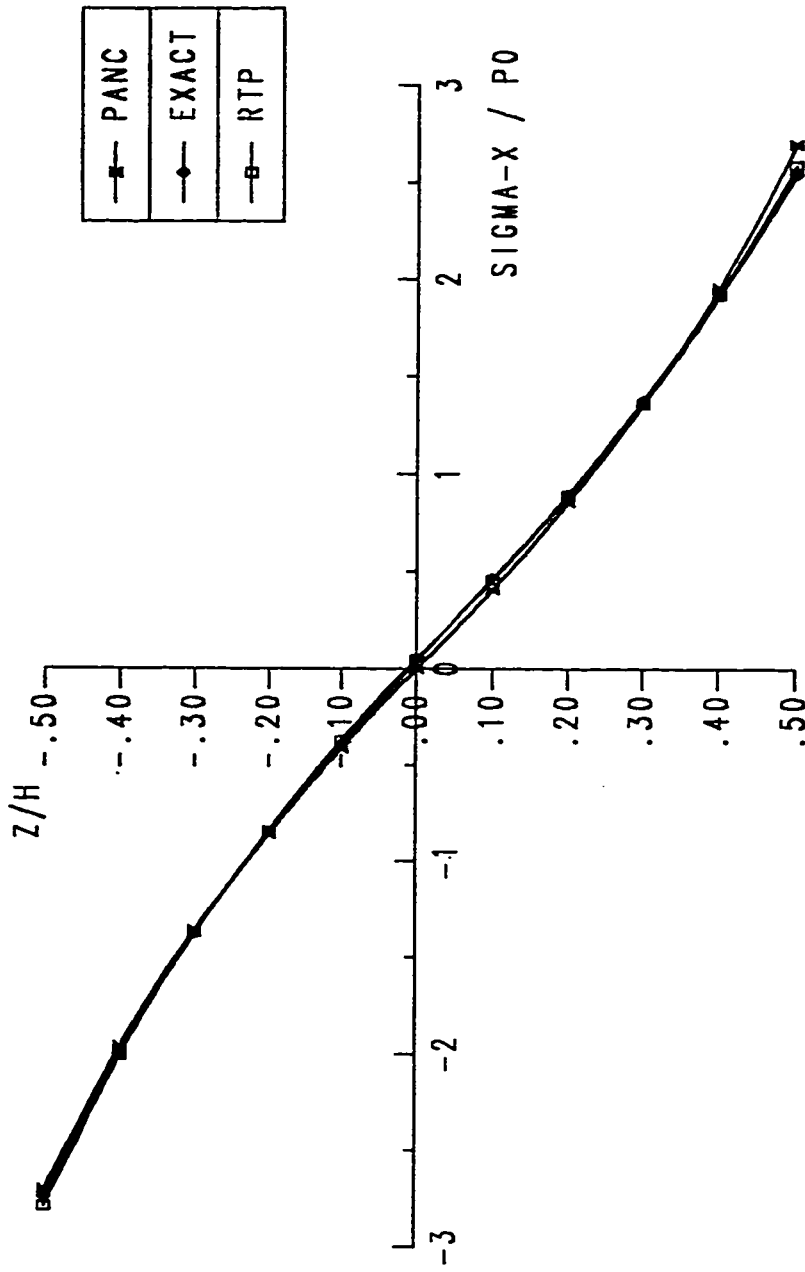


FIG.5.5 : MAX. NORMAL STRESS SIGMA-X VS Z/H ( H/L=1.0, P=P0 \* SIN(PI \* X/L) )

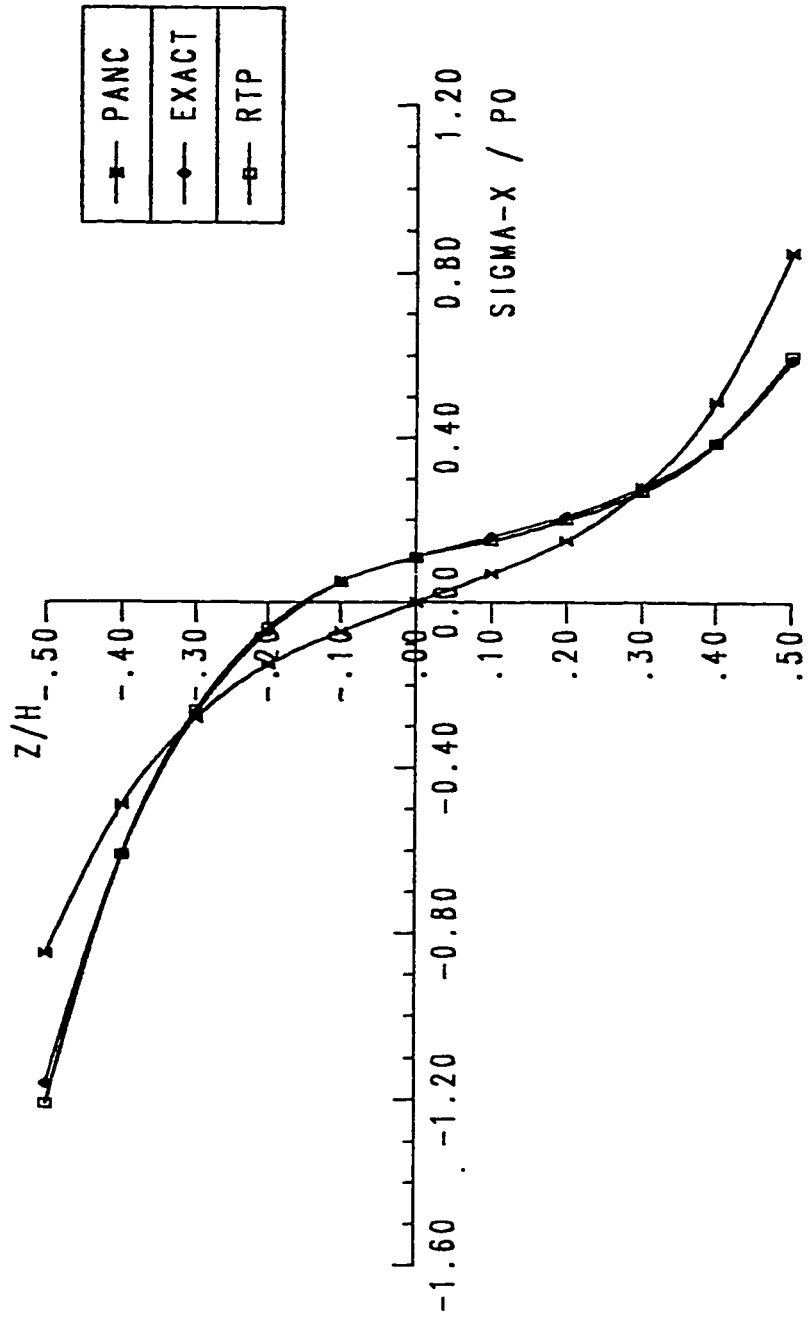


FIG.5.6 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.5, P=P0\*SIN(PI\*X/L) )

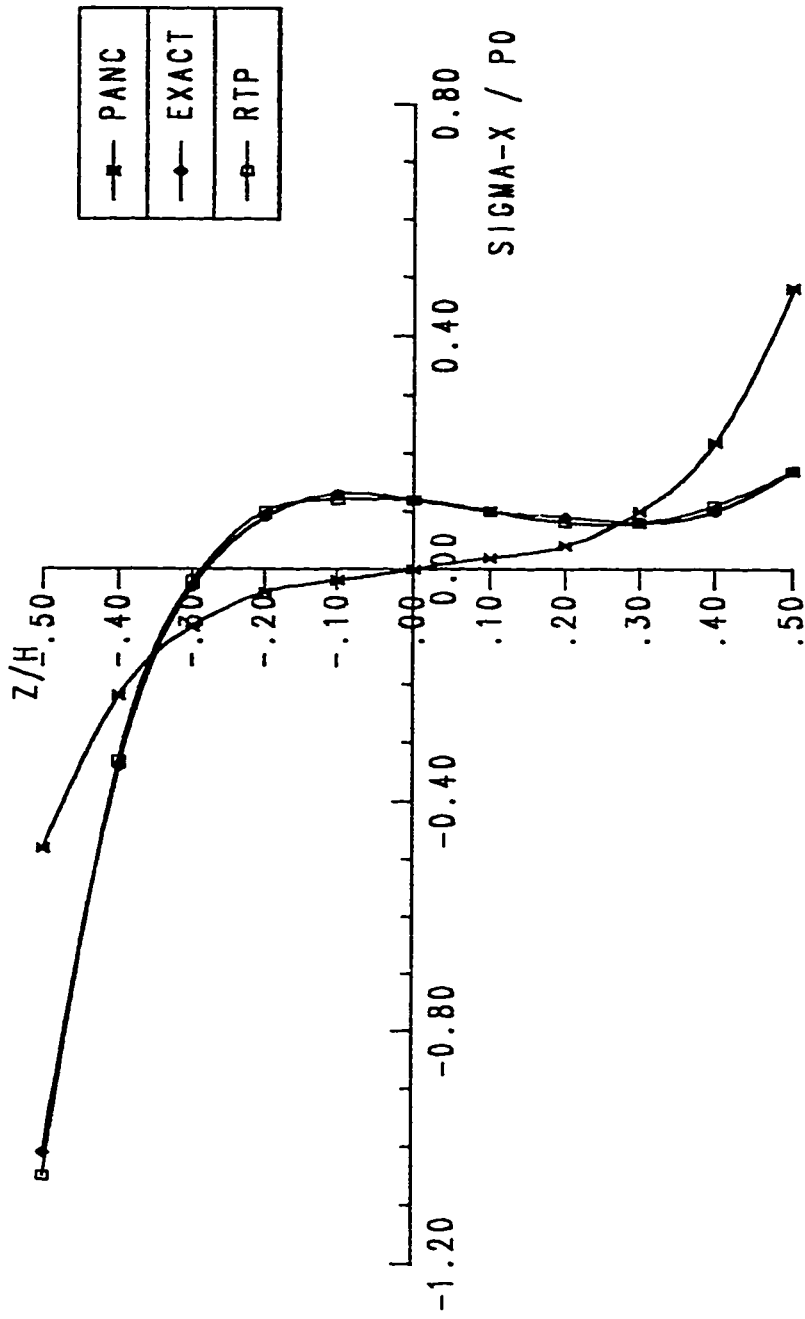




FIG.5.7 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.0, P=P0 \* SIN(PI \* X/L) )

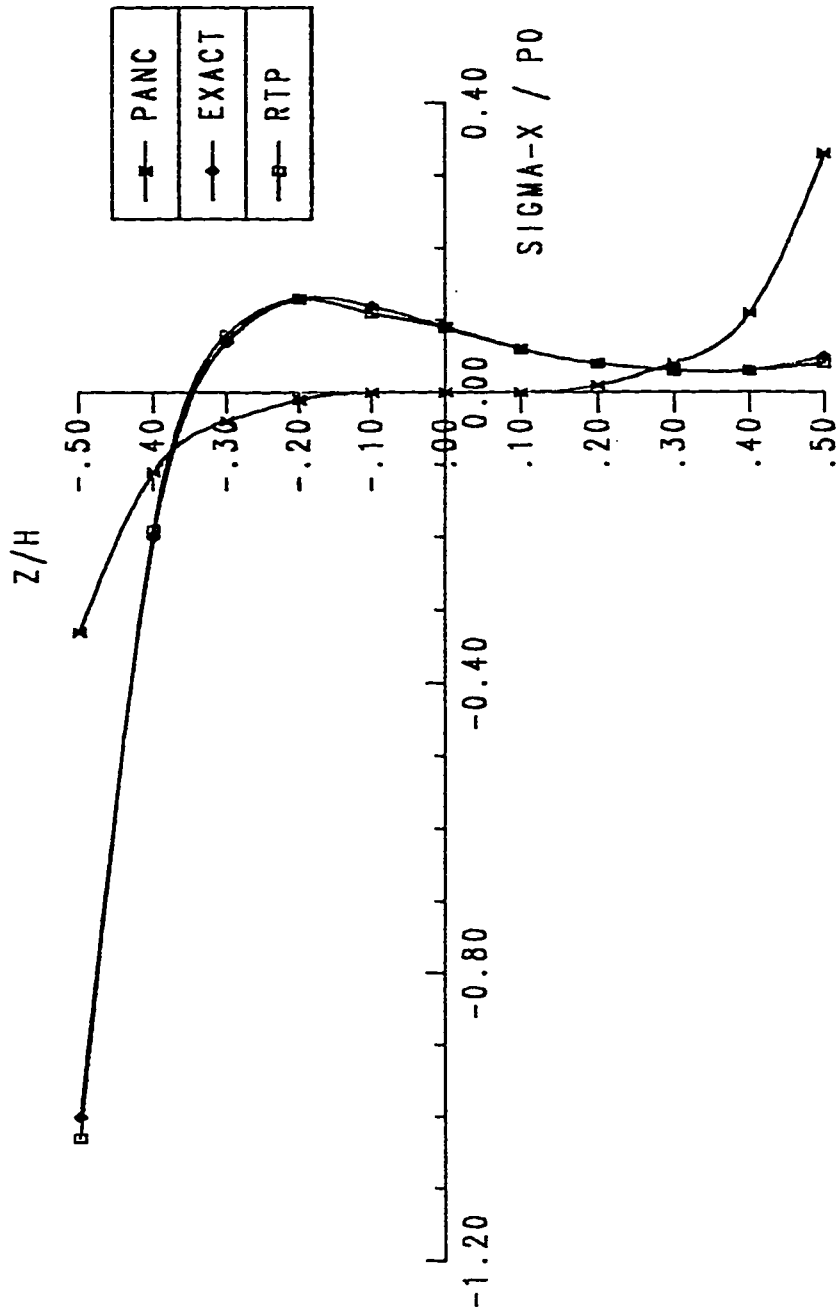


FIG.5.8 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.5, P=P0\*SIN(PI\*X/L) )

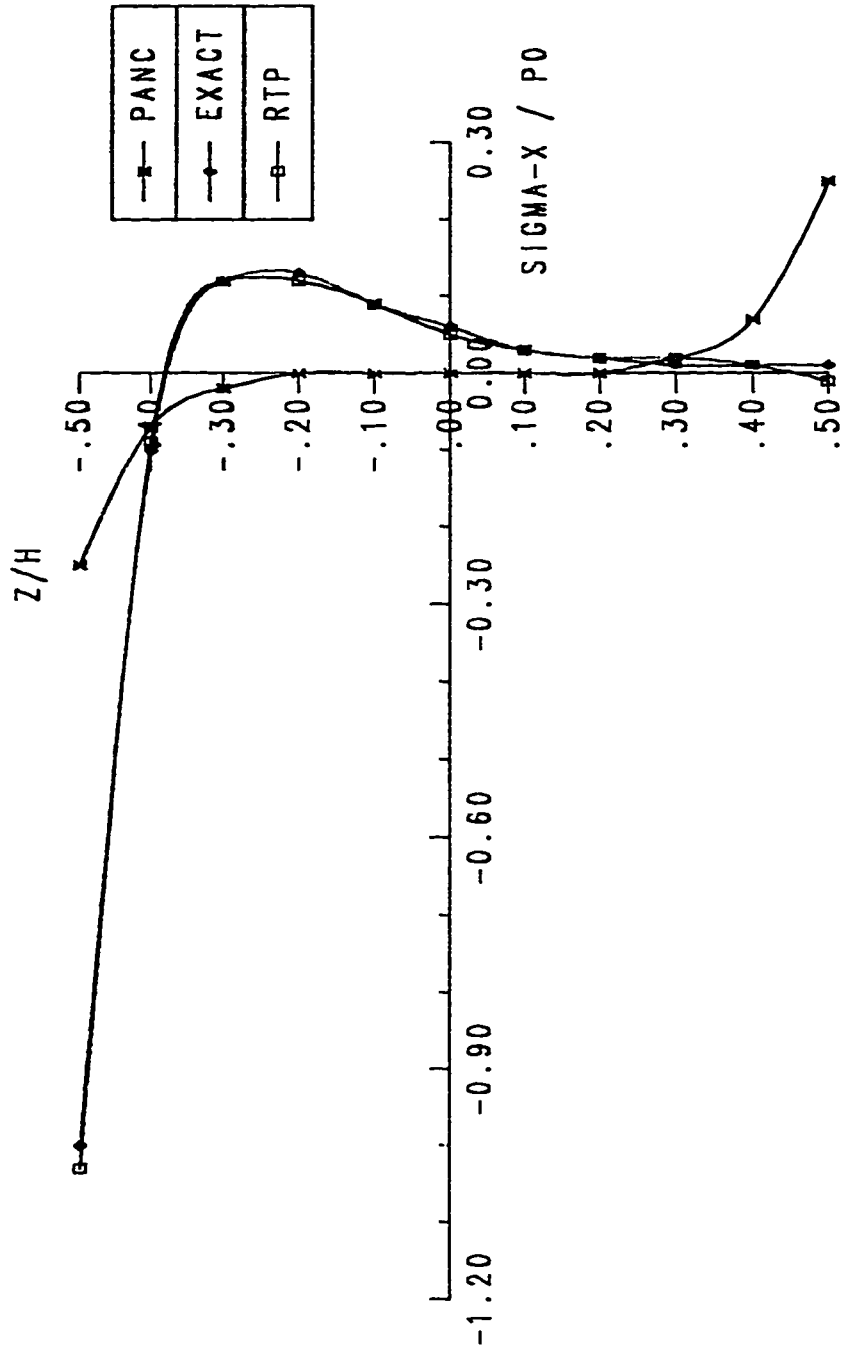


FIG.5.9 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=3.0, P=P0\*SIN(PI\*X/L) )

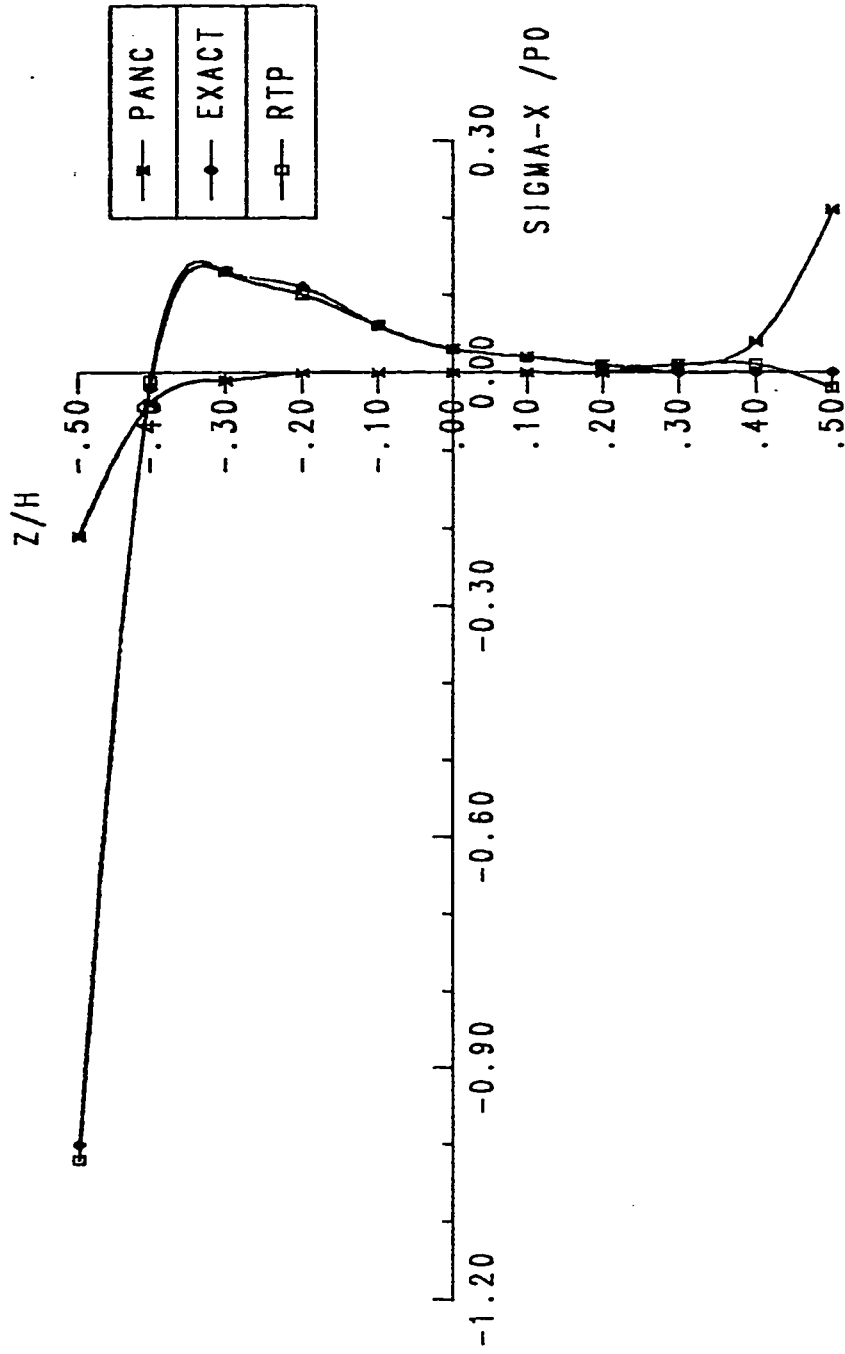


FIG. 5.10 : DEFLECTION COEFFICIENT K VS H/L  
 { K = MAX. W0 (REFINED THEORY)/MAX. W0 (CLASSICAL THEORY) }  
 { P = UNIFORM LOAD }

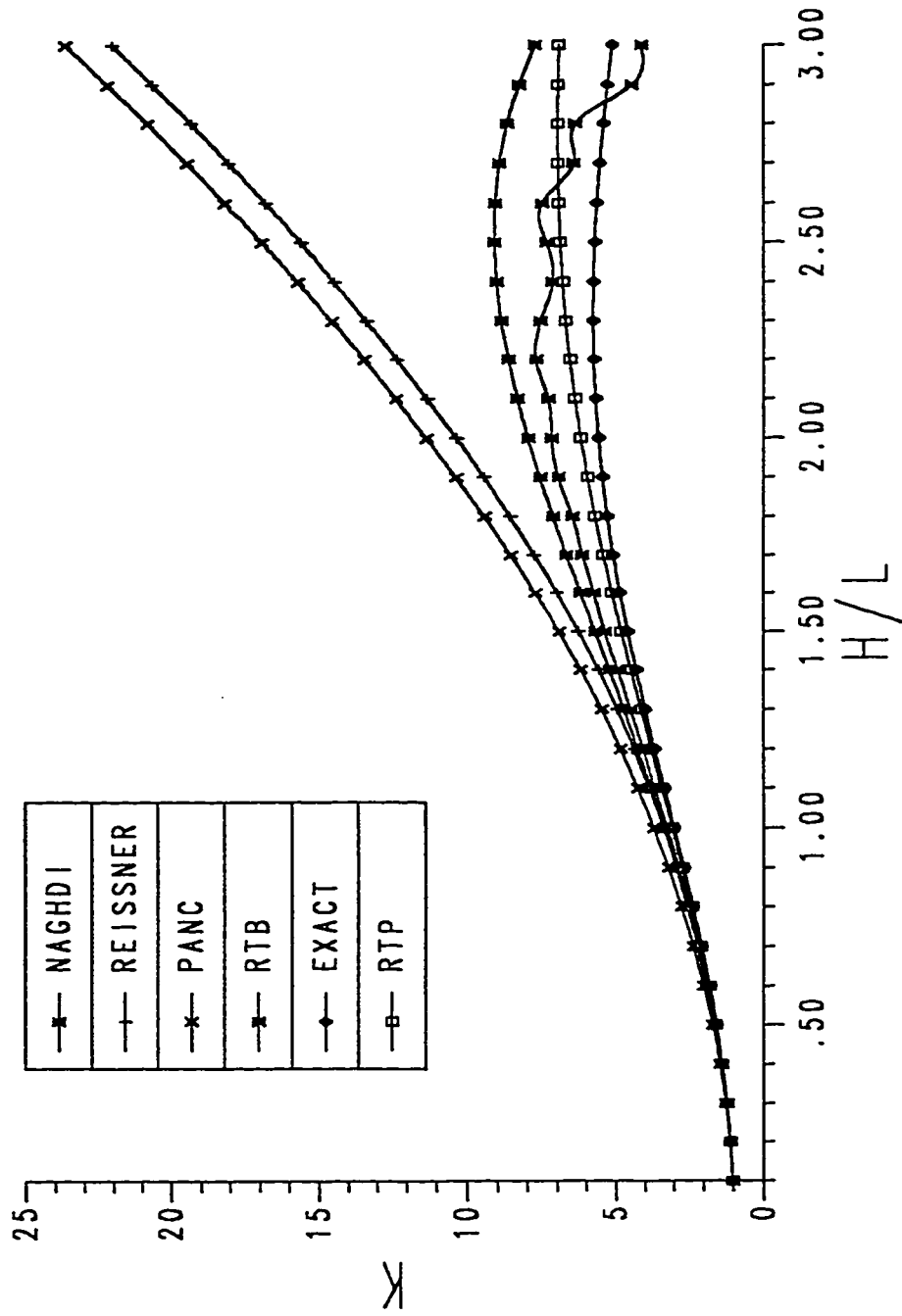


FIG.5.11: MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.1, UNIFORM LOAD)

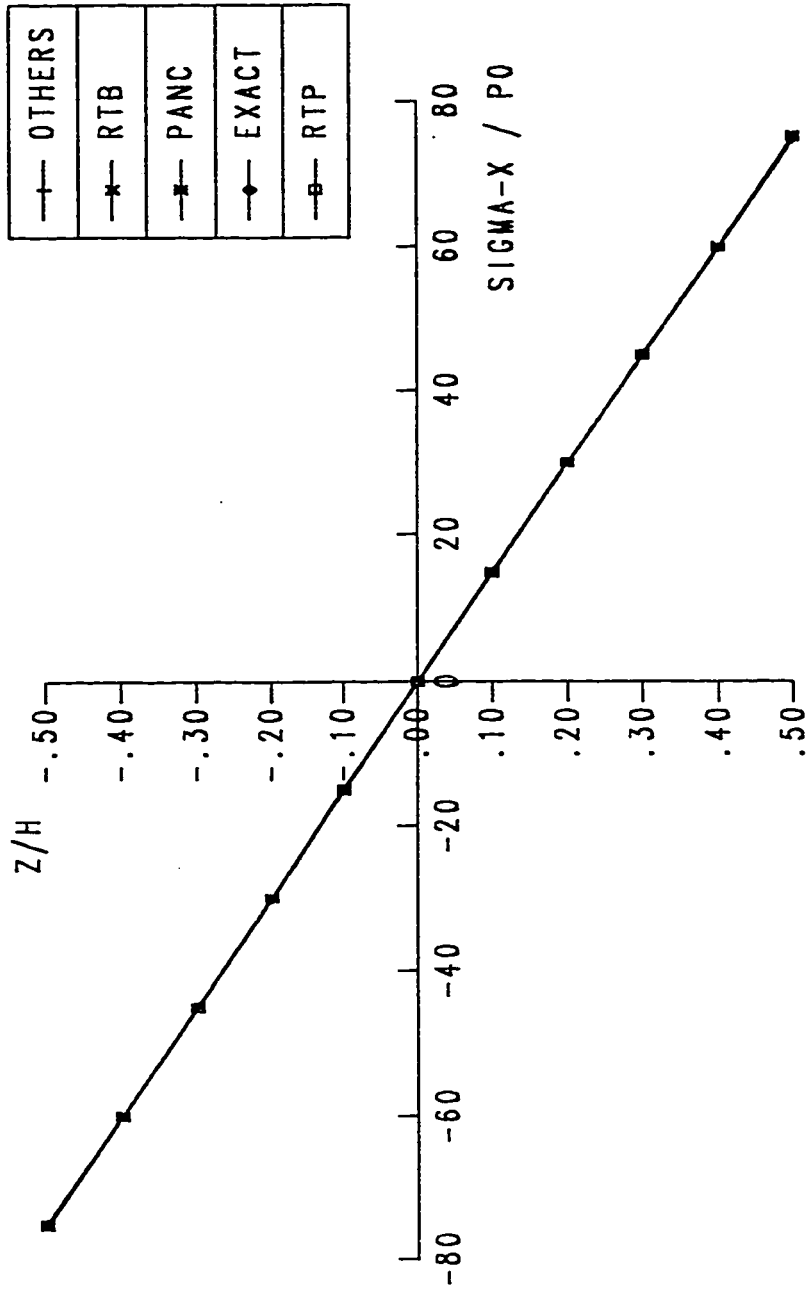


FIG.5.12 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.3, UNIFORM LOAD)

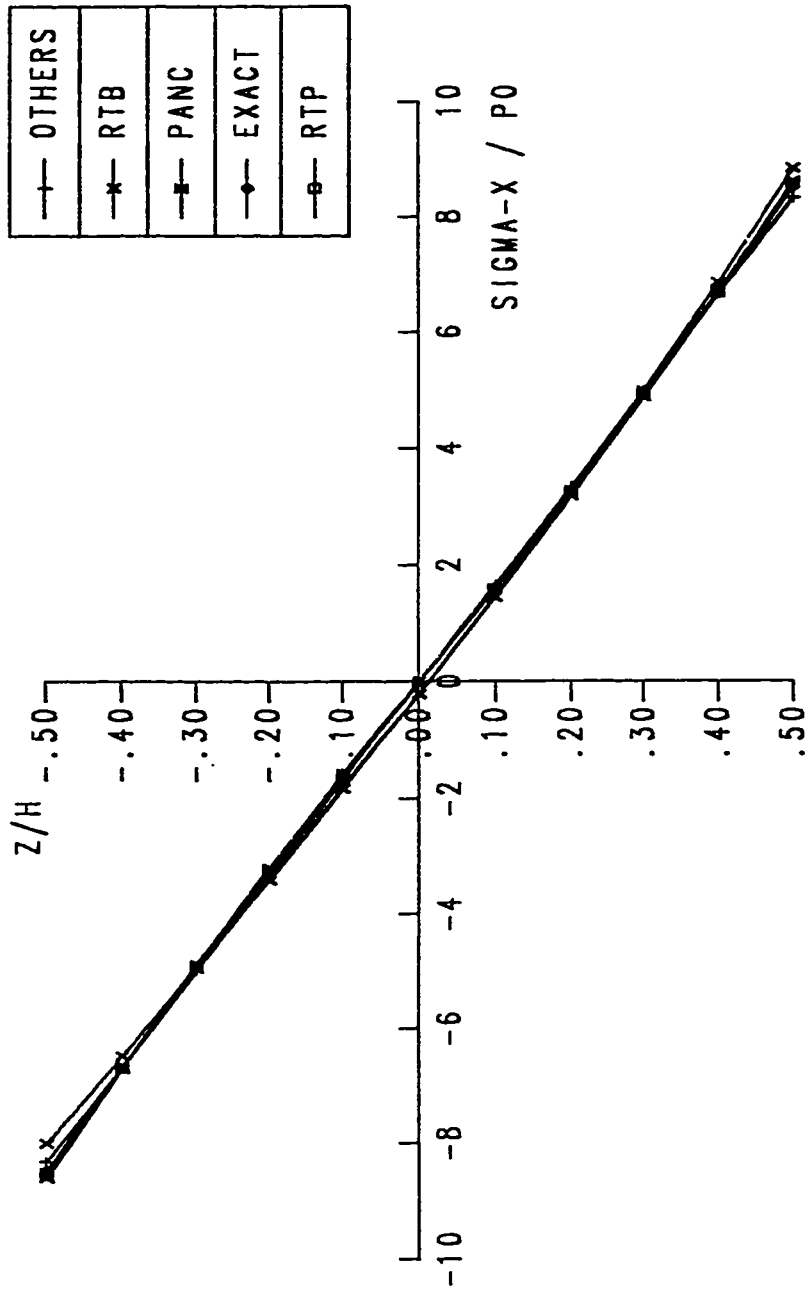


FIG.5.13 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.5, UNIFORM LOAD)

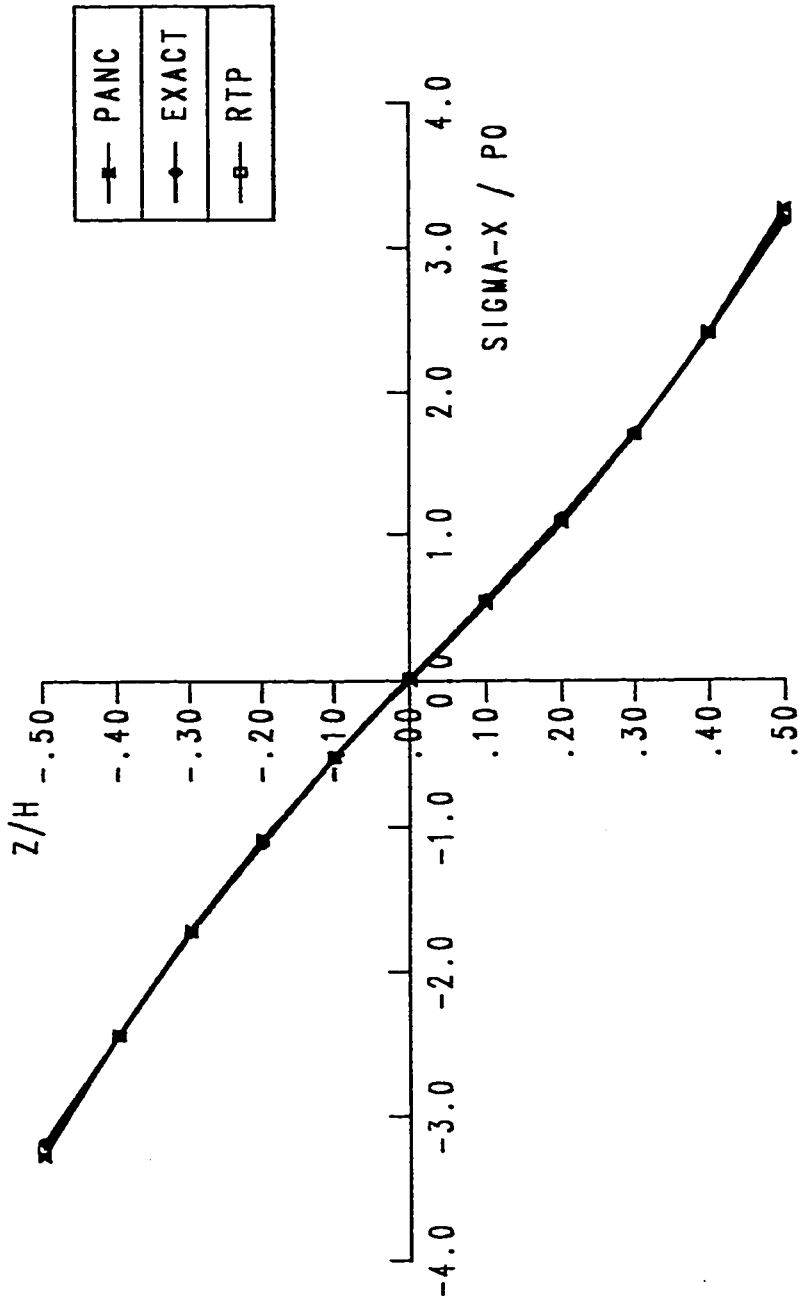


FIG. 5.14 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.0, UNIFORM LOAD)

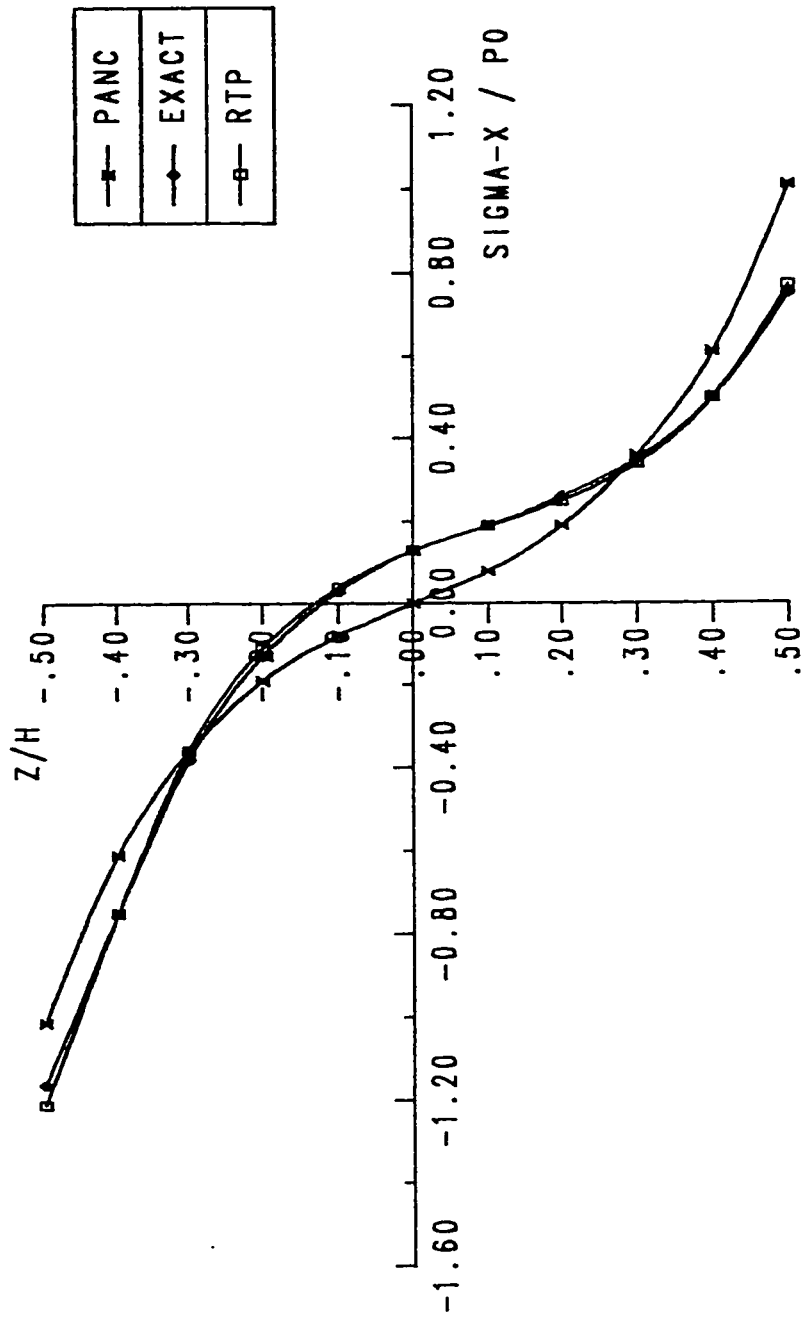




FIG. 5.15 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.5, UNIFORM LOAD)

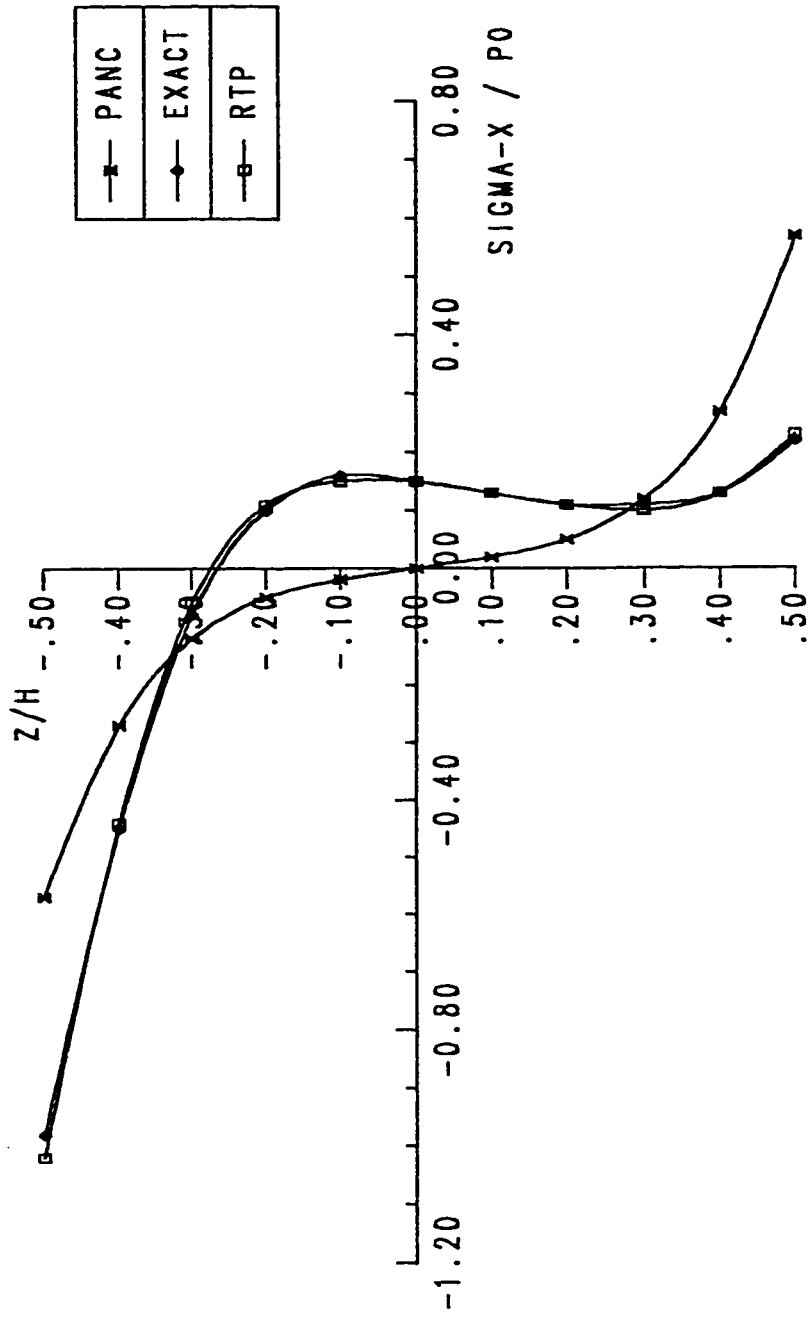


FIG. 5.16 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.0, UNIFORM LOAD)

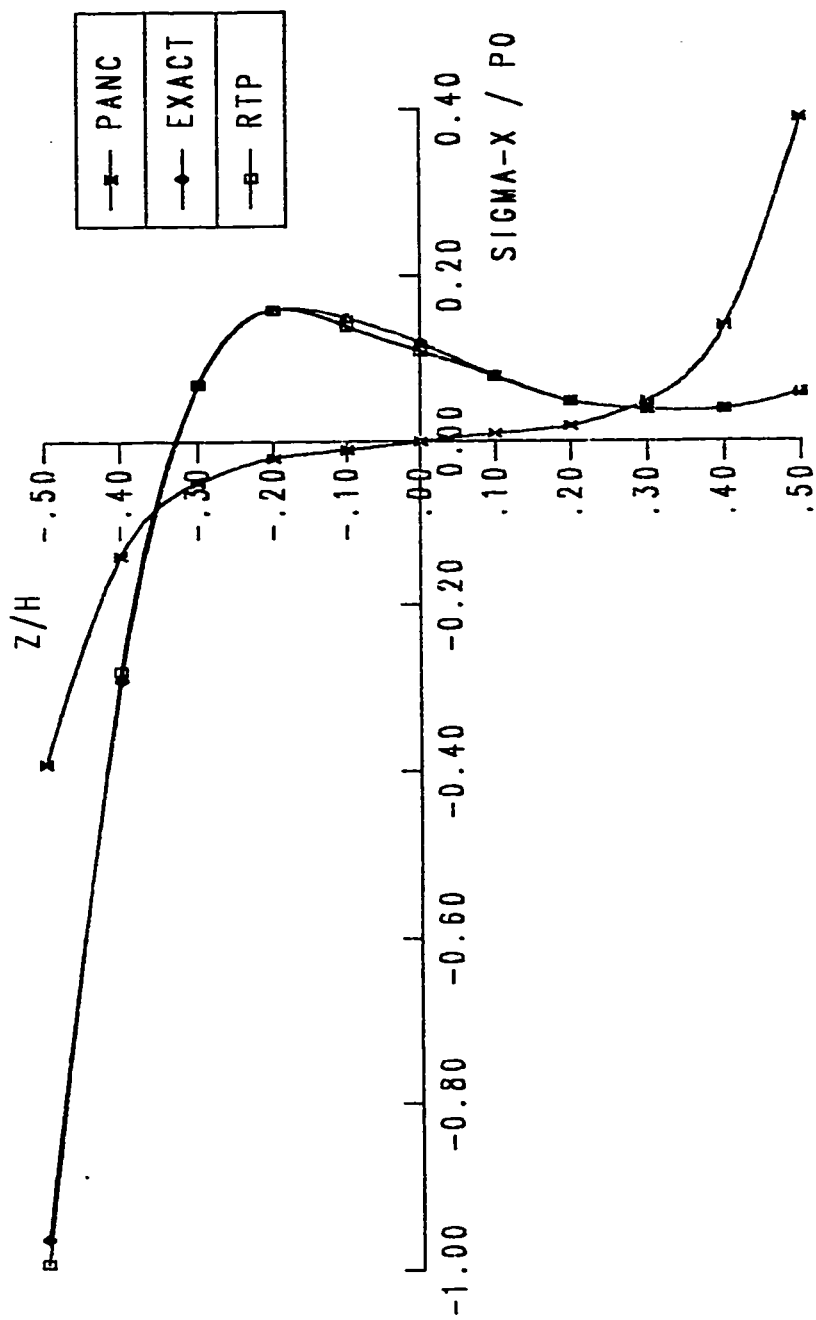


FIG. 5.17: MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.5, UNIFORM LOAD)

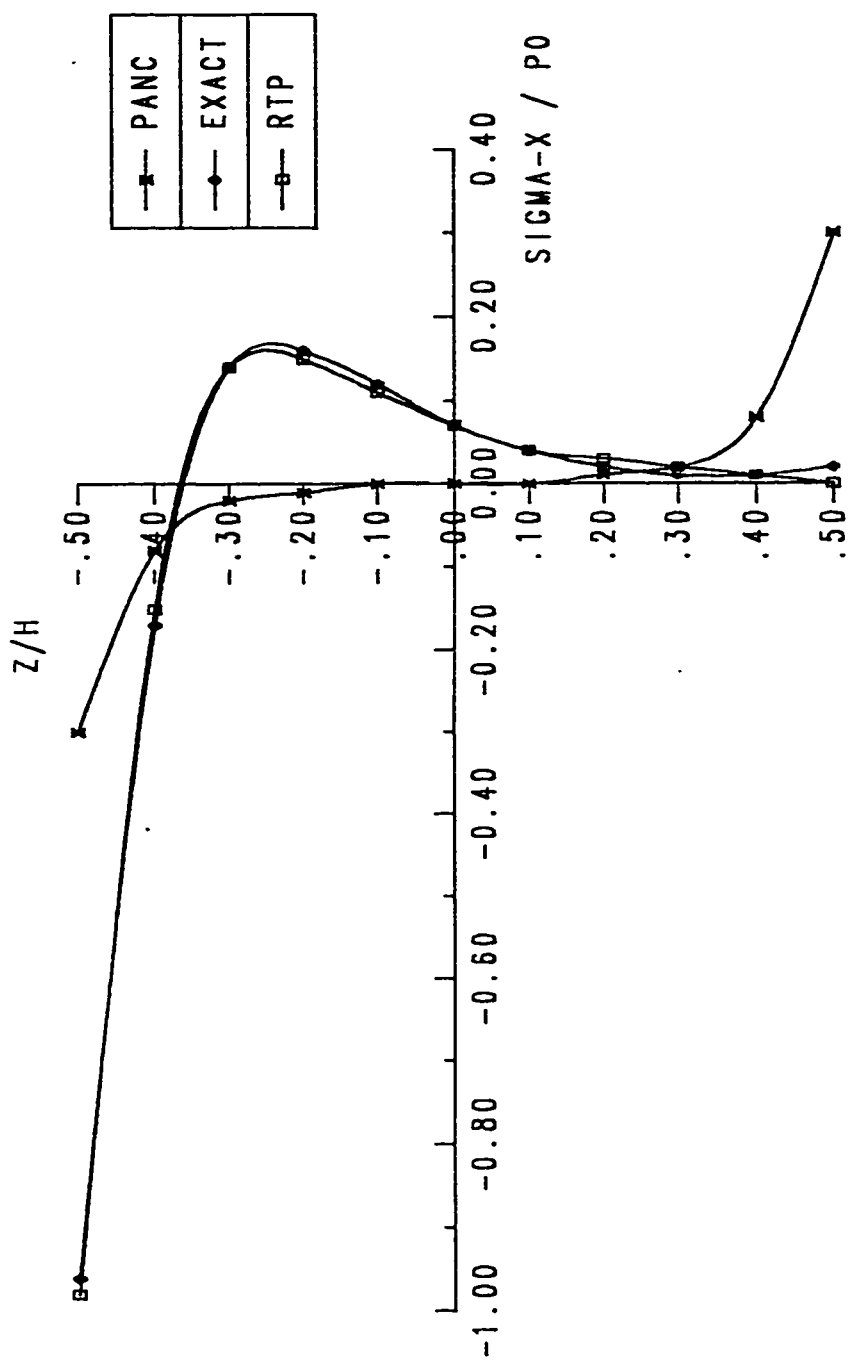


FIG. 5.18 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=3.0, UNIFORM LOAD)

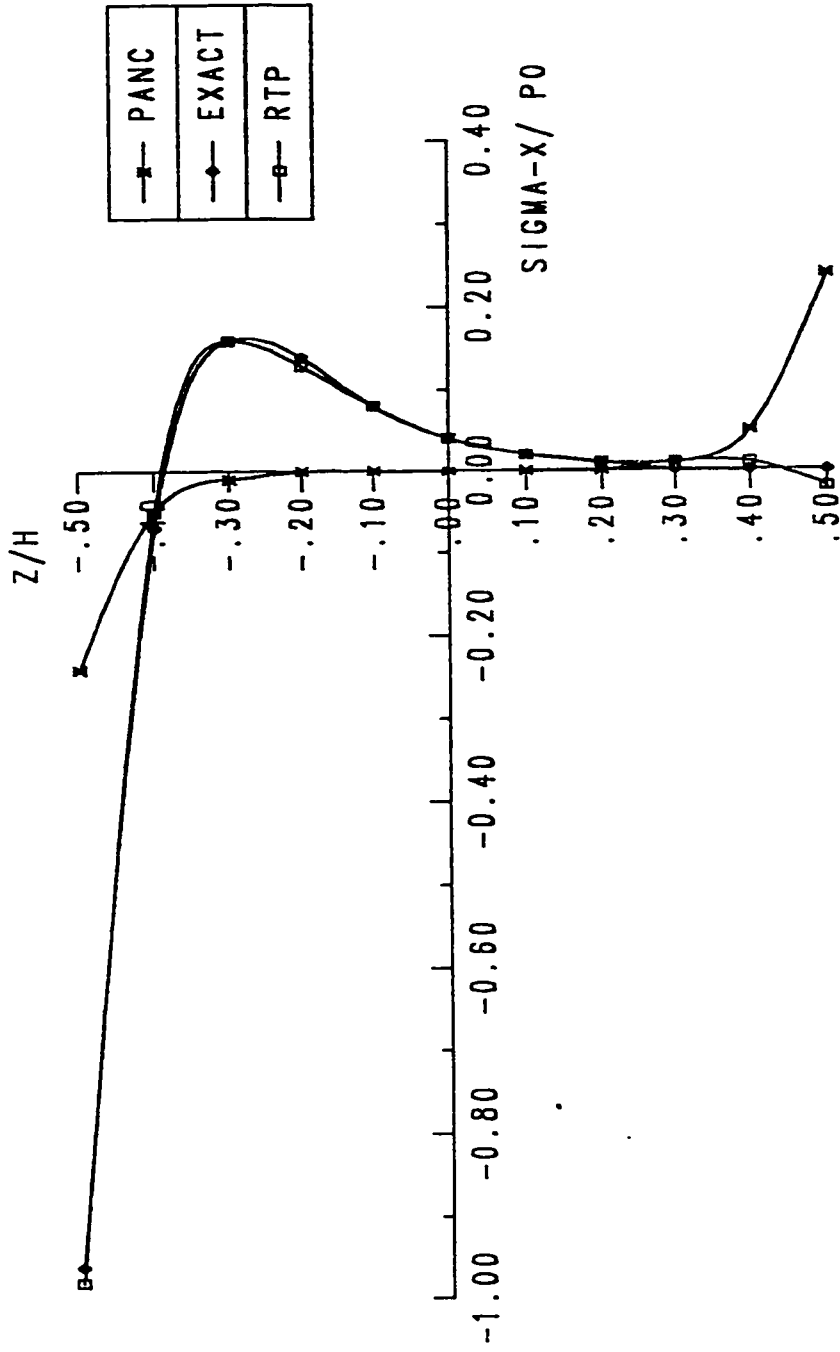


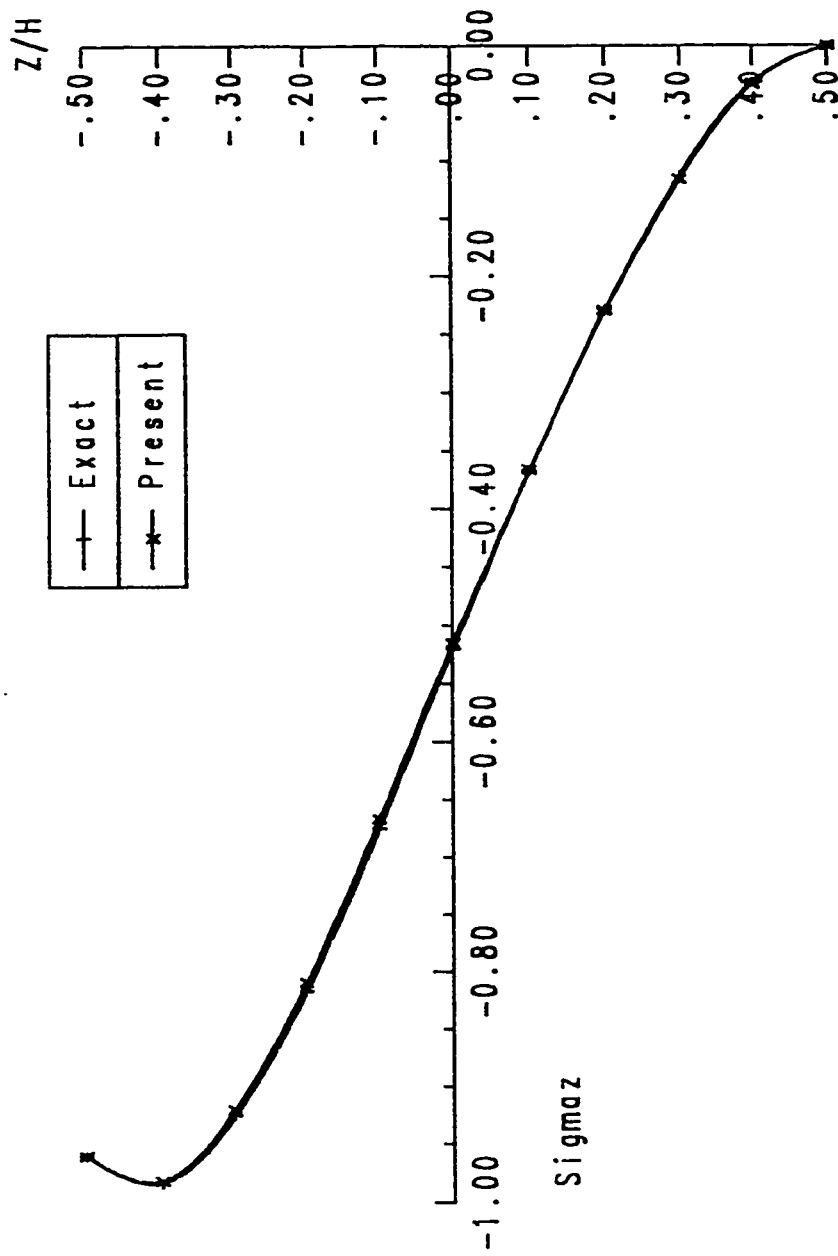
FIG. 5.19 : Max. Normal Stress  $\sigma_{max}$  Vs  $Z/H$  ( $H/L=1.0$ , Uniform Load)

FIG. 5.20 : Max. Normal Stress  $\sigma_{max}$  Vs  $Z/H$  ( $H/L=2.0$ , Uniform Load)

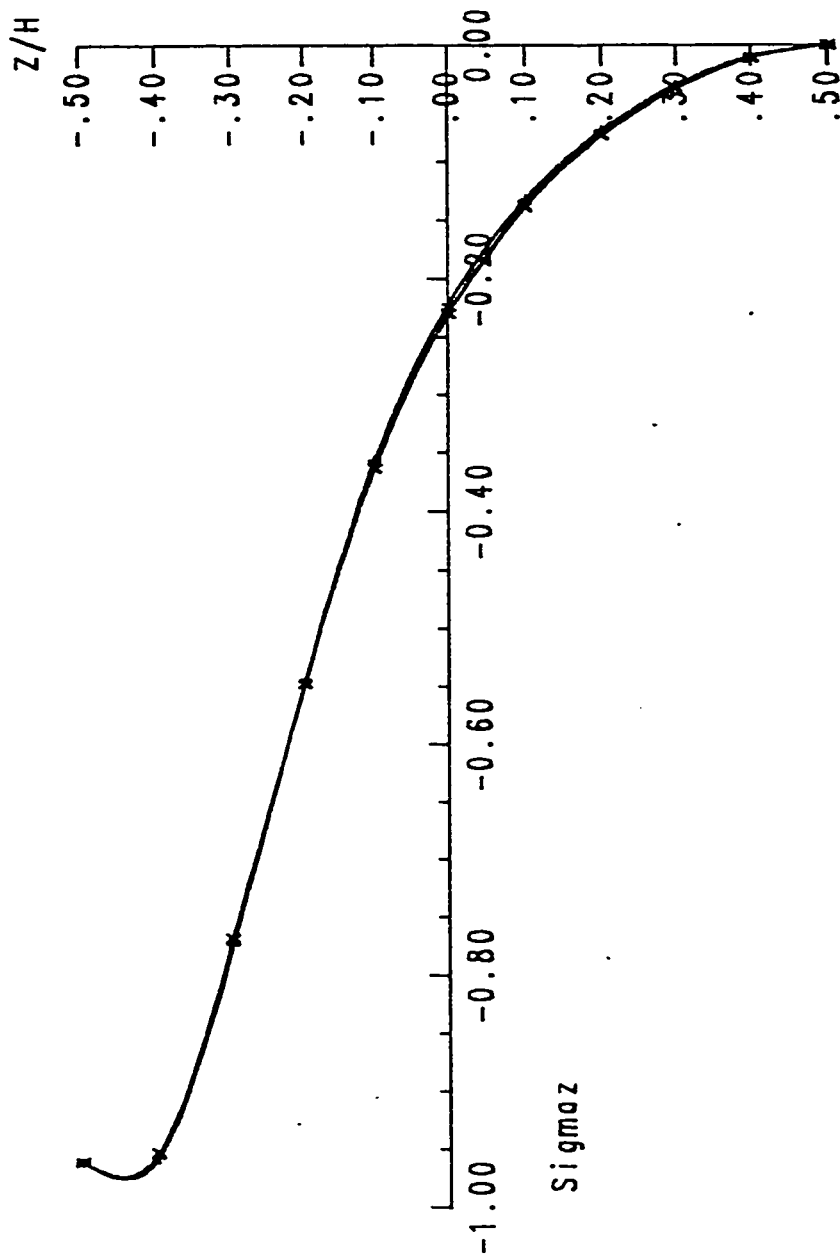


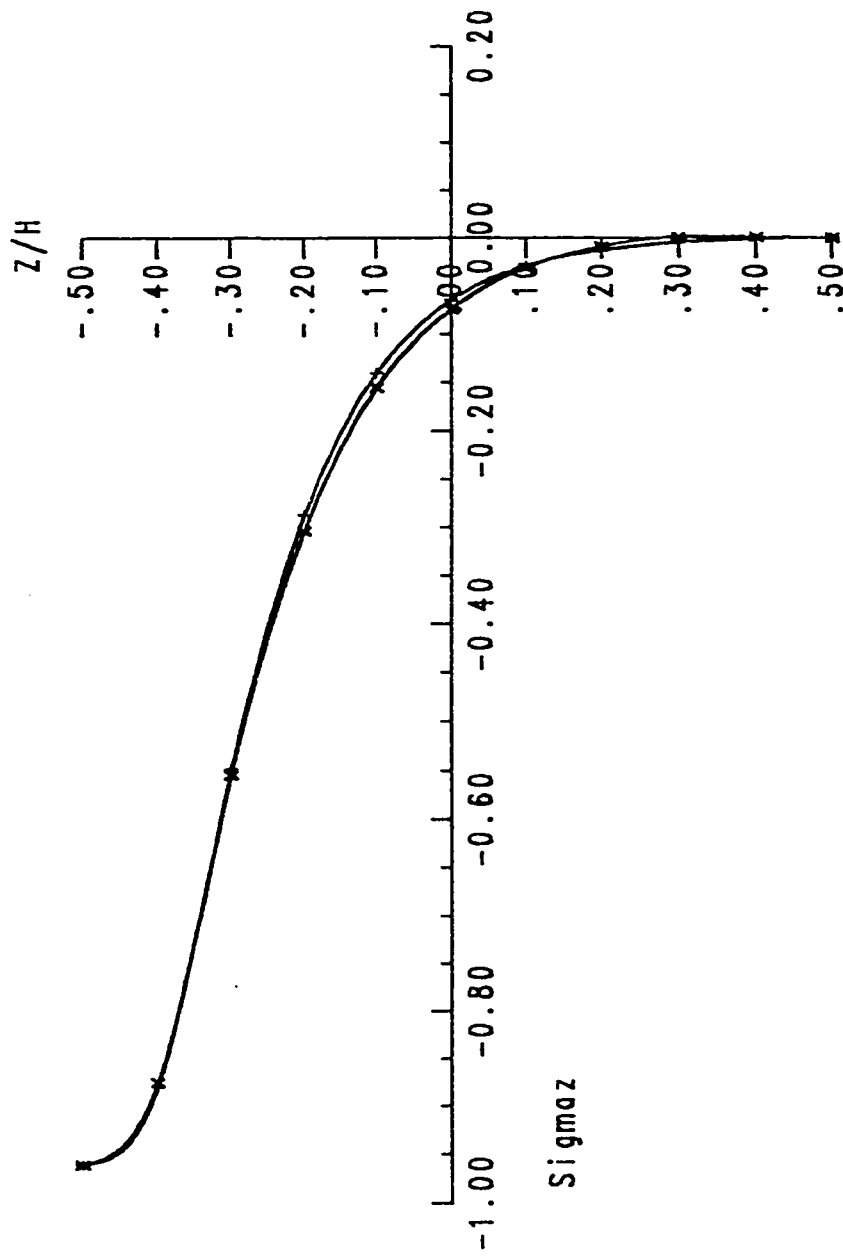
FIG. 5.21 : Max. Normal Stress  $\sigma_{maz}$  Vs  $Z/H$  ( $H/L=3.0$ , Uniform Load)

FIG. 5.22 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.005-1)

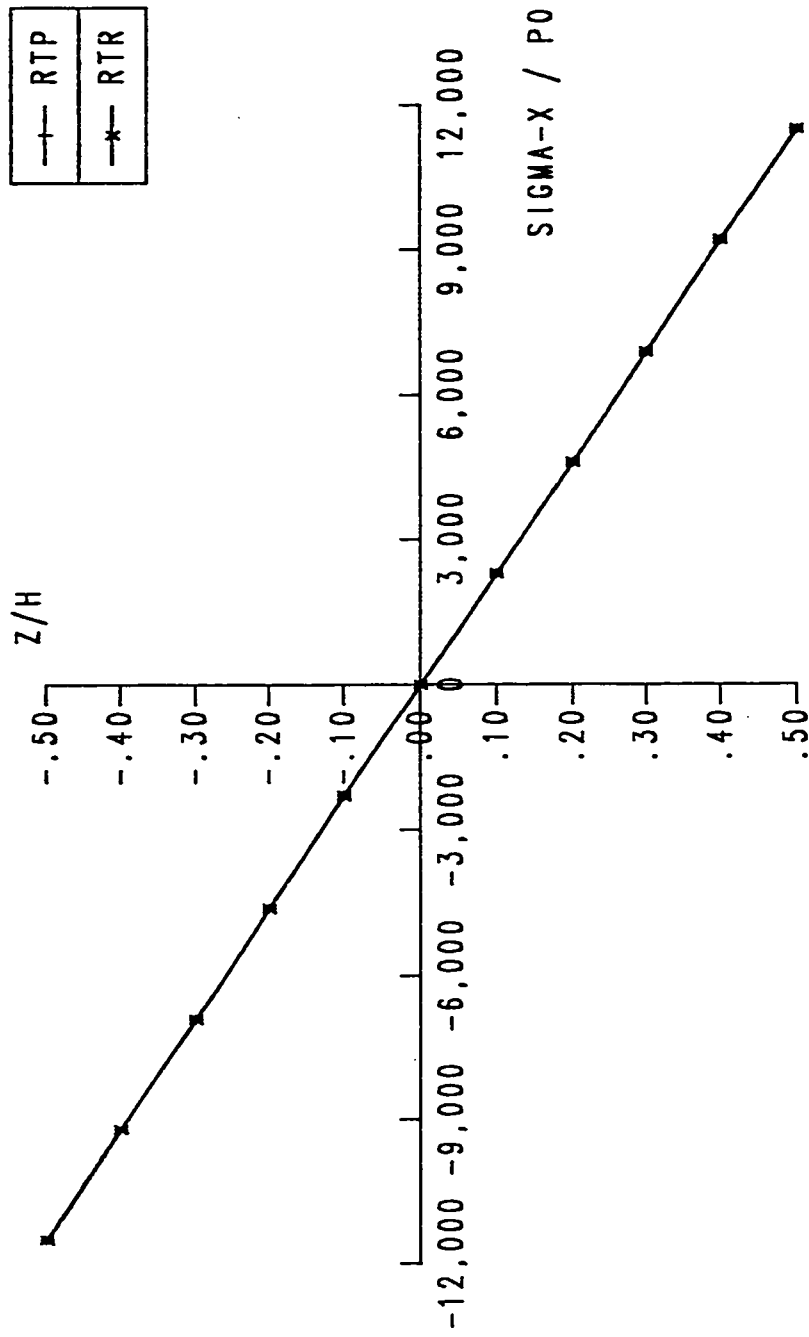




FIG. 5.23 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.01-1)

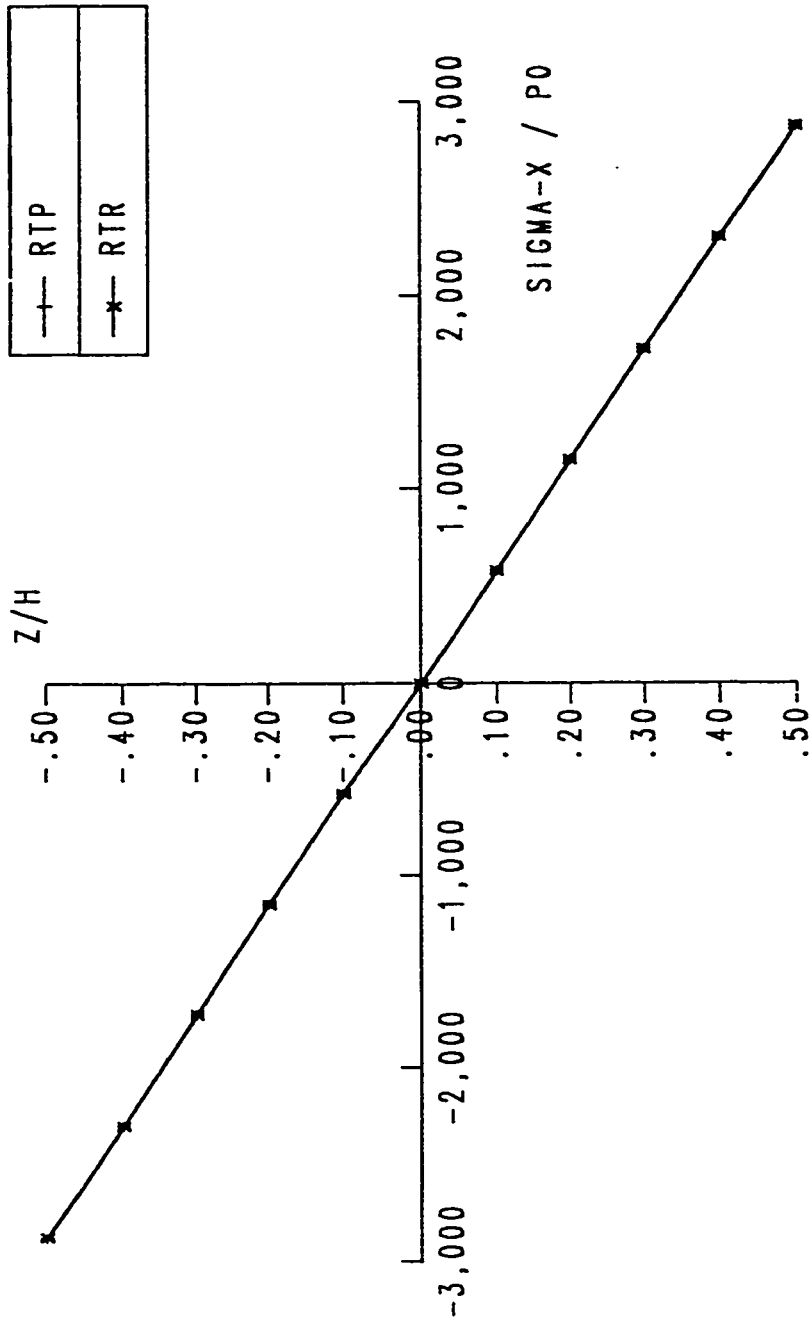


FIG. 5.24 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.05-1)

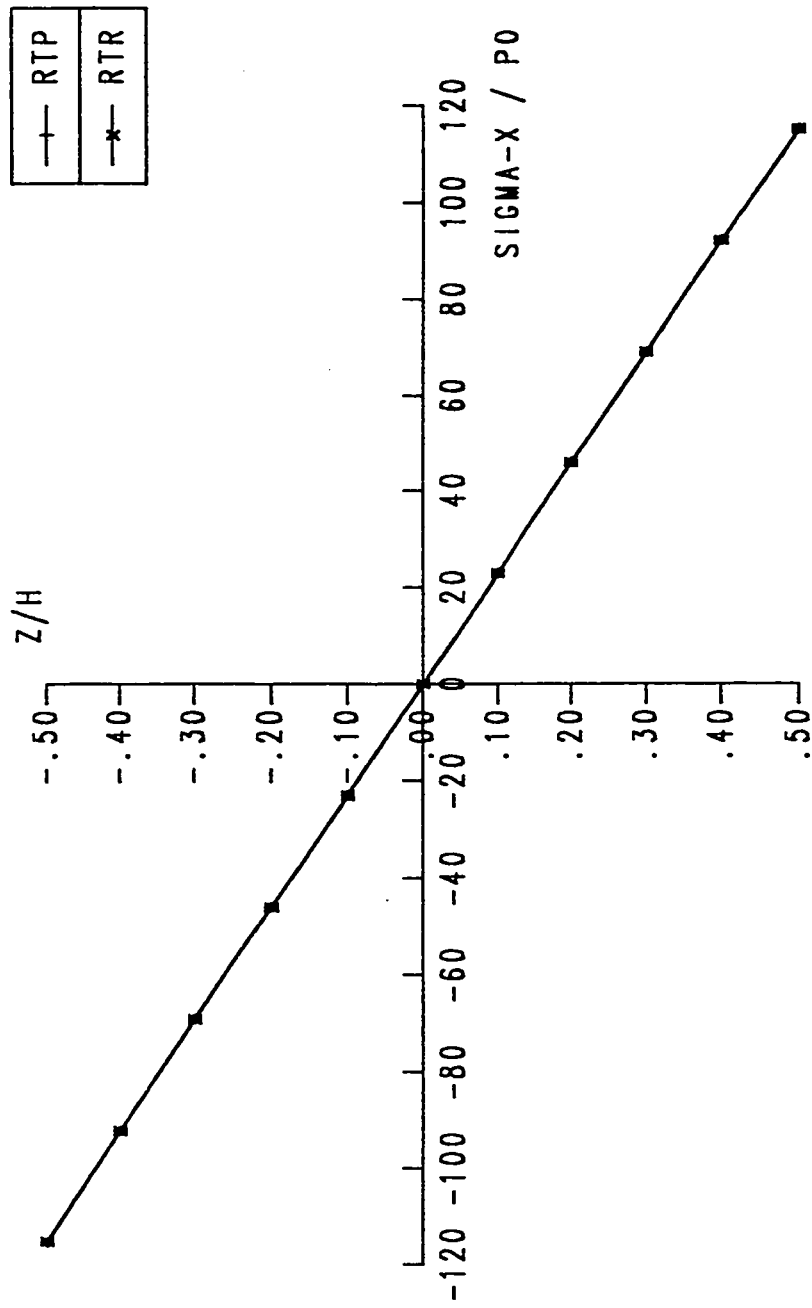


FIG. 5.25 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.1-1)

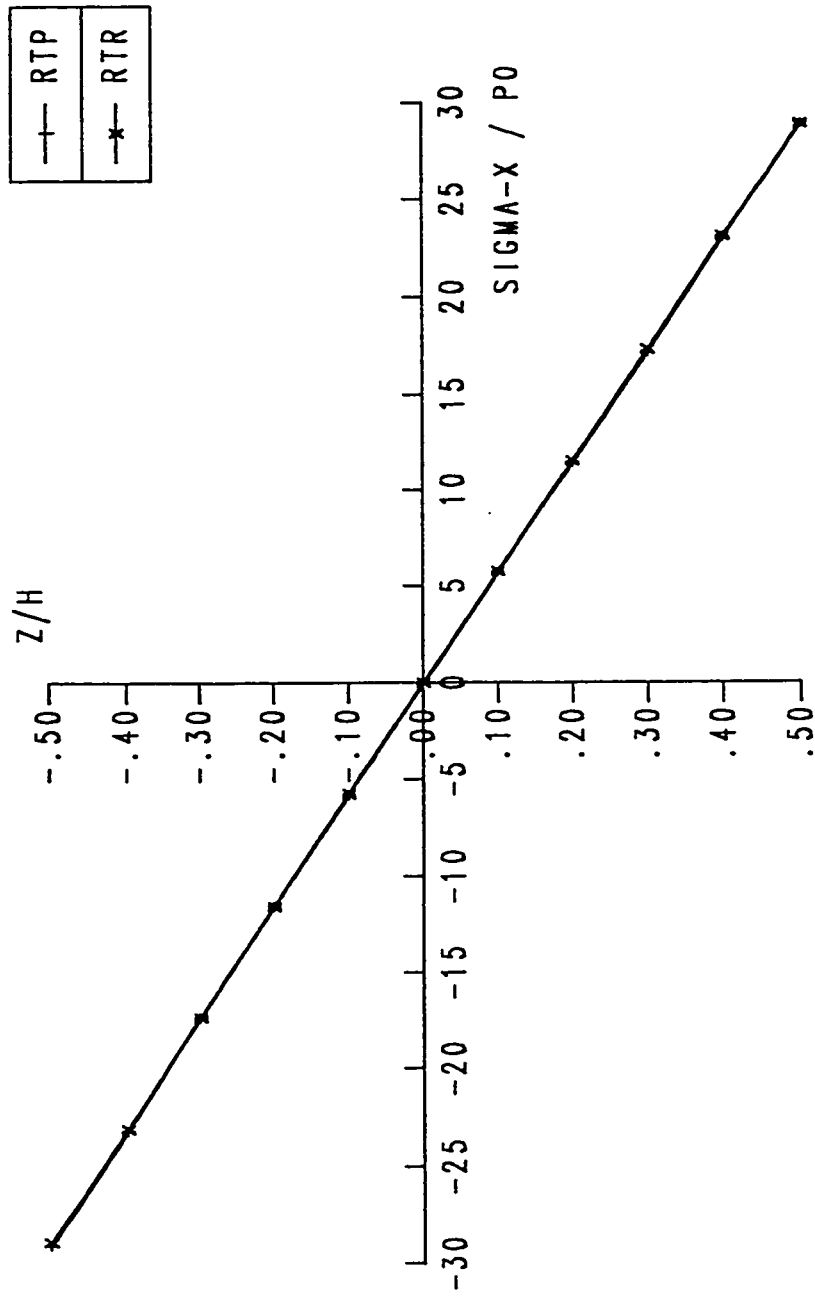


FIG. 5.26 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.2-1)

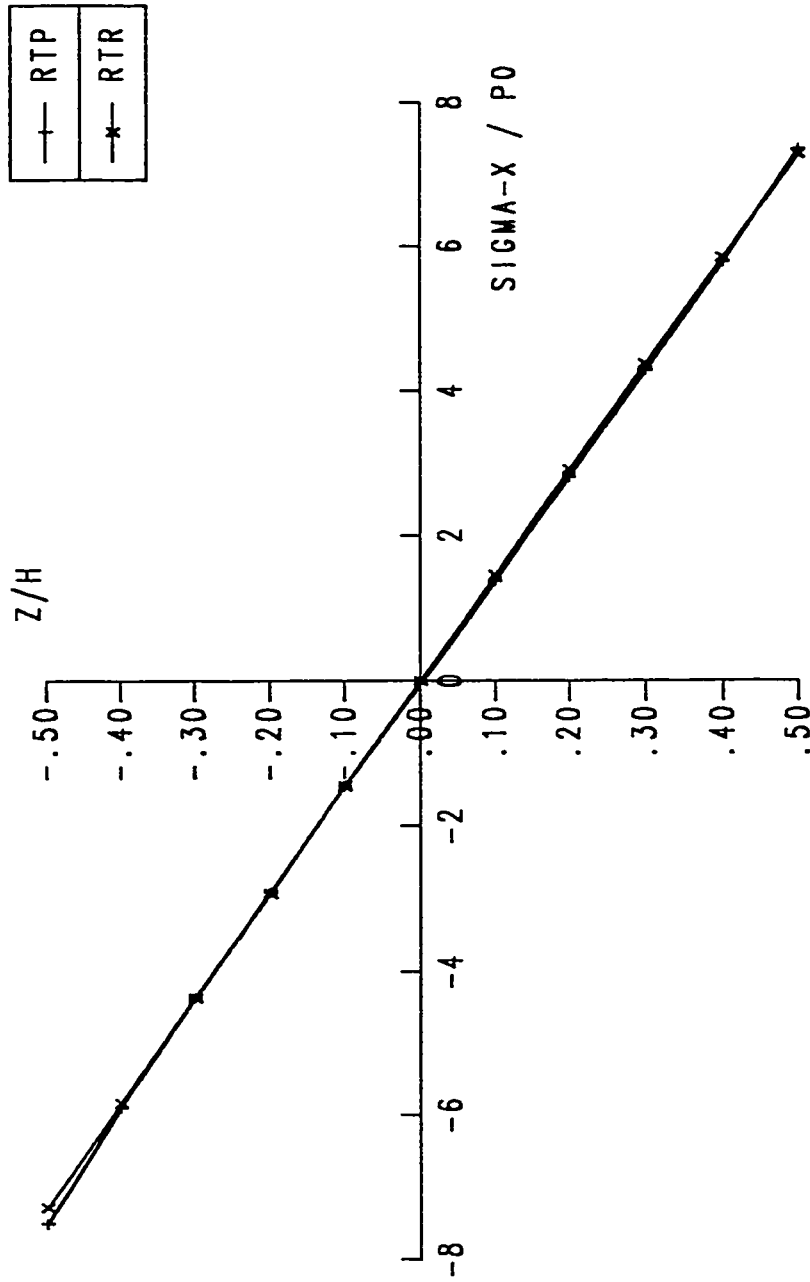


FIG. 5.27 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.3-1)

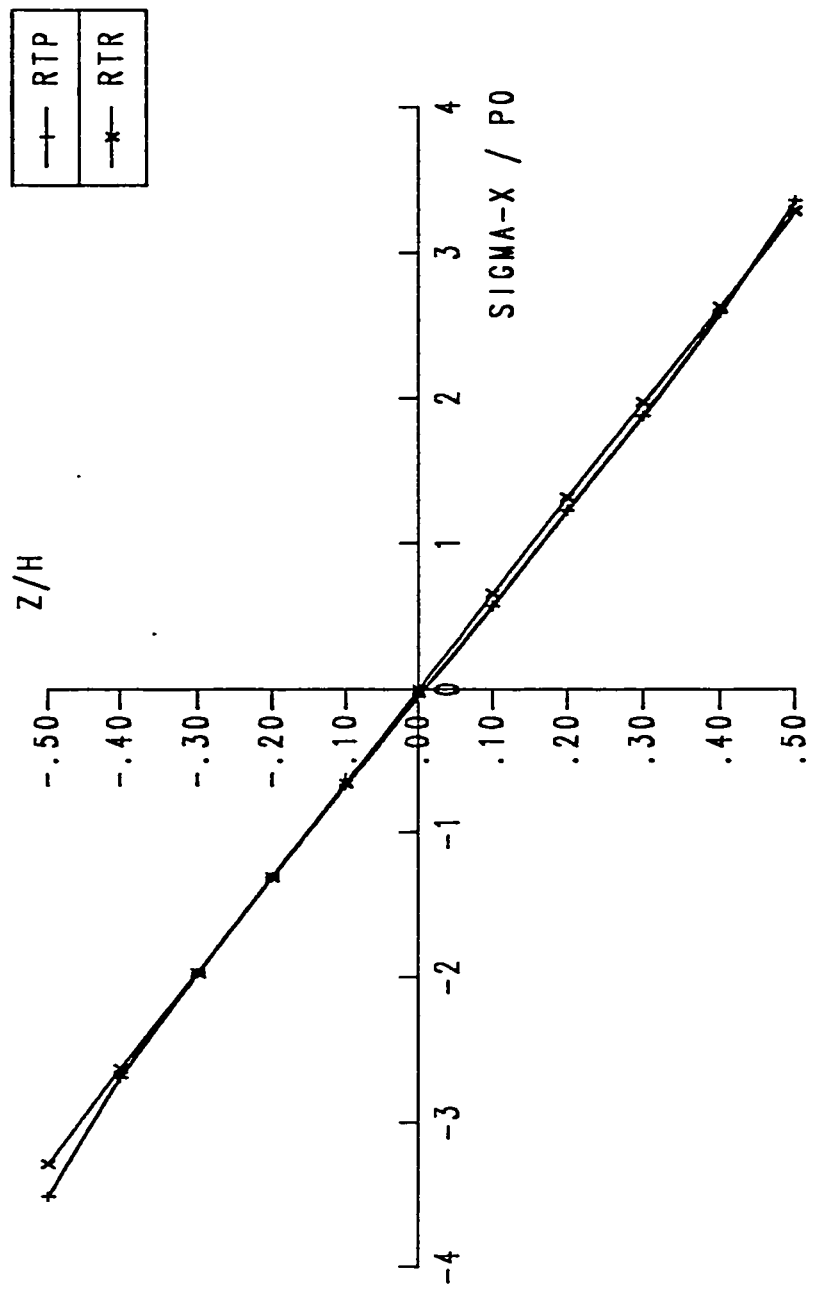


FIG. 5.28 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.5-1)

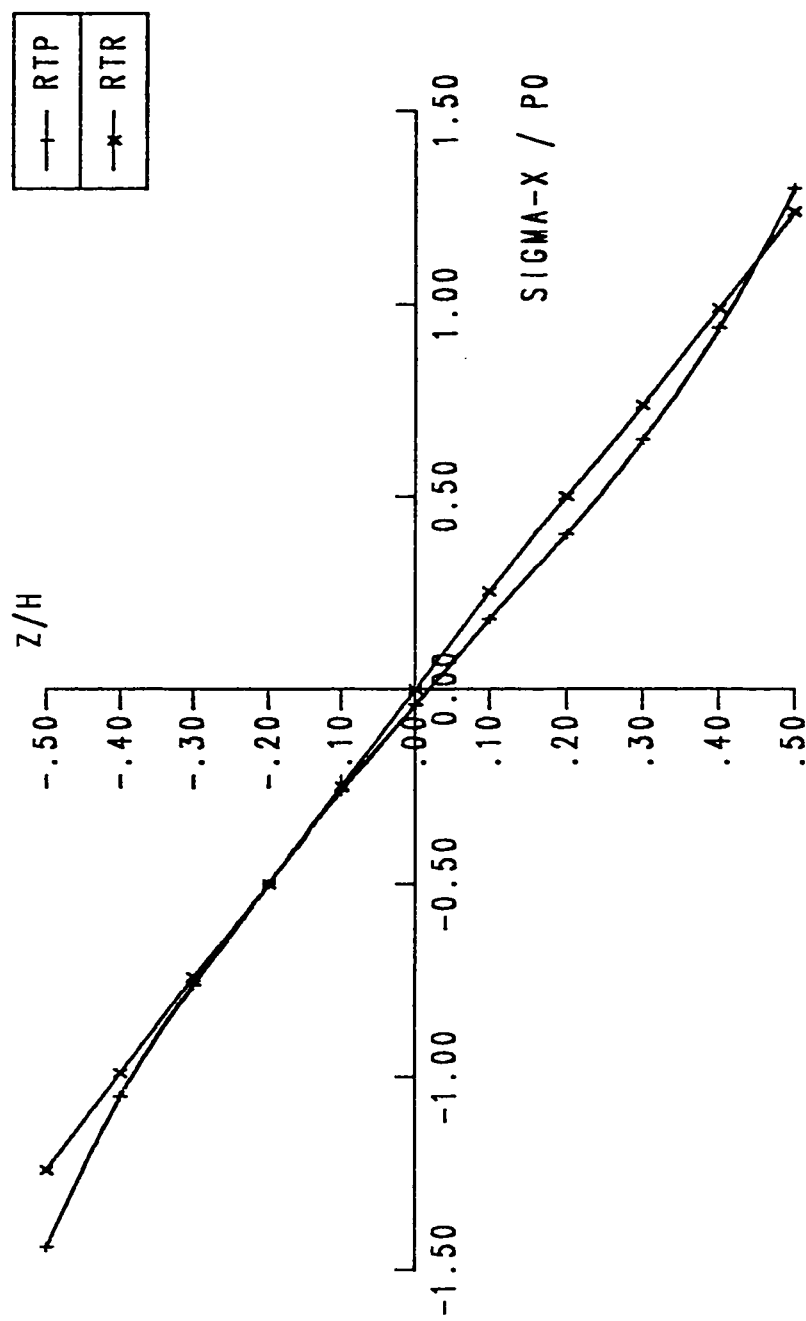


FIG. 5.29 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.7-1)

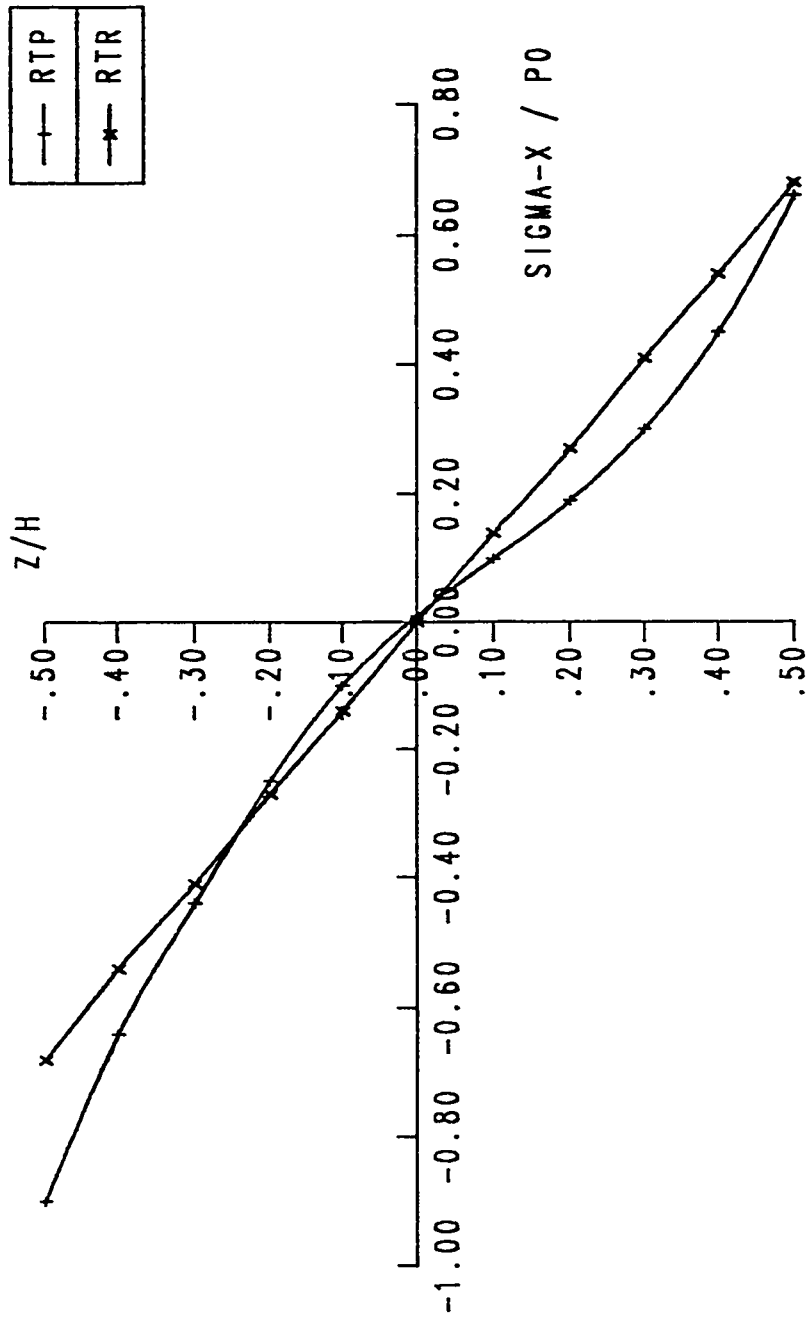


FIG. 5.30 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS1.-I)

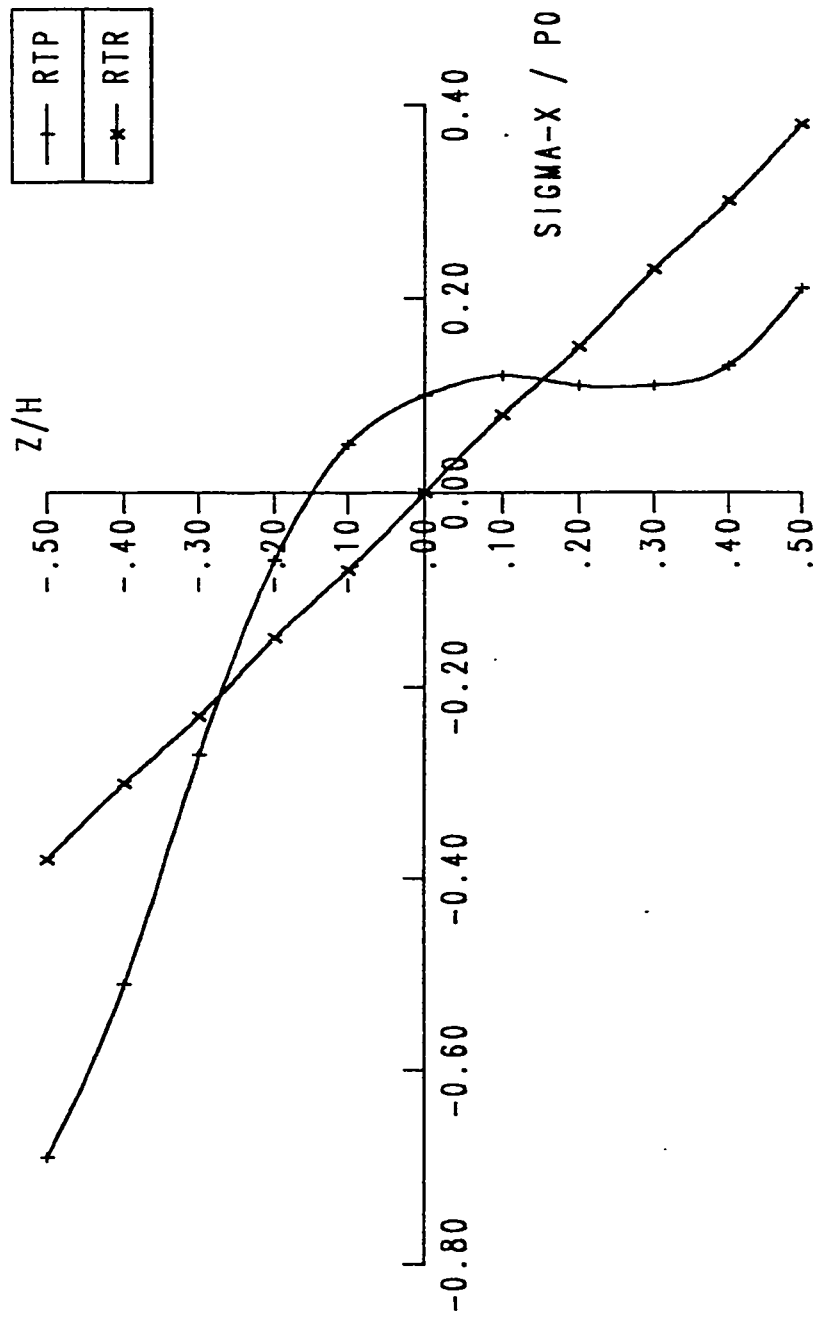




FIG. 5.31 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.005-11)

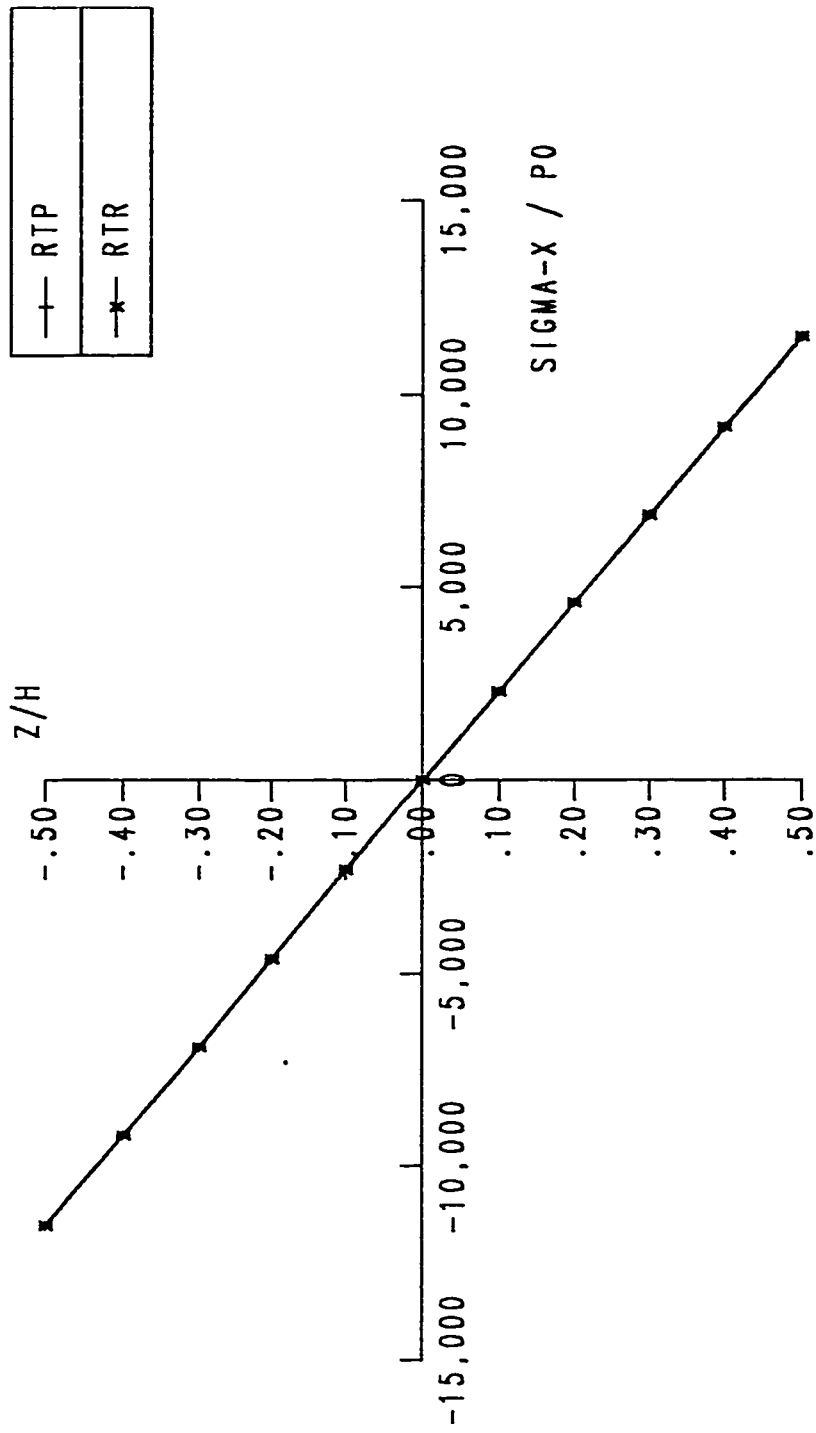


FIG. 5.32 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.01-11)

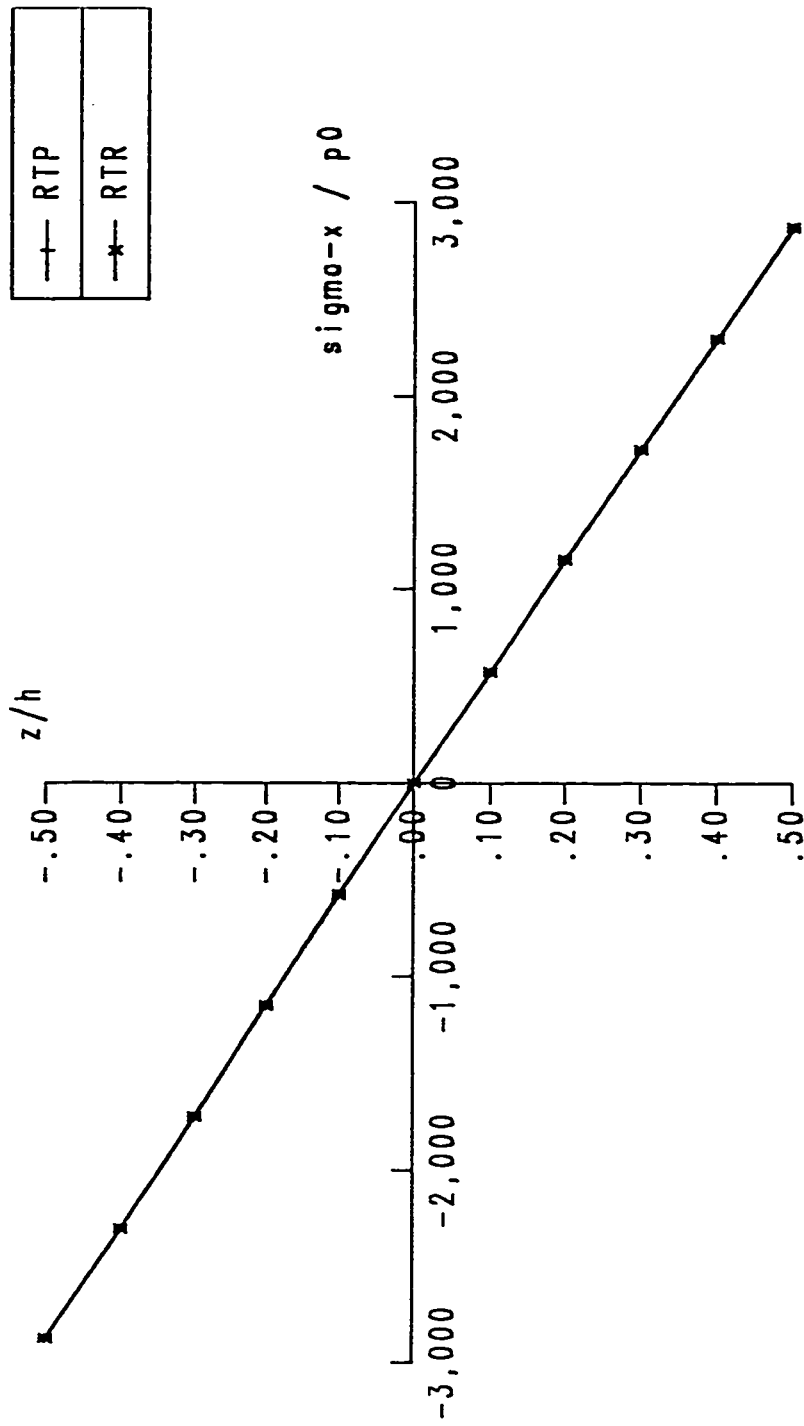


FIG. 5.33 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.05-11)

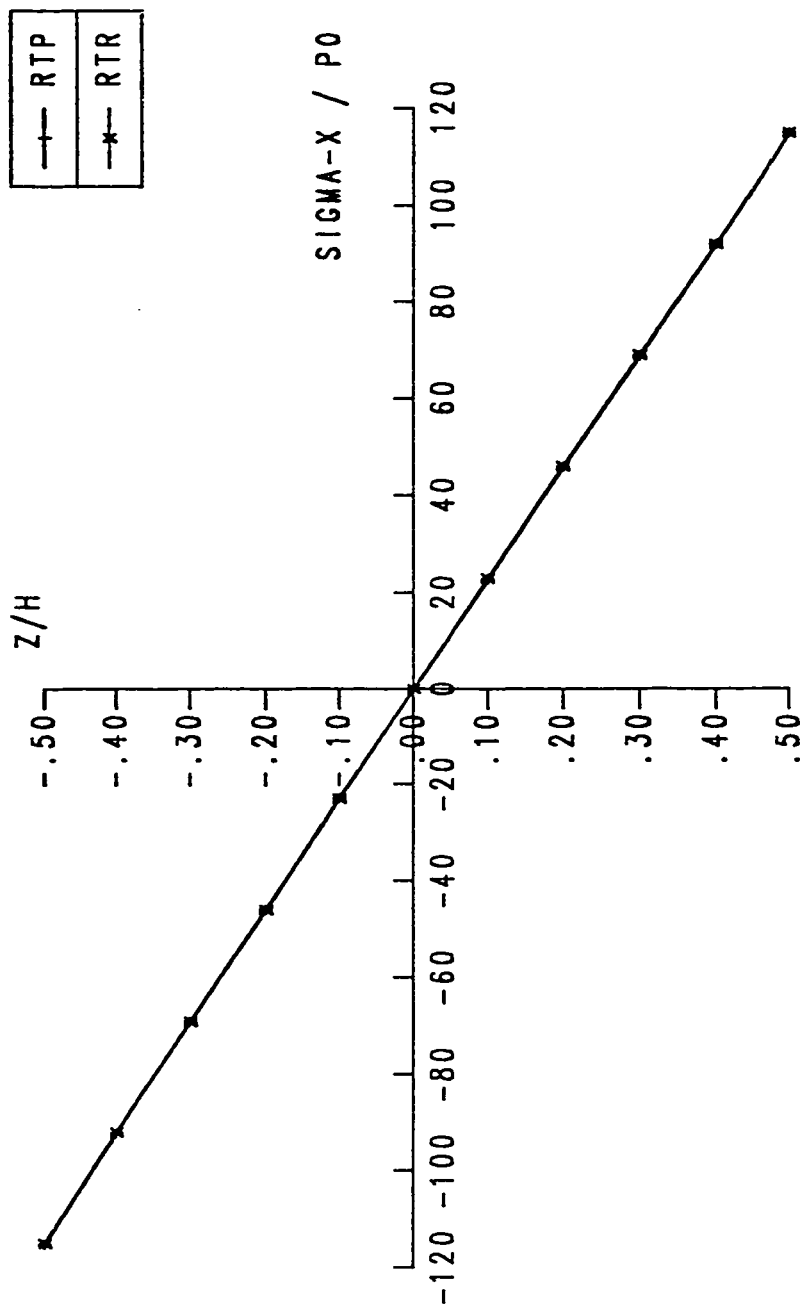


FIG. 5.34 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.1-11)

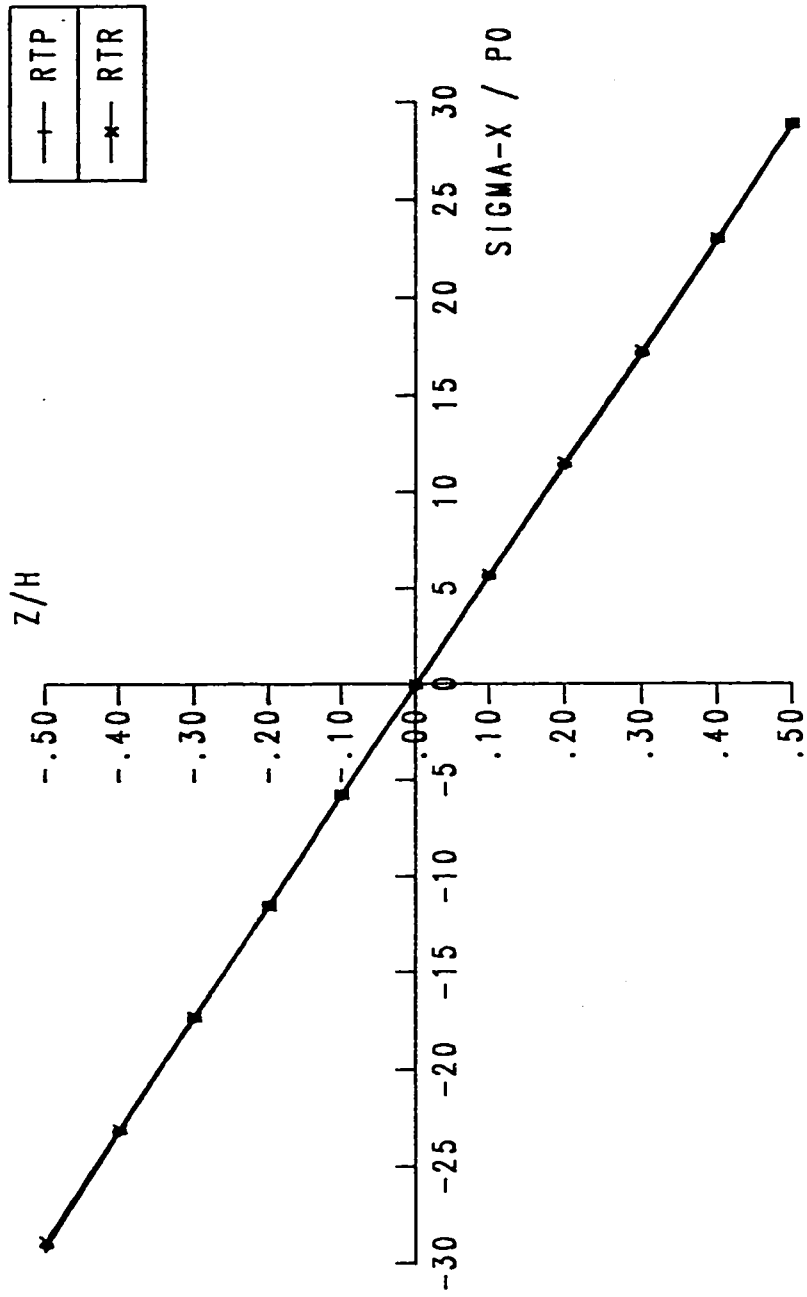


FIG. 5.35 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.2-11)

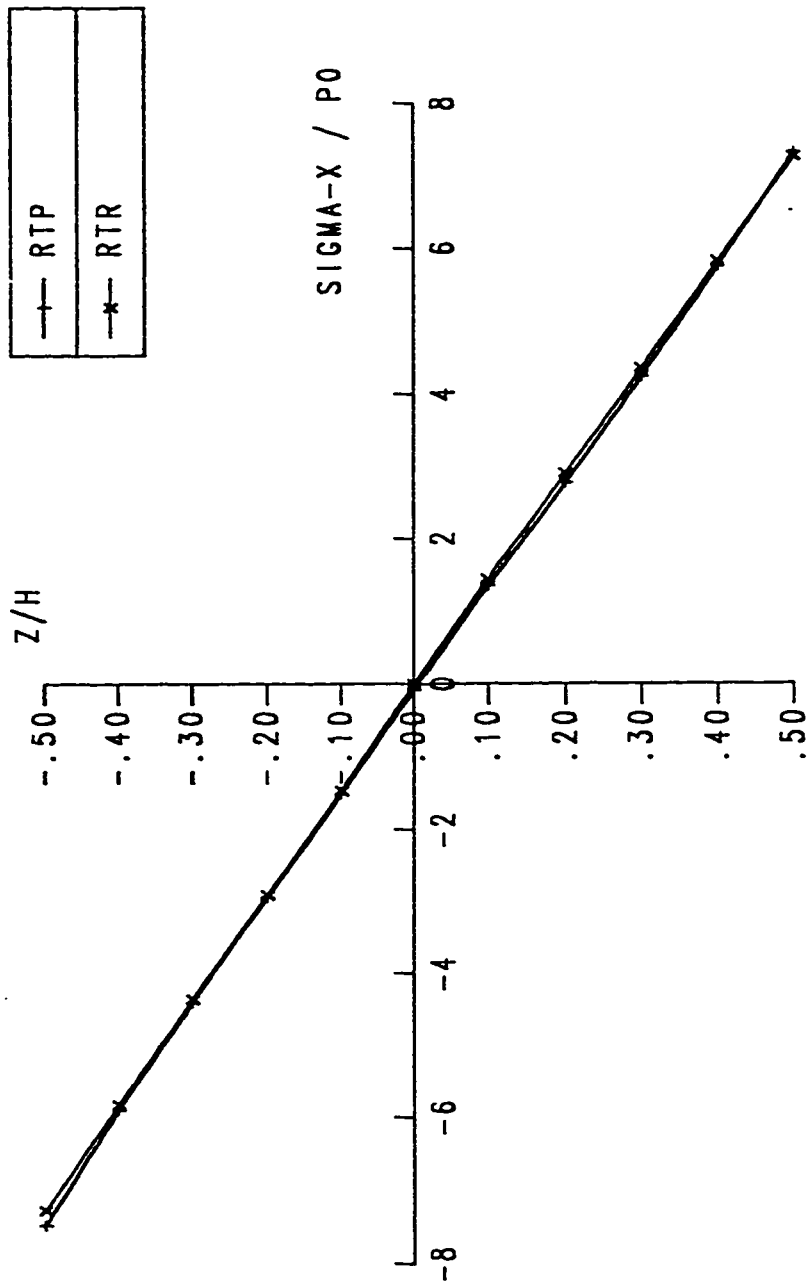


FIG. 5.36 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.3-11)

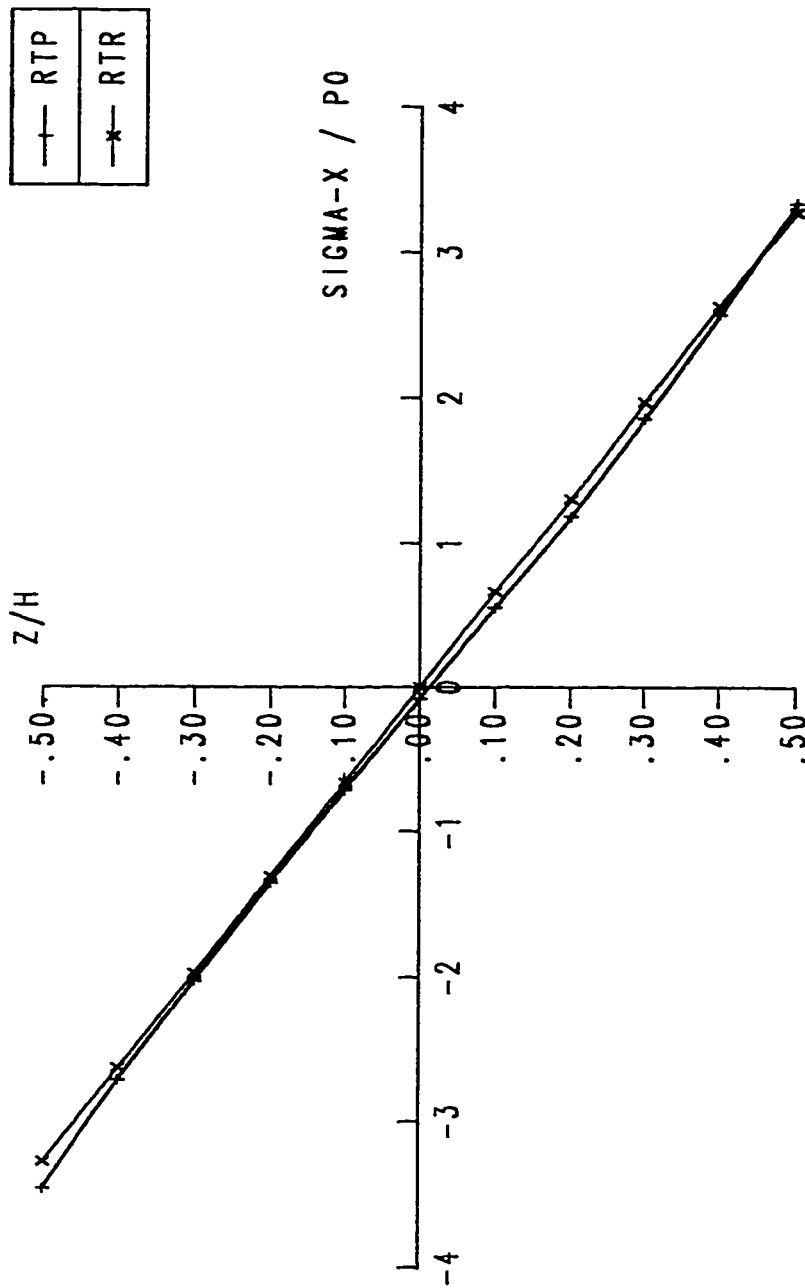


FIG. 5.37 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.4-11)

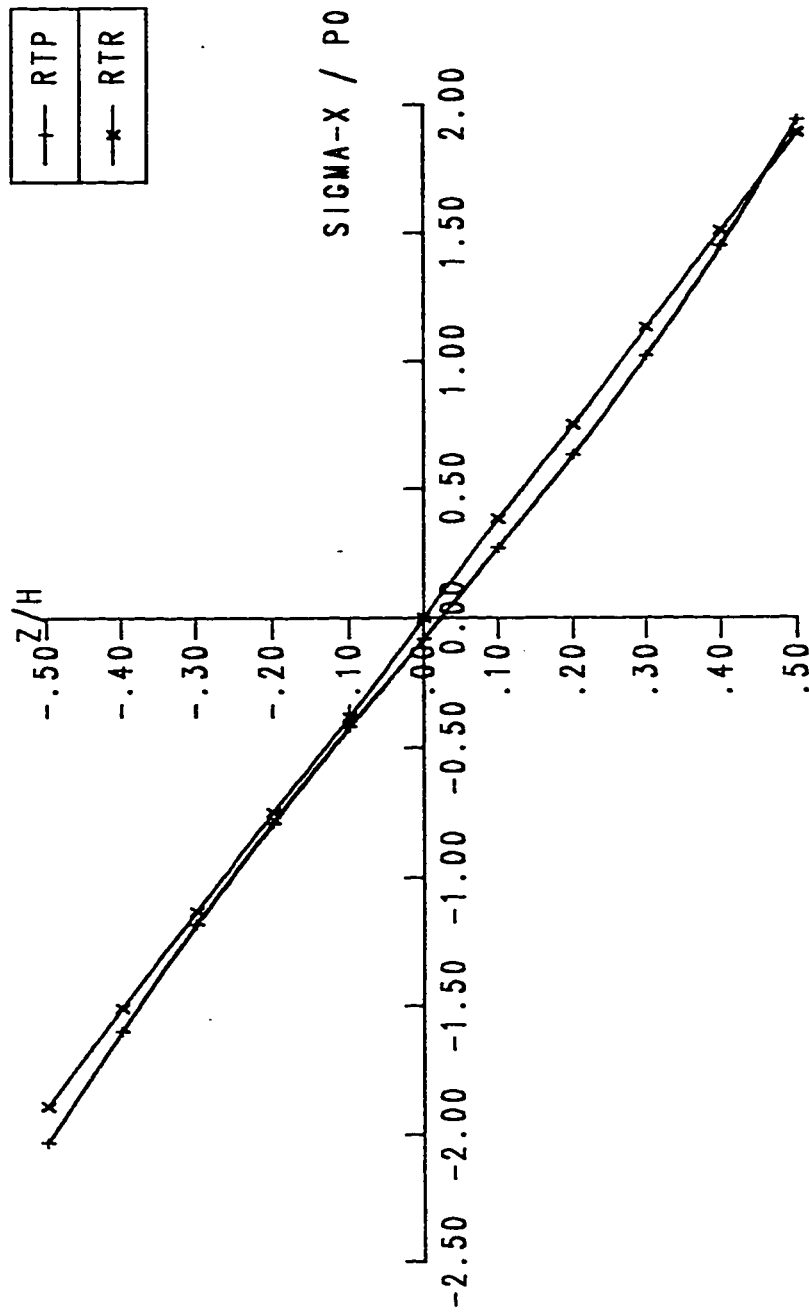


FIG. 5.38 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.5-11)

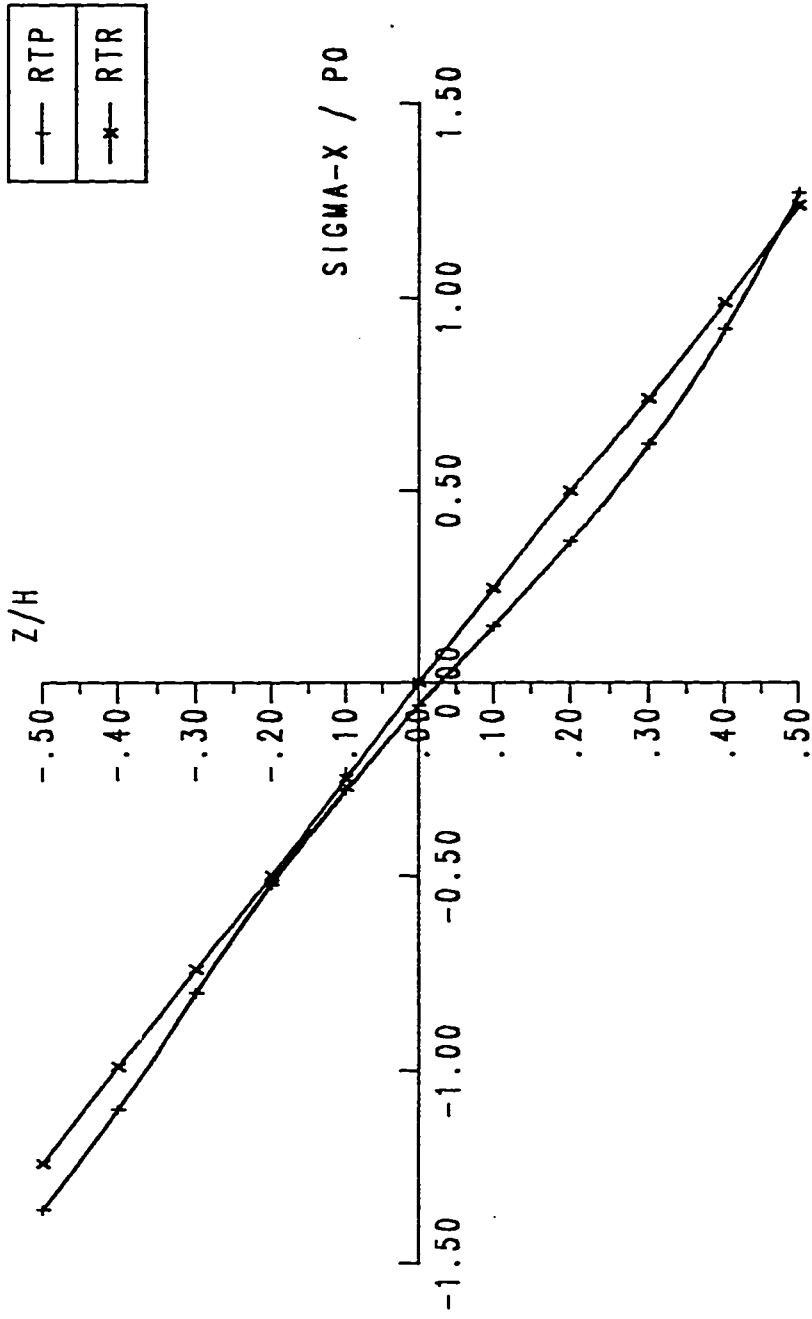




FIG. 5.39 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.6-11)

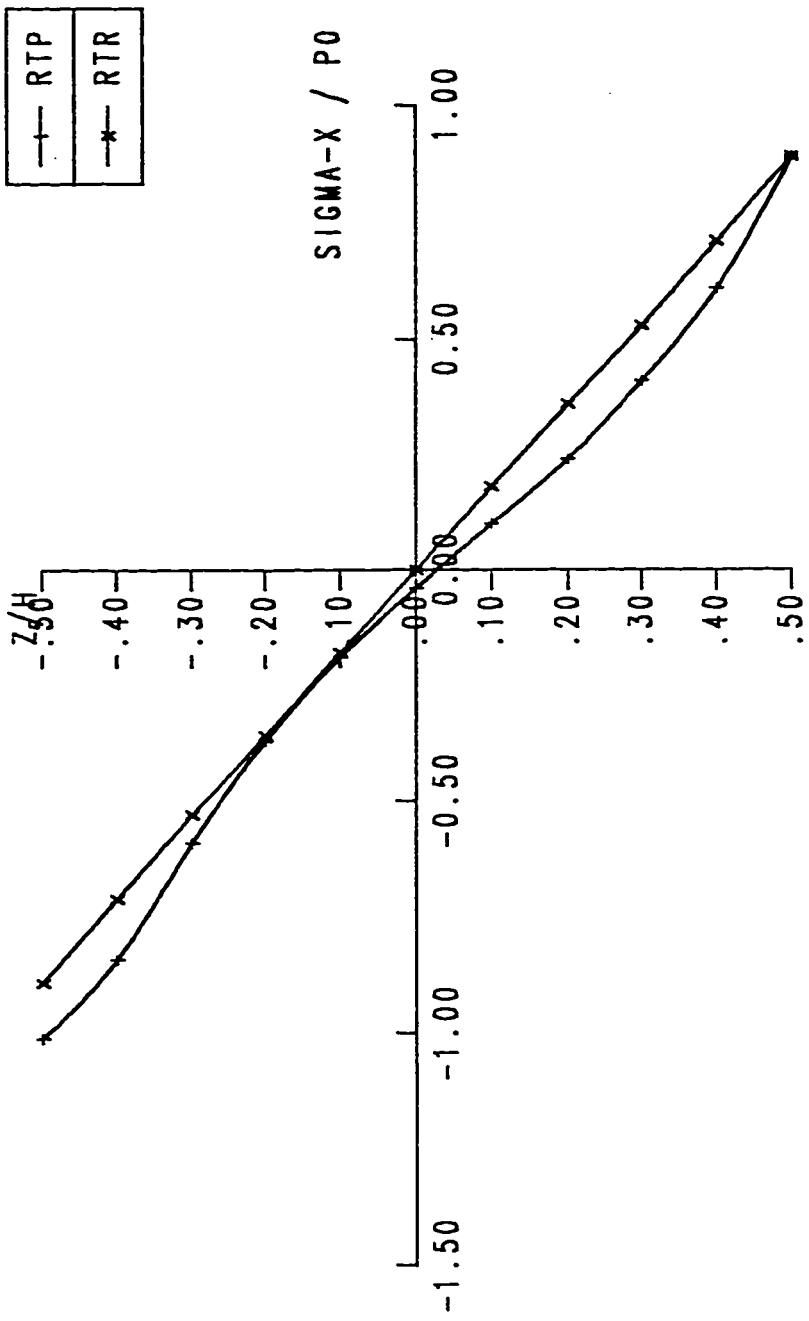


FIG. 5.40 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.7-11)

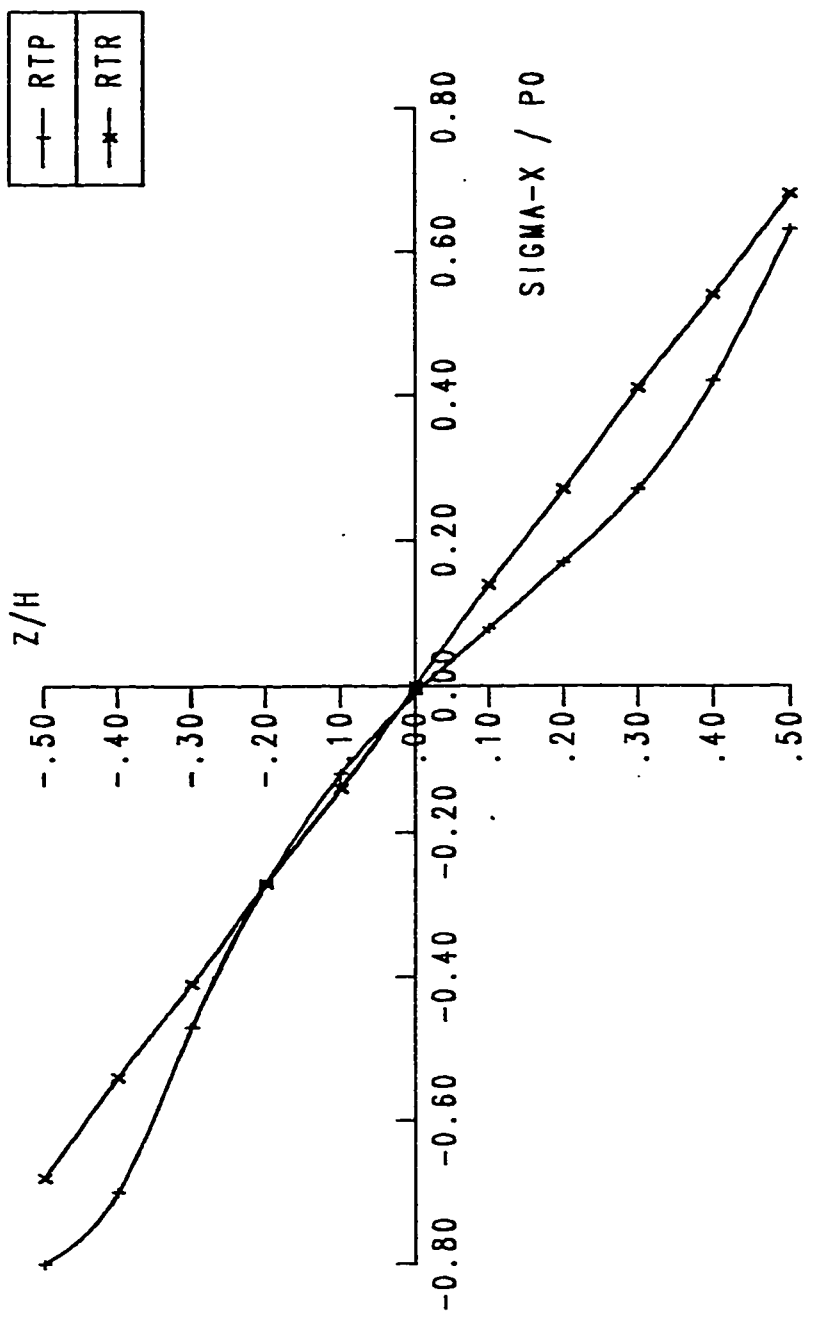


FIG. 5.41 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.8-11)

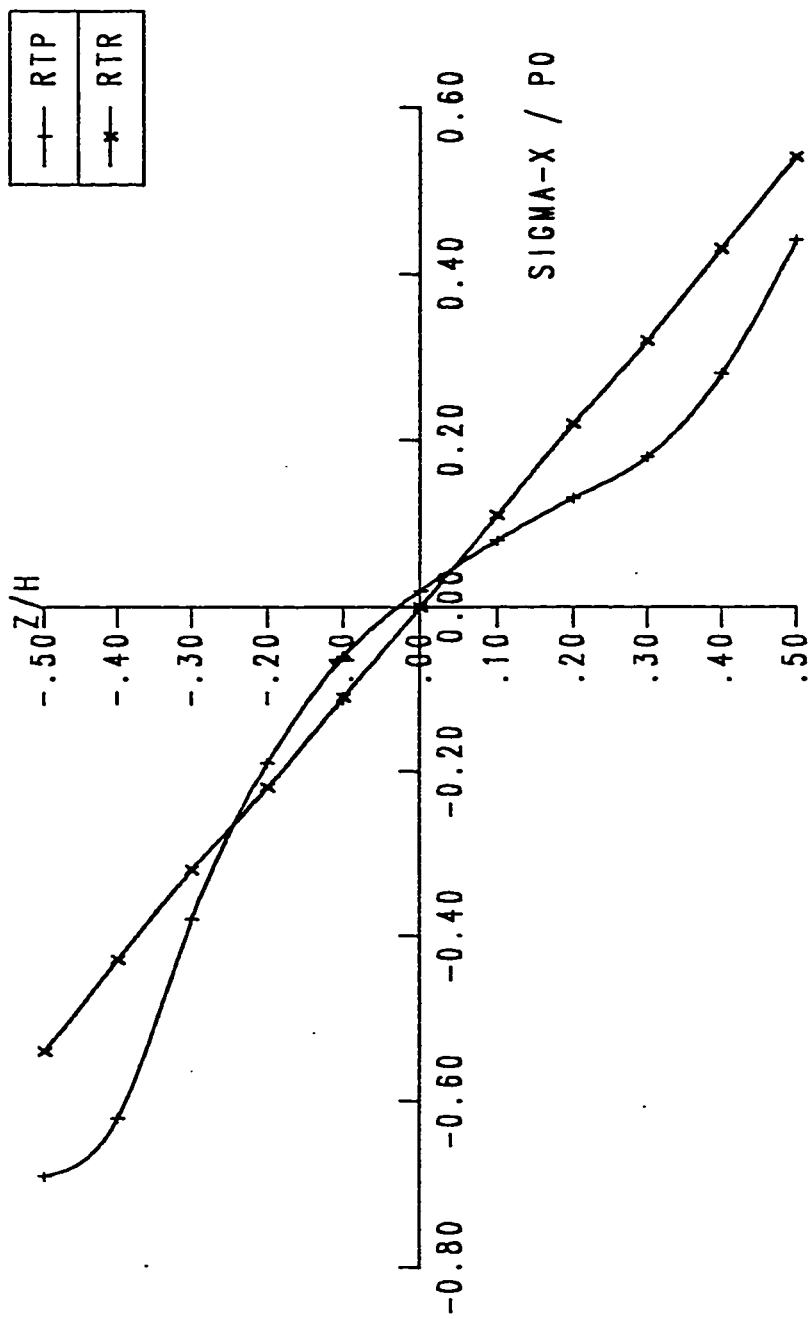


FIG. 5.42 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.9-11)

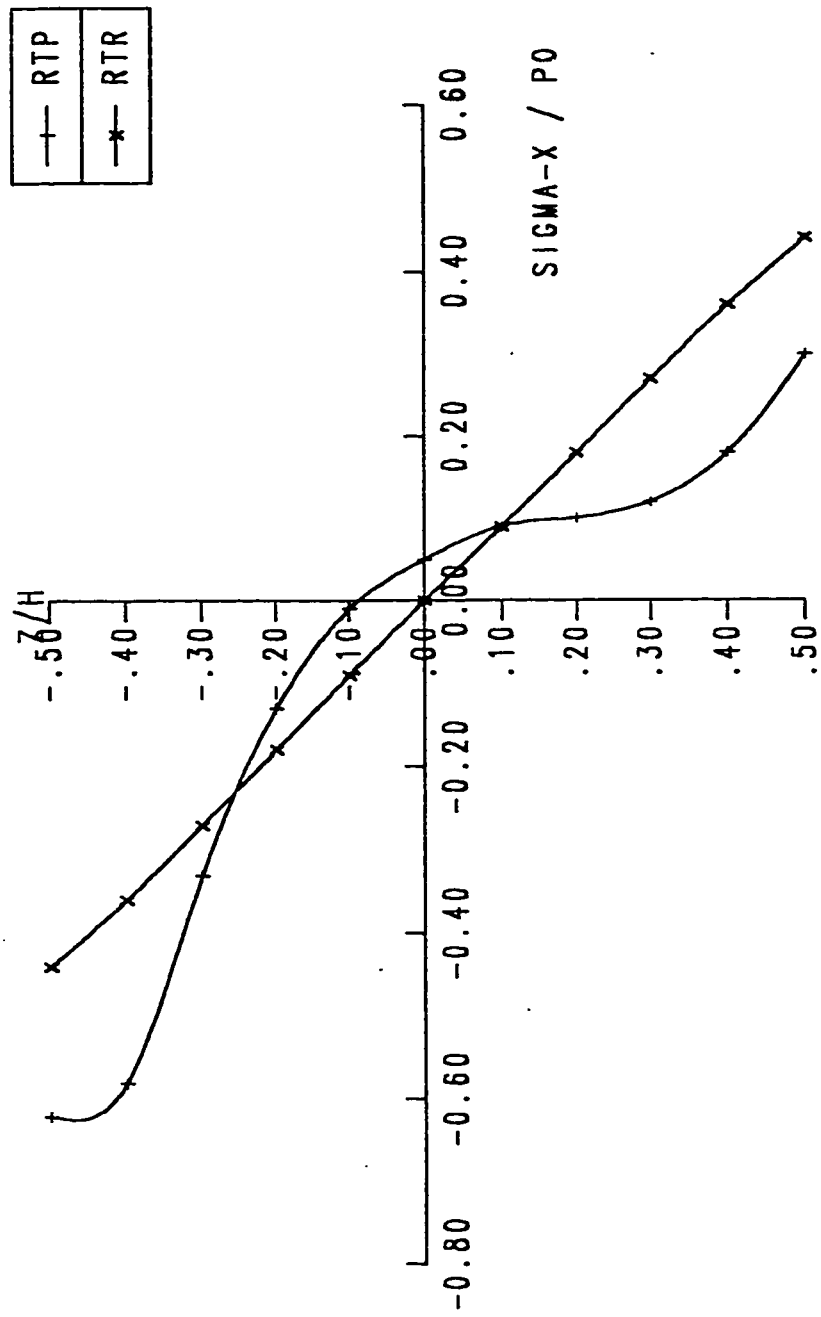


FIG. 5.43 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS1.-II)

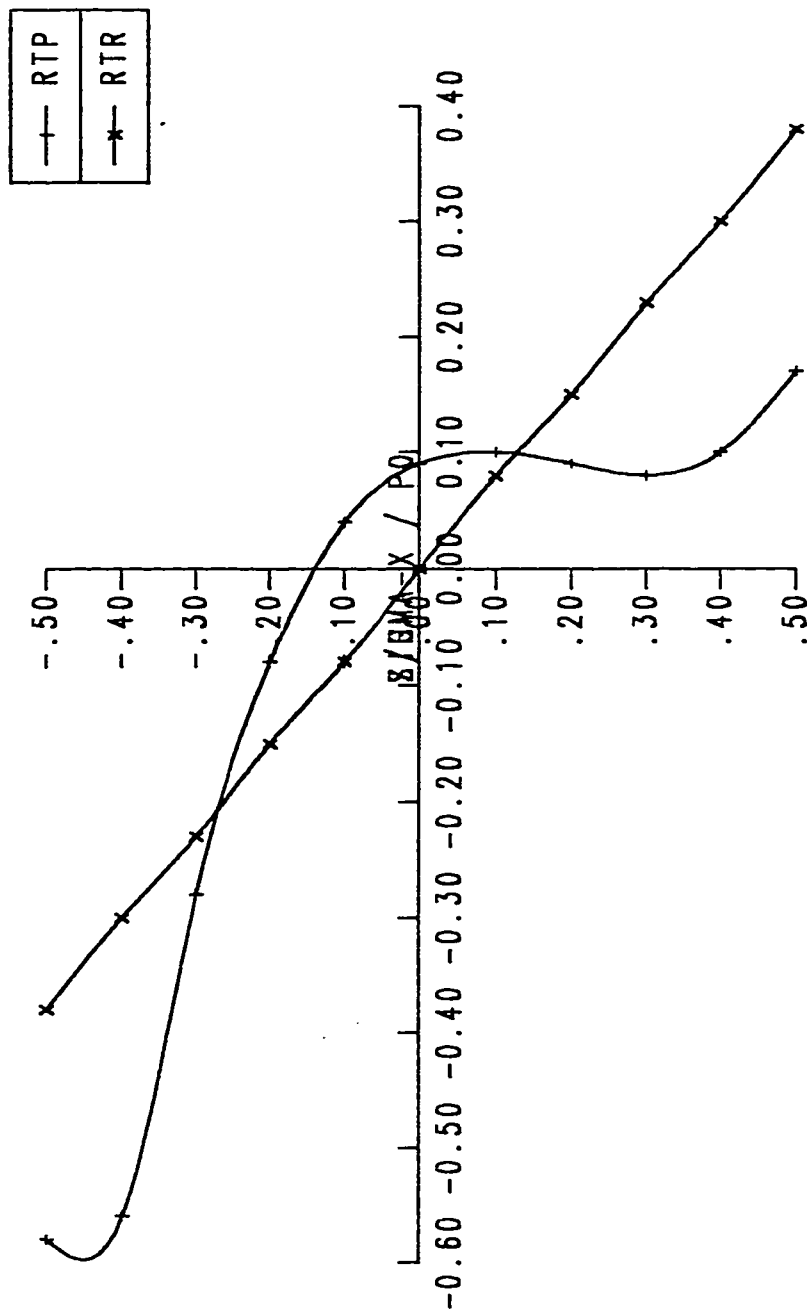


FIG. 5.44 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.1-11)

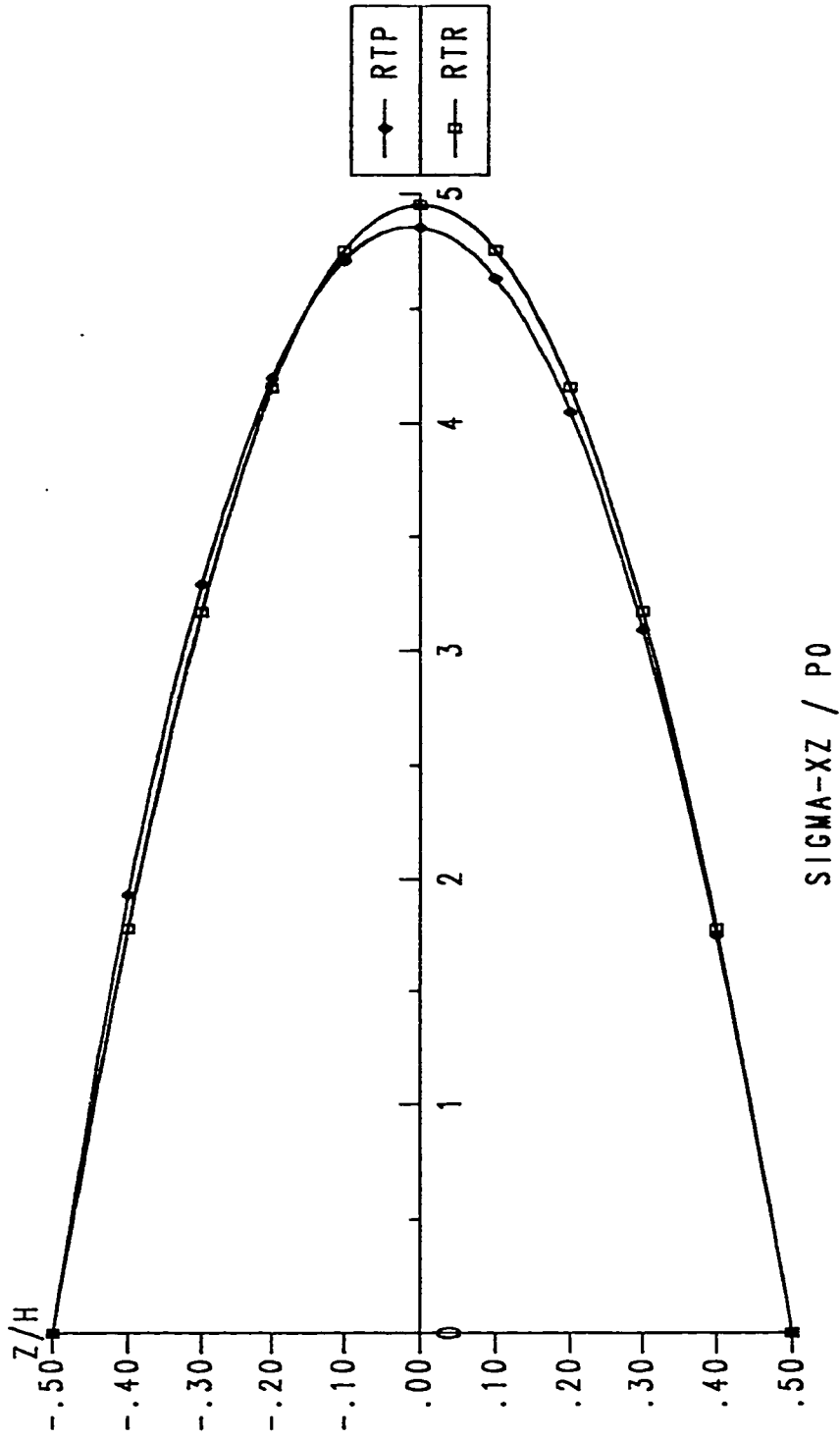


FIG. 5.45 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.3-11)

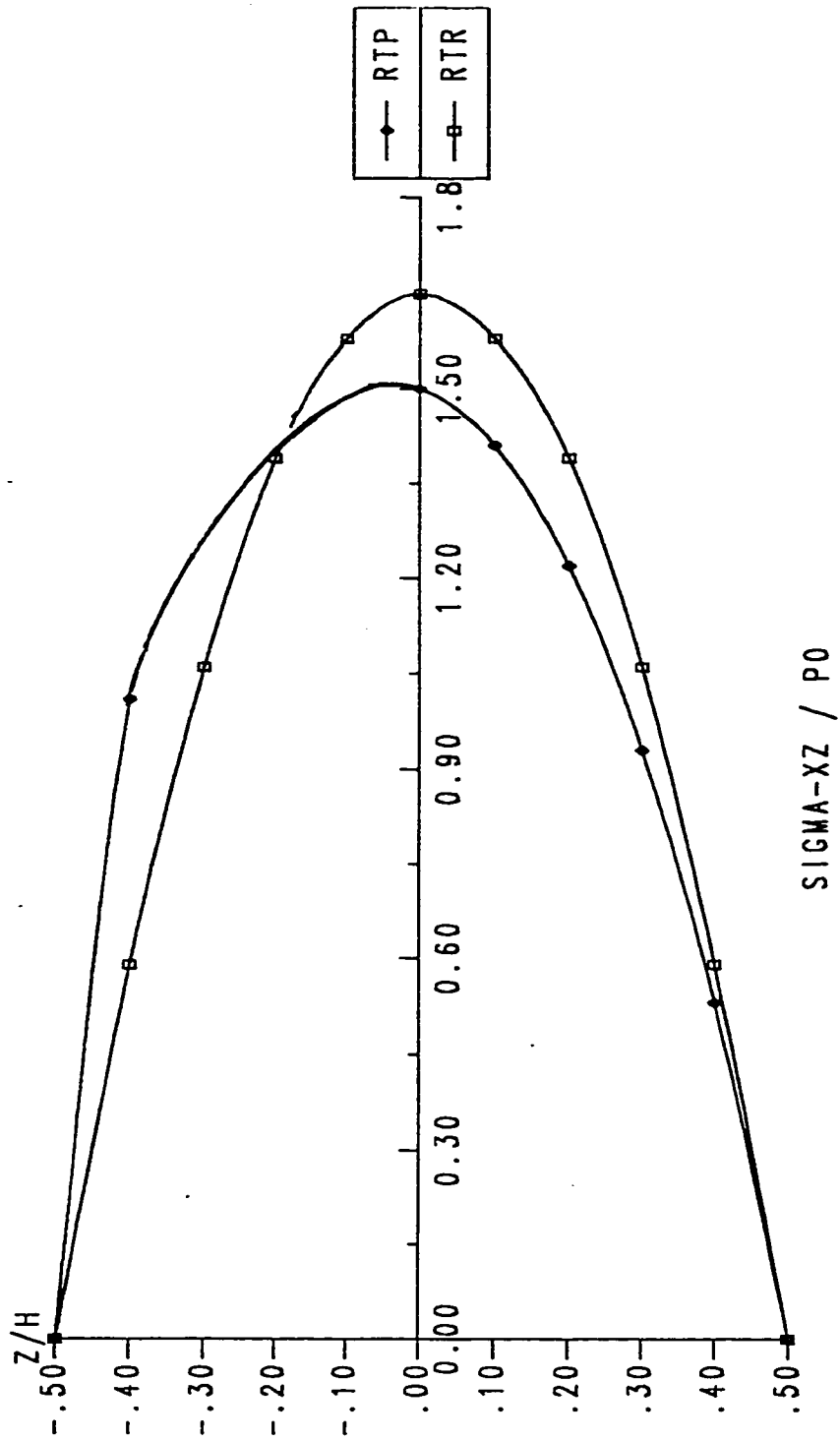


FIG. 5.46 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.5-11)

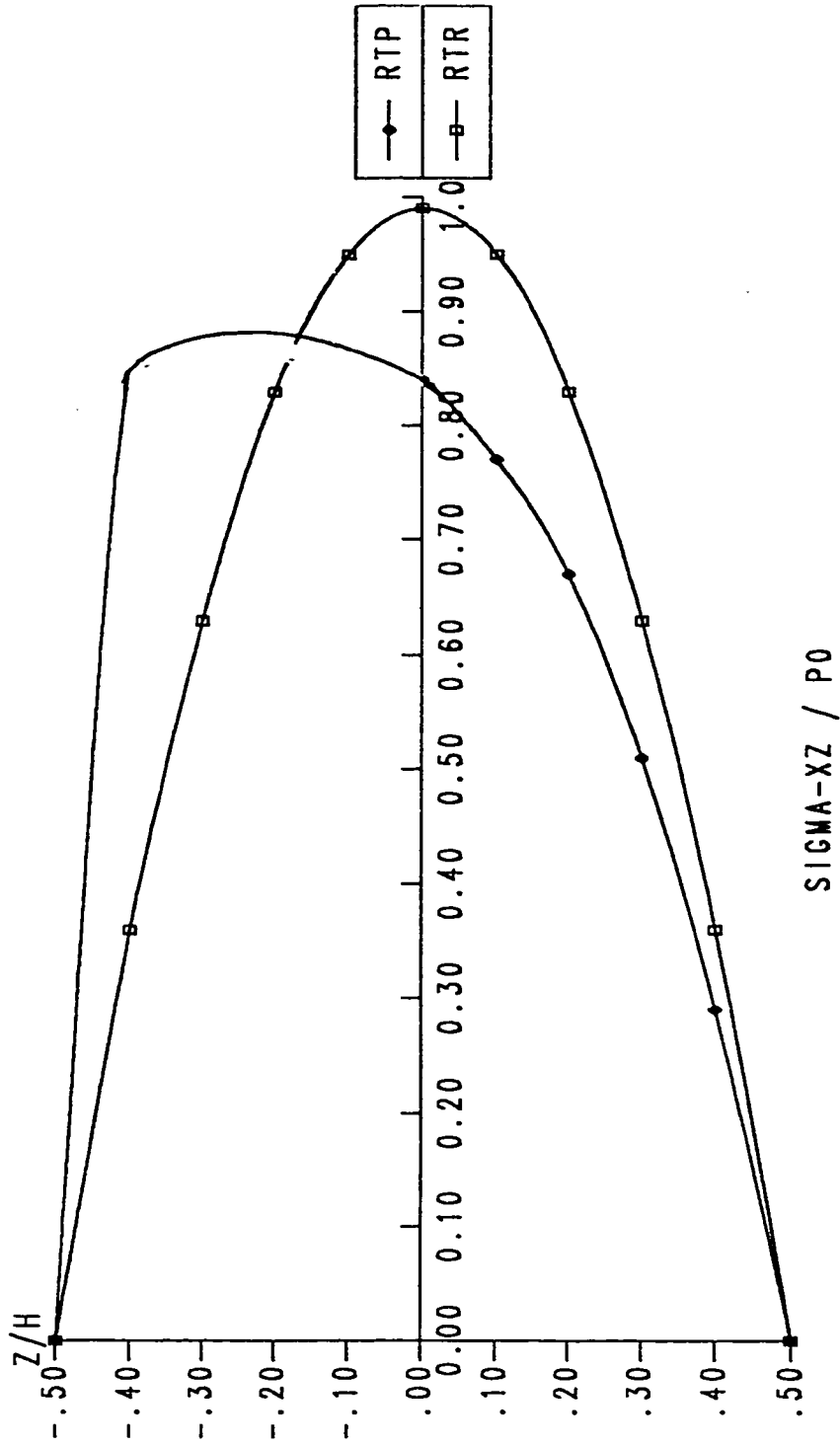




FIG. 5.47 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS1.-II)

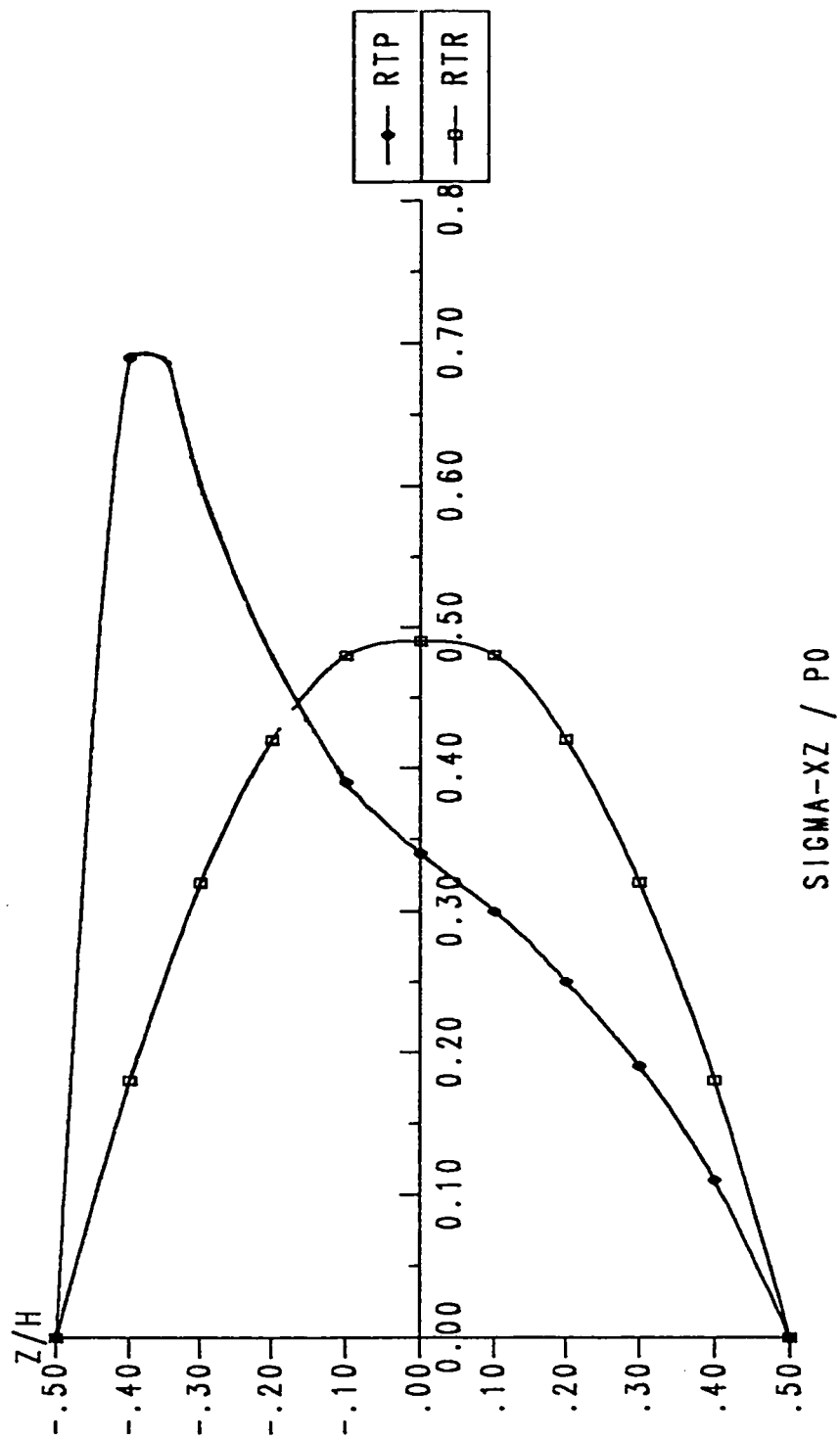


FIG. 5.48 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.005-1)

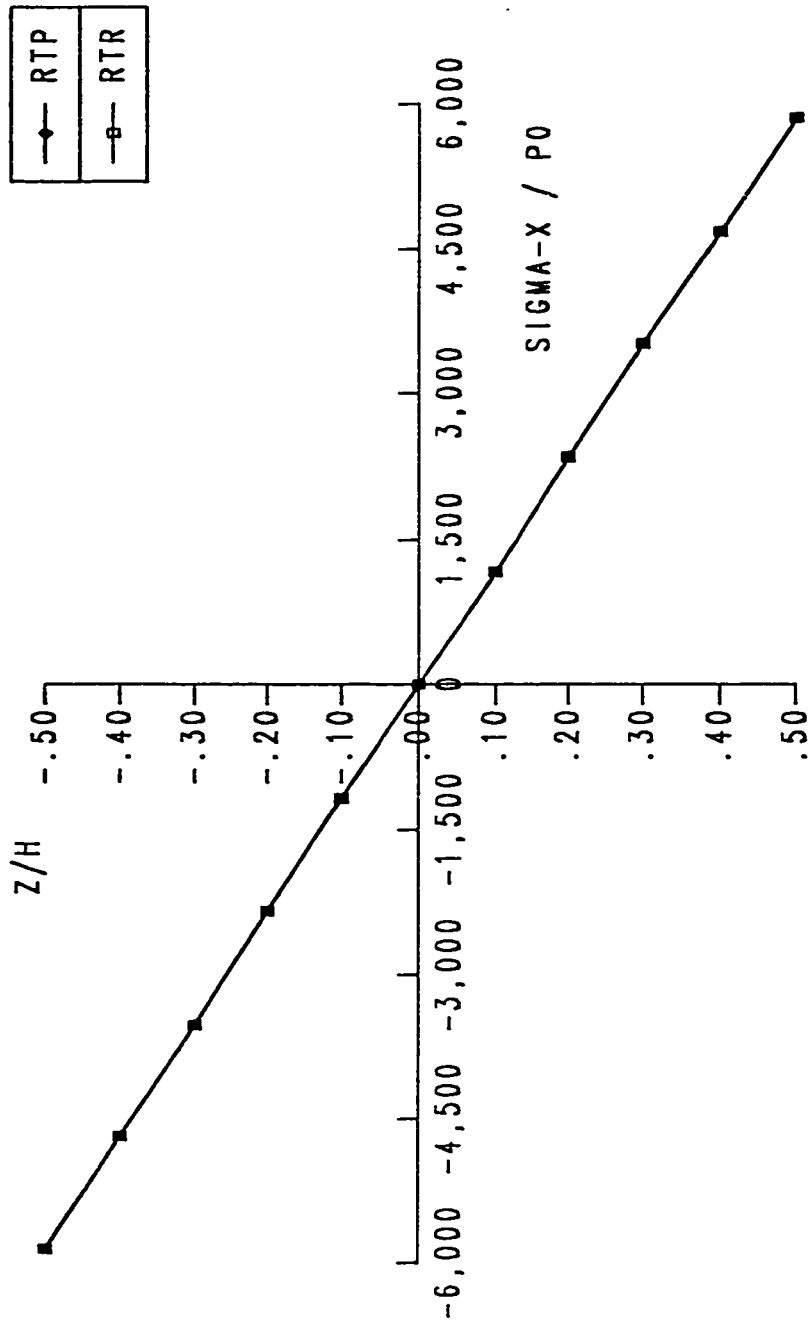


FIG. 5.49 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.1-1)

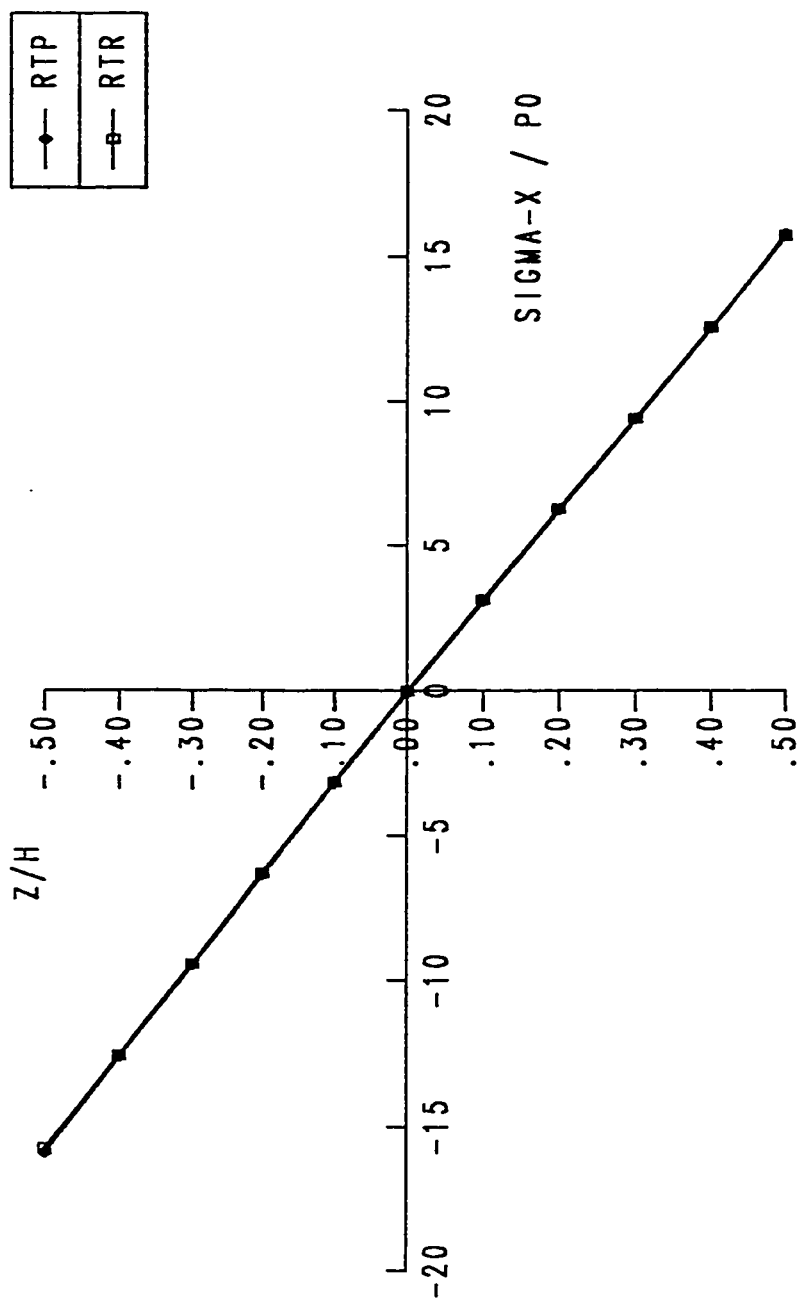


FIG. 5.50 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.3-1)

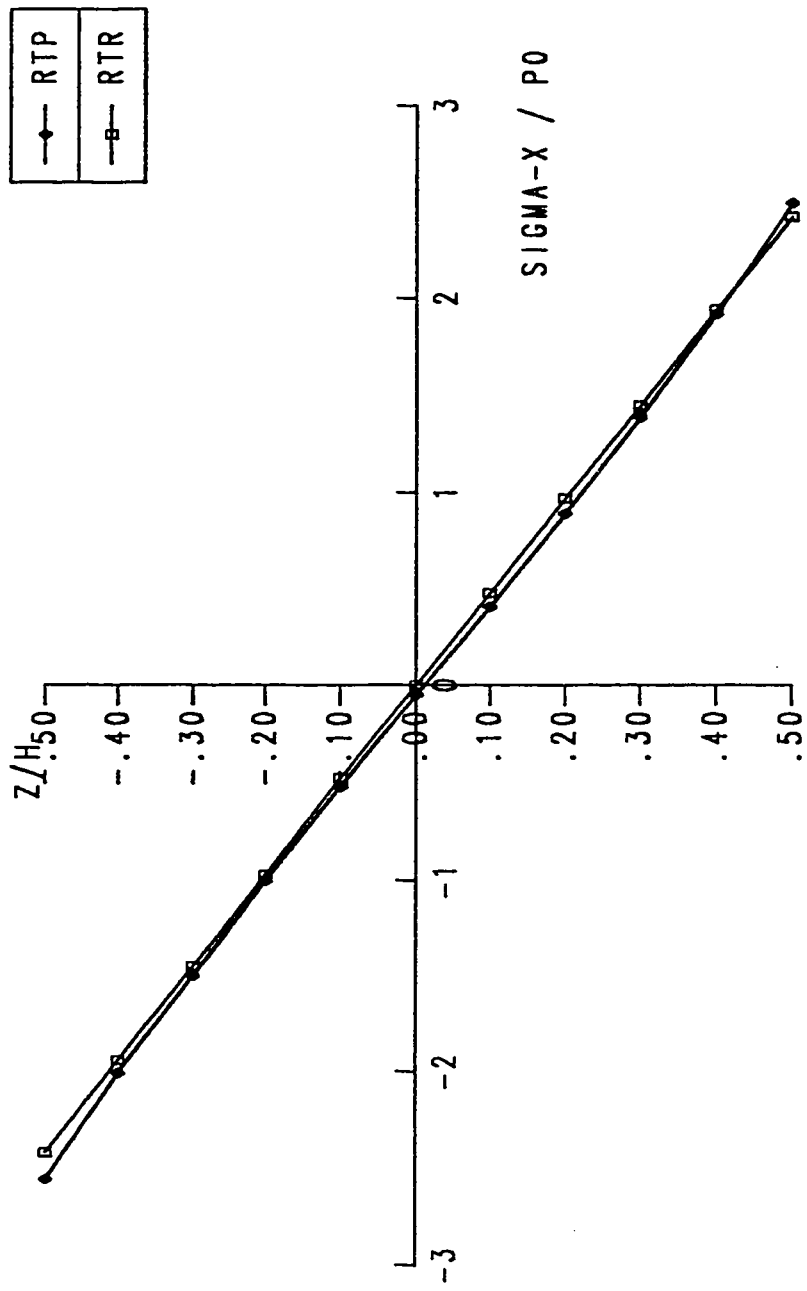


FIG. 5.51 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.5-1)

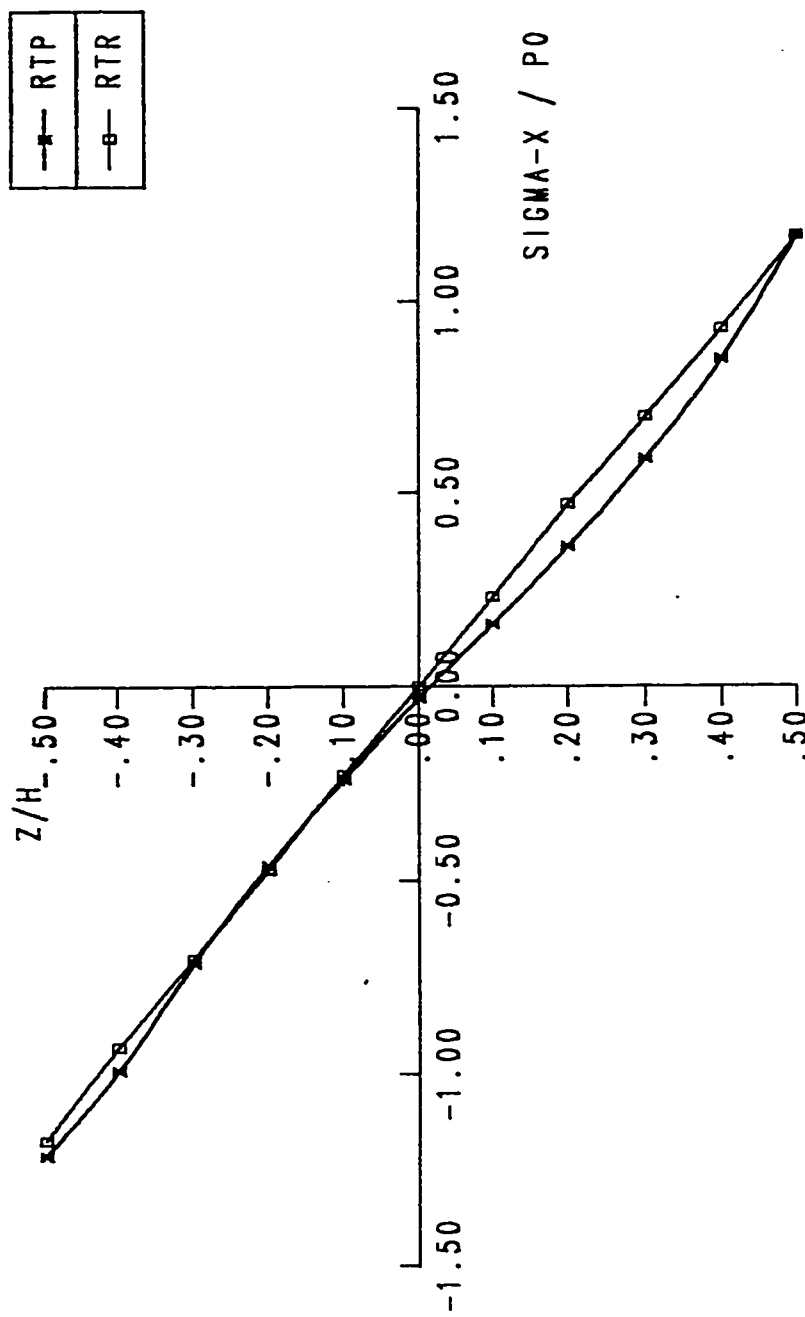


FIG. 5.52 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.7-1)

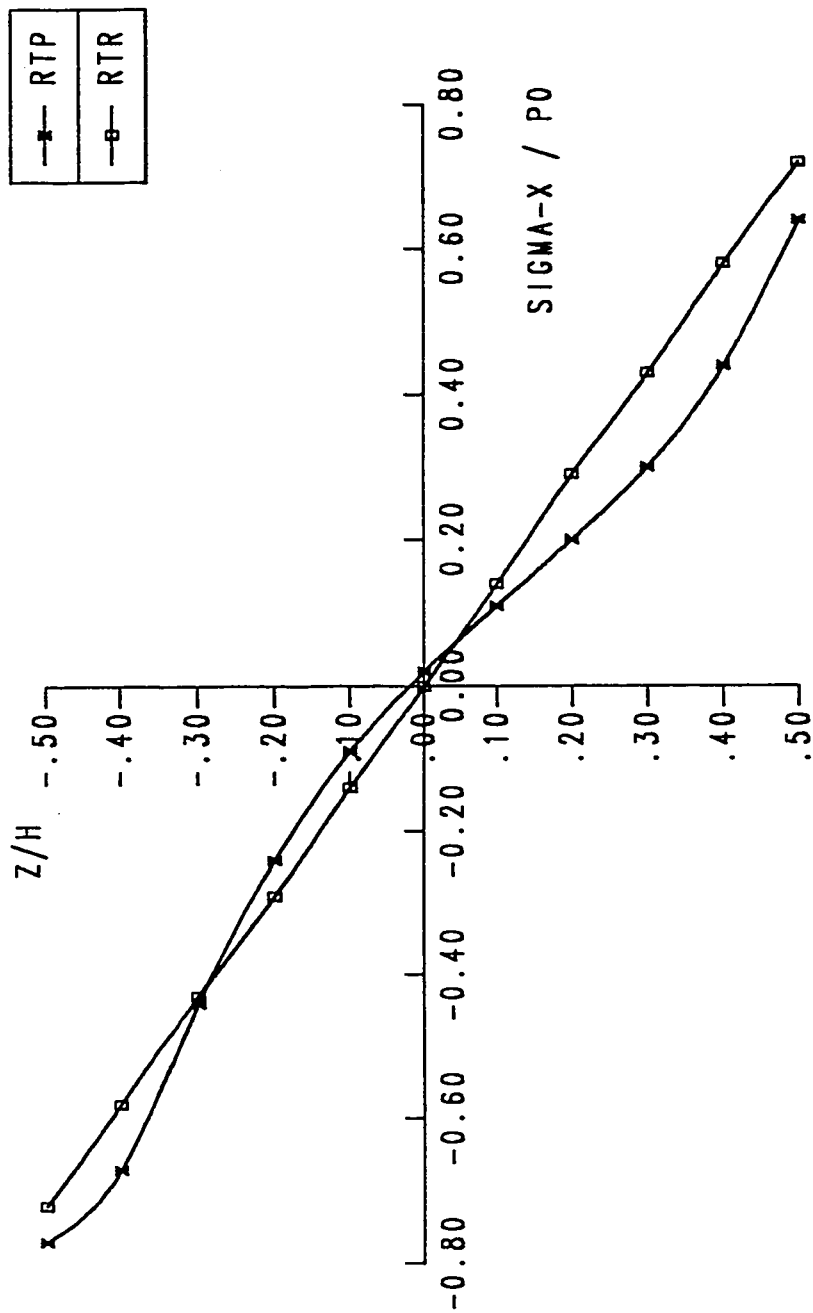


FIG. 5.53 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC1.-1)

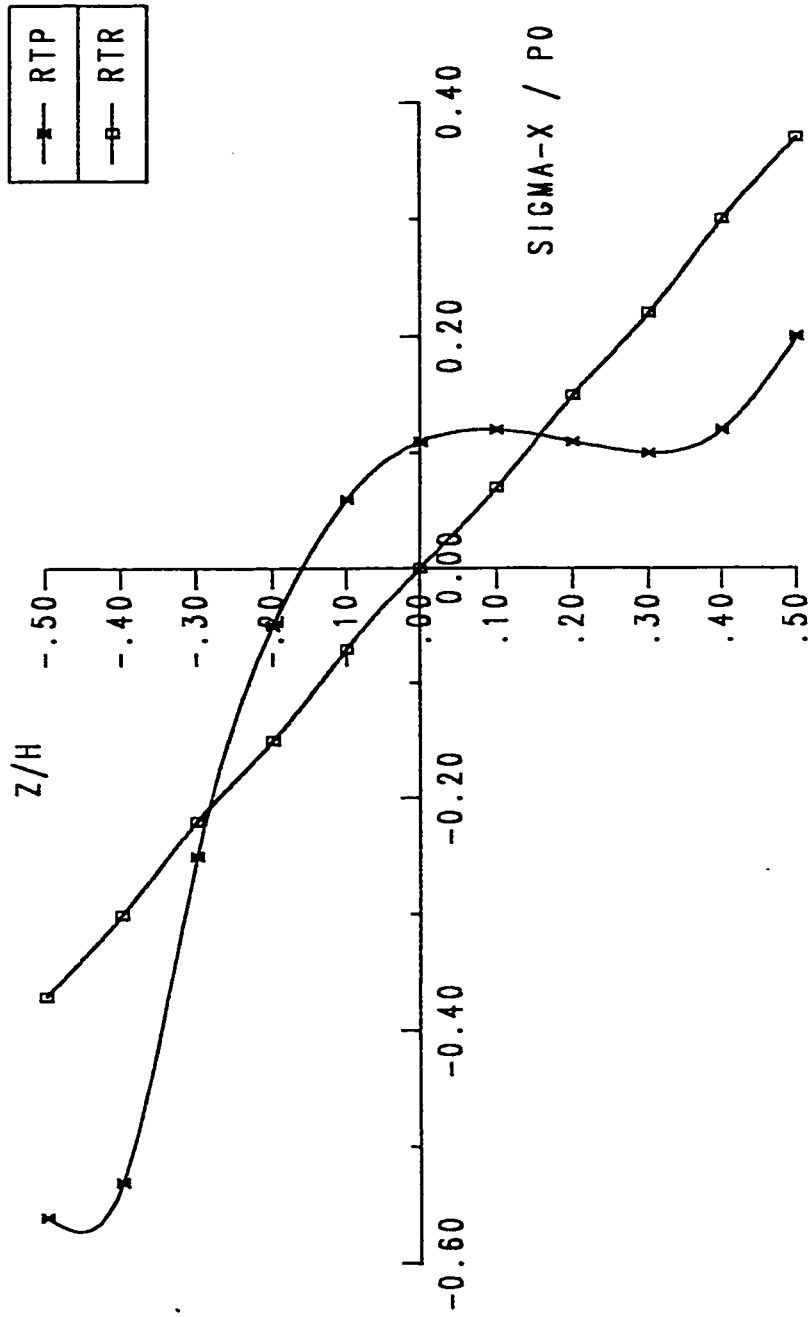


FIG. 5.54 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.005-11)

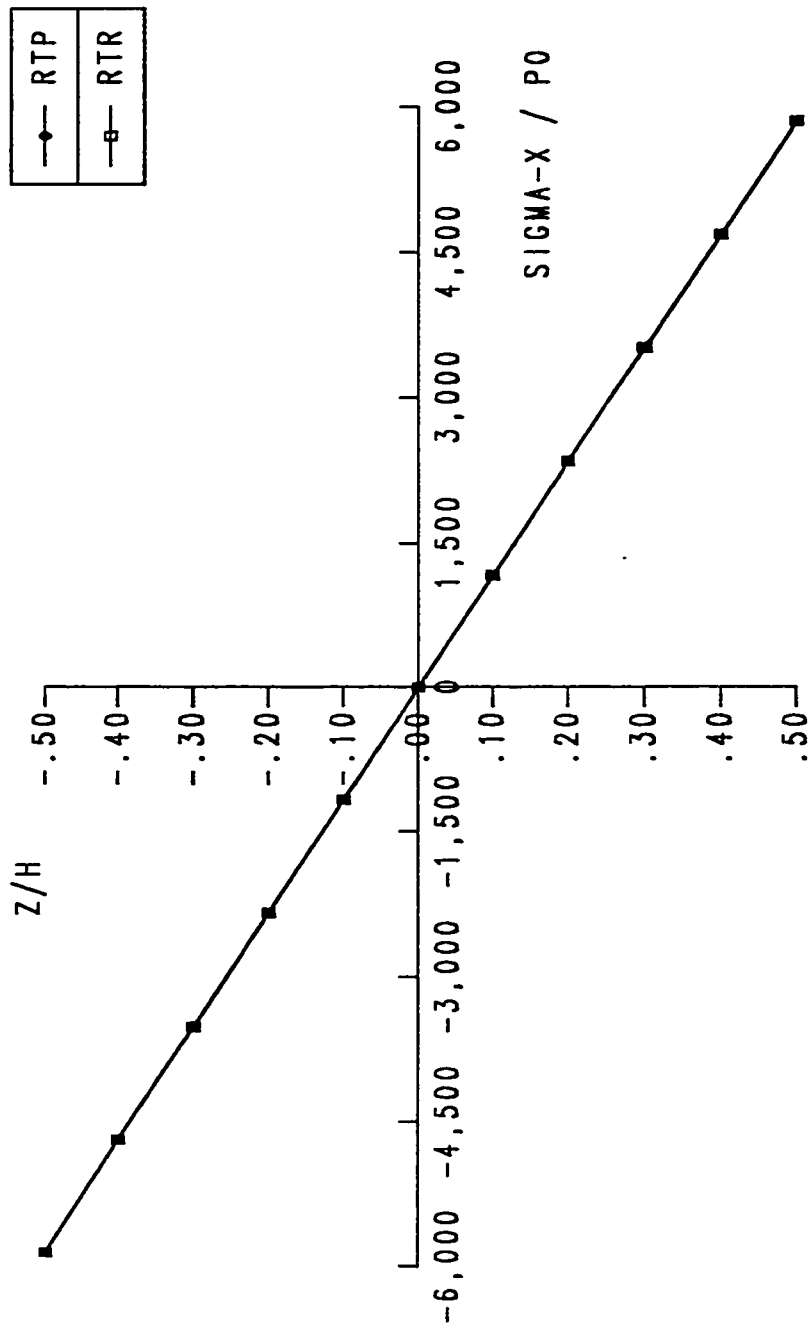




FIG. 5.55 : MAX. NORMALSTRESS SIGMA-X VS Z/H (SC.1-11)

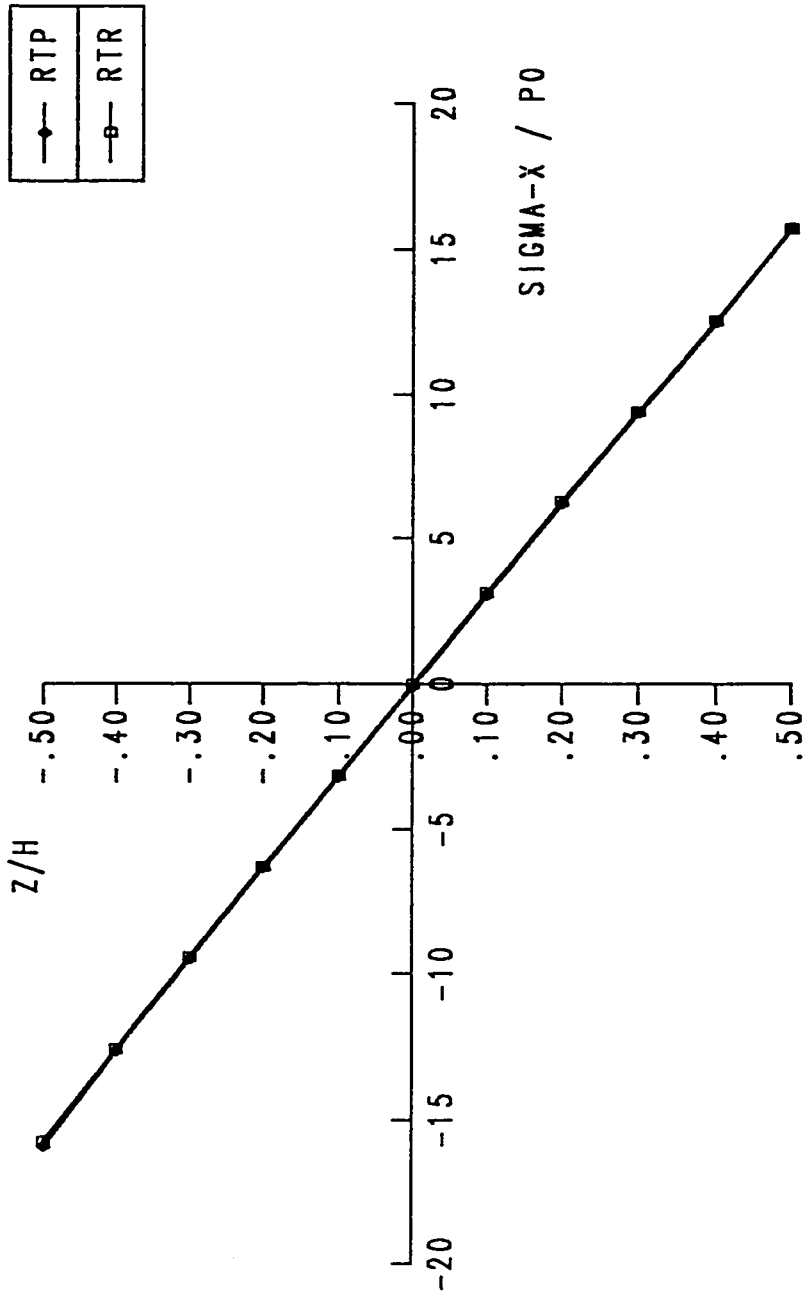


FIG. 5.56 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.3-11)

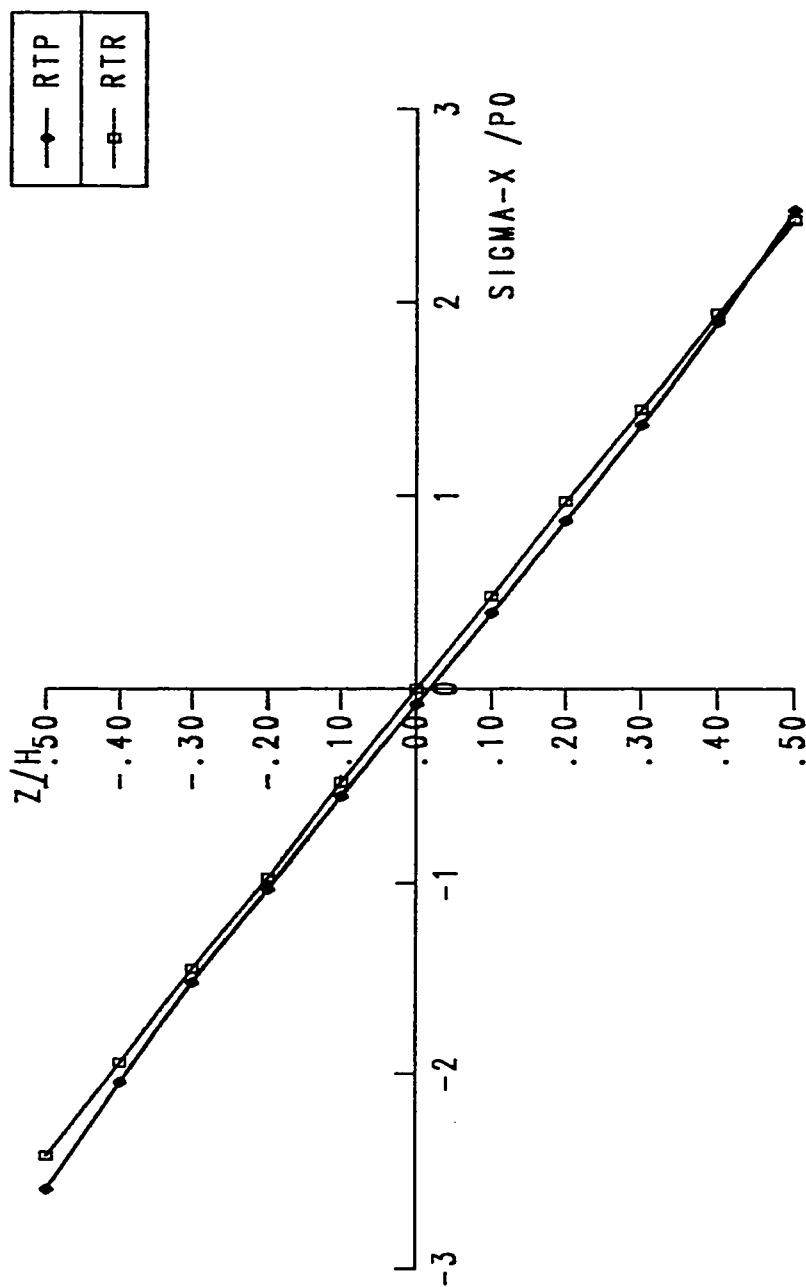


FIG. 5.57 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.5-11)

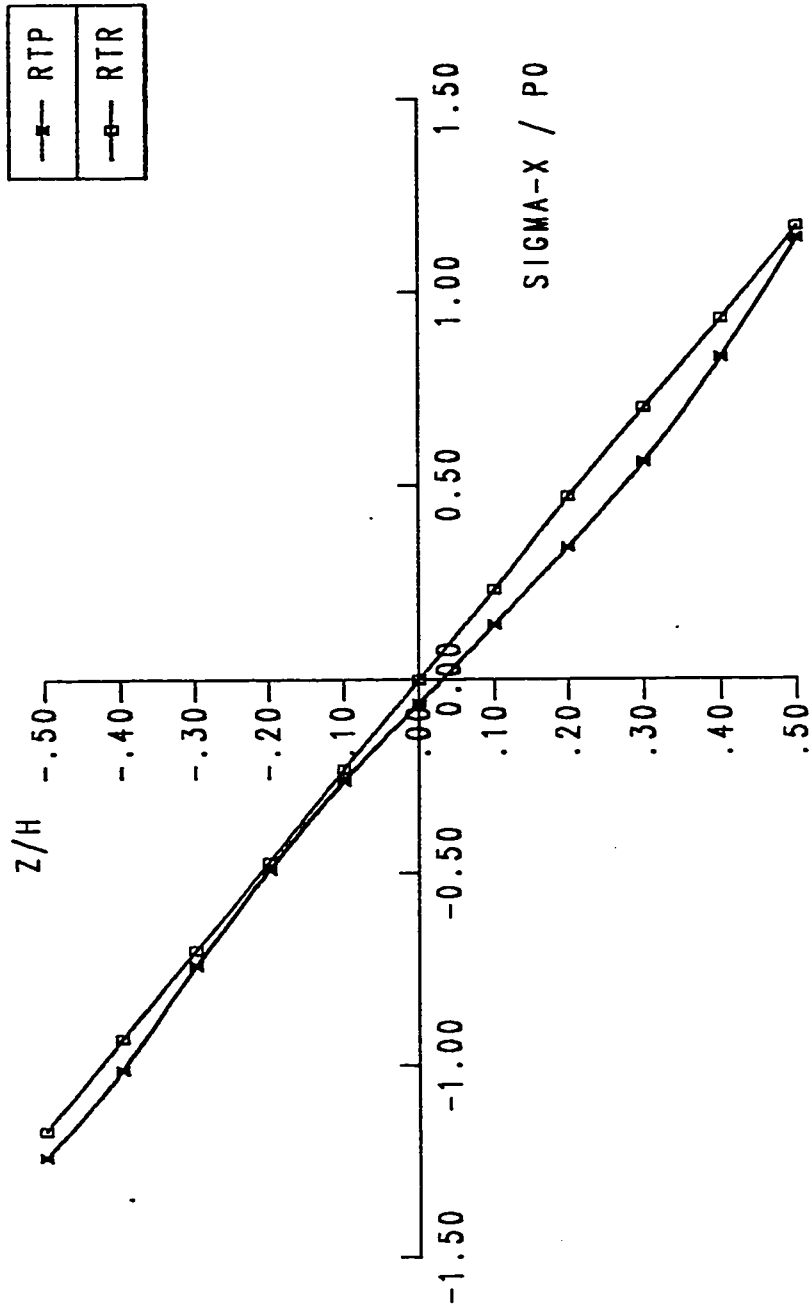


FIG. 5.58 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.7-11)

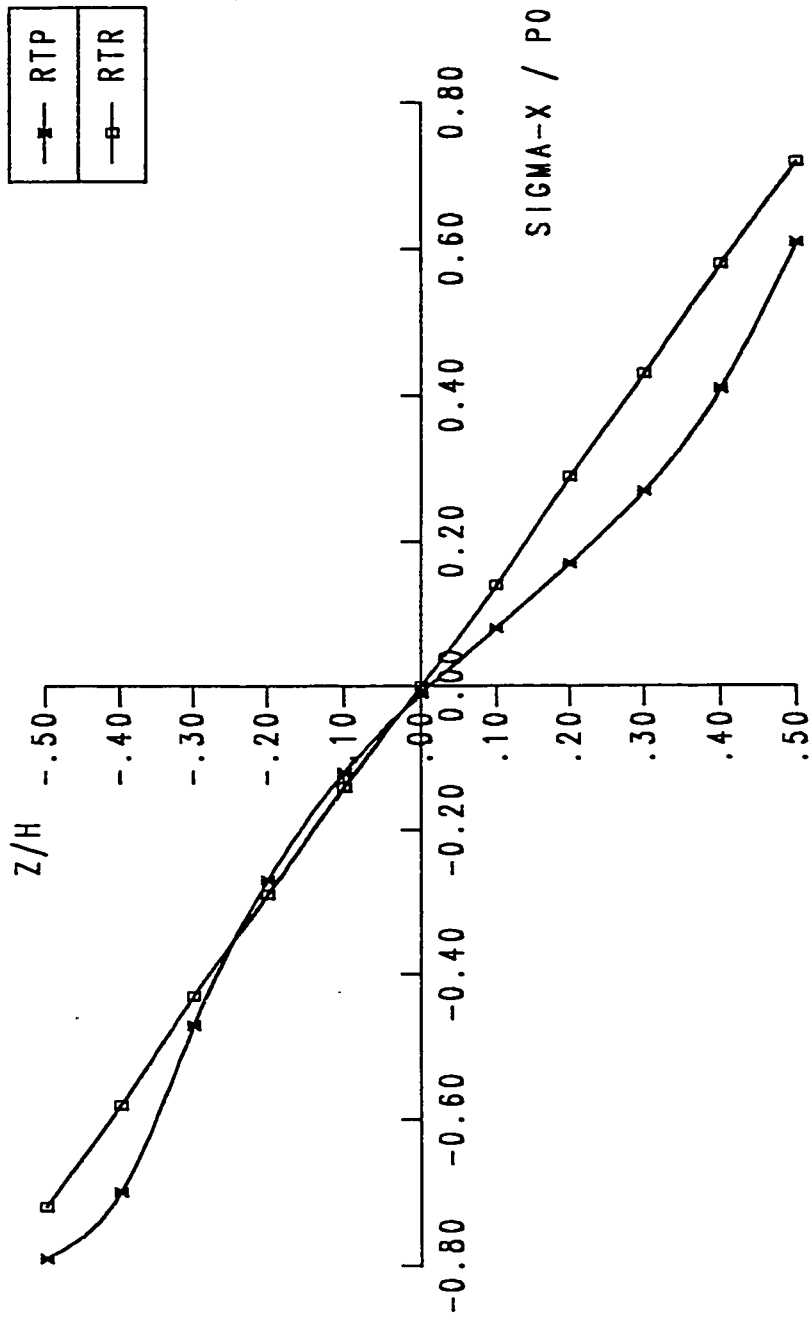


FIG. 5.59 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC1.-11)

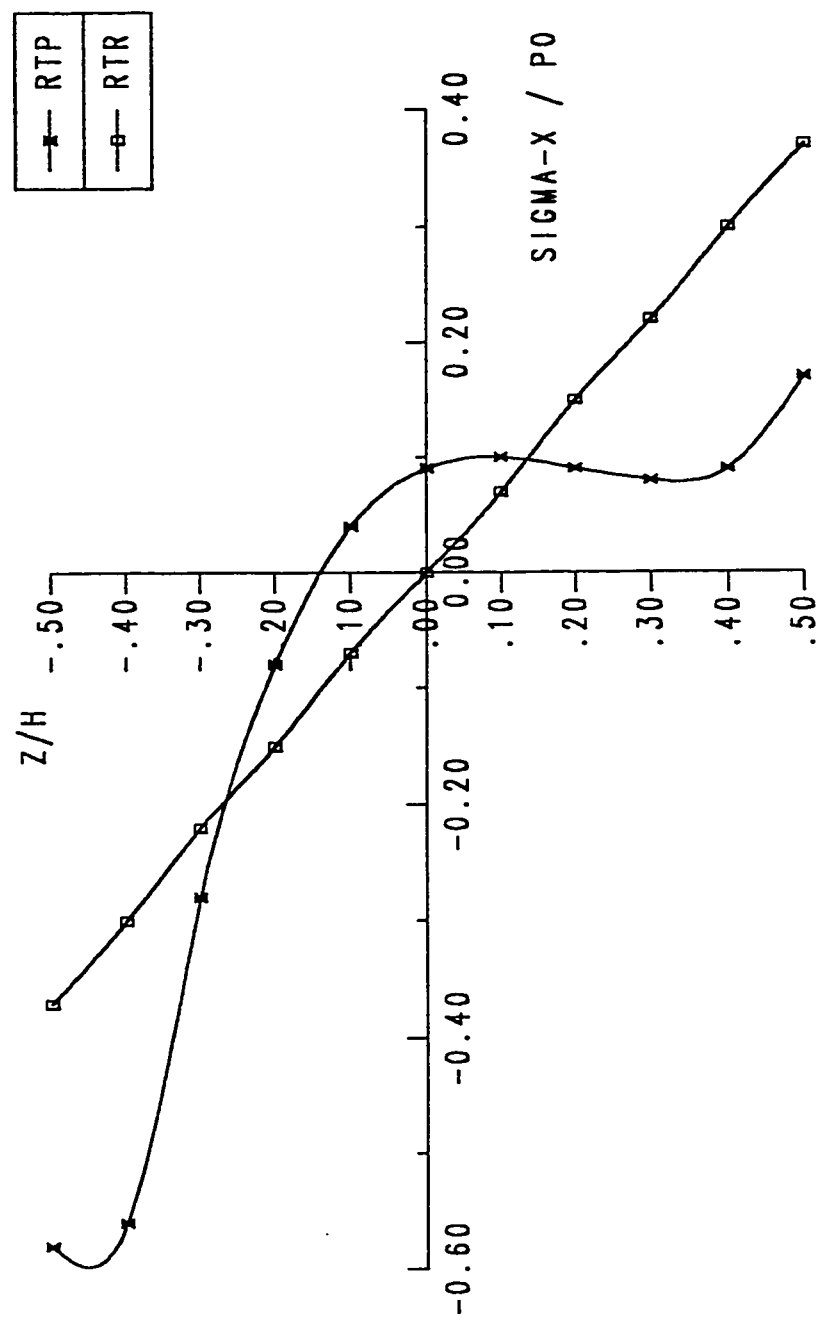


FIG. 5.60 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.005-I)

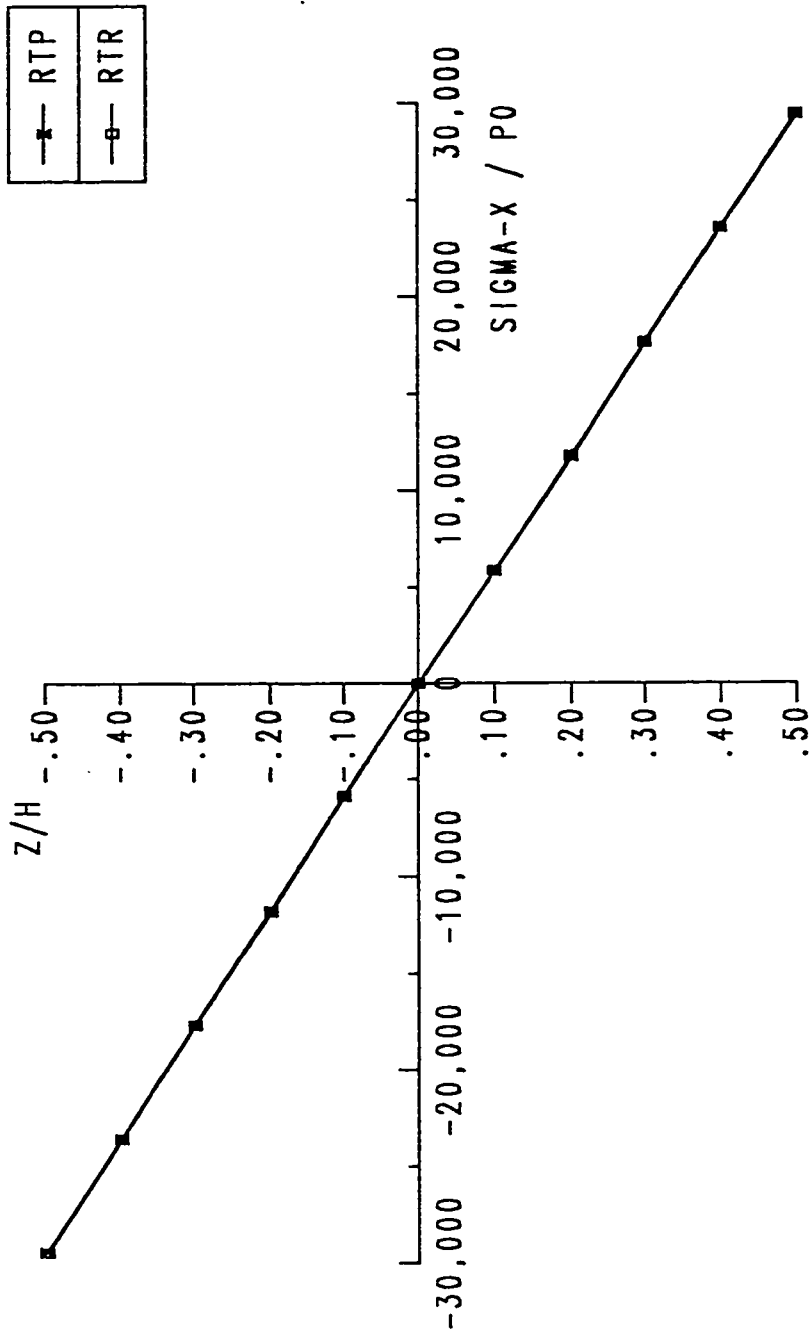


FIG. 5.61 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.1-1)

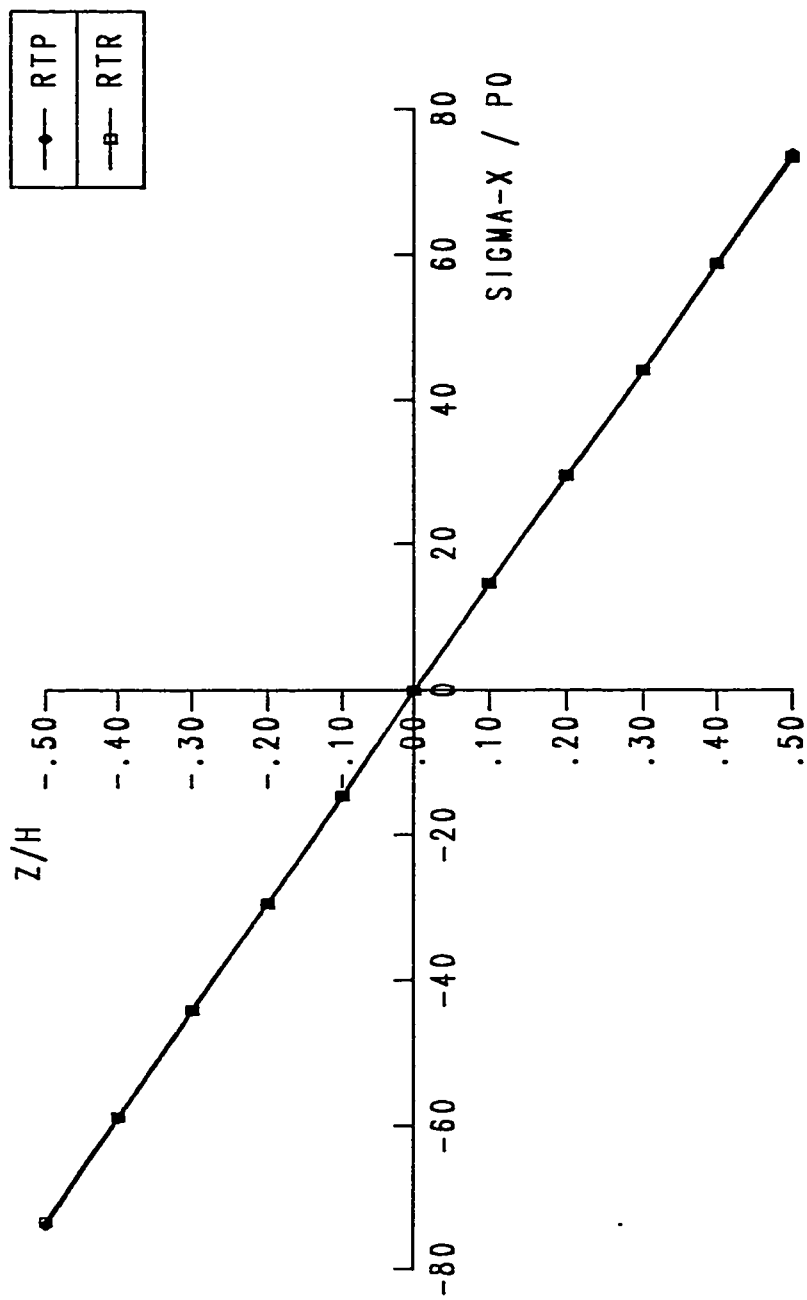


FIG. 5.62 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.3-1)

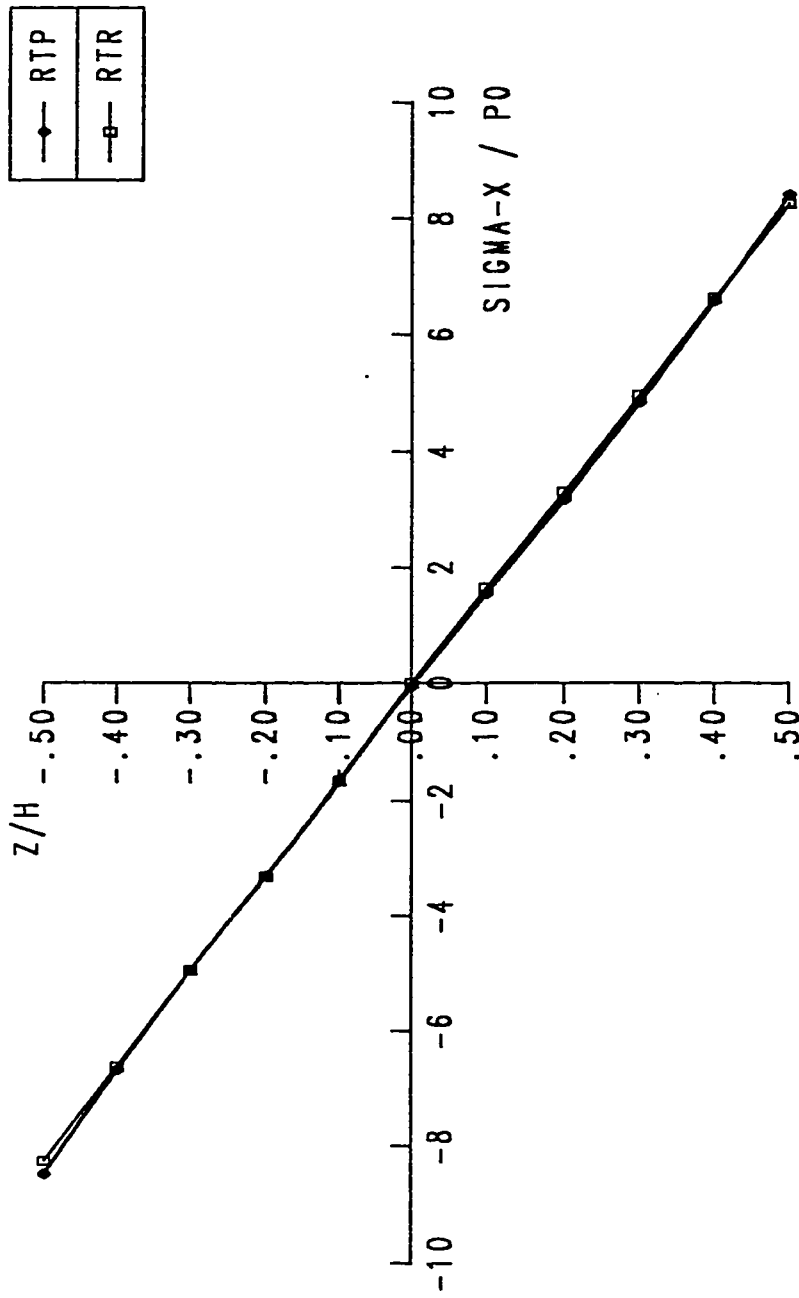




FIG. 5.63 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.5-1)

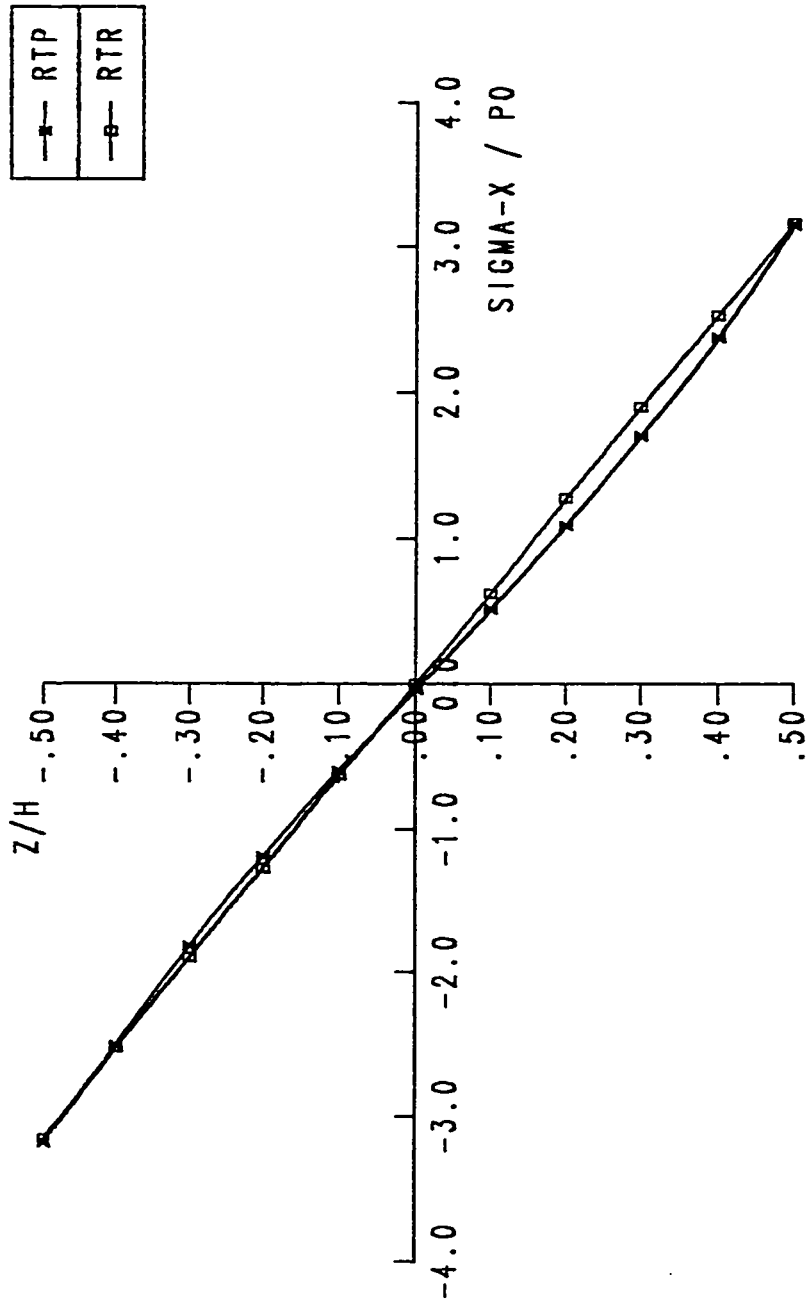


FIG. 5.64 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.7-1)

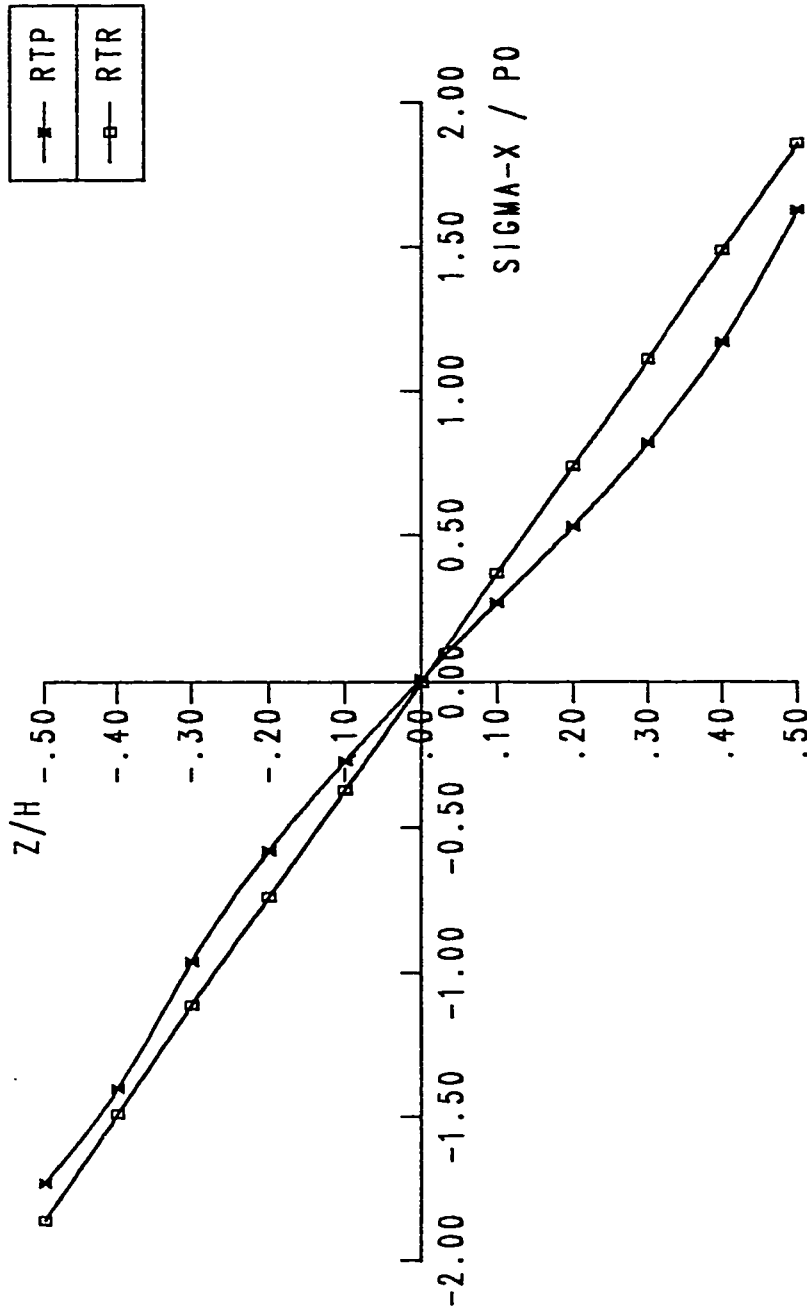


FIG. 5.65 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF1.-I)

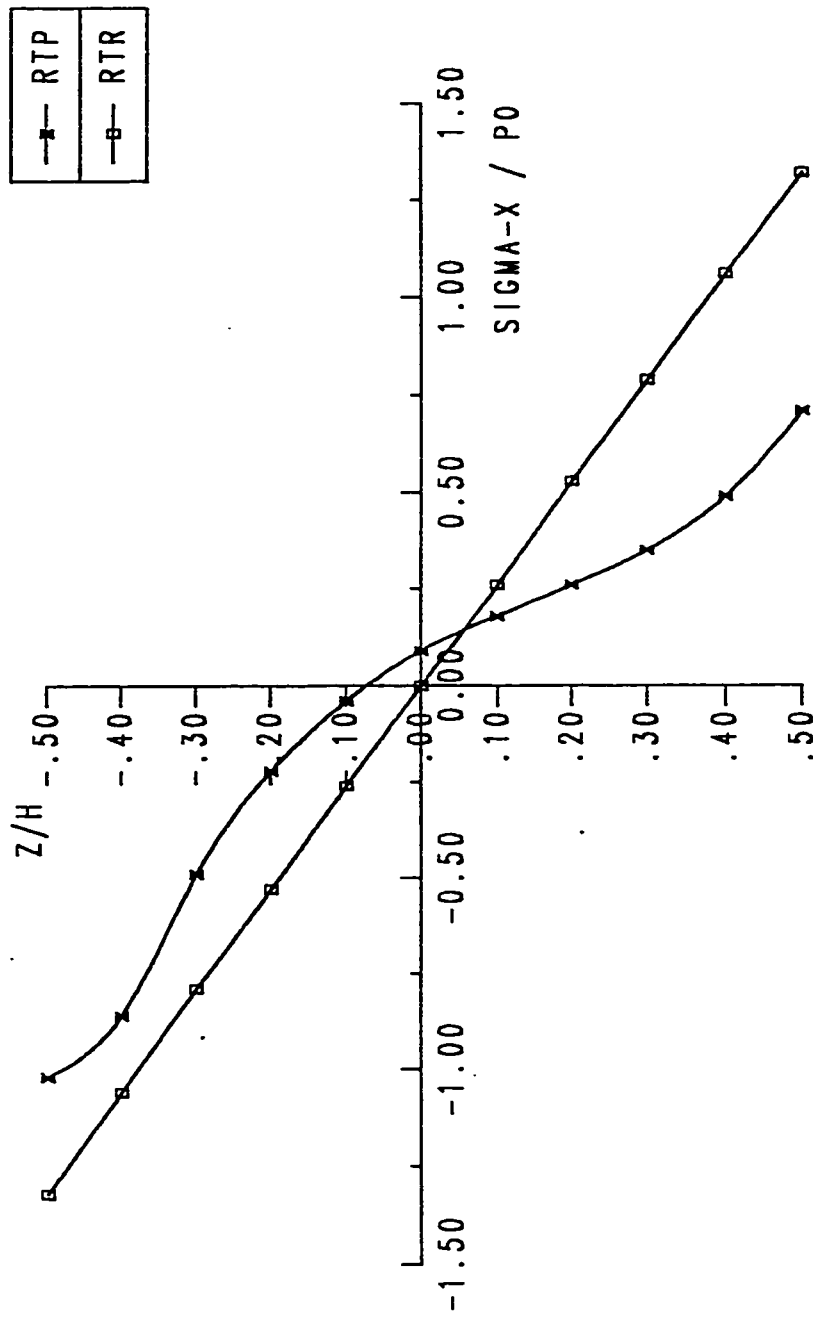


FIG. 5.66 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.005-11)

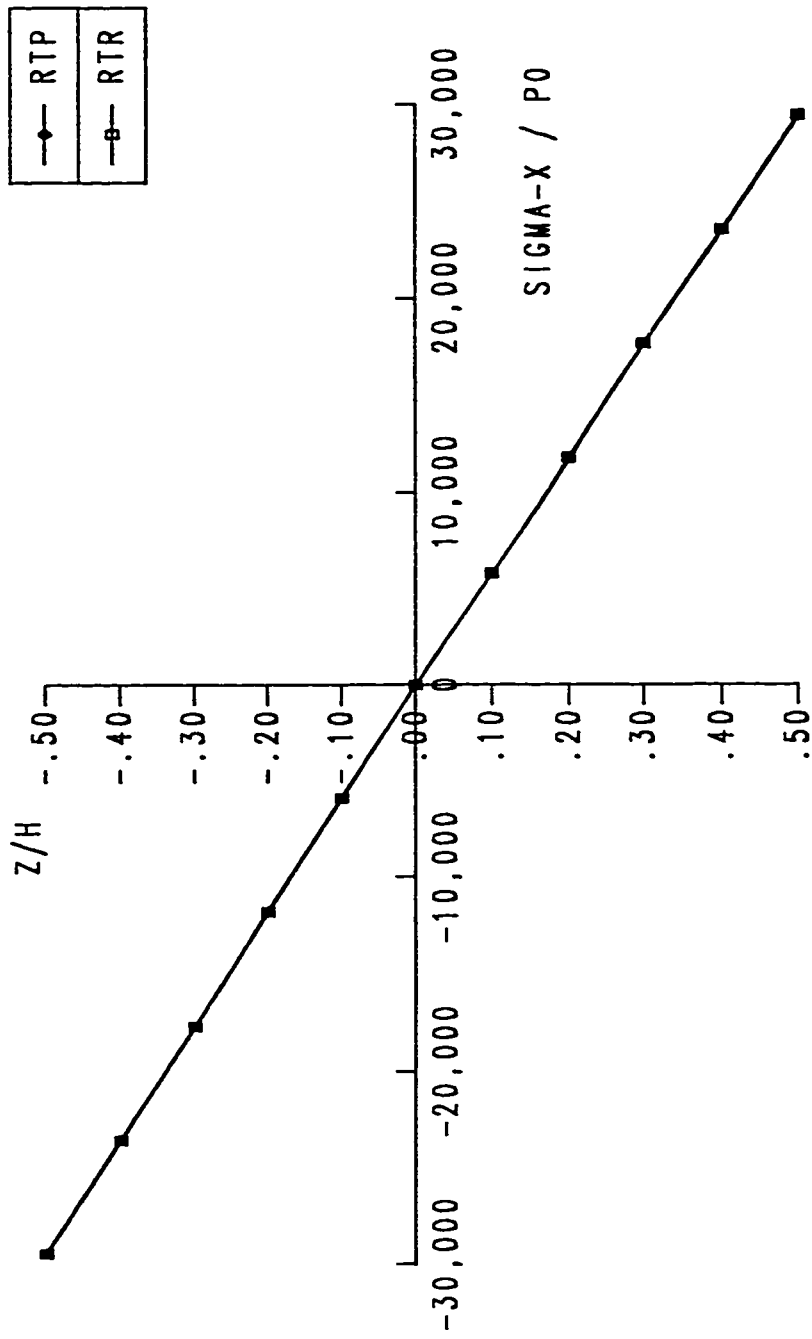


FIG. 5.67 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.1-11)

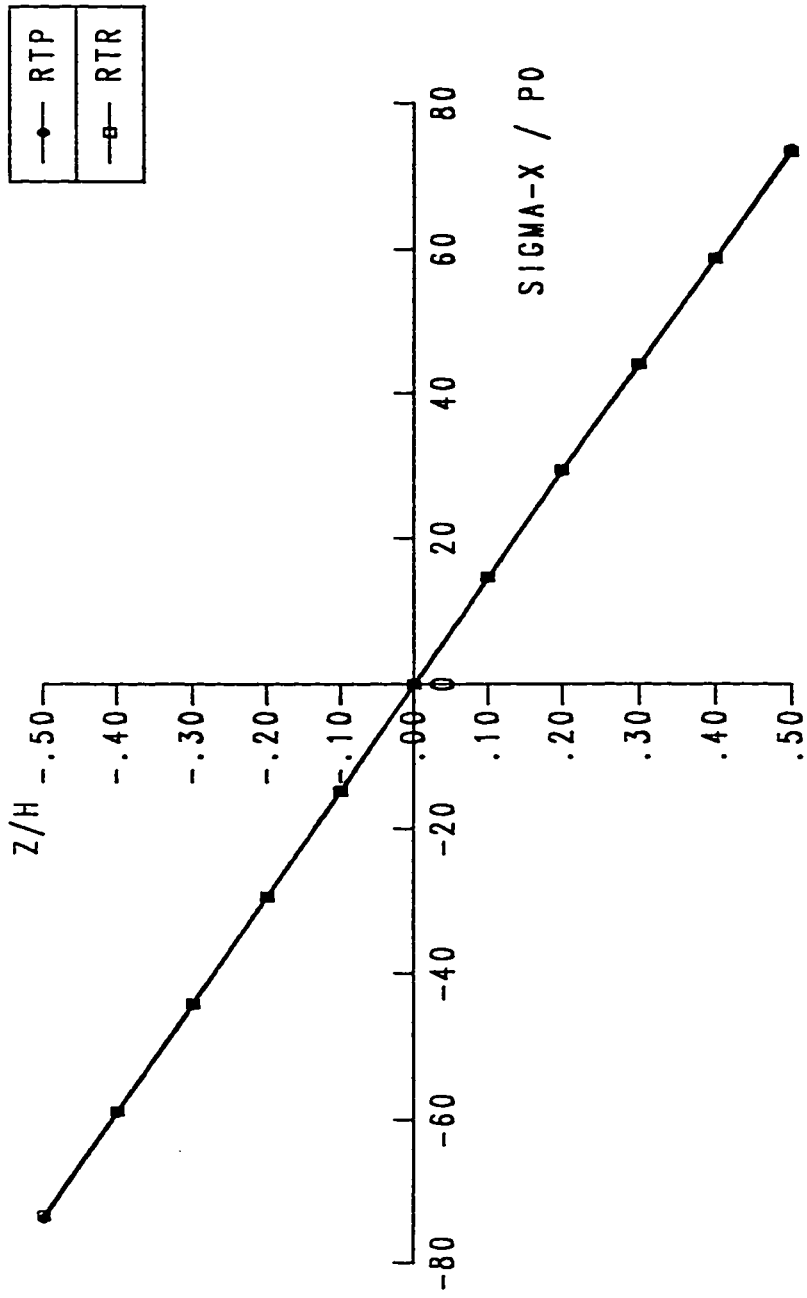


FIG. 5.68 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.3-11)

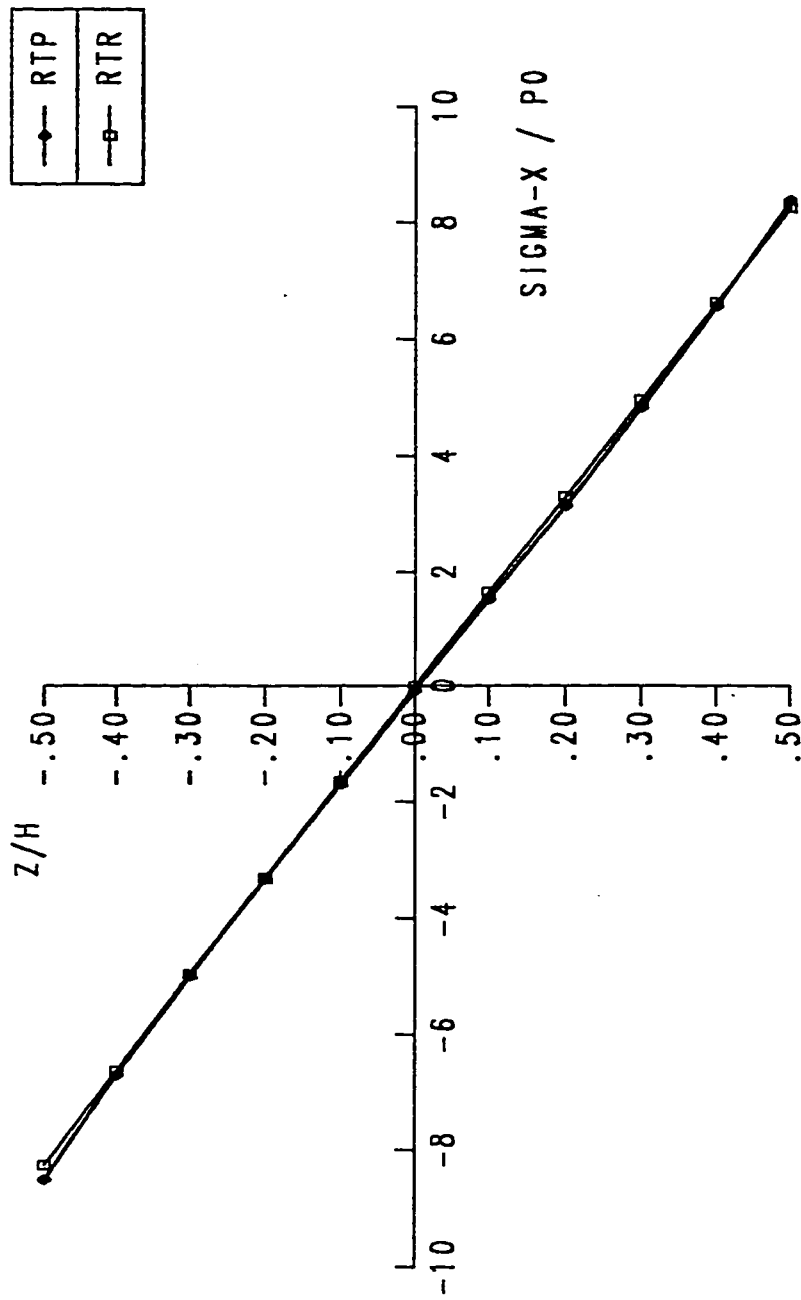


FIG. 5.69 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.5-11)

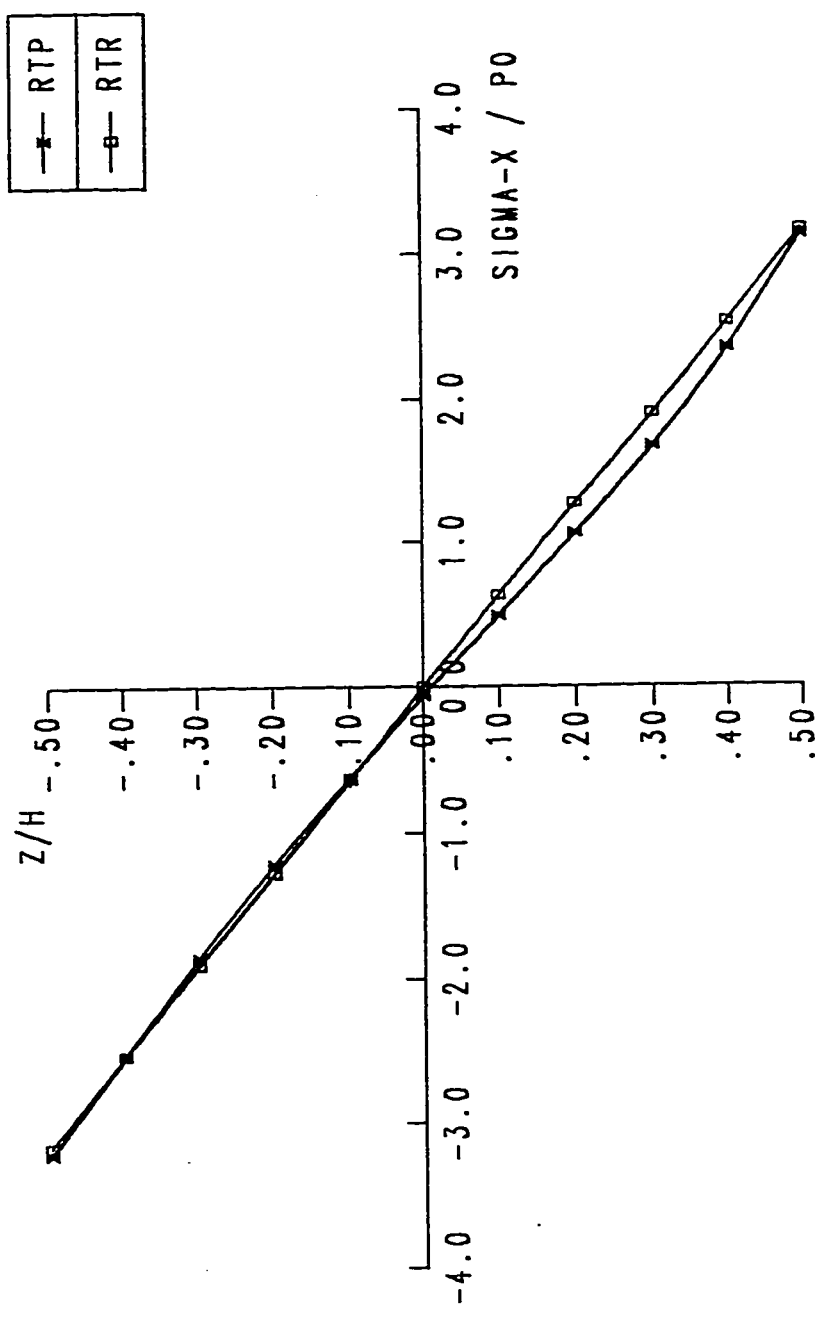


FIG. 5.70 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.7-11)

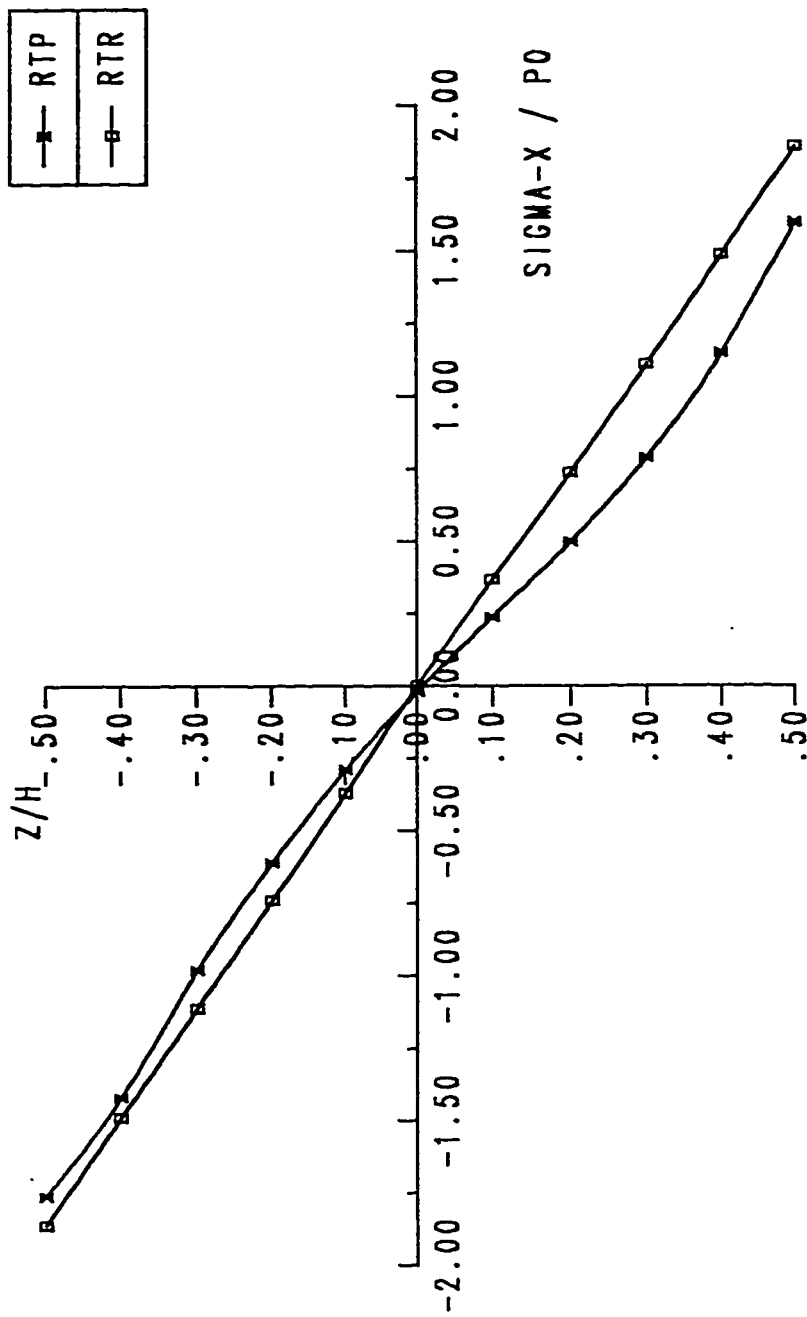




FIG. 5.71 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF1.-II)

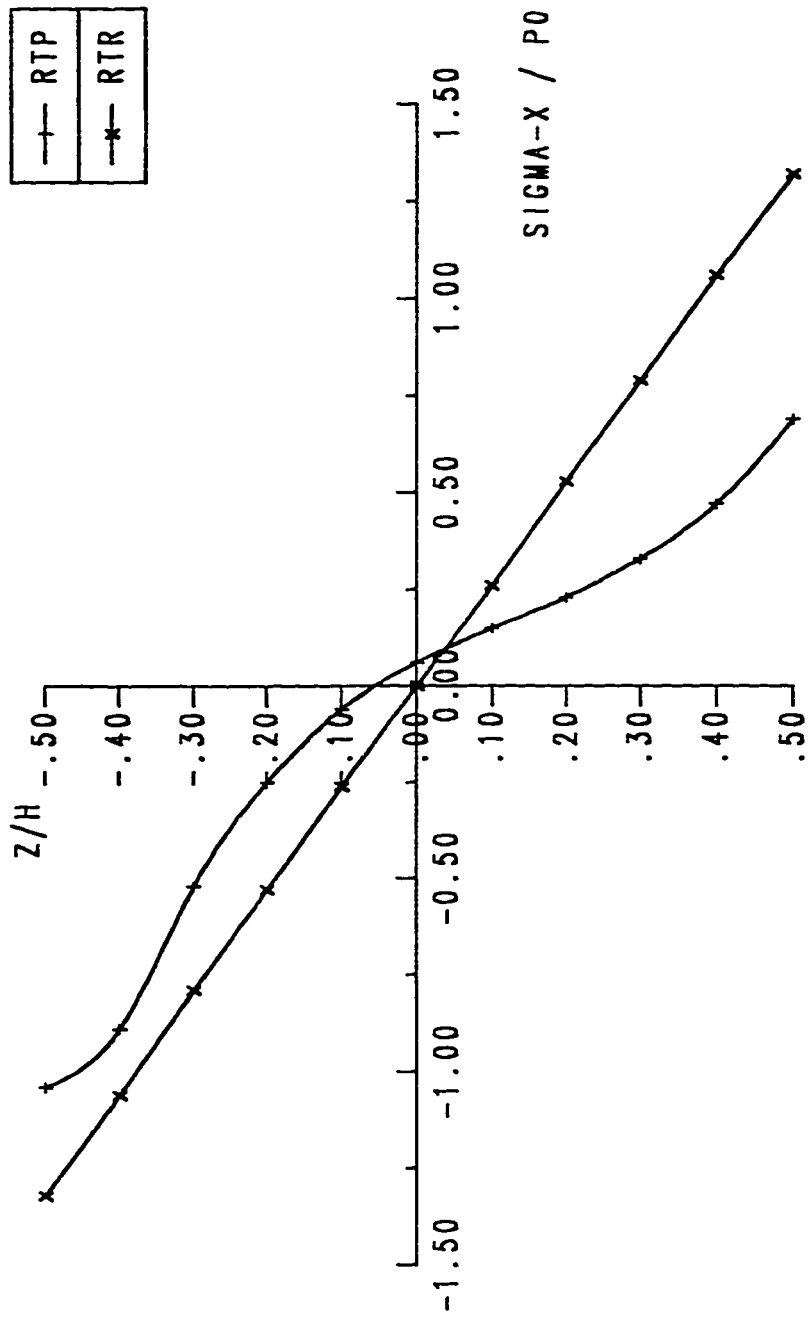


FIG. 72 : Max. Normal Stress Sigma-X VS Z/H (SS.1-11, Strip Load, Width = 0.2 a)

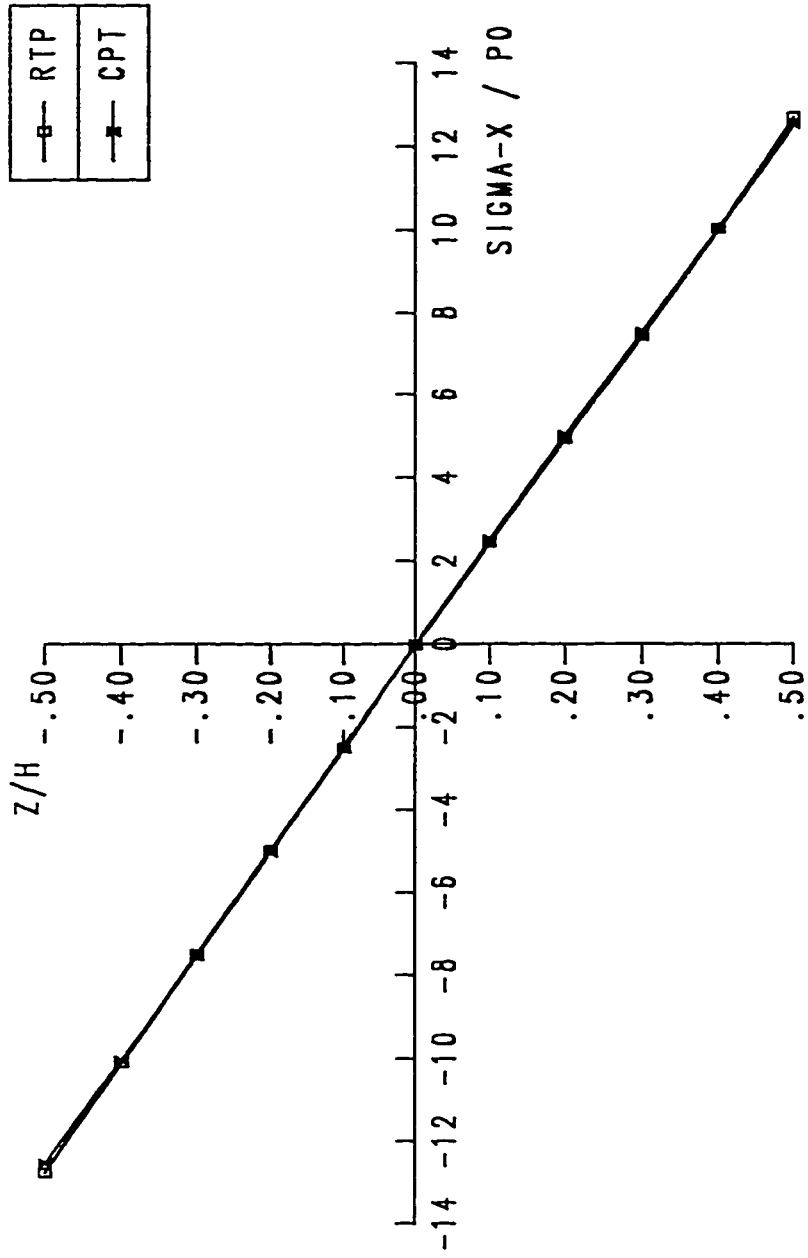


FIG. 73 : Max. Normal Stress Sigma-X VS Z/H (SS.3-11, Strip Load, Width = 0.2 a)

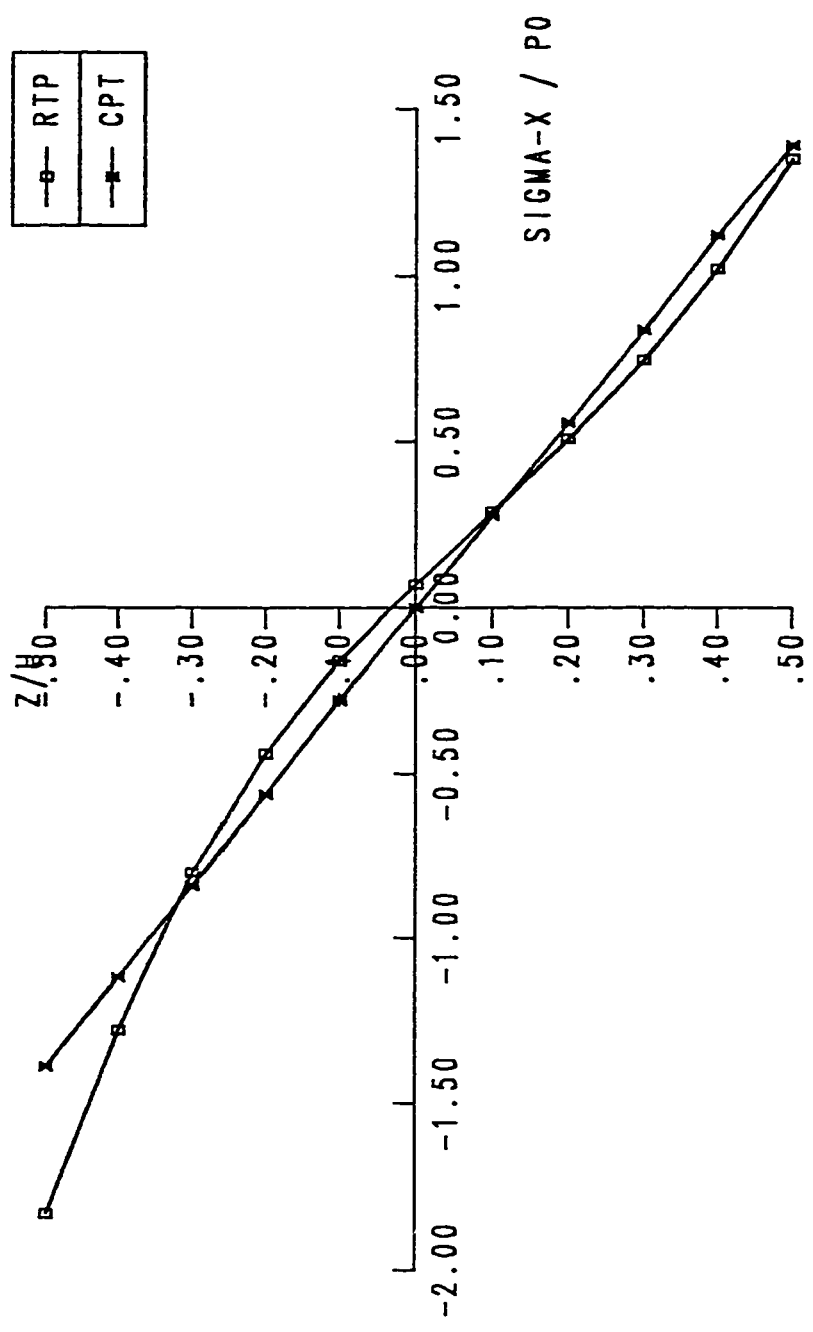


FIG. 74 : Max. Normal Stress Sigma-X VS Z/H (SS.5-11, Strip Load, Width = 0.2 a)

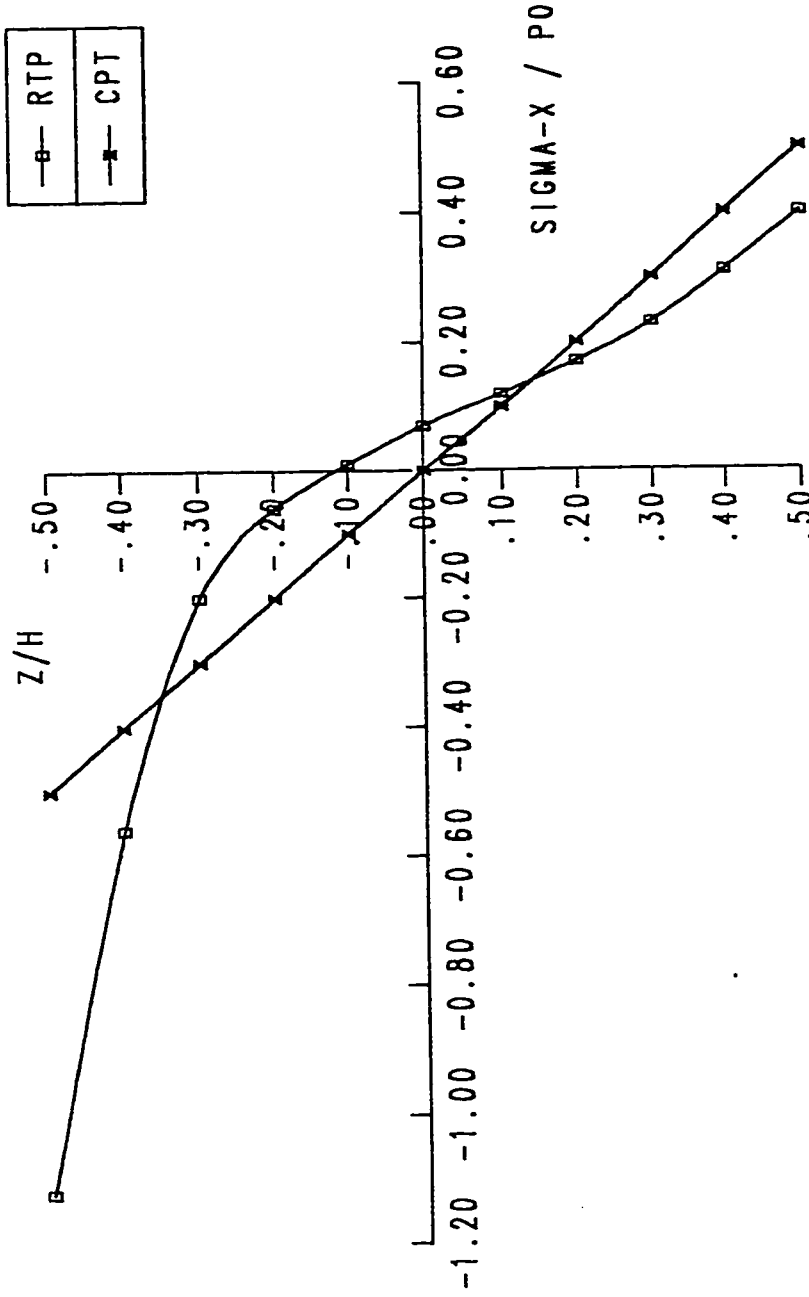


FIG. 75 : Max. Normal Stress  $\sigma_x$  VS  $Z/H$  (SS.7-II, Strip Load, Width = 0.2 a)

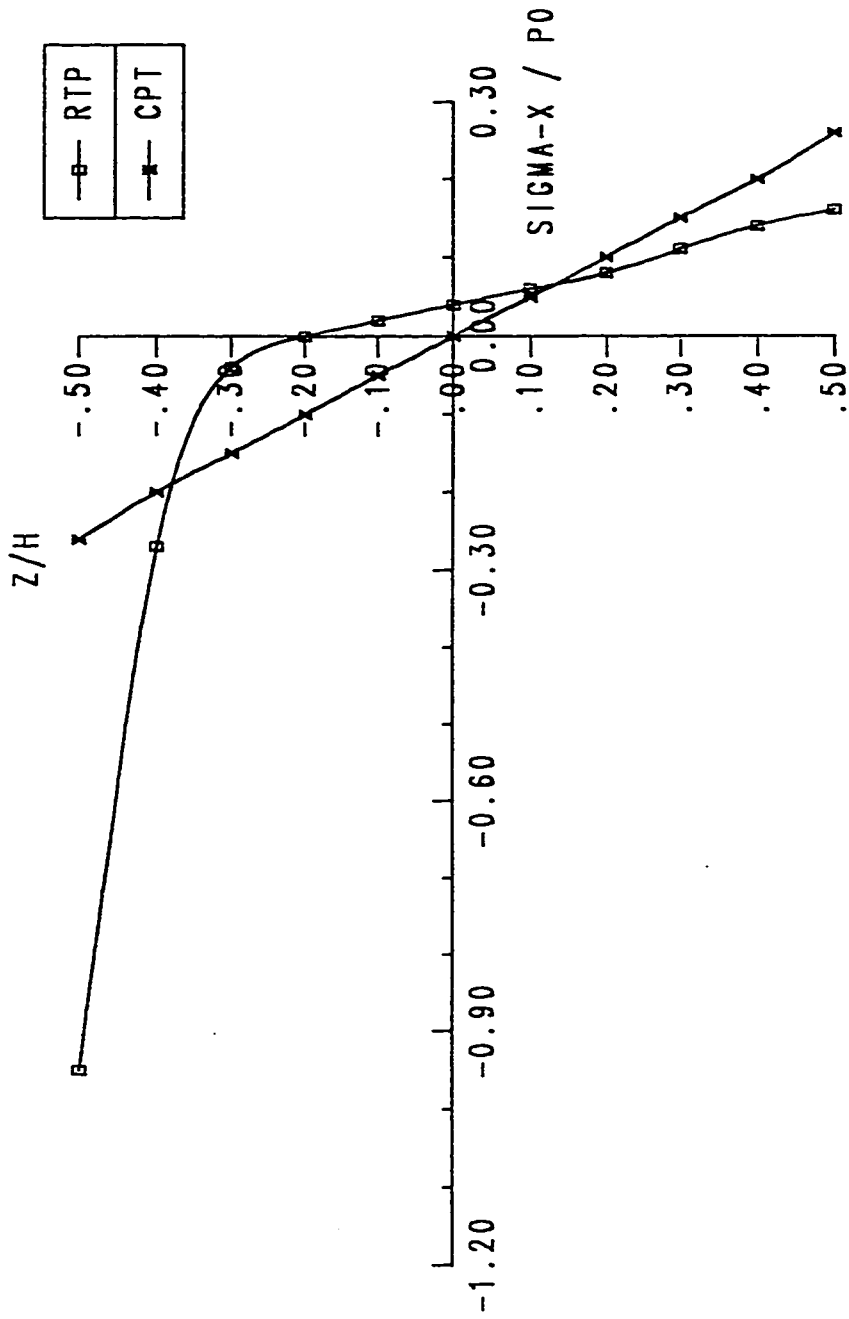


FIG. 76 : Max. Normal Stress Sigma-X VS Z/H (SS1.-II, Strip Load, Width = 0.2 a)

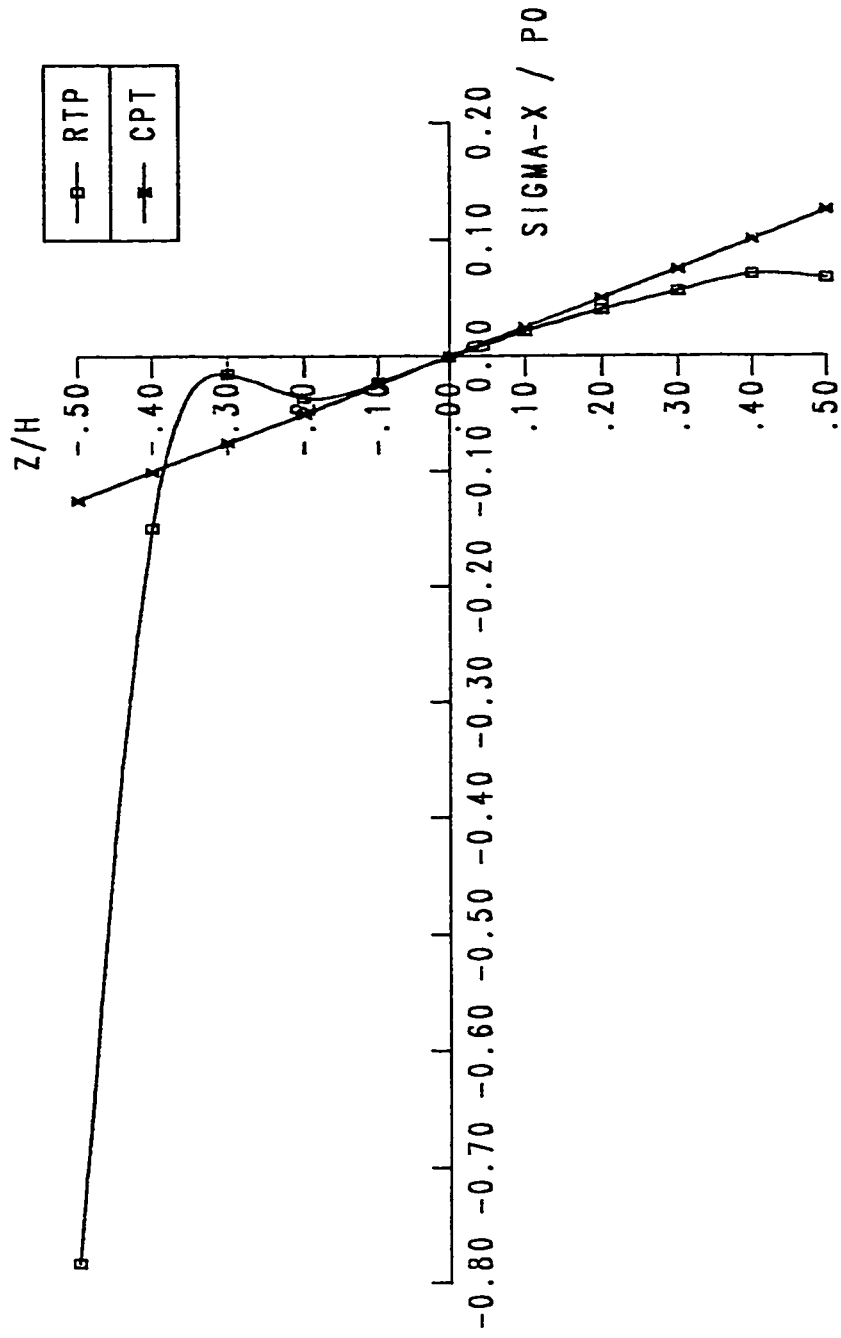


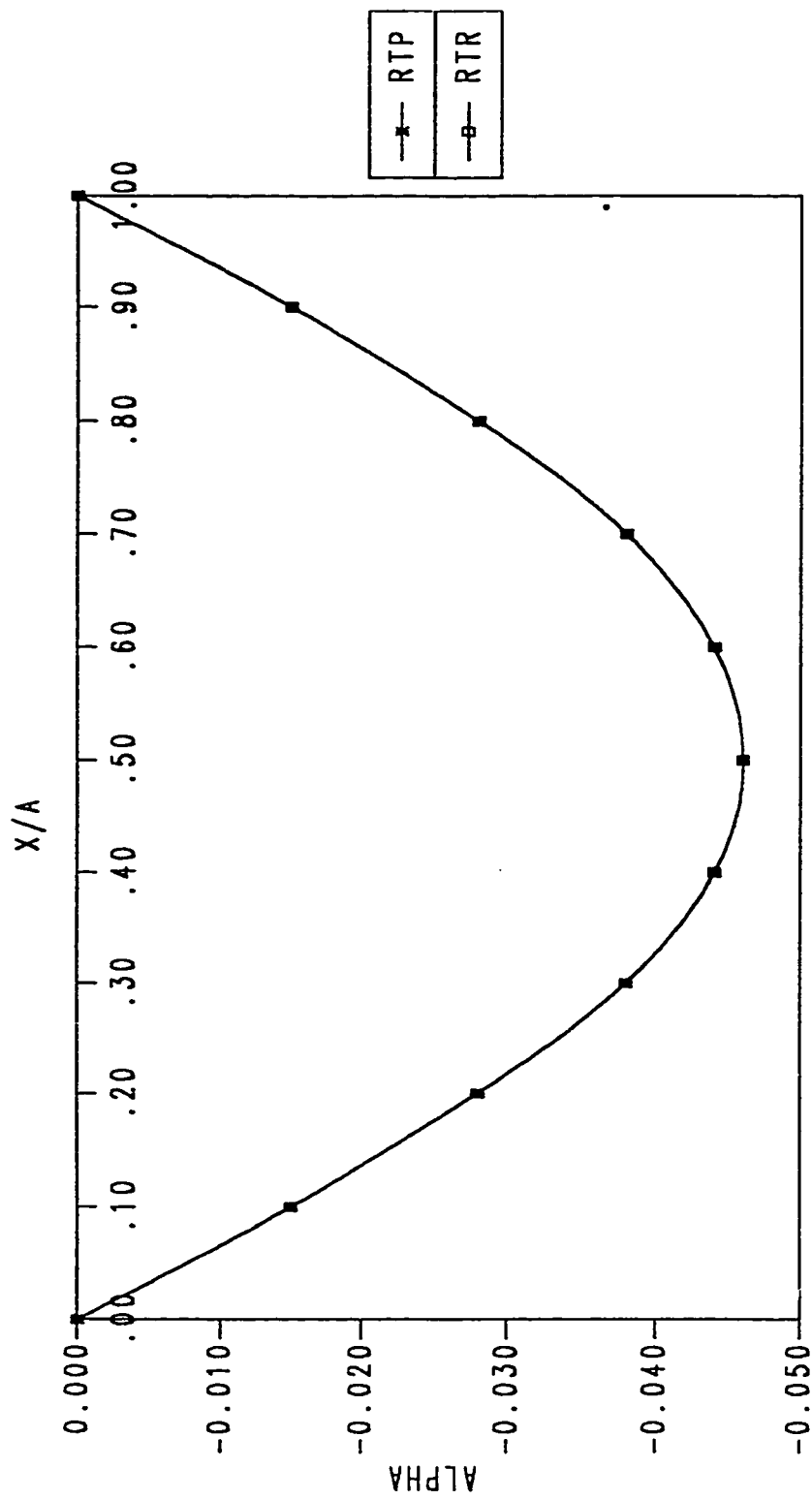
FIG. 5.77 : DEFLECTION OF TOP SURFACE OF PLATE AT  $Y=0.0$  (SS.1-11)

FIG. 5.78 : DEFLECTION OF TOP SURFACE OF PLATE AT  $Y=0.0$  (SS.5-11)

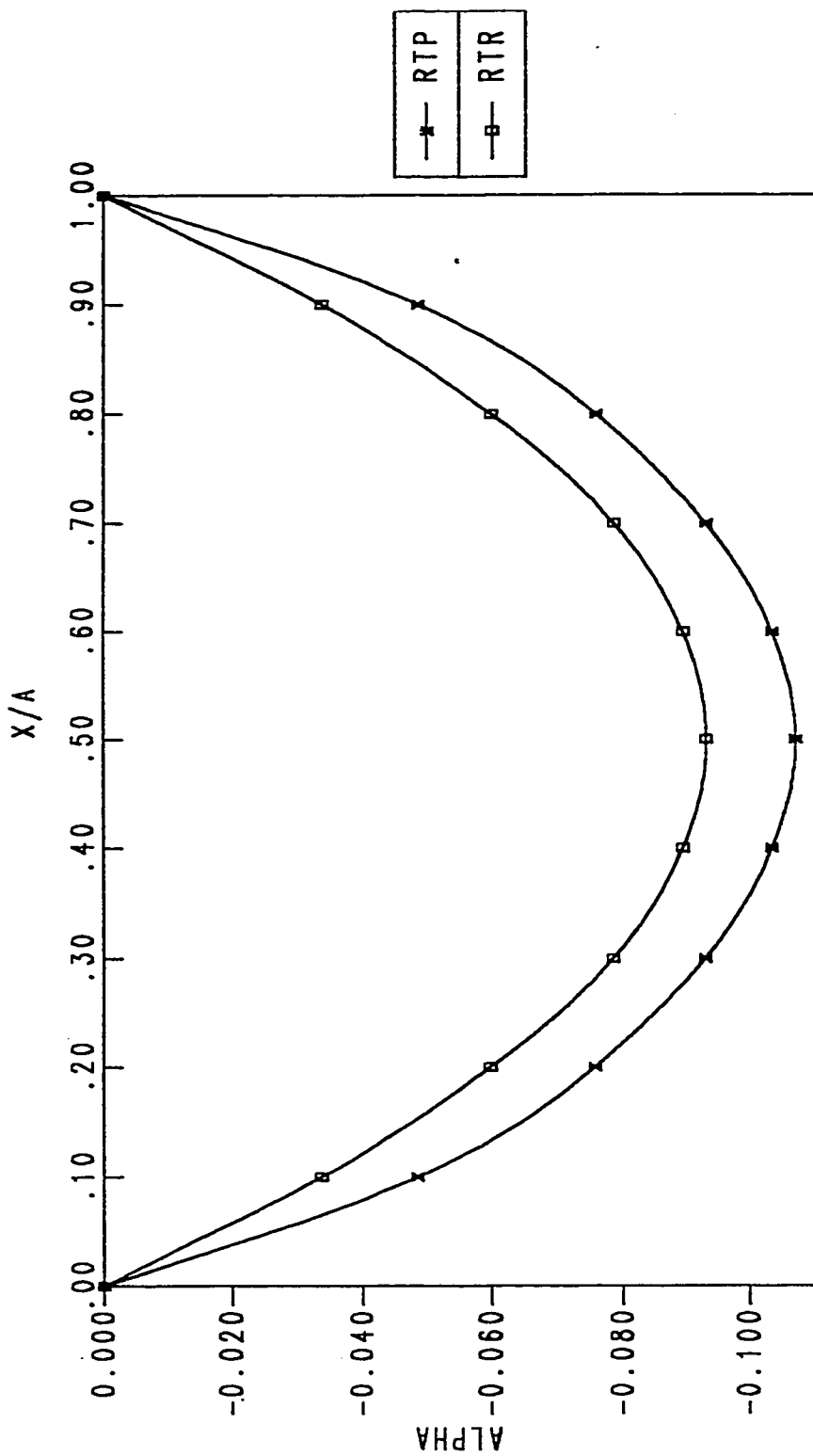




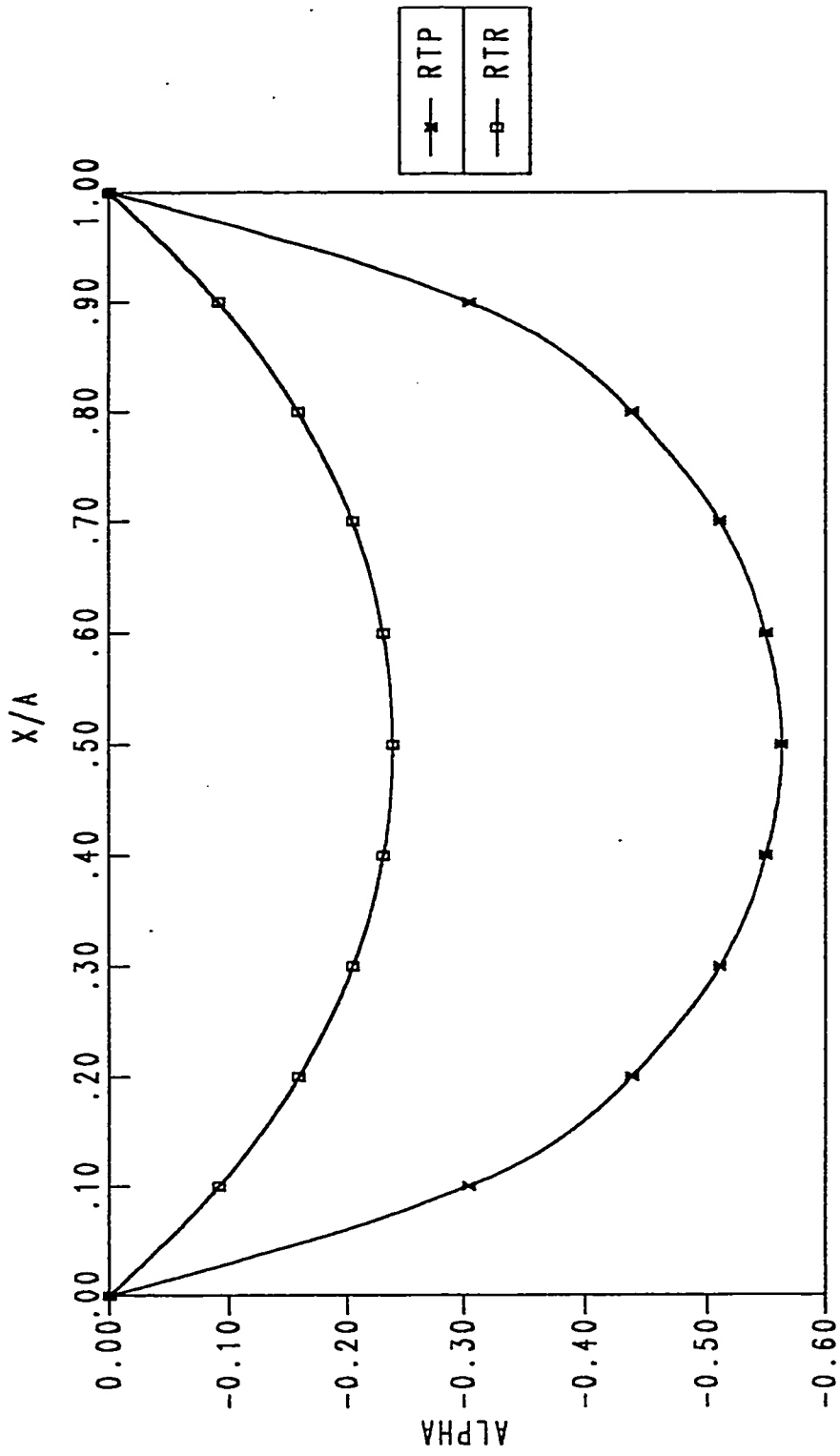
FIG. 79 : DEFLECTION OF TOP SURFACE OF PLATE AT  $Y=0.0$  (SS1.-II)

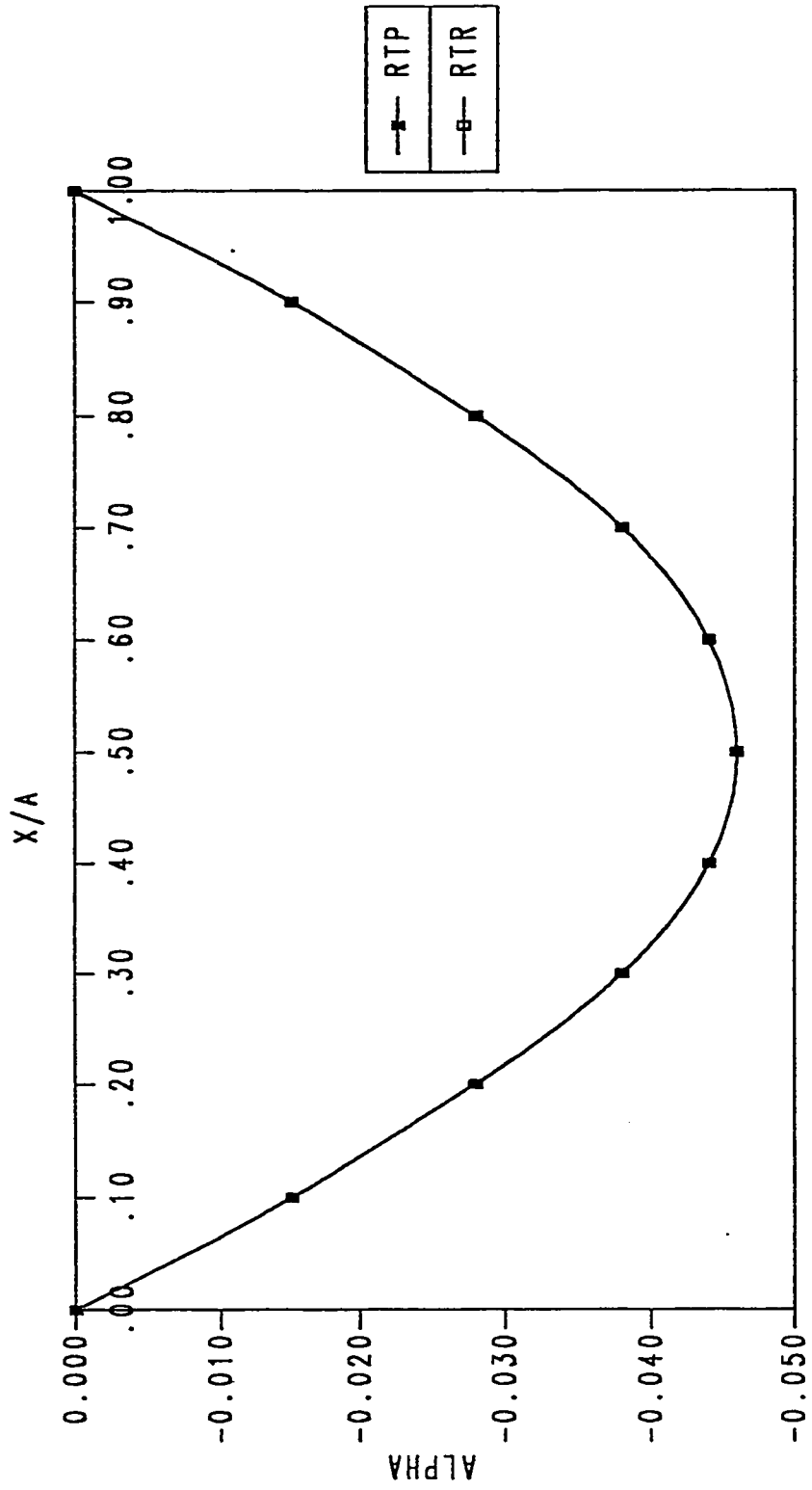
FIG. 5.80 : DEFLECTION OF MID SURFACE OF PLATE AT  $Y=0.0$  (SS.1-II)

FIG. 5.81 : DEFLECTION OF MID. SURFACE OF PLATE AT  $Y=0.0$  (SS.5-II)

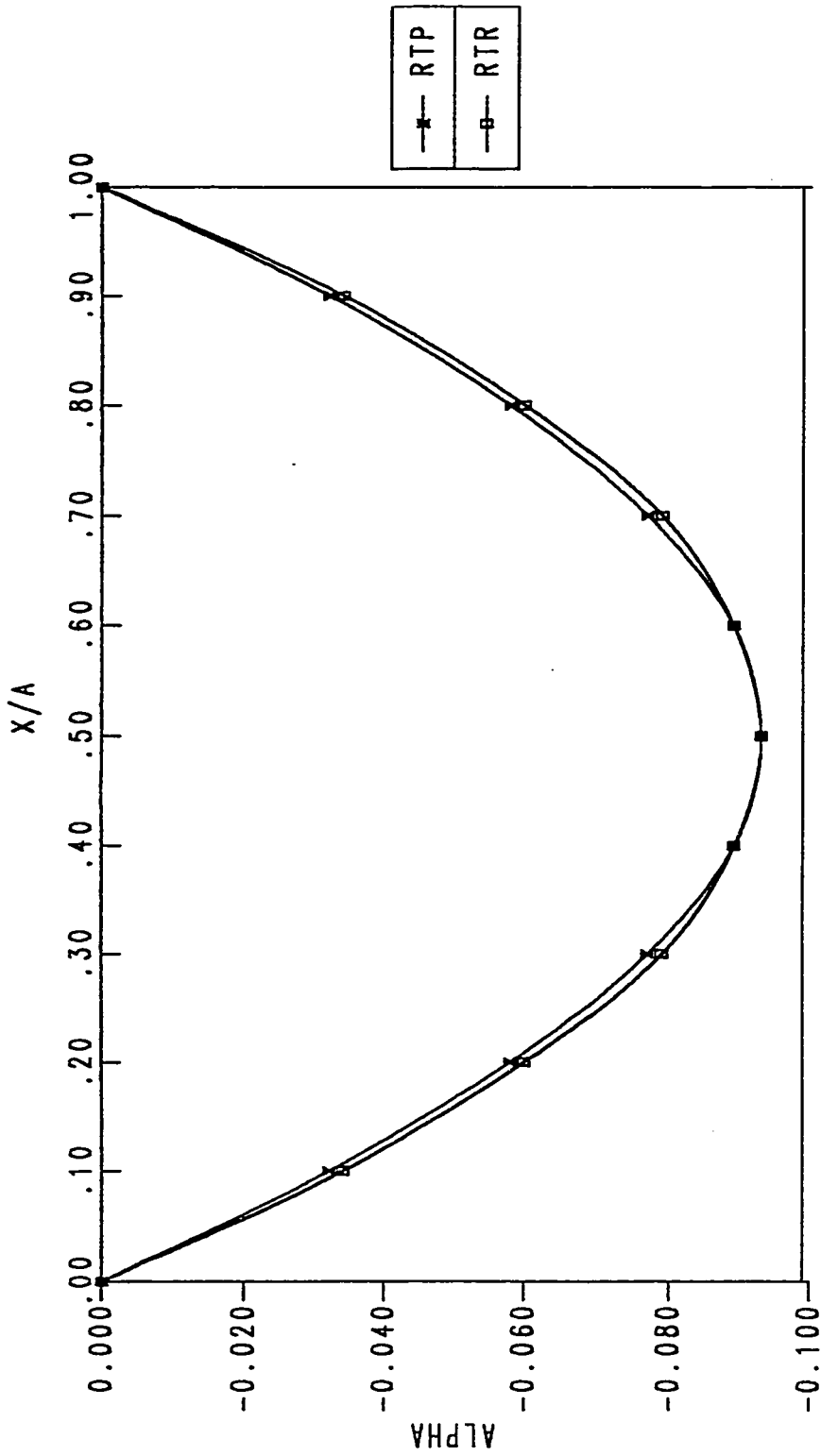


FIG. 5.82 : DEFLECTION OF MID. SURFACE OF PLATE AT  $Y=0.0$  (SS1.-II)

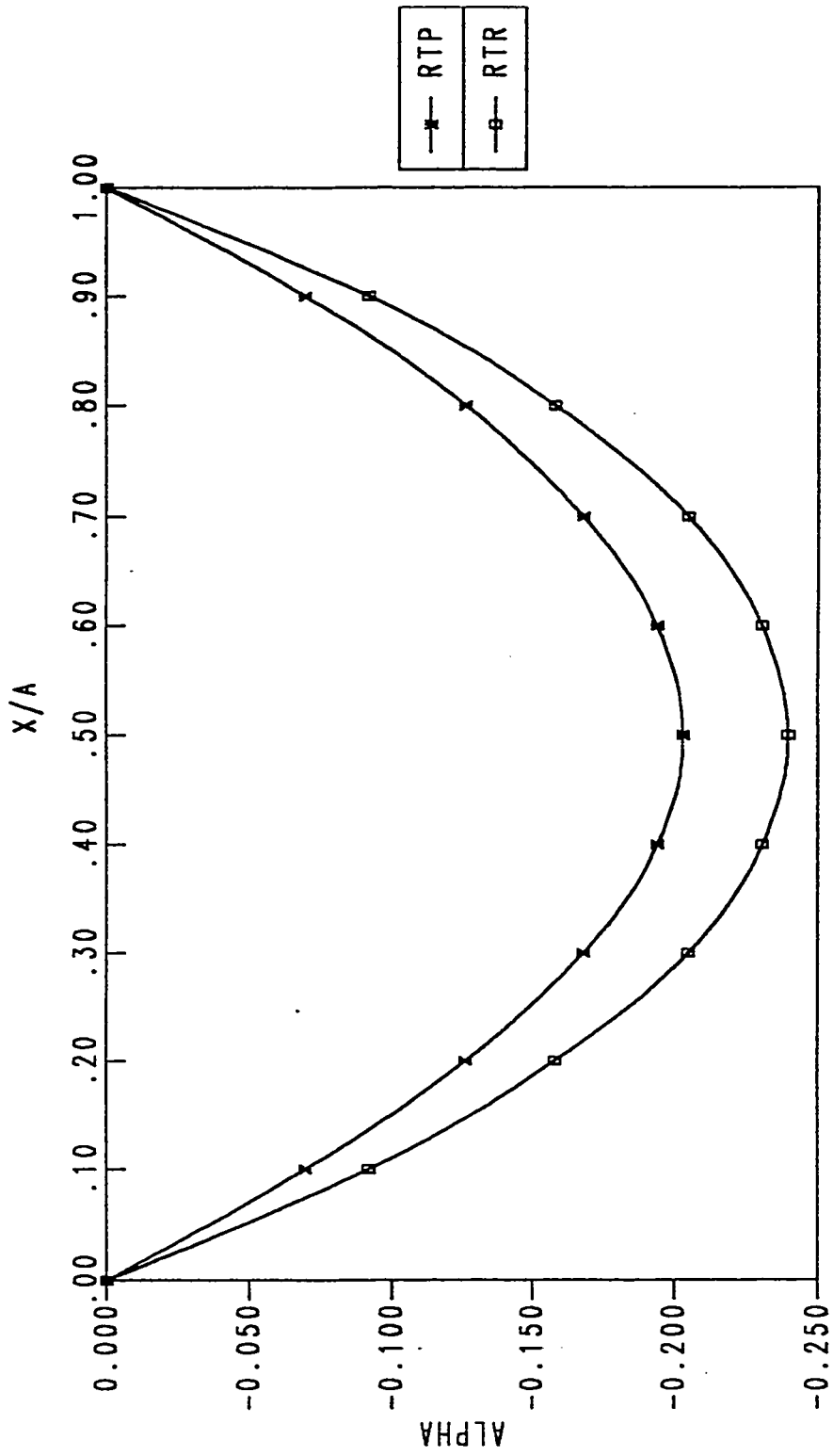


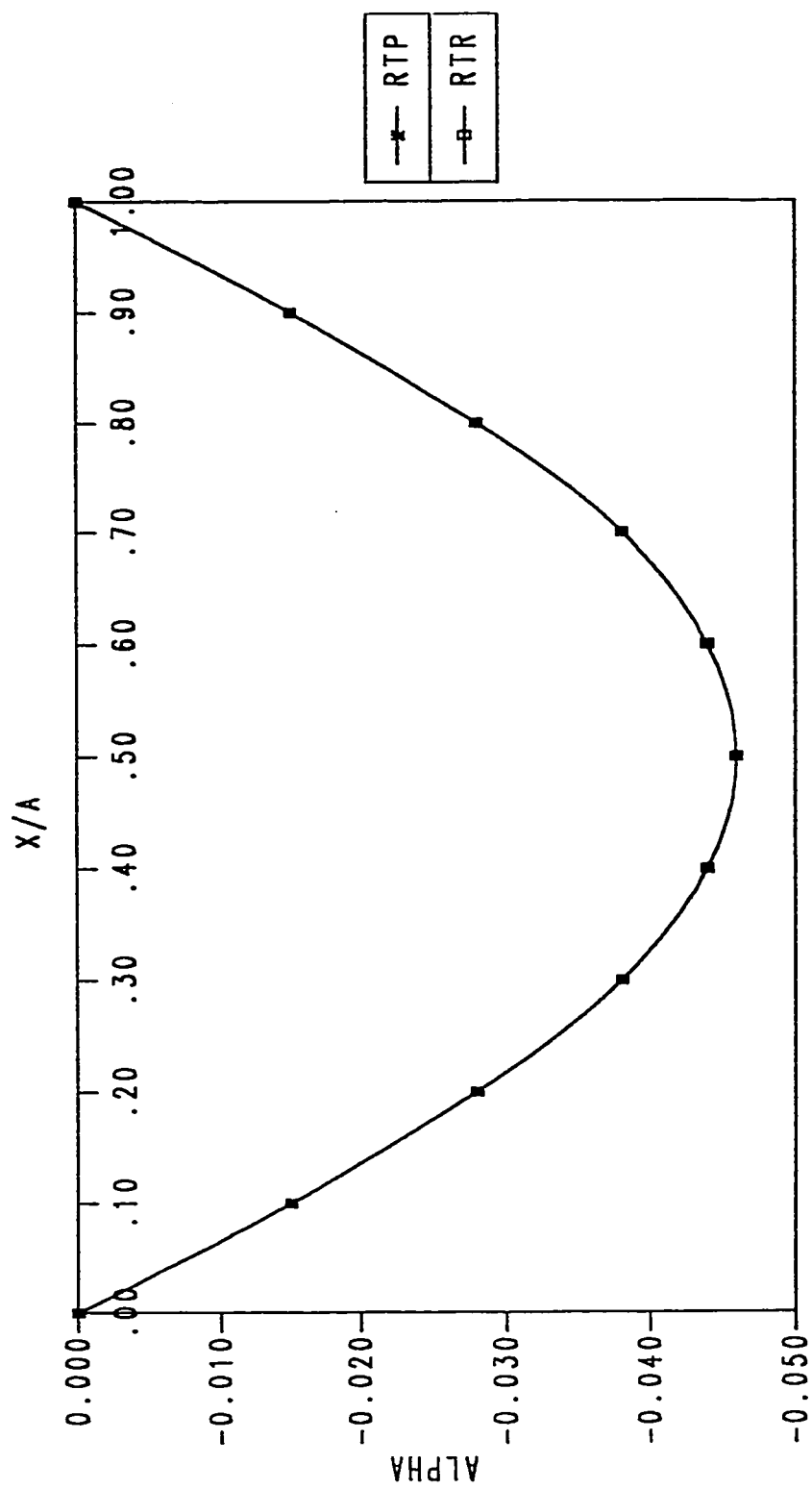
FIG. 5.83 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT  $Y=0.0$  (SS.1-11)

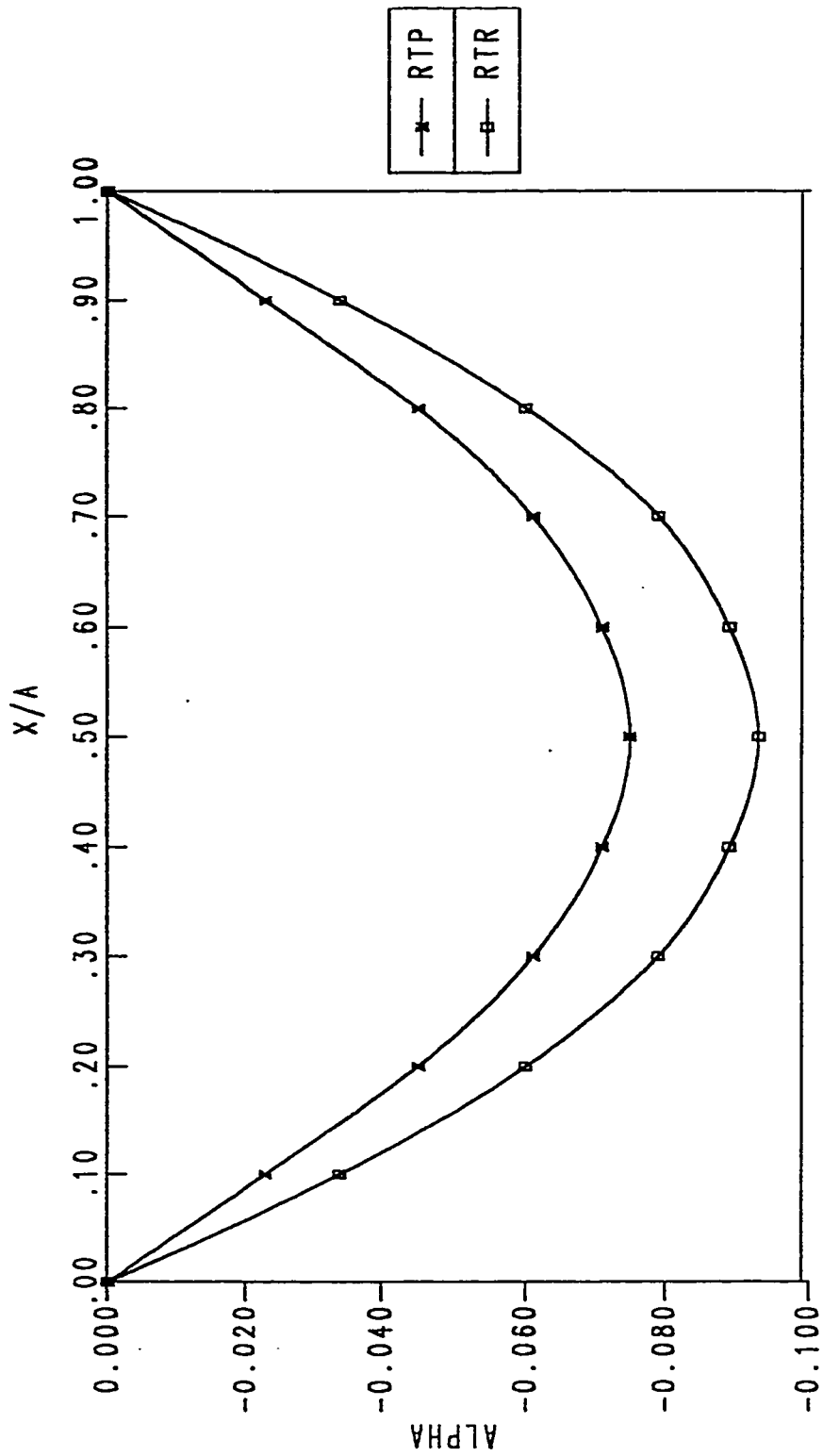
FIG. 5.84 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT  $Y=0.0$  (SS.5-II)

FIG. 5.85 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT  $Y=0.0$  (SS1.-II)

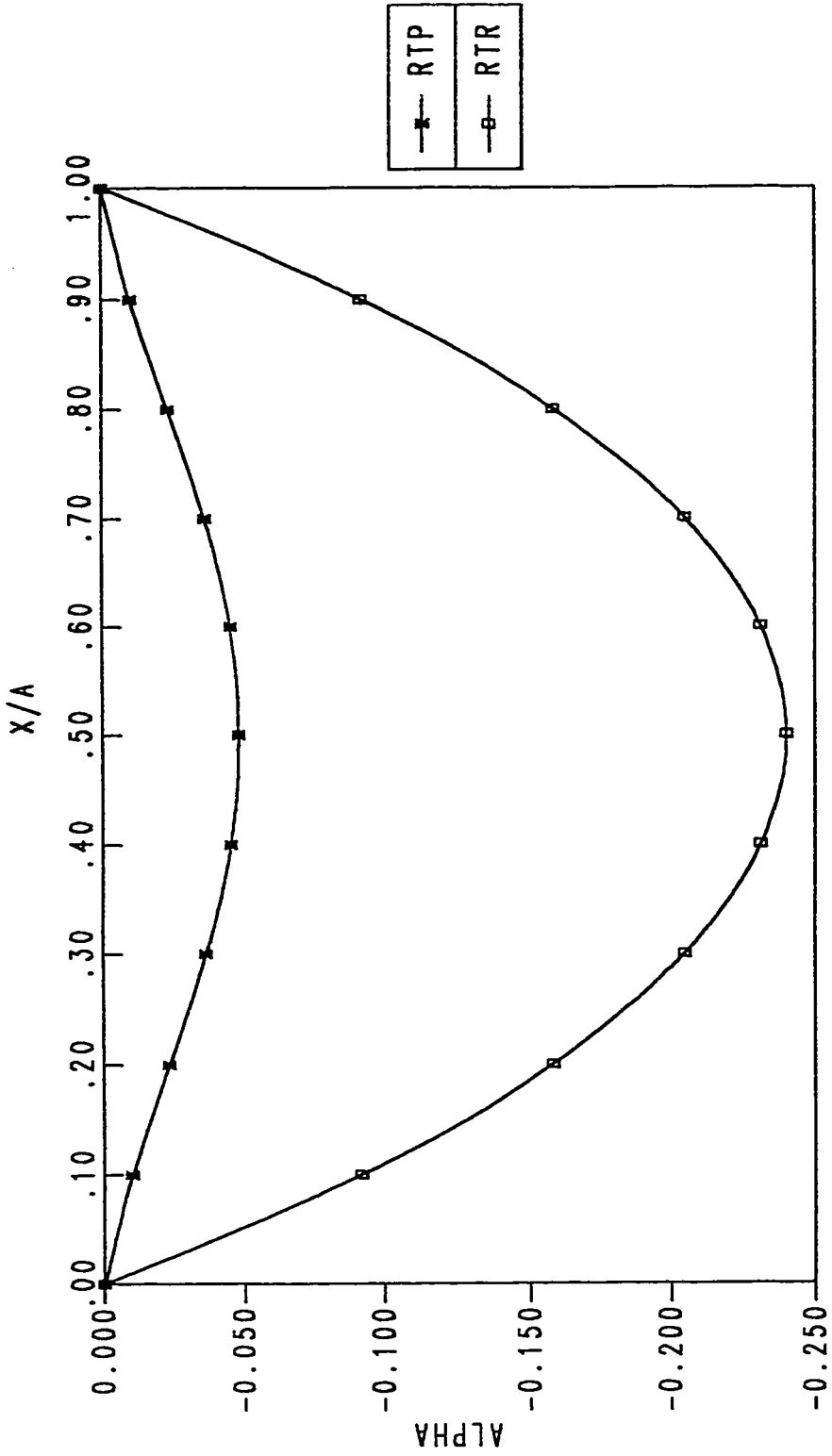


FIG. 86 : Deflection Of TOP, MID., & BOTTOM SURFACES At  $Y = 0.0$  (SS.1-11)

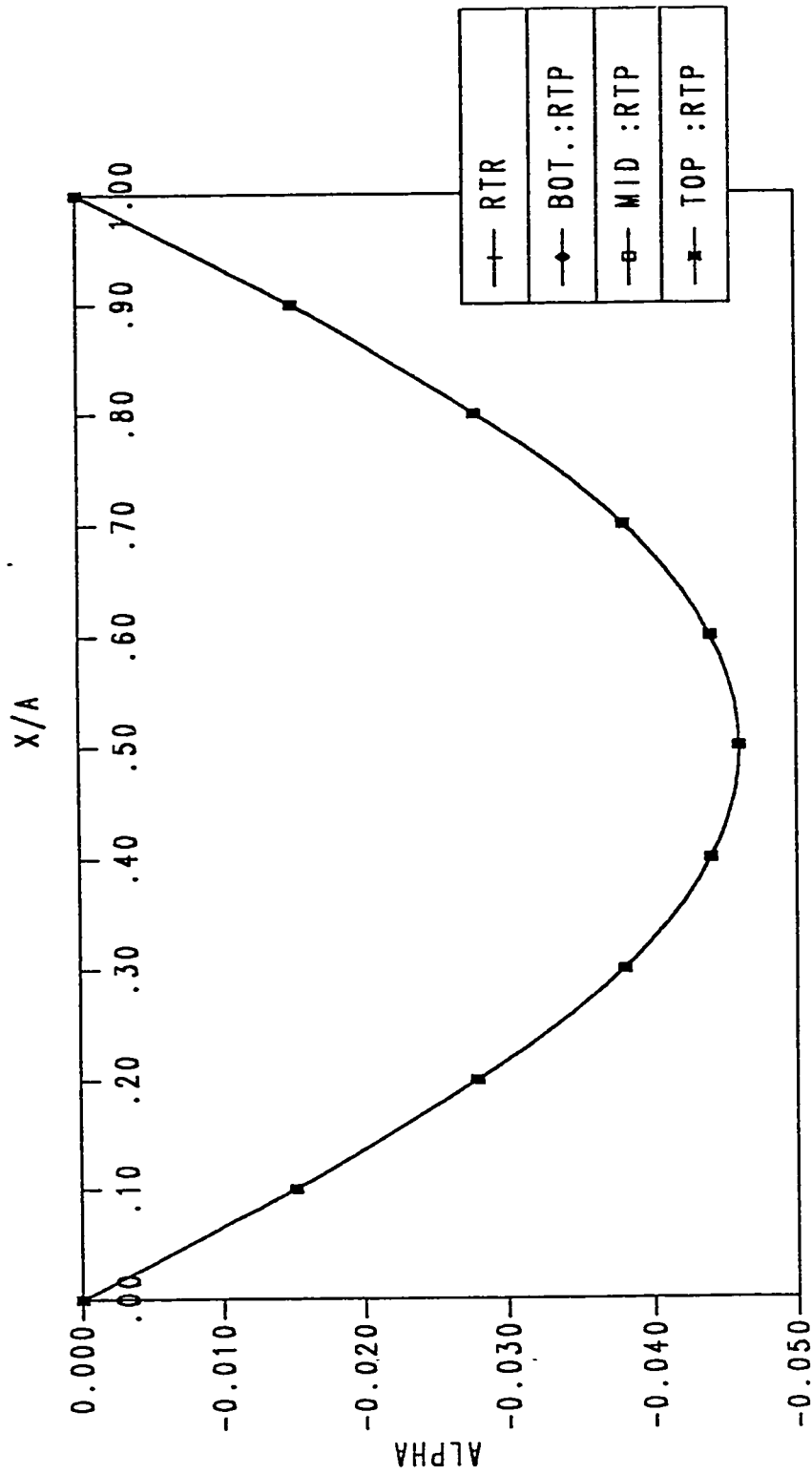




FIG. 87 : DEFLECTION OF TOP, MID, & BOT. SURFACES OF PLATE AT  $Y=0.0$  (SS.5-II)

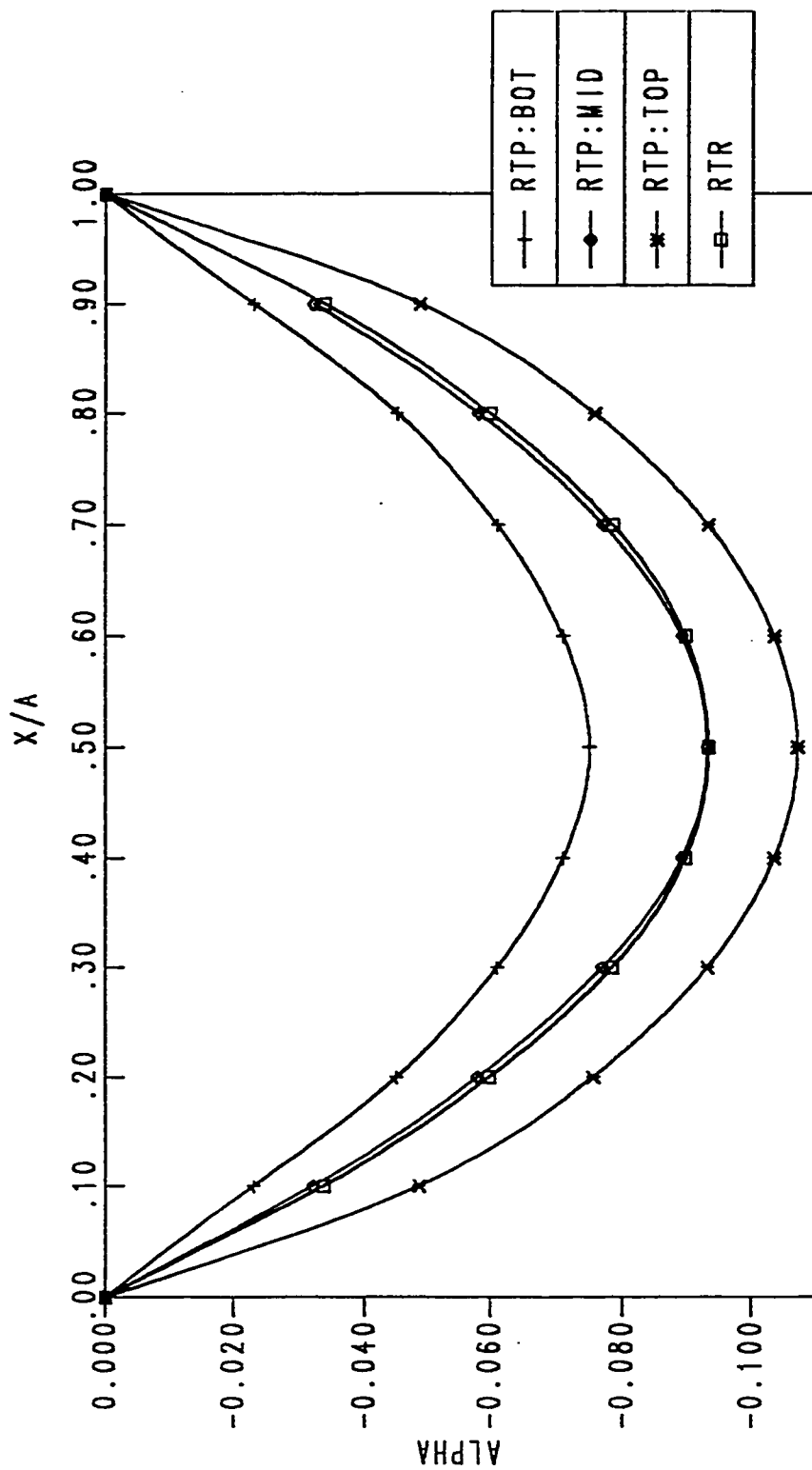


FIG. 88 : DEFLECTION OF TOP, MID, & BOT SURFACES OF PLATE AT  $Y=0.0$  (SS1.-II)

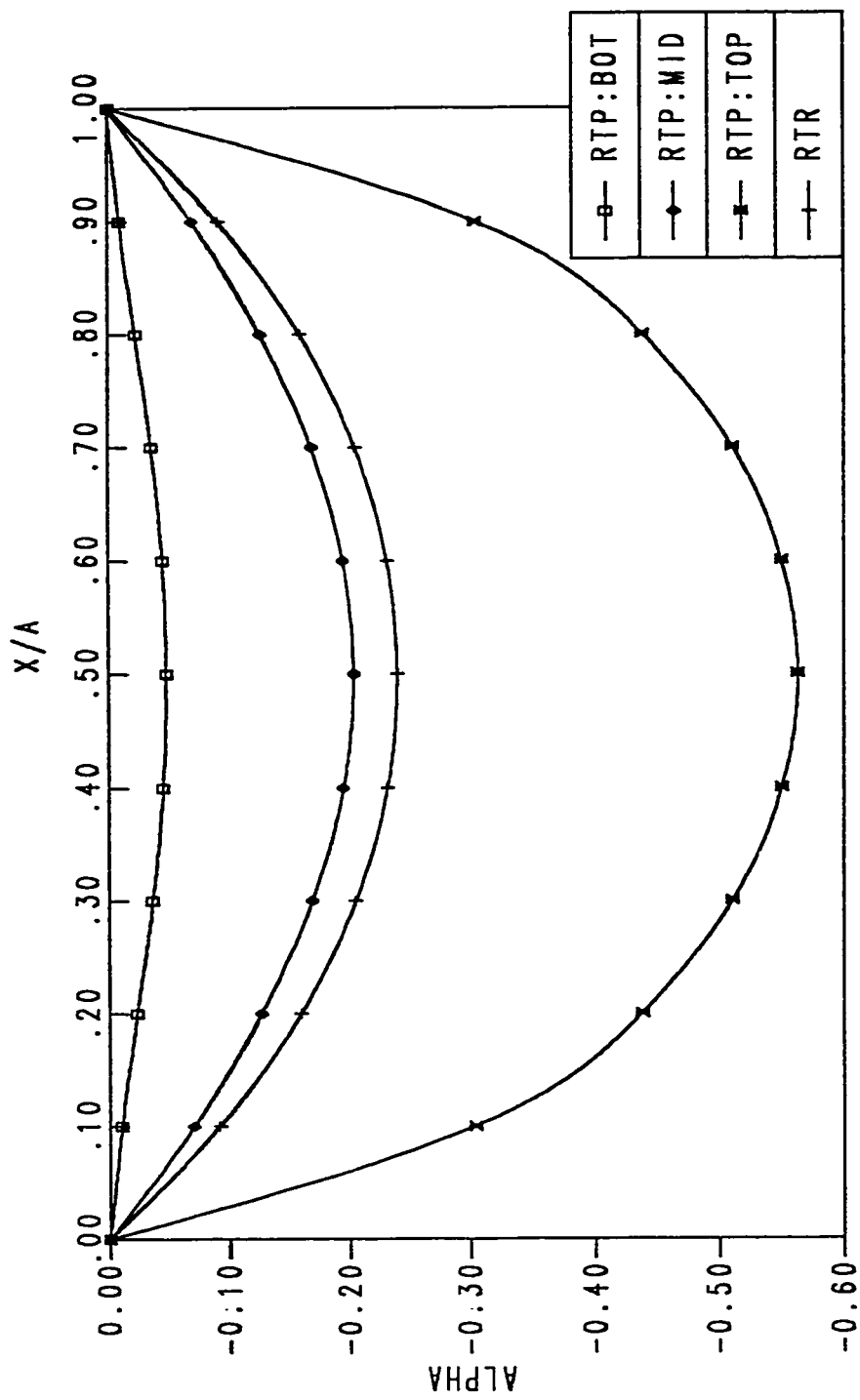


FIG. 5.89 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS.1)

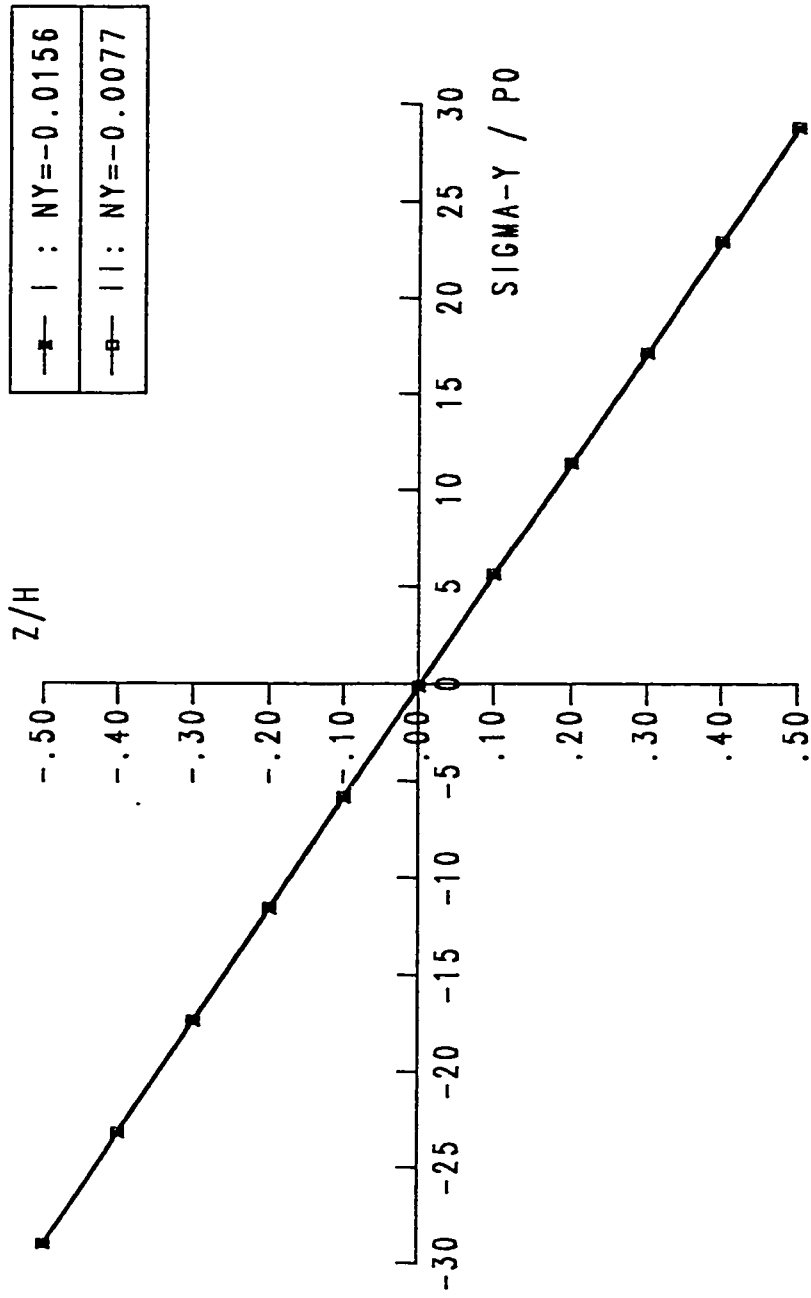


FIG. 5.90 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS.5)

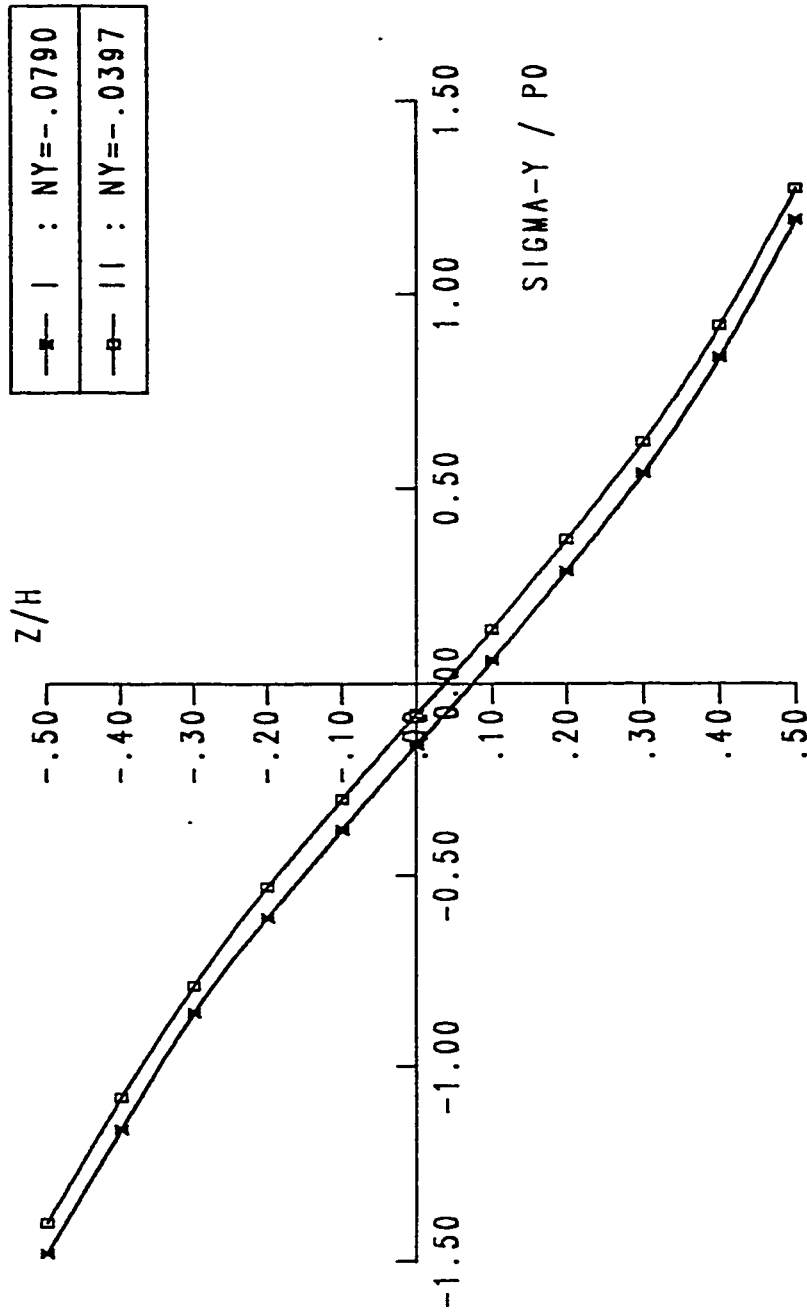


FIG. 5.91 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS1.0)

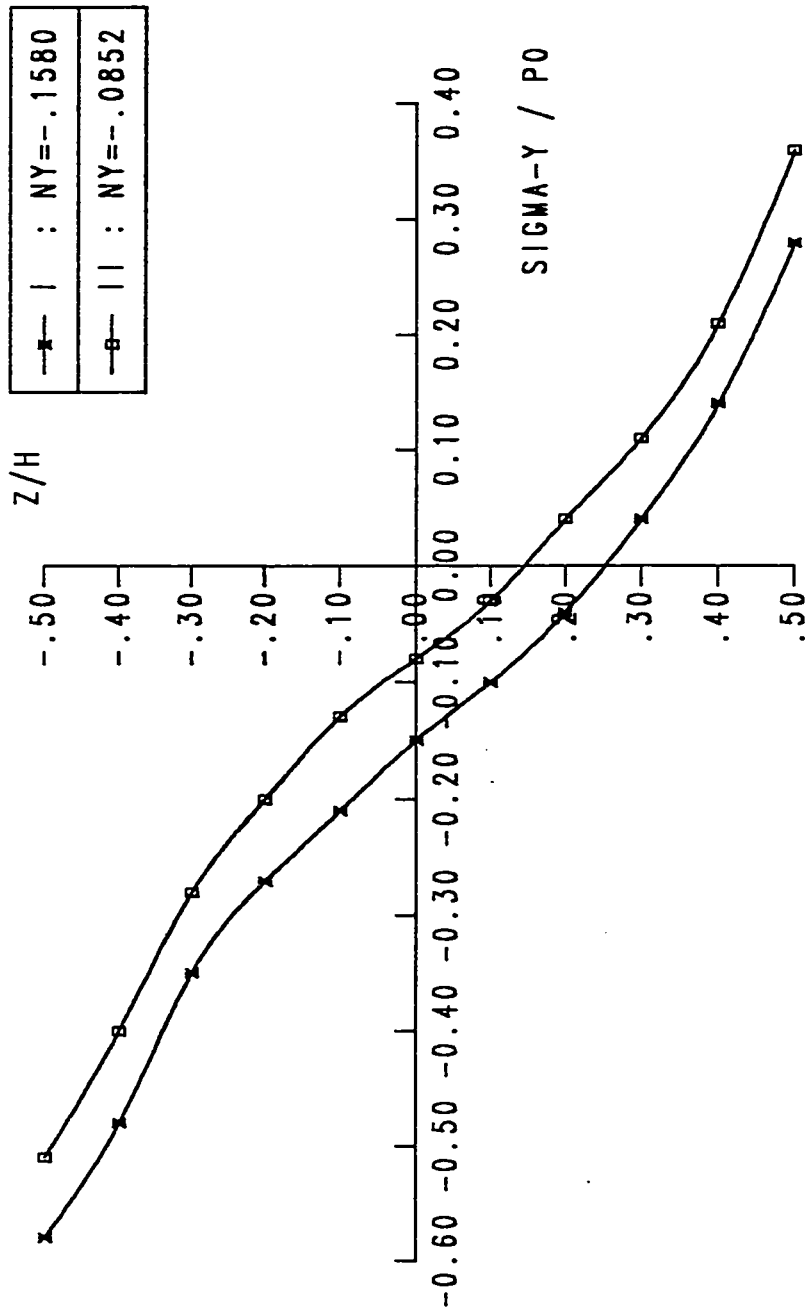


Table 5.1 Coefficient  $\alpha$  for the Center  
Deflection of a Uniformly Loaded  
Simply Supported Square Plate

h/a	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$
0.005	0.044009	0.044366	0.04433	0.044366
0.01	0.044149	0.044380	0.04434	0.044380
0.05	0.044789	0.044849	0.04481	0.044849
0.1	0.046294	0.046315	0.04625	0.046314
0.2	0.052171	0.052176	0.05194	0.052157
0.3	0.061946	0.061946	-	0.061867
0.4	0.075619	0.075623	0.07474	0.075312
0.5	0.093229	0.093207	-	0.092448
0.6	0.11463	0.11470	0.10853	0.11314
0.7	0.14008	0.14010	-	0.13717
0.8	0.16941	0.16941	0.15682	0.16426
0.9	0.20220	0.20262	-	0.19428
1.0	0.24024	0.23975	0.21982	0.22679

NOTE :  $\alpha = 0.04433$   
By CPT : Classical Plate Theory (For All h/a Ratios)

$\alpha_1$  = FEM : Goma'a and Baluch  
 $\alpha_2$  = RTR : Refined Theory (Reissner)  
 $\alpha_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\alpha_4$  = RTP : Refined Theory (Present)

$w = \alpha(pa^4/Eh^3)$ ,  $\mu = 0.3$

Table 5.2 Coefficient  $\beta$  for the Center  
Resultant Moment  $M_x$   
of a Uniformly Loaded Simply Supported Square Plate

$h/a$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
0.005	0.047477	0.047890	0.0479	0.047890
0.01	0.047659	0.047892	0.0479	0.047892
0.05	0.048072	0.047928	0.0492	0.047927
0.1	0.048285	0.048040	0.0512	0.048042
0.2	0.048776	0.048490	0.0534	0.048509
0.3	0.049549	0.049240	—	0.049339
0.4	0.050623	0.050290	0.0559	0.050284
0.5	0.052003	0.051640	—	0.051500
0.6	0.053689	0.053290	0.0640	0.052949
0.7	0.055682	0.055240	—	0.054611
0.8	0.057980	0.057490	0.0776	0.056460
0.9	0.063496	0.060040	—	0.058593
1.0	0.063496	0.062890	0.0964	0.060833

NOTE :  $\beta = 0.0479$   
By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\beta_1$  = FEM : Goma'a and Baluch  
 $\beta_2$  = RTR : Refined Theory (Reissner)  
 $\beta_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\beta_4$  = RTP : Refined Theory (Present)

$M_x = \beta pa^2$ ,  $\mu = 0.3$

Table 5.3 Coefficient  $\gamma$  for the Center  
Resultant Moment  $M_y$   
of a Uniformly Loaded Simply Supported Square Plate

$h/a$	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_4$
0.005	0.047477	0.047888	0.0479	0.047888
0.01	0.047659	0.047889	0.0479	0.047889
0.05	0.048072	0.047927	0.0492	0.047927
0.1	0.048285	0.048045	0.0512	0.048043
0.2	0.048776	0.048517	0.0534	0.048498
0.3	0.049549	0.049303	—	0.049203
0.4	0.050623	0.050405	0.0559	0.050179
0.5	0.052003	0.051821	—	0.051418
0.6	0.053689	0.053552	0.0640	0.052952
0.7	0.055682	0.055597	—	0.054787
0.8	0.057980	0.057957	0.0776	0.056923
0.9	0.063496	0.060632	—	0.059369
1.0	0.063496	0.063621	0.0964	0.062159

NOTE :  $\gamma = 0.0479$   
By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\gamma_1$  = FEM : Goma'a and Baluch  
 $\gamma_2$  = RTR : Refined Theory (Reissner)  
 $\gamma_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\gamma_4$  = RTP : Refined Theory (Present)

$M_y = \gamma pa^2, \mu = 0.3$



Table 5.4 Coefficient  $\alpha$  for the Center Deflection of a Uniformly Loaded Simple/Clamped Square Plate

$h/a$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$
0.005	0.0018120	0.0019179	0.00190	0.0019179
0.01	0.0018369	0.0019201	0.00188	0.0019201
0.05	0.0019672	0.0019901	0.00176	0.0019908
0.1	0.002194	0.002201	0.00166	0.002206
0.2	0.002980	0.002982	0.00158	0.003005
0.3	0.004163	0.004165	-	0.004197
0.4	0.005696	0.005697	0.00166	0.005703
0.5	0.007562	0.007565	-	0.007499
0.6	0.009763	0.009772	0.00182	0.009583
0.7	0.012314	0.012323	-	0.011966
0.8	0.015206	0.015227	0.00203	0.014653
0.9	0.018517	0.018490	-	0.017647
1.0	0.022100	0.022116	0.00231	0.020946

NOTE :  $\alpha = 0.0192$

By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\alpha_1$  = FEM : Goma'a and Baluch

$\alpha_2$  = RTR : Refined Theory (Reissner)

$\alpha_3$  = RTB : Refined Theory (Voyiadjis and Baluch)

$\alpha_4$  = RTP : Refined Theory (Present)

$w = \alpha pa^4/D, \mu = 0.3$

Table 5.5 Coefficient  $\beta$  for the Center  
Resultant Moment  $M_x$   
of a Uniformly Loaded Simple/Clamped Square Plate

$h/a$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
0.005	0.023429	0.024396	0.0242	0.024396
0.01	0.023643	0.024410	0.0241	0.024410
0.05	0.024784	0.024864	0.0261	0.024871
0.1	0.034170	0.026196	0.0243	0.026250
0.2	0.035011	0.030675	0.0216	0.030959
0.3	0.036073	0.036367	-	0.036721
0.4	0.037652	0.042456	0.0210	0.042240
0.5	0.040033	0.048551	-	0.046993
0.6	0.043359	0.054290	0.0279	0.050899
0.7	0.047662	0.059191	-	0.054050
0.8	0.052928	0.062634	0.0411	0.056579
0.9	0.059129	0.063861	-	0.058617
1.0	0.066235	0.061985	0.0596	0.060273

NOTE :  $\beta = 0.0244$   
By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\beta_1$  = FEM : Goma'a and Baluch  
 $\beta_2$  = RTR : Refined Theory (Reissner)  
 $\beta_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\beta_4$  = RTP : Refined Theory (Present)

$M_x = \beta pa^2$ ,  $\mu = 0.3$

Table 5.6 Coefficient  $\gamma$  for the Center  
Resultant Moment  $M_y$   
of a Uniformly Loaded Simple/Clamped Square Plate

$h/a$	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_4$
0.005	0.031950	0.033247	0.0331	0.033247
0.01	0.032372	0.033250	0.0330	0.033250
0.05	0.033628	0.033345	0.0334	0.033350
0.1	0.02631	0.033045	0.0321	0.033639
0.2	0.03089	0.034373	0.0295	0.034647
0.3	0.03652	0.035469	—	0.036160
0.4	0.04206	0.037119	0.0269	0.038228
0.5	0.04699	0.039583	—	0.040927
0.6	0.05121	0.042990	0.0322	0.044288
0.7	0.05484	0.047370	—	0.048312
0.8	0.05803	0.052712	0.0444	0.052987
0.9	0.06090	0.058996	—	0.058297
1.0	0.06357	0.066206	0.0623	0.064220

NOTE :  $\gamma = 0.0332$   
By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\gamma_1$  = FEM : Goma'a and Baluch  
 $\gamma_2$  = RTR : Refined Theory (Reissner)  
 $\gamma_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\gamma_4$  = RTP : Refined Theory (Present)

$M_y = \gamma pa^2$ ,  $\mu = 0.3$

Table 5.7 Coefficient  $\alpha$  for the Center Deflection  
of a Simple/Free Square Plate

$h/a$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$
0.005	0.013127	0.013095	0.01309	0.013094
0.010	0.013294	0.013098	0.01309	0.013097
0.050	0.013956	0.013174	0.01310	0.013169
0.1	0.013495	0.013407	0.01312	0.013397
0.2	0.014469	0.014328	0.01326	0.014299
0.3	0.016016	0.015859	-	0.015786
0.4	0.018163	0.017999	0.01352	0.017830
0.5	0.020913	0.020748	-	0.020406
0.6	0.024278	0.024105	0.01395	0.023487
0.7	0.028229	0.028072	-	0.027053
0.8	0.032819	0.032648	0.01457	0.031090
0.9	0.037981	0.037834	-	0.035588
1.0	0.043800	0.043629	0.01527	0.040542

NOTE :  $\alpha = 0.01377$

By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\alpha_1$  = FEM : Goma'a and Baluch

$\alpha_2$  = RTR : Refined Theory (Reissner)

$\alpha_3$  = RTB : Refined Theory (Voyiadjis and Baluch)

$\alpha_4$  = RTP : Refined Theory (Present)

$w = \alpha pa^4/D, \mu = 0.3$

Table 5.8 Coefficient  $\beta$  for the Center Resultant Moment  $M_x$  of a Uniformly Loaded Simple/Free Square Plate

$h/a$	$\beta_1$	$\beta_2$	$\beta_3$	$\beta_4$
0.005	0.12002	0.12274	0.1225	0.12255
0.01	0.12027	0.12294	0.1225	0.12255
0.05	0.12320	0.12465	0.1228	0.12260
0.1	0.12442	0.12246	0.1240	0.12275
0.2	0.12547	0.12287	0.1252	0.12332
0.3	0.12645	0.12411	—	0.12414
0.4	0.12765	0.12683	0.1270	0.12506
0.5	0.12901	0.13180	—	0.12601
0.6	0.13048	0.13980	0.1313	0.12704
0.7	0.13202	0.15165	—	0.12823
0.8	0.13364	0.16826	0.1386	0.12964
0.9	0.13534	0.19066	—	0.13134
1.0	0.13713	0.21999	0.1489	0.13338

NOTE :  $\beta = 0.1235$   
By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\beta_1$  = FEM : Goma'a and Baluch  
 $\beta_2$  = RTR : Refined Theory (Reissner)  
 $\beta_3$  = RTB : Refined Theory (Voyiadjis and Baluch)  
 $\beta_4$  = RTP : Refined Theory (Present)

$M_x = \beta pa^2$ ,  $\mu = 0.3$

Table 5.9 Coefficient  $\gamma$  for the Center  
Resultant Moment  $M_y$   
of a Uniformly Loaded Simple/Free Square Plate

$h/a$	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_4$
0.005	0.026176	0.027227	0.0271	0.027080
0.01	0.026190	0.027376	0.0272	0.027081
0.05	0.026586	0.028660	0.0275	0.027115
0.1	0.026193	0.025831	0.0283	0.027222
0.2	0.024942	0.024414	0.0299	0.027644
0.3	0.023540	0.022757	-	0.028323
0.4	0.022057	0.021013	0.0324	0.029241
0.5	0.020622	0.019373	-	0.030409
0.6	0.019316	0.017927	0.0358	0.031861
0.7	0.018163	0.016687	-	0.033635
0.8	0.017150	0.015628	0.0399	0.035764
0.9	0.016253	0.014707	-	0.038268
1.0	0.015445	0.013882	0.0478	0.041153

NOTE :  $\gamma = 0.0102$

By CPT : Classical Plate Theory (For All  $h/a$  Ratios)

$\gamma_1$  = FEM : Goma'a and Baluch

$\gamma_2$  = RTR : Refined Theory (Reissner)

$\gamma_3$  = RTB : Refined Theory (Voyiadjis and Baluch)

$\gamma_4$  = RTP : Refined Theory (Present)

$M_y = \gamma pa^2$ ,  $\mu = 0.3$

Table 5.10 Coefficient  $\alpha$  for the Center Deflection of a Simply Supported Square Plate with a Line Load at  $x = a/2$

$h/a$	$\alpha_1$	$\alpha_2$
0.005	0.073601	0.073620
0.01	0.073601	0.073653
0.05	0.073601	0.074700
0.1	0.073601	0.077939
0.2	0.073601	0.090682
0.3	0.073601	0.11144
0.4	0.073601	0.14031
0.5	0.073601	0.17695
0.6	0.073601	0.22124
0.7	0.073601	0.13717
0.8	0.073601	0.33156
0.9	0.073601	0.39619
1.0	0.073601	0.46787
$\alpha_1 = \text{CPT : Classical Plate Theory}$ $\alpha_2 = \text{RTP : Refined Theory (Present)}$ $w = \alpha(pa^3/Eh^3), \mu = 0.3$		

Table 5.11 Coefficient  $\beta$  for the Center Resultant Moment  $M_x$  of a Simply Supported Square Plate with a Line Load at  $x = a/2$

$h/a$	$\beta_1$	$\beta_2$
0.005	0.127	0.12405
0.01	0.127	0.12405
0.05	0.127	0.12386
0.1	0.127	0.12378
0.2	0.127	0.12505
0.3	0.127	0.12758
0.4	0.127	0.13200
0.5	0.127	0.13737
0.6	0.127	0.14366
0.7	0.127	0.15071
0.8	0.127	0.15843
0.9	0.127	0.16630
1.0	0.127	0.17515
$\beta_1 =$ CPT : Classical Plate Theory $\beta_2 =$ RTP : Refined Theory (Present) $M_x = \beta pa, \mu = 0.3$		



Table 5.12 Coefficient  $\gamma$  for the center  
Resultant Moment  $M_y$  of a Simply Supported Square Plate  
With A Line Load at  $x = a/2$

$h/a$	$\gamma_1$	$\gamma_2$
0.005	0.092	0.091064
0.01	0.092	0.091129
0.05	0.092	0.093099
0.1	0.092	0.098766
0.2	0.092	0.11682
0.3	0.092	0.14017
0.4	0.092	0.16671
0.5	0.092	0.19565
0.6	0.092	0.22639
0.7	0.092	0.25854
0.8	0.092	0.29179
0.9	0.092	0.32599
1.0	0.092	0.36103
$\gamma_1$ = CPT : Classical Plate Theory $\gamma_2$ = RTP : Refined Theory (Present) $M_y = \gamma pa, \mu = 0.3$		

Table 5.13 Coefficient  $\alpha$  for the Center Deflection of a Simply Supported Square Plate with a Strip Load (Width = 0.2 a) Centered at  $x = a/2$

$h/a$	$\alpha_1$	$\alpha_2$
0.005	0.014368	0.014368
0.01	0.014368	0.014373
0.05	0.014368	0.014558
0.1	0.014368	0.015132
0.2	0.014368	0.017402
0.3	0.014368	0.021106
0.4	0.014368	0.026252
0.5	0.014368	0.032779
0.6	0.014368	0.040661
0.7	0.014368	0.049839
0.8	0.014368	0.060240
0.9	0.014368	0.071676
1.0	0.014368	0.084327
$\alpha_1 =$ CPT : Classical Plate Theory $\alpha_2 =$ RTP : Refined Theory (Present) $w = \alpha(P_0 a^4 / Eh^3), \mu = 0.3$		

Table 5.14 Coefficient  $\beta$  for the center  
Resultant Moment  $M_x$  of a Simply Supported Square Plate  
With A Strip Load (Width =  $0.2a$ ) Centered At  $x = a/2$

$h/a$	$\beta_1$	$\beta_2$
0.005	0.020914	0.020914
0.01	0.020914	0.020915
0.05	0.020914	0.020925
0.1	0.020914	0.020940
0.2	0.020914	0.020969
0.3	0.020914	0.020999
0.4	0.020914	0.021387
0.5	0.020914	0.021848
0.6	0.020914	0.022456
0.7	0.020914	0.023192
0.8	0.020914	0.024043
0.9	0.020914	0.024925
1.0	0.020914	0.025983
$\beta_1 =$ CPT : Classical Plate Theory $\beta_2 =$ RTP : Refined Theory (Present) $M_x = \beta P_0 a^2, \mu = 0.3$		

Table 5.15 Coefficient  $\beta$  for the center Resultant Moment  $M_y$  of a Simply Supported Square Plate With A Strip Load (Width = 0.2a) Centered At  $x = a/2$

$h/a$	$\beta_1$	$\beta_2$
0.005	0.016841	0.016841
0.01	0.016841	0.016843
0.05	0.016841	0.016904
0.1	0.016841	0.017087
0.2	0.016841	0.017807
0.3	0.016841	0.019017
0.4	0.016841	0.020633
0.5	0.016841	0.022608
0.6	0.016841	0.024875
0.7	0.016841	0.027378
0.8	0.016841	0.030075
0.9	0.016841	0.032941
1.0	0.016841	0.035958
$\beta_1 =$ CPT : Classical Plate Theory $\beta_2 =$ RTP : Refined Theory (Present) $M_x = \beta P_0 a^2, \mu = 0.3$		

**Table 5.16 Total Distributed Reaction R Along Edges Of  
A Uniformly Loaded Square Plate**

$h/a$	$\alpha_1$	$\alpha_2$	$\alpha_3$
0.005	-1.02	-1.02	-1.02
0.01	-1.02	-1.02	-1.02
0.05	-1.02	-1.02	-1.02
0.1	-1.02	-1.03	-1.03
0.2	-1.03	-1.03	-1.04
0.3	-1.07	-1.04	-1.05
0.4	-1.07	-1.05	-1.07
0.5	-1.08	-1.05	-1.08
0.6	-1.09	-1.05	-1.08
0.7	-1.09	-1.05	-1.08
0.8	-1.09	-1.05	-1.09
0.9	-1.11	-1.05	-1.09
1.0	-1.09	-1.06	-1.09
$\alpha_1 =$ SIMPLY SUPPORTED SQUARE PLATE. $\alpha_2 =$ SIMPLY SUPPORTED / CLAMPED SQUARE PLATE. $\alpha_3 =$ SIMPLY SUPPORTED / FREE SQUARE PLATE. $R = \alpha (P_0) , \mu = 0.3$			

## APPENDIX

## A-1 DERIVATION OF EQUATION (4-28) :

Equations (4-25) to (4-27), can be expressed in the form :

$$a_{11} \bar{w} + a_{12} \varphi_x + a_{13} \varphi_y = c_1 P \quad (\text{A-1})$$

$$a_{21} \bar{w} + a_{22} \varphi_x + a_{23} \varphi_y = c_2 P \quad (\text{A-2})$$

$$a_{31} \bar{w} + a_{32} \varphi_x + a_{33} \varphi_y = c_3 P \quad (\text{A-3})$$

Where :

$$a_{11} = a \frac{\partial}{\partial x} \Delta - S \frac{\partial}{\partial x} \quad (\text{A-4.1})$$

$$a_{12} = b \left( 2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - S \quad (\text{A-4.2})$$

$$a_{13} = b \frac{\partial^2}{\partial x \partial y} \quad (\text{A-4.3})$$

$$a_{21} = a \frac{\partial}{\partial y} \Delta - S \frac{\partial}{\partial y} \quad (\text{A-4.4})$$

$$a_{22} = b \frac{\partial^2}{\partial x \partial y} \quad (\text{A-4.5})$$

$$a_{23} = b \left( 2 \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) - S \quad (\text{A-4.6})$$

$$a_{31} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (\text{A-4.7})$$

$$a_{32} = \frac{\partial}{\partial x} \quad (\text{A-4.8})$$

$$a_{33} = \frac{\partial}{\partial y} \quad (\text{A-4.9})$$

$$c_1 = c \frac{\partial}{\partial x} \quad (\text{A-4.10})$$

$$c_2 = c \frac{\partial}{\partial y} \quad (\text{A-4.11})$$

$$c_3 = \frac{-1}{S} \quad (\text{A-4.12})$$

$$a = -D + \frac{h^3 F_1 S}{6} \quad (\text{A-4.13})$$

$$c = \mu \frac{h^3 F_1}{12(1 - \mu)} \quad (\text{A-4.14})$$

To obtain the governing differential equation for  $\bar{w}$ , we write :

$$\bar{w} = \frac{\begin{vmatrix} c_1 p & a_{11} & a_{13} \\ c_2 p & a_{22} & a_{23} \\ c_3 p & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

or :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \{\bar{w}\} = \begin{vmatrix} c_1 p & a_{11} & a_{13} \\ c_2 p & a_{22} & a_{23} \\ c_3 p & a_{32} & a_{33} \end{vmatrix} \{p\} \quad (A-5)$$

By expanding the operators determinants in equation (A-5), we get for this equation:

$$\{(2b^2 - ab)\Delta^3 + (aS - 2bS)\Delta^2\}\{\bar{w}\} = \{A\Delta^2 + B\Delta + C\}\{p\}$$

or :

$$M'\Delta^3\bar{w} + N'\Delta^2\bar{w} = A\Delta^2p + B\Delta p + Cp \quad (A-6)$$

Thus equation (4-28) is proved .



**A-2 DERIVATION OF THE FUNCTION  $Y_m(y)$  IN EQN. 4-37 :**

Substituting equation (4-36) in equation (4-32), we get :

$$\begin{aligned}
 & M' \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \\
 & \quad \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \{ \bar{w}_2 \} + \\
 & \quad N' \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \{ \bar{w}_2 \} = 0 \quad (A-7)
 \end{aligned}$$

OR :

$$\begin{aligned}
 & M' \left( \frac{\partial^6}{\partial x^6} + 3 \frac{\partial^6}{\partial x^4 \partial y^2} + 3 \frac{\partial^6}{\partial x^2 \partial y^4} + \frac{\partial^6}{\partial y^6} \right) \{ \bar{w}_2 \} + \\
 & \quad N' \left( \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) \{ \bar{w}_2 \} = 0
 \end{aligned}$$

OR :

$$\begin{aligned}
 & M' \left( -\alpha_m^6 Y_m + 3\alpha_m^4 Y_m(y) - 3\alpha_m^2 Y_m^{(iv)}(y) + Y_m^{(vi)}(y) \right) + \\
 & \quad N' \left( \alpha_m^4 Y_m - 2\alpha_m^2 Y_m(y) + Y_m^{(iv)}(y) \right) = 0
 \end{aligned}$$

Rearranging the above equation , we get :

$$\begin{aligned}
 & Y_m^{(vi)} - \left( 3\alpha_m^2 - \frac{N'}{M'} \right) Y_m^{(iv)} + \alpha_m^2 \left( 3\alpha_m^2 - 2 \frac{N'}{M'} \right) Y_m \\
 & - \alpha_m^4 \left( \alpha_m^2 - \frac{N'}{M'} \right) Y_m = 0 \quad (A-8)
 \end{aligned}$$

The characteristic equation for the above differential equation is :

$$r^6 - \left( 3\alpha_m^2 - \frac{N^1}{M^1} \right) r^4 + \alpha_m^2 \left( 3\alpha_m^2 - 2\frac{N^1}{M^1} \right) r^2 - \alpha_m^4 \left( \alpha_m^2 - \frac{N^1}{M^1} \right) = 0 \quad (\text{A-9})$$

A root for the above equation is:  $\pm \alpha_m$

Thus equation (A-9) can be rewritten as:

$$\left( r^2 - \alpha_m^2 \right) \left\{ \left( r^2 - \alpha_m^2 \right) \left( r^2 - \left( \alpha_m^2 - \frac{N^1}{M^1} \right) \right) \right\} = 0 \quad \text{From the}$$

above equation, the roots for equation (A-9) are :

$$\pm \alpha_m, \pm \alpha_m, \pm \sqrt{\alpha_m^2 - \frac{N^1}{M^1}}$$

OR :

$$\pm \alpha_m, \pm \alpha_m, \pm \gamma_m$$

where :

$$\gamma_m^2 = \alpha_m^2 - \frac{N^1}{M^1} \quad (\text{A-10})$$

Therefore we get for  $Y_m(y)$  :

$$Y_m(y) = c_1 e^{-\alpha_m y} + c_2 y e^{-\alpha_m y} + c_3 e^{\alpha_m y} + c_4 y e^{\alpha_m y} + c_5 e^{-\gamma_m y} + c_6 e^{\gamma_m y} \quad (\text{A-11})$$

Since :

$$\sinh(y) = \frac{e^y - e^{-y}}{2}$$

$$\cosh(y) = \frac{e^y + e^{-y}}{2} \text{ And :}$$

$$e^y = \sinh(y) + \cosh(y)$$

$$e^{-y} = \cosh(y) - \sinh(y)$$

Then equation (A-11) can be rewritten as :

$$\begin{aligned} Y_m(y) = & A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + C_m \sinh \alpha_m y \\ & + D_m \alpha_m y \cosh \alpha_m y + E_m \cosh \gamma_m y \\ & + F_m \sinh \gamma_m y \end{aligned} \quad (\text{A-13})$$

Thus equation 4-37 is proved .

### A-3 DERIVATION OF THE PARTICULAR SOLUTIONS FOR THE BENDING PROBLEM :

To get the particular solutions for this case , the dependent variables may be assumed to be of the form :

$$w_o = \sum w_{oo} \sin \alpha_m x$$

$$u_o = \sum u_{oo} \cos \alpha_m x$$

$$v_o = \sum v_{oo} \sin \alpha_m x$$

$$Q_x = \sum Q_{ox} \cos \alpha_m x$$

$$Q_y = \sum Q_{oy} \sin \alpha_m x$$

$$\phi_x = \sum \phi_{ox} \cos \alpha_m x$$

$$\phi_y = \sum \phi_{oy} \sin \alpha_m x$$

$$M_x = \sum M_{ox} \cos \alpha_m x$$

$$M_y = \sum M_{oy} \sin \alpha_m x$$

$$M_{xy} = \sum M_{oxy} \cos \alpha_m x$$

$$p = \sum M_{oxy} \cos \alpha_m x \tag{A-14}$$

Substituting equations (A-14) into equation (3-7), we get :

$$\frac{dQ_x}{dx} = -p$$

or :

$$Q_{ox} = \frac{P_m}{\alpha_m} \quad (A-15)$$

Let :

$$M_m = M_{ox} + M_{oy}$$

Then , from equations (3-27.1), (3-27.2), and (A-14), we get :

$$M_m = P_m \left[ \frac{1 + \mu}{\alpha_m^2} + \frac{\mu h^3 F_1}{12} \right] \quad (A-16)$$

From the governing equation for  $w_o$  (equation (3-35)) we get :

$$w_{oo} = \frac{P_m}{\alpha_m^4 D} \left( 1 + \frac{(2 - \mu) h^3 \alpha_m^2 F_1}{12(1 - \mu)} - \frac{\alpha_m^4 D}{N} \right. \\ \left. + \frac{\mu h^2 \alpha_m^2}{40(1 - \mu)} + \frac{\mu^2 h^5 \alpha_m^4 F_1}{480(1 - \mu^2)} \right) \quad (A-17)$$

From equation (3-27.4), we get for  $\phi_{ox}$

$$\phi_{ox} = \left\{ -w_m \alpha_m + \frac{1}{S} \frac{P_m}{\alpha_m} - \frac{1}{N} \alpha_m P_m \right. \\ \left. - \frac{1}{N} \alpha_m P_m + \frac{1}{R} \alpha_m M_m \right\} \quad (A-18)$$

From equation (3-27.5), we get for  $\phi_{oy}$

$$\phi_{oy} = 0 \quad (A-19)$$

From the equilibrium equation:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x$$

we get :

$$\frac{dM_x}{dx} = Q_x$$

From which and with equation (A-15) for  $Q_{ox}$  , we get for  $M_{ox}$  :

$$M_{ox} = \frac{P_m}{\alpha_m^2} \quad (A-20)$$

From equations (3-27.1), (A-18), and (A-20) , we get for  $\phi_{ox}$ :

$$\phi_{ox} = P_m \left[ \frac{\mu(1 + \mu)}{E\alpha_m} - \frac{1}{\alpha_m^3 D} \right] \quad (A-21)$$

Similarly by using equation (3-34), we get for  $Q_{oy}$  :

$$Q_{oy} = 0 \quad (A-22)$$

And from the equilibrium equation:

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y$$

we get :

$$M_{oxy} = 0$$

(A-23)

**A-4 PHYSICAL INTERPRETATION FOR THE AVERAGE DISPLACEMENTS  $\bar{w}$ ,  $\bar{u}$ ,  $\bar{v}$ , AND AVERAGE ROTATIONS  $\phi_x$  and  $\phi_y$  :**

For convenience in formulation and analysis, average displacements  $\bar{w}$ ,  $\bar{u}$ ,  $\bar{v}$ , and average rotations  $\phi_x$  and  $\phi_y$  are introduced. This is similar to introducing moment stress resultants which are actually average stresses :

$$\{\text{Exact Stresses : } \sigma_x, \sigma_y, \dots$$

$$\{\text{Average Stresses : } M_x, M_y, \dots$$

Similarly :

$$\{\text{Exact Displacements : } u, v, w$$

$$\{\text{Average Displacements : } \bar{u}, \bar{v}, \text{ and } \bar{w}$$

The average displacement  $\bar{u}$  is defined as follows :

$$\bar{u} = \frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} u \, dz \quad (\text{A-24})$$

And similarly :



$$\bar{v} = \frac{1}{h} \int_{-\frac{h}{2}}^{+\frac{h}{2}} v \, dz \quad (\text{A-25})$$

Equating work of the transverse shear stress  $\tau_{xz}$  due to displacement  $w$  to the work of the transverse shear resultant  $Q_x$  due to average displacement  $\bar{w}$ , one has :

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \tau_{xz} w \, dz = Q_x \bar{w} \quad (\text{A-26})$$

On substituting for  $\tau_{xz}$  and  $w$  from equations (3.3) and (3.15), respectively yields for the  $\bar{w}$  the expression :

$$\bar{w} = w_o + \frac{p}{N} - \frac{M}{R} \quad (\text{A-27})$$

The same result would be obtained if one were to use the work of  $\tau_{yz}$  stresses.

Defining the average rotations of sections  $x = \text{constant}$ ,  $y = \text{constant}$  by  $\psi_x$  and  $\psi_y$ , respectively, one may equate the work of the resultant couple on the average rotation to the work of the corresponding stresses  $\sigma_x$ ,  $\sigma_y$ , on the displacements  $u$  and  $v$  and expressed as :

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x u \, dz = M_x \psi_x \quad (\text{A-28})$$

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y v \, dz = M_y \psi_y \quad (\text{A-29})$$

The stress expressions to be used for  $\sigma_x$ ,  $\sigma_y$  are the initial linear

$$\text{variations } (\sigma_x = \frac{12M_x}{h^3} z, \quad \sigma_y = \frac{12M_y}{h^3} z)$$

On substituting the linear form of  $\sigma_x$ , and  $u$  into equation (A-28) and integrating the results, an expression for  $\psi_x$  is obtained as :

$$\psi_x = - \frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N} \frac{\partial p}{\partial x} + \frac{1}{R} \frac{\partial M}{\partial x} \quad (\text{A-30})$$

Similarly an expression for  $\psi_y$  is obtained as :

$$\psi_y = - \frac{\partial w_o}{\partial y} + \frac{Q_y}{S} - \frac{1}{N} \frac{\partial p}{\partial y} + \frac{1}{R} \frac{\partial M}{\partial y} \quad (\text{A-31})$$

On comparison of equations (A-30) and (A-3.27.4), one notes that

$$\psi_x = \phi_x,$$

i.e. :

$\phi_x$  is the rotation of a vertical element  $x = \text{constant}$  of the plate .

Also on comparison of equations (A-31) and (3.27.5), one notes that

$$\psi_y = \phi_y ,$$

i.e. :

$\phi_y$  is the rotation of a vertical element  $y = \text{constant}$  of the plate .

## A-5 PROGRAM LISTING

## A-5.1 PROGRAM DISS2 LISTING :

C		DIS00010
C		DIS00020
C		DIS00030
C		DIS00040
C		DIS00050
C		DIS00060
C		DIS00070
C	*****	DIS00080
C		DIS00090
C	PROGRAM FOR THE ANALYSIS OF THICK PLATE BENDING PROBLEMS	DIS00100
C	USING LEVY METHOD	DIS00110
C		DIS00120
C	PROGRAM WRITTEN BY : AMMAR KHALEEL HAFEZ MOHAMMED	DIS00130
C	IN DHAHRAN , SAUDI ARABIA.	DIS00140
C		DIS00150
C	*****	DIS00160
C		DIS00170
	IMPLICIT REAL*8 (A-H,O-Z)	DIS00180
	DOUBLE PRECISION NU,KPD,K4,K5	DIS00190
	DATA NU,0.30, BAR/1.00,	DIS00200
	. MTERM/25, IBOUND/1, IPLANE/1, ISTRES/4, IPLOT/2, IDEF/2,	DIS00210
	. IPRINT/2, NPLATE/ 1, MPLATE/13, IZMAX/11,	DIS00220
	. X/0.50, Y/0.00, Z1/0.50, UU/0.200, ILOAD/1/	DIS00230
C	*****	DIS00240
18	FORMAT('*****	DIS00250
	*****')	DIS00260
	GO TO (170,171) IPLANE	DIS00270
170	WRITE(6,175) IPLANE	DIS00280
175	FORMAT('IPLANE = ',I2,2X,' : EDGE AT Y = +-B/2 IS NOT ALLOWED TO STRET	DIS00290
	.CH IN THE Y-DIRECTION )	DIS00300
	GO TO 177	DIS00310
171	WRITE(6,176) IPLANE	DIS00320
176	FORMAT('IPLANE = ',I2,2X,' : EDGE AT Y = +-B/2 IS ALLOWED TO STRETCH	DIS00330
	. IN THE Y-DIRECTION )	DIS00340
177	CONTINUE	DIS00350
	GO TO (70,71,72) IBOUND	DIS00360
70	WRITE(6,73) IBOUND	DIS00370
	GO TO 76	DIS00380
71	WRITE(6,74) IBOUND	DIS00390
	GO TO 76	DIS00400
72	WRITE(6,75) IBOUND	DIS00410
73	FORMAT('IBOUND = ',I2,2X,' : PLATE SIMPLY SUPPORTED AT Y = +, - B/2)	DIS00420
74	FORMAT('IBOUND = ',I2,2X,' : PLATE CLAMPED AT Y = +, - B/2)	DIS00430
75	FORMAT('IBOUND = ',I2,2X,' : PLATE FREE AT Y = +, - B/2)	

76 CONTINUE	DIS00440
GO TO (400,401,501) ILOAD	DIS00450
	DIS00460
400 WRITE(6,402)	DIS00470
GO TO 404	DIS00480
402 FORMAT(' LOAD : UNIFORM LOAD ')	DIS00490
401 WRITE(6,403) ZI	DIS00500
GO TO 404	DIS00510
403 FORMAT(' LOAD : LINE LOAD APPLIED AT ZI = ',F8.2)	DIS00520
501 WRITE(6,503) UU,ZI	DIS00530
503 FORMAT('LOAD : STRIP LOAD ,WIDTH = ',F8.3,',CENTERED AT ZI = ',F8.3)	DIS00540
404 WRITE(6,188) NU	DIS00550
188 FORMAT('NU = ',F6.3)	DIS00560
WRITE(6,101) BAR	DIS00570
WRITE(6,122) MTERM	DIS00580
122 FORMAT('M = 1,3,5,,,,',I2)	DIS00590
101 FORMAT('B/A = ',F10.2)	DIS00600
C PI=22.0/7.0	DIS00610
PI=-1.00	DIS00620
PI=DARCOS(PI)	DIS00630
GO TO (490,491,491) IDEF	DIS00640
490 CONTINUE	DIS00650
WRITE(6,141)	DIS00660
WRITE(6,492) X,Y	DIS00670
WRITE(6,141)	DIS00680
492 FORMAT('DEFLECTIONS,X-M,Y-MOM : ARE EVALUATED AT X = ',F8.2,2X,	DIS00690
,Y = ',F8.2)	DIS00700
C	DIS00710
GO TO 435	DIS00720
491 CONTINUE	DIS00730
GO TO (370,371,373,373,435) ISTRES	DIS00740
370 CONTINUE	DIS00750
WRITE(6,141)	DIS00760
WRITE(6,183) X,Y	DIS00770
183 FORMAT('NOTE : SIGMAX,SIGMAY,& SIGMAZ ARE EVALUATED AT (' ,F4.1,'A,	DIS00780
,'F4.1,'B,Z) ' )	DIS00790
WRITE(6,141)	DIS00800
C WRITE(6,331)	DIS00810
331 FORMAT('SIGMAX ',6X,'SIGMAY ',4X,'SIGMAZ ')	DIS00820
GO TO 435	DIS00830
371 WRITE(6,141)	DIS00840
WRITE(6,180)	DIS00850
WRITE(6,181)	DIS00860
WRITE(6,182)	DIS00870
180 FORMAT('NOTE : SIGMAXY IS EVALUATED AT (0 ,B/2,Z)')	DIS00880
181 FORMAT('NOTE : SIGMAXZ IS EVALUATED AT (0 , 0 ,Z)')	DIS00890
182 FORMAT('NOTE : SIGMAYZ IS EVALUATED AT (A/2,B/2,Z)')	DIS00900
WRITE(6,141)	DIS00910
WRITE(6,372)	DIS00920
372 FORMAT('SIGMAXY ',6X,'SIGMAXZ ',4X,'SIGMAYZ ')	DIS00930

GO TO 435	DIS00940
373 CONTINUE	DIS00950
WRITE(6,141)	DIS00960
WRITE(6,437) X,Y	DIS00970
C WRITE(6,497)	DIS00980
C WRITE(6,466) X,Y	DIS00990
C WRITE(6,479)	DIS01000
WRITE(5,141)	DIS01010
497 FORMAT(7X,'Z',8X,'SIGZ-B',5X,'SIGZ-P')	DIS01020
C497 FORMAT(7X,'Z',8X,'SIGXZR',5X,'SIGXZB',7X,'SIGXZP')	DIS01030
C497 FORMAT(7X,'H/A',8X,'XSHERR',5X,'XSHERB',7X,'XSHERP')	DIS01040
479 FORMAT(7X,'HAR',5X,'XYMOMR',7X,'XYMOMP')	DIS01050
C479 FORMAT(5X,'H/A',6X,'TOTAL INPLANE FORCE NY')	DIS01060
C WRITE(6,440)	DIS01070
C437 FORMAT('NOTE :SIGMA-X IS EVALUATED BY DIFFERENT REFINED THEORIES')	DIS01080
C437 FORMAT('NOTE : SIGY , SIGXY , SIGYZ : ARE EVLUATED AT FREE END:	DIS01090
437 FORMAT('NOTE : STRESSES ARE EVALUATED AT X = ',F8.2,2X,'Y = ',F8.2)	DIS01100
466 FORMAT('NOTE : A CHECK FOR TOTAL LOAD ON PLATE')	DIS01110
C437 FORMAT('NOTE : NU IS EVLUATED AT X=0.5*A,Y=0.0')	DIS01120
C440 FORMAT('Z ',10X,'NU REISSNER',6X,'NU PRESENT')	DIS01130
C437 FORMAT('NOTE : W(X,Y,Z) IS EVLUATED AT X=0.5*A,Y=0.0')	DIS01140
C440 FORMAT(5X,'X ',10X,'W REISSNER',6X,'W PRESENT', ' AT Z/H = 0.0')	DIS01150
C440 FORMAT('Z ',10X,'W REISSNER',6X,'W PRESENT')	DIS01160
435 CONTINUE	DIS01170
C*****	DIS01180
DO 300 IPLATE=NPLATE,MPLATE	DIS01190
C*****	DIS01200
IF(IPLATE.GT.4) GO TO 134	DIS01210
GO TO (130,131,132,133) IPLATE	DIS01220
130 HAR=0.005	DIS01230
GO TO 136	DIS01240
131 HAR=0.010	DIS01250
GO TO 136	DIS01260
132 HAR=0.050	DIS01270
GO TO 136	DIS01280
133 HAR=0.100	DIS01290
GO TO 136	DIS01300
C134 HAR=0.200*(IPLATE-4)	DIS01310
134 HAR=0.100*(IPLATE-3)	DIS01320
IF(IPLATE.GT.13) GO TO 184	DIS01330
GO TO 136	DIS01340
184 HAR= IPLATE-12.0	DIS01350
136 CONTINUE	DIS01360
WRITE(6,141)	DIS01370
WRITE(6,367) HAR	DIS01380
367 FORMAT('H/A = ',F6.3)	DIS01390
WRITE(6,477)	DIS01400
C477 FORMAT(7X,'H/A',8X,'YSHERR',5X,'YSHERB',7X,'YSHERP')	DIS01410
C477 FORMAT(7X,'Z',8X,'SIGXYR',5X,'SIGXYB',7X,'SIGXYP')	DIS01420
477 FORMAT(7X,'Z',9X,'SIGXR',6X,'SIGXB',7X,'SIGXP')	DIS01430

C477	FORMAT(7X,'Z',5X,'SIGMA-X(B)',5X,'SIGMA-X(P)')	DIS01440
141	FORMAT('*****')	DIS01450
	WRITE(6,141)	DIS01460
	Z = - 0.600000	DIS01470
C	*****	DIS01480
	DO 250 IZ=1,IZMAX	DIS01490
C	*****	DIS01500
	Z = Z + 0.100000	DIS01510
C	*****	DIS01520
	DO 200 IBALCH = 1,2	DIS01530
C	*****	DIS01540
	WBAR = 0.0	DIS01550
	WBARE = 0.0	DIS01560
	WBARR = 0.0	DIS01570
	WBARRE = 0.0	DIS01580
	XMOM = 0.0	DIS01590
	YMOM = 0.0	DIS01600
	XYMOM = 0.0	DIS01610
	XSHER = 0.0	DIS01620
	YSHER = 0.0	DIS01630
	WR = 0.0	DIS01640
	WP = 0.0	DIS01650
	EPSXP = 0.0	DIS01660
	EPSYP = 0.0	DIS01670
	EPSZP = 0.0	DIS01680
	EPSXR = 0.0	DIS01690
	EPSYR = 0.0	DIS01700
	EPSZR = 0.0	DIS01710
	APLOAD = 0.0	DIS01720
C		DIS01730
	XMOMR = 0.0	DIS01740
	YMOMR = 0.0	DIS01750
	XYMOMR = 0.0	DIS01760
	VXR = 0.0	DIS01770
	VYR = 0.0	DIS01780
	W0 = 0.0	DIS01790
	XMPYM = 0.0	DIS01800
C		DIS01810
	GO TO (340,341) IBALCH	DIS01820
340	XSTB = 0.0	DIS01830
	YSTB = 0.0	DIS01840
	ZSTB = 0.0	DIS01850
	XYSTB = 0.0	DIS01860
	XZSTB = 0.0	DIS01870
	YZSTB = 0.0	DIS01880
	GO TO 342	DIS01890
C		DIS01900
341	XSTP = 0.0	DIS01910
	YSTP = 0.0	DIS01920
		DIS01930

7STP=0.0	DIS01940
XYSTP=0.0	DIS01950
X7STP=0.0	DIS01960
YZSTP=0.0	DIS01970
YNYP=0.0	DIS01980
C	DIS01990
342 XSTR=0.0	DIS02000
YSTR=0.0	DIS02010
ZSTR=0.0	DIS02020
KYSTR=0.0	DIS02030
KZSTR=0.0	DIS02040
YZSTR=0.0	DIS02050
XLOADR=0.0	DIS02060
XLOADP=0.0	DIS02070
C*****	DIS02080
DO 100 M=1,MTERM,2	DIS02090
C*****	DIS02100
IF(HAR.LT.0.10)GO TO 222	DIS02110
ITHIICK=2	DIS02120
GO TO 223	DIS02130
222 ITHIICK=1	DIS02140
223 CONTINUE	DIS02150
GO TO (112,113,113) IPRINT	DIS02160
112 WRITE(6,18)	DIS02170
WRITE(6,17) M	DIS02180
17 FORMAT(' M = ',I2)	DIS02190
113 CONTINUE	DIS02200
GO TO (150,151) IBALCH	DIS02210
150 F1=6./5.	DIS02220
F2=-1./2.	DIS02230
F4=-1./48.	DIS02240
F3=39./1120.	DIS02250
GO TO 152	DIS02260
151 CONTINUE	DIS02270
CALL DISS(M,NU,HAR,ALPHA,A1,A2,A3,A4,A5,F1,F2,F3,F4,	DIS02280
Z,F1Z,F1ZP,F2Z,F3Z)	DIS02290
152 CONTINUE	DIS02300
CALL POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,IHAR2,	DIS02310
HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,	DIS02320
X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)	DIS02330
CALL BENDNG(IBOUND,ITHIICK,M,NU,HAR,AP,APB,GAMB,KPD,UU,	DIS02340
BAR,BETA,BETAP,A,B,EE,IPRINT,F1,X,Y,ZI,ILOAD)	DIS02350
CALL FORCES(IBOUND,ITHIICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,	DIS02360
BETA,BETAP,A,B,EE,WPAR,WPAE,XM,YM,IPRINT,X,Y,	DIS02370
ZI,UU,ILOAD,XYM,QX,QY)	DIS02380
CALL REISS(M,IBOUND,ITHIICK,NU,HAR,BAR,AP,APB,GAMB,F1,	DIS02390
WPARR,XMR,YMR,XYMR,WPARRE,VX,VY,SIGXR,SIGYR,UU,	DIS02400
SIGZR,SIGXYR,SIGXZR,SIGYZR,X,Y,Z,ZI,ILOAD,F1ZR)	DIS02410
CALL XPLANE(M,IPLANE,IBOUND,NU,HAR,BAR,AP,APB,C1,C2,UP,	DIS02420
XK4,X,Y,ZI,UU,ILOAD,F1,F2)	DIS02430



CALL	STRESS(IBOUND,ITHICK,M,IJAR,BAR,NU,AP,APB,GAMB,KPD,F1,	DIS02440
	F2,F3,F4,F1Z,F2Z,F3Z,BETA,BETAP,A,B,EE,C1,C2,	DIS02450
	UP,XK4,SIGX,SIGY,SIGZ,SIGXY,SIGXZ,SIGYZ,IBALCHI,	DIS02460
	X,Y,Z,ZI,UU,ILOAD,FIZP,QX,QY,YNV)	DIS02470
	WBAR = WBAR + WPAR	DIS02480
	WBARE = WBARE + WPARE	DIS02490
	WBARR = WBARR + WPARR	DIS02500
	WBARRE = WBARRE + WPARRE	DIS02510
	XMOM = XMOM + XM	DIS02520
	YMOM = YMOM + YM	DIS02530
	XYMOM = XYMOM + XYM	DIS02540
	XSHER = XSHER + QX	DIS02550
	YSHER = YSHER + QY	DIS02560
C		DIS02570
C		DIS02580
	APLOAD = APLOAD + PM*DSIN(APX)	DIS02590
C	WRITE(6,330) HAR,PM,APLOAD	DIS02600
C		DIS02610
C		DIS02620
	YNYP = YNYP + YNY	DIS02630
C		DIS02640
	XMOMR = XMOMR + XMR	DIS02650
	YMOMR = YMOMR + YMR	DIS02660
	XYMOMR = XYMOMR + XYMR	DIS02670
	VXR = VXR + VX	DIS02680
	VYR = VYR + VY	DIS02690
	XNU2 = 12.*(1.-NU**2.)	DIS02700
C		DIS02710
	GO TO (35,36,40)IBOUND	DIS02720
35	ALFAI = WBAR	DIS02730
	ALFAIR = WBARR	DIS02740
	GO TO 37	DIS02750
36	ALFAI = WBAR/XNU2	DIS02760
	ALFAIR = WBARR/XNU2	DIS02770
	GO TO 37	DIS02780
40	ALFAI = WBAR/XNU2	DIS02790
	ALFAIE = WBARE/XNU2	DIS02800
C40	ALFAI = WBAR	DIS02810
C	ALFAIE = WBARE	DIS02820
	ALFAIR = WBARR/XNU2	DIS02830
	ALFARE = WBARRE/XNU2	DIS02840
37	BETA1 = XMOM	DIS02850
	GAMA1 = YMOM	DIS02860
	BETAIR = XMOMR	DIS02870
	GAMAIR = YMOMR	DIS02880
C		DIS02890
	GO TO (114,115,115) IPRINT	DIS02900
114	WRITE(6,125) ALFAI,BETA1,GAMA1	DIS02910
125	FORMAT('ALFAI =',E12.5,3X,'BETA1 =',E12.5,3X,'GAMA1 =',E12.5)	DIS02920
C		DIS02930

C NOTE :	DIS02940
C PNR = P/N	DIS02950
C RMR = M/R	DIS02960
C WHERE :	DIS02970
C M = M + M	DIS02980
C X Y	DIS02990
C	DIS03000
115 XMPYM = XM + YM	DIS03010
K4 = 4.*F3*HAR**4./AP	DIS03020
K5 = 3.*NU/10.*XMPYM*HAR**2.	DIS03030
GO TO (30,31,31)IBOUND	DIS03040
31 K4 = K4/XNU2	DIS03050
K5 = K5/XNU2	DIS03060
C WPAR = WPAR/XNU2	DIS03070
30 PNR = K4	DIS03080
RMR = K5	DIS03090
W0 = W0 + WPAR/XNU2-PNR + RMR	DIS03100
GO TO (116,117,117) IPRINT	DIS03110
116 CONTINUE	DIS03120
WRITE(6,102) W0	DIS03130
102 FORMAT('W0 =',E12.5)	DIS03140
117 CONTINUE	DIS03150
C*****	DIS03160
C ==> IPLANE : IS AN INDICATOR WHETHER THE EDGE AT (X,+ -B/2)	DIS03170
C IS OR NOT ALLOWED TO STRETCH IN THE Y-DIRECTION .	DIS03180
C IF :	DIS03190
C IPLANE = 1 ==> EDGE IS NOT ALLOWED TO STRETCH IN THE Y-DIRECTION .	DIS03200
C IPLANE = 2 ==> EDGE IS ALLOWED TO STRETCH IN THE Y-DIRECTION .	DIS03210
C	DIS03220
C ==> NOTE : IPRINT : INDICATOR WHETHER TO PRINT INTERMEDIATE	DIS03230
C RESULTS FOR FORCES & DEFLECTION OR NOT	DIS03240
C IPRINT = 1 PRINT INTERMEDIATE RESULTS .	DIS03250
C IPRINT = 2 DO NOT PRINT INTERMEDIATE RESULTS .	DIS03260
C IPRINT = 3 DO NOT PRINT FINAL RESULTS .	DIS03270
C	DIS03280
C ==> NOTE : IDEF : INDICATOR WHETHER TO PRINT INTERMEDIATE	DIS03290
C RESULTS FOR DEFLECTION OR NOT	DIS03300
C IDEF = 1 PRINT INTERMEDIATE RESULTS .	DIS03310
C IDEF = 2 DO NOT PRINT INTERMEDIATE RESULTS .	DIS03320
C	DIS03330
C	DIS03340
C ==> NOTE : ISTRES : INDICATOR WHETHER TO PRINT INTERMEDIATE	DIS03350
C RESULTS FOR STRESSES OR NOT	DIS03360
C ISTRES = 1 PRINT INTERMEDIATE RESULTS .	DIS03370
C ISTRES = 2 DO NOT PRINT INTERMEDIATE RESULTS .	DIS03380
C	DIS03390
C ==> NOTE : IPLOT : INDICATOR WHETHER TO PRINT RESULTS	DIS03400
C FOR PLOTTING PURPOSES OR NOT	DIS03410
C IPLOT = 1 PRINT RESULTS .	DIS03420
C IPLOT = 2 DO NOT PRINT RESULTS .	DIS03430

```

C.....
XSTR = XSTR + SIGXR
YSTR = YSTR + SIGYR
ZSTR = ZSTR + SIGZR
XYSTR = XYSTR + SIGXYR
XZSTR = XZSTR + SIGXZR
YZSTR = YZSTR + SIGYZR
C
F2R = -.1/4.*{ 2.*Z - 3.*Z**2 + 2.*Z**4 }
F3R = 39./1120.
XMPYMR = XMR + YMR
WR = WR + PM*HAR4*(F2R-F3R)*DSIN(APX)
      + 3.*NU*HAR2*XMPYMR*(1./10.-2.*Z**2) + WPARR
C
EPSX = SIGXR -NU*( SIGYR + SIGZR )
EPSY = SIGYR -NU*( SIGXR + SIGZR )
EPSZ = PM*F1ZR*DSIN(APX) - 12.*NU*XMPYMR/IIAR2*Z
EPSXR = EPSXR + EPSX
EPSYR = EPSYR + EPSY
EPSZR = EPSZR + EPSZ
C
GO TO (332,333) IBALCH
332 XSTB = XSTB + SIGX
YSTB = YSTB + SIGY
ZSTB = ZSTB + SIGZ
XYSTB = XYSTB + SIGXY
XZSTB = XZSTB + SIGXZ
YZSTB = YZSTB + SIGYZ
GO TO 190
333 XSTP = XSTP + SIGX
YSTP = YSTP + SIGY
ZSTP = ZSTP + SIGZ
XYSTP = XYSTP + SIGXY
XZSTP = XZSTP + SIGXZ
YZSTP = YZSTP + SIGYZ
C
XMPYMP = XM + YM
WP = WP + PM*HAR4*(F2Z-F3)*DSIN(APX)
      + 3.*NU*HAR2*XMPYMP*(1./10.-2.*Z**2) + WPAR
C
EPSX = SIGX -NU*( SIGY + SIGZ )
EPSY = SIGY -NU*( SIGX + SIGZ )
EPSZ = PM*F1Z*DSIN(APX) - 12.*NU*XMPYMP/IIAR2*Z
EPSXP = EPSXP + EPSX
EPSYP = EPSYP + EPSY
EPSZP = EPSZP + EPSZ
XK22 = (1.-NU)/(1. + NU)
GO TO (441,442) ITHICK
441 EFSIN = 0.0
GO TO 443

```

DIS03440  
DIS03450  
DIS03460  
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DIS03900  
DIS03910  
DIS03920  
DIS03930



C*****	DIS04440
XSTR = XSTR / HAR2	DIS04450
YSTR = YSTR / HAR2	DIS04460
ZSTR = ZSTR / HAR2	DIS04470
XYSTR = XYSTR / HAR2	DIS04480
XZSTR = XZSTR / HAR	DIS04490
YZSTR = YZSTR / HAR	DIS04500
C	DIS04510
XSTB = XSTB / HAR2	DIS04520
YSTB = YSTB / HAR2	DIS04530
ZSTB = ZSTB / HAR2	DIS04540
XYSTB = XYSTB / HAR2	DIS04550
XZSTB = XZSTB / HAR	DIS04560
YZSTB = YZSTB / HAR	DIS04570
C	DIS04580
XSTP = XSTP / HAR2	DIS04590
YSTP = YSTP / HAR2	DIS04600
ZSTP = ZSTP / HAR2	DIS04610
XYSTP = XYSTP / HAR2	DIS04620
XZSTP = XZSTP / HAR	DIS04630
YZSTP = YZSTP / HAR	DIS04640
C	DIS04650
GO TO (514,515) IPLOT	DIS04660
514 CONTINUE	DIS04670
C WRITE(6,330) XYSTR,XYSTB,XYSTP	DIS04680
C WRITE(6,530) Z,XSTR,XSTB,XSTP	DIS04690
WRITE(6,530) Z,YSTR,YSTB,YSTP	DIS04700
C WRITE(6,530) Z,ZSTR,ZSTB,ZSTP	DIS04710
GO TO 439	DIS04720
530 FORMAT(8(F10.2,2X))	DIS04730
515 CONTINUE	DIS04740
GO TO (360,361,439,438) ISTRES	DIS04750
360 CONTINUE	DIS04760
WRITE(6,335)	DIS04770
WRITE(6,325) Z	DIS04780
WRITE(6,335)	DIS04790
WRITE(6,330) XSTR,YSTR,ZSTR	DIS04800
WRITE(6,330) XSTB,YSTB,ZSTB	DIS04810
WRITE(6,330) XSTP,YSTP,ZSTP	DIS04820
GO TO 439	DIS04830
330 FORMAT(6(F12.5,2X))	DIS04840
361 CONTINUE	DIS04850
WRITE(6,335)	DIS04860
WRITE(6,325) Z	DIS04870
WRITE(6,335)	DIS04880
WRITE(6,330) XYSTR,XZSTR,YZSTR	DIS04890
WRITE(6,330) XYSTB,XZSTB,YZSTB	DIS04900
WRITE(6,330) XYSTP,XZSTP,YZSTP	DIS04910
335 FORMAT('*****')	DIS04920
325 FORMAT('Z/II = ',F8.5)	DIS04930

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GO TO 439
438 CONTINUE
C WRITE(6,335)
C WRITE(6,325) Z
C WRITE(6,335)
C WRITE(6,330) HAR,YNYP
C WRITE(6,330) HAR,XYMOMR,XYMOMP
C WRITE(6,530) Z,ZSTB,ZSTP
C WRITE(6,530) Z,XYSTR,XYSTB,XYSTP
C WRITE(6,530) Z,YSTR,YSTB,YSTP
  WRITE(6,330) Z,XSTR,XSTB,XSTP
C WRITE(6,330) YSTR,XYSTR,YZSTR
C WRITE(6,330) YSTB,XYSTB,YZSTB
C WRITE(6,330) YSTP,XYSTP,YZSTP
C WRITE(6,330) HAR,VXR,XSHERB,XSHERP
C WRITE(6,330) HAR,APLOAD
C WRITE(6,478) HAR,XLOADP
478 FORMAT('H/A = ',F8.4,2X,'TOTAL REACTION ALONG EDGES OF PLATE = ',
.F8.2)
C WRITE(6,330) Z,WBARR,WP
C XNUR = DABS(EPXSR/EPXZR)
C XNUP = DABS(EPXSP/EPXZP)
C WRITE(6,530) Z,XNUR,XNUP
439 CONTINUE
C*****
250 CONTINUE
C*****
300 CONTINUE
C*****
  WRITE(6,18)
  STOP
  END
C
C*****
C*** END OF MAIN PROGRAM ***
C*****
C*****
C*****
C
C*** SUBROUTINE * XPLANE * TO FIND SOLUTION OF THE IN-PLANE PROBLEM
C *****
C IE : TO DETERMINE THE CONSTANTS C1 AND C2 IN THE EXPRESSION
C
C FOR THE INPLANE DISPLACEMENTS UBAR & VBAR .
C
SUBROUTINE XPLANE(M,IPLANE,IBOUND,NU,IHAR,IBAR,AP,APB,C1,C2,UP,
  XK4,X,Y,ZI,UU,ILOAD,F1,F2)
  IMPLICIT REAL*8(A-H,O-Z)
  DOUBLE PRECISION NU
DIS04940
DIS04950
DIS04960
DIS04970
DIS04980
DIS04990
DIS05000
DIS05010
DIS05020
DIS05030
DIS05040
DIS05050
DIS05060
DIS05070
DIS05080
DIS05090
DIS05100
DIS05110
DIS05120
DIS05130
DIS05140
DIS05150
DIS05160
DIS05170
DIS05180
DIS05190
DIS05200
DIS05210
DIS05220
DIS05230
DIS05240
DIS05250
DIS05260
DIS05270
DIS05280
DIS05290
DIS05300
DIS05310
DIS05320
DIS05330
DIS05340
DIS05350
DIS05360
DIS05370
DIS05380
DIS05390
DIS05400
DIS05410
DIS05420
DIS05430

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C*****
CALL    POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,IAR2,
.       IAR3,HAR4,HAR5,IAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
.       X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C*****
UP = PM*NU*(1. + NU)*F2/AP
XK1 = (1.-NU)/2.
XK2 = (1. + NU)/2.
XK4 = (1. + XK1)/XK2
XK7 = -1./DSINH(APB)*( APB-XK4*DSINH(APB) )
XK8 = XK7*DCOSH(APB) + APB*DSINH(APB)
GO TO (1,2) IPLANE
1  C2 = -UP/XK8
   C1 = XK7*C2
   GO TO 3
2  A11 = DCOSH(APB)
   A12 = APB*DSINH(APB)
   A21 = (1.-NU)*DCOSH(APB)
   A22 = (1.-XK4)*DCOSH(APB) + (1.-NU)*APB*DSINH(APB)
   R1 = -UP
   R2 = -(1.-NU)*UP
   C1 = (A22*R1-A12*R2)/(A11*A22-A12*A21)
   C2 = (A11*R2-A21*R1)/(A11*A22-A12*A21)
3  RETURN
   END
C*****
C*****
C*****
C
C*** SUBROUTINE * STRESS * TO EVALUATE:
C      *****
C          THE STRESSES SIGX,SIGY,SIGZ,SIGXY,SIGXZ,&
C          SIGYZ
C      AT A SPECIFIED POINT(X,Y) IN THE PLATE AND
C      ACCORDING TO THE SPECIFIED BOUNDARY CONDITIONS
C
SUBROUTINE STRESS(BOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
.       F2,F3,F4,F1Z,F2Z,F3Z,BETA,BETAP,A,B,EE,C1,C2,
.       UP,XK4,SIGX,SIGY,SIGZ,SIGXY,SIGXZ,SIGYZ,IBALCH,
.       X,Y,Z,ZI,UU,ILOAD,F1ZP,QX,QY,YN)
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION  NU,KPD
C*****
CALL    POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,IAR2,
.       IAR3,HAR4,HAR5,IAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
.       X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C*****
XNU2 = (1.-NU**2)
XNU1 = NU/12./(1.-NU)
XNUP1 = NU + 1.

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DIS05440  
DIS05450  
DIS05460  
DIS05470  
DIS05480  
DIS05490  
DIS05500  
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DIS05700  
DIS05710  
DIS05720  
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DIS05900  
DIS05910  
DIS05920  
DIS05930

	XNUM1 = NU-1.	DIS05940
	XK1 = 6.*(1.-NU)/F1/HAR2	DIS05950
	XK22 = (1.-NU)/(1.+NU)	DIS05960
C		DIS05970
	APY1 = APY	DIS05980
	APX1 = APX	DIS05990
	GAMY1 = GAMY	DIS06000
	XK7 = -NU*NU*F1/6./XNUM1	DIS06010
	XK8 = -NU/12./XNUM1	DIS06020
	GO TO (333,334) ITHICK	DIS06030
333	EEBAR1 = 0.0	DIS06040
	EESIN = 0.0	DIS06050
	EECOS = 0.0	DIS06060
	GO TO 335	DIS06070
334	EEBAR1 = EE*DSINH(GAMY)	DIS06080
	EESIN = EE*DSINH(GAMY)	DIS06090
	EECOS = EE*DCOSH(GAMY)	DIS06100
		DIS06110
C		DIS06120
335	CONTINUE	DIS06130
	GO TO (20,21) IBALCH	DIS06140
20	G1 = Z/4. - 5.*Z**3/3.	DIS06150
	G2 = -3./10.*Z + 2.*Z**3	DIS06160
	G3 = 5./4.*Z - 5./3.*Z**3	DIS06170
	G4 = - 1./48. - Z*(-336.*NU**2-195.*NU + 195.)/5600./XNUM1	DIS06180
	. + Z**2/4. - Z**3*(8.*NU**2 + 5.*NU-5.)/20./XNUM1	DIS06190
	. + Z**5/10.	DIS06200
	F1Z = -1./4.*(2. - 6.*Z + 8.*Z**3)	DIS06210
	F1ZP = 3./2.*(1. - (2.*Z)**2)	DIS06220
C	WRITE(6,50) G1,G2,G3,G4	DIS06230
	GO TO 22	DIS06240
21	G1 = (F1Z - F2)/F1 - Z	DIS06250
	G2 = 2.*Z**3 - .30*Z	DIS06260
	G3 = (F1Z - F2)/F1	DIS06270
	G4 = -F3Z + F3*Z + F4	DIS06280
C		DIS06290
22	CONTINUE	DIS06300
	DWDX2 = - A*AP2*DCOSH(APY) - B*AP2*APY*DSINH(APY)	DIS06310
	. - EECOS*AP2 - AP2*BETA	DIS06320
	DWDY2 = A*AP2*DCOSH(APY) + B*(AP2*APY*DSINH(APY) + 2.*AP2*DCOSH(APY))	DIS06330
	. + EECOS*GAMA2/HAR2	DIS06340
	DWDXY = A*AP2*DSINH(APY) + B*(AP2*APY*DCOSH(APY) + AP2*DSINH(APY))	DIS06350
	. + EEBAR1*AP*DSQRT(GAMA2)/HAR	DIS06360
	XI1 = DWDX2 + NU*DWDY2	DIS06370
	XJ1 = NU*DWDX2 + DWDY2	DIS06380
	XLI = DWDXY	DIS06390
		DIS06400
C		DIS06410
	DFIX3 = -A*AP4*HAR2*DCOSH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06420
	. *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY) )	DIS06430
	. + EECOS*XK22*AP4*HAR2 + AP4*HAR2*BETA	DIS06440



DFIXY2 = A*AP4*HAR2*DCOSH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06440
. *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)	DIS06450
. + 2.*AP4*HAR2*DCOSH(APY) )	DIS06460
. - EECOS*XK22*AP2*GAMA2	DIS06470
DFIY3 = -A*AP4*HAR2*DCOSH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06480
. *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)	DIS06490
. + 4.*AP4*HAR2*DCOSH(APY) )	DIS06500
. + EECOS*XK22*GAMA2*GAMA2/HAR2	DIS06510
DFIYX2 = A*AP4*HAR2*DCOSH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06520
. *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)	DIS06530
. + 2.*AP4*HAR2*DCOSH(APY) )	DIS06540
. - EECOS*XK22*AP2*GAMA2	DIS06550
XI2 = DFIX3 + DFIYX2 + NU*DFIXY2 + NU*DFIY3	DIS06560
XJ2 = NU*DFIX3 + NU*DFIYX2 - DFIXY2 + DFIY3	DIS06570
C	DIS06580
DFIX2Y = A*AP4*HAR2*DSINH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06590
. *AP4*HAR2*DSINH(APY) + APY*AP4*HAR2*DCOSH(APY)	DIS06600
. + AP4*HAR2*DSINH(APY) )	DIS06610
. - EEBAR1*XK22*AP3*HAR*DSQRT(GAMA2)	DIS06620
DFIY2X = -A*AP4*HAR2*DSINH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1	DIS06630
. *AP4*HAR2*DSINH(APY) + APY*AP4*HAR2*DCOSH(APY)	DIS06640
. + 3.*AP4*HAR2*DSINH(APY) )	DIS06650
. + EEBAR1*XK22*AP/HAR*GAMA2*DSQRT(GAMA2)	DIS06660
XL2 = DFIX2Y + DFIY2X	DIS06670
C	DIS06680
DFIXDX = A*AP2*DCOSH(APY) - B*(-2.*F1*AP4*HAR2/6./XNUM1*DCOSH(APY)	DIS06690
. + APY*AP2*DSINH(APY) )	DIS06700
. - EECOS*XK22*AP2 - AP2*BETAP	DIS06710
DFIYDY = -A*AP2*DCOSH(APY) - B*(-2.*F1*AP4*HAR2/6./XNUM1*DCOSH(APY)	DIS06720
. + APY*AP2*DSINH(APY) + 2.*AP2*DCOSH(APY) )	DIS06730
. + EECOS*XK22*GAMA2/HAR2	DIS06740
XI3 = DFIXDX + NU*DFIYDY	DIS06750
XJ3 = NU*DFIXDX + DFIYDY	DIS06760
C	DIS06770
DFIXDY = -A*AP2*DSINH(APY) - B*(-2.*F1*AP4*HAR2/6./XNUM1*DSINH(APY)	DIS06780
. + APY*AP2*DCOSH(APY) + AP2*DSINH(APY) )	DIS06790
. + EEBAR1*XK22*AP*DSQRT(GAMA2)/HAR	DIS06800
DFIYDX = -A*AP2*DSINH(APY) - B*(-2.*F1*AP4*HAR2/6./XNUM1*DSINH(APY)	DIS06810
. + APY*AP2*DCOSH(APY) + AP2*DSINH(APY) )	DIS06820
. + EEBAR1*XK22*AP*DSQRT(GAMA2)/HAR	DIS06830
XL3 = DFIXDY + DFIYDX	DIS06840
C	DIS06850
XI4 = -AP2*HAR4*PM	DIS06860
XJ4 = NU*XI4	DIS06870
XL4 = 0.0	DIS06880
C	DIS06890
DUDX = -C1*AP*DCOSH(APY) - C2*APY*AP*DSINH(APY) - AP*UP	DIS06900
DVDY = C1*AP*DCOSH(APY)	DIS06910
. + C2*( APY*AP*DSINH(APY) + (1.-XK4)*AP*DCOSH(APY) )	DIS06920
XI5 = DUDX + NU*DVDY	DIS06930

	XJ5 = NU*DUDX + DVDY	DIS06940
	XJSBAR = XJ5	DIS06950
	XI5 = XI5*HAR2	DIS06960
	XJ5 = XJ5*HAR2	DIS06970
C		DIS06980
	DUDY = C1*AP*DSINH(APY)	DIS06990
	+ C2*( APY*AP*DCOSH(APY) + AP*DSINH(APY) )	DIS07000
	DVDX = C1*AP*DSINH(APY)	DIS07010
	+ C2*( APY*AP*DCOSH(APY) - XK4*AP*DSINH(APY) )	DIS07020
	XL5 = DUDY + DVDX	DIS07030
	XL5 = XL5*HAR2/2	DIS07040
C		DIS07050
	XI6 = -NU*PM*F1Z*HAR2/XNUM1	DIS07060
	XJ6 = XI6	DIS07070
	XL6 = XL4	DIS07080
C		DIS07090
	XI7 = PM*NU**2*AP2*F1*HAR4/6./XNUM1	DIS07100
	XJ7 = NU*X17	DIS07110
C		DIS07120
83	CONTINUE	DIS07130
	GO TO (24,25) IBALCH	DIS07140
24	X17 = 0.0	DIS07150
	XJ7 = 0.0	DIS07160
25	CONTINUE	DIS07170
	SIGX = 1./XNU2*( XI1*G1 + XK8*XI2*G2 + XI3*G3 + XI4*G4	DIS07180
	+ XI7*G2 + XI5 ) + X16	DIS07190
C		DIS07200
	SIGY = 1./XNU2*( XJ1*G1 + XK8*XJ2*G2 + XJ3*G3 + XJ4*G4	DIS07210
	+ XJ7*G2 + XJ5 ) + XJ6	DIS07220
C		DIS07230
	SIGXY = 1./XNUP1*( XL1*G1 + XK8*XL2*G2 + XL3*G3/2.	DIS07240
	+ XL4*G4 + XL6*G2 + XL5 )	DIS07250
C		DIS07260
C	YNY = C1*(1.-NU)*AP*DCOSH(APY) + C2*( (1.-XK4)*AP*DCOSH(APY)	DIS07270
C	+ (1.-NU)*APY*AP*DSINH(APY) )	DIS07280
C	+ (1.-NU)*AP*UP	DIS07290
	YNY = 1./XNU2*XJSBAR - NU/XNUM1*PM*F2	DIS07300
C		DIS07310
	SIGX = SIGX*DSIN(APX)	DIS07320
	SIGY = SIGY*DSIN(APX)	DIS07330
	SIGZ = HAR2*PM*F1Z*DSIN(APX)	DIS07340
	SIGXY = SIGXY*DCOS(APX)	DIS07350
	SIGXZ = QX*F1ZP	DIS07360
	SIGYZ = QY*F1ZP	DIS07370
	YNY = YNY*HAR*DSIN(APX)	DIS07380
	RETURN	DIS07390
	END	DIS07400
C		DIS07410
C		DIS07420
C		DIS07430

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C
C*** SUBROUTINE REISS TO FIND SOLUTION OF THE PROBLEM
C      USING REISSNER'S SHEAR DEFORMATION
C      THEORY.
C
SUBROUTINE REISS(M,IBOUND,ITHIICK,NU,HAR,BAR,AP,APB,GAMB,F1,
.      WPARR,XMR,YMR,XYMR,WPARRE,VX,VY,SIGXR,SIGYR,UU,
.      SIGZR,SIGXYR,SIGXZR,SIGYZR,X,Y,Z,ZI,LOAD,F1ZR)
IMPLICIT REAL*8(A-H,O-Z)
DOUBLE PRECISION NU
XK=(2.-NU)/(1.-NU)
C*****
C      CALL    POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
.      HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
.      X,Y,APX,APY,GAMY,F1,PM,LOAD,ZI,UU)
C*****
C
C== => EVALUATE THE CONSTANTS : C4,C5,C6
C FOR THE VARIOUS B.C.'S .
C
GAM2=AP2*HAR2+10.
GAMY=Y*BAR/HAR*DSQRT(GAM2)
XK=(2.-NU)/(1.-NU)
C
APY1=APY
APX1=APX
GAMY1=GAMY
APY2=0.0
APX2=0.0
GAMY2=0.0
GO TO (2,3,4) IBOUND
C
C *-----*
C*** SIMPLY SUPPORTED PLATE AT Y = +, - B/2 ***
C *-----*
C
2 CONTINUE
C4=0.0
C4SH1=0.0
C4SH2=0.0
C4SH3=0.0
C6=1./2./DCOSH(APB)
C5=-1./DCOSH(APB)*(1.+XK*HAR2*AP2/10.+APB*DTANH(APB)/2.)
GO TO 5
C *-----*
C*** CLAMPED PLATE AT Y = +, - B/2 ***
C *-----*
3 R1=-2.*HAR3*AP3/5./DSQRT(GAM2)/(1.-NU)*DCOSH(APB)
. *DTANH(GAMB)-APB*DSINH(APB)*DTANH(APB)+APB*DCOSH(APB)
. +(1.+2.*HAR2*AP2/5./(1.-NU))*DSINH(APB)

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DIS07440  
DIS07450  
DIS07460  
DIS07470  
DIS07480  
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DIS07660  
DIS07670  
DIS07680  
DIS07690  
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DIS07810  
DIS07820  
DIS07830  
DIS07840  
DIS07850  
DIS07860  
DIS07870  
DIS07880  
DIS07890  
DIS07900  
DIS07910  
DIS07920  
DIS07930

R2 = -HAR3*AP3/5./DSQRT(GAM2)/(1.-NU)*DTANH(GAMB)	DIS07940
. + DTANH(APB)*(1.+XK*HAR2*AP2/10.)	DIS07950
C6 = R2,R1	DIS07960
C5 = -1./DCOSH(APB)*(1.+XK*HAR2*AP2/10.+APB*DSINH(APB)*C6)	DIS07970
C4SH2 = 4./AP2*(2.*C6*DCOSH(APB)-1.)	DIS07980
GO TO (6,7) ITHICK	DIS07990
6 C4 = 0.0	DIS08000
C4SH1 = 0.0	DIS08010
C4SH3 = 0.0	DIS08020
GO TO 5	DIS08030
7 C4 = 4./5.*HAR2*AP/DCOSH(GAMB)*(2.*C6*DCOSH(APB)-1.)	DIS08040
C4SH1 = 4./AP2*HAR/DSQRT(GAM2)/DCOSH(GAMB)*DSINH(GAMY)	DIS08050
. *(2.*C6*DCOSH(APB)-1.)	DIS08060
C4SH3 = 4./AP*HAR*DSQRT(GAM2)*(2.*C6*DCOSH(APB)-1.)*DCOSH(APY)	DIS08070
. /DCOSH(APB)	DIS08080
GO TO 5	DIS08090
C * _____*	DIS08100
C*** FREE PLATE AT Y = +, - B/2 ***	DIS08110
C * _____*	DIS08120
4 R5 = DSQRT(HAR2*AP2+10.)	DIS08130
R4 = 1./HAR*AP*R5	DIS08140
R3 = 2.*HAR2*AP2/5.*(1.-R4*DTANH(APB)/DTANH(GAMB))+3.+NU	DIS08150
. -2.*APB*(1.-NU)/DSINH(2.*APB)	DIS08160
C6 = NU*(GAM2)/10./R3/DCOSH(APB)	DIS08170
C5 = C6*(1.-NU)*(1.+NU-(1.-NU)*APB/DTANH(APB))	DIS08180
C	DIS08190
C	DIS08200
GO TO (8,9) ITHICK	DIS08210
8 C4 = 0.0	DIS08220
C C4SH1 = 8./AP3*DSINH(APB)*C6	DIS08230
C4SH1 = 0.0	DIS08240
C4SH2 = 0.0	DIS08250
C C4SH3 = 0.0	DIS08260
C4SH3 = 8.*(DSINH(APY)*C6/AP2)	DIS08270
GO TO 5	DIS08280
9 C4 = 8./5.*HAR*AP2*R5*DSINH(APB)/DSINH(GAMB)*C6	DIS08290
C4SH1 = 8./AP3/DSINH(GAMB)*DSINH(APB)*C6*DSINH(GAMY)	DIS08300
C4SH2 = 8./HAR*AP*DSQRT(GAM2)/AP2/DSINH(GAMB)*DSINH(APB)*C6	DIS08310
. *DCOSH(GAMY)	DIS08320
C C4SH3 = 8./AP2*DSINH(GAMB)*DSINH(APB)*DSINH(GAMY)*C6	DIS08330
C4SH3 = 8.*(DSINH(APY)*C6/AP2)	DIS08340
C	DIS08350
C	DIS08360
5 APD2 = AP/2.	DIS08370
GO TO (60,61) ITHICK	DIS08380
60 C4COS = 0.0	DIS08390
GO TO (100,100,101) IBOUND	DIS08400
101 C4COS = 8./5.*HAR/AP2*R5*DSINH(APB)/DTANH(GAMB)*C6	DIS08410
100 C4SIN = 0.0	DIS08420
GO TO 62	DIS08430

61	C4COS = C4*DCOSH(GAMY)	DIS08440
	C4SIN = C4*DSINH(GAMY)	DIS08450
62	CONTINUE	DIS08460
C		DIS08470
	WPARR = 48.*(1.-NU**2)*(1./AP5*(C5*DCOSH(APY) + C6*APY*DSINH(APY) + 1.)	DIS08480
	+ XK*HAR2/AP3/10.)	DIS08490
C		DIS08500
	WPARRE = 48.*(1.-NU**2)*(1./AP5*(C5*DCOSH(APB) + C6*APB*DSINH(APB)	DIS08510
	+ 1.) + XK*HAR2/AP3/10.)	DIS08520
C		DIS08530
	XMR = C6*8./AP3*(HAR2*AP2/5.-NU)*DCOSH(APY)	DIS08540
	+ 4./AP3*(1.-NU)*C6*APY*DSINH(APY)	DIS08550
	+ 4./AP3*(1.-NU)*C5*DCOSH(APY)	DIS08560
	+ C4COS + 4./AP3*(HAR2*AP2*NU/10./(1.-NU) + 1.)	DIS08570
	- PM*NU*HAR2/10./(1.-NU)	DIS08580
C		DIS08590
	YMR = -C6*8./AP3*(HAR2*AP2/5.+1.)*DCOSH(APY)	DIS08600
	- 4./AP3*(1.-NU)*C6*APY*DSINH(APY)	DIS08610
	- 4./AP3*(1.-NU)*C5*DCOSH(APY)	DIS08620
	+ C4COS + 4.*NU/AP3*(HAR2*AP2*XK/10.+1.)	DIS08630
	- PM*NU*HAR2/10./(1.-NU)	DIS08640
C		DIS08650
	XYMR = -C6*4./AP3*(1.-NU)*APB*DCOSH(APY) - 4./AP3*(1.-NU)*(C5	DIS08660
	+ C6)*DSINH(APY) + C4SH1	DIS08670
C		DIS08680
	VX = -4.*(2.*DCOSH(APY)*C6-1.)/AP2 + C4SH2	DIS08690
C		DIS08700
	VY = -8.*(DSINH(APY1)*C6/AP2) + C4SH3	DIS08710
C		DIS08720
C		DIS08730
	WPARR = WPARR*DSIN(APX)	DIS08740
	WPARRE = WPARRE*DSIN(APX)	DIS08750
	XMR = XMR*DSIN(APX)	DIS08760
	YMR = YMR*DSIN(APX)	DIS08770
	XYMR = XYMR*DCOS(APX)	DIS08780
	VX = VX*DCOS(APX)	DIS08790
	VY = VY*DSIN(APX)	DIS08800
C		DIS08810
	FIZR = -1./4.*(2.-3.*(2.*Z) + 8.*Z**3)	DIS08820
C		DIS08830
	SIGXR = 12.*XMR*Z	DIS08840
	SIGYR = 12.*YMR*Z	DIS08850
	SIGZR = HAR2*FIZR*PM	DIS08860
	SIGZR = SIGZR*DSIN(APX)	DIS08870
	SIGXYR = 12.*XYMR*Z	DIS08880
	SIGXZR = 3./2.*VX*(1.-4.*Z**2)	DIS08890
	SIGYZR = 3./2.*VY*(1.-4.*Z**2)	DIS08900
C		DIS08910
	RETURN	DIS08920
	END	DIS08930

```

C_____ DIS08940
C_____ DIS08950
C_____ DIS08960
C DIS08970
C DIS08980
C DIS08990
C*** SUBROUTINE BOUND TO EVALUATE THE COEFFICIENT MATRIX DIS09000
C ACCORDING TO THE SPECIFIED BOUNDARY DIS09010
C CONDITIONS DIS09020
C IBOUND : IS AN INDICATOR TO TELL WHAT BOUNDARY CONDITION DIS09030
C ,FOR THE EDGE AT Y = +, - B/2. , IS DIS09040
C BEING CONSIDERED AS FOLLOWS : DIS09050
C IBOUND = 1 == == > INDICATES SIMPLY SUPPORTED EDGE DIS09060
C IBOUND = 2 == == > INDICATES CLAMPED EDGE DIS09070
C IBOUND = 3 == == > INDICATES FREE EDGE DIS09080
C DIS09090
C SUBROUTINE BOUND(M,IBOUND,ITHICK,NU,IHAR,BAR,AP,APB,GAMB,F1, DIS09100
C PK,BETA,BETAP,AMAT,RH,IFPR,FIX,X,Y) DIS09110
C IMPLICIT REAL*8(A-H,O-Z) DIS09120
C DOUBLE PRECISION NU DIS09130
C DIMENSION AMAT(3,3),RH(3),IFPR(3),FIX(3) DIS09140
C***** DIS09150
C CALL POWERS(M,IHAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,IHAR2, DIS09160
C IHAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2, DIS09170
C X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU) DIS09180
C***** DIS09190
C AHR=1./IHAR DIS09200
C XK1=6.*(1.-NU)/F1/HAR2 DIS09210
C XNU2=12.*(1.-NU**2.) DIS09220
C XK22=(1.-NU)/(1.+NU) DIS09230
C GO TO (2,3,4) IBOUND DIS09240
C *_____ DIS09250
C*** SIMPLY SUPPORTED PLATE AT Y = +, - B/2 *** DIS09260
C *_____ DIS09270
C DIS09280
C == == > WBAR(X,+ -B/2)=0.0 DIS09290
C DIS09300
C 2 AMAT(1,1)=DCOSH(APB) DIS09310
C AMAT(1,2)=APB*DSINH(APB) DIS09320
C AMAT(1,3)=1./DTANH(GAMB) DIS09330
C DIS09340
C == == > MY(X,+ -B/2)=0.0 DIS09350
C DIS09360
C AMAT(2,1)=AP2*DCOSH(APB) DIS09370
C AMAT(2,2)=2.*AP2*DCOSH(APB) DIS09380
C + AP2*APB*DSINH(APB) DIS09390
C AMAT(2,3)=+(AP2*IHAR2+12./F1)/DTANH(GAMB) DIS09400
C DIS09410
C == == > QX(X,+ -B/2)=0.0 DIS09420
C DIS09430
C == == NOTE : THE ABOVE B.C. COMES FROM THE B.C.:

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C          PHIX(X,+ -B/2)=DW/DX + QX/S          DIS09440
C          AND SINCE W(X,+ -B/2) = 0.0 THEN      DIS09450
C          DW/DX = 0.0 == => QX(X,+ -B/2)/S = 0.0 DIS09460
C          OR SIMPLY : QX(X,+ -B/2) = 0.0        DIS09470
C                                                  DIS09480
C          AMAT(3,1)=0.0                          DIS09490
C          AMAT(3,2)= 2.*AP3*DCOSH(APB)           DIS09500
C          AMAT(3,3)=-2.*XK1*AP/(1.+NU)/DTANH(GAMB) DIS09510
C                                                  DIS09520
C          RII(1)=-BETA                           DIS09530
C          RII(2)= PK*(1.-2./NU)                  DIS09540
C          RII(3)= + XK1*AP*(BETA + BETAP)        DIS09550
C          GO TO 11                               DIS09560
C *-----*                                     DIS09570
C*** CLAMPED PLATE AT Y = +, - B/2 ***          DIS09580
C *-----*                                     DIS09590
C                                                  DIS09600
C == => WBAR(X,+ -B/2)=0.0                     DIS09610
C                                                  DIS09620
C          3 AMAT(1,1)=DCOSH(APB)                 DIS09630
C          AMAT(1,2)= APB*DSINH(APB)              DIS09640
C          AMAT(1,3)= 1./DTANH(GAMB)              DIS09650
C                                                  DIS09660
C == => PHIY(X,+ -B/2)=0.0                     DIS09670
C                                                  DIS09680
C          AMAT(2,1)= AP*DSINH(APB)               DIS09690
C          AMAT(2,2)=(F1*AP3*HAR2/3./(1.-NU)+ AP)*DSINII(APB) DIS09700
C          + APB*AP*DCOSH(APB)                   DIS09710
C          AMAT(2,3)=-{1.-NU}/(1.+NU)/HAR*DSQRT(AP2*HAR2+ 12./F1) DIS09720
C                                                  DIS09730
C == => DQY/DY + P = 0.0 ; AT Y = +- B/2.      DIS09740
C                                                  DIS09750
C                                                  DIS09760
C == == == NOTE : THE ABOVE B.C. COMES FROM THE EQUILIBRIUM EQN. DIS09770
C          DQX/DX + DQY/DY + P = 0.0             DIS09780
C          SINCE FROM THE B.C. :                 DIS09790
C          PHIX(X,+ -B/2)=DW/DX + QX/S          DIS09800
C          AND SINCE W(X,+ -B/2) = 0.0 THEN      DIS09810
C          DW/DX = 0.0 == => QX(X,+ -B/2)/S = 0.0 DIS09820
C          AND ALSO : DQX/DX = 0.0               DIS09830
C                                                  DIS09840
C          AMAT(3,1)=0.0                          DIS09850
C          AMAT(3,2)= 2.*AP4*DCOSH(APB)           DIS09860
C          AMAT(3,3)=-2.*XK1/(1.+NU)/HAR2*(AP2*HAR2+ 12./F1)/DTANHII(GAMB) DIS09870
C                                                  DIS09880
C          RII(1)=-BETA                           DIS09890
C          RII(2)=0.0                             DIS09900
C          RII(3)= + XNU2*4./AP                   DIS09910
C          GO TO 11                               DIS09920
C *-----*                                     DIS09930

```

```

C*** FREE PLATE AT Y = +, - B/2 ***
C *-----*
C
C == > MY(X, + -B/2) = 0.0
C
4 CONTINUE
  AMAT(1,1) = +(1.-NU)*AP2*DCOSH(APB)
  AMAT(1,2) = +(1.-NU)*APB*AP2*DSINH(APB)
  .          + F1*AP4*HAR2/3.*DCOSH(APB)
  .          + 2.*AP2*DCOSH(APB)
  AMAT(1,3) = -XK22*(GAMA2/HAR2-NU*AP2)/DTANH(GAMB)
C
C == > VY(X, + -B/2) = 0.0
C THE ABOVE EQN. IS OBTAINED FROM THE EQN. :
C VY = QY - DMXY/DX
C SINCE AT Y = + -B/2. :
C DMXY/DX = 0.0 & QY = 0.0
C*****
  AMAT(2,1) = (1.-NU)*AP3*DSINH(APB)
  AMAT(2,2) = -(1.+NU)*AP3*DSINH(APB)
  .          + (1.-NU)*APB*AP3*DCOSH(APB)
  AMAT(2,3) = + ((1.-NU)*AP2 + 12.*XK22/F1/HAR2)
  .          /HAR*DSQRT(GAMA2)
C*****
C
C == > QY(X, + -B/2) = 0.0
C THE ABOVE EQN. IS OBTAINED FROM THE EQN. :
C DMY/DY - DMXY/DX = QY
C NOTING THAT :
C 1) DMXY/DX = 0.0 ( SINCE MXY(X, + -B/2.) = 0.0 )
C 2) D2W/DXY = 0.0 ( SINCE DMXY = D2W/DXY = 0.0 )
C**** SEE CHAPTER 4 FOR MORE DETAILS ****
C*****
  AMAT(3,1) = (1.-NU)*AP3*DSINH(APB)
  AMAT(3,2) = -2.*AP3*DSINH(APB)
  .          + (1.-NU)*AP3*DSINH(APB)
  .          + (1.-NU)*APB*AP3*DCOSH(APB)
  AMAT(3,3) = -(1.-2.*XK22)*12./F1/HAR2/HAR*DSQRT(GAMA2)
  .          + ((1.-NU)*AP2 )
  .          /HAR*DSQRT(GAMA2)
C*****
C
C AMAT(3,1) = 0.0
C AMAT(3,2) = + 2.*AP3*DSINH(APB)
C AMAT(3,3) = + (1.-2.*XK22)*12./F1/HAR2/HAR*DSQRT(GAMA2)
C
C AMAT(3,1) = 0.0
C AMAT(3,2) = -2.*AP3*DSINH(APB)
C AMAT(3,3) = + 12.*XK22/F1/HAR2/HAR*DSQRT(GAMA2)
C

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DIS09940  
DIS09950  
DIS09960  
DIS09970  
DIS09980  
DIS09990  
DIS10000  
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DIS10020  
DIS10030  
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DIS10080  
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DIS10100  
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DIS10140  
DIS10150  
DIS10160  
DIS10170  
DIS10180  
DIS10190  
DIS10200  
DIS10210  
DIS10220  
DIS10230  
DIS10240  
DIS10250  
DIS10260  
DIS10270  
DIS10280  
DIS10290  
DIS10300  
DIS10310  
DIS10320  
DIS10330  
DIS10340  
DIS10350  
DIS10360  
DIS10370  
DIS10380  
DIS10390  
DIS10400  
DIS10410  
DIS10420  
DIS10430



RII(1) = -NU*AP2*BETAP + PK	DIS10440
RII(2) = 0.0	DIS10450
RII(3) = 0.0	DIS10460
GO TO (11,11) ITHICK	DIS10470
17 CONTINUE	DIS10480
IFPR(3) = 1	DIS10490
FIX(3) = 0.0	DIS10500
11 CONTINUE	DIS10510
RETURN	DIS10520
END	DIS10530
C	DIS10540
C_____	DIS10550
C_____	DIS10560
C_____	DIS10570
C	DIS10580
C*** SUBROUTINE POWERS TO EVALUATE THE POWERS OF : ALPHA , B/A , H/A	DIS10590
C	DIS10600
SUBROUTINE POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,IIAR2,	DIS10610
.          HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,	DIS10620
.          X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)	DIS10630
IMPLICIT REAL*8(A-H,O-Z)	DIS10640
PI = -1.00	DIS10650
PI = DARCOS(PI)	DIS10660
ALPHA = M*PI	DIS10670
AP = ALPHA	DIS10680
AP2 = AP**2.	DIS10690
AP3 = AP**3.	DIS10700
AP4 = AP**4.	DIS10710
AP5 = AP**5.	DIS10720
AP6 = AP**6.	DIS10730
HAR2 = HAR**2.	DIS10740
HAR3 = HAR**3.	DIS10750
HAR4 = HAR**4.	DIS10760
HAR5 = HAR**5.	DIS10770
BAR2 = BAR**2.	DIS10780
BAR3 = BAR**3.	DIS10790
BAR4 = BAR**4.	DIS10800
BAR5 = BAR**5.	DIS10810
GAMA2 = AP2*HAR2 + 12./F1	DIS10820
APX = AP*X	DIS10830
APY = AP*BAR*Y	DIS10840
GAMY = Y*BAR*DSQRT(GAMA2)/IIAR	DIS10850
GO TO (50,51,52) ILOAD	DIS10860
50 PM = 4./AP	DIS10870
GO TO 53	DIS10880
51 APZI = AP*ZI	DIS10890
PM = 2.*DSIN(APZI)	DIS10900
GO TO 53	DIS10910
52 APZI = AP*ZI	DIS10920
APU = AP*UU/2.	DIS10930

```

      PM = 4./AP*DSIN(APZI)*DSIN(APU)
53 CONTINUE
      RETURN
      END
C
C-----
C-----
C-----
C
C*** SUBROUTINE " BENDNG " TO EVALUATE THE CONSTANTS A(M),B(M), & E(M)
C      *****
C      ACCORDING TO THE SPECIFIED BOUNDARY CONDITIONS
C
      SUBROUTINE BENDNG(BOUND,ITHICK,M,NU,HAR,AP,APB,GAMB,KPD,UU,
      BAR,BETA,BETAP,A,B,EE,IPRINT,F1,X,Y,ZI,ILOAD)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION NU,K1,K2,K11,K12,K13,K14,
      KPD
      DIMENSION AMAT(3,3),SOLT(3),RH(3),IFPR(3),FIX(3),
      BMAT(3,3)
C
C*** P0 : IS THE VALUE OF THE UNIFORMLY DISTRIBUTED LOAD ON THE PLATE
C
C   P = 4.0*P0/(M*PI)
C-----
      CALL POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
      HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
      X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C-----
      XK1 = 6.*(1.-NU)/F1/HAR2
      XK22 = (1.-NU)/(1. + NU)
      XNU2 = 12.*(1.-NU**2.)
      APB = (AP/2.)*BAR
      AHR = 1./HAR
C
      GAMB = .5*BAR*AHR*DSQRT(GAMA2)
C
      K11 = (2.-NU)*(1 + NU)*AP4*HAR4*F1/144.
      K12 = (3.-2.*NU)*(1. + NU)*AP2*HAR2*F1/12.
      K13 = (1.-NU**2.)
      K14 = AP4*(AP2*HAR2*F1/12. + 1.0)
      K1 = 12.0*PM*(K11 + K12 + K13)/K14
      BETA = K1
      K2 = 2.*NU*(1. + NU)*F1**2.*HAR4/3./AP/(1.-NU)*AP/4.*PM
      BETAP = -(K1-K2)/(AP2*HAR2*F1/6./(1.-NU) + 1.0)
      PK = PM*NU*(1. + NU)*F1*HAR2
C
C-----{ NOTE: PK = K*P*H**2. }-----
C
      BETA = (AA*AP**4.-BB*AP**2. + CC)/(-MP*AP**6. + NP*AP**4.)*P

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DIS10940  
DIS10950  
DIS10960  
DIS10970  
DIS10980  
DIS10990  
DIS11000  
DIS11010  
DIS11020  
DIS11030  
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DIS11050  
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DIS11080  
DIS11090  
DIS11100  
DIS11110  
DIS11120  
DIS11130  
DIS11140  
DIS11150  
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DIS11190  
DIS11200  
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DIS11300  
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DIS11320  
DIS11330  
DIS11340  
DIS11350  
DIS11360  
DIS11370  
DIS11380  
DIS11390  
DIS11400  
DIS11410  
DIS11420  
DIS11430

C	BETAP = -AP*(BETA-KAP*P*D.S)/U2	DIS11440
C	GAMA = DSQRT(AP**2.-NP/MP)	DIS11450
C	*****	DIS11460
	GO TO (676,677,677) IPRINT	DIS11470
676	WRITE(6,101) ALPHA	DIS11480
	WRITE(6,110) GAMB	DIS11490
C	WRITE(6,309) K11,K12	DIS11500
C	WRITE(6,311) K13,K14	DIS11510
	WRITE(6,111) BETA	DIS11520
	WRITE(6,310) K2	DIS11530
	WRITE(6,112) BETAP	DIS11540
677	CONTINUE	DIS11550
101	FORMAT('ALPHA = ',E15.5)	DIS11560
C309	FORMAT('K11 = ',E10.5,2X,'K12 = ',E10.5)	DIS11570
C311	FORMAT('K13 = ',E10.5,2X,'K14 = ',E10.5)	DIS11580
310	FORMAT('K2 = ',E10.5)	DIS11590
110	FORMAT('GAMB = ',E15.5)	DIS11600
111	FORMAT('BETA = ',E15.5)	DIS11610
112	FORMAT('BETAP = ',E15.5)	DIS11620
C	*****	DIS11630
	N = 3	DIS11640
	NEQNS = N	DIS11650
	DO 64 I = 1,N	DIS11660
	RH(I) = 0.0	DIS11670
	IFPR(I) = 0	DIS11680
	FIX(I) = 0.0	DIS11690
	DO 64 J = 1,N	DIS11700
64	AMAT(I,J) = 0.0	DIS11710
C		DIS11720
	CALL BOUND(M,IBOUND,ITHICK,NU,IHAR,BAR,AP,APB,GAMB,FI,	DIS11730
	PK,BETA,BETAP,AMAT,RH,IFPR,FIX,X,Y)	DIS11740
C	*****	DIS11750
	DO 315 I = 1,N	DIS11760
	DO 315 J = 1,N	DIS11770
315	BMAT(I,J) = AMAT(I,J)	DIS11780
C	*****	DIS11790
C	WRITE(6,228) M	DIS11800
C228	FORMAT('M = ',I2,3X,'COEFFICIENT MATRIX BEFORE MODIFICATION')	DIS11810
C	DO 121 I = 1,N	DIS11820
C	WRITE(6,122) (AMAT(I,J),J = 1,N)	DIS11830
C121	CONTINUE	DIS11840
122	FORMAT(3(E12.5,2X))	DIS11850
C	DO 123 I = 1,N	DIS11860
C	WRITE(6,124) RH(I)	DIS11870
C123	CONTINUE	DIS11880
C	*****	DIS11890
C		DIS11900
	GO TO (672,673,673) IPRINT	DIS11910
672	WRITE(6,227) M	DIS11920
227	FORMAT('M = ',I2,3X,'COEFFICIENT MATRIX AFTER MODIFICATION')	DIS11930

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DO 723 I=1,N                                DIS11940
WRITE(6,122) (AMAT(I,J1),J1=1,N)           DIS11950
723 CONTINUE                                DIS11960
DO 226 I=1,N                                DIS11970
WRITE(6,124) RH(I)                          DIS11980
226 CONTINUE                                DIS11990
673 CONTINUE                                DIS12000
124 FORMAT(E12.5)                           DIS12010
C*****                                     DIS12020
CALL GREDU (NEQNS,AMAT,FIX,RH,IFPR)         DIS12030
CALL BAKSU (NEQNS,AMAT,FIX,RH,IFPR,SOLT)    DIS12040
C CALL GREDUC (NEQNS,AMAT,FIX,RH,IFPR)      DIS12050
C CALL BAKSUB (NEQNS,AMAT,FIX,RH,IFPR,SOLT) DIS12060
C*****                                     DIS12070
C CALL JORDAN(NEQNS,AMAT,RH,SOLT)           DIS12080
C CALL DLSARG(N,AMAT,N,RH,I,SOLT)           DIS12090
C CALL DLSLRG(N,AMAT,N,RH,I,SOLT)           DIS12100
C*****                                     DIS12110
GO TO (674,675,675) IPRINT                  DIS12120
674 WRITE(6,229) M                           DIS12130
229 FORMAT('M = ',I2,3X,'COEFFICIENT MATRIX AFTER SOLUTION') DIS12140
DO 224 I=1,N                                DIS12150
WRITE(6,122) (AMAT(I,J1),J1=1,N)           DIS12160
224 CONTINUE                                DIS12170
DO 525 I=1,N                                DIS12180
WRITE(6,124) RH(I)                          DIS12190
525 CONTINUE                                DIS12200
675 CONTINUE                                DIS12210
C*****                                     DIS12220
A = SOLT(1)                                  DIS12230
B = SOLT(2)                                  DIS12240
EE = SOLT(3)                                  DIS12250
C*****                                     DIS12260
GO TO (205,206,206) IPRINT                   DIS12270
205 RII1 = AMAT(1,1)*A + AMAT(1,2)*B + AMAT(1,3)*EE DIS12280
RII2 = AMAT(2,1)*A + AMAT(2,2)*B + AMAT(2,3)*EE DIS12290
RII3 = AMAT(3,1)*A + AMAT(3,2)*B + AMAT(3,3)*EE DIS12300
WRITE(6,316) RII1,RII2,RII3                 DIS12310
316 FORMAT('RII1 = ',E12.5,2X,'RII2 = ',E12.5,2X,'RII3 = ',E12.5) DIS12320
206 CONTINUE                                DIS12330
C*****                                     DIS12340
GO TO (665,203) ITHICK                       DIS12350
665 CONTINUE                                DIS12360
FE = 0.0                                     DIS12370
GO TO 204                                     DIS12380
203 CONTINUE                                DIS12390
EE = EE/DSINII(GAMB)                         DIS12400
204 CONTINUE                                DIS12410
C*****                                     DIS12420
C BETAP = BETAP*AP2                          DIS12430

```

KPD = PK	DIS12440
GO TO (678,679,679) IPRINT	DIS12450
678 WRITE(6,27) A,B,EE	DIS12460
WRITE(6,312) KPD	DIS12470
WRITE(6,112) BETAP	DIS12480
27 FORMAT('A =',E12.4,2X,'B =',E12.4,2X,'EE =',E12.4)	DIS12490
312 FORMAT('KPD =',E10.5)	DIS12500
C	DIS12510
679 CONTINUE	DIS12520
RETURN	DIS12530
END	DIS12540
C	DIS12550
C_____	DIS12560
C_____	DIS12570
C_____	DIS12580
C	DIS12590
C*** SUBROUTINE ' FORCES ' TO EVALUATE:	DIS12600
C	DIS12610
C	DIS12620
C	DIS12630
C	DIS12640
C	DIS12650
C	DIS12660
C	DIS12670
C	DIS12680
C	DIS12690
C	DIS12700
C	DIS12710
C	DIS12720
C	DIS12730
C	DIS12740
C	DIS12750
C	DIS12760
C	DIS12770
C	DIS12780
C	DIS12790
C	DIS12800
C	DIS12810
C	DIS12820
C	DIS12830
C	DIS12840
C	DIS12850
C	DIS12860
C	DIS12870
C	DIS12880
C	DIS12890
C	DIS12900
C	DIS12910
C	DIS12920
C	DIS12930

```

SUBROUTINE FORCES(BOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
    BETA,BETAP,A,B,EE,WPAR,WPARE,XM,YM,IPRINT,X,Y,
    ZI,UU,ILOAD,XYM,QX,QY)
    IMPLICIT REAL*8(A-H,O-Z)
    DOUBLE PRECISION NU,KPD
    *****
    CALL POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
    HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
    X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
    *****
    APD2 = AP/2.0
    XK1 = 6.*(1.-NU)/F1/HAR2
    XNU2 = 12.*(1.-NU**2.)
    XK22 = (1.-NU)/(1. + NU)
    APY1 = Y*AP
    APX1 = X*AP
    GAMY1 = GAMY
    C
    APY2 = 0.0
    APX2 = 0.0
    GAMY2 = 0.0
    C
    GO TO (180,181) ITHICK
180 EFSIN = 0.0
    EFCOS = 0.0

```

FFBAR1 = 0.0	DIS12940
FFBAR2 = 0.0	DIS12950
FFBAR3 = 0.0	DIS12960
GO TO 183	DIS12970
181 EFSIN = EE*DSINH(GAMY)	DIS12980
EECOS = EE*DCOSH(GAMY)	DIS12990
EEBAR1 = EE*DCOSII(GAMY)	DIS13000
FFBAR2 = EE*DSINH(GAMY)	DIS13010
183 CONTINUE	DIS13020
C	DIS13030
WPARE = A*DCOSH(APY) + B*APY*DSINII(APY) + EECOS + BETA	DIS13040
WPARE = 0.0	DIS13050
C	DIS13060
GO TO (162,162,163)IBOUND	DIS13070
163 CONTINUE	DIS13080
GO TO (160,161) ITHICK	DIS13090
160 WPARE = A*DCOSH(APB) + B*APB*DSINH(APB) + BETA	DIS13100
GO TO 162	DIS13110
161 WPARE = A*DCOSH(APB) + B*APB*DSINH(APB) + EEBAR2 + BETA	DIS13120
WPARE = WPARE*DSIN(APX)	DIS13130
C	DIS13140
C	DIS13150
162 CONTINUE	DIS13160
C	DIS13170
XM = (1.-NU)*AP2*DCOSH(APY)*A + B*(2.*F1*HAR2*AP4/6.*DCOSII(APY)	DIS13180
. - 2.*NU*AP2*DCOSH(APY) + (1.-NU)*APY*AP2*DSINH(APY))	DIS13190
. - XK22*(AP2 - NU/HAR2*GAMA2)*EECOS	DIS13200
. - AP2*BETAP + KPD	DIS13210
C	DIS13220
YM = -(1.-NU)*AP2*DCOSII(APY)*A + B*(-2.*F1*HAR2*AP4/6.*DCOSII(APY)	DIS13230
. - 2.*AP2*DCOSH(APY) - (1.-NU)*APY*AP2*DSINII(APY))	DIS13240
. + XK22*(-NU*AP2 + 1./HAR2*GAMA2)*EECOS	DIS13250
. - NU*AP2*BETAP + KPD	DIS13260
C	DIS13270
XYM = 2.*AP2*DSINH(APY)*A + B*(4.*F1*HAR2*AP4/6./(1.-NU)*DSINII(APY)	DIS13280
. + 2.*AP2*DSINH(APY) + 2*APY*AP2*DCOSH(APY))	DIS13290
. - 2.*XK22*AP/HAR*DSQRT(GAMA2)*EEBAR2	DIS13300
XYM = XYM/24./(1. + NU)	DIS13310
C	DIS13320
QX = 1./XNU2*(-2.*AP3*DCOSII(APY)*B + 12.*XK22*AP/HAR2/F1	DIS13330
. *EEBAR1 + XK1*AP*(BETA + BETAP))	DIS13340
C	DIS13350
GO TO (100,100,101) IBOUND	DIS13360
100 QY = 1./XNU2*(-2.*AP3*DSINII(APY)*B + 12.*XK22/HAR2/F1	DIS13370
. /HAR*DSQRT(GAMA2)*EEBAR2)	DIS13380
GO TO 102	DIS13390
101 QY = 1./XNU2*((1.-NU)*AP3*DSINH(APB)*A	DIS13400
. + B*(-(1. + NU)*AP3*DSINII(APB)	DIS13410
. + (1.-NU)*APB*AP3*DCOSII(APB))	DIS13420
. + EFSIN*(-12./F1/HAR2*(1.-2.*XK22) + (1.-NU)*AP2)	DIS13430

```

      *DSQRT(GAMA2):HAR )
C   XYM = A*AP2*DSINH(APY) + B*(AP2*APY*DCOSH(APY) + AP2*DSINH(APY))
C   + EEBAR1*AP*DSQRT(GAMA2)/HAR
C   XYM = XYM/24./(1. + NU)
C
102 CONTINUE
      WPAR = WPAR*DSIN(APX)
      XM = XM/XNU2*DSIN(APX)
      YM = YM/XNU2*DSIN(APX)
      XYM = XYM*DCOS(APX)
      QX = QX*DCOS(APX)
      QY = QY*DSIN(APX)
      RETURN
      END
C
C
C
C
C
C
C
C
C
C*** SUBROUTINE DISS TO EVALUATE THE FUNCTION F1(Z) AND ALL
C      RELATED FUNCTIONS AND CONSTANTS
C
      SUBROUTINE DISS(M,NU,HAR,ALPHA,A1,A2,A3,A4,A5,F1,F2,F3,F4,
      Z,F1Z,F1ZP,F2Z,F3Z)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION NU
      DIMENSION AMAT(3,3),RH(3),IFPR(3),FIX(3),SOLT(3)
C
      NEQNS = 3
      PI = 22.0/7.0
      ALPHA = M*PI
      AP = ALPHA
      HAR2 = HAR*HAR
      AP2 = AP**2
      AP4 = AP**4
C
      AA = AP2*(2.-NU)/(1.-NU)
      BB = AP4/(1.-NU**2)
      DD = DSQRT(AA**2-.4.*BB)
      A = DSQRT(.5*(AA + DD) )
      B = DSQRT(.5*(AA-DD) )
      AH = A*HAR
      AH2 = A*HAR/2.
      BH = B*HAR
      BH2 = B*HAR/2.
      AZ = AH*Z
      BZ = BH*Z

```

DIS13440  
DIS13450  
DIS13460  
DIS13470  
DIS13480  
DIS13490  
DIS13500  
DIS13510  
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DIS13600  
DIS13610  
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DIS13680  
DIS13690  
DIS13700  
DIS13710  
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DIS13800  
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DIS13900  
DIS13910  
DIS13920  
DIS13930

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C
AMAT(1,1)=1.0
AMAT(1,2)=2*DSINH(AH2)
AMAT(1,3)=2*DSINH(BH2)
AMAT(2,1)=1.0
AMAT(2,2)=AH*DCOSH(AH2)
AMAT(2,3)=BH*DCOSH(BH2)
AMAT(3,1)=1.-NU**2.
AMAT(3,2)=12.*NU**2.*(2.*DSINH(AH2)/AH**2.-DCOSH(AH2)/AH)
AMAT(3,3)=12.*NU**2.*(2.*DSINH(BH2)/BH**2.-DCOSH(BH2)/BH)
DO 10 I=1,NEQNS
RH(I)=0.0
FIX(I)=0.0
10 IFPR(I)=0.0
RH(I)=1.0
CALL GREDUC (NEQNS,AMAT,FIX,RH,IFPR)
CALL BAKSUB (NEQNS,AMAT,FIX,RH,IFPR.SOLT)
A1=SOLT(1)
A3=SOLT(2)
A5=SOLT(3)
A2=0.5*( AH/BH*DSINH(AH2)/DTANH(BH2) - DCOSH(AH2) )
A4=-A2*AH/BH*DSINH(AH2)/DSINH(BH2)
C*****
C1=-(A3/AH + A5/BH)
C2=2.*(1.+NU)/AP2/HAR**2.*( A2 + A4 ) - ( A2/AH**2. + A4/BH**2. )
C WRITE(6,52) C1,C2
C52 FORMAT('C1 =',E15.6,2X,'C2 =',E15.6)
C
C
F1 = A1 - 12.*A3*( 2./AH**2.*DSINH(AH2) - DCOSH(AH2)/AH )
. - 12.*A5*( 2./BH**2.*DSINH(BH2) - DCOSH(BH2)/BH )
C
F31=A1/40. + C1
F32=12./AH**3.*A3
F33=DCOSH(AH2)-2.*DSINH(AH2)/AH
F34=12./BH**3.*A5
F35=DCOSH(BH2)-2.*DSINH(BH2)/BH
F3 = F31 + F32*F33 + F34*F35
C
F2 = 2./AH*DSINH(AH2)*A2 + 2./BH*DSINH(BH2)*A4
C
F4 = A2*2./AH**3)*DSINH(AH2)
. + A4*2./BH**3)*DSINH(BH2) + C2
C
C WRITE(6,60) AH,BH
C60 FORMAT('AH =',E12.4,2X,'BH =',E12.4)
C WRITE(6,12) A1
C WRITE(6,13) A2
C WRITE(6,14) A3
C WRITE(6,15) A4
DIS13940
DIS13950
DIS13960
DIS13970
DIS13980
DIS13990
DIS14000
DIS14010
DIS14020
DIS14030
DIS14040
DIS14050
DIS14060
DIS14070
DIS14080
DIS14090
DIS14100
DIS14110
DIS14120
DIS14130
DIS14140
DIS14150
DIS14160
DIS14170
DIS14180
DIS14190
DIS14200
DIS14210
DIS14220
DIS14230
DIS14240
DIS14250
DIS14260
DIS14270
DIS14280
DIS14290
DIS14300
DIS14310
DIS14320
DIS14330
DIS14340
DIS14350
DIS14360
DIS14370
DIS14380
DIS14390
DIS14400
DIS14410
DIS14420
DIS14430

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C WRITE(6,16) A5 DIS14440
C12 FORMAT('A1',2X,'=',F20.6) DIS14450
C13 FORMAT('A2',2X,'=',F20.6) DIS14460
C14 FORMAT('A3',2X,'=',F20.6) DIS14470
C15 FORMAT('A4',2X,'=',F20.6) DIS14480
C16 FORMAT('A5',2X,'=',F20.6) DIS14490
C WRITE(6,42) F1 DIS14500
C WRITE(6,43) F2 DIS14510
C WRITE(6,44) F3 DIS14520
C WRITE(6,45) F4 DIS14530
C42 FORMAT('PRESENT WORK F1',3X,'=',E15.5) DIS14540
C43 FORMAT('PRESENT WORK F2',3X,'=',E15.5) DIS14550
C44 FORMAT('PRESENT WORK F3',3X,'=',E15.5) DIS14560
C45 FORMAT('PRESENT WORK F4',3X,'=',E15.5) DIS14570
C DIS14580
  F1Z = A1*Z + A2*DCOSH(AZ) + A3*DSINH(AZ) DIS14590
  + A4*DCOSH(BZ) + A5*DSINH(BZ) DIS14600
C DIS14610
  F1ZP = A1 + A2*AH*DSINH(AZ) + A3*AH*DCOSH(AZ) DIS14620
  + A4*BH*DSINH(BZ) + A5*BH*DCOSH(BZ) DIS14630
C DIS14640
  F2Z = Z**2/2.*A1 + A2*DSINH(AZ)/AH + A3*DCOSH(AZ)*AH DIS14650
  + A4*DSINH(BZ)/BH + A5*DCOSH(BZ)/BH + C1 DIS14660
C DIS14670
  F3Z = Z**3/6.*A1 + A2*DCOSH(AZ)/AH**2 DIS14680
  + A3*DSINH(AZ)/AH**2 DIS14690
  + A4*DCOSH(BZ)/BH**2 DIS14700
  + A5*DSINH(BZ)/BH**2 DIS14710
  + C1*Z + C2 DIS14720
  RETURN DIS14730
  END DIS14740
C DIS14750
C***** DIS14760
C DIS14770
C DIS14780
C DIS14790
C DIS14800
C SUBROUTINE GREduc (NEQNS,ASTIF,FIXED,ASLOD,IFPRE) DIS14810
  IMPLICIT REAL*8(A-H,O-Z) DIS14820
  DIMENSION ASLOD(3),ASTIF(3,3), DIS14830
  FIXED(3),IFPRE(3) DIS14840
C DIS14850
C == => NOTE : NEQNS : NUMBER OF EQUATIONS TO BE SOLVED = N DIS14860
C ASTIF(N,N) : COEFFICIENT MATRIX DIS14870
C FIXED(N) : VECTOR OF PRESCRIBED ( OR KNOWN ) VARIABLES; DIS14880
C FIXED(N) DIS14890
C ASLOD(N) : VECTOR OF R.H.S. OF THE EQUATIONS; ASLOD(N). DIS14900
C IFPRE(N) : VECTOR INDICATING WHETHER A VARIABLE IS DIS14910
C PRESCRIBED OR NOT ; IF : DIS14920
C IFPRE(I) = 0 == => VARIABLE #I IS NOT PRESCRIBED DIS14930

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C	IFPRE(I)=1 == => VARIABLE #I IS PRESCRIBED	DIS14940
C		DIS14950
C		DIS14960
	NEQNS=3	DIS14970
	DO 50 IEQNS = 1,NEQNS	DIS14980
	IF(IFPRE(IEQNS).GT.0)GO TO 30	DIS14990
C---	CHOOSING THE BIGGEST	DIS15000
	JCOLS = IEQNS	DIS15010
	IF(JCOLS.EQ.NEQNS) GO TO 50	DIS15020
	RMAX = 0.0	DIS15030
	DO 101 IROWS = IEQNS,NEQNS	DIS15040
	R = ASTIF(IROWS,JCOLS)	DIS15050
	IF(DABS(R).LE.DABS(RMAX)) GO TO 101	DIS15060
	RMAX = R	DIS15070
	IBIG = IROWS	DIS15080
101	CONTINUE	DIS15090
	IF (IBIG.EQ.IEQNS) GO TO 500	DIS15100
C---	INTERCHANGING ROWS	DIS15110
	SHIFT2 = ASLOD(IEQNS)	DIS15120
	ASLOD(IEQNS) = ASLOD(IBIG)	DIS15130
	ASLOD(IBIG) = SHIFT2	DIS15140
	DO 103 J = 1,NEQNS	DIS15150
	SHIFT1 = ASTIF(IEQNS,J)	DIS15160
	ASTIF(IEQNS,J) = ASTIF(IBIG,J)	DIS15170
	ASTIF(IBIG,J) = SHIFT1	DIS15180
103	CONTINUE	DIS15190
C---	REDUCE EQUATIONS	DIS15200
500	PIVOT = ASTIF(IEQNS,IEQNS)	DIS15210
	IF(DABS(PIVOT).LT.1.0E-50)GO TO 60	DIS15220
	IF(IEQNS.EQ.NEQNS)GO TO 50	DIS15230
	IEQN1 = IEQNS + 1	DIS15240
	DO 20 IROWS = IEQN1,NEQNS	DIS15250
	FACTR = ASTIF(IROWS,IEQNS)/PIVOT	DIS15260
	IF(FACTR.EQ.0.0)GO TO 20	DIS15270
	DO 10 ICOLS = IEQNS,NEQNS	DIS15280
	ASTIF(IROWS,ICOLS) = ASTIF(IROWS,ICOLS) - FACTR * ASTIF(IEQNS,ICOLS)	DIS15290
10	CONTINUE	DIS15300
	ASLOD(IROWS) = ASLOD(IROWS) - FACTR * ASLOD(IEQNS)	DIS15310
C	WRITE(6,124) ASLOD(IROWS)	DIS15320
C		DIS15330
20	CONTINUE	DIS15340
C		DIS15350
C		DIS15360
C	WRITE(6,229) IEQNS	DIS15370
C229	FORMAT('IEQNS = ',I2,'COEFFICIENT MATRIX AFTER SOLUTION')	DIS15380
C	DO 224 I = 1,NEQNS	DIS15390
C	WRITE(6,122) (ASTIF(I,J),J = 1,NEQNS)	DIS15400
C224	CONTINUE	DIS15410
C	DO 225 I = 1,NEQNS	DIS15420
C	WRITE(6,124) ASLOD(I)	DIS15430

C225 CONTINUE	DIS15440
C122 FORMAT(3(E12.5,2X))	DIS15450
C124 FORMAT(E12.5)	DIS15460
C	DIS15470
C	DIS15480
GO TO 50	DIS15490
C	DIS15500
C ADJUST RHS(LOADS) FOR	DIS15510
C PRESCRIBED DISPLACEMENTS	DIS15520
C	DIS15530
30 DO 40 IROWS = IEQNS,NEQNS	DIS15540
ASLOD(IROWS) = ASLOD(IROWS)-ASTIF(IROWS,IEQNS)*FIXED(IEQNS)	DIS15550
ASTIF(IROWS,IEQNS) = 0.0	DIS15560
40 CONTINUE	DIS15570
C	DIS15580
C	DIS15590
C WRITE(6,229) IEQNS	DIS15600
C DO 324 I = 1,NEQNS	DIS15610
C WRITE(6,122) (ASTIF(I,J1),J1 = 1,NEQNS)	DIS15620
C324 CONTINUE	DIS15630
C DO 325 I = 1,NEQNS	DIS15640
C WRITE(6,124) ASLOD(I)	DIS15650
C325 CONTINUE	DIS15660
C	DIS15670
GO TO 50	DIS15680
60 PRINT 100	DIS15690
100 FORMAT(5X,15HINCORRECT PIVOT)	DIS15700
STOP	DIS15710
50 CONTINUE	DIS15720
RETURN	DIS15730
END	DIS15740
CC	DIS15750
C-----	DIS15760
C BACK-SUBSTITUTION ROUTINE	DIS15770
C-----	DIS15780
C	DIS15790
SUBROUTINE BAKSUB (NEQNS,ASTIF,FIXED,ASLOD,IFPRE,DISPL.)	DIS15800
C	DIS15810
IMPLICIT REAL*8(A-H,O-Z)	DIS15820
DIMENSION ASTIF(NEQNS,NEQNS),IFPRE(NEQNS),	DIS15830
FIXED(NEQNS),DISPL(NEQNS),ASLOD(NEQNS)	DIS15840
C	DIS15850
NEQNI = NEQNS + 1	DIS15860
DO 30 IEQNS = 1,NEQNS	DIS15870
NBACK = NEQNI-IEQNS	DIS15880
PIVOT = ASTIF(NBACK,NBACK)	DIS15890
RFSID = ASLOD(NBACK)	DIS15900
IF(NBACK.EQ.NEQNS)GO TO 20	DIS15910
NBAC1 = NBACK + 1	DIS15920
DO 10 ICOLS = NBAC1,NEQNS	DIS15930

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RESID = RESID-ASTIF(NBACK,ICOLS)*DISPL(ICOLS) DIS15940
10 CONTINUE DIS15950
20 IF(IFPRE(NBACK).LE.0) DIS15960
   *DISPL(NBACK) = RESID/ASTIF(NBACK,NBACK) DIS15970
C *DISPL(NBACK) = RESID;PIVOT DIS15980
  IF(IFPRE(NBACK).GT.0)DISPL(NBACK) = FIXED(NBACK) DIS15990
C IF(IFPRE(NBACK).GT.0)REACT(NBACK) = -RESID DIS16000
30 CONTINUE DIS16010
  RETURN DIS16020
  END DIS16030
C DIS16040
C----- DIS16050
C*** GAUSS-JORDAN REDUCTION ROUTINE DIS16060
C----- DIS16070
C DIS16080
  SUBROUTINE JORDAN(NEQNS,ASTIF,ASLOD,SOL) DIS16090
  IMPLICIT REAL*8(A-H,O-Z) DIS16100
  DIMENSION ASLOD(NEQNS),ASTIF(NEQNS,NEQNS),SOL(NEQNS) DIS16110
  DO 30 IEQNS = 1,NEQNS DIS16120
  PIVOT = ASTIF(IEQNS,IEQNS) DIS16130
  DO 20 IROWS = 1,NEQNS DIS16140
  FACTR = ASTIF(IROWS,IEQNS)/PIVOT DIS16150
  IF(IROWS.EQ.IEQNS.OR.FACTR.EQ.0.0) GO TO 20 DIS16160
  DO 10 ICOLS = 1,NEQNS DIS16170
  ASTIF(IROWS,ICOLS) = ASTIF(IROWS,ICOLS)-FACTR*ASTIF(IEQNS,ICOLS) DIS16180
10 CONTINUE DIS16190
  ASLOD(IROWS) = ASLOD(IROWS)-FACTR*ASLOD(IEQNS) DIS16200
20 CONTINUE DIS16210
30 CONTINUE DIS16220
  DO 40 IEQNS = 1,NEQNS DIS16230
  SOL(IEQNS) = ASLOD(IEQNS)/ASTIF(IEQNS,IEQNS) DIS16240
40 CONTINUE DIS16250
  RETURN DIS16260
  END DIS16270
C DIS16280
C----- DIS16290
C DIS16300
C----- DIS16310
C DIS16320
  SUBROUTINE GREDU (NEQNS,ASTIF,FIXED,ASLOD,IFPRE) DIS16330
  IMPLICIT REAL*8(A-H,O-Z) DIS16340
  DIMENSION ASLOD(NEQNS),ASTIF(NEQNS,NEQNS), DIS16350
  FIXED(NEQNS),IFPRE(NEQNS) DIS16360
C DIS16370
C GAUSSIAN REDUCTION ROUTINE DIS16380
C DIS16390
  DO 50 IEQNS = 1,NEQNS DIS16400
  IF(IFPRE(IEQNS).EQ.1) GO TO 30 DIS16410
C DIS16420
C REDUCE EQUATIONS DIS16430

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C		DIS16440
	PIVOT = ASTIF(IEQNS,IEQNS)	DIS16450
	IF(DABS(PIVOT).LT.1.0E-50) GO TO 60	DIS16460
	IF(IEQNS.EQ.NEQNS) GO TO 50	DIS16470
	IEQN1 = IEQNS + 1	DIS16480
	DO 20 IROWS = IEQN1,NEQNS	DIS16490
	FACTR = ASTIF(IROWS,IEQNS)/PIVOT	DIS16500
	IF(FACTR.EQ.0.0) GO TO 20	DIS16510
	DO 10 ICOLS = IEQNS,NEQNS	DIS16520
	ASTIF(IROWS,ICOLS) = ASTIF(IROWS,ICOLS) - FACTR * ASTIF(IEQNS,ICOLS)	DIS16530
10	CONTINUE	DIS16540
	ASLOD(IROWS) = ASLOD(IROWS) - FACTR * ASLOD(IEQNS)	DIS16550
20	CONTINUE	DIS16560
		DIS16570
C		DIS16580
C		DIS16590
C	WRITE(6,229) IEQNS	DIS16600
C229	FORMAT('IEQNS = ',I2,' COEFFICIENT MATRIX AFTER SOLUTION')	DIS16610
C	DO 224 I = 1,NEQNS	DIS16620
C	WRITE(6,122) (ASTIF(I,J1),J1 = 1,NEQNS)	DIS16630
C224	CONTINUE	DIS16640
C	DO 225 I = 1,NEQNS	DIS16650
C	WRITE(6,124) ASLOD(I)	DIS16660
C225	CONTINUE	DIS16670
C122	FORMAT(3(E12.5,2X))	DIS16680
C124	FORMAT(E12.5)	DIS16690
C		DIS16700
C		DIS16710
	GO TO 50	DIS16720
C		DIS16730
C	ADJUST RHS(LOADS) FOR PRESCRIBED DISPLACEMENTS	DIS16740
C		DIS16750
30	DO 40 IROWS = IEQNS,NEQNS	DIS16760
	ASLOD(IROWS) = ASLOD(IROWS) - ASTIF(IROWS,IEQNS) * FIXED(IEQNS)	DIS16770
	ASTIF(IROWS,IEQNS) = 0.0	DIS16780
40	CONTINUE	DIS16790
	GO TO 50	DIS16800
60	WRITE(6,900) PIVOT,IEQNS	DIS16810
900	FORMAT(5X,18HINCORRECT PIVOT = ,E20.6,5X,13HEQUATION NO. ,I5)	DIS16820
	STOP	DIS16830
50	CONTINUE	DIS16840
	RETURN	DIS16850
	END	DIS16860
C		DIS16870
C	-----	DIS16880
C		DIS16890
C	-----	DIS16900
C		DIS16910
	SUBROUTINE BAKSU (NEQNS, ASTIF, FIXED, ASLOD, IFPRE,XDISP)	DIS16920
	IMPLICIT REAL*8(A-H,O-Z)	DIS16930
	DIMENSION ASTIF(NEQNS,NEQNS),IFPRE(NEQNS),	

```
      .      FIXED(NEQNS),XDISP(NEQNS),ASLOD(NEQNS)      DIS16940
C      BACK-SUBSTITUTION ROUTINE      DIS16950
C      NEQN1 = NEQNS + 1      DIS16960
C      DO 30 IEQNS = 1,NEQNS      DIS16970
      NBACK = NEQN1 - IEQNS      DIS16980
      PIVOT = ASTIF(NBACK,NBACK)      DIS16990
      RESID = ASLOD(NBACK)      DIS17000
      IF(NBACK.EQ.NEQNS) GO TO 20      DIS17010
      NBAC1 = NBACK + 1      DIS17020
      DO 10 ICOLS = NBAC1,NEQNS      DIS17030
      RESID = RESID - ASTIF(NBACK,ICOLS)*XDISP(ICOLS)      DIS17040
10 CONTINUE      DIS17050
20 IF(IFPRE(NBACK).EQ.0) XDISP(NBACK) = RESID/PIVOT      DIS17060
   IF(IFPRE(NBACK).EQ.1) XDISP(NBACK) = FIXED(NBACK)      DIS17070
30 CONTINUE      DIS17080
   RETURN      DIS17090
   END      DIS17100
      DIS17110
      DIS17120
```

## A-5.2 PROGRAM DISS4 LISTING :

```

C*****
C*****
C
C PROGRAM DISS4 : TO FIND SOLUTION ( DEFLECTION & STRESSES )
C
C      IN THE CASE OF CYLINDRICAL BENDING
C DONE BY AMMAR KHALEEL HAFEDH MOHAMMED ( IN PH.D DISSERTATION )
C
C*****
      IMPLICIT REAL*(A-H,O-Z)
      DOUBLE PRECISION  NU,NUP1,NUSM1,NUM1,LH2,LA,K,N,LAMDA,
      .      INCREM
      DATA NU/0.30D0,E/1.0D0,HAR/0.00,INCREM/0.500,
      .      NPLATE/6 ,NTERM/15,MP/15,
      .      IPRINT/2,IDEF/2,ISTRES/2,IBAL/2,ISIGZ/1,IFOUR/2/
C
C
      NUP1 = NU + 1.D0
      NUSM1 = 1.0 - NU**2
      NUM1 = 1.0 - NU
      G = E/(2.D0*(1.D0 + NU))
      PI = 22.D0/7.D0
C
C NOTE : MP = IS AN INDICATER TO TELL AT WHAT "M" VALUE WE WANT RESULTS
C      TO BE PRINTED
C      NPLATE = IS AN INDICATER TO TELL US FOR HOW MANY PLATE RATIOS W
C      WANT THE RESULTS
C
C      IDEF = IS AN INDICATER FOR PRINTING DEFLECTION RESULTS
C IF IDEF = 1 : PRINT DEFLECTIONS
C IF IDEF = 2 : DO NOT PRINT DEFLECTIONS
C
C      ISTRES = IS AN INDICATER FOR PRINTING STRESS SIGMAX
C IF ISTRES = 1 : PRINT STRESSES
C IF ISTRES = 2 : DO NOT PRINT STRESSES
C
C
C      ISIGZ = IS AN INDICATER FOR PRINTING STRESS SIGMAZ
C IF ISIGZ = 1 : PRINT STRESSES
C IF ISIGZ = 2 : DO NOT PRINT STRESSES
C
C      IPRINT = IS AN INDICATER FOR PRINTING INTERMEDIATE RESULTS
C IF IPRINT = 1 : PRINT INTERMEDIATE RESULTS
C IF IPRINT = 2 : DO NOT PRINT INTERMEDIATE RESULTS
C
      WRITE(6,210)
210 FORMAT('CYLINDRICAL BENDING ')

```

DIS00010  
DIS00020  
DIS00030  
DIS00040  
DIS00050  
DIS00060  
DIS00070  
DIS00080  
DIS00090  
DIS00100  
DIS00110  
DIS00120  
DIS00130  
DIS00140  
DIS00150  
DIS00160  
DIS00170  
DIS00180  
DIS00190  
DIS00200  
DIS00210  
DIS00220  
DIS00230  
DIS00240  
DIS00250  
DIS00260  
DIS00270  
DIS00280  
DIS00290  
DIS00300  
DIS00310  
DIS00320  
DIS00330  
DIS00340  
DIS00350  
DIS00360  
DIS00370  
DIS00380  
DIS00390  
DIS00400  
DIS00410  
DIS00420  
DIS00430  
DIS00440  
DIS00450  
DIS00460

```

        GO TO (212,213) IFOUR                                DIS00470
212 WRITE(6,211)                                           DIS00480
211 FORMAT('LOAD P0 = SIN(PI*X/L)')                        DIS00490
        GO TO 215                                           DIS00500
213 WRITE(6,214)                                           DIS00510
214 FORMAT('LOAD P0 = UNIFORM LOAD')                      DIS00520
215 ABAR = 1.D0                                           DIS00530
        WRITE(6,188) NU                                     DIS00540
        WRITE(6,18)                                        DIS00550
188 FORMAT('NU = ',F6.2)                                   DIS00560
        GO TO (561,562) IDEF                                DIS00570
561 WRITE(6,102)                                           DIS00580
101 FORMAT('*****')                                       DIS00590
102 FORMAT('          DEFLECTIONS ')                      DIS00600
        WRITE(6,101)                                       DIS00610
        WRITE(6,556)                                       DIS00620
        GO TO 564                                           DIS00630
562 GO TO (565,564) ISTRES                                 DIS00640
565 WRITE(6,103)                                           DIS00650
        WRITE(6,101)                                       DIS00660
        WRITE(6,555)                                       DIS00670
555 FORMAT(7X,'Z/H',8X,'RTP',6X,'EXACT',8X,'PANC',8X,'RTB',8X,'OTHERS'
        .)                                               DIS00680
556 FORMAT(5X,' H',6X,'RTP',7X,'EXACT',6X,'RTB',6X,'PANC',6X,'REISS'
        .,6X,'NAGHDI')                                     DIS00690
564 GO TO (406,407) ISIGZ                                  DIS00700
406 WRITE(6,408)                                           DIS00710
        WRITE(6,409)                                       DIS00720
408 FORMAT('* H * PRESENT * PRESENT * EXACT * EXACT *')  DIS00730
409 FORMAT('* * SIGMAX * SIGMAZ * SIGMAX * SIGMAZ*')     DIS00740
407 CONTINUE                                              DIS00750
C*****                                                  DIS00760
        DO 200 I = 1,NPLATE                                 DIS00770
C*****                                                  DIS00780
        IF(I.LE.30) GO TO 31                               DIS00790
        GO TO 32                                           DIS00800
31 HIR = HAR + INCREM                                     DIS00810
C                                                         DIS00820
C                                                         DIS00830
C NOTE : INCREM IS THE INCREMENT IN THE A/H RATIO      DIS00840
C                                                         DIS00850
C                                                         DIS00860
C                                                         DIS00870
C                                                         DIS00880
        AIIR = 1.D0/HIR                                     DIS00890
        GO TO 33                                           DIS00900
32 IF(I.EQ.31) AHR = 0.D0                                  DIS00910
        IF(I.GE.31) GO TO 34                               DIS00920
        AIIR = AIIR + 2.D0                                  DIS00930
        GO TO 33                                           DIS00940
34 AHR = AIIR + 100.D0                                    DIS00950
33 HI = ABAR/AIIR                                         DIS00960

```



	GO TO (800,801) ISTRES	DIS00970
800	WRITE(6,101)	DIS00980
	WRITE(6,25) H	DIS00990
	WRITE(6,101)	DIS01000
801	GO TO (404,405) ISIGZ	DIS01010
404	WRITE(6,101)	DIS01020
	WRITE(6,25) H	DIS01030
	WRITE(6,101)	DIS01040
405	D = E*H**3/(12.D0*(1.D0-NU**2))	DIS01050
	WCT = 0.D0	DIS01060
	WMT = 0.D0	DIS01070
	WOPANC = 0.D0	DIS01080
	WREIS = 0.D0	DIS01090
	WNAGD = 0.D0	DIS01100
	WPT = 0.D0	DIS01110
	WBT = 0.D0	DIS01120
	WOCHEK = 0.D0	DIS01130
	WOEXAK = 0.D0	DIS01140
C		DIS01150
C		DIS01160
	NPOINT = 11	DIS01170
C	*****	DIS01180
	DO 100 J = 1,NPOINT	DIS01190
C	*****	DIS01200
C		DIS01210
	SIGMAP = 0.D0	DIS01220
	SIGMPA = 0.D0	DIS01230
	SIGMAE = 0.D0	DIS01240
	SIGMAB = 0.D0	DIS01250
	SIGMAM = 0.D0	DIS01260
	SIGMAO = 0.D0	DIS01270
	SIGZP = 0.D0	DIS01280
	SIGZE = 0.D0	DIS01290
C		DIS01300
	IF(J.EQ.1)GO TO 222	DIS01310
	GO TO 223	DIS01320
222	Z = -0.50*H	DIS01330
223	ZH = Z/H	DIS01340
C		DIS01350
C		DIS01360
C	*****	DIS01370
	DO 10 M = 1,NTERM,2	DIS01380
C	*****	DIS01390
C		DIS01400
C		DIS01410
C	PRESENT WORK : DEFLECTION	DIS01420
C		DIS01430
C		DIS01440
	ALPHA = M*PI/ABAR	DIS01450
	AP = ALPHA	DIS01460

	APII2 = ALPHA*H/2.	DIS01470
	APB = ALPIIA**2	DIS01480
	APBS = ALPIIA**4	DIS01490
	AA = APB*(2.D0-NU)/(1.D0-NU)	DIS01500
	BB = APBS/(1.D0-NU**2)	DIS01510
	DD = DSQRT(AA**2-4.D0*BB)	DIS01520
	A = DSQRT(.5D0*(AA + DD))	DIS01530
	B = DSQRT(.5D0*(AA-DD))	DIS01540
	AH = A*H	DIS01550
	AH2 = A*H/2.D0	DIS01560
	BH = B*H	DIS01570
	BH2 = B*H/2.D0	DIS01580
C		DIS01590
C		DIS01600
C		DIS01610
	A11 = H	DIS01620
	A12 = 2*DSINH(AH2)	DIS01630
	A13 = 2*DSINH(BH2)	DIS01640
	A21 = 1.D0	DIS01650
	A22 = A*DCOSH(AH2)	DIS01660
	A23 = B*DCOSH(BH2)	DIS01670
	A31 = 1.0D0-NU**2	DIS01680
	A32 = (12.*NU**2/H**3)*(2.*DSINH(AH2)/A**2-H*DCOSH(AH2), A)	DIS01690
	A33 = (12.*NU**2/H**3)*(2.*DSINH(BH2)/B**2-H*DCOSH(BH2), B)	DIS01700
	B11 = 1.D0	DIS01710
	B22 = 0.D0	DIS01720
	B33 = 0.D0	DIS01730
	D11 = A22*A33-A23*A32	DIS01740
	D12 = A21*A33-A23*A31	DIS01750
	D13 = A21*A32-A22*A31	DIS01760
	D22 = B22*A33-A23*B33	DIS01770
	D23 = B22*A32-A22*B33	DIS01780
	D33 = A21*B33-B22*A31	DIS01790
	DET1 = A11*D11-A12*D12 + A13*D13	DIS01800
	DET2 = B11*D11-A12*D22 + A13*D23	DIS01810
	DET3 = A11*D22-B11*D12 + A13*D33	DIS01820
	DET4 = -A11*D23-A12*D33 + B11*D13	DIS01830
	A1 = DET2/DET1	DIS01840
	A3 = DET3/DET1	DIS01850
	A5 = DET4/DET1	DIS01860
	DCOT = 1.D0/DTANH(BH2)	DIS01870
C		DIS01880
C		DIS01890
	A2 = 0.5D0/( A*DSINH(AH2)/( DTANH(BH2)*B ) - DCOSH(AH2) )	DIS01900
	A4 = -1.D0*DSINH(AH2)*A*A2/( B*DSINH(BH2) )	DIS01910
C		DIS01920
	GO TO (500,501) IPRINT	DIS01930
500	WRITE(6,18)	DIS01940
	WRITE(6,24) AHR	DIS01950
	WRITE(6,25) H	DIS01960

WRITE(6,111) ZH	DIS01970
WRITE(6,17) M	DIS01980
WRITE(6,18)	DIS01990
WRITE(6,102)	DIS02000
24 FORMAT(' A/H = ',F8.2)	DIS02010
25 FORMAT('                    II = ',F10.4)	DIS02020
WRITE(6,101)	DIS02030
17 FORMAT('            M = ',I2)	DIS02040
111 FORMAT('            Z/H = ',F6.2)	DIS02050
C WRITE(6,12) A1	DIS02060
C WRITE(6,13) A2	DIS02070
C WRITE(6,14) A3	DIS02080
C WRITE(6,15) A4	DIS02090
C WRITE(6,16) A5	DIS02100
12 FORMAT('A1 = ',E15.6)	DIS02110
13 FORMAT('A2 = ',E15.6)	DIS02120
14 FORMAT('A3 = ',E15.6)	DIS02130
15 FORMAT('A4 = ',E15.6)	DIS02140
16 FORMAT('A5 = ',E15.6)	DIS02150
18 FORMAT('*****')	DIS02160
501 F1 = A1-(12./H**3)*A3*(2./A**2*DSINH(AH2)-H*DCOSH(AH2)/	DIS02170
A)-(12./H**3)*A5*(2./B**2*DSINH(BH2)-H*DCOSH(BH2)/B)	DIS02180
C1 = -(A3/A + A5/B)	DIS02190
C2 = 2.D0*(1.D0 + NU)/APB*(A2 + A4)-(A2/A**2 + A4/	DIS02200
B**2)	DIS02210
C	DIS02220
C WRITE(6,552) C1	DIS02230
C WRITE(6,553) C2	DIS02240
552 FORMAT('C1 = ',F20.6)	DIS02250
553 FORMAT('C2 = ',F20.6)	DIS02260
C	DIS02270
F31 = (H**2/40.)*A1 + C1	DIS02280
F32 = (12./H**3)*(A3/A**2)	DIS02290
F33 = (H/A)*DCOSH(AH2)-2.*DSINH(AH2)/A**2	DIS02300
F34 = (12./H**3)*(A5/B**2)	DIS02310
F35 = (H/B)*DCOSH(BH2)-2.*DSINH(BH2)/B**2	DIS02320
F3 = F31 + F32*F33 + F34*F35	DIS02330
C	DIS02340
C	DIS02350
F2 = 2./A*DSINH(AH2)*A2 + 2./B*DSINH(BH2)*A4	DIS02360
F4 = 2./A**3*DSINH(AH2)*A2 + 2./B**3*DSINH(BH2)*A4	DIS02370
+ C2*H	DIS02380
C	DIS02390
C	DIS02400
GO TO (600,601) IPRINT	DIS02410
600 WRITE(6,42) F1	DIS02420
WRITE(6,51) F1B	DIS02430
WRITE(6,43) F3	DIS02440
WRITE(6,52) F3B	DIS02450
WRITE(6,53) F2	DIS02460

```

WRITE(6,54) F4
42 FORMAT('PRESENT WORK F1 = ',E15.6)
43 FORMAT('PRESENT WORK F3 = ',E15.6)
51 FORMAT('BALUCH WORK F1B = ',E15.6)
52 FORMAT('BALUCH WORK F3B = ',E15.6)
53 FORMAT('PRESENT WORK F2 = ',E15.6)
54 FORMAT('PRESENT WORK F4 = ',E15.6)
601 S = G/F1
    N = E/F3
    R = 10.*E*H/(3.*NU)
    GO TO (230,231) IFOUR
230 P = 1.0
    GO TO 232
231 P = 4.D0/(M*PI)
232 AM = M*PI/2.D0
C   AM = ALPHA*ABAR/2.
C   W00 = P/(APBS*D)
C   W01 = 1.0 + (2.-NU)*H**3*APB*F1/(12.*(1.-NU))-APBS*D/N
C   W02 = NU*H**2*APB/(40.*(1.-NU))
C   W03 = NU**2*H**5*APBS*F1/(480.*(1.-NU)**2)
C   W04 = NU**2*H**5*APBS*F1/(240.*(1.-NU)**2*(1.+NU))
C   W0 = W00*(W01 + W02-W03 + W04)*DSIN(AM)
    WOCHEC = P/(BB*E)*( AA*A1-BB*C1-A3*(A**3
.   - A*AA + BB/A ) - A5*( B**3-B*AA
.   + BB/B ) ) *DSIN(AM)
C   + BB/B ) ) *P*C1/E ) *DSIN(AM)
    BMCHEK = H**3*P*A1/(12.*NU)
    BM = D*P*(NUP1/(D*APB)-NU*NUP1**2*F1/E + 2.*NU*NUP1*F1/E)
    BM = BMCHEK
    W0 = ( P*( 1./((APBS*D)+(2.-NU)*NUP1*F1/(APB*E)-1./N )
;   + BM/R ) *DSIN(AM)
C ;   + BM/R -P*C1/E ) *DSIN(AM)
    WOCHEK = WOCHEC + WOCHEK
    WMT = WMT + W0
    WC = P*DSIN(AM)/(APBS*D)
    WCT = WCT + WC
    WRP = WMT/WCT
    WRCHEK = WOCHEK/WCT
C
C
C EXACT SOLUTION
C
C
    R3 = -P*DSINH(APH2)/(2.*APB*( DSINH(APH2)*DCOSH(APH2) + APH2 ))
    R4 = -P*DCOSH(APH2)/(2.*APB*( DSINH(APH2)*DCOSH(APH2)-APH2 ))
    R1 = -R4*(APH2*DTANH(APH2) + 1.)
    R2 = -R3*(APH2/DTANH(APH2) + 1.)
C
C
C NOTE: IN EXACT SOLUTION ( E ) WILL BE REPLACED BY ( E/(1.-NU**2) )

```

DIS02470  
DIS02480  
DIS02490  
DIS02500  
DIS02510  
DIS02520  
DIS02530  
DIS02540  
DIS02550  
DIS02560  
DIS02570  
DIS02580  
DIS02590  
DIS02600  
DIS02610  
DIS02620  
DIS02630  
DIS02640  
DIS02650  
DIS02660  
DIS02670  
DIS02680  
DIS02690  
DIS02700  
DIS02710  
DIS02720  
DIS02730  
DIS02740  
DIS02750  
DIS02760  
DIS02770  
DIS02780  
DIS02790  
DIS02800  
DIS02810  
DIS02820  
DIS02830  
DIS02840  
DIS02850  
DIS02860  
DIS02870  
DIS02880  
DIS02890  
DIS02900  
DIS02910  
DIS02920  
DIS02930  
DIS02940  
DIS02950  
DIS02960

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C DIS02970
C DIS02980
EEXAC = E/NUSMI DIS02990
W0EXAC = (R4*AP*DSIN(AM)/EEXAC)*(2. + NUP1*APII2*DTANH(APH2)) DIS03000
W0EXAK = W0EXAC + W0EXAK DIS03010
C WREXAC = W0EXAK/WCT DIS03020
WREXAC = DABS(W0EXAK/WCT) DIS03030
C DIS03040
C DIS03050
C PANC'S WORK DIS03060
C DIS03070
C DIS03080
LAMDA = ALPHA*DSQRT(2./(1.-NU)) DIS03090
LA = LAMDA DIS03100
LH2 = LAMDA*H/2. DIS03110
K = 2.*E*(LH2-DTANH(LH2))/(LAMDA**3*(1.-NU**2)) DIS03120
WPANC = P*DSIN(AM)/(APBS*K) DIS03130
W0PANC = WPANC + W0PANC DIS03140
WRPANC = W0PANC/WCT DIS03150
C DIS03160
C END OF PANC'S WORK DIS03170
C DIS03180
C***** DIS03190
C DIS03200
C DIS03210
C BALUCH'S WORK DIS03220
C DIS03230
C***** DIS03240
C DIS03250
C1B = 0.D0 DIS03260
C2B = -NUP1/APB DIS03270
F1B = 6.D0/(5.D0*H) DIS03280
F3B = 39.D0*H/1120.D0 DIS03290
C F3B = 39.D0*H/1120.D0 + C1B DIS03300
SB = G/F1B DIS03310
NB = E/F3B DIS03320
C W00 = P/(APBS*D) DIS03330
C W01 = 1.0 + (2.-NU)*H**3*APB*F1/(12.*(1.-NU))-APBS*D/N DIS03340
C W02 = NU*H**2*APB/(40.*(1.-NU)) DIS03350
C W03 = NU**2*H**5*APBS*F1/(480.*(1.-NU)**2) DIS03360
C W04 = NU**2*H**5*APBS*F1/(240.*(1.-NU)**2*(1. + NU)) DIS03370
C W0B = W00*(W01 + W02 - W03 + W04)*DSIN(AM) DIS03380
BMB = D*P*((1. + NU)/(D*APB) - NU*(1. + NU)**2*F1B/E + 2.*NU*
; (1. + NU)*F1B/E) DIS03390
W0B = (P*(1./(APBS*D) + (2.-NU)*(1. + NU)*F1B/(APB*E) - 1./NB)
; + BMB/R)*DSIN(AM) DIS03410
WBT = WBT + W0B DIS03430
WRB = WBT/WCT DIS03440
C***** DIS03450
C DIS03460

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C		DIS03470
C	REISSNER SHEAR DEFORMATION THEORY	DIS03480
C		DIS03490
C		DIS03500
C	*****	DIS03510
	WREISS=(1.+APB*H**2*(2.-NU)/(10.*(1.-NU)))*P/(APBS*D)*	DIS03520
	; DSIN(AM)	DIS03530
	WREIS=WREISS+WREIS	DIS03540
	WRREIS=WREIS/WCT	DIS03550
C	*****	DIS03560
C		DIS03570
C		DIS03580
C	NAGHDI-ESSENBURG TRANSVERSE NORMAL STRAIN THEORY	DIS03590
C		DIS03600
C		DIS03610
C	*****	DIS03620
	WNAGDI=(1.D0+(8.-3.*NU*(1.-NU))*H**2*APB/(40.*(1.-NU))	DIS03630
	; -3.*APBS*H**4/1120.)*P/(APBS*D)*DSIN(AM)	DIS03640
	WNAGD=WNAGDI+WNAGD	DIS03650
	WRNAGD=WNAGD/WCT	DIS03660
C		DIS03670
C	WRITE(6,19) WCT	DIS03680
C	WRITE(6,21) WMT	DIS03690
	GO TO (672,503) IPRINT	DIS03700
672	IF(M.GE.MP) GO TO 544	DIS03710
	GO TO 503	DIS03720
544	WRITE(6,22) WRP	DIS03730
C	WRITE(6,64) WRCHEK	DIS03740
	WRITE(6,72) WREXAC	DIS03750
	WRITE(6,56) WRB	DIS03760
	WRITE(6,27) WRPANC	DIS03770
	WRITE(6,29) WRREIS	DIS03780
	WRITE(6,41) WRNAGD	DIS03790
C	WRITE(6,67) BM	DIS03800
C	WRITE(6,68) BMCHEK	DIS03810
C19	FORMAT(' ', 'W ,CLASSICAL THEORY , WCT =', F15.6)	DIS03820
C21	FORMAT(' ', 'W ,MODIFIED THEORY , WMT =', F15.6)	DIS03830
333	FORMAT(' ', 'BBBBBBBBBBBBBBBBBBBB C2B =', F15.6)	DIS03840
22	FORMAT(' ', 'PRESENT WORK RATIO ; WRP =', F15.6)	DIS03850
72	FORMAT(' ', 'EXACT SOLUTION RATIO; WREXAC =', F15.6)	DIS03860
64	FORMAT(' ', 'PRESENT WORK RATIO ; WRCHEK =', F15.6)	DIS03870
C67	FORMAT(' ', 'BM =', F20.6)	DIS03880
C68	FORMAT(' ', 'BMCHEK =', F20.6)	DIS03890
56	FORMAT(' ', 'BALUCH RATIO ; WRB =', F15.6)	DIS03900
27	FORMAT(' ', 'WRPANC =', F15.6)	DIS03910
29	FORMAT(' ', 'WRREIS =', F15.6)	DIS03920
41	FORMAT(' ', 'WRNAGD =', F15.6)	DIS03930
C		DIS03940
C		DIS03950
C	PRESENT WORK - STRESSES : SIGMAX	DIS03960

```

C
C
WRITE(6,101)
WRITE(6,103)
WRITE(6,101)
103 FORMAT('          STRESSES  ')
503 AZ = A*Z
    BZ = B*Z
    F1Z = A1*Z + A2*DCOSH(AZ) + A3*DSINH(AZ) + A4*DCOSH(BZ)
        + A5*DSINH(BZ)
    F3Z = (A1/6)*Z**3 + A2/A**2*DCOSH(AZ) + A3/A**2*
        DSINH(AZ) + A4/B**2*DCOSH(BZ) + A5/B**2*DSINH(BZ)
        - C1*Z - C2
C W2P = W0/DSIN(AM) + P*C1/E
    W2P = W0/DSIN(AM)
    SIGXP = ( (E*APB*W2P;NUSM1)*Z - P*(2.D0-NU)/NUM1*F1Z
        + (P*APB/NUSM1)*F3Z - (2.*NU*APB*BM/(H**3
        *NUSM1))*Z**3 - P/(NUSM1*H)*( (NU**2-NU-2.)*F2 +
        APB*F4 ) ) * DSIN(AM)
    SIGMAP = SIGMAP + SIGXP
C SIGXP = SIGMAP
C
C PRESENT WORK - STRESSES : SIGMAZ
C
    SIGZP = SIGZP + P*F1Z*DSIN(AM)
C
C
C EXACT SOLUTION - STRESSES : SIGMAX
C
C
    APZ = -ALPHA*Z
    SIGXE = APB*( R1*DSINH(APZ) + R2*DCOSH(APZ) + R3*(2.*
        DCOSH(APZ) + APZ*DSINH(APZ)) + R4*(2.*DSINH(APZ)
        + APZ*DCOSH(APZ)) ) * DSIN(AM)
    SIGMAE = SIGMAE + SIGXE
C SIGXE = SIGMAE
C
C EXACT SOLUTION - STRESSES : SIGMAX
    SIGZE = SIGZE - APB*DSIN(AM)*( R1*DSINH(APZ) + R2*DCOSH(APZ)
        + R3*APZ*DSINH(APZ) + R4*APZ*DCOSH(APZ) )
C
C
C
C PANC'S SOLUTION : STRESSES
C
C
    F1ZP = 0.5 * ( LA*Z*DCOSH(LH2)-DSINH(LA*Z) ) / ( LH2*
;    DCOSH(LH2)-DSINH(LH2) )
    W2PA = WPANC/DSIN(AM)
    SIGXPA = ( (E*APB*W2PA;NUSM1)*Z - ( 2.*P/NUM1 ) * F1ZP )

```

DIS03970  
DIS03980  
DIS03990  
DIS04000  
DIS04010  
DIS04020  
DIS04030  
DIS04040  
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DIS04080  
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DIS04100  
DIS04110  
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DIS04130  
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DIS04150  
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DIS04360  
DIS04370  
DIS04380  
DIS04390  
DIS04400  
DIS04410  
DIS04420  
DIS04430  
DIS04440  
DIS04450  
DIS04460

```

;      *DSIN(AM)
SIGMPA = SIGMPA + SIGXPA
C SIGXPA = SIGMPA
C
C
C BALUCH'S SOLUTION : STRESSES
C
C
SIGB0 = APB*H**2/(48.*NUSM1)
SIGB1 = 12./(APB*H**2) - 3.0/5.0
; + APB*I1**2*(168.*NU**2-195.)/(5600.*NUSM1)
SIGB2 = -APB*H**2/(4.*NUSM1)
SIGB3 = 4. + (APB*I1**2/NUSM1)*(5.-4.*NU**2),20.
SIGB4 = -APB*H**2/(10.*NUSM1)
SIGXB = ( SIGB0 + SIGB1*ZH + SIGB2*ZH**2 + SIGB3*ZH**3
; + SIGB4*ZH**5 ) * P * DSIN(AM)
SIGMAB = SIGMAB + SIGXB
GO TO(592,593)IBAL
592 WRITE(6,18)
WRITE(6,25) H
WRITE(6,111) ZH
WRITE(6,17) M
WRITE(6,590) APB,AM,P
WRITE(6,591) SIGB0,SIGB1,SIGB2,SIGB3,SIGB4,SIGXB,SIGMAB
590 FORMAT(3F10.5)
591 FORMAT(7F10.5)
C SIGXB = SIGMAB
C
C
C BALUCH'S MODIFIED SOLUTION : STRESSES
C
C
593 F1ZB = -0.5D0 + (3./2.)*ZH - 2.*ZH**3
C F3ZB = -0.25*Z**2 + 0.25/H*Z**3 - 0.1/H**3*Z**5
F3ZB = -0.25*Z**2 + 0.25/H*Z**3 - 0.1/H**3*Z**5 + C1B*Z + C2B
F2B = -0.5*H
C F4B = -I1**3/48.
F4B = -H**3/48. + C2B*H
W2B = W0B/DSIN(AM)
SIGXBM = ( (E*APB*W2B/NUSM1)*Z - P*(2.D0-NU) NUSM1*F1ZB
; + (P*APB/NUSM1)*F3ZB -(2.*NU*APB*BMB (I1**3
; *NUSM1))*Z**3 - P/(NUSM1*I1)*( (NU**2-NU-2.)*F2B +
; APB*F4B ) ) * DSIN(AM)
SIGMAM = SIGMAM + SIGXBM
C SIGXBM = SIGMAM
C
C
C STRESSE BY OTHER PLATE THEORIES :
C KIRCHOFF THIN PLATE , REISSNER SHEAR
C DEFORMATION PLATE THEORY, AND NAGHDI-ESENBERG

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DIS04470

DIS04480

DIS04490

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DIS04960



C	TRANSVERSE NORMAL STRAIN THEORY	DIS04970
C		DIS04980
C		DIS04990
	SIGXO = ( 12.*P/(APB*H**2) ) *ZII*DSIN(AM)	DIS05000
	SIGMAO = SIGMAO + SIGXO	DIS05010
C	SIGXO = SIGMAO	DIS05020
C		DIS05030
	GO TO (670,10) IPRINT	DIS05040
670	IF(M.GE.MP) GO TO 644	DIS05050
	GO TO 10	DIS05060
644	WRITE(6,104) SIGMAP	DIS05070
	WRITE(6,105) SIGMAE	DIS05080
	WRITE(6,106) SIGMPA	DIS05090
	WRITE(6,107) SIGMAB	DIS05100
C	WRITE(6,109) SIGMBM	DIS05110
	WRITE(6,108) SIGMAO	DIS05120
104	FORMAT( ' ',PRESENT WORK : SIGMAP = ',F15.6)	DIS05130
105	FORMAT( ' ',EXACT SOLUTION : SIGMAE = ',F15.6)	DIS05140
106	FORMAT( ' ',PANC WORK : SIGMPA = ',F15.6)	DIS05150
107	FORMAT( ' ',BALUCH WORK : SIGMAB = ',F15.6)	DIS05160
109	FORMAT( ' ',BALUCH MOD. WORK: SIGMBM = ',F15.6)	DIS05170
108	FORMAT( ' ',OTHER THEORIES : SIGMAO = ',F15.6)	DIS05180
C		DIS05190
10	CONTINUE	DIS05200
C		DIS05210
	GO TO (667,668) ISTRES	DIS05220
667	CONTINUE	DIS05230
C	WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA,SIGMAB	DIS05240
	WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA,SIGMAB,SIGMAO	DIS05250
C	WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA	DIS05260
C	WRITE(6,558) ZH,SIGMAP,SIGMAE	DIS05270
558	FORMAT(7(F10.2,2X))	DIS05280
C668	IF(I.EQ.31) AHR = 0.D0	DIS05290
668	GO TO (400,401) ISIGZ	DIS05300
400	WRITE(6,402) ZH,SIGZP,SIGZE	DIS05310
402	FORMAT(3(F10.3,2X))	DIS05320
C		DIS05330
401	Z = Z + 0.1 * H	DIS05340
100	CONTINUE	DIS05350
C		DIS05360
	GO TO (502,200) IDEF	DIS05370
502	CONTINUE	DIS05380
C	WRITE(6,225) H,WRP,WREXAC,WRB,WRPANC	DIS05390
	WRITE(6,225) H,WRP,WREXAC,WRB,WRPANC,WRREIS,WRNAGD	DIS05400
C	WRITE(6,225) H,WRP,WREXAC,WRPANC	DIS05410
225	FORMAT(7(F8.2,2X))	DIS05420
200	CONTINUE	DIS05430
	STOP	DIS05460
	END	DIS05470

## REFERENCES

1. S. Timoshenko and S. Woinowsky-Kreiger , Theory of Plates and Shells (McGraw-Hill, New York, 2nd ed., NY, 1959)
2. Reissner, E. "The Effects of Transverse Shear Deformation on the Bending of Elastic Plates", Journal of Applied Mechanics, Vol. 12, No. 2, Transactions ASME, Vol. 67, June 1946, 69-77.
3. Hildebrand, F.B., Reissner, E. and Thomas, G. B., "Notes on the Foundation of the Theory of Small Displacements of Orthotropic Shells", NACA Technical Note No. 1833, 1949.
4. Reissner, E., "Stress-Strain Relations in Theory of Thin Elastic Shells", Journal of Math and Physics, Vol. 31, 1952, 109-119 .
5. Naghdi, P.M., "On the Theory of Thin Elastic Shells", Quarterly of Applied Mathematics, Vol. 14, 1957, 369-380.
6. Essenburg, F., "On the Significance of the Inclusion of the Effect of Transverse Normal Strain in Problems Involving Beams with Surface Constraints," Journal of Applied Mechanics, Vol. 41, No.2, Transactions ASME, Vol. 97, Series E, March 1975, 127-132.

7. Kromm A., "Verallgemeinerte Theorie der Plattensstatik," Ing-Arch.21, 1953.
8. Kromm A., "Uber die Randquerkrafte bei gestutzten Platten," Z. angew. Math. Mech. 35, 1955.
9. Panc, V., "Theories of Elastic Plates", Noordhoff International Publishing, Leyden, The Netherlands, 1975.
10. Baluch, M. H., Voyiadjis, G. Z. and Azad, A. K., "A Refined Theory for Isotropic Plates", Transactions of CSME, Vol. 8, No. 1, 1984, 21-27.
11. Voyiadjis, G.Z., Baluch, M.H., and Chi, W.k., "Effects of Shear and Normal Strain on Plate Bending", Journal of Engineering Mechanics, ASCE, Vol. 111, No. 9, September, 1985, pp. 1130-1143.
12. Salerno, V.L., and Goldberg, M.A., "Effect of Shear Deformation on the Bending of Rectangular Plates", Journal of Applied Mechanics, Vol. 27, Mar., 1960, pp. 54-58.
13. Goma'a, Said Taha Khalil, "Finite Element Modelling For The Bending Of Thick Plates", M.Sc. Thesis, King Fahd University Of Petroleum And Minerals, Dhahran, Saudi Arabia, June 1989.
14. S. Timoshenko and J.N. Goodier , Theory of Elasticity, (McGraw-Hill, New York, 2nd ed., NY, 1951)

15. Little, R.W., *Elasticity*, Prentice-Hall, Englewood Cliffs, N.J., 1973, 109-112.
16. Baluch, M.H. and Voyiadjis, G.S., "A Refined Theory for the Bending of Plates", 15th International Congress of Theoretical and Applied Mechanics, Toronto, Canada, August 1980.
17. Hencky, H., "Über die Berücksichtigung der Schubverzerrungen in Ebenen Platten," *Ing.-Arch*, 16, 1947.
18. Green, A.E., "On Reissner's Theory of Bending of Elastic Plates," *Quarterly of Applied Mathematics*, Vol. 7, 1949, 223.
19. Panc, V., "Verscharfte Theorie der Elastischen Platte", *Ing.-Arch*, 33, 1964.
20. Ambartsunyan, S.A., *Theory of Anisotropic Plates*, Progress in Materials Science Series, Volume II, Technomic Publication, Conn., 1970.
21. Whitney, J.M. and Pagano, N.J., "Shear Deformation in Heterogeneous Anisotropic Plates", *Journal of Applied Mechanics*, Vol. 37, *Transactions ASME*, Vol. 92, Series E, 1970, 1031-1036.
22. Whitney, J.M., "Stress Analysis of Thick Laminated Composite and Sandwich Plates", *Journal Composite Materials*, Vol. 6, 1972, 426-440.

23. Whitmey, J.M. and Sun C.T., "A Higher Order Theory for Extensional Motion of Laminated Composites", *Journal of Sound and Vibration*, Vol. 30, 1973, 85-97.
24. Nelson, R.B. and Lords, D.R., "A Refined Theory of Laminated Orthotropic Plates", *Journal of Applied Mechanics*, Vol. 41, *Transactions ASME*, Vol. 96, Series E,
25. Reissner, E., "On Transverse Bending of Plates, Including the Effect of Transverse Shear Deformation", *International Journal of Solids and Structures*, Vol. 11, 1975, 569-573.
26. Lo, K.H., Christensen, R.M. and Wu, E.M., "A High Order Theory of Plate Deformation - Part 1 : Homogeneous Plates", *Journal of Applied Mechanics*, Vol. 44, *Transactions ASME*, Vol. 99, Series E, 1977, 663-668.
27. Lo, K.H., Christensen, R.M. and Wu, E.M., "A High Order Theory of Plate Deformation - Part 2 : Laminated Plates", *Journal of Applied Mechanics*, Vol. 44, *Transactions ASME*, Vol. 99, Series E, 1977, 669-676.

28. Voyiadjis, G. Z. and Baluch, M.H., "Refinements in the Bending of Plates with One Plane of Elastic Symmetry", Proceedings of the International Symposium on the Mechanical Behavior of Structured Media," Studies in Applied Mechanics, 5, May 1981, Carleton University, Ottawa, Canada.
29. Voyiadjis, G.Z. and Baluch, M.H., "Refined Theory for Flexural Motions of Isotropic Elastic Plates", Journal of Sound and Vibration, Vol. 76(1), 1981, 57-64.