# $H_\infty$ State Feedback with Closed-Loop Transfer Recovery ($H_\infty$ / CLTR) Design Method

by

Hussain Naser Al-Dawaish

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
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In

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**King Fahd University of Petroleum and Minerals (Saudi Arabia), 1991**

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**This thesis written by HUSSAIN NASER AL-DAWAISH under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of *MASTER OF SCIENCE in ELECTRICAL ENGINEERING***

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**TO**

**MY PARENTS**

**AND TO**

**MY WIFE**

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# خلاصة الرسالة


عنوان البحث : طريقة تصميم حالات الـ ($H_\infty$) ذات التغذية المرتدة مع استرجاع انتقال الحلقة المغلقة .

اسم الطالب : حسين ناصر الدويش

التخصص : هندسة كهربائية .

تاريخ التخرجة : محرم ١٤١٢هـ / يوليه ١٩٩١م .

في هذه الرسالة قمنا بتطوير طريقة جديدة لتصميم أنظمة التحكم . وهذه الطريقة تتكون من خطوتين : ١) تصميم نظام تحكم يعتمد على حالات الـ ($H_\infty$) ذات التغذية المرتدة . ٢) استرجاع الأداء المنجز باستخدام راصد الحالات . والمنظم الناتج من عملية التصميم هذه سوف يؤكد الثبات الداخلي وتحت ظروف معينة يخفض إلى الحد الأدنى معيار الـ ($H_\infty$) لتابع الانتقال ذو الحلقة المغلقة . والعمليات الحسابية لإيجاد المنظم بهذه الطريقة سوف تنقص إلى حل معادلة واحدة جبرية ( ريكاتية ) بدلاً من حل معادلتين جبريتين ( ريكاتية ) إذا أستخدمنا طريقة الـ ($H_\infty$) . أما بالنسبة لأنظمة الوقت المنفصل فإن التحويل الثنائي الخطي سوف يستخدم لتحويل مسألة الوقت المنفصل إلى مسألة وقت متصل حيث يتم إنجاز العمليات الحسابية باستخدام طرق التصميم الخاصة بأنظمة الوقت المتصل ثم تحول النتائج إلى الوقت المنفصل .

أيضاً لقد قدمنا طريقة استرجاع باستخدام مراقب الحالة الحالية بالنسبة لأنظمة الوقت المنفصل والتي يكون وقت المعالجة للمنظم كبيراً .

بالنسبة لأنظمة الوقت المنفصل أحادية المدخل والمخرج ذات الطور الأدنى ، لقد طورنا طريقة تصميم مبسّطة لطريقة ( غاوسي الخطية التربيعية مع استرجاع تابع الانتقال ) . في هذه الطريقة أوضحنا بأن العمليات الحسابية لإيجاد المنظم سوف تقل إلى اختيار الأصفار للمنظم أما أقطاب المنظم فسوف تكون هي أصفار النظام المراد التحكم فيه .

أمثلة تصميم عديدة استخدمت لبيان التطورات النظرية لهذه الرسالة .


**درجة الماجستير في العلوم**

**جامعة الملك فهد للبترول والمعادن**

**الظهران ، المملكة العربية السعودية**

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## ABSTRACT

Title : $H_\infty$ STATE-FEEDBACK WITH CLOSED-LOOP TRANSFER RECOVERY ($H_\infty$/CLTR) DESIGN METHOD

By : Hussain Naser Al-Dawaish

Major Field : Electrical Engineering

Date : July 1991

In this thesis, a new technique for control system design is developed. It consists of two steps: 1) design of an $H_\infty$ state feedback control law; 2) recovering the achievable performance using a state observer. The resulting controller will ensure internal stability and under certain conditions minimizes the $H_\infty$-norm of the closed-loop transfer function. The controller computations will reduce to solving one algebraic $H_\infty$ Ricatti equation compared to two algebraic $H_\infty$ Riccati equations using the $H_\infty$ design method.

For discrete time systems, a recovery procedure for the LQG/LQR method using current estimator for compensators with large processing time is introduced.

A simplified LQG/LTR frequency domain design method for SISO discrete time minimum phase systems is developed. For this method, it is shown that the controller computations will reduce to selecting the zeros of the controller, and the poles of the controller will be the same as the zeros of the plant.

Several design examples are used to illustrate the theoretical developments of this thesis.

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# CHAPTER 1

# INTRODUCTION

The last two decades have brought major developments in the design techniques of multivariable feedback systems that can stabilize a plant despite its model uncertainty and can also reduce the sensitivity of the systems to external plant disturbances. The most known are the Linear Quadratic Gaussian with Loop Transfer Recovery (LQG/LTR) and the $H_\infty$ design techniques. The objective of any design method is to find a compensator to meet some design specifications. These specifications relate to nominal stability, stability-robustness to modeling errors and good performance.

An exposition of the LQG/LTR design method was given by Doyle and Stein [12]. The philosophy of the LQG/LTR design method is to design a state feedback loop with desirable performance and robustness properties assuming that all the states are available for feedback (this is called loop shaping). Then, to recover those properties using a state observer with appropriately selected gains which will provide an estimate for the states of the system (this is called loop transfer recovery). Specifically, if the plant is minimum-phase, then the state feedback loop can be approximated (recovered) over a large frequency range by the loop with the observer-based compensator. Since minimum-phase requirement is critical for good recovery, this is a drawback to the application of this method. However, experience has been that the recovery procedure still works but the convergence breaks down near the frequencies where the non-minimum phase zeros are. Moreover, loop transfer recover procedures for non-minimum phase systems have been proposed [15], [13]. The LQG/LTR methodology has been applied to many MIMO feasibility studies involving helicopters [1], submarines [24] and engines [29].

The application of LQG/LTR has been extended to design compensators for discrete-time systems. Basically, the same design methodology was attempted [28]. The results were inferior to that of the continuous-time case, because of the fact that discrete-time feedback systems do not possess all the attractive features which are present in the continuous-time case. For example, stability margins and sensitivity properties are not guaranteed. Moreover, loop shaping for discrete-time systems is more involved. The loop-shaping procedures used for discrete-time systems [28] are still not as effective as its continuous counterpart. Thus, there is a need to consider other methods (e.g. $H_\infty$ method) of designing suitable state feedback gains that result in loops with desired shapes.

Loop transfer recovery techniques for discrete-time systems distinguish two cases for the compensator processing time. When the processing time is negligible (compared to the sampling time) current estimators can be used which result in perfect recovery [28]. On the other hand, it is more appropriate to use prediction estimators when the processing time is significant [26]. Here, we will show that current estimators can also be used for the case of large processing time by introducing a unit-step delay in the control loop.

It is shown in [9] that for minimum-phase continuous-time SISO systems, the LQG/LTR design method centers on locating the zeros of the controller such that the system has desired specifications and closed-loop stable. The poles of the controller will be the same as the zeros of the plant. In this thesis, we will show that this procedure will be also applicable to discrete-time systems and will result in perfect recovery under the condition of minimum-phase and negligible processing time.

A new approach to feedback design is the $H_\infty$ design method which was first introduced by Zames [8]. Mathematically the $H_\infty$ control problem is to minimize a weighted infinity norm of some closed-loop transfer function or a combination of transfer functions over the set of controllers that satisfy the internal stability requirements. In a typical $H_\infty$ control problem, the plant would incorporate additional frequency dependent weights which are selected to express particular stability and performance specifications relevant to the design objectives. Selecting the weights in the $H_\infty$ design method is an important part that requires engineering judgement and experience. Many frequency domain control design problems can be formulated as $H_\infty$ control problem, for example, sensitivity minimization problem, stability-robustness problem (the general $H_\infty$ control problem) [15].

Frequency domain approaches have been initially proposed to solve the $H_\infty$ control problem [5] [15], but the computation of the controller is rather involved. Recently, considerable interest has been focused on state-space approaches for the solution of the $H_\infty$ control problems, which reduced the general $H_\infty$ control problem to solution of two algebraic Riccati equations [13].

$H_\infty$ control problem with state feedback has been studied by several researchers in recent years [18] [19] [23] [13]. It is shown that the $H_\infty$ state feedback gain for which the closed-loop system is stable and the closed-loop transfer function is minimized, can be obtained by solving one algebraic Riccati equation. An interesting result [23] shows that if the measured outputs are the state of the system, then the infimum of the $H_\infty$ norm of the closed-loop transfer function using linear static state feedback equals the infimum of the $H_\infty$ norm of the closed-loop transfer function over all stabilizing dynamic state feedback controllers. This result motivates us to use a hybrid control structure consisting of $H_\infty$ state feedback gain and state estimator for the case when all the states are not available for feedback. This new design method developed here consists of two steps: assuming all the states are available for feedback, design an $H_\infty$ state feedback control which will minimize the $H_\infty$ norm of the state feedback closed-loop transfer function, then recover the achievable performance using a state estimator. This method will be called $H_\infty$ state feedback with closed-loop transfer recovery ($H_\infty$/CLTR) and it will ensure internal stability and that the $H_\infty$ norm of the closed-loop transfer function for stable plants is minimized. Moreover, the computations of an $H_\infty$/CLTR controller will involve one algebraic Riccati equation only.

An explicit development of $H_\infty$ control problem for discrete-time system is not yet available, except for some work for $H_\infty$ state feedback [17] [10]. However, these suffer from some computational difficulties related to the solution of a Riccati equation. Therefore, the bilinear transformation will be used to transfer discrete-time problem into continuous-time one, carry out the computations using continuous $H_\infty$/CLTR design method, then transform back the controller.

## 1.1 Thesis Contributions

- A new technique for control system design is developed. This design technique consists of two steps: 1) assuming that all the states are available for feedback, design $H_\infty$ state feedback control law which will minimize the $H_\infty$ norm of the state feedback closed-loop transfer function; 2) recover the state feedback closed-loop transfer function using state observer. The resulting controller will ensure internal stability and for stable plants minimizes the $H_\infty$ norm of the closed-loop transfer function

- A recovery procedure using current estimator for discrete time systems with large compensator processing time is introduced.

A simplified LQG/LTR frequency domain design method for SISO minimum phase discrete time systems is developed.

## 1.2 Organization of the Thesis

**Chapter 1. Introduction.**

**Chapter 2. Continuous Time Feedback Control Systems and the $H_\infty$/CLTR Design Method**

In this chapter basic definitions of continuous time feedback control systems and controller synthesis techniques known as LQG/LTR and $H_\infty$ will be briefly reviewed. The $H_\infty$ state feedback with closed loop transfer recover ($H_\infty$/CLTR) design method is introduced. This new design method developed here uses the $H_\infty$ state feedback gains combined with state observer to design the compensator.

**Chapter 3. Discrete Time Feedback Control Systems and the $H_\infty$/CLTR Design Method**

This chapter examines the same issues as chapter 2 but for discrete time systems. A recovery procedure using current estimators for compensators with large processing time is presented. The $H_\infty$/CLTR method explained in chapter 2 can be used with bilinear transformation to design compensators for discrete time systems. A simplified LQG/LTR frequency domain design method for SISO minimum-phase systems is introduced.

**Chapter 4. Examples**

Chapter 4 presents several examples that illustrate compensator design techniques for continuous and discrete time systems.

Chapter 5. Conclusion.

## 1.3 Notation and Definitions

**Notation**

| | |
|---|---|
| $R$ | real Euclidean spaces |
| $R^{nxm}$ | space of *nxm* real matrices |
| $C$ | complex Euclidean spaces |
| $C^{nxm}$ | space of *nxm* complex matrices |
| $A^T$ | transpose of matrix $A$ |
| $A^H$ | complex conjugate transpose of a matrix $A$ |
| $I$ | identity matrix |
| $tr(A)$ | the sum of the diagonal elements of $A$ |
| $\bar{\sigma}(A)$ | the largest singular value of $A$ |
| $\underline{\sigma}(A)$ | the smallest singular value of $A$ |
| $\rho(A)$ | the spectral radius of $A$ |
| $\lVert A\rVert_F^2$ | the Frobenius norm of $A$, defined by $\lVert A\rVert_F^2 \underline{\Delta} tr(A^T A)$ |
| $\lVert G\rVert_\infty$ | the $H_\infty$-norm of $G$ defined as: $\sup_{\omega\in R}\bar{\sigma}(G\,j\omega))$ |
| $L^{-1}\{\cdot\}(t)$ | inverse Laplace transform of $\cdot$ |
| $Z^{-1}\{\cdot\}(k)$ | inverse *z*-transform of $\cdot$ |
| $\lvert\cdot\rvert$ | absolute value of $\cdot$ |
| SISO | Single Input Single Output |
| MIMO | Multiple Input Multiple Output |
| KBF | Kalman-Bucy Filter |
| LQR | Linear Quadratic Regulator |
| LQG | Linear Quadratic Gaussian |
| LTR | Loop Transfer Recovery |

LQG/LTR Linear Quadratic Gaussian with Loop Transfer Recovery

$H_\infty$/CLTR $H_\infty$ state feedback with Closed Loop Transfer Recovery

LTI Linear Time Invariant

**Definitions**

- $H_2$ norm: The $H_2$ norm is defined in the frequency domain for a stable transfer matrix *G(s)* as

$$\|G(s)\|_2 := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} trace[G(jw)^* \dot{G}(jw)]dw \right)^{1/2}$$

- $H_\infty$ norm : The $H_\infty$ norm is defined in the frequency domain for a stable transfer matrix *G(s)* as

$$\|G(s)\|_\infty := \sup_{w \in R} \bar{\sigma}[G(jw)]$$

- The transfer function of a system with state space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is given by

$$G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
$$= C(sI - A)^{-1}B + D$$

**Matrix Inversion Lemma**

**If $A$ and $C$ are nonsingular $mxm$ and $nxn$ matrices, respectively. Then**

$$[A+BCD]^{-1} = A^{-1} - A^{-1}B[DA^{-1}B + C^{-1}]^{-1}DA^{-1}$$

# CHAPTER 2

# CONTINUOUS TIME FEEDBACK CONTROL SYSTEMS AND THE $H_\infty$/CLTR DESIGN METHOD

## 2.1 Introduction

In this chapter basic definitions of continuous time feedback control systems are briefly reviewed. This includes plant model, uncertainties, type of inputs, control objectives, and performance and robustness specifications. Also, in this chapter we consider some recent design techniques for multivariable systems. The particular techniques we shall be concerned with here are the linear quadratic Gaussian with loop transfer recovery (LQG/LTR) and the $H_\infty$ optimal control which is relatively a new approach to feedback design. We shall state the main results of these methods. For more details, we refer the reader to [12], [20], [15], [28] for LQG/LTR method and [15], [5], [13] for $H_\infty$ method and [24], [1], [29], [21] for application of these techniques to specific design examples. Also in this chapter, a new technique is developed for control system design. In this technique we combine the $H_\infty$ state-feedback method described in [18] with the Closed-Loop Transfer Recovery (CLTR). This method will be called $H_\infty$/CLTR technique.

## 2.2 Control System Description

### 2.2.1 Classical Feedback System

The block diagram of the standard feedback control system is shown in Fig. 2.1. It consists of the plant $G(s)$, compensator $K(s)$ forced by command input $(r)$, measurement noise $(n)$ and disturbance $(d)$. The measured variable $(y)$ is corrupted

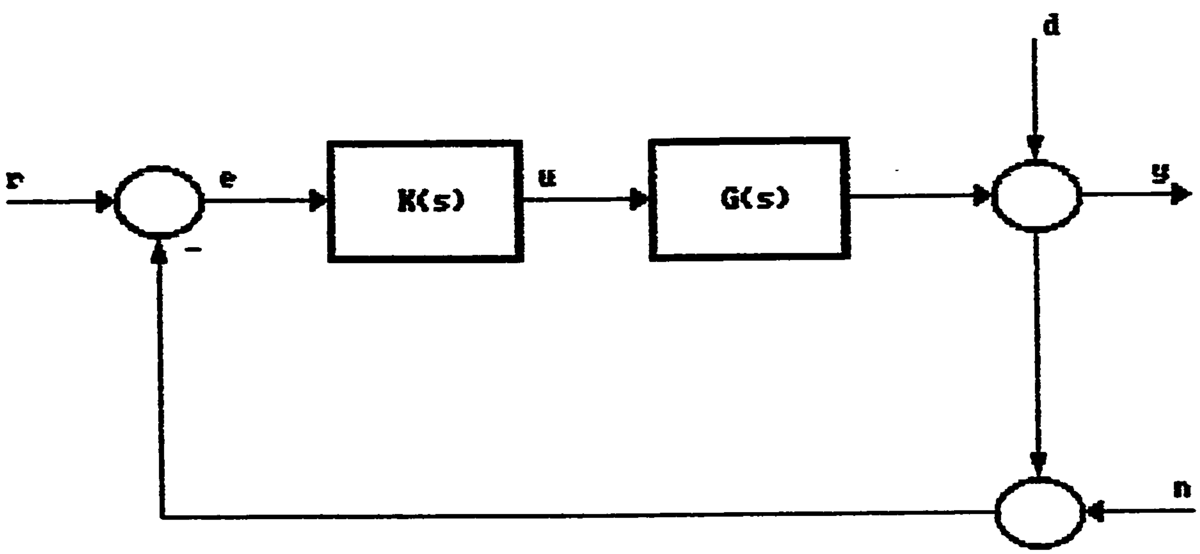



Fig.2.1 : Standard Feedback Control System

by the noise *(n)*. The compensator *K(s)* determines the plant input *u* on the basis of the error *e (e=r-y)*. The objective of the control system is to keep the output *(y)* close to the reference *(r)*.

The dynamical behaviour of the plant is modelled in the time domain by a linear time-invariant system as

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.2.1-1)$$

$$y(t) = Cx(t) + Du(t) \quad (2.2.1-2)$$

where $x \in R^n, y \in R^m, u \in R^p$ are the state output, and input vectors respectively and *A,B,C,* and *D* are constant matrices of appropriate dimensions. The transfer function of the plant is $G(s) = C(sI - A)^{-1}B + D$.

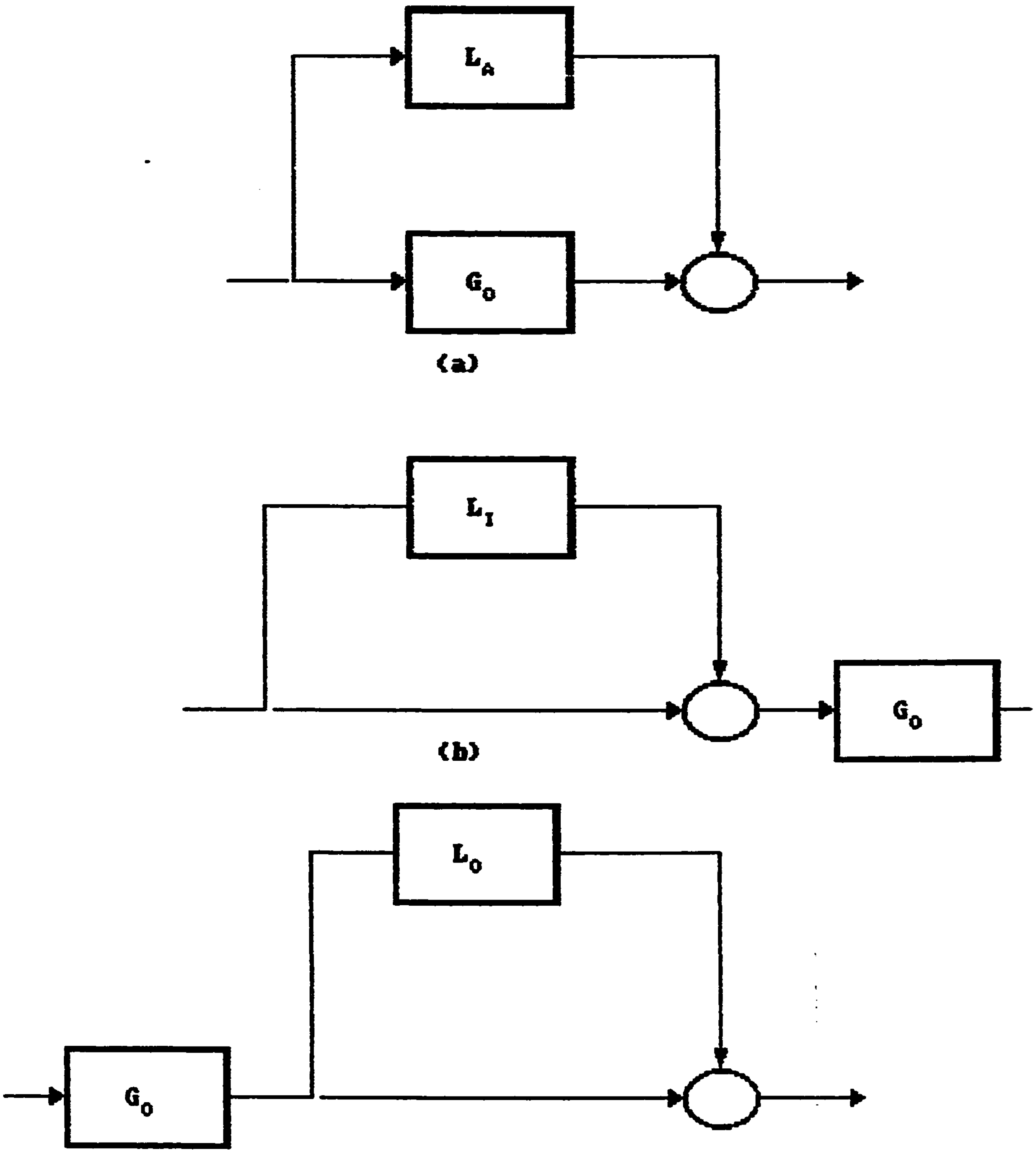
## 2.2.2 Plant Model and Uncertainty Description

The dynamical behaviour of a plant is described not only by a single linear time-invariant model but by a family of models. This family can be parametrized in many different ways. The most common types of uncertainty structures [22] are shown in Fig. 2.2.

In Fig. 2.2a the actual model *G(s)* is assumed to be related to the nominal plant $G_o(s)$ by an additive uncertainty, i.e.

$$G(s) = G_o(s) + L_a(s) \quad ;where \quad L_a(s) = G(s) - G_o(s) \qquad (2.2.2-1)$$

where in Fig. 2.2b, the actual model is related to the nominal plant by a multiplicative




(c)

**Fig.2.2 : Types of Uncertainty.(a) Additive.(b) Multiplicative at the Input.(c) Multiplicative at the Output**

input uncertainty, i.e.

$$G(s) = G_o(s)\ [I + L_i(s)]\ ;$$

*where* $$L_i(s) = G_o^{-1}(s)[G(s) - G_o(s)] \qquad (2.2.2-2)$$

and in Fig. 2.2c by a multiplicative output uncertainty, i.e.

$$G(s) = [I + L_o(s)]\ G_o(s)\ ;where\ \ L_o(s) = [G(s) - G_o(s)]G_o^{-1}(s) \qquad (2.2.2-3)$$

The uncertainty is typically constrained by the following magnitude bounds.

$$\bar{\sigma}(L_a(jw)) \le l_a(w) \ ,w \ge 0 \qquad (2.2.2-4)$$

$$\bar{\sigma}(L_i(jw)) \le l_i(w) \ ,w \ge 0 \qquad (2.2.2-5)$$

$$\bar{\sigma}(L_o(jw)) \le l_0(w) \ ,w \ge 0 \qquad (2.2.2-6)$$

The main sources of uncertainty are

* parameter variations
* order reduction
* time delays
* neglected non-linearities

### 2.2.3 Input Specifications

The inputs are all external signals entering the feedback loop at some point. For the control system design, the inputs have to be specified. The specifications include a description of the magnitude, energy and frequency content of the signals. These specifications are usually given as a weighted $L^2$ norms [28].

$$\left\| L^{-1}\{W_r(s)r(s)\}\right\|_2 \le 1$$

$$\left\| L^{-1}\{W_d(s)d(s)\}\right\|_2 \le 1$$

$$\left\| L^{-1}\{W_n(s)n(s)\}\right\|_2 \le 1$$

where $W_r$, $W_d$, and $W_n$ are appropriate weighting that reflect the magnitude and frequency content of the commands (*r*), disturbances (*d*), and noise (*n*) respectively.

## 2.3 Control Objectives

In order for the compensator to work well on the actual plant the following objectives have to be met:

* nominal stability
* nominal performance
* robust stability

### 2.3.1 Sensitivity and Complementary Sensitivity Functions

The most important relationships between inputs and outputs in Fig. 2.1 are

$$y = (I + GK)^{-1} d \quad ; for \quad n = 0 \qquad (2.3.1-1)$$

$$e = (I + GK)^{-1} (d - r) \quad ; for \quad n = 0 \qquad (2.3.1-2)$$

$$y = GK(I + GK)^{-1} (r - n) \quad ; for \quad d = 0 \qquad (2.3.1-3)$$

The sensitivity function is defined as

$$S(s)=(I+GK)^{-1} \qquad (2.3.1-4)$$

and the complementary sensitivity is defined as

$$T(s)=GK(I+GK)^{-1} \qquad (2.3.1-5)$$

and thus,

$$S(s)+T(s)=I \qquad (2.3.1-6)$$

The sensitivity function *S(s)* relates the external inputs *(d-r)* to the error *(e)*. It also expresses the affect of the disturbance *(d)* on the output *(y)*. The performance of the feedback control system can be judged by the behaviour of the sensitivity function. To eliminate the effect of the disturbance *(d)* on the output *y*, we need to make *S(s)* as small as possible, if *GK* is strictly proper (which is always the case for physical systems) then

$$\lim_{s\to\infty} S(s)=\lim_{s\to\infty}(I+GK)^{-1}=I \qquad (2.3.1-7)$$

Thus, *S(s)* can be made small only over a finite frequency range, and *S(s)* can not be made equal to zero due to the physical limitations.

The complementary sensitivity *T(s)* relates the reference *(r)* to the output *(y)*. Therefore, to have the output *(y)* follows the reference input *(r)*, *T(s)* should be made unity. However, because of (2.3.1-7)

$$\lim_{s\to\infty} Gk(I+Gk)^{-1}=0$$

Thus *T(s)* can be made equal to unity over a finite frequency range only.

The complementary sensitivity *T(s)* also describes the effect of the measurement noise *(n)* on the output *(y)*. Thus *T(s)* should be made as small as possible to supress the noise. This shows the trade-offs inherent in feedback design, i.e.

* reference input tracking and disturbance rejection which require $S \approx 0, T \approx I$

* suppresion of measurement noise which require $S \approx I, T \approx 0$

2.3.2 Nominal Stability

The minimal requirement on the compensator is to stabilize the closed-loop nominal system. The conditions for nominal stability can be derived in terms of the internal stability of the system. A linear time invariant system is internally stable if the transfer function between any two points of the control system is stable (i.e.,have all poles on the open left half plane (LHP) ) [22].For example, for the system in Fig.2.3 the system is stable if and only all elements in the 2x2 transfer matrix (2.3.2.-1) have all their poles in the open LHP.

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} G_o K(I+G_o K)^{-1} & (I+G_o K)^{-1} G_o \\ K(I+G_o K)^{-1} & -K(I+G_o K)^{-1} G_o \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \qquad (2.3.2-1)$$

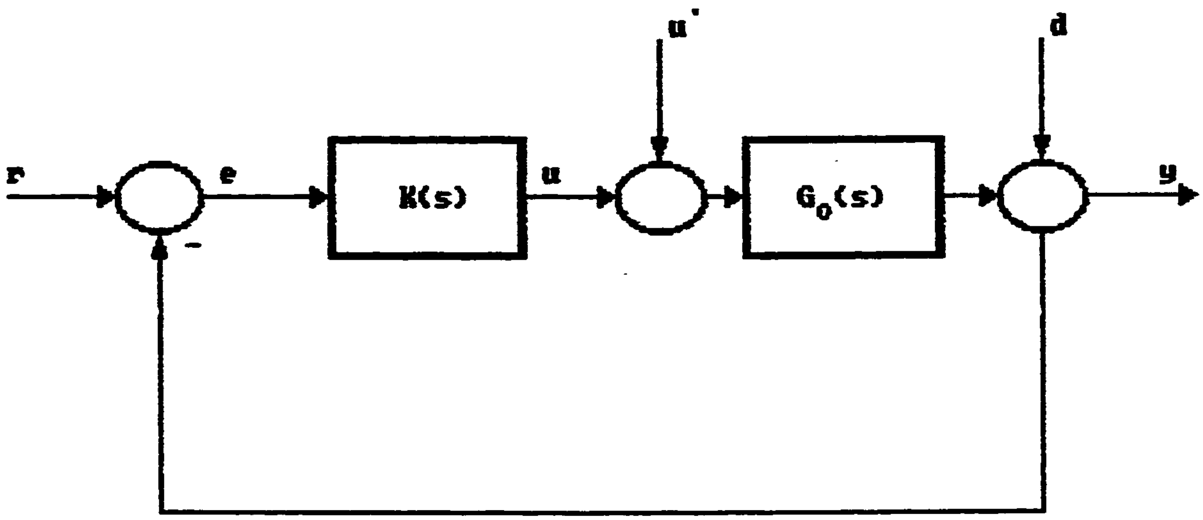



Fig.2.3 : Internal Stability Block Diagram

### 2.3.3 Nominal Performance

The most basic objective of a feedback compensator is to keep the error between the plant output *(y)* and the reference *(r)* small when the overall system is affected by the external signals *(r)*, *(n)*, and *(d)*. The singular values of $S_o$ ($S_o$ is the sensitivity function for the nominal plant $G_o$) determine the disturbance attenuation since $S_o$ is the closed-loop transfer function from disturbance *(d)* to the plant output *(y)*. Thus a disturbance attenuation performance specification may be written as [22].

$$\bar{\sigma}(S_o(jw)) \le |W^{-1}(jw)| \qquad (2.3.3-1)$$

where $|W^{-1}(jw)|$ is the desired disturbance attenuation factor. Allowing $W(jw)$ to depend on frequency $w$ enables one to specify a different attenuation factor for each frequency $w$. Note that $\bar{\sigma}(G_oK)$ is small for high frequencies and therefore

$$\bar{\sigma}(S_o) = \bar{\sigma}(I + G_oK)^{-1} \cong I \qquad w \ \ large$$

Thus, tight performance specifications are only meaningful in the low frequency range where $G_oK$ is large, then the performance specification reduces to

$$\underline{\sigma}(I + G_oK) \approx \underline{\sigma}(G_oK) \ge |W(jw)| \qquad w \ \ small \ \ (2.3.2-2)$$

### 2.3.4 Robust Stability

Assume that all the plants $G$ in the family of plants describing the system have the same number of right half plane (RHP) poles and that a particular compensator $K$ stabilizes the nominal plant $G_o$. Then, the system is robustly stable with the compensator $K$ if and only if the complementary sensitivity functions $T_o$ for the nominal plant $G_o$ satisfies the following Bound (assuming that the uncertainty is multiplicative at the system output, (i.e. $G = (I + L_o)G_o$ and $\bar{\sigma}(L_o) \leq l_o$) [22].

$$\bar{\sigma}(T_o)l_o \leq 1 \quad , \forall w \Leftrightarrow \|T_o l_o\|_\infty \leq 1 \qquad (2.3.4-1)$$

Note that for high frequencies $G_o K$ is small and

$$\bar{\sigma}(T_o) = \bar{\sigma}(G_o K(I + G_o K)^{-1}) \equiv \bar{\sigma}(G_o K)$$

therefore (2.3.4-1) becomes

$$\bar{\sigma}(G_o K) \leq l_o^{-1} \qquad w \text{ large} \qquad (2.3.4-2)$$

Thus, the loop gain $\bar{\sigma}(G_o K)$ has to be shaped to fall below the uncertainty bound $l_o^{-1}$. If the uncertainty is multiplicative at the input such that

$$G = G_o(I + L_i) \text{ and } \bar{\sigma}(L_i) \leq l_i$$

then the system is robustly stable with compensator $K$ if and only if

$$\bar{\sigma}(KG_o) \leq l_i^{-1} \qquad w \text{ large} \qquad (2.3.4-3)$$

Thus, the loop gain $\bar{\sigma}(KG_o)$, which is generally not equal to $\bar{\sigma}(G_o K)$, has to be shaped to fall below the uncertainty bound $l_i^{-1}$.

### 2.3.5 Summary

The design specifications can be reduced to bounds on the maximum and minimum singular values of the loop gain $G_o K$ (assuming multiplicative output uncertainty) as

$$\underline{\sigma}(G_o(jw)K(jw)) \geq |W_1(jw)| \quad w \text{ small} \quad (2.3.5-1)$$

$$\underline{\sigma}(G_o(jw)K(jw)) \leq |W_2^{-1}(jw)| \quad w \text{ large} \quad (2.3.5-2)$$

or as bounds on the sensitivity and complementary sensitivity functions as

$$\bar{\sigma}(S_o(jw)) \leq |W_1^{-1}(jw)|$$

$$\bar{\sigma}(T_o(jw)) \leq |W_2^{-1}(jw)|$$

if the system is nominally stable, these conditions will ensure good performance and robust stability. These conditions are depicted graphically in Fig. 2.4.

## 2.4 LQG/LTR Design Method

The LQG/LTR design method assumes that a nominal model is available, which is linear time-invariant and is represented by a state differential and output equations

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (2.4-1)$$

$$y(t) = Cx(t) \qquad (2.4-2)$$

where $x \in R^n$ is the plant states, $u \in R^m$ is the plant input, $y \in R^m$ is the plant output, and $A, B, C$ are constant real matrices of appropriate dimensions. The transfer function of the nominal system is given by

$$G_o(s) = C(sI - A)^{-1}B \qquad (2.4-3)$$

This design procedure attempts to find a compensator *K(s)* which will stabilize the nominal closed-loop system of Fig. 2.1. and satisfy performance and robustness conditions (2.3.5-1) and (2.3.5-2).

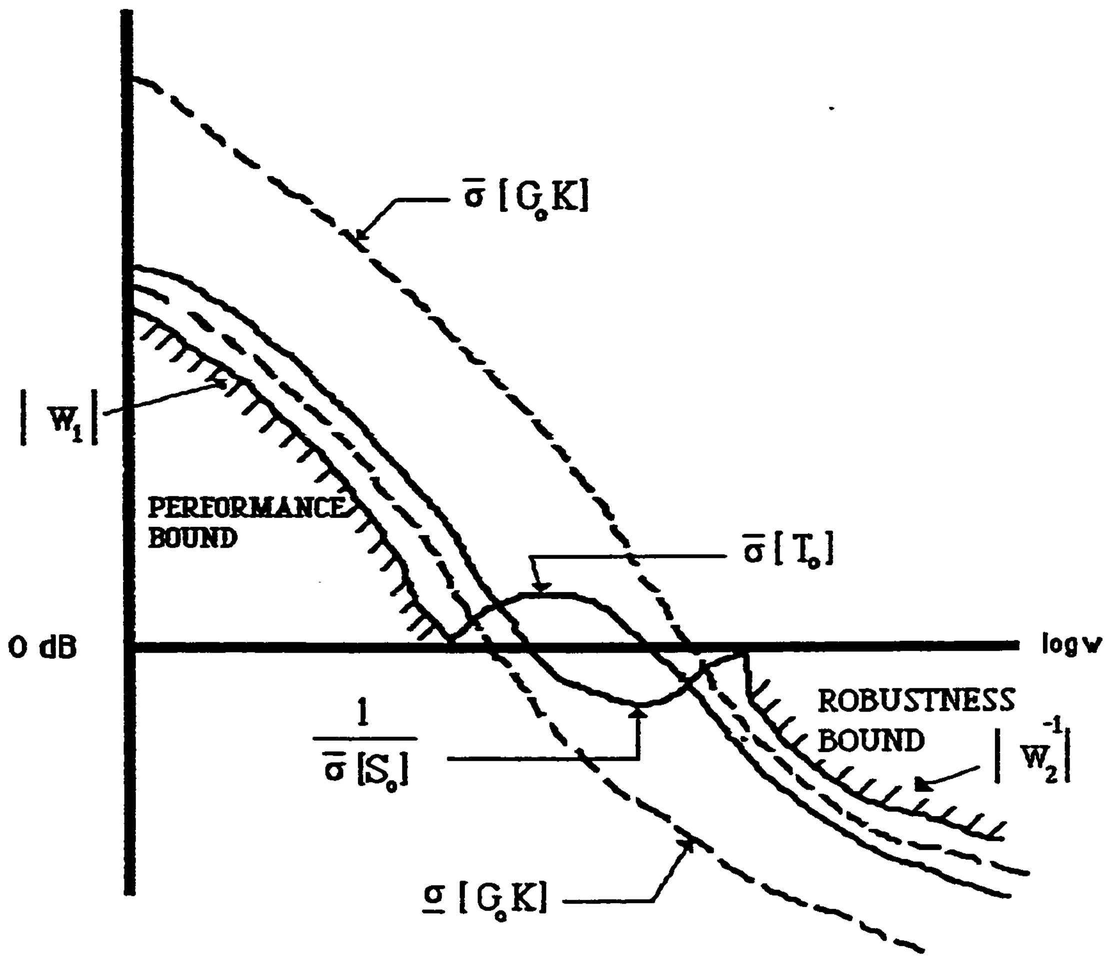



Fig. 2.4: Performance and Robustness specifications

The internal LQG/LTR compensator is shown in Fig. 2.5. where a full state linear feedback gain $K_c$ operates on an estimate of the plant states provided by a linear filter with gain $K_f$. The design is accomplished by determining the free parameters $K_c$ and $K_f$ so that the compensator *K(s)* is constructed.

Since the plant and compensator operate on vector signals, their transfer functions do not commute, and an analysis point of reference must be chosen. All design objectives and model uncertainties must be expressed in terms of bounds on signals that occur at this point reference. The designer must choose one of two possible points of reference; these are the plant input node, denoted by *(u)*, and the plant output node, denoted by *(y)* (Fig. 2.5).

The LQG/LTR design method involves two basic steps (assuming that the input node is used as the reference point): a full-state feedback design with loop transfer function which has the desired loop shape and then an approximation of this full-state loop transfer function with a realizable LQG controller using a recovery procedure.

### 2.4.1 Linear Quadratic Regulator (LQR)

The first step in designing LQG/LTR compensator is to design a full-state feedback system with loop transfer function $L(s) = K_c(sI - A)^{-1}B$ to produce the target feedback loop shown in Fig. 2.6.

*L(s)* should satisfy the performance and robustness constraints of Fig. 2.4. $K_c$ is found by solving the following linear quadratic regulator problem

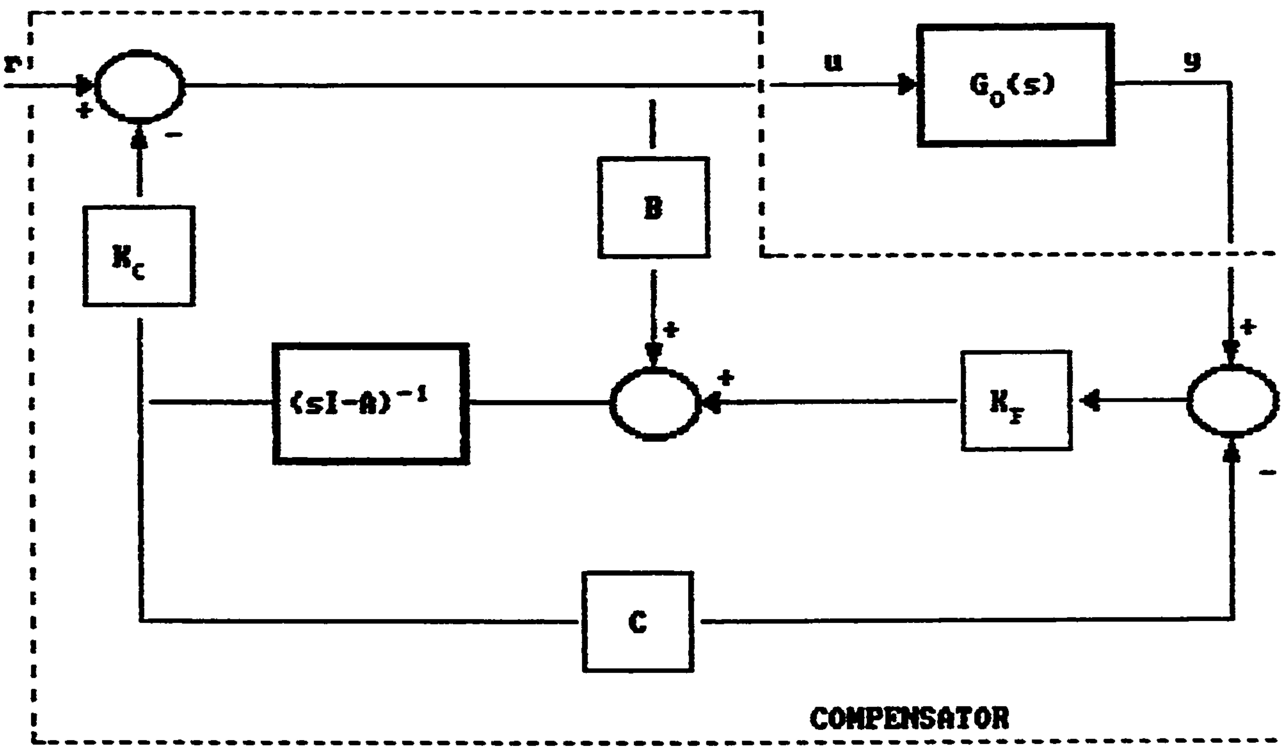



Fig.2.5 : LQG/LTR Compensator

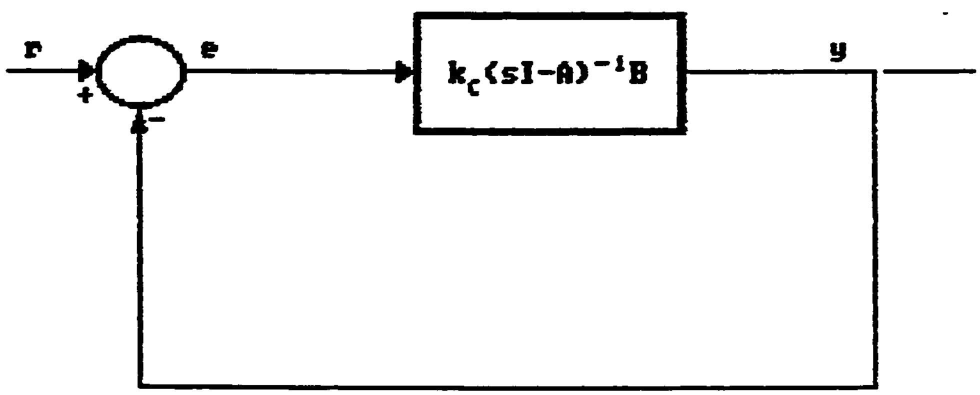



**Fig.2.6 : Target Feedback Loop**

minimize $J=\frac{1}{2}\int_0^{\infty}(x^TQx+u^TRu)dt \quad Q\geq 0 \quad R>0$

subject to $\dot{x}=Ax+Bu \quad x\in R^n \;, \; u\in R^m$

The solution to this optimization problem is given by:

$$u=-K_c x$$

where $K_c$ is defined by the following Riccati equation

$$A^TP_c+P_cA+Q-P_cBR^{-1}B^TP_c=0 \qquad (2.4-4)$$

$$K_c=R^{-1}B^TP_c \qquad (2.4-5)$$

where it is assumed that *(A,B)* is stabilizable and *(A,H)* is detectable. Also, there is no loss of generality in taking $Q=H^TH$, and $R=\rho I$ where $H$ is an mxn matrix [32].

Shaping the singular values of loop transfer function $L(s)=K_c(SI-A)^{-1}B$ is based on the following Kalman equality equation [12]

$$[I+L(jw)]^HR[I+L(jw)]=R+W^H(jw)W(jw) \qquad (2.4-6)$$

where $W(s)=H(sI-A)^{-1}B$

The design parameters $\rho$ and $H$ influence the loop transfer function *L(s)* in the following way [12]:

$$\sigma_i[L(j\omega)] \approx \frac{1}{\sqrt{\rho}}\ \sigma_i[W(j\omega)] \qquad (2.4-7)$$

Since loop transfer function depends on the original plant dynamics, additional dynamics may be appended to the original plant to achieve certain objectives. For example integrators may be appended to the original plant to achieve zero steady state error.

The performance specifications (Fig. 2.4) can be achieved by choosing $H$ such that at low frequencies

$$W(j\omega) \approx \frac{1}{j\omega}I$$

At high frequencies, under the assumption of minimum phase on $H(sI - A)^{-1}B$, it is shown in [12] that the gain $K_c$ behaves asymptotically as

$$\sqrt{\rho}\ \ K_c \rightarrow WH \quad as \quad \rho \rightarrow o$$

from which it follows that the maximum asymptotic crossover frequency of the loops is [12]

$$\omega_{c,\max} = \frac{\bar{\sigma}[HB]}{\sqrt{\rho}} \qquad (2.4-8)$$

Thus, from (2.4-8) we can select $\rho$ to adjust the crossover frequency consistent with the stability robustness constraints.

### 2.4.2 Loop Transfer Recovery

The second step in designing LQG/LTR compensator is to construct an approximation of the full-state loop transfer function *L(s)* of section 2.4.1 with a realizable compensator using a recovery procedure. The loop transfer function *L(s)* is only an intermediate design function since it assumes that all the states are available for feedback. Thus the second step of the design is to provide an estimate of the states by processing the output measurements using state estimators. There are many LTR procedures which all can be formulated in terms of Luenberger-observer [4]. Also, it should be pointed out that the state feedback design of section 2.4.1 can be performed completely independent of the specific LTR procedure chosen.

#### 2.4.2.1 The Luenberger Observer

Let the LTI plant model be represented by

$$\dot{x} = Ax + Bu \qquad , x \in R^n \quad , u \in R^m$$

$$y = Cx \qquad , y \in R^m$$

with $n > m$, $(C,A)$ observable and $(C,B)$ of full rank.

Assume the plant is controlled by an observer-based compensator with state feedback

$$u = -K_c \hat{x}$$

where $K_c$ is the state feedback gain and $\hat{x}$ is the state estimate. The states are estimated using a Luenberger observer [4].

$$\dot{z} = Dz + Gu + Ey$$

$$w = K_c\hat{x} = Pz + Vy \qquad (2.4.2.1-1)$$

where $z$ is the observer state vector.

The observer matrices *T,D,E,G,P,V* satisfy [4]

i) $D$ is the stability matrix

ii) $TA - DT = EC$

iii) $G = TB$

iiii) $K_c = PT + VC$

condition (ii) and (iii) imply that the observer error

$$e(t) = z(t) - Tx(t)$$

satisfies $\dot{e}(t) = De(t)$

and $\lim_{t \to \infty} e(t) = 0$

because $D$ is a stability matrix. Based on the equations for the plant and compensator, it is possible to determine the following transfer function of the compensator *K(s)*

$$K(s) = V + P(sI - D + GP)^{-1}(E - GV)$$

The exact LTR at the input loop breaking point is defined as

$$K(jw)G_o(jw)=K_c(jwI-A)^{-1}B \quad \text{for all } w$$

This condition is equivalent to $B=0$, thus it can be met only asymptotically. The recovery error matrix is defined as [6]

$$E(s)=K_c(sI-A)^{-1}B-K(s)G_o(s)$$

*E(s)* can be written as

$$E(s)=M(s)(I+M(s))^{-1}(I+L(s))$$

where

$$M(s)=P(sI-D)^{-1}G$$

The transfer matrix *M(s)* is called the recovery matrix [6]. *M(s)* is of great significance because it describes the mismatch between the actual and the desired transfer function. It can also be used to compare the different LTR procedures. The exact recovery is obtained if and if

$$E(jw)=0 \qquad \text{for all } w$$

or

$$M(jw)=0 \qquad \text{for all } w$$

2.4.2.2 The Full-Order Observer

The full-order observer is the most used observer type and it appears from the Luenberger observer by the following selection of the matrices in the Luenberger observer [4]

$$
\begin{aligned}
D &= A - FC \\
G &= B \\
P &= K_c \\
E &= F \\
V &= 0 \\
T &= I
\end{aligned}
$$

where F is the observer gain matrix.

The recovery matrix is given by

$$M(s) = K_c(sI - A + FC)^{-1}B$$

The Kalman Buey Filter (KBF) is a full-order observer and is defined by the following equations [4]

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - \hat{y})$$

$$\hat{y} = Cx$$

where $K_f$ is the filter gain defined by the following Riccati equation

$$AP_f + P_f A^T + R_w - P_f C^T R_v^{-1} C P_f = 0$$

$$K_f = P_f C^T R_v^{-1}$$

and the compensator transfer function is

$$K(s) = K_c(sI - A + BK_c + K_f C)^{-1} K_f$$

Comparing with the Luenberger observer, we get

$$D = A - K_f C$$
$$G = B$$
$$P = K_c$$
$$E = K_f$$
$$V = 0$$
$$T = I$$

Based on these matrices, the recovery matrix is $M(s) = K_c(sI - A + K_f C)^{-1} B$. Now in order to approximate the loop transfer matrix *L(s)* of section (2.4.1), the parameters $R_w$ and $R_v$ without loss of generality can be taken as $R_w = \Gamma\Gamma^T$ and $R_v = I$, where $\Gamma = qB$. It is shown [12] that

$$\lim_{q \to \infty} K(s)G(s) = L(s) \quad \text{(point wise in } s\text{)}$$

2.4.2.3 The Minimal-Order observer

let the plant defined by the LTI model

$$\dot{x} = Ax + Bu \quad ; x \in R^n \quad , u \in R^r$$

$$y = Cx \qquad , y \in R^m$$

and the matrices A, B and C be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad , B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$C = [I_m \quad o]$$

Then the minimal-order observer is given by [16]

$$D = A_{22} - V_2 \quad A_{12}$$

$$G = B_2 - V_2 \quad B_1$$

$$P = K_c \begin{bmatrix} o & \\ & I_{n-m} \end{bmatrix}$$

$$E = A_{21} - V_2 \quad A_{11} + A_{22} \quad V_2 - V_2 \quad A_{12} \quad V_2$$

$$V = K_c \begin{bmatrix} I_m \\ V_2 \end{bmatrix}$$

$$K_c = [K_1 \quad K_2]$$

$$T = [-V_2 \quad I]$$

where D, G, P, E and V are the Luenberger observer matrices, and $V_2$ is the minimal-order observer gain. The recovery matrix of the minimal-order observer is given by

$$M(s) = K_2(sI - A_{22} + V_2 + A_{12})^{-1}(B_2 - V_2 \quad B_1)$$

## 2.5. $H_\infty$ Design Method

### 2.5.1 Introduction

This section is concerned with $H_\infty$ optimal control. This relatively new approach to feedback design is briefly explained. In this section, we shall use the representation shown in Fig. 2.7, which is called the Linear Fractional Transformation (LFT) of *K(s)* and *P(s)*. *P(s)* is related to the nominal plant and some frequency weighting functions as will be shown in the next sections. The signal $w$ of Fig. 2.7 contains all external inputs, including disturbances, sensor noise, and commands; the output $z$ is an error signal; $y$ is the measured variables; and $u$ is the control input. A state-space algorithm to design the compensator K(s) which is described in [13] will be briefly summarized. Suppose that *P(s)* is partitioned as

$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \qquad (2.5.1-1)$$

so that

$$z=P_{11}(s)w + P_{12}(s)u \qquad (2.5.1-2)$$

$$y=P_{21}(s)w + P_{22}(s)u \qquad (2.5.1-3)$$

Then we can write

$$z = [P_{11}(s)+P_{12}(s)K(s)(I-P_{22}(s)K(s))^{-1}P_{21}(s)]w \qquad (2.5.1-4)$$

Define $F(P,K)=[P_{11}(s)+P_{12}(s)K(s)(I-P_{22}(s)K(s))^{-1}P_{21}(s)]$

Then $z=F(P,K)w$

By suitably defining *P(s)*, a number of practical design problems can be put into form [15]

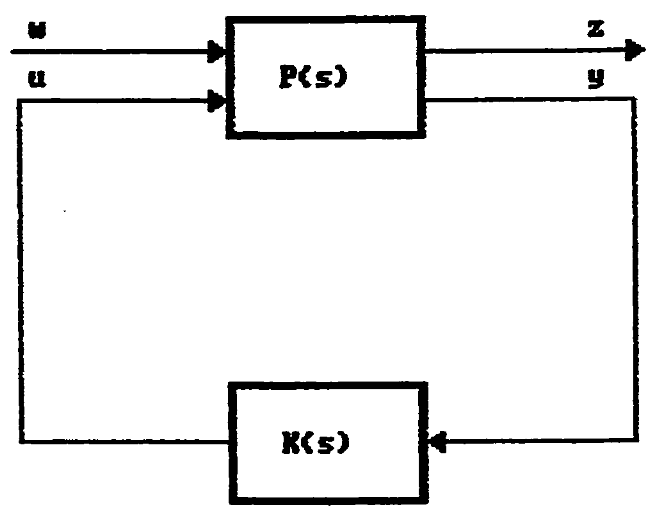



Fig.2.7 : Linear Fractional Transformation

$$\text{minimize} \quad \| F(P,K) \|_\infty$$

where the minimization is over all realizable controllers *K(s)* which stabilize the closed-loop. This is known as the $H_\infty$-optimization problem. In this section we will show different $H_\infty$ problems and the state-space algorithms for the general $H_\infty$ problem.

### 2.5.2 Sensitivity Minimization Problem

The sensitivity minimization or equivalently the disturbance attenuation problem can be written as (section 2.3)

$$\| W_1 S_o \|_\infty \leq \gamma$$

where $S_o = (I + G_o(s)K(s))^{-1}$

and $W_1(s)$ is an appropriate frequency weighting.

Since we require that

$$F(P,K) = W_1(I + G_oK)^{-1}$$

and $W_1(I+G_oK)^{-1} = W_1[I - G_oK(I+G_oK)^{-1}]$ (2.5.2-1)

comparing (2.5.2-1) and (2.5.1-4) we get

$$P_{11}(s) = W_1(s) \;,\; P_{12}(s) = -W_1(s)G_o(s) \quad P_{21}(s) = I \;,\; P_{22}(s) = -G_o(s)$$

### 2.5.3 Robust Stability Problem

The requirement for robust stability can be written as (section 2.3)

$$\| W_2 T_o \|_\infty \leq \gamma$$

where $T_o = G_o(s)K(s)(I + G_o(s)K(s))^{-1}$ and $W_2(s)$ is an appropriate frequency weighting.

Since we require that

$$F(P,K)=W_2(s)G_o(s)K(s)(I+G_o(s)K(s))^{-1} \quad (2.5.3-1)$$

Comparing (2.5.3-1) and (2.5.1-4) we get

$P_{11}(s)=0$ , $P_{12}(s)=W_2(s)G_o(s)$ , $P_{21}=I$ , $P_{22}(s)=-G_o(s)$

### 2.5.4 Mixed Performance and Robustness Problem (General $H_\infty$ Problem)

Now, suppose that we wish to obtain good performance and to maintain stability in the presence of plant uncertainties. This objective can be achieved if

$$\left\| \begin{matrix} W_1 S_o \\ W_2 T_o \end{matrix} \right\|_\infty \leq \gamma$$

To put this into standard form, we require that

$$F(P,K) = \begin{bmatrix} W_1 S_o \\ W_2 T_o \end{bmatrix}$$

from section (2.5.3) and (2.5.2), we see that we should choose

$$P_{11}(s)=\begin{bmatrix} W_1(s) \\ o \end{bmatrix}, P_{12}(s)=\begin{bmatrix} -W_1(s)G_o(s) \\ W_2(s)G_o(s) \end{bmatrix}, P_{21}(s)=I \quad , P_{22}(s)=-G_o(s)$$

The feedback structure with the frequency weighting is shown in Fig. 2.8

### 2.5.5 State-Space Solutions to mixed sensitivity and robustness $H_\infty$ Problem

In this section, a state-space approach for all controllers solving a mixed sensitivity and robustness $H_\infty$ problem [13] is provided. For a given scalar $\gamma > 0$, we want to find

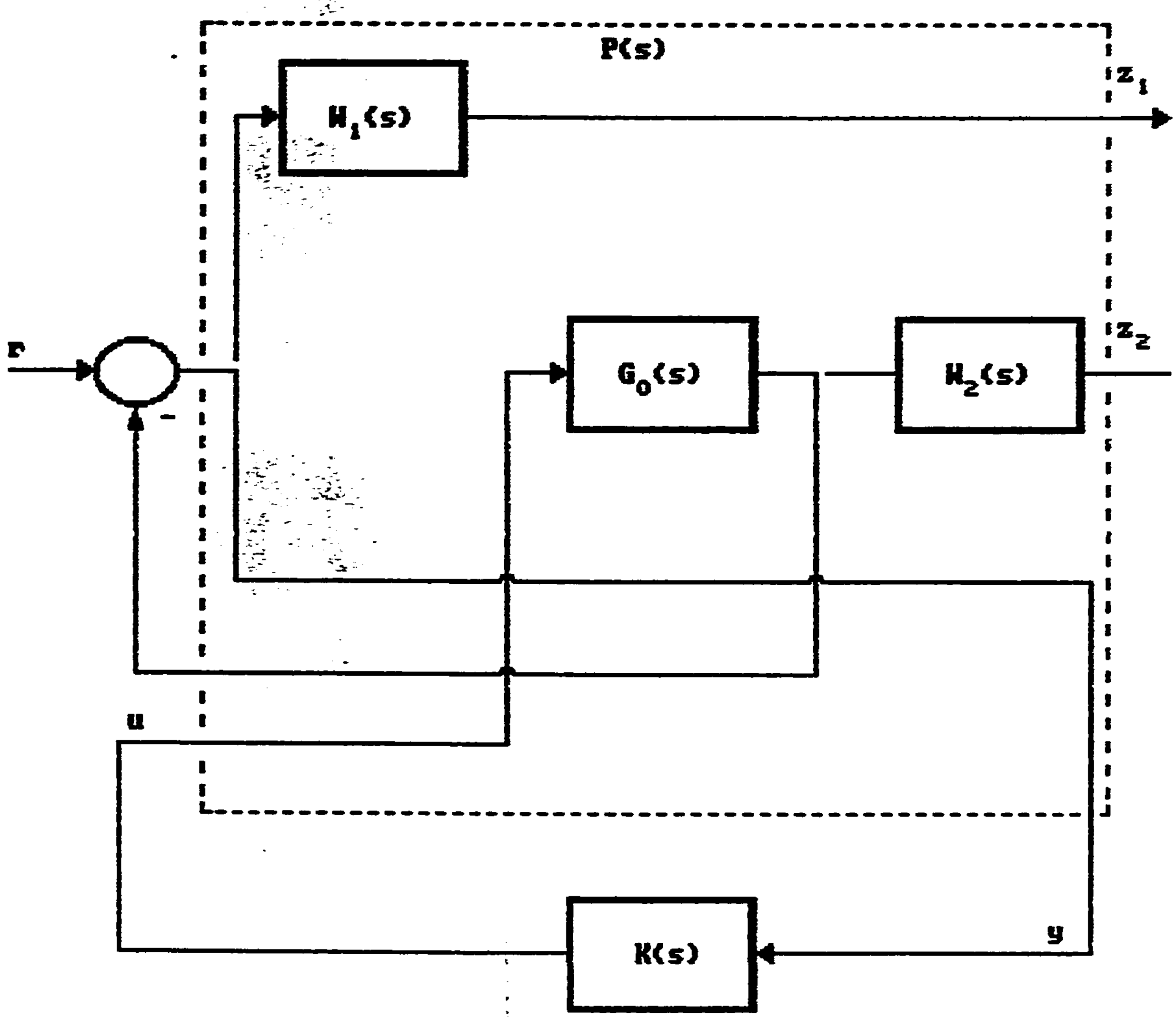



**Fig.2.8 : Weighted Performance and Robustness Problem Block Diagram**

all controllers such that the $H_\infty$ norm of the closed-loop transfer function is strictly less than $\gamma$. Necessary conditions for the existence of such controller is that the stabilizing solutions to two algebraic Riccati equations are positive semi-definite and the spectral radius of their product is less than $\gamma^2$ [13].

### 2.5.5.1 Plant Augmentation

The augmented plant *P(s)* is given by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} W_1(s) & -W_1 G_o(s) \\ O & W_2(s) G_o(s) \\ I & -G_o(s) \end{bmatrix}$$

The state-space realizations of the plant and the weighting functions are given by

$$G_o(s) := \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix}, \quad W_1(s) := \begin{bmatrix} A_{w1} & B_{w1} \\ C_{w1} & D_{w1} \end{bmatrix}$$

$$W_2(s) := \begin{bmatrix} A_{w2} & B_{w2} \\ C_{w2} & D_{w2} \end{bmatrix}$$

and the realization of the augmented plant *P(s)* is given by [15]

$$P(s) := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

with the respective matrices

$$A = \begin{bmatrix} A_p & 0 & 0 \\ -B_{w1} C_p & A_{w1} & 0 \\ B_{w2} C_p & 0 & A_{w2} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ B_{wl} \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ -B_{wl}D_p \\ B_{w2}D_p \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -D_{wl}C_p & C_{wl} & 0 \\ D_{wl}C_p & 0 & C_{w2} \end{bmatrix}, \quad C_2 = [-C_p \quad 0 \quad 0]$$

$$D_{11} = \begin{bmatrix} D_{wl} \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} -D_{wl}D_p \\ D_{w2}D_p \end{bmatrix}, \quad D_{21} = I \quad , D_{22} = -D_p$$

The weighted performance and robustness problem block diagram is shown in Fig. 2.8.

### 2.5.5.2 Controller Design Algorithm

The realization of the transfer matrix $P(s)$ is taken to be of the form

$$P(s) := \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$$

The following assumption are made

i) $(A, B_1)$ is stabilizable and $(C_1, A)$ is detectable.

ii) $(A, B_2)$ is stabilizable and $(C_2, A)$ is detectable.

iii) $D_{12}^T[C_1 \quad D_{12}] = [0 \quad I]$.

iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$

The problem considered here is to find all admissible controller $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$, where $T_{zw}$ is the closed-loop transfer function given by

$$T_{zw}(s) = P_{11}(s) + P_{12}(s)K(s)(I - P_{22}(s)K(s))^{-1}P_{21}(s)$$

The $H_\infty$ solution involves two algebraic Riccati equation

$$A^T X + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 = 0 \qquad (2.5.5.2-1)$$

$$AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T = 0 \qquad (2.5.5.2-2)$$

There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ if the following three conditions hold.

i) The solution $X$ of equation (2.5.5.2-1) is positive semi-definite.

ii) The solution $Y$ of equation (2.5.5.2.-2) is positive semi-definite.

iii) The spectral radius of the product $XY$ is less than $\gamma^2 (\rho(XY) < \gamma^2)$.

When these conditions hold, one such controller is [13]

$$K(s) = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix}$$

where

$$\hat{A} = A + \gamma^{-2}B_1B_1^TX + B_2\hat{C} - \hat{B}C_2$$

$$\hat{B} = (I - \gamma^{-2}YX)^{-1}YC_2^T$$

$$\hat{C} = -B_2^TX$$

To find the controller which minimize $\|T_{zw}\|_\infty$, it is necessary to use the algorithm iteratively, reducing $\rho(XY)$ until the limiting value $\gamma_0$ is reached or until one or other of the two Riccati equations (2.5.5.2-1) and (2.5.5.2-2) fails to have a positive semi-definite solution.

## 2.6 $H_\infty$ State-Feedback with Closed-Loop Transfer Recovery ($H_\infty$/CLTR) Design Method

In section (2.5), it is shown that the design of an $H_\infty$ controller for the mixed performance and robustness problem involves solving two algebraic Riccati equations iteratively until certain performance and robustness specifications are satisfied. In [23] it is shown that if all the states of the plant can be measured, then the infimum of the norm of the closed-loop transfer function using linear static state-feedback equals the infimum of the norm of the closed-loop transfer function over all stabilizing dynamic state-feedback controllers. The design of an $H_\infty$ state-feedback control law can be done by solving one algebraic Riccati equation iteratively [18]. These facts motivates us to use a hybrid control structure consisting of state estimator and $H_\infty$ state-feedback gains for the case when all the states are not available for feedback. The design procedure developed in this section consists of two steps:

1) Assuming that all the states are available for feedback, design an $H_\infty$ state-feedback control which will make the $H_\infty$ norm of the state-feedback closed-loop transfer function less than same constant $\gamma > 0$.

2) Recover the achievable performance, using state estimator. This means that we want the closed-loop transfer function with the new control law (state feedback gains and state estimator) to be equal to the state-feedback closed-loop transfer function of step (1).

This design method will ensure internal stability and, under certain conditions, the $H_\infty$ norm of the closed-loop transfer function is less than $\gamma$. Moreover, the computations of an $H_\infty$/CLTR controller will be reduced to solving one algebraic Riccati equation.

### 2.6.1 Problem Formulation

Consider the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \qquad (2.6.1-1)$$

where $x \in R^n$ is the state, $w \in R^k$ is the disturbance, $u \in R^r$ is the control, $z \in R^l$ is the error, and $y \in R^m$ is the measured output. The weighting functions are assumed to be absorbed in the plant description.

We assume that $(A,B_2)$ is stabilizable, $(A,C_2)$ is detectable and $D_{22} = 0$, the controller will be based on a state feedback:

$$u = -K_c \hat{x} \qquad (2.6.1-2)$$

where $\hat{x}$ is the estimate of the state $x$ and $K_c$ is the $H_\infty$ state-feedback gain.

And a full-order observer

$$\dot{\hat{x}} = (A - K_f C_2)\hat{x} + K_f y + B_2 u \qquad (2.6.1-3)$$

where $K_f$ is the observer gain and let the closed-loop system denoted by $T_c(s)$, be formed by the plant (2.6.1-1) with the controller (2.6.1-2) and (2.6.1-3). Then our main concern is to find an observer-based controller with which the resulting closed-loop system is internally stable and satisfies $\|T_c(s)\|_\infty < \gamma$. The $H_\infty$/CLTR controller structure is shown in Fig. 2.9.

### 2.6.2 $H_\infty$ State Feedback Control

Consider the system (2.6.1-1), with the assumptions that $C_2 = I$ , $D_{21} = 0$ , and $D_{22} = 0$, the $H_\infty$ state-feedback gain $K_c$ can be found by the following procedure from [18].

Let a constant $\gamma > 0$ be such that $Y^2 I - D_{11}^T D_{11} > 0$

suppose rank $D_{12} =: \; i \le j$

let $U \in R^{jxi}$ and $\Sigma_2 \in R^{ixm}$ be any matrices such that

rank $(\Sigma_2) =$ rank $(U) = i$, and $D_{12} = U\Sigma_2$

let $\Phi \in R^{(m-i)xm}$ be such that $\Phi\Sigma_2^T = 0 \quad (\Phi = 0 \;\text{ if }\; i = m)$

let $\Xi_H := \Sigma_2^T(\Sigma_2\Sigma_2^T)^{-1}\{U^T R U\}^{-1}(\Sigma_2\Sigma_2^T)^{-1}\Sigma_2$

where $R := I + D_{11}(\gamma^2 I - D_{11}^T D_{11})^{-1} D_{11}^T$ and let

$$A_H = A + B_1(\gamma^2 I - D_{11}^T D_{11})^{-1} D_{11}^T C_1$$

$$B_H = B_2 + B_1(\gamma^2 - D_{11}^T D_{11})^{-1} D_{11}^T D_{12}$$

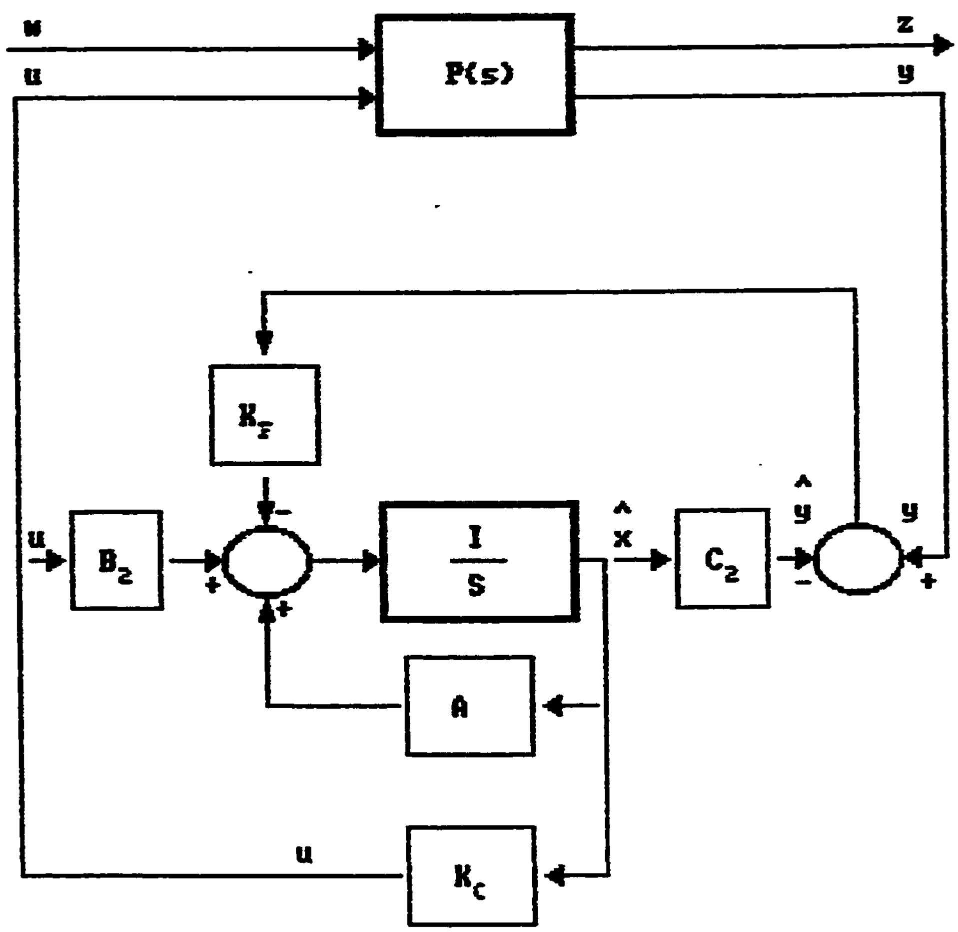



Fig.2.9 : H /CLTR Controller Structure

$$C_H = \{I + D_{11}(\gamma^2 I - D_{11}^T D_{11})^{-1} D_{11}\}^{\frac{1}{2}} C_1$$

$$D_H = B_1(\gamma^2 I - D_{11}^T D_{11})^{\frac{-1}{2}}$$

$$E_H = \{I + D_{11}(\gamma^2 I - D_{11}^T D_{11})^{-1} D_{11}^T\}^{\frac{1}{2}} D_{12}$$

Using the above definitions, the system (2.6.1-1)is said to satisfy the following algebraic Riccati equation with constant $\gamma$ if (for any $Q>0$) there exists $\epsilon > 0$ such that the Riccati equation

$$A_H - B_H \Xi E_H C_H\}^T X + X\{A_H - B_H \Xi E_H C_H\} + X D_H D_H^T X - X B_H \Xi B_H^T X$$

$$-\frac{1}{\epsilon} X B_H \Phi^T \Phi B_H^T X + C_H^T \{I - E_H \Xi_H E_H^T\} C_H + \epsilon Q = 0 \qquad (2.6.2-1)$$

has a positive definite solution X and the $H_\infty$ state feedback given by

$$K_c = \left\{\frac{1}{2\epsilon}\Phi^T \Phi + \Xi_H\right\} B_H^T X + \Xi_H E_H^T C_H \qquad (2.6.2-2)$$

will ensure that $A$-$B_2K_c$ is a stability matrix and the state-feedback closed-loop function

$$T_{cs}(s) = (C_1 - D_{12}K_c)(sI - A + B_2 K_c)^{-1} B_1 + D_{11} \quad (2.6.2-3)$$

satisfies the following $H_\infty$ norm bound

$$\| T_{cs} \|_\infty < \gamma$$

The following algorithm can be used to find the state-feedback gain $K_c$.

(1) Given $A, B_1, B_1, C_1, D_{11}, D_{12}$, compute $U, \Sigma_2, \Phi$ and let $Q=I$.

(2) let $\gamma > 0$ be such that $\gamma^2 I - D_{11}^T D_{11} > 0$.
Compute $A_H, B_H, C_H, D_H, F_H, E_H$.

(3) let $\epsilon = 1$.

(4) Solve (2.6.2-1). If $X>0$, go to (5). Otherwise, if (2.6.2-1) does not have a positive definite symmetric solution even for a sufficiently small $\epsilon (<< 1)$, then increase $\gamma$ and go to (2).

(5) If $\gamma <$ specified performance level, compute $K_c$ as in (2.6.2-2).
Otherwise, decrease $\gamma$, and go to (2).

### 2.6.3 Full-Order State Observer

The full-order state-observer which is given by

$$\dot{\hat{x}} = (A - K_f C_2)\hat{x} + K_f (y - C_2 \hat{x}) + B_2 u$$

can be used to provide an estimate for the states of the system described in (2.6.1-1). First, we need to calculate the closed-loop transfer function $T_c(s)$. From (2.6.1-1), (2.6.1-2) and (2.6.1-3), it follows that

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -B_2 K_c \\ K_f C_2 & A - K_f C_2 - B_2 K_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 \\ K_f D_{21} \end{bmatrix} w \quad (2.6.3-1)$$

$$z = [C_1 \quad -D_{12} K_c] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_{11} w \quad (2.6.3-2)$$

let the error between the state $x$ and the estimate $\hat{x}$ given be

$$e = x - \hat{x} \qquad (2.6.3-3)$$

Then, applying the equivalence transformation

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} -I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix}$$

to (2.6.3-1) and (2.6.3-2) yields

$$\begin{bmatrix} \dot{e} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A - K_f C_2 & 0 \\ -K_f C_2 & A - B_2 K_c \end{bmatrix} \begin{bmatrix} e \\ \hat{x} \end{bmatrix} + \begin{bmatrix} -B_1 + K_f D_{21} \\ K_f D_{21} \end{bmatrix} w \qquad (2.6.3-4)$$

$$= [-C_1 \quad C_1 - D_{12} K_c] \begin{bmatrix} e \\ \hat{x} \end{bmatrix} + D_{11} w \qquad (2.6.3-5)$$

From (2.6.3-4), we can see that the closed-loop system will be internally stable if the matrices $A - K_f C_2$ and $A - B_2 K_c$ are stability matrices. The matrix $A - B_2 K_c$ is stable from section (2.6.2) and the matrix $A - K_f C_2$ is to be made stable by the proper choice of observer gain $K_f$.

From (2.6.3-4) and (2.6.3-5), $T_c(s)$ is given by [33]

$$T_c(s) = D_{11} + (C_1 - D_{12}K_c)(sI - A + B_2K_c)^{-1}B_1$$
$$+ C_1(sI - A + K_fC_2)^{-1}(B_1 - K_fD_{21})$$
$$-(C_1 - D_{12}K_c)(sI - A + B_2K_c)^{-1}(sI - A)$$
$$\times(sI - A + K_fC_2)^{-1}(B_1 - K_fD_{21}) \qquad (2.6.3-9)$$

Comparing (2.6.3-9) and (2.6.2-3), $T_c(s)$ can be written as

$$T_c(s) = T_{cs}(s) + \alpha(s) \quad (B_1 - K_fD_{21})$$

$$\alpha(s) = C_1(sI - A + K_fC_2)^{-1}(B_1 - K_fD_{21})$$

where $-(C_1 - D_{12}K_c)(sI - A + B_2K_c)^{-1}(sI - A)$

$$\times(sI - A + K_fC_2)^{-1}$$

From section (2.5.5.1) we have $D_{21} = I$, thus

$$T_c(s) = T_{cs}(s) + \alpha(s) \quad (B_1 - K_f) \qquad (2.6.3-10)$$

A perfect recovery (i.e. $T_c(s) = T_{cs}(s)$) is possible if $K_f = B_1$.

Now, we investigate the stability of the matrix $A - B_1C_2$. From section (2.5.5.1) we have

$$B_1 = \begin{bmatrix} 0 \\ B_{w1} \\ 0 \end{bmatrix}, C_2 = [-C_p \quad 0 \quad 0] \quad, A = \begin{bmatrix} A_p & 0 & 0 \\ -B_{w1}C_p & A_{w1} & 0 \\ B_{w2}C_p & 0 & A_{w2} \end{bmatrix}$$

Thus, the matrix

$$A - B_1C_2 = \begin{bmatrix} A_p & 0 & 0 \\ -2B_{w1}C_p & A_{w1} & 0 \\ B_{w2}C_p & 0 & A_{w2} \end{bmatrix}.$$

The matrix $A - B_1C_2$ will be stable if and only if the matrices $A_p, A_{w1}$ and $A_{w2}$ are stable. The matrices $A_{w1}, A_{w2}$ can be made stable by the appropriate choice of the weighting functions $W_1(s)$ and $W_2(s)$. Thus, the requirement that $A - B_1C_2$ is stable will be such that $A_p$ is stable. $A_p$ is stable if the nominal plant transfer function $G_o(s)$ is stable. Therefore, $K_f$ can be chosen to be equal to $B_1$ for stable plants and will result in perfect recovery. Using (2.6.1-2) and (2.6.1-3), the compensator *K(s)* is given by

$$K(s) = -K_c(sI - A + B_2K_c + K_fC_2)^{-1}K_f$$

If the unstability of the plant is caused by poles at the origin, the matrix $A_p$ can be replaced by a new matrix $\bar{A}_p$, where $\bar{A}_p = A_p$, except for the zero eigenvalues are replaced by ones at $\lambda$, where $\lambda$ is a small negative number. Changing the zero eigenvalues of $A_p$ to a small negative number will make the matrix $\bar{A}_p$ stable, so that this design method can be applied directly and will have a very little effect on the behavior of the plant except at very low frequencies.

If the unstability of the plant is not caused by poles at the origin, only approximate recovery of the state-feedback transfer function is possible. Looking at the closed-loop transfer function which is given by

$$T_c(s) = T_{cs}(s) + \alpha(s)L \quad ; \quad where \quad L = B_1 - K_f$$

an approximate recovery can be achieved if $L$ is minimized subject to the constraint that $A - K_f C_2$ is stable. The minimization of $L$ can be solved by minimizing $\|L\|_F^2$.

The state-space realization $(A_p, B_p, C_p)$ of unstable plants can always be written in the following form

$$A_p = \begin{bmatrix} A_u & 0 \\ 0 & A_s \end{bmatrix} \quad ; \quad B_p = \begin{bmatrix} B_u \\ B_s \end{bmatrix}$$

$$C_p = [C_u \quad C_s]$$

Where $(A_u, B_u, C_u)$ is the state space realization of the unstable part of the plant, and $(A_s, B_s, C_s)$ is the state space realization of the stable part of the plant.

Thus, $A$, $B_1$ and $C_2$ matrices of the augmented system (section 2.5.5.1)are given by

$$A = \begin{bmatrix} A_u & 0 & 0 & 0 \\ 0 & A_s & 0 & 0 \\ -B_{wl}C_u & -B_{wl}C_s & A_{wl} & 0 \\ B_{w2}C_u & B_{w2}C_s & 0 & A_{w2} \end{bmatrix} , B_1 = \begin{bmatrix} 0 \\ 0 \\ B_{wl} \\ 0 \end{bmatrix}$$

$$C_2 = [-C_u \quad -C_s \quad 0 \quad 0 \;]$$

Now, since $A_s$ is stable and $A_{w1}$ and $A_{w2}$ can be selected to be stable, $L$ can be chosen as

$$L = B_1 - K_f = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ B_{w1} \\ 0 \end{bmatrix} - \begin{bmatrix} K_{f1} \\ 0 \\ B_{w1} \\ 0 \end{bmatrix}$$

and the problem of minimizing $\|L\|_F^2$, will reduce to minimizing $\|L_1\|_F^2$, which is equivalent to minimizing $\|K_{f1}\|_F^2$. The matrix $A - K_f C_2$ is

$$\begin{bmatrix} A_u - K_{f1}C_u & -K_{f1}C_s & 0 & 0 \\ 0 & A_s & 0 & 0 \\ -2B_{w1}C_p & 0 & A_{w1} & 0 \\ 0 & 0 & 0 & 3A_{w2} \end{bmatrix}$$

which is stable if $A_u - K_{f1}C_u$ is stable. Thus the observer design for unstable plants will be : Find $K_{f1}$ which satisfies $A_u - K_{f1}C_u$ is stable and $\|K_{f1}\|_F^2$ is minimized. A procedure which solves this problem is given in appendix A.

# Chapter 3

## DISCRETE TIME FEEDBACK CONTROL SYSTEMS AND THE $H_\infty$/CLTR DESIGN METHOD

### 3.1 INTRODUCTION

In the previous chapter, useful design techniques for continuous time systems are described. Here, some basic definitions of feedback discrete time control systems are considered. These definitions relate to plant model, uncertainties, type of inputs, control objectives, performance and robustness specifications. The LQG/LTR design method for discrete time systems is reviewed. It is shown how this method differs for discrete systems and how it is affected by compensator processing time. A recovery procedure using current estimator [7] for the case when the compensator processing time is large (compared to the sampling time) will be introduced. A simplified LQG/LTR frequency domain approach for compensator design for SISO minimum-phase discrete systems is developed. The discrete-time $H_\infty$ state-feedback control has not been studied extensively except for the work in [17] and [10], which employ an algebraic Riccati equation for the solution of the $H_\infty$ state-feedback problem. However, in order to solve the Riccati equation, an iterative algorithm is used. This is inefficient from a computational point of view. Moreover, these algorithms as expericned by the author fail to converge to a solution even for very simple low order examples. Therefore, the $H_\infty$/CLTR design method for discrete-time system will utilize the frequency domain bilinear transformation to transform the discrete-time problem into a continuous-time one, carry out the computation using continuous-time $H_\infty$/CLTR and then transform back the solution.

## 3.2 Control System Description

### 3.2.1 Classical Feedback System

The block diagram of the standard discrete-time feedback control system is shown in Fig.3.1.It consists of the plant *G(z)*, compensator *K(z)* forced by command input *(r)*, measurement noise *(n)* and disturbance *(d)*. The measured variable *(y)* is corrupted by the noise *(n)*. The compensator *K(z)* determines the plant input *(u)* on the basis of the error *(e)* *(e=r-y)*.The objective of the control system is to keep the output *(y)* close to the reference *(r)*.

The dynamical behaviour of the plant is modelled in the time domain by a linear time-invariant system as

$$x_{i+1} = Ax_i + Bu_i \qquad (3.2.1-1)$$

$$y_i = Cx_i + Du_i \qquad (3.2.1-2)$$

where $x_i \in R^n$ , $y_i \in R^m$ , $u_i \in R^p$ are the state, output, and input vectors respectively and *A,B,C* and *D* are constant matrices of appropriate dimensions. The transfer function of the plant is $G(z) = C(zI - A)^{-1}B + D$

### 3.2.2. Plant Model and Uncertainty Description

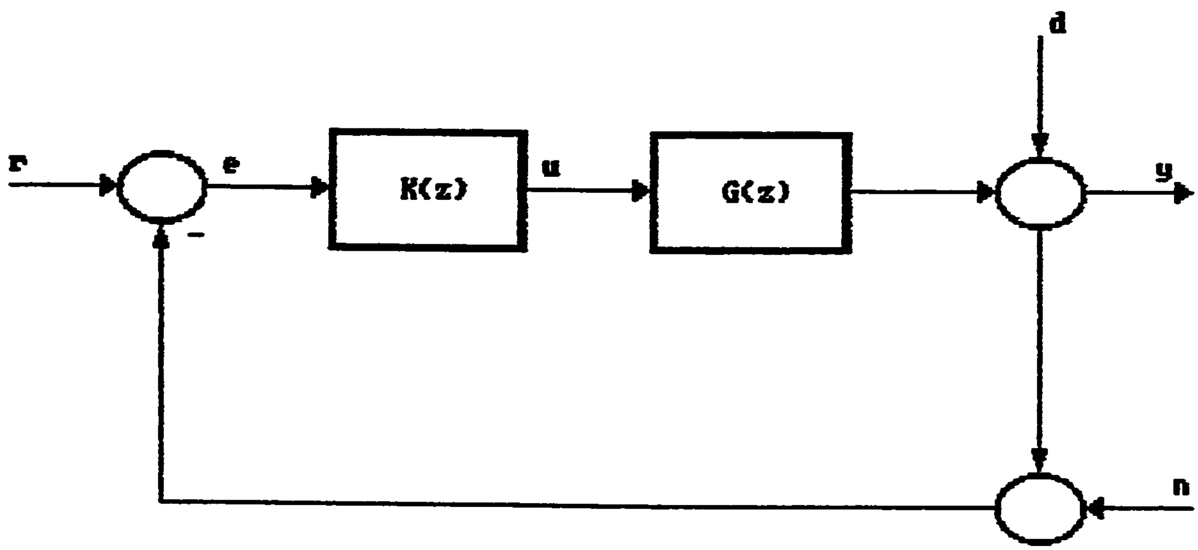



Fig.3.1 : Standard Feedback Control System

The dynamic behaviour of a plant is described not only by a single linear time-invariant model but by a family of models. This family of plants can be represented by a nominal model which has the same number of unstable poles as any member of the family of plants. The uncertainty can be parametrized by many different ways. As in section 2.2, the uncertainty can be described by [28]

$$G(z) = G_o(z) + L_a(z)$$

$$\bar{\sigma}[L_a(e^{j\theta})] < l_a(\theta) \quad ,0 \leq \theta \leq \pi$$

$$G(z) = G_o(z)[I + L_i(z)]$$

$$\bar{\sigma}[L_i(e^{j\theta})] < l_{mi}(\theta) \quad ,0 \leq \theta \leq \pi$$

$$G(z) = [I + L_o(z)]G_o(z)$$

$$\bar{\sigma}[L_o(e^{j\theta})] < l_o(\theta) \quad ,0 \leq \theta \leq \pi$$

### 3.2.3 Inputs Specifications

The inputs are all external signals entering the feedback loop at some point. For the control system design, the inputs have to be specified. The specifications include a description of the magnitude, energy and frequency content of the signals. These specifications are usually given as a weighted $L^2$ norms [28]

$$\left| Z^{-1}\{W_r(z)r(z)\} \right|_{L^2} \leq 1$$

$$\left| Z^{-1}\{W_d(z)d(z)\} \right|_{L^2} \leq 1$$

$$\left| Z^{-1}\{W_n(z)n(z)\} \right|_{L^2} \leq 1$$

where $W_r$, $W_d$ and $W_n$ are appropriate weightings, that reflect the magnitude and frequency content of the commands *(r)*, disturbance *(d)*, and noise *(n)* respectively.

## 3.3 Control Objectives

In order for the compensator to work well on the actual plant the following objectives have to be met

* Nominal stability
* Nominal performance
* Robust stability

### 3.3.1 Nominal Stability

The minimal requirement on the compensator is to stabilize the closed-loop nominal system. The conditions for nominal stability can be derived in terms of the internal stability of the system. A linear time-invariant system is internally stable if the transfer functions between any two inputs of the control system are stable (i.e., have all poles inside the unit circle) [22]. As in section 2.3, the system shown in Fig. 3.2 is internally stable if and only if all the elements in the transfer matrix (3.3.1-1) have all poles inside the unit circle.

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} G_o K(I+G_o K)^{-1} & (I+G_o K)^{-1} G_o \\ K(I+G_o K)^{-1} & -K(I+G_o K)^{-1} G_o \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} \qquad (3.3.1-1)$$

### 3.3.2 Nominal Performance

The most basic objective of a feedback compensator is to keep the error between the

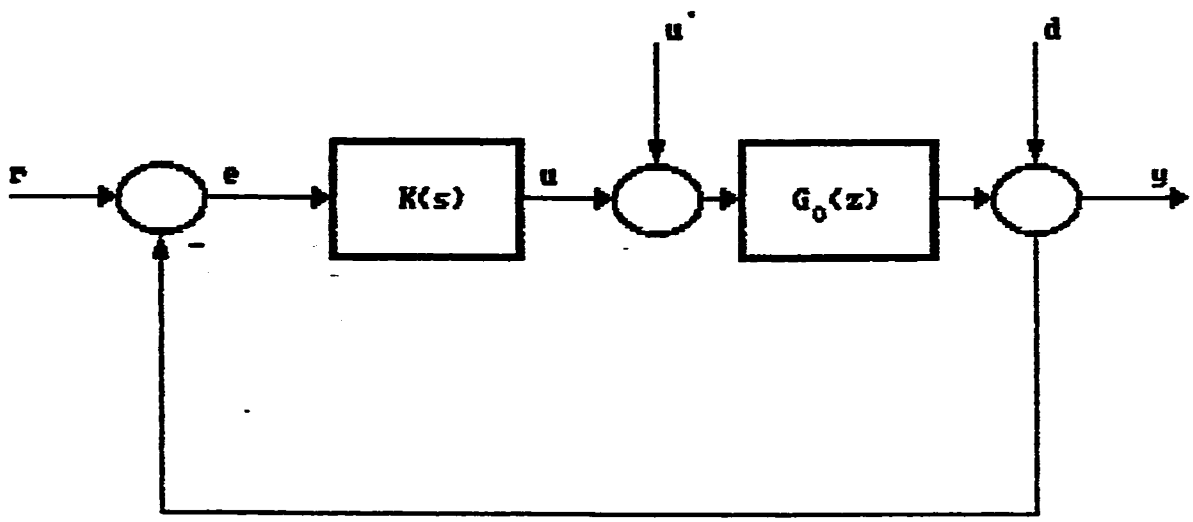



**Fig.3.2 : Internal Stability Block Diagram**

plant output $y$ and the reference $r$ small when the overall system is affected by external signals $r$, $n$ and $d$. From Fig. 3.1 the error can be written as

$$e = r - y = (I + G_o K)^{-1}(r - d) + (I + G_o K)^{-1} G_o K n \qquad (3.3.2-1)$$

In order to quantify performance, a measure of smallness for the error has to be defined. Using (3.3.2-1), the performance specifications can be written as [28]

$$\left\| W_e (I + G_o K)^{-1} W_r^{-1} \right\|_\infty \leq 1 \qquad (3.3.2-2)$$

$$\left\| W_e (I + G_o K)^{-1} W_d^{-1} \right\|_\infty \leq 1 \qquad (3.3.2-3)$$

$$\left\| W_e (I + G_o K)^{-1} G_o K W_n^{-1} \right\|_\infty \leq 1 \qquad (3.3.2-4)$$

The above conditions are satisfied if

$$\underline{\sigma}[I + G_o K] \geq f_r(\theta) \qquad (3.3.2-5)$$

$$\underline{\sigma}[I + G_o K] \geq f_d(\theta) \qquad (3.3.2-6)$$

$$\underline{\sigma}[I + (G_o K)^{-1}] \geq \frac{1}{f_n(\theta)} \qquad (3.3.2-7)$$

$$f_i(\theta) = \frac{\bar{\sigma}[W_e(e^{j\theta})]}{\underline{\sigma}[W_i(e^{j\theta})]} \qquad , i = r, d, n$$

Measurement noise is often small and as we will see in the next section, the plant uncertainty put a more restrictive bound on $\underline{\sigma}[I + (G_o K)^{-1})]$, this will reduce (3.3.2-5) - (3.3.2-7) to

$$\underline{\sigma}[I + G_o K] \geq f(\theta)$$

### 3.3.3 Robust Stability

Assume that all the plant $G$ in the family of plants describing the system have the same number of unstable poles (poles in $|z| \geq 1$) and that a particular compensator $K$ stabilizes the nominal plant $G_o$. Then [28]

a) if the uncertainty is additive and if

$$\left\| L_a(z)[I + G_o(z)K(z)]^{-1} \right\|_\infty < 1$$

or

$$\left\| [1 + G_o(z)K(z)] L_a(z) \right\|_\infty < 1$$

the closed loop system will be robustly stable.

b) if the uncertainty is multiplicative at the system input and if

$$\left\| [I + G_o(z)K(z)]^{-1} G_o(z)K(z)L_i(z) \right\|_\infty < 1$$

or

$$\underline{\sigma}\left[ I + (G_o(e^{i\theta})K(e^{i\theta}))^{-1} \right] > l_i(\theta) \qquad 0 \leq \theta \leq \pi$$

the closed-loop system will be robustly stable.

c) if the uncertainty is multiplicative at the system output and if

$$\left\| L_o(z)G_o(z)K(z)[I+G_o(z)K(z)]^{-1} \right\|_\infty < 1$$

or

$$\underline{\sigma}\left[I+(G_o(e^{j\theta})K(e^{j\theta}))^{-1}\right] > l(\theta)_o \quad 0 \le \theta \le \pi$$

### 3.3.4 SUMMARY

The design specifications can be reduced to bounds on minimum singular values of the return difference $I+G_oK$ and the inverse return difference $I+(G_oK)^{-1}$ (assuming multiplicative output uncertainty) as

$$\underline{\sigma}[I+G_oK] \ \ge f(\theta) \quad 0 \le \theta \le \pi \qquad (3.3.4-1)$$

$$\underline{\sigma}[I+(G_oK)^{-1}] \ \ge l_o(\theta) \quad 0 \le \theta \le \pi \qquad (3.3.4-2)$$

These bounds can be transformed into bounds on the sensitivity and complementary sensitivity functions as

$$\underline{\sigma}[S^{-1}] \ \ge f(\theta)$$

where $S$ is the sensitivity function

using

$$\underline{\sigma}[S^{-1}] = \frac{1}{\bar{\sigma}[S]}$$

we get

$$\bar{\sigma}[S] \ \le f^{-1}(\theta) \qquad (3.3.4-3)$$

Also

$$\underline{\sigma}[T^{-1}] \ge l_o(\theta)$$

Where $T$ is the complementary sensitivity functions and as above we get

$$\bar{\sigma}[T] \le l_o^{-1}(\theta) \qquad (3.3.4-4)$$

If the system is nominally stable, these conditions will ensure good performance and robust stability. These conditions are depicted graphically in Figs. 3.3 and 3.4.

## 3.4 LQG/LTR Design Method for Discrete Time Systems

The LQG/LTR design method assumes that a nominal model is available, which is finite-dimensional linear time-invariant and is represented by a state difference and output equations

$$x_{i+1} = Ax_i + Bu_i$$

$$y_i = Cx_i \qquad i \ge 0$$

where $x_i \in R^n$ is the plant states, $u_i \in R^m$ is the plant inputs, $y_i \in R^m$ is the plant output, and *A,B,C* are constant real matrices of appropriate dimensions, the transfer function of the nominal plant is given by $G_o(z) = C(zI - A)^{-1}B$.

The design method attempts to find a controller *K(z)* which will stabilize the closed loop system (Fig. 3.1) and satisfies the performance and robustness constraints (Figs. 3.3 and 3.4).

Since the plant and compensator operate in vector signals, their transfer functions do not commute, and an analysis point of reference must be chosen. All design objectives and model uncertainties must be expressed in terms of bounds on signals that occur at this point of reference. The designer must choose one of two possible points of reference. These are the plant input node denoted by *u*, or the plant output node denoted by *y* (Fig. 3.1).

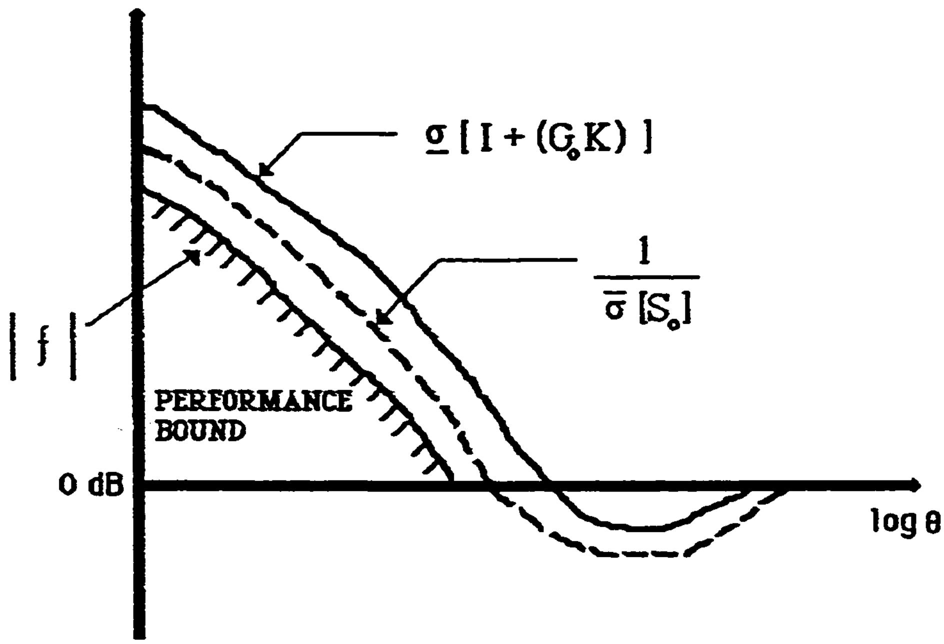



Fig. 3.3: Performance specifications on the output return difference and sensitivity function

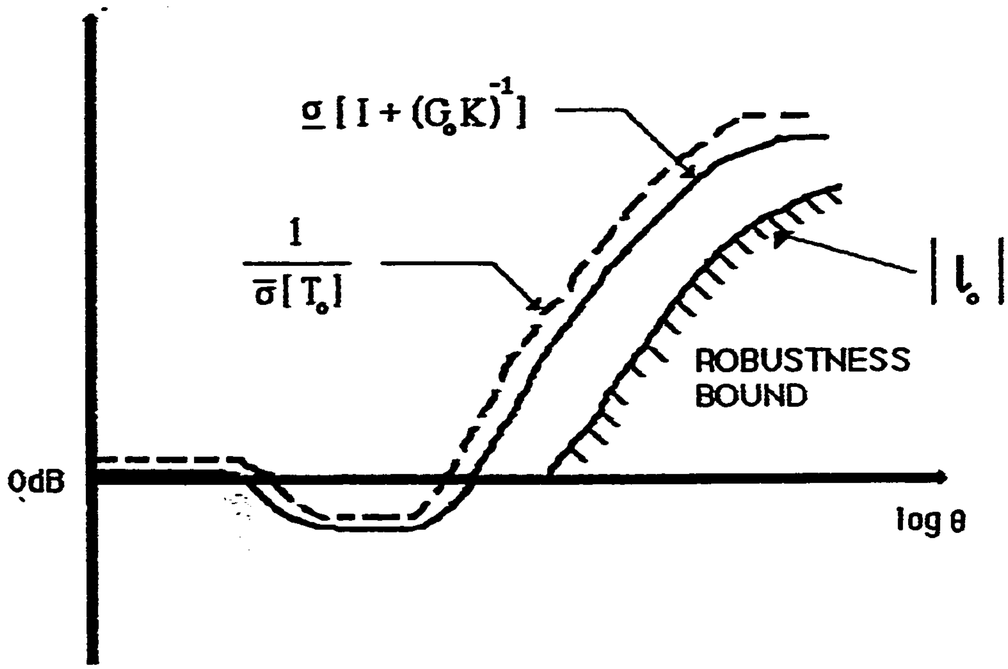



**Fig 3.4: Robustness Specifications on the output inverse return difference and complementary sensitivity function**

The compensator processing time is an important factor in discrete time systems. Using the control law $u_i = -K_c x_i$ (the same control law as in continuous systems) may not be realizable. This is because, if the dynamic order of the plant is high, time delay due to the computation time of the compensator will be large. If the control law $u_i = -K_c x_i$ is used when the compensator processing time is large, the performance of the system will be greatly deteriorated or the system may become unstable [27]. Therefore, two cases can be distinguished.

### 3.4.1 Compensator Processing Time is Negligible

The LQG/LTR design method for compensator with negligible processing involves two basic steps (assuming that the input node is used as the reference point).

1) A full-state feedback system is designed such that $I + L(z)$ and $I + L^{-1}(z)$, where $L(z) = K_c(zI - A)^{-1}B$, have the desired shapes of Fig.3.3 and Fig respectively. 3.4. $K_c$ is found by solving the following linear quadratic regular (LQR) problem [32].

$$\text{minimize } J = \frac{1}{2}\sum_{i=0}^{\infty} x_i Q x_i + u_i R u_i \quad , Q \geq 0 , R \geq 0$$

$$\text{subject to} \quad x_{i+1} = A x_i + B u_i \qquad x \in R^n \quad , u \in R^m$$

The solution to this optimization problem is given by

$$u_t = -K_c x_t$$

where $K_c$ is the full-state LQR gain defined by the following discrete Riccati equation

$$A^T P_c A - P_c + Q - A^T P_c B(R + B^T P_c B)^{-1} B^T P_c A = 0$$

$$K_c = (R + B^T P_c B)^{-1} B^T P_c A$$

where it is assumed that *(A,B)* is stabilizable and $(A, H_w)$ is detectable and $Q = H_w^T H_w$, $H_w \in R^{mxn}$. Shaping the singular values of the return difference *I+L(z)* and the inverse return difference $I + L^{-1}(z)$ is based on the following discrete Kalman equality

$$[I + L(z)]^H \quad (R + B^T P_c B)[I + T(z)] = R + W^H(z)W(z) \qquad (3.4.1-1)$$

where

$$W(z) = H_w(zI - A)^{-1} B \qquad (3.4.1-2)$$

Comparing (3.4.1-1)and (2.4.1-6)we see that they are very similar except for the term $B^T P_c B$, it is the presence of this term that causes the discrete LQR to have inferior properties as compared to the continuous time LQR. The $B^T P_c B$ term prevents the separation of the loop transfer function terms from the LQR weighting matrices (this is because $P_c$ depends on $R$ and $Q$). This separation was possible in continuous time LQR and it led to a way to select the weighting matrices so that the LQR loop transfer function has the desired loop shape.

The following loop shaping procedure for multi-input multi-output systems is described in [28]:

Letting $R = \rho I$ and taking the determinant of 3.4.1-1,we get

$$|\,det(I + L(z))\,|^2 \beta_2 = det(\rho I + W^H(z)W(z))$$

where $\beta^2 = det(\rho I + B^T PB)$ and

$$det(I + L(z)) = det(I + K_c(zI - A)^{-1})B)$$

$$= det((zI - A)^{-1})det(zI - A + BK_c) = \frac{A_c(z)}{A(z)}$$

where

$$A(z) = det(zI - A) = z^n + \alpha_1 z^{n-1} ... + \alpha_n$$

$$A_c(z) = det(zI - A + BK_c) = z^n + \alpha_{cl} z^{n-1} ... + \alpha_{cn}$$

i.e. $A(z)$ and $A_c(z)$ are the open loop and closed loop characteristics polynomials, respectively. Also $A(z)$ can be written as

$$A(z) = \prod_{i=1}^{n} (z - \pi_i), \quad 0 < |\pi_i| \leq 1, \quad i = 1,2,....,q \quad \text{and} \quad |\pi_i| > 1, \quad i = q+1,....,n$$

For large values of $\rho$

let
$$\gamma = \left\{ \begin{matrix} 1, & q = n \\ \prod_{i=q+1}^{n} \pi_i, & q < n \end{matrix} \right\}$$

Thus when $\underline{\sigma}[W]$ is large, we have

$$\frac{1}{\gamma^2} \left( 1 + \frac{1}{\rho} \sigma_i^2 [W(z)] \right) \leq \sigma_i^2 [I + L(z)] \leq \left( 1 + \frac{1}{\rho} \sigma_i^2 [W(Lz)] \right) \qquad (3.4.1-3)$$

Now assuming $W(z)$ is minimum phase and $H_w B = \ldots = H_w A^{k-2} B = 0$, and $det(H_w A^{k-1}) \neq 0$,,

Thus for very small values of ρ we have

$$\frac{\sigma_i^2[W(z)]}{\bar{\sigma}^2[H_w A^{k-1} B]} \leq \sigma_i^2[I + L(z)] \leq \frac{\sigma_i^2[W(z)]}{\underline{\sigma}^2[H_w A^{k-1} B]} \qquad (3.4.1-4)$$

Equations (3.4.1-3 to 3.4.1-4)show the effect of ρ and $H_w$ on the return difference and they give an idea on how to select them to satisfy the loop shape constraints imposed on the return difference. Also from the above it is clear that the loop transfer function depends on the original plant dynamics. Thus it may be necessary to append additional dynamics. For example, to achieve zero steady state error may require additional integrators in the plant.

To manipulate the singular values of $I + L^{-1}(z)$. It is shown in [28] that a good stability margin can not be obtained by using small values of ρ. Thus to get a large values of $\underline{\sigma}[I + L^{-1}(z)]$ at high frequencies, large values of ρ should be used.

Thus for large values of ρ we have [28]

$$\underline{\sigma}[I + L^{-1}(z)] \approx \rho \underline{\sigma}[B^T P_1 A (zI - A)^{-1} B \qquad (3.4.1-5)$$

Where $P_1$ is the solution to the following Lyapunov equation

$$A^T P_1 A - P_1 + H_w^T H_w = 0$$

2) The loop transfer function $L(z)$ of step (1) is only an intermediate design function since it assumes that all the states are available for feedback. The loop transfer function $L(z)$ should be recovered by a realizable LQG controller. The following LTR procedure is described in [28].

A KBF is defined by the following equation

$$\hat{x}_{i+1|i} = A\hat{x}_{i|i-1} + Bu_i + K_f^P(y_i - C\hat{x}_{i|i-1}) \qquad (3.4.1-3)$$

where $\hat{x}_{i|i-1}$ is the least squares estimate of $X_i$ given $\{y_0, y_1, \ldots, y_{i-1}\}$, and the filter gain $K_f^P$ is defined by the following equations:

$$AP_fA^T - P_f + R_w - AP_fC^T(R_v + CP_fC^T)^{-1}C\ P_fA^T = 0 \qquad (3.4.1-4)$$

$$K_f^f = P_fC^T(R_v + CP_fC^T)^{-1}$$

$$K_f^P = A\ K_f^f$$

It is shown in [28] that if $G_o(z)$ is minimum phase, and $CB$ is full rank, then letting $R_v = 0$ and $R_w = BB^T$ in (3.4.1-3), then $K_f^f = B(CB)^{-1}$ and we get a perfect recovery (i.e. $K(z)G_o(z) = L(z)$), and the compensator transfer becomes

$$K(z) = z\ \overline{K}_c[zI - A + K_f^PC]^{-1}\ K_f^P \qquad (3.4.1-5)$$

where $\overline{K}_c$ is related to $K_c$ by

$$K_c = A\overline{K}_c$$

Here we will show the mechanism by which the perfect recovery is obtained. Using the matrix inversion lemma (section 1.3), (3.4.1-5) can be written as

$$K(z) = z\overline{K}_c(zI - A)^{-1}\{I + K_f^p C(zI - A)^{-1}\}^{-1} K_f^p$$

substituting $\overline{K}_c = K_c A^{-1}$ (we are assuming that the discrete time system is obtained by sampling a continuous time system), and $K_f^p = AK_f^f$, then

$$K(z) = zK_c(zI - A)^{-1}A^{-1}\{I + AK_f^f C(zI - A)^{-1}\}^{-1} AK_f^f$$

$$= zK_c(zI - A)^{-1}\{I + K_f^f C(zI - A)^{-1}A\}^{-1} K_f^f$$

using the matrix inversion lemma

$$K(z) = zK_c(zI - A)^{-1}K_f^f\{I - C[zI - A + AK_f^f C]^{-1} AK_f^f\}$$

substituting $K_f^f = B(CB)^{-1}$, we get

$$K(z) = \{K_c(zI - A)^{-1}B\{z(CB)^{-1}\{I - C(zI - A + K_f^p C)^{-1} K_f^p\}$$

$$= L(z)z(CB)^{-1}\{I - C(zI - A + K_f^p C)^{-1} K_f^p\} \qquad (3.4.1-6)$$

Now, let

$$R(z) = z(CB)^{-1}\{I - C(zI - A + K_f^p C)^{-1} K_f^p\}$$

$$= z(CB)^{-1}\{I + C(zI - A)^{-1}K_f^p\}^{-1}$$

$$= z\{CB + C(zI - A)^{-1}K_f^p CB\}^{-1}$$

noting that $K_f^p CB = AK_f^f CB = AB(CB)^{-1}CB = AB$, thus

$$R(z) = z\{C[I + (zI - A)^{-1}A]B\}^{-1}$$
$$= z\{C(zI - A)^{-1}[(zI - A) + A]B\}^{-1}$$
$$= z\{zC(zI - A)^{-1}B\}^{-1}$$
$$= z\{zG(z)\}^{-1} \qquad (3.4.1-7)$$

Now from (3.4.1-6) and (3.4.1-7), *K(z)* is given by

$$K(z) = \{zL(z)\} \quad \{zG(z)\}^{-1} \qquad (3.4.1-8)$$

Which shows how perfect recovery is obtained (i.e. $K(z)G(z) = L(z)$).

### 3.4.2 Compensator Processing Time is Not Negligible

The problem of large compensator processing time can be solved by introducing a unit step delay in the controller. This will result in the following optimal control law for the quadratic cost function [27]

$$u_i = -K_c \ [Ax_{i-1} + B \ \ u_{i-1}] \qquad i \geq 1 \qquad (3.4.2-1)$$

the structure of this regulator is shown in Fig. 3.5. The loop transfer functions at the input of the plant in Fig. 3.5 is $L(z) = K_c(zI + BK_c)^{-1}A(zI - A)^{-1}B$

The first step in the LQG/LTR method is to chose $K_c$ such that $I + L(z)$ and $I + L^{-1}(z)$ satisfy the performance and robustness constraints (Figs. 3.3 and 3.4). The loop transfer function *L(z)* is only an intermediate design function since it assumes that all the states of the plant are available for feedback. The loop transfer function *L(z)* should be recovered by a realizable LQG controller. The

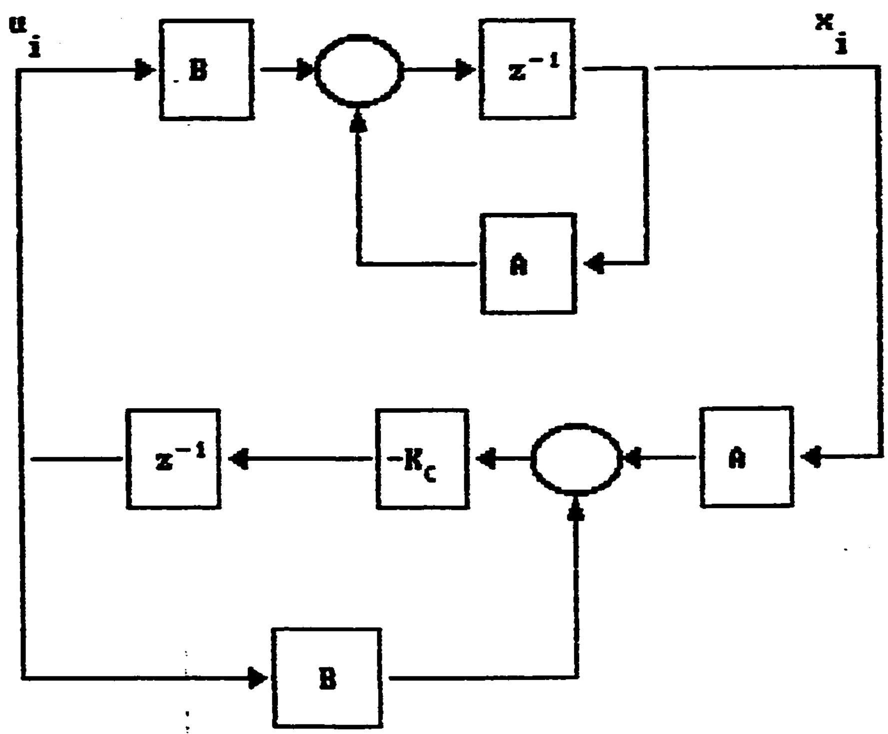



Fig.3.5 : State Feedback Control with A Unit-Step Control

internal LQG/LTR compensator is shown in Fig. 3.6, where the current estimator is used to provide an estimate for the states of the system. Here, a procedure will be described to recover $L(z)$.

The KBF of section (3.4.1) can be decomposed into two parts known as measurement-update and time-update [28].

Measurement-update:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_f^f(y_i - C\hat{x}_{i|i-1}) \qquad (3.4.2-2)$$

Time-update:

$$\hat{x}_{i+1|i} = A\ \hat{x}_{i|i} + Bu_i \qquad (3.4.2-3)$$

(3.4.2-3) can be written as

$$\hat{x}_{i+1|i+1} = (I - K_f^f C)\hat{x}_{i+1|i} + K_f^f y_{i+1} \qquad (3.4.2-4)$$

let $S_{i+1} = \hat{x}_{i+1|i+1}$ and using (3.4.2-4) and (3.4.2-3) we get

$$S_{i+1} = (I - K_f^f C)A\ \ S_i + (I - K_f^f C)BU_i + K_f^f y_{i+1}$$

Thus, we can write

$$[zI - (I - K_f^f C)A]\ \ S = (I - K_f^f C)BU + zK_f^f\ \ Y \qquad (3.4.2-5)$$

The control law $u_i = -K_c[A\hat{x}_{i-1|i-1} + Bu_{i-1}]$ can be written as

$$U = K_c(zI + BK_c)^{-1} A\hat{X} \qquad (3.4.2-6)$$

Using (3.4.2-5) and (3.4.2-6) and noting that $\hat{X} = S$, we get

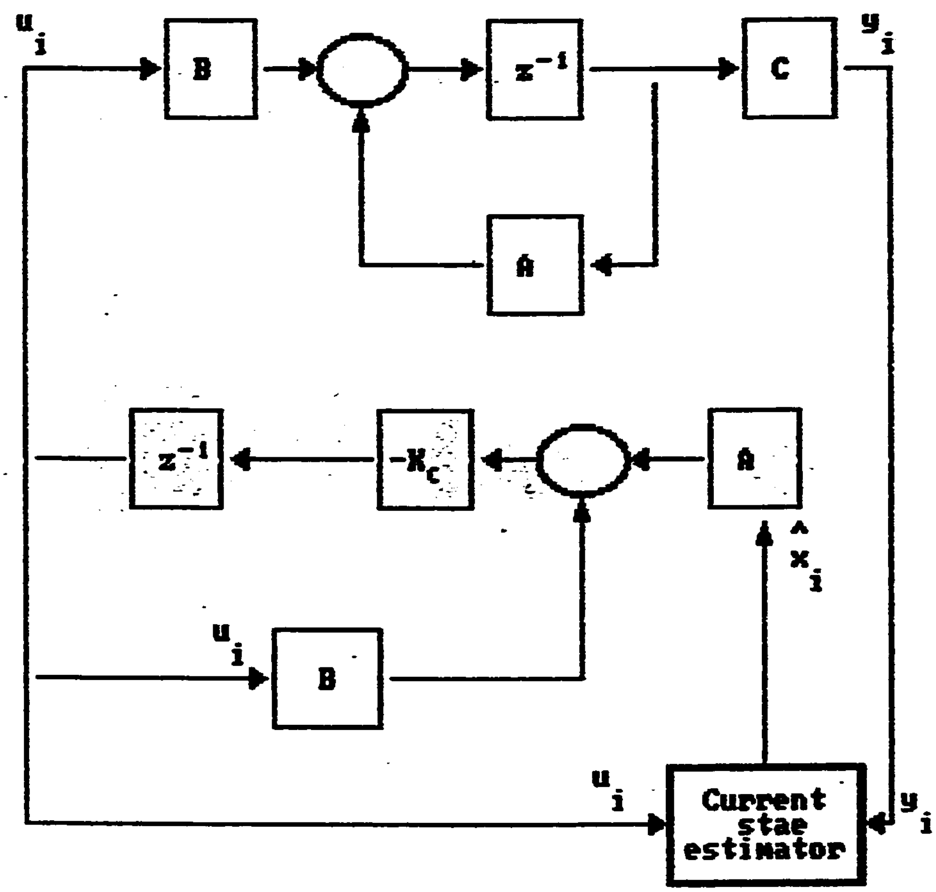



**Fig.3.6 : LQG/LTR Compensator**

$$\{zI+(I-K_f^fC)[K_c(zI+BK_c)^{-1}A-A]\}\ S=zK_f^fY$$

Thus

$$\{I-(CI-K_f^fC)(zI+BK_c)^{-1}A\}S=K_f^fY \qquad (3.4.2-7)$$

From 3.4.2-6,3.4.2-7and noting that $K_f^p = AK_f^f$ we get

$$U=K_c(zI+BK_c)^{-1}\ \{I-(A-K_f^pC)(zI+BK_c)^{-1}\}^{-1}K_f^pY$$

$$=K_c[zI-A+BK_c+K_f^pC]^{-1}K_f^pY$$

Thus, the compensator $K(z)$ is given by

$$K(z)=K_c(zI-A+BK_c+K_f^pC)^{-1}K_f^p$$

if $G_o(z)$ is minimum phase, and $CB$ is a full rank, and letting $R_v=0$ and $R_w=BB^T$ in the Riccati equation of the filter, we get [28]

$$K_f^p=AB(CB)^{-1} \qquad (3.4.2-8)$$

and the recovery is perfect i.e. $K(z)G_o(z)=L(z)$.

The following proof of the above result is described in [26]

Let $R(z)=K(z)G_o(z)=K_c(zI-A+BK_c+K_f^pC)^{-1}K_f^pC(zI-A)^{-1}B$

where $K_f^p$ is given by (3.4.2-8).$R(z)$ can be written as

$$R(z)=K_c(zI+BK_c)^{-1}\ \{I-A[I-B(CB)^{-1}C](zI+BK_c)^{-1}\}^{-1}$$

$$\times\ AB(CB)^{-1}C(zI-A)^{-1}B \qquad (3.4.2-9)$$

Noting that

$$[I - B(CB)^{-1}C]B = 0 \qquad (3.4.2-8)$$

and $$(zI + BK_c)^{-1} = z^{-1}I - z^{-1}B(zI + K_cB)^{-1}K_c \qquad (3.4.2-10)$$

we have

$$R(z) = K_c(zI + BK_c)^{-1}\{I - z^{-1}A[I - B(CB)^{-1}C]\}^{-1}AB$$
$$\times \quad (CB)^{-1}C(zI - A)^{-1}B \qquad (3.4.2-12)$$

Define

$$Q = I - B(CB)^{-1}C \qquad (3.4.2-13)$$

then

$$R(z) = K_c(zI + BK_c)^{-1}A(I - z^{-1}QA)^{-1}(I - Q)(zI - A)^{-1}B \quad (3.4.2-14)$$

Furthermore, the portion containing $Q$ in (3.4.2-13) can be written as

$$(I - z^{-1}QA)^{-1}(I - Q) = (I - z^{-1}QA)^{-1}[(I - z^{-1}QA) + (z^{-1}QA - Q)]$$
$$= I - z^{-1}(I - z^{-1}QA)^{-1}Q(zI - A) \qquad (3.4.2-15)$$

Noting that $QB=0$, from (3.4.2-10) - (3.4.2-15) it follows that

$$R(z) = K_c(zI + BK_c)^{-1}A(zI - A)^{-1}B$$
$$= L(z)$$

### 3.5 Simplified LQG/LTR Design Method for SISO Discrete-Time Systems

The perfect recovery for MIMO Discrete systems obtained in section (3.4.1) is possible only for square minimum-phase systems satisfying det [CB] $\neq 0$. The requirement that det [CB] $\neq 0$. ensures that the nominal system transfer function $G_o(z)$ has the maximum possible number *(n-m)* (where *n* is the order of the system and *m* is the number of inputs) of finite zeros and minimum possible number *(m)* of infinite zeros [14]. The compensator transfer function is given by (section 3.4.1).

$$K(z) = z\overline{K}_c(zI - A + K_f^p c)^{-1} \quad K_f^p \qquad (3.5-1)$$

By choosing $R_\omega = BB'$ and $R_v = 0$ in the KBF Riccati equation (3.4.1-4), the poles of the compensator *K(z)* which are the eigen values of $A - K_f^p C$, will be located at the *(n-m)* minimum-phase zeros of the plant $G_o(z)$ and the remainder *m* at the origin [14]. The mechanism by which the perfect recovery is obtained is that the *(n-m)* poles of *K(z)* will cancel the *(n-m)* minimum-phase zeros of $G_o(z)$ and the *(m)* origin poles of *K(z)* will cancel the *m* origin zeros introduced by the factor *z* in (3.5.1).

The perfect recovery conditions if transformed to SISO systems will be that the nominal system transfer function should have *(n-1)* minimum-phase zeros. Thus, $G_o(z)$ can be written as

$$G_o(z) = \frac{(z+z_{g1})(z+z_{g2})----(z+z_{gn-1})}{(z+p_{g1})(z+p_{g2})----(z+p_{gn})}; \quad |z_{gi}| < 1, i = 1, \ldots, n-1$$

and the compensator *K(z)*

$$K(z) = \frac{(z+z_{k1})(z+z_{k2})----(z+z_{kn-1})}{(z+z_{g1})(z+z_{g2})----(z+z_{gn-1})}$$

note that the poles of *K(z)* are the zeros of $G_o(z)$.

Therefore, the design for SISO systems centers on locating *(n-1)* zeros for the compensator *K(z)*, and the design can be accomplished as follows:

i) Choose *(n-1)* zeros such that the closed-loop system of Fig. 3.7 is stable. The transfer function *L(z)* in Fig. 3.7 is given by

$$L(z) = \frac{(z+z_{k1})(z+z_{k2})----(z+z_{kn-1})}{(z+p_{g1})(z+p_{g2})----(z+p_{gn})}$$

where

$z_{ki}, i = 1, —, n-1$ are the chosen zeros.

$p_{gi}, i = 1, —, n$ are the poles of $G_o(z)$.

Also, the return difference $I + L(z)$ and the inverse return difference $I + L^{-1}(z)$ should satisfy the performance and robustness specifications given in section (3.4).

ii) The compensator *K(z)* will be given by

$$K(z) = \frac{(z+z_{k1})(z+z_{k2})----(z+z_{kn-1})}{(z+z_{g1})(z+z_{g2})----(z+z_{gn-1})}$$

$z_{ki}, i = 1, —, n-1$ are the chosen zeros.

$z_{gi}, i = 1, —, n-1$ are the zeros of $G_o(z)$.

and thus, perfect recovery is obtained (i.e. $K(z)G_o(z) = L(z)$).

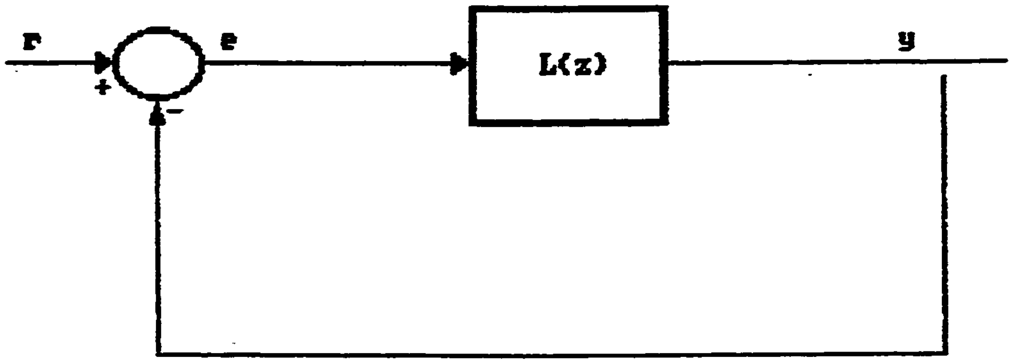



**Fig.3.7 : Loop Transfer Function**

## 3.6 $H_\infty$/CLTR Design Method for Discrete-Time Systems

The problem of $H_\infty$-state feedback design for discrete-time systems has been considered in [10] and [17]. But for these methods there are some computational difficulties related to the solution of the derived Riccati equations. The proposed solutions to the derived Riccati equations is iterative. However, it did not converge to a solution even for trivial systems (2nd order). The well-known bilinear transformation between the $z$ and $s$ planes can be used to convert between discrete and continuous-time models. An obvious approach for solving a discrete-time problem is to transform it into a continuous-time one. Carry out the computations using continuous-time algorithms and then transform back the solution.

### 3.6.1 Bilinear Transformation Properties

The method of discrete-time controller design to be discussed later uses the bilinear transformation, and so some properties of the bilinear transformation will be briefly reviewed.

The bilinear transformation is a bijective map between the $z$-plane and the $w$-plane. When converting discrete system to pseudo-continuous systems by the bilinear transformation, the following substitution is made

$$z \leftarrow \frac{1+\alpha w}{1-\alpha w}$$

where $\alpha$ is a positive constant. For, conversion of a $w$-plane system into a $z$-plane system the following substitution is employed

$$w \leftarrow \left(\frac{1}{\alpha}\right) \frac{z-1}{z+1}$$

when the bilinear transformation is used with sampled-data systems, $\alpha$ is usually set to $T/2$, where $T$ is the sampling period.

Now let the discrete-time system transfer function be

$$G(z) = H(zI - \Phi)^{-1}\Gamma + J$$

and the pseudo-continuous system transfer function be

$$G(w) = C(wI - A)^{-1}B + D$$

then we have

$$A = \left(\frac{1}{\alpha}\right) (\Phi + I)^{-1} (\Phi - I)$$

$$B = \sqrt{\frac{2}{\alpha}} (\Phi + I)^{-1}\Gamma$$

$$C = \sqrt{\frac{2}{\alpha}} H(\Phi + I)^{-1}$$

$$D = J - H(\Phi + I)^{-1}\Gamma$$

$$\Phi = (I - \alpha A)^{-1} (I + \alpha A)$$

$$\Gamma = \sqrt{2\alpha} (I - \alpha A)^{-1}B$$

$$H = \sqrt{2\alpha} C(I - \alpha A)^{-1}$$

$$J = D + \alpha C(I - \alpha A)^{-1}B$$

### 3.6.2 Controller Design in the $w$-plane

The following discrete controller design algorithm is proposed. It is assumed, to start with, that we have a discrete nominal plant $G_o(z)$ and bounds on the sensitivy

($S(z)$) and complementary sensitivity ($T(z)$) functions.

The algorithm is

1) Map $G(z)$ into the $w$-plane using the bilinear transformation.
2) Map the bound on $S(z)$ and $T(z)$ into the $W$-plane.
3) Proceed with the design method given in section (2.6) as if it were in the $s$-plane.
4) After the design is done, map the controller back into the $z$-plane via the bilinear transformation.

## 3.7 Design Considerations for Sampled-Data Systems

In the previous sections the discussion centered on discrete time systems. However, the discrete time system will be obtained by sampling a continuous time system and we are concerned about robustness and performance of the sampled-data system of Fig. 3.8. Thus we are interested in knowing how the continuous plant description of section 2.2.2can be transformed into an equivalent description on the resulting discrete time system.

Here equivalence is in the sense that if the discrete time system is stable in the face of discrete uncertainty, then the sampled-data system will be stable in the face of plant uncertainty. Another issue is how the performance requirements as described in section 2.3.3can be translated into requirements on the discrete time system.

Consider plant $G(s)$. It is seen in section 2.2.2that one way of describing the plant is

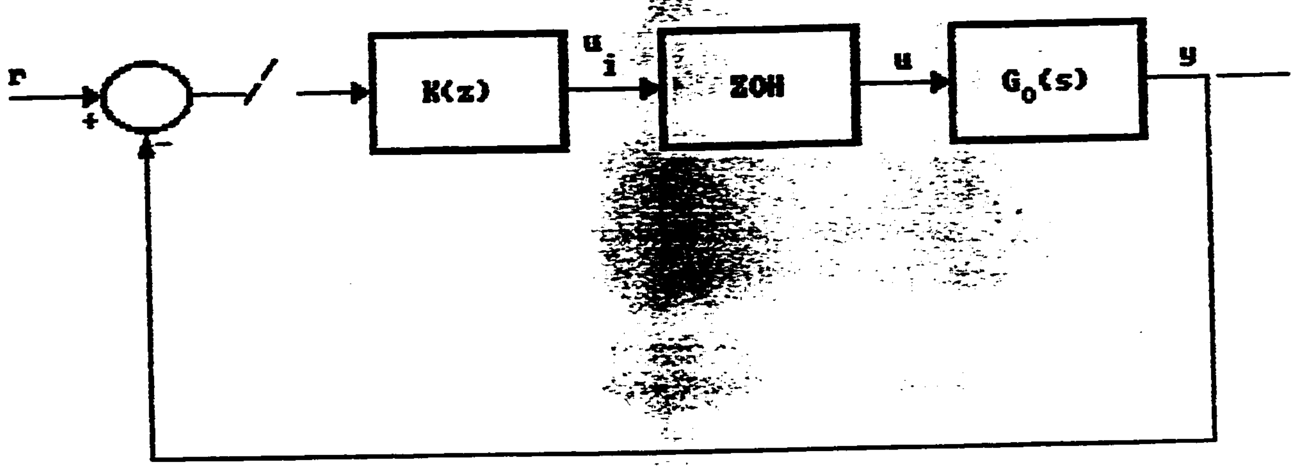



Fig.3.8 : Sampeled-Data Control System

$$G(s) = G_o(s) + L_\alpha(s)$$

Where $G_o(s)$ is a nominal plant, $L_\alpha(s)$ is an additive uncertainty which is characterized by the following

$$\bar{\sigma}(L_\alpha(j\omega)) \le l_\alpha(\omega) \quad , \omega \ge 0$$

and *G(s)* and $G_o(s)$ have the same number of unstable poles. However, in many practical cases more canbe said about the uncertainty. For example, to name a few, paramater variations, model reduction. In such cases, the plant can be described as

$$G(s) = \{G(s,\beta) \mid \beta \in S\}$$

where the S represents say the values of the parameters.Then it is a simple matter to get the discrete equivalent of the continuous time plant. Namely

$$G(z) = \{G(z,\beta) \mid \beta \in S\}$$

and *G(z)* can be represented as

$$G(z) = G_o(z) + L_\alpha(z)$$

where $G(z,\beta)$ is the *ZOH* equivalent of $G(s,\beta)$, $G_o(z)$ is the *ZOH* equivalent of $G_o(z)$and $L_\alpha(z)$ is an additive uncertainty which is characterized by the following

$$\bar{\sigma}[L_a(e^{j\theta})] < l_a(\theta), \qquad 0 \le \theta \le \pi$$

$l_a(\theta)$ can be computed as $l_a(\theta) = \max_{\beta \in S} \quad \bar{\sigma}[G(e^{j\theta},\beta) - G_o(e^{j\theta})]$

This approach is used in the design of a discrete controller for a double-mass spring which the spring constant and the damping coefficient are uncertain (section 4.1).

The above approach is not applicable if what is known about the actual plant is a nominal plant and a bound on the largest singular value of perturbation (either additive or multiplicative) as described in section 2.2.2.For this case a comprehensive treatment is given in [30]. The main result in [30] is that if, for example, the plant is described by

$$G(s) = G_o(s)[I + L_t(s)$$

where

$$\bar{\sigma}[L_t(jw)] < l_t(w), \qquad w \ge 0$$

and if aliasing is negligible, then the equivalent discrete time system is given by

$$G(z) = G_o(z)[I + L_t(z)]$$

$$\bar{\sigma}[L_t(e^{jwT} < l_t(w), \qquad w \le \frac{\pi}{T}$$

# CHAPTER 4

# EXAMPLES

## 4.1 Introduction

The purpose of this chapter is to illustrate by example the design techniques discussed in the previous chapters. This chapter includes continuous and discrete controller design examples for SISO systems and MIMO systems.

## 4.2 Double Mass-Spring System - LQG/LTR Design Method

Consider the system of Fig. 4.1.It consists of two unit masses which are connected by a spring and a damper with an uncertain spring constant and damping coefficient:

$$k_{min} = 1 \le k \le k_{max} = 2$$

$$b_{min} = .1 \le b \le b_{max} = .5$$

It is desired to control the position of the second mass with a force applied to the first mass. This is a control problem with two uncertain parameters but a known model.

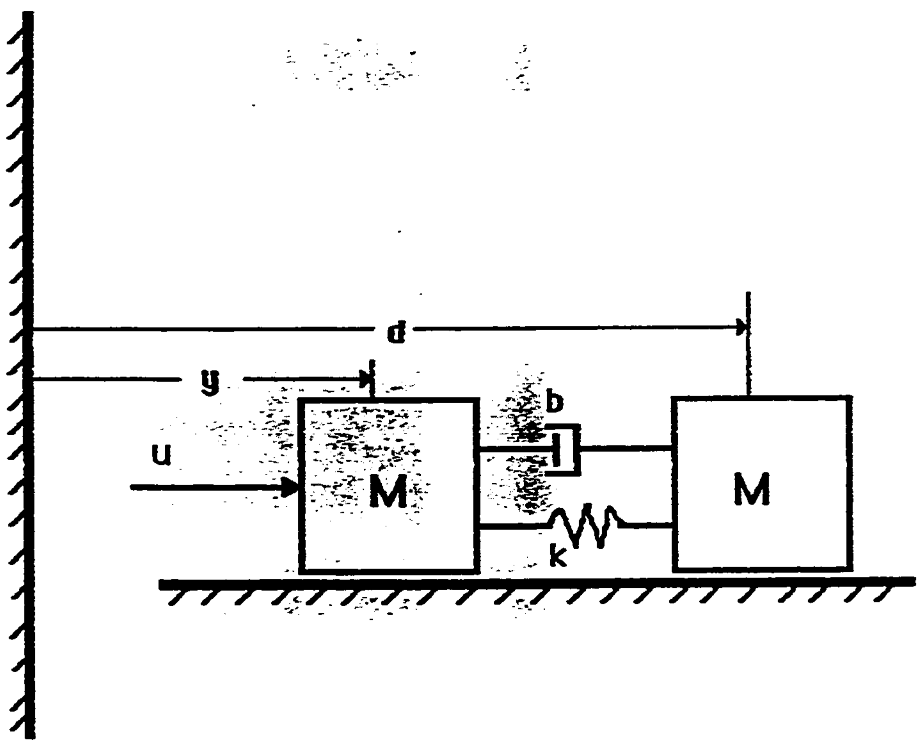



Fig. 4.1 : Double Mass-Spring System

### 4.2.1 The Plant Model

The equations of motion derived from Fig. 4-1 are [7]

$$\ddot{y} + (\dot{y} - \dot{d})b + (y - d)k = u$$
$$\ddot{d} + (\dot{d} - \dot{y})b + (d - y)k = 0$$

which, when transformed, become,

$$\begin{bmatrix} s^2 + bs + k & -(bs + k) \\ -(bs + k) & s^2 + bs + k \end{bmatrix} \begin{bmatrix} y(s) \\ d(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(s)$$

Controlling the position of the second mass with input applied to the first mass is a SISO control problem with the transfer function

$$\frac{d(s)}{u(s)} = \frac{bs + k}{s^2(s^2 + 2bs + 2k)}$$

This will be called the real plant.

### 4.2.2. Nominal Model and Uncertainties

We will deal with the parameter uncertainty as a model uncertainty. Because this is SISO system we can reflect all the uncertainties to the input or the output of the plant. The variation in the spring constant and damping coefficient is treated as a multiplicative uncertainty as follows

$$G(s) = (1 + \Delta_m(s))G_o(s) = G_o(s)(1 + \Delta_m(s))$$

where $G(s)$ is the actual transfer function, $G_o(s)$ is the nominal transfer function, and $\Delta_m(s)$ is the multiplicative uncertainty.

Choose the nominal model as

$$G_o(s)=\frac{b_o s+k_o}{s^2(s^2+2b_o s+2k_o)}$$

where $k_{min}\le k_o\le k_{max}$, and $b_{min}\le b_o\le b_{max}$

the uncertainty is bounded as follows

$$|\Delta_m(jw)|<\Delta(w) \qquad ,w\ge 0$$

where $\Delta(w)=\max_{1\le k\le 2}\left\{\max_{1\le b\le .5}\left|\frac{G(jw)-G_o(jw)}{G_o(jw)}\right|\right\} \quad ,w\ge 0$

A plot of $\Delta(w)$ is shown in Fig. 4.2.

The controller-canonical state-space realization of the nominal plant is:

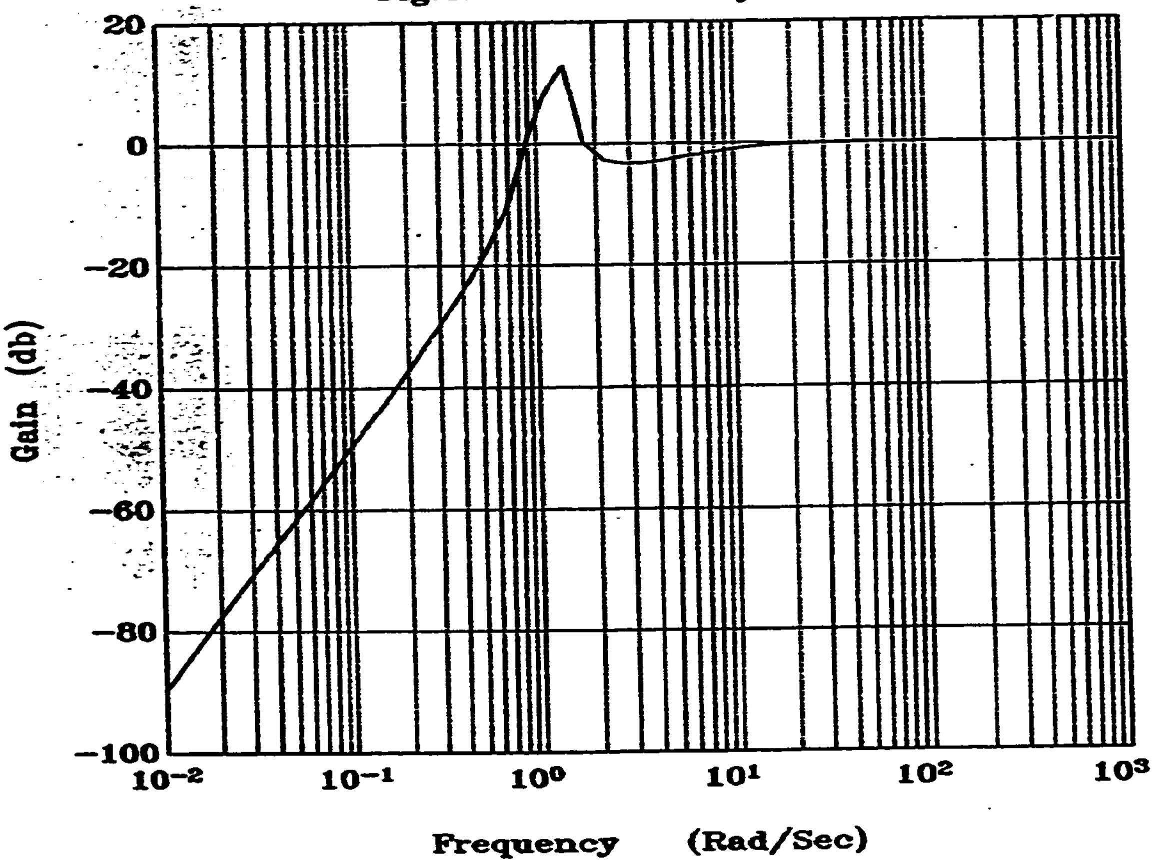
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -2k_o & -2b_o\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 1\end{bmatrix}u$$

$$= Ax+Bu$$

$$d=[k_o\ \ b_o\ \ 0\ \ 0]\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = Cx$$

The values of $k_o$ and $b_o$ are selected to be 1.5 and .25 respectively.

Fig.4.2 : Uncertainty Bound



### 4.2.3 LQG/LTR Controller Design

Here the design is done by selecting an LQR gain matrix $K_c$ such that the LQR loop transfer function $L(s) = K_c(sI - A)^{-1}B$ satisfies the loop shape constraints of section (2.4). The LQR gain matrix is found, as shown in section 2.4, by solving the regulator Riccati equation with the parameters $\rho$ and $H$. Here $H$ and $\rho$ are chosen as

$$H = [0 \quad 1.0 \quad .25 \quad 1.5] \quad ; \quad \rho = 1.0$$

The corresponding LQR gain matrix is

$$K_c = [.2659 \quad .1018 \quad .4077 \quad .0530]$$

Figure 4.3 shows the LQR loop transfer function and the uncertainty bound which shows that the state feedback design will stabilize the plant for all admissible values of the uncertainties.

Now, we need to design a compensator *K(s)* such that:

$K(jw)G_o(jw) \rightarrow K_c(jwI - A)^{-1}B$ for those frequencies $w$ where the sensitivity and robust stabilization requirements are relevant. Using the results of section 2.4, *K(s)* is given by

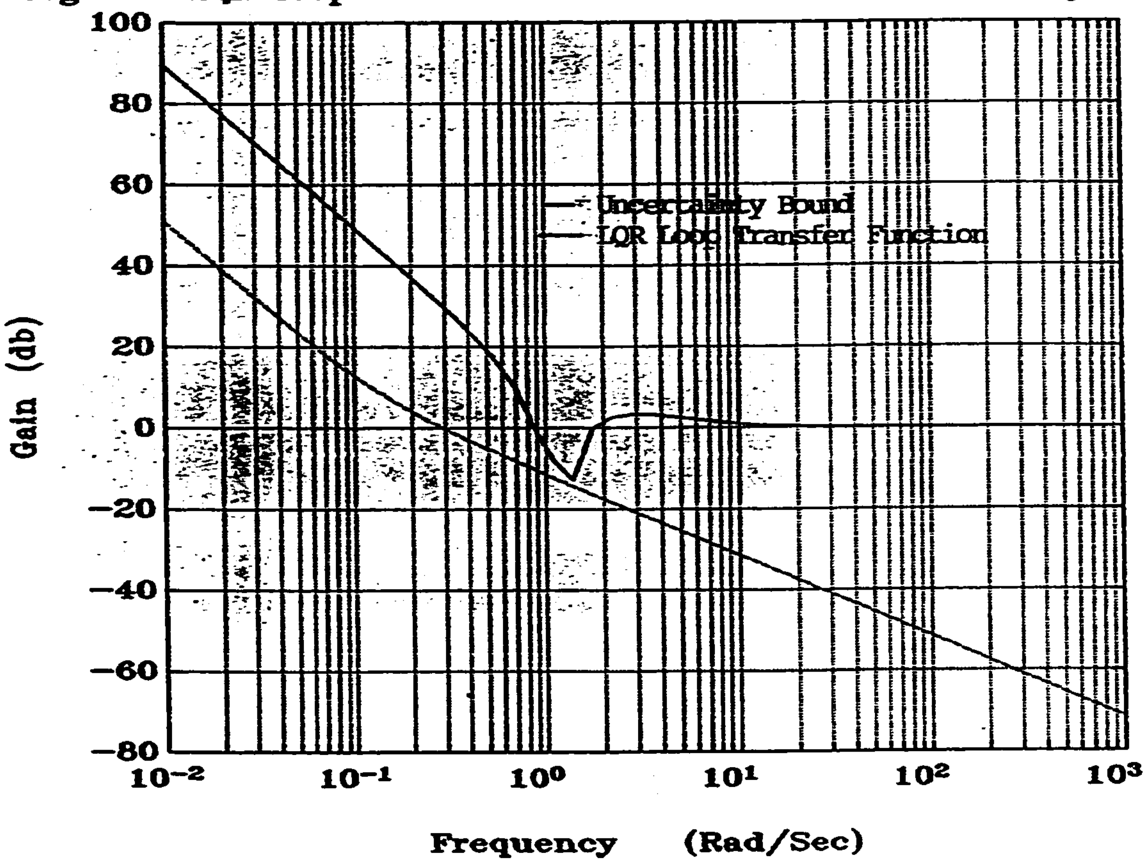
$$K(s) = K_c(SI - A + BK_c + K_f C)^{-1} K_f$$

where the KBF gain $K_f$ is given by

$$K_f = [9.9920x10^8 \quad 3.1735x10^6 \quad 5.0386x10^3 \quad 4.000]^T$$

The poles and zeros of the compensator are:

Fig.4.3 : LQR Loop Transfer Function and the Uncertainty Bound



$$poles \quad 1.0x10^2 \begin{bmatrix} -3.1498+5.4572i \\ -3.1498-5.4572i \\ -6.3022 \\ -0.0600 \end{bmatrix}$$

$$zeros \quad \begin{bmatrix} -.1250-1.2183i \\ -.1250+1.2183i \\ -.1329 \end{bmatrix}$$

The designed and recovered loop transfer functions are shown in Fig. 4.4. Also shown the recovery is perfect at low frequencies. The time-responses to a step-input for different $k$'s and $b$'s are given in Fig. 4.5.

### 4.3 Double Mass-Spring System-Discrete LQG/LTR Controller

#### 4.3.1 Discrete System Description

Selecting the sampling period to be $T_s = 1.0$ sec., the *ZOH* equivalent of the nominal system given in section 4.2 is

$$G_o(z) = \frac{.5z+2}{z^4+2.6093z^3+2.9973z^2-2.1669z+.7788}$$

Note that the system is non-minimum phase. The variation of the spring constant and damping coefficient can be treated as a multiplicative uncertainty as follows. Let $k$ be the spring constant and $b$ be the damping coefficient, then the actual system is a member of the following set

Fig.4.4 : Designed and Recovered Loop Transfer Functions

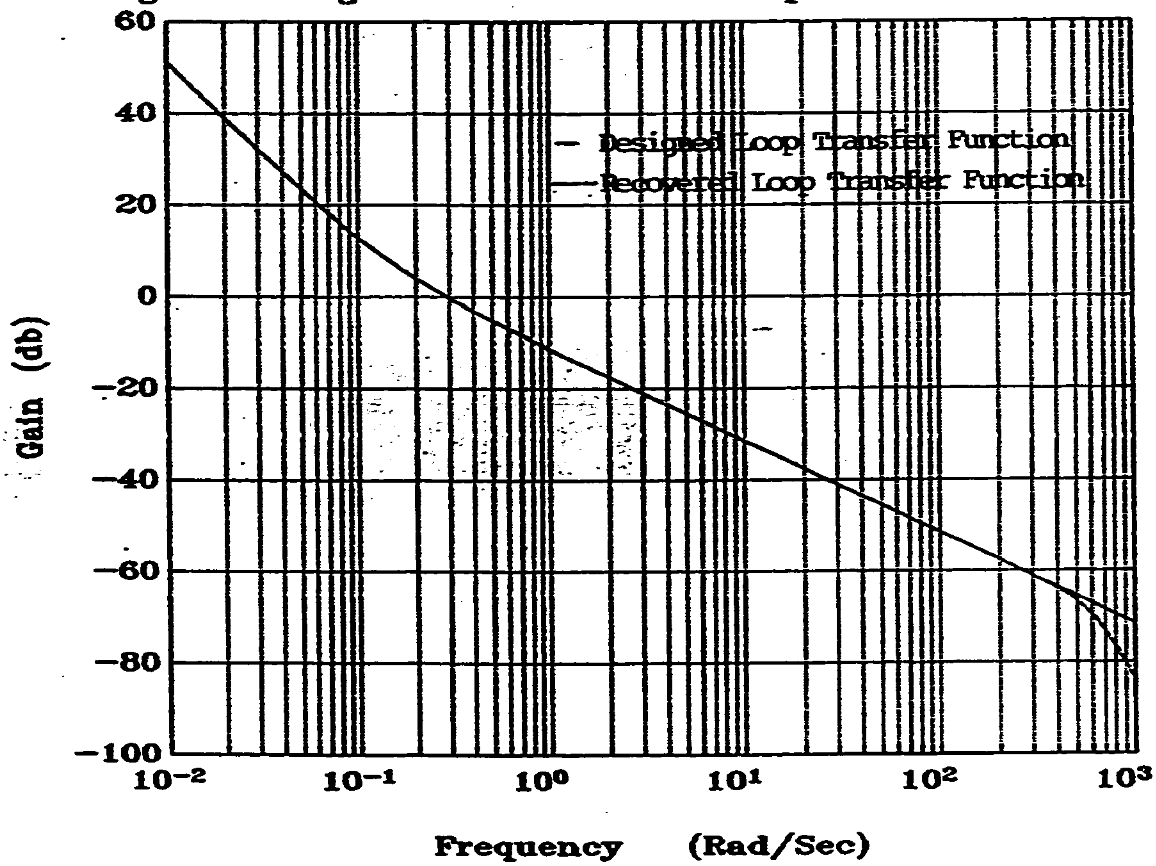
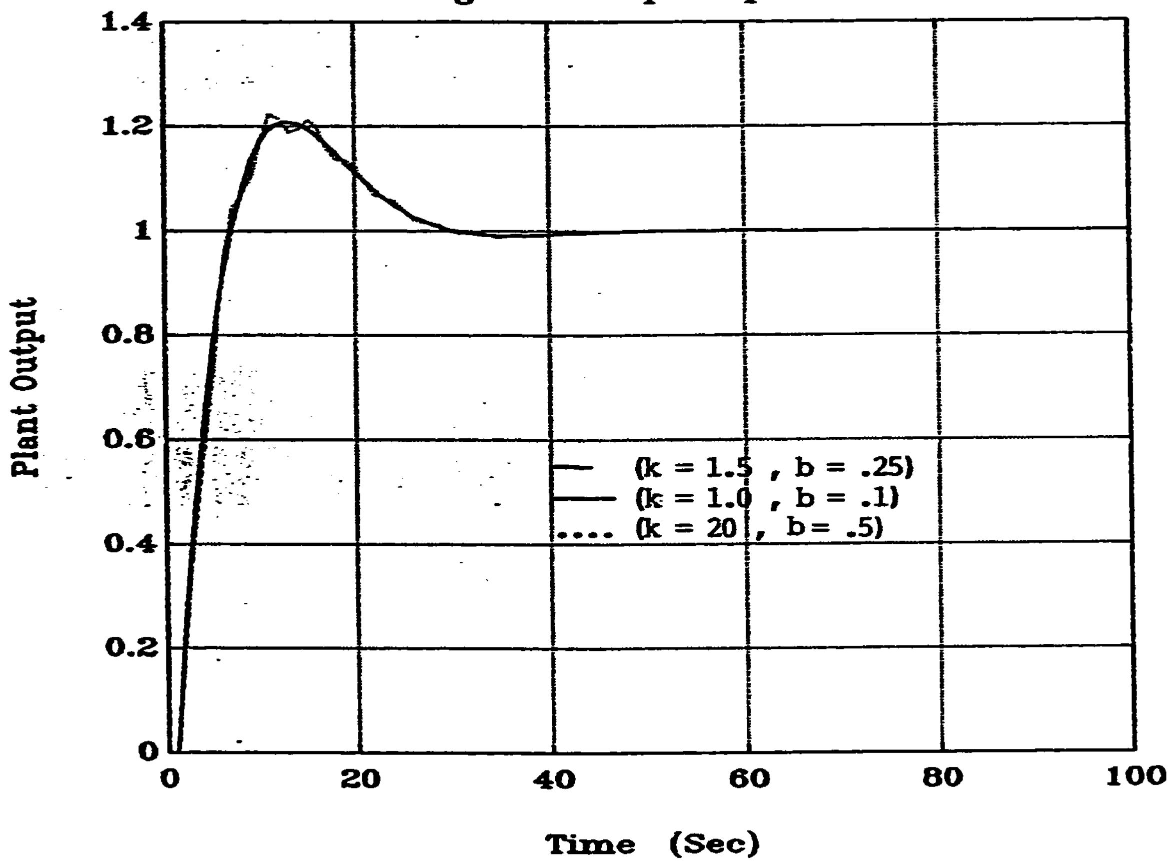


Fig.4.5 : Step Response



$$\{G_o(z,k,b) \mid 1 \le k \le 2, \quad .1 \le b \le .5\}$$

where $G_o(z,k,b)$ is the *ZOH* equivalent of $G_o(s,k,b)$, *let* $G_T(z)$ be the actual discrete time system, and $G_o(z)$ be the nominal discrete time system, then we have

$$G_T(z) = [1 + L_m(z)]G_o(z)$$

where

$$|L_m(e^{j\theta})| < l_m(\theta) \quad , \quad 0 \le \theta \le \pi$$

and $$l_m(\theta) = \max_{1 \le k \le 2} \left\{ \max_{.1 \le b \le .5} \left| \frac{G(e^{j\theta},k,b) - G_o(e^{j\theta})}{G_o(e^{j\theta})} \right| \right\} \quad \theta \le \theta \le \pi$$

a plot of $l_m(\theta)$ is shown in Fig. 4.6.

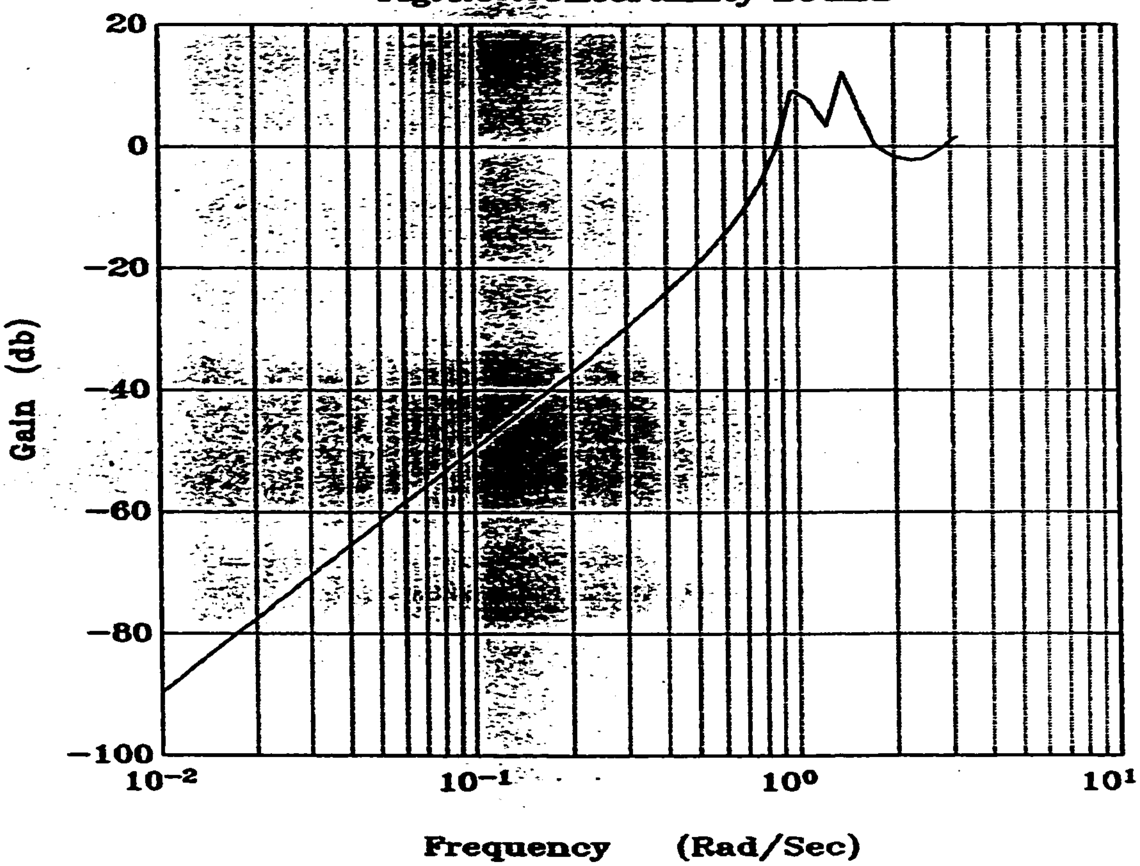
### 4.3.2 Discrete LQG/LTR Compensator Design

Next, the discrete LQG/LTR design method is applied to the nominal system. The design process consists of two steps as in section 3.4 (assuming negligible processing time).

Step 1: Design the target loop transfer function $L(z) = K_c(zI - A)B$, where $K_c$ is the LQR gain matrix, such $L(z)$ satisfies the loop shape constraints. In this case, the constraints are that $|1 + L^{-1}(e^{j\theta})| > l_m(e^{j\theta})$, zero steady state error, and that the bandwidth is as large as possible. The LQR gain matrix is found, as shown in section 3.4, by solving the regulator Riccati equation with the parameters ρ and *H*. Here *H* and ρ are chosen as

$$H = [0 \quad 1.0 \quad -.6093 \quad .7788], \rho = 10$$

Fig.4.6 - Uncertainty Bound



Figures 4.7and 4.8show the LQR return difference and inverse-return difference with the multiplicative uncertainty bound which shows that the condition for robust stability is satisfied. The corresponding LQR gain matrix is

$$K_c = [.2647 \quad -.3951 \quad .3486 \quad -.1821]$$

Step 2: Recover the designated target loop transfer function using the results of section 3.4, the following compensator is designed

$$K(z) = \frac{.3349z^3 - .5047z^2 + .444z - .2341}{z^3 + .6705z^2 + .328z - .0007}$$

Figures 4.9 and 4.10 show the return difference and inverse-return difference of $K(z)G_o(z)$. As seen in the figures, the recovery is not complete. This is due to the fact that $G_o(z)$ is non-minimum phase. However, the system satisfies the uncertainty requirements $(|\,1 + (K(e^{j\theta})G_o(e^{j\theta}))^{-1}\,| > l_m(e^{j\theta}))$.

The time-responses to a step-input for different $k$'s and $b$'s are given in Fig. 11.

### 4.3 Hydraulic Actuator

Consider the SISO hydraulic actuator described in [25]. The nominal transfer function of the system is given by

$$G_o(s) = \frac{9000}{s^3 + 30s^2 + 700s + 1000}$$

the system is stable and the poles of $G_o(s)$ are

Fig.4.7 : LQR Return Difference

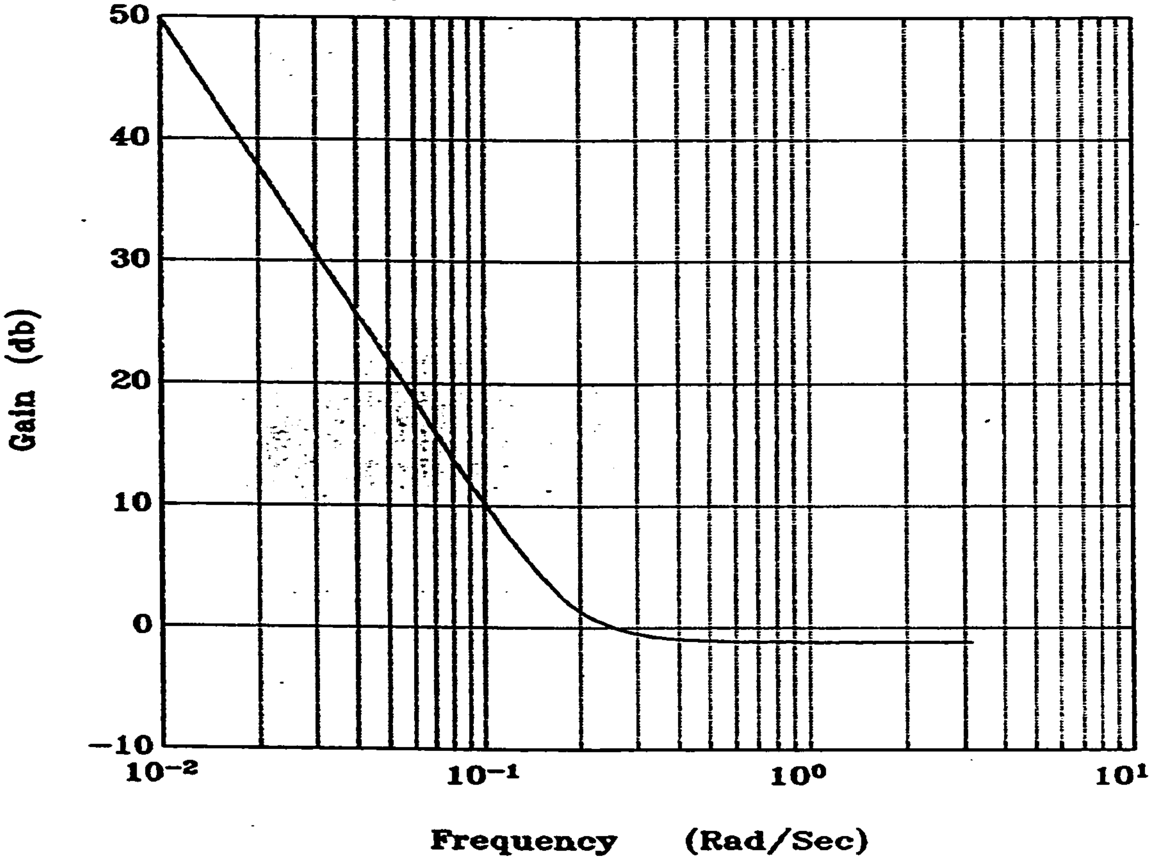


Fig.4.8 :LQR Inverse Return Difference and Uncertainty Bound

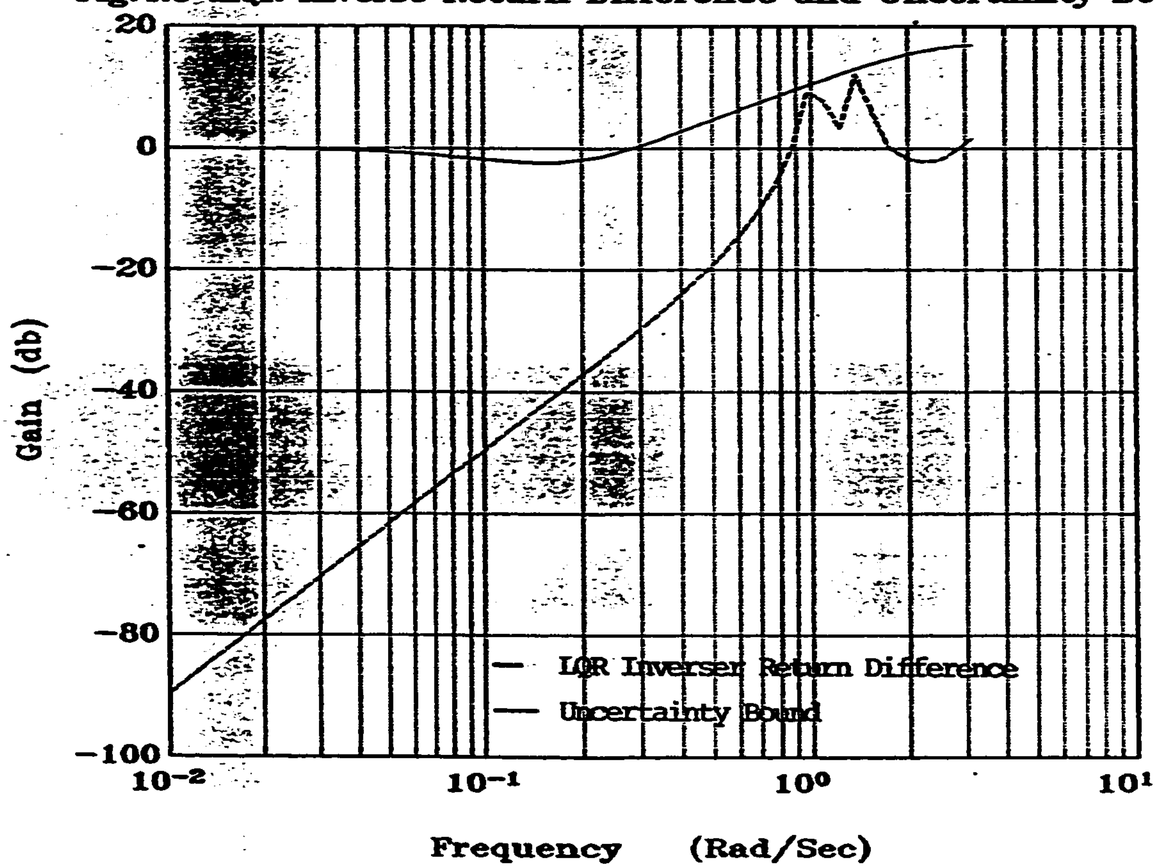


Fig.4.9 : Designed and Recovered Return Difference

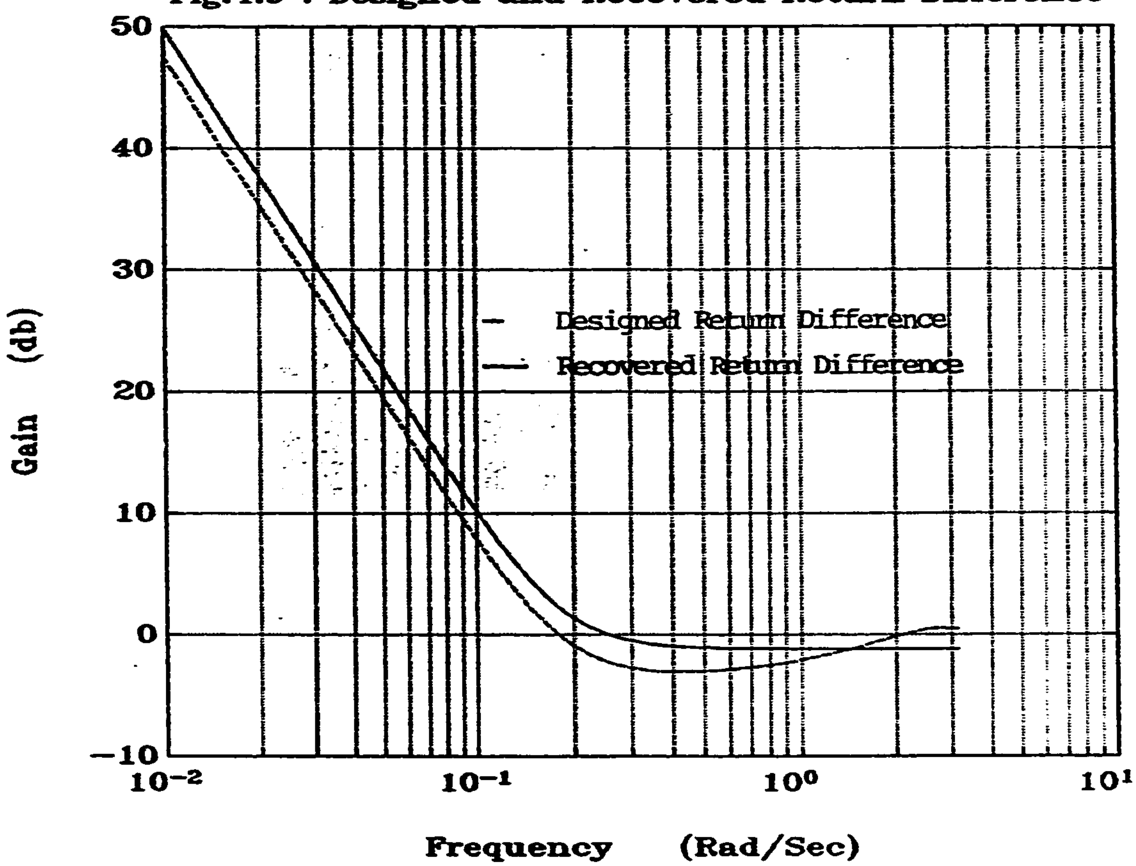
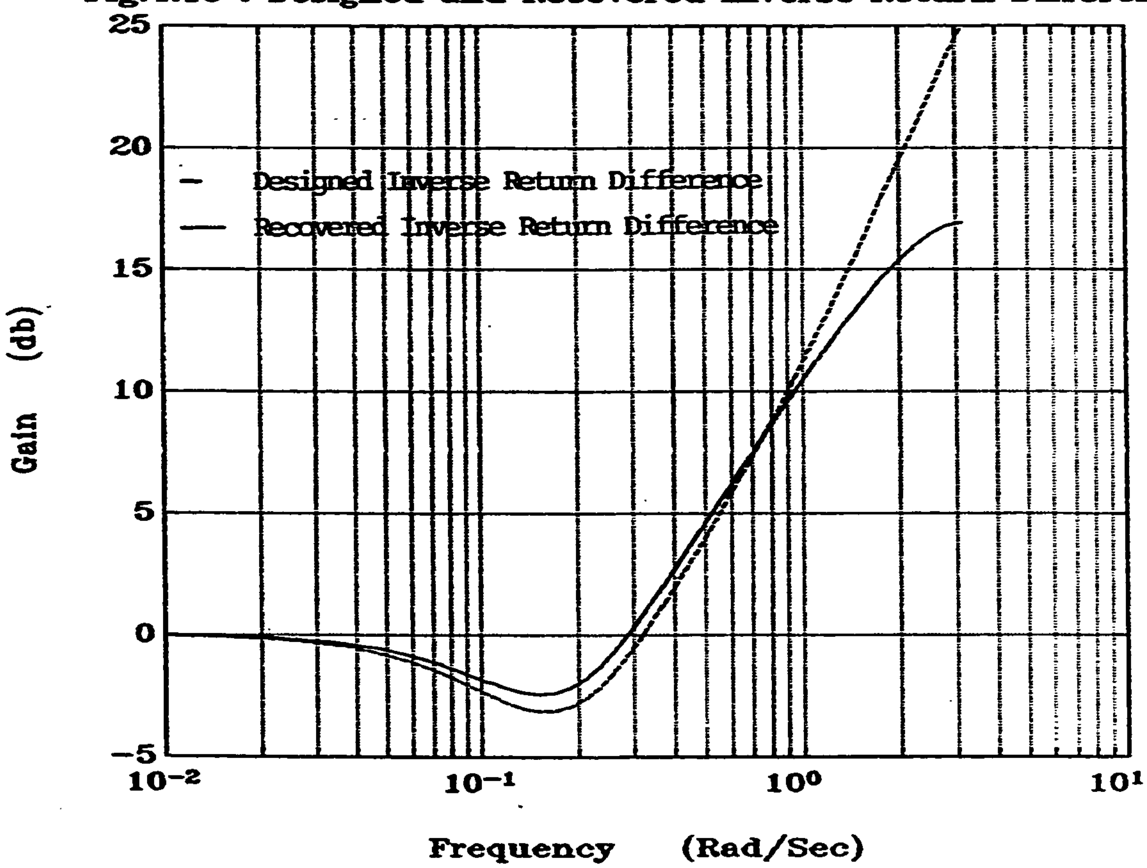


Fig.4.10 : Designed and Recovered Inverse Return Difference



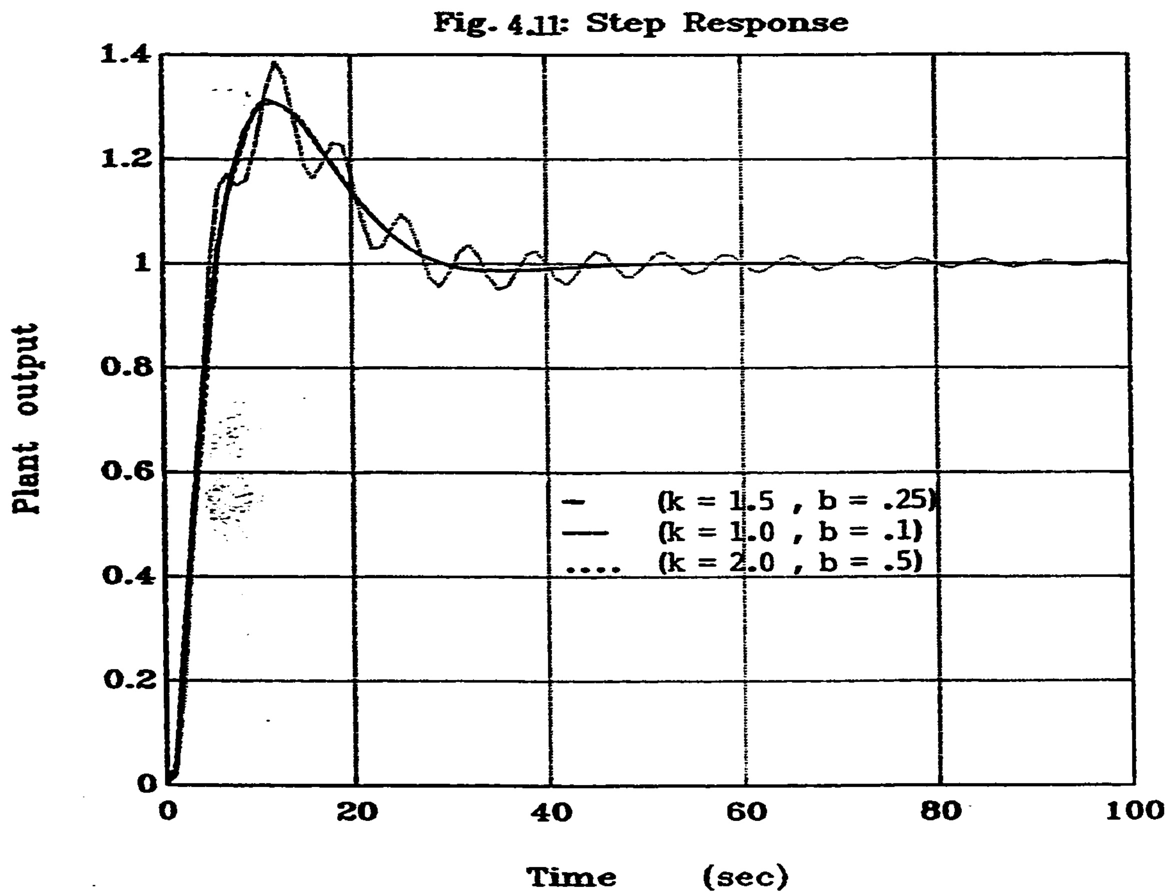


Fig. 4.11: Step Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Plant output
0
20
40
60
80
100
Time (sec)
(k = 1.5 , b = .25)
(k = 1.0 , b = .1)
(k = 2.0 , b = .5)

$$\begin{bmatrix} -14.239+21.305i \\ -14.239-21.305i \\ -1.5229 \end{bmatrix}$$

It is desired to use the LQG/LTR method described in section (2.4) to design a compensator and compare the resulting compensator performance with the one designed using $H_\infty$ design method described in section (2.5) and the $H_\infty$/CLTR design method of section (2.6).

### 4.3.1 Design Specifications

In this example, it is necessary to have a closed loop bandwidth of 30 (rad/sec). This will ensure that the singular values of the complementary sensitivity function be attenuated at frequencies beyond $w = 30$ (rad/sec). Thus, the system will have sufficient stability margin to tolerate variations in the loop transfer function which may arise from unmodeled dynamics at and beyond this frequency. Also, we want to minimize the singular values of the sensitivity function as much as possible, this ensures good attenuation of plant disturbances and insensitivity to small plant variations.

### 4.3.2 LQG/LTR Design

Here we want to design an LQG/LTR compensator which has the property that the closed-loop system will have zero-steady error to arbitrary constant (step) commands and/or disturbances. This specification implies that we must have an integrator in the input channel. Therefore the transfer function of the nominal system augmented with one integrator becomes

$$G_a(s) = \frac{1}{s} G_o(s)$$

and $G_a(s)$ will be used in the following design steps. The state-space realization *(A,B,C)* of $G_a(s)$ is given in table 1.

The first step of the LQG/LTR procedure is to shape the state-feedback loop transfer function $L(s) = K_c(sI - A)^{-1} B$ such that *L(s)* satisfy the design specifications. The following state-feedback was found to satisfy such specifications.

$$K_c = [28.2905 \quad .1750 \quad -3.8196 \quad -6.1719]$$

The frequency response of *L(s)* is shown in Fig. 12. The second step of the design procedure is to use a KBF to provide an estimate for the states. The KBF gain should be chosen such that

$$K(s)G_a(s) \rightarrow K_c(sI - A)^{-1} B$$

for the frequencies of interest. The optimal KBF gain was found to be

$$K_f = [1.0x10^7 \quad 4.4325x10^4 \quad 109.0398 \quad .1557]^T$$

and thus the compensator *K(s)* will be

$$K(s) = \frac{1}{s} K_c(sI - A + BK_c + K_f C)^{-1} K_f$$

The frequency response of the loop transfer functions $K(s)G_a(s)$ is shown in Fig. 13. The results of the design can be seen from Fig. 14 and 15 for the sensitivity and complementary sensitivity functions where the bandwidth is found to be close to the

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -30 & -700 & -1000 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad 0 \quad 9000]$$

Table 4.1 : State Space Matrices for $G_a(s)$

Fig.4.12 State Feedback Loop Transfer Function

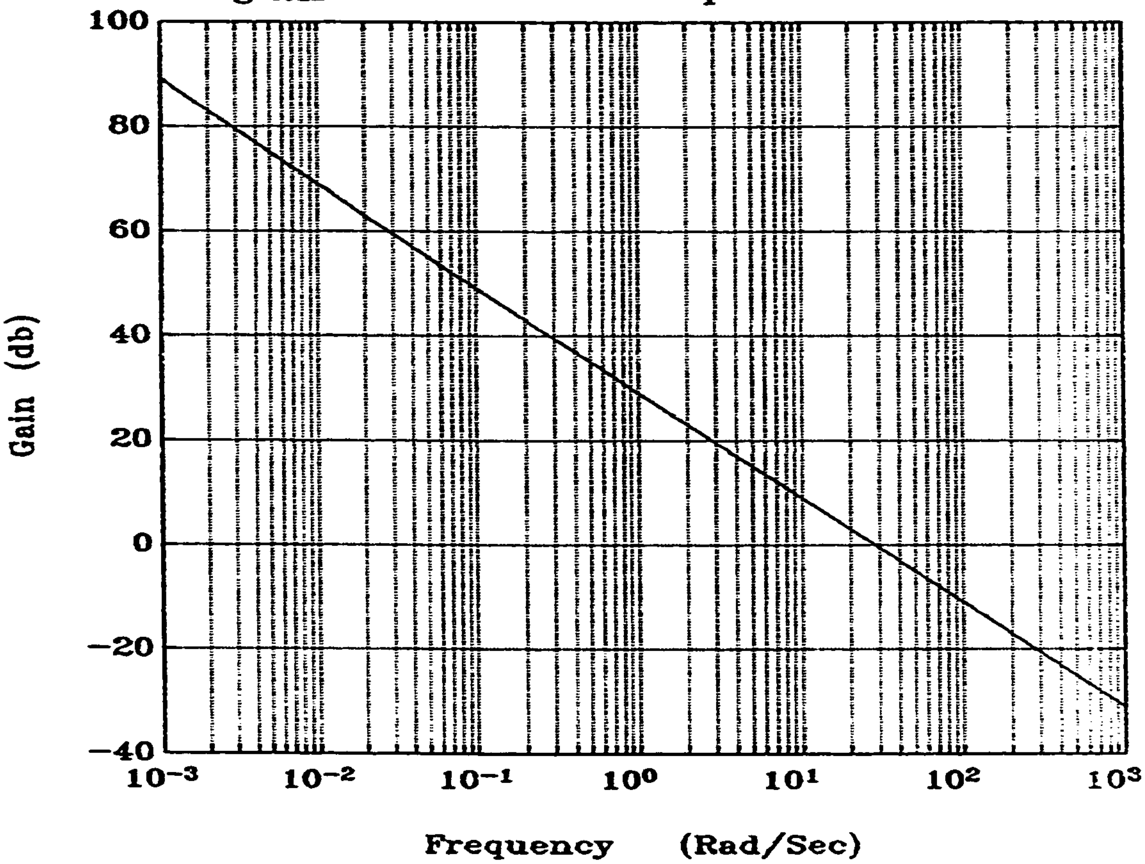


Fig.4.13: Designed and Recovered Loop Transfer Functions

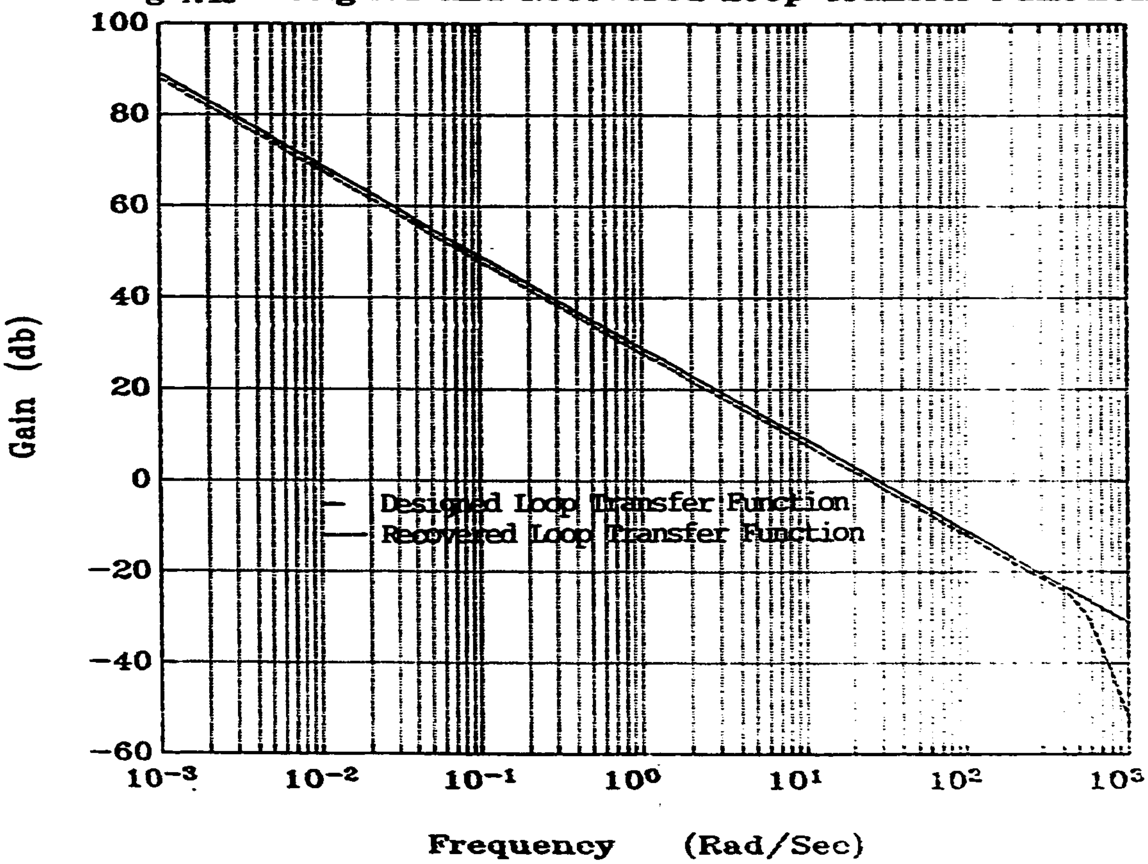


Fig.4.14: Sensitivity Function

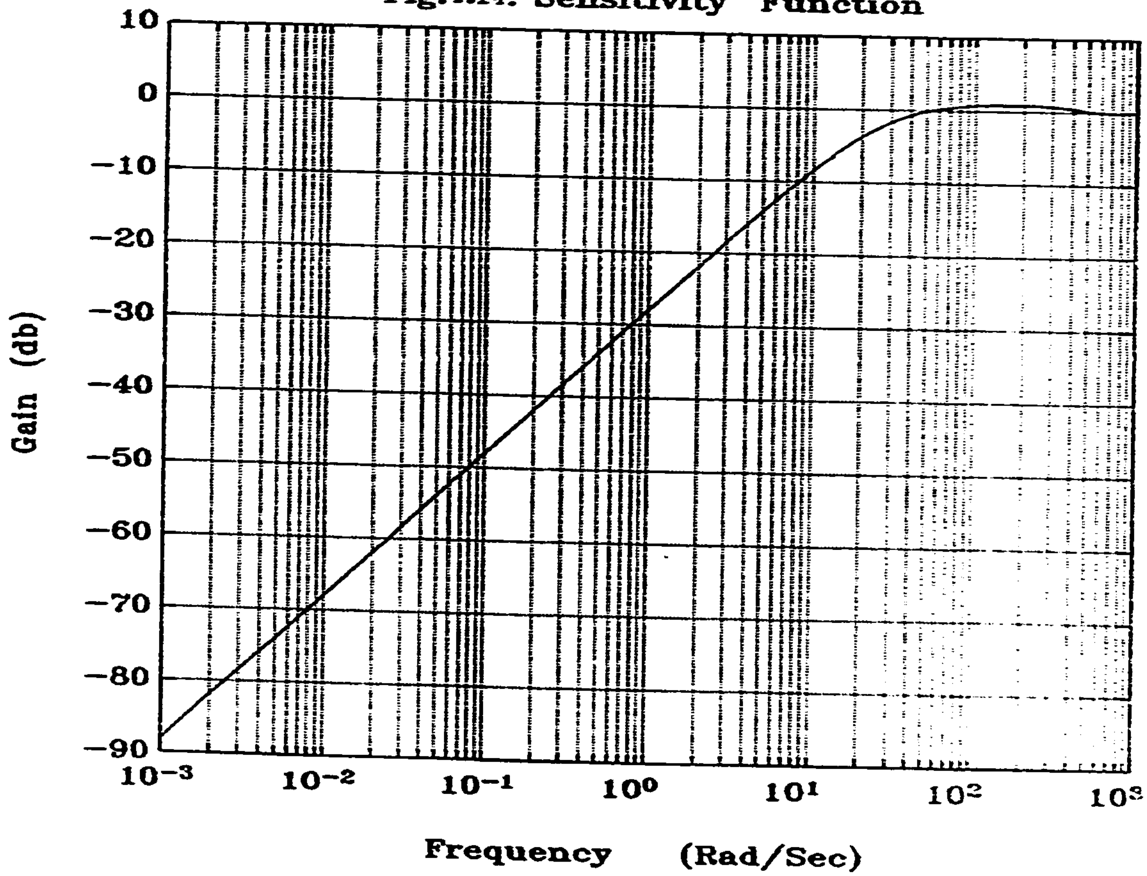
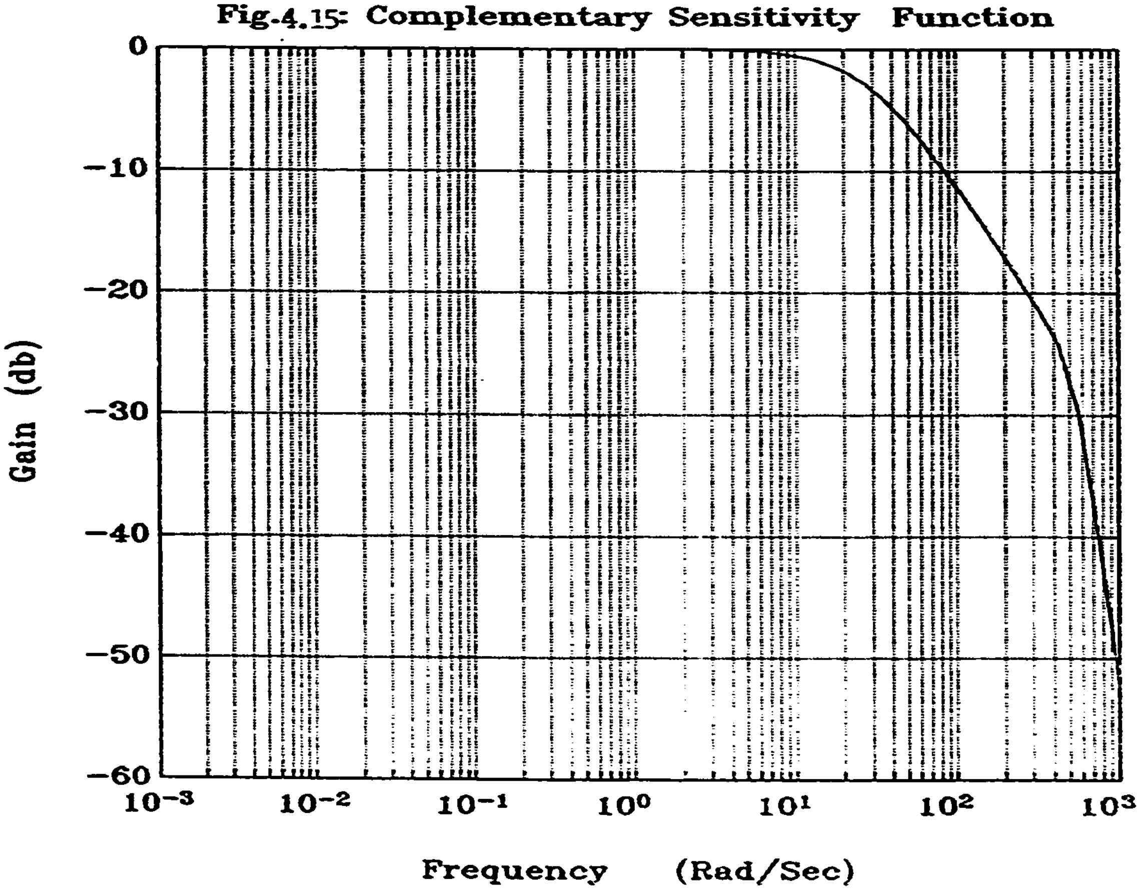


Fig.4.15: Complementary Sensitivity Function



desired value. The graphs also shows the minimization of the sensitivity function.

### 4.3.3. $H_\infty$ Design

The $H_\infty$ design procedure starts by selecting some weighting functions which reflect the desired performance and robustness specifications. The design specifications of section (4.3.1) can be achieved by selecting the following weights on the sensitivity and complementary sensitivity respectively

$$W_1(s) = \gamma \frac{5}{s + .001}$$

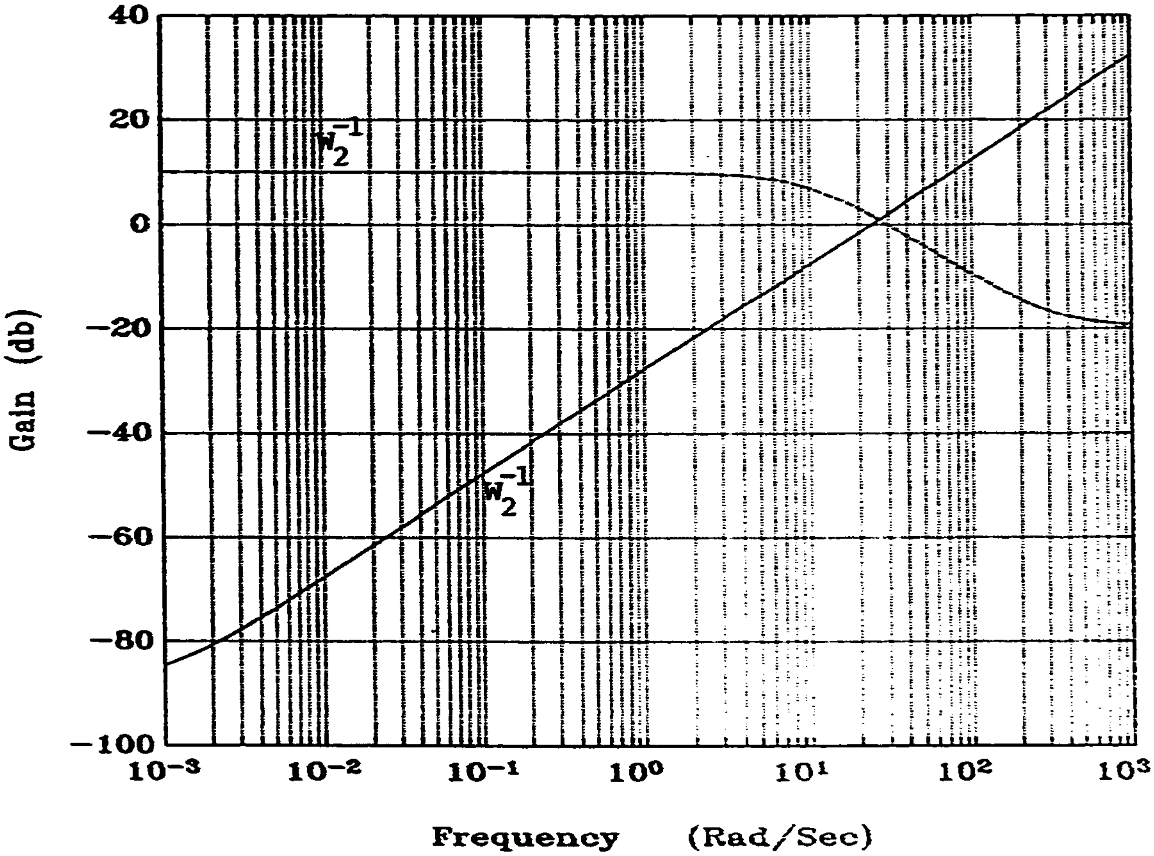
where $\gamma$ is a constant which can be adjusted until the optimal solution is achieved.

$$W_2(s) = \frac{9.5(s+10)}{(s+300)}$$

The plots of the singular value of $W_1(s)(for \quad \gamma = 2.0)$, and $W_2(s)$ are shown in Fig. 16.

The plant should be augmented with weights as in section (2.5). The state-space realization of the augmented system is given in table 4.2. Using the computer program ROBUST-CONTROL TOOLBOX [25] for several increasing values of $\gamma$ until the computer program responds that no solution exists for any larger $\gamma$, the desired robustness and performance specifications were achieved by using $\gamma = 2.0$. The results are shown in Figs. 17, 18 and 19. Fig. 17 clearly indicates that the frequency resposnse

Fig.4.16: Weighting Functions



$$A = \begin{bmatrix} 3.0000e+001 & -7.0000e+002 & 0 & 3.1853e+001 & -9.9949e+002 \\ 1.0000e+000 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.0000e-003 & 2.8668e+002 & -8.9954e+003 \\ 0 & -3.1853e-002 & 0 & -1.3166e+001 & -9.0004e+003 \\ 0 & 9.9949e-001 & 0 & -4.1958e-001 & -2.8683e+002 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 2.4000e+002 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & .0 & 0 & 3.0166e+001 & 8.5487e+004 \end{bmatrix}$$

$$C_2 = [0 \quad 0 \quad 0 \quad 2.8668e+002 \quad -8.9954e+003]$$

$$\left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 1.0000e-002 \\ 0 & 0 \\ \hline 1.0000e+000 & 0 \end{array}\right]$$

Table 4.2 : State Space Matrices of the Augmented System

Fig.4.17: Closed Loop Function

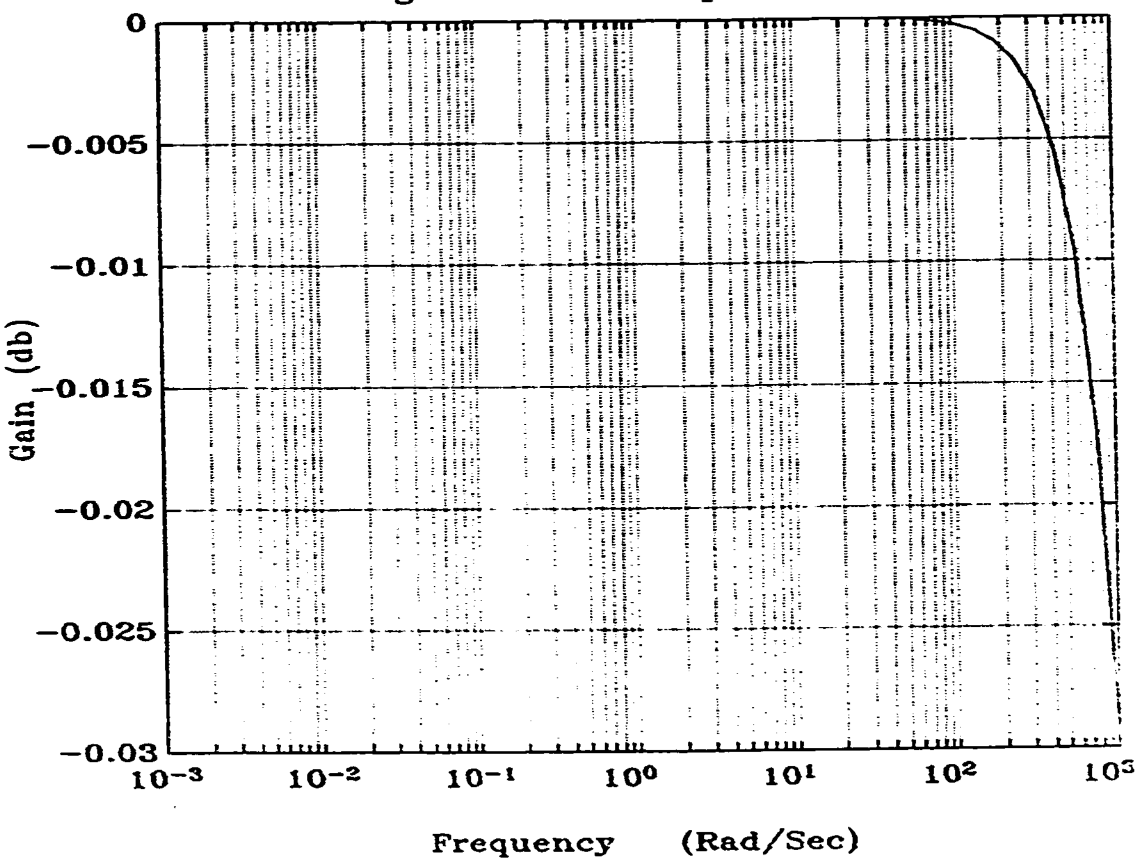


Fig.4.18: Sensitivity Function

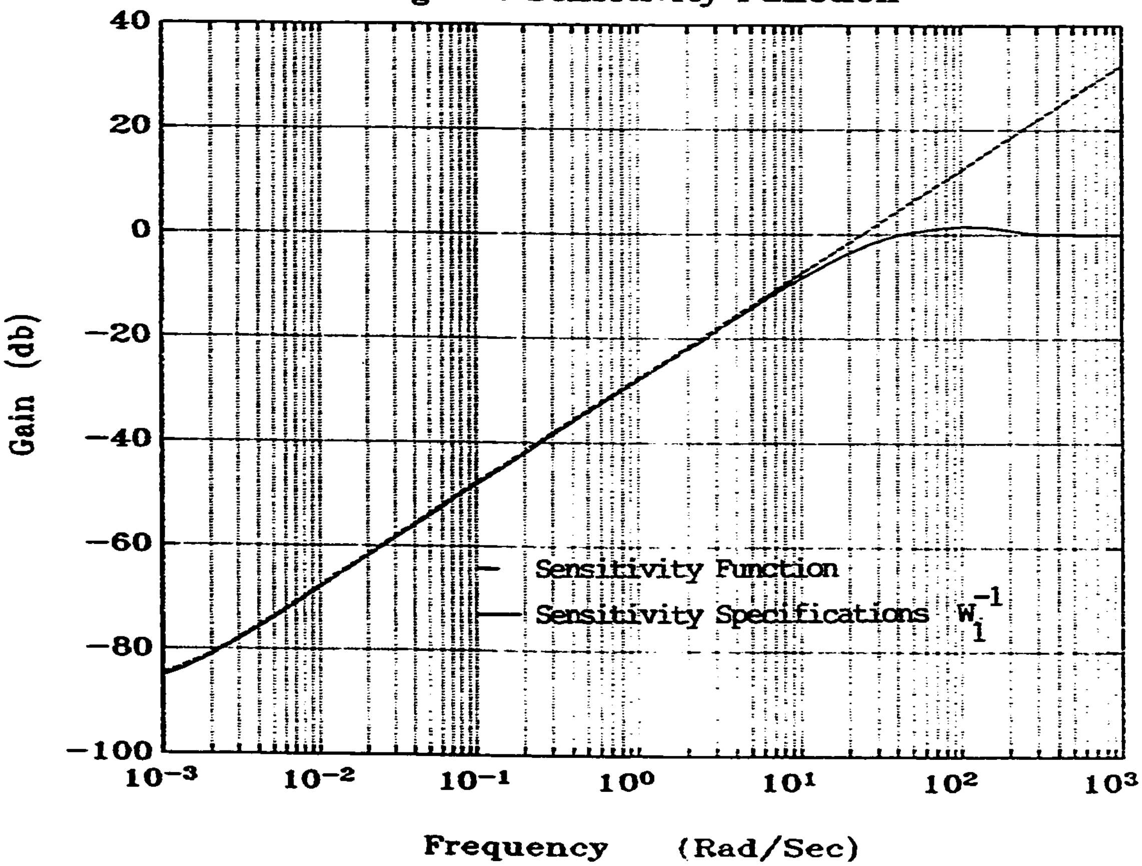
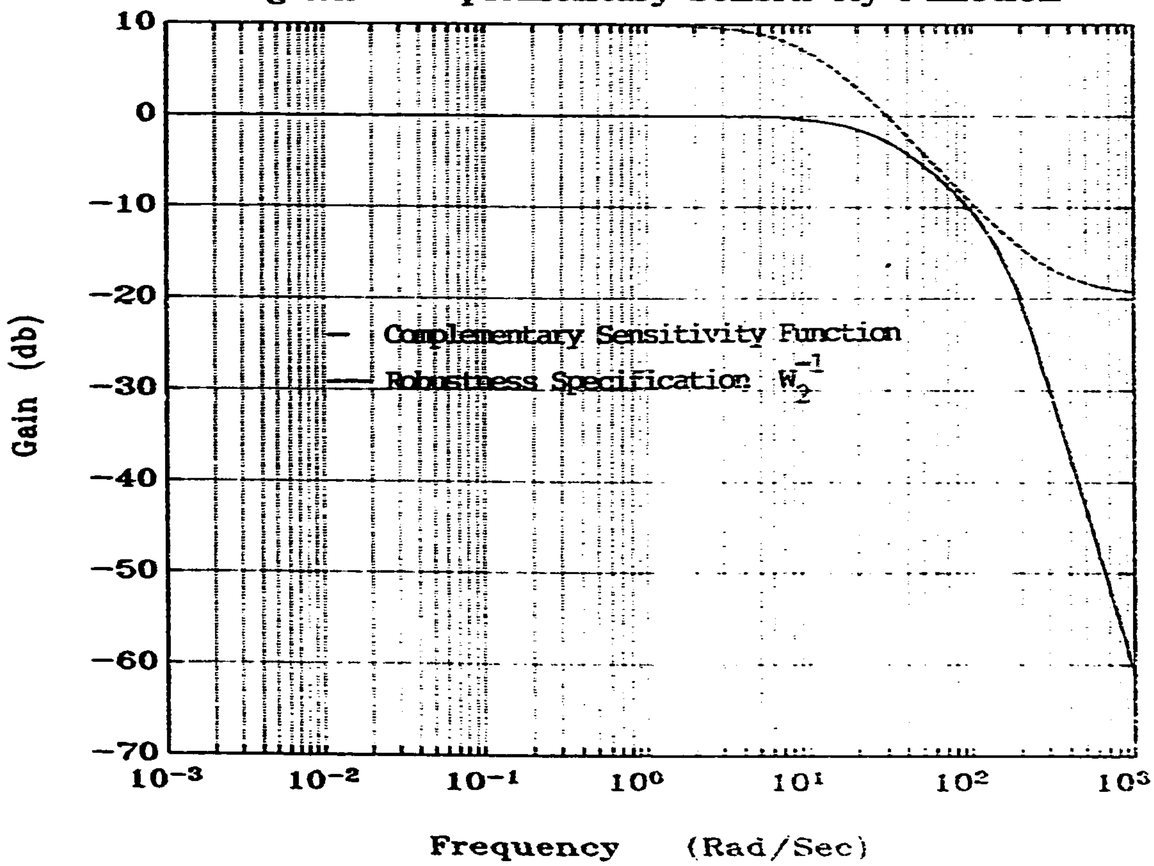


Fig.4.19 Complementary Sensitivity Function



of the closed-loop transfer function is almost one for the frequencies of interest. Figs. 18 and 19 show that the sensitivity and the complementary sensitivity functions are moved close to their respective upper limits $W_1^{-1}$ and $W_2^{-1}$ respectively. The final $H_\infty$ control law has five states, is stable and has the realization

$$K(s) = C_K(sI - A_K)^{-1} B_K$$

The matrices $A_K, B_K$ and $C_K$ are given in table 4.3.

### 4.3.4 $H_\infty$/CLTR Compensator Design

Here, an $H_\infty$/CLTR compensator will be considered. The same weightings $W_1(s)$ and $W_2(s)$ will be used here again. This design is accomplished in two steps. The first step is to find the state-feedback gain where all the states of the augmented system (table 4.2) are assumed to be available for feedback. The resulting state-feedback gain is given by

$$K_c = 1 \times 10^8 [.0001 \quad 0.350 \quad -.0134 \quad -.1520 \quad 4.5005]$$

Since the plant is stable the observer gain $K_f$ can be selected to equal to $B_1$ (section 2.6). Thus, $K_f$ is given by

$$K_f = [0 \quad 0 \quad 1 \quad 0 \quad 0]^T$$

This will give the following compensator

$$K(s) = -K_c(sI - A + B_2 K_c + K_f C_2)^{-1} K_f$$

$$A_k = \begin{bmatrix} -1.2394e+004 & -3.1781e+006 & 1.2135e+006 & 1.3784e+007 & -4.0823e+008 \\ 1.0000e+000 & -5.5737e-032 & -2.6667e-032 & 1.2727e-016 & 5.2578e-015 \\ 0 & 0 & -1.0000e-003 & 0 & 0 \\ 2.3323e-031 & -3.1853e-002 & -2.0171e-029 & -1.3166e+001 & -9.0004e+003 \\ 1.1586e-033 & 9.9949e-001 & 6.5482e-032 & -4.1958e-001 & -2.8683e+002 \end{bmatrix}$$

$$B_k = \begin{bmatrix} -6.6646e-019 \\ -4.3256e-019 \\ 1.0000e+000 \\ -2.1495e-017 \\ 2.1037e-020 \end{bmatrix}$$

$$C_k = [-1.2364e+004 \quad -3.1774e+006 \quad 1.2135e+006 \quad 1.3784e+007 \quad -4.0823e+008]$$

Table 4.3 : State Space Matrices for *K(s)*

Fig. 20 shows the closed-loop transfer function for the state-feedback system and the recovered closed-loop transfer function, which appeared to be identical indicating perfect recovery. The sensitivity and complementary sensitivity functions are shown in Figs. 21 and 22 where they have been pushed flat against their respective limits $W_1^{-1}$ and $W_2^{-1}$.

### 4.3.5 Comparison Between the Designs

The capability of each design method to meet both stability margin and disturbance attenuation requirements can be used as a basis for comparison between the different design methods. In this example the $H_\infty$ compensator and $H_\infty$/CLTR compensator have satisfied the design specifications. The bandwidth requirement was not satisfied using LQG/LTR design. The best bandwidth was achieved using LQG/LTR was 10 rad/sec, in contrast to the both $H_\infty$ designs which produced a bandwidth of 30 rad/sec. The order of the compensator was the same in all designs.

## 4.4 Hydraulic Actuator - Discrete $H_\infty$/CLTR Controller

The design method for discrete $H_\infty$/CLTR controller described in section (3.6) will be used here to design an $H_\infty$/CLTR discrete controller for the hydraulic actuator given in section (4.3). The *ZOH* of the plant transfer function $G_o(s)$ is

$$G_o(z) = \frac{-.075z^2 + .553z + .295}{z^3 - 2.68z^2 + 2.42z - .741}$$

Now, $G_o(z)$ is transformed to the $W$-plane using the bilinear transformation. The design will be done as if the system is in the $s$-plane, and the same weighting functions $W_1(s)$ and $W_2(s)$ of section (4.3) will be used here again. The state-space

Fig. 4.20:Designed and recovered Closed-Loop Transfer Functions

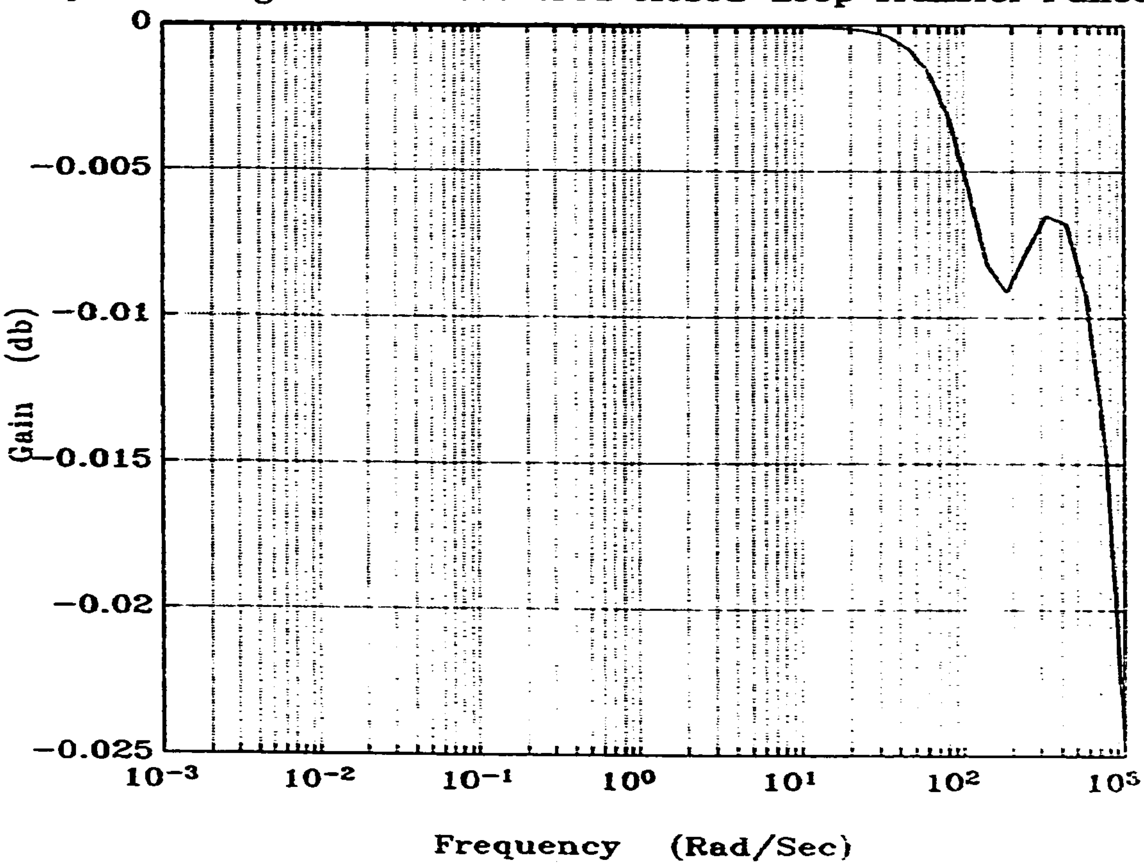


Fig.4.21: Sensitivity Function

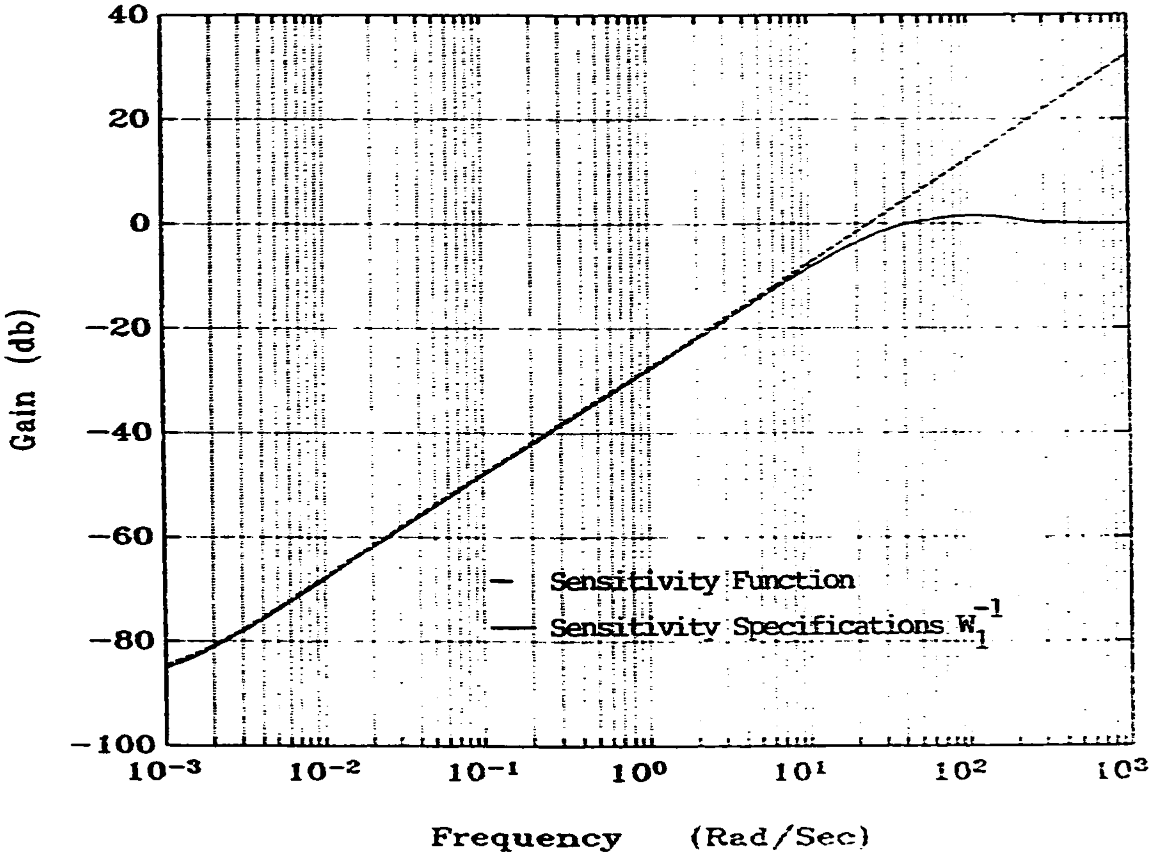
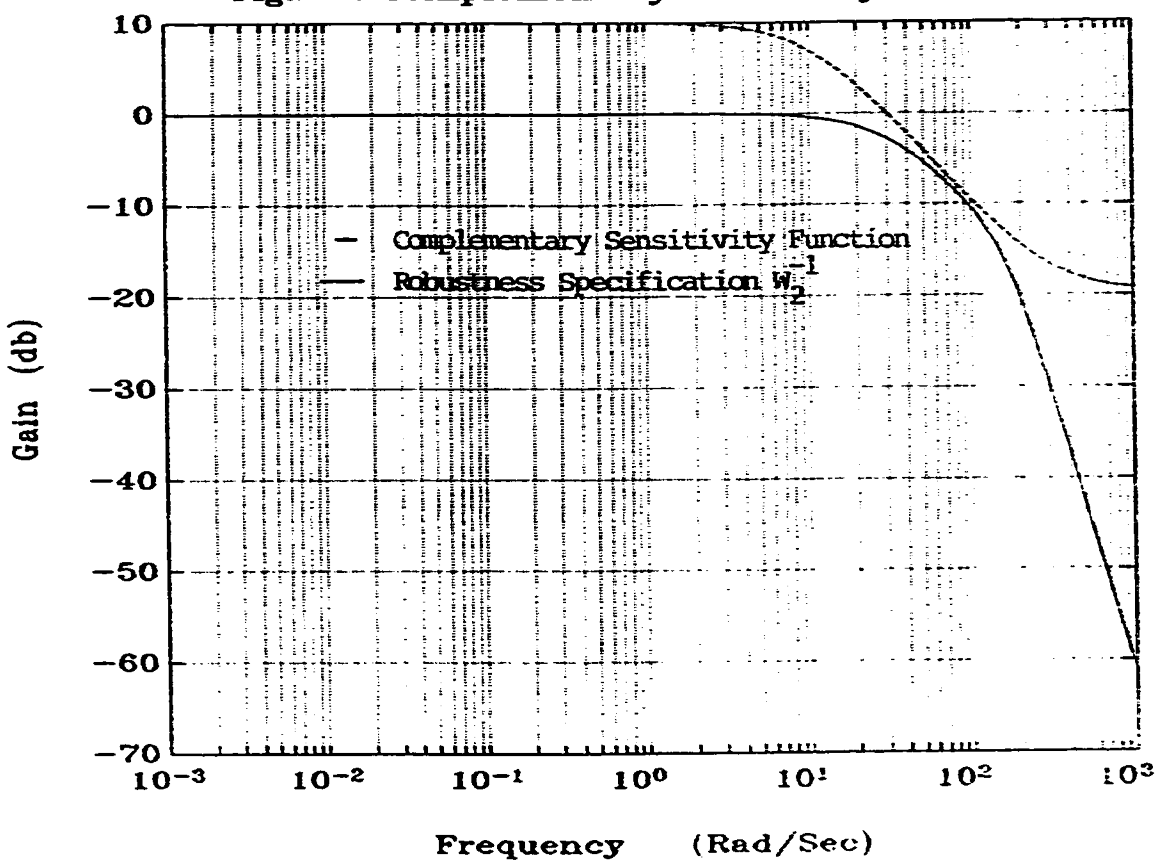


Fig.4.22: Complementary Sensitivity Function



realization of the augmented system is given in table 4.4.The state feedback gain which satisfy the requirement that the magnitude of the state feedback closed-loop transfer function is equal to one for the frequencies of interest is given by

$$K_c = 1\times 10^4[2.96 \quad -3.10 \quad 1.119 \quad .099 \quad -.095 \quad .002]$$

Since the system is stable, $K_f = B_1$

$$K_f = [-2.47 \quad -.67 \quad 2.12 \quad .127 \quad -.095 \quad -.002]^T$$

The controller $K(s)$ is given by

$$K(s) = -K_c(sI - A + B_2K_c + K_fC_2)^{-1}K_f$$

The results of the design are shown in Figs. 23, 24, and 25. Fig. 23 shows the magnitude of the state feedback closed-loop transfer function and the recovered closed-loop transfer function which are shown to be identical indicating perfect recovery. The sensitivity and complementary sensitivity functions are shown in Figs. 24 and 25 where they have been pushed flat against these limits $W^{-1}$ and $W_2^{-1}$ respectively.

The last step of the design is to use bilinear transformation to transform the compensator $K(s)$ into the z-plane. The state space realization of the compensator $K(z)$ is given in table 4.5.

$$A=\begin{bmatrix} -1.8338e-001 & 5.6447e-001 & -1.9953e-001 & 1.1236e-002 & 1.9220e-003 & -1.0285e-004 \\ -5.6277e-001 & -9.1745e-001 & 9.7945e-001 & -5.8763e-002 & -7.8623e-003 & -6.1710e-005 \\ -1.8690e-001 & -9.5530e-001 & -2.4363e+000 & 2.3515e-001 & 1.8500e-002 & -6.1665e-003 \\ -7.4214e-002 & 3.2741e-001 & 1.7236e+000 & -1.4479e+001 & 1.7100e+001 & -9.2634e-001 \\ 4.8057e-002 & 2.1360e-001 & 1.1491e+000 & -2.6411e+001 & 1.4104e+001 & -6.0311e+000 \\ -1.1705e-002 & -5.1787e-002 & -2.7570e-001 & 4.5827e+000 & 6.8348e+000 & -3.0015e+002 \end{bmatrix}$$

$$B_1=\begin{bmatrix} -2.4730e+000 \\ -6.6997e-001 \\ 2.1180e+000 \\ 1.2738e-001 \\ -9.5108e-002 \\ -1.9654e-003 \end{bmatrix} \qquad B_2=\begin{bmatrix} 1.7890e+001 \\ 2.0588e+001 \\ 9.8147e+000 \\ -3.6721e+000 \\ -2.4022e+000 \\ 5.8162e-001 \end{bmatrix}$$

$$C_1^T=\begin{bmatrix} -1.8059e+001 & 6.0018e-002 \\ 2.0596e+001 & 1.1877e-001 \\ -1.0039e+001 & 1.6576e-001 \\ 5.9508e-001 & 6.3333e-001 \\ 9.0546e-002 & 2.1304e+000 \\ -2.9396e-003 & 5.8696e-001 \end{bmatrix} \qquad C_2^T=\begin{bmatrix} -1.9083e-001 \\ -3.0940e-001 \\ -5.6135e-002 \\ -3.5700e+000 \\ 1.1104e+000 \\ -1.6592e-002 \end{bmatrix}$$

$$\left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} 1.6667e-001 & -6.2372e-005 \\ 0 & 3.5528e-003 \\ \hline 1.0000e+000 & -3.7423e-004 \end{array}\right]$$

Table 4.4 : State Space Matrices of the Augmented System

Fig. 4.23:Designed and Recovered Closed−Loop Transfer Function

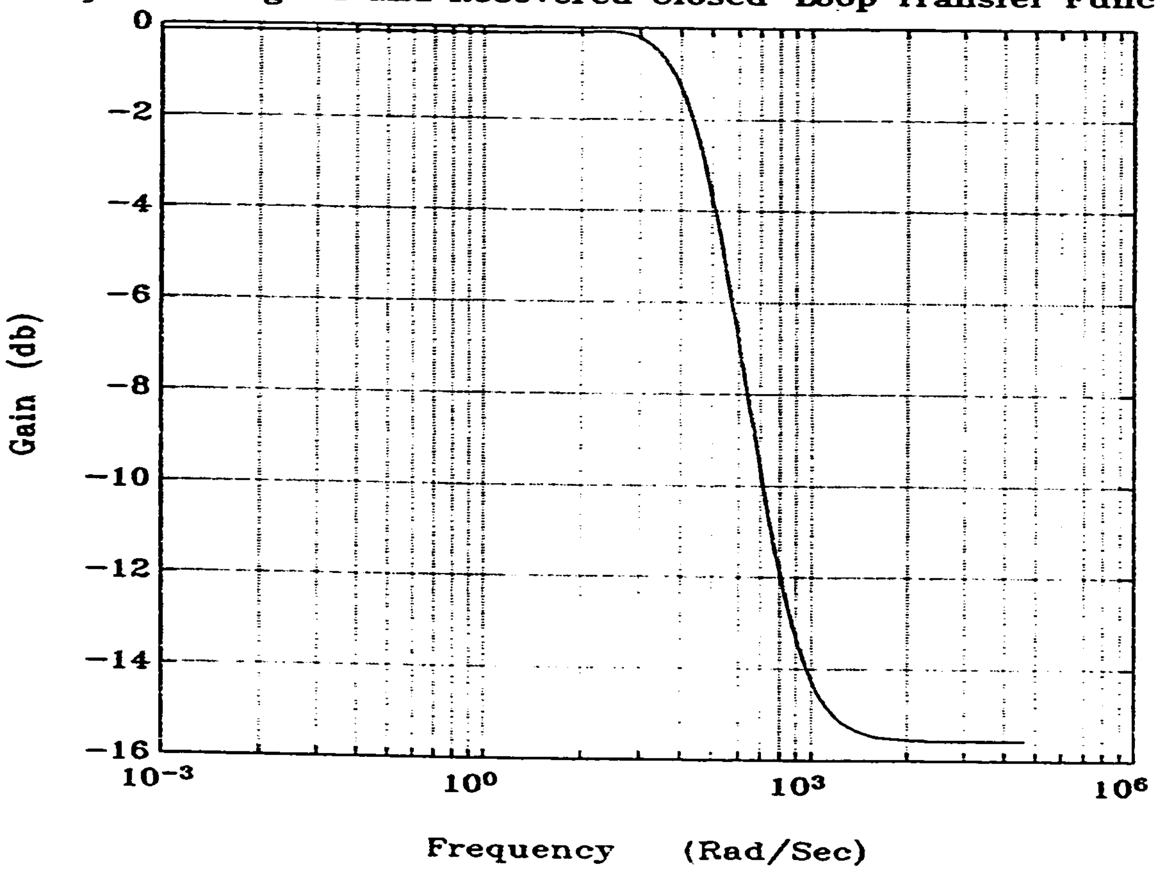


Fig.4.24: Sensitivity Function

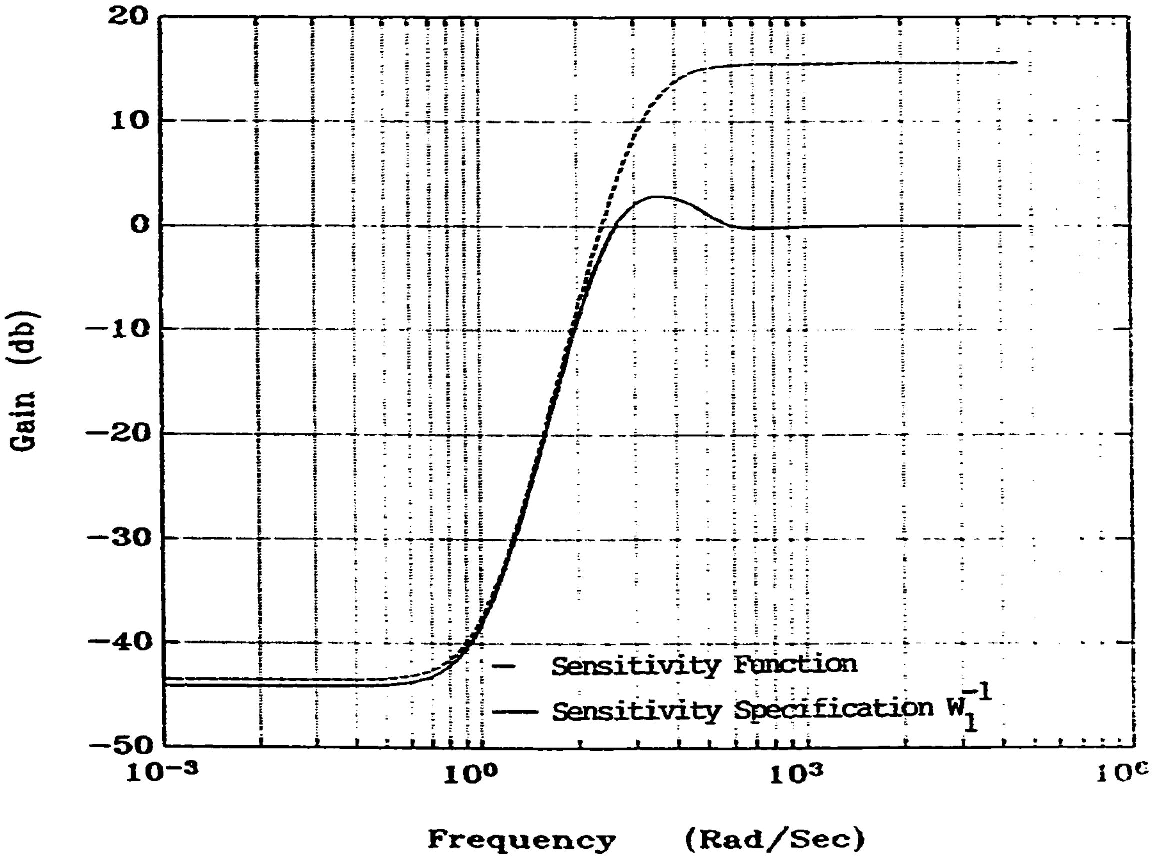
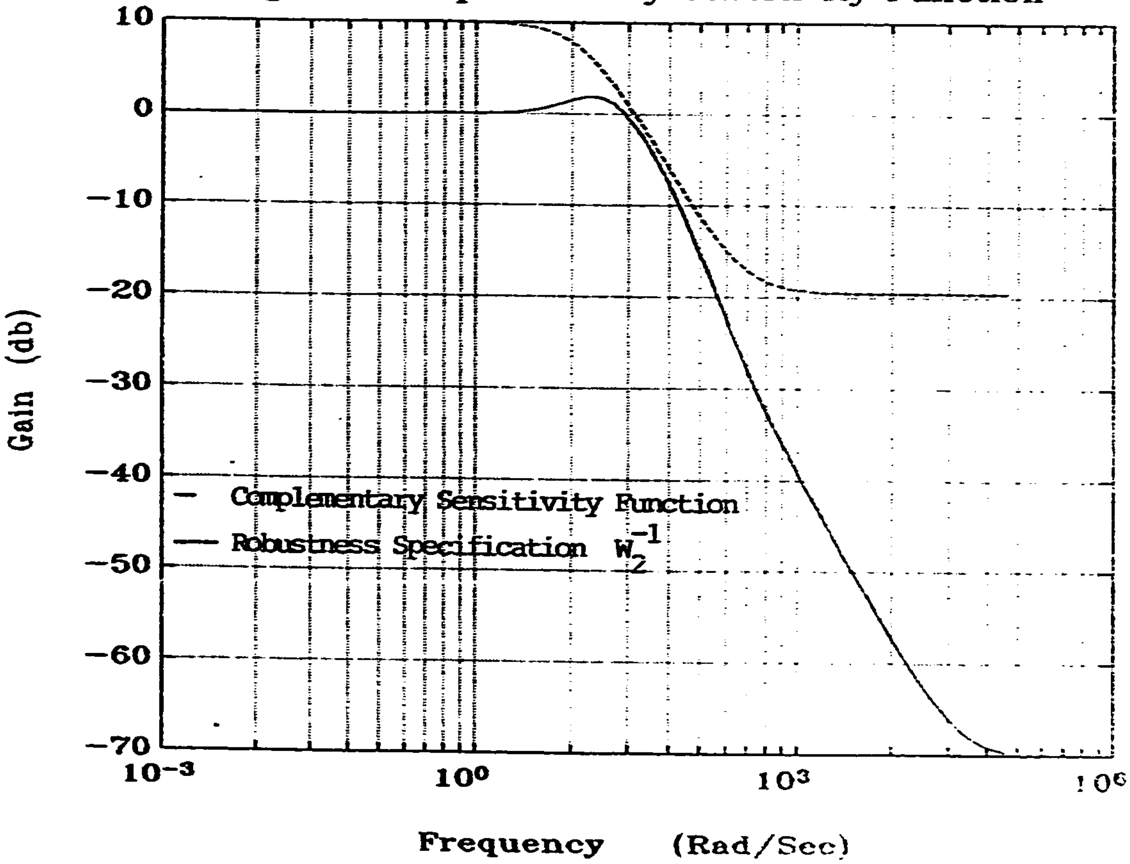


Fig.4.25: Complementary Sensitivity Function



$$A = \begin{bmatrix} -6.747e+002 & 7.0323e+002 & -2.4783e+002 & -1.2394e+001 & 1.5802e+001 & -2.8887e-001 \\ -7.7262e+002 & 8.0505e+002 & -2.8335e+002 & 1.4097e+001 & 1.8051e+001 & -3.3008e-001 \\ -3.6117e+002 & 3.7587e+002 & -1.3148e+002 & -6.5053e+000 & 8.4204e+000 & -1.5411e-001 \\ 1.3007e+002 & -1.3536e+002 & 4.7720e+001 & 3.2205e+000 & -2.8917e+000 & 5.0374e-002 \\ 6.5880e+001 & -6.8560e+001 & 2.4170e+001 & 9.6868e-001 & -6.8875e-001 & 6.2540e-003 \\ -6.3821e+000 & 6.6416e+000 & -2.3415e+000 & -1.0252e-001 & 1.7571e-001 & -2.0332e-001 \end{bmatrix}$$

$$B = \begin{bmatrix} 6.6702e+002 \\ 7.6702e+002 \\ 3.6336e+002 \\ -1.2909e+002 \\ -6.5702e+002 \\ 6.3427e+000 \end{bmatrix} \qquad C^T = \begin{bmatrix} -1.8852e+001 \\ 1.9619e+001 \\ -6.9141e+000 \\ -3.4342e-001 \\ 4.4034e-001 \\ -8.0526e-003 \end{bmatrix}$$

$$D = [1.8747e+001]$$

Table 4.5 : State Space Matrices for *K(z)*

## 4.5 Large Space Structure

In this example, we want to design a controller using LQG/LTR, $H_\infty$ and $H_\infty$/CLTR design methods for the LSS described in [21]. A 4-state reduced order model from 116-state original model will be used in the design. The state-space realization $(A_g, B_g, C_g)$ of the reduced model is given in [25].

where

$$A_g = \begin{bmatrix} -.99 & .005 & .4899 & 1.9219 \\ .009 & .9876 & 1.9010 & -.4918 \\ -.4961 & -1.9005 & .511.7030 & 4.9716 \\ -1.9215 & .4907 & -7.7879 & -398.3118 \end{bmatrix}$$

$$B_g = \begin{bmatrix} .7827 & -.6140 \\ .6130 & .7826 \\ .7835 & .5960 \\ .6069 & .7878 \end{bmatrix}$$

$$C_g = \begin{bmatrix} .7829 & .6128 & -.7816 & -6061 \\ -.6144 & .7820 & -.5984 & .7884 \end{bmatrix}$$

### 4.5.1 Design Specifications

The design specifications are [25]

1) Robustness Specifications:

-20 db/decade roll-off above 2000 rad/sec.

2) Performance Specifications:

Minimization of the sensitivity function.

Those specifications lead to the following weighting functions for performance and robustness respectively [25]

$$W_1(s) = \gamma \begin{bmatrix} \frac{\left(1+\frac{s}{5000}\right)^2}{.01\left(1+\frac{s}{100}\right)^2} & 0 \\ 0 & \frac{\left(1+\frac{s}{5000}\right)^2}{.01\left(1+\frac{s}{100}\right)^2} \end{bmatrix}$$

where $\gamma$ goes from one to 1.5.

$$W_2(s) = \begin{bmatrix} \frac{s}{2000} & 0 \\ 0 & \frac{s}{2000} \end{bmatrix}$$

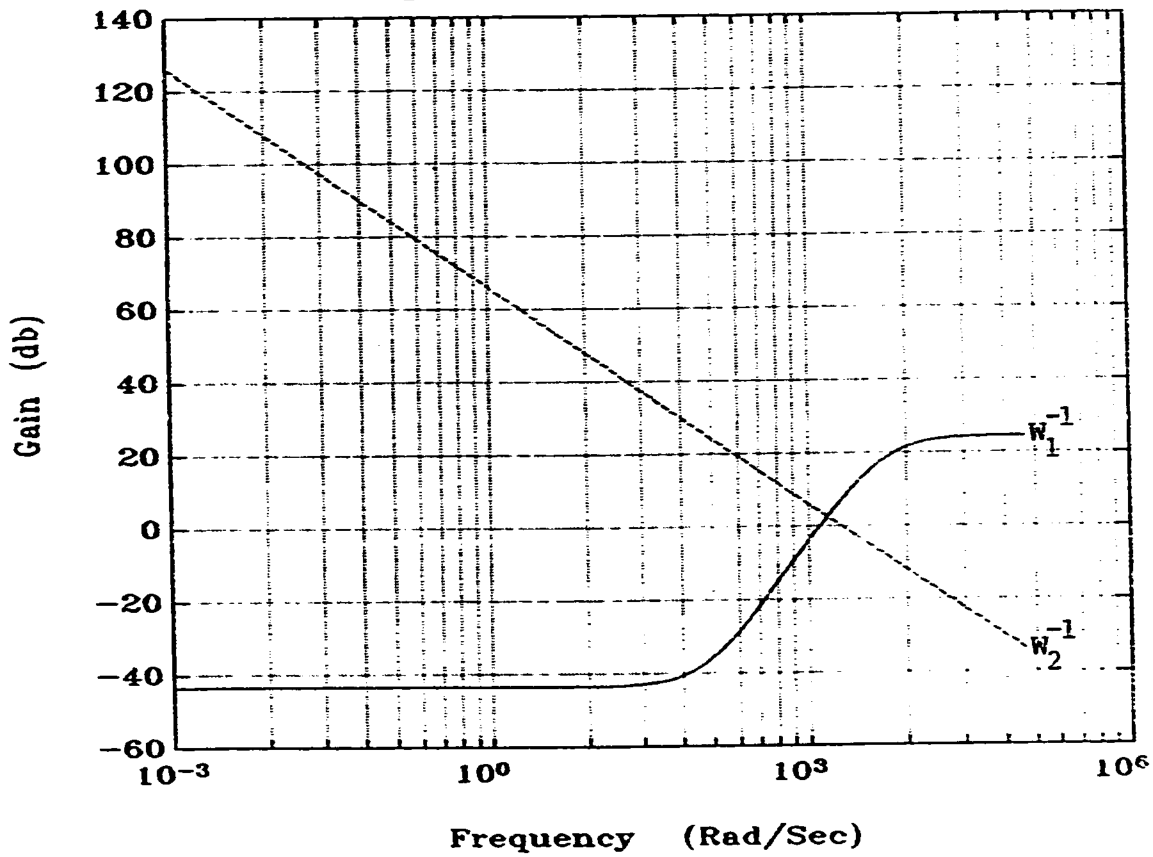
A plot of the singular values of $W_1(s)$ (for $\gamma = 1.5$) and $W_2(s)$ is shown in Fig. 26.

### 4.5.2 LQG/LTR Design

In his section, an LQG/LTR compensator is to be designed. The closed-loop feedback system should have zero steady-state error to arbitrary constant (step) commands and/or disturbances. This specification implies that we must have an integrator in each input channel. Also, we would like to have all loop singular values to be identical at both low and high frequencies. This requirement makes the system has about the same speed of response in all directions. This system will be augmented with two integrator, and is given by

$$A_a = \begin{bmatrix} A_g & B_g \\ 0_{2\times4} & 0_{2\times2} \end{bmatrix}, B_a = \begin{bmatrix} 0_{4\times2} \\ I_{2\times2} \end{bmatrix}, C_a = [C_g \quad 0_{2\times2}]$$

Fig.4.26 : Weighting Functions



To balance the loop singular values, we will choose the weighting matrices of the LQR problem of section 2.4 as [20]

$$H = \begin{bmatrix} H_l \\ H_h \end{bmatrix}$$

where

$$H_l = -[C_g A_g^{-1} B_g]^{-1} \quad \text{and} \quad H_h = C_g^T [C_g C_g^T]^{-1}$$

and $R = \rho I$, where $\rho$ is selected to adjust the crossover frequency consistent with the stability-robustness constraints.

This choice of the weighting matrices will make all singular values of the state-feedback loop transfer function $L(s) = K_c(sI - A_a)^{-1} B_a$ roll-off at -20 db/dec at both low and high frequencies. The state-feedback gain is given by

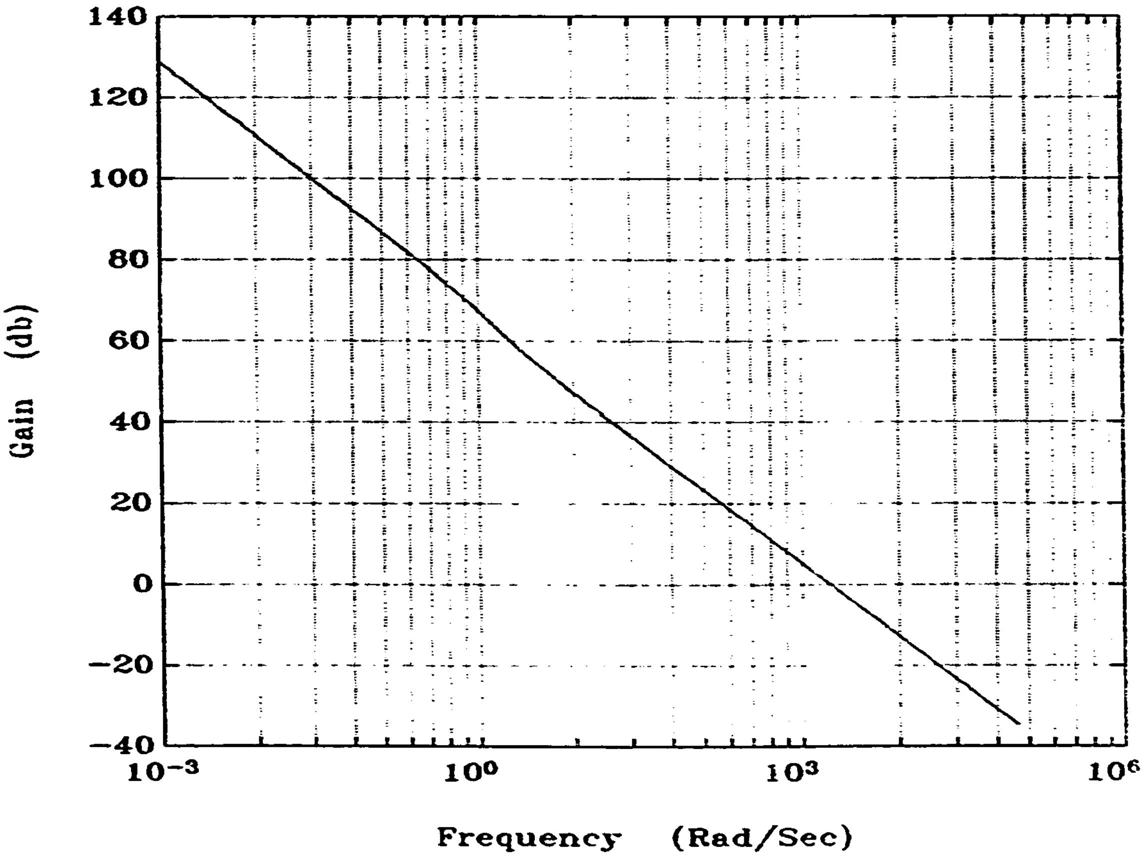
$$K_c = 1.0x10^3 \begin{bmatrix} 1.8311 & 0.0006 \\ 0.0006 & 1.8308 \\ 0.7236 & -0.5655 \\ 0.5734 & 0.7290 \\ -0.6206 & -0.4797 \\ -0.4594 & 0.5960 \end{bmatrix}^T$$

The singular values of *L(s)* are shown in Fig. 27 where it is shown that the two singular values are identical and satisfy the robustness bound $W_2(s)$. The next step is to design the compensator *K(s)* such that *L(s)* is recovered. Using the results of section 2.4

$$K(s) = K_c(sI - A_a + B_a K_c + K_f C_a)^{-1} K_f$$

where the filter gain $K_f$ is given by

Fig.4.27 : State Feedback Transfer Function



$$K_f = 1.0x10^{10} \begin{bmatrix} 1.001 & -0.0118 \\ 0.119 & 0.9999 \\ 0.0045 & -0.0035 \\ 0.0039 & 0.0047 \\ 0.0014 & 0.0010 \\ 0.0007 & -0.0010 \end{bmatrix}$$

The singular values of the loop transfer function $K(s)G_\alpha(s)$ $(G_\alpha(s) = C_\alpha(sI - A)^{-1} \; B_\alpha)$ are shown in Fig. 28 which clearly shows that it approximate *L(s)*.

The sensitivity and complementary sensitivity are shown in Figs. 29 and 30. There it shows the minimization of the sensitivity function and the robustness bound is satisfied by the complementary sensitivity function.

### 4.5.3 $H_\infty$ Design

Here, we consider the design of an $H_\infty$ compensator for the LSS using the ROBUST CONTROL TOOLBOX [25]. The first step in this design is to form the augmented system *p(s)* using the nominal model *G(s)* and the weightings $W_1(s)$ and $W_2(s)$ as in section 2.5. The state-space realization of *p(s)* is given in Appendix B. The design parameter $\gamma$ will be increased from one until the computer program responds that no solution exists for any larger $\gamma$. The optimal $\gamma$ was found to be 1.5. The design results are shown in Figs. 31, 32 and 33. Fig. 31 shows that the closed-loop singular values are almost one for all frequencies of interest. The sensitivity and complementary sensitivity are shown in Figs. 32 and 33 where it is shown that they have been pushed flat against these limits $W_1^{-1}$ and $W_2^{-1}$. The compensator has eight states , is stable and proper, and has the realization

$$K(s) = C_k(sI - A_k)^{-1} B_k$$

**Fig.28 : Designed and Recovered Loop Transfer Functions**

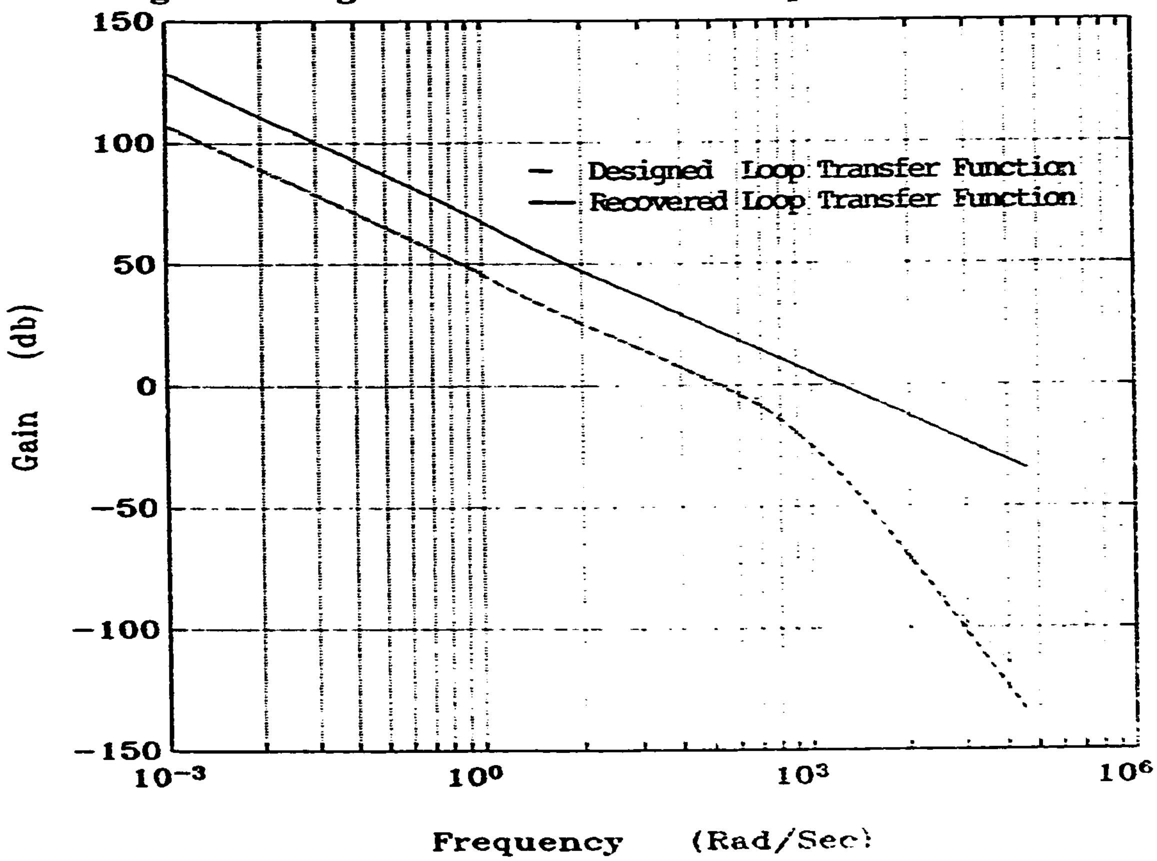


Fig.29 : Sensitivity Function

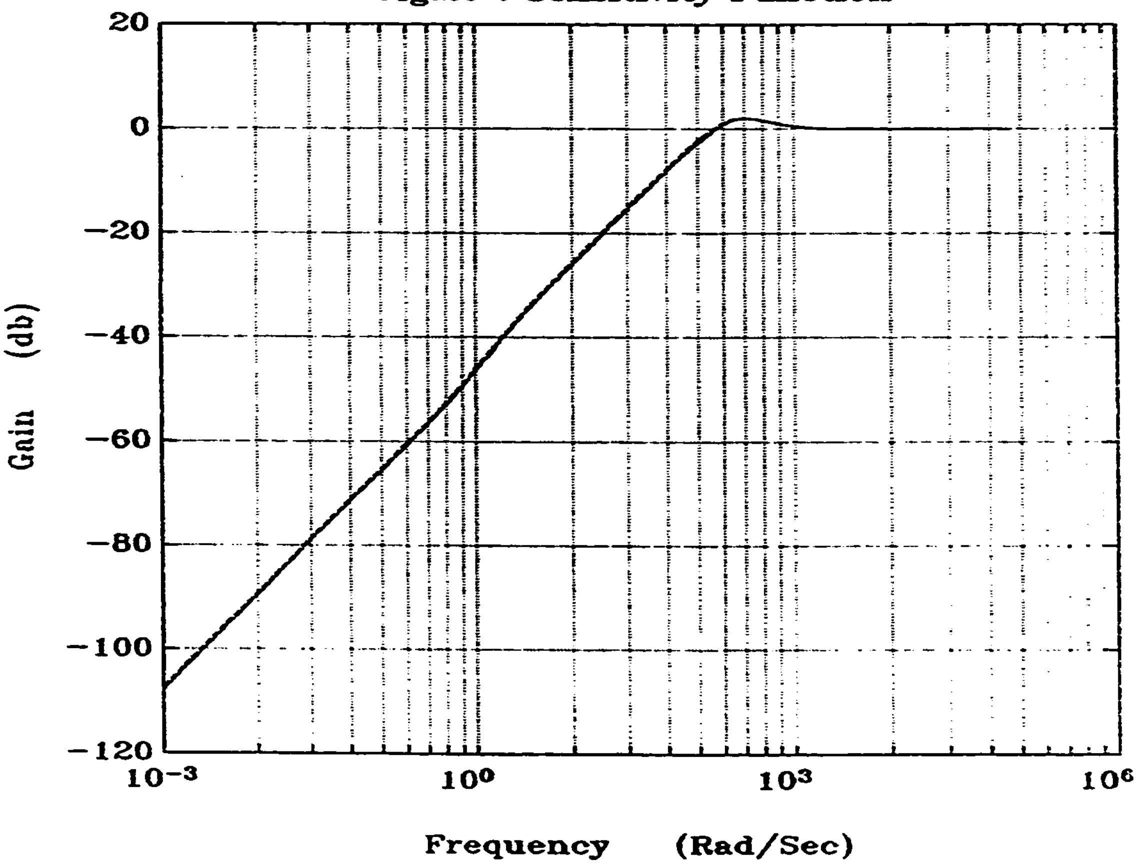


Fig.4.30 : Complementary Sensitivity Function

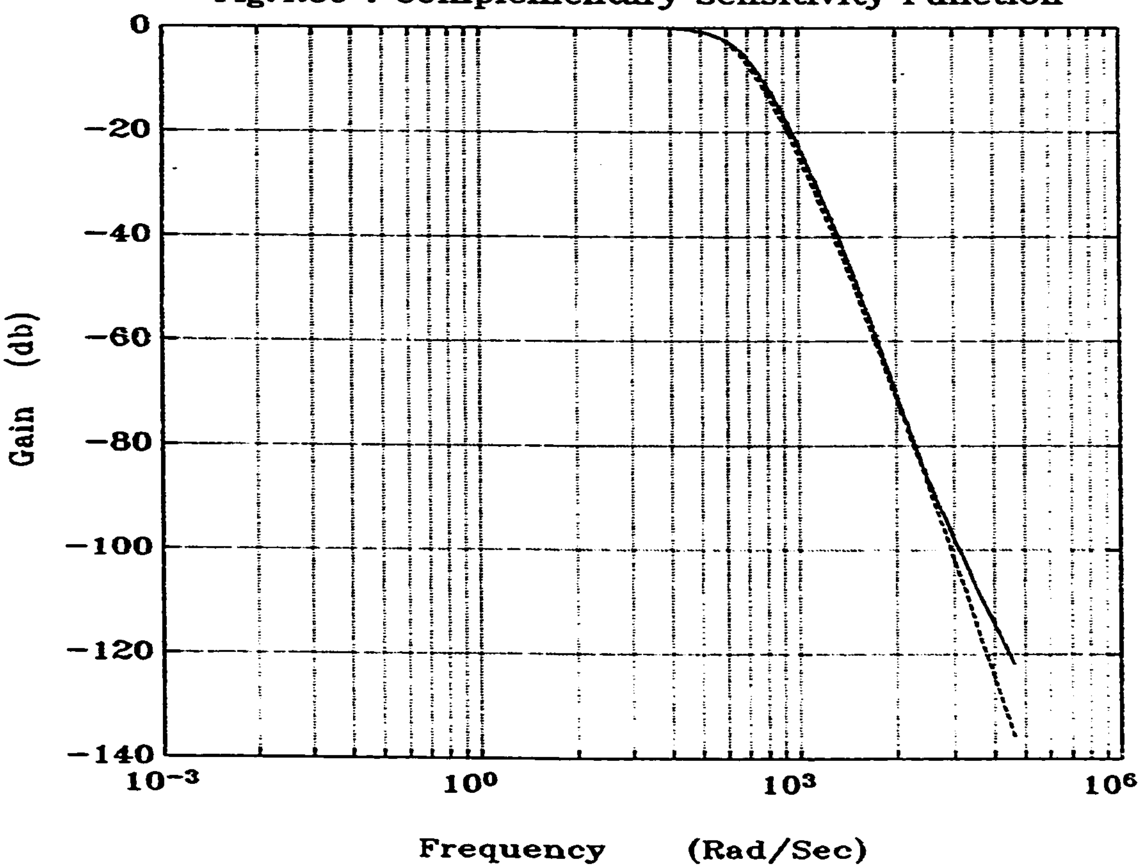


Fig.4.31 : Closed–Loop Transfer Function

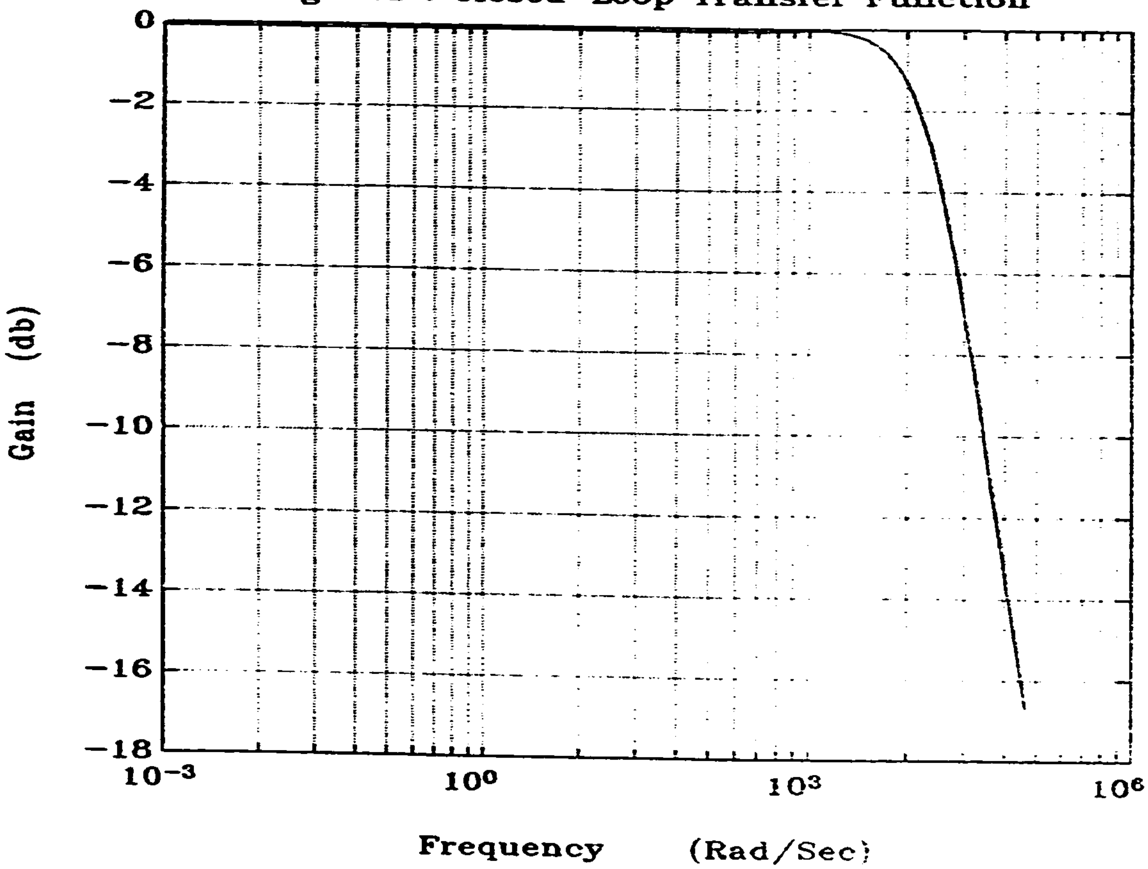
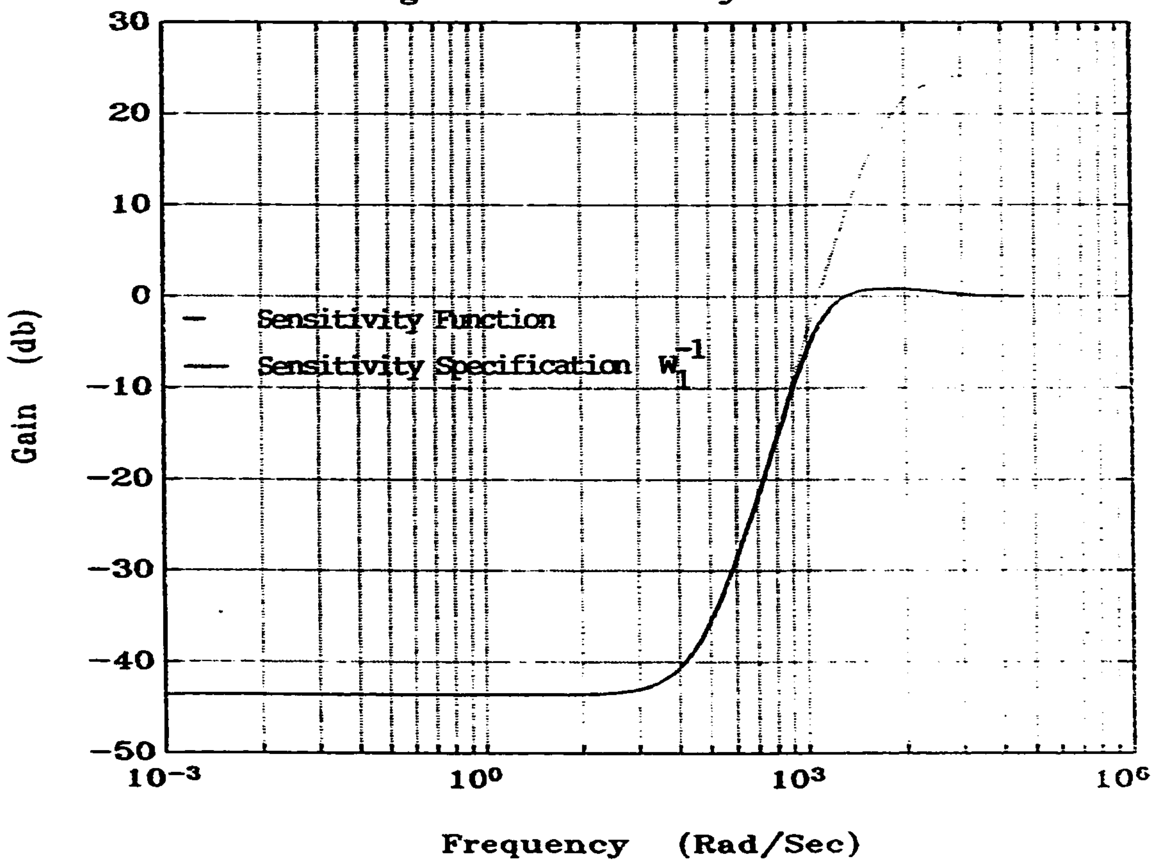


Fig.4.32 : Sensitivity Function



where $A_k, B_k, C_k$ are given in Appendix B.

### 4.5.4 $H_\infty$/CLTR Design

The design of $H_\infty$/CLTR compensator involves two basic steps. The first step is to find the state-feedback gain for the augmented system given in table 6. The closed-loop transfer function of the state-feedback system should equal to unity for the frequencies of interest. The optimal state-feedback gain was found to be

$$K_c = 1.0x10^9 \begin{bmatrix} 1.8666 & -1.3926 \\ -0.3510 & 0.1914 \\ 0.1602 & -0.0880 \\ 0.8344 & -0.6226 \\ 1.5027 & -1.1166 \\ -0.1605 & 0.0639 \\ -0.0047 & 0.0022 \\ -0.0250 & 0.0186 \end{bmatrix}$$

Since the plant is stable, the observer gain $K_f$ can be selected to be equal to $B_1$ (section 2.6). Thus, $K_f$ is given by

$$K_f = \begin{bmatrix} -40.5808 & 54.4904 \\ 54.4967 & 40.5855 \\ -22.6707 & -16.7456 \\ -16.7562 & 22.6844 \\ 46.1146 & -52.6214 \\ -52.6213 & -46.1145 \\ -0.0035 & -0.0030 \\ -0.0015 & 0.0016 \end{bmatrix}$$

This will give the following compensator

$$K(s) = -K_c(sI - A + K_f C_2 + B_2 K_c) K_f$$

Fig. 34 shows the singular values of the state-feedback closed-loop transfer function and the recovered singular values of the closed-loop transfer function which are identical indicating perfect recovery. The singular values of the sensitivity and complementary sensitivity functions are shown in Figs. 35 and 36 where they have been pushed flat against their respective limits $W_1^{-1}$ and $W_2^{-1}$

### 4.5.5. Comparison Between the Designs

The two compensators designed using $H_\infty$ method and $H_\infty$/CLTR method have met the stability robustness and performance specifications. The bandwidth requirement was satisfied using these two compensators which is 2000 rad/sec. In contrast, the best that was achieved using LQG/LTR compensator was a bandwidth of only 300 rad/sec.

Fig.4.33 : Complementary Sensitivity Function

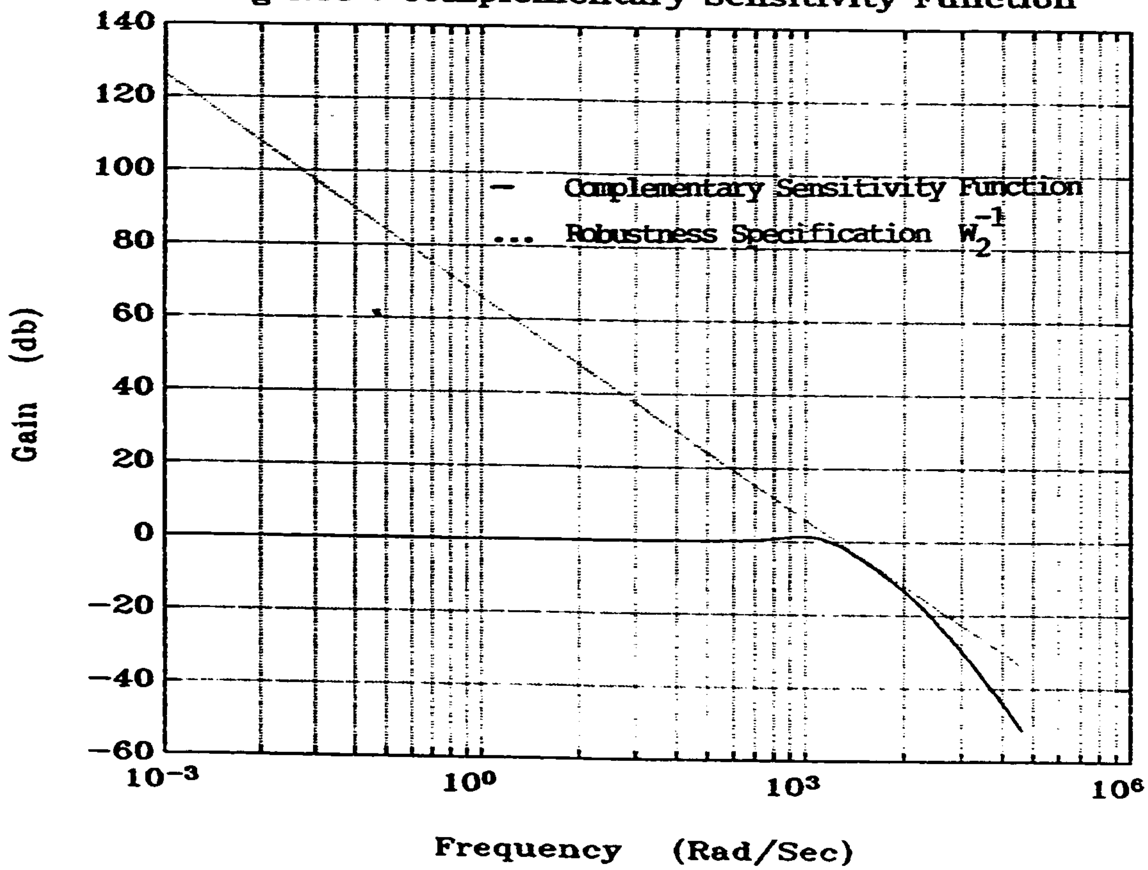


Fig.4.34:Designed and Recovered Closed-Loop Transfer Function

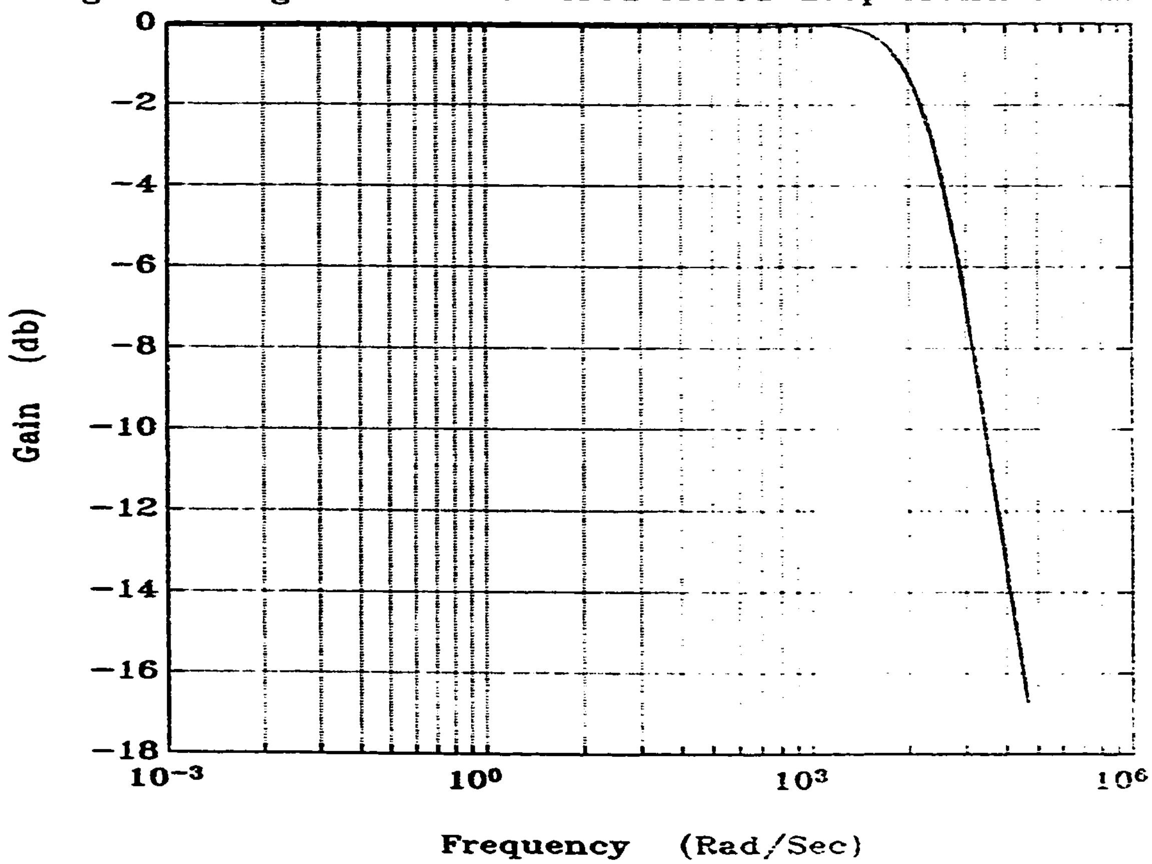


Fig.4.35 : Sensitivity Function

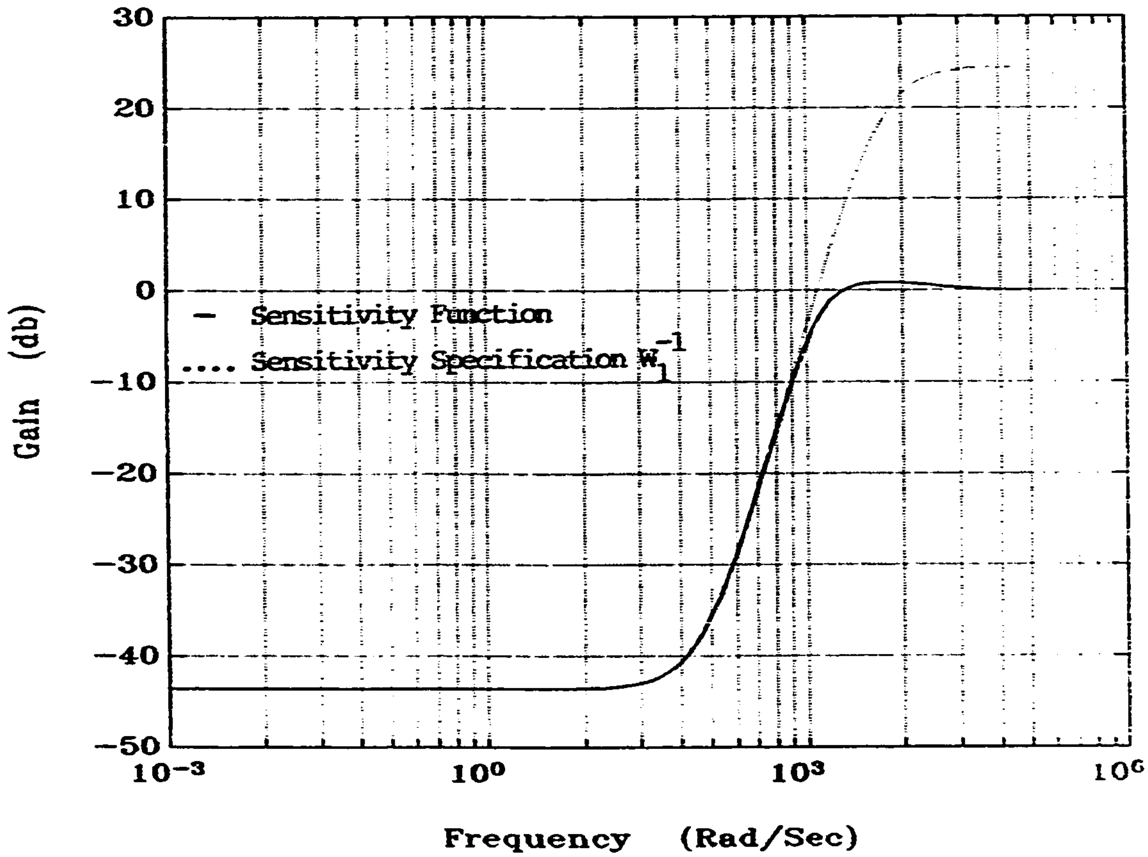
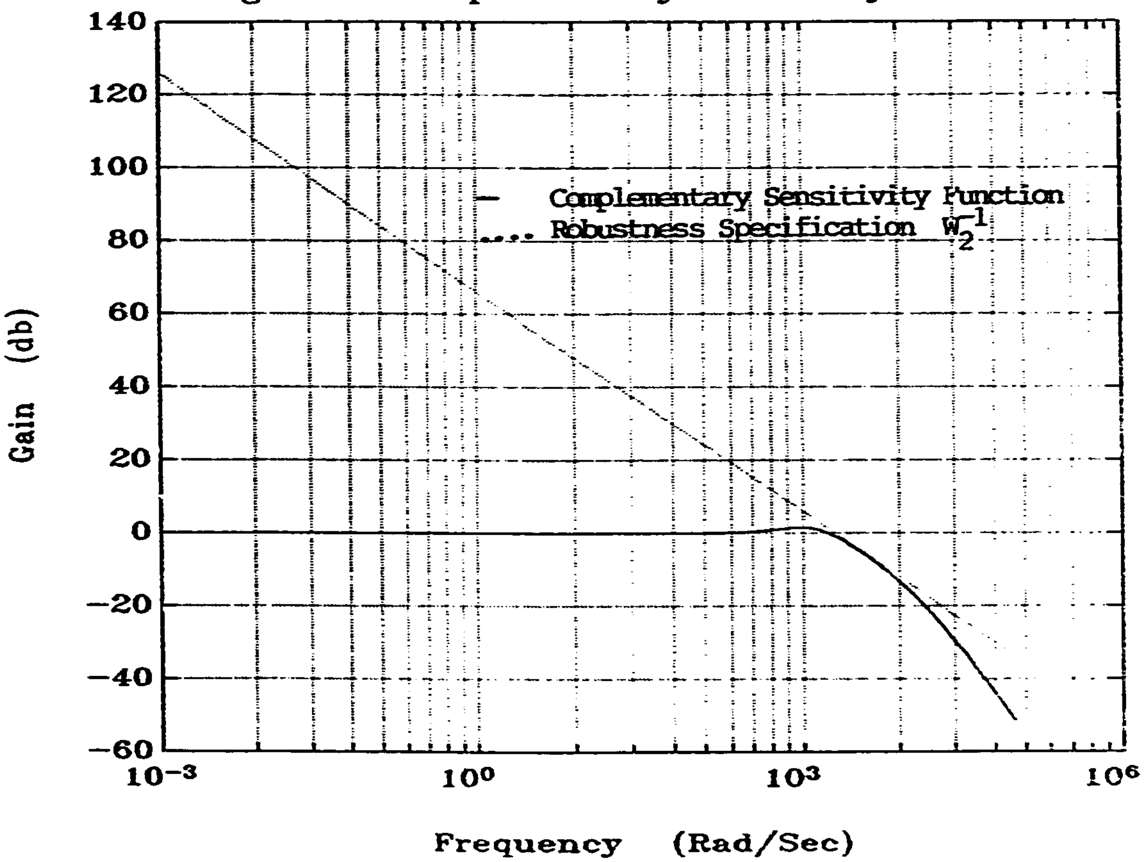


Fig.4.36 : Complementary Sensitivity Function



# CHAPTER 5

# CONCLUSION AND RECOMMENDATIONS

## 5.1 Conclusions

In this thesis, a new technique for control system design is developed. This design technique consists of two steps: 1) design an $H_\infty$ state feedback control, so that the closed-loop transfer function is minimized; 2) recover the achievable performance using state observer. The resulting controller will ensure internal stability and minimize the closed loop transfer function. For discrete time systems the bilinear transformation can be used to transform the discrete time problem into continuous time one, carry out the computations using continuous time techniques, and transform back the solution.

A recovery procedure using current estimator for discrete time systems with large compensator processing time is introduced.

A simplified LQG/LTR frequency domain design method for SISO minimum phase discrete system is presented.

Several design examples have been used to illustrate the theoretical developments of this thesis. These examples include continuous and discrete time control problems.

## 5.2 Recommendations

The following points can be considered for future work:

* Discrete-time solution for the $H_\infty$ state-feedback so that the $H_\infty$/CLTR can be implemented in the discrete-time and design can be done for discrete-time systems without using the bilinear tmasformation.

* Loop-shaping procedure for the state feedback loop transfer function of section (3.4.2) which is given by

$$L(z) = K_c(zI + BK_c)^{-1}A(zI - A)^{-1}B.$$

* For sampled-data systems; how the performance and robustness specifications can be transformed from s-plane to z-plane ?

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# APPENDIX A

## Observer Design

The problem of minimizing $\|K_f\|_F^2$ (where $K_f$ is the observer gain) can be formulated as follows. Let the state space representation of the design system given by $(A \in R^{nxn}, B \in R^{nxp}, C \in R^{mxn}$. Let $(\bar{A}, \bar{C}) = (Q^{-1}AQ, CQ)$ be the observable canonical form by introducing a state similarity transformation as in [3]

$$\bar{X}(t) = Q^{-1}x(t) \qquad (A-1)$$

where

$$CQ = (0 \quad C_2), |C_1| \neq 0 \qquad (A-2)$$

and

$$Q^{-1}AQ\left[\frac{I_{n-m}}{0}\right] = \begin{bmatrix} 0 & \cdots & 0 \\ \bar{I}_{v-1} & & \\ & & 0 \\ 0 & & \bar{I}_1 \end{bmatrix} \qquad (A-3)$$

where $n_i, i = 1, n-1$ indicates the number of the observability indexes which are greater than $i$ and $v$ is the greatest observability index.
and

$$\bar{I}_i = \begin{bmatrix} 0 \\ I_{ni} \end{bmatrix} \} \; n_{i-1} \quad \text{, and} \quad i = 1 \ldots, v = 1 \qquad (A-4)$$

The general state observer is given by

$$\dot{z}(t) = Fz(t) + K_f y(t) + Hu(t) \qquad (A-5)$$

$$\hat{x}(t) = T^{-1}z(t)$$

The solution of (A-5) which estimates $\bar{x}(t)$ is shown in [3]

F = a Jordan formed matrix (A-6)

$$K_f = (T\bar{A} - FT)\begin{bmatrix} 0 \\ I_m \end{bmatrix} C_1^{-1} \qquad (A-7)$$

$$H = TQ^{-1}B \qquad (A-8)$$

And $T$ is formed by the row vectors of the form

$$t_{ij}^T = \left(0..0:0..0,1,0..0:0:0..0,\lambda_i,0..0:...:0..0,\lambda^{v_j-1},0..0\right) \qquad (A-9)$$

where the non-zero entries of $t_{ij}^T$ are at the $jth$ position from the right of each block and $\lambda_i$ is the $ith$ eigen value of $F$. The format (A-9) is for distinctly real $\lambda$'s. To ensure the non-singularity of $T$, we may require that $\upsilon_p$ number of $j$'s equal to $p$, $p = 1,\ldots,m$

Since $T$ is an explicit function of the observer poles ($\lambda$'s), and $K_f$ depends on $T$ and $F$ by equation (A-7), we can minimize $\|K_f\|_F^2$ by choosing the eigen structure $T$ and $F$.

### A.1 Example

Let $\bar{A}, \bar{C}$ be in the observable canonical form given by

$$\bar{A} = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}, \bar{c} = [0 \quad 1]$$

Because the system is single output, $T$ has no freedom in its forms except the poles $\lambda_1, \lambda_2$.

From (A-9)

$$\begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix} \quad if \quad \lambda_1 \neq \lambda_2 \quad F = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \lambda_1 \\ 0 & 1 \end{bmatrix} \quad if \quad \lambda_1 = \lambda_2 \quad F = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{bmatrix}$$

from (A-7) and for the case when $\lambda_1 \neq \lambda_2$

$$K_f(T\bar{A} - FT)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_f = \begin{bmatrix} -6 + 5\lambda_1 - \lambda_1^2 \\ -6 + 5\lambda_2 - \lambda_2^2 \end{bmatrix}$$

Thus $\lambda_1$ and $\lambda_2$ can be chosen to be any negative real number and such that $\|K_f\|_F^2$ is minimized. This can be solved by nonlinear programming methods.

# APPENDIX b

## DATA FOR THE LARGE SPACE STRUCTURE

The state space realization of the augmented system for the large space structure design example discussed in section (4.5) is given by:

$$A = \begin{bmatrix} -2.4523e+001 & 3.2248e-004 & -6.2501e-002 & -1.6091e+001 \\ -3.4589e-004 & -2.4529e+001 & 1.6087e+001 & -6.2561e-002 \\ -3.0551e-002 & 7.7662e+000 & -6.5680e+000 & -2.8572e-004 \\ -7.7701e+000 & -3.0405e-002 & 2.2037e-004 & -6.5741e+000 \\ 5.9096e+001 & -4.7074e+000 & 2.9958e+000 & 3.5869e+001 \\ 4.7067e+000 & 5.9105e+001 & 3.5855e+001 & 2.9969e+000 \\ 9.1236e-004 & 1.4677e-001 & 4.3794e-001 & -3.9326e-003 \\ -1.2053e-001 & 1.5458e-003 & 5.8382e-003 & 3.6120e-001 \end{bmatrix}$$

$$\begin{bmatrix} -5.9216e+001 & -4.7161e+000 & -8.2370e-002 & 1.6025e+000 \\ 4.7169e+000 & -5.9226e+001 & -1.9671e+000 & -6.0396e-002 \\ -3.1416e+000 & 3.7595e+001 & 1.1829e+000 & 4.0093e-002 \\ -3.7614e+001 & -3.1431e+000 & -5.4201e-002 & 9.6419e-001 \\ -1.6990e+002 & -1.5480e-004 & 4.993e-001 & 1.0807e+001 \\ 1.9318e-004 & -1.6990e+002 & -1.3268e+001 & 4.5207e-001 \\ -1.1938e-003 & 1.9568e-002 & -3.1171e+002 & 5.5893e+000 \\ -8.0289e-003 & -3.1578e-004 & -6.6203e+000 & -3.9828e+002 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -4.0581e+001 & 5.4490e+001 \\ 5.4497e+001 & 4.0586e+001 \\ -2.2671e+001 & -1.6746e+001 \\ -1.6756e+001 & 2.2684e+001 \\ 4.6115e+001 & -5.2621e+001 \\ -5.2621e+001 & -4.6114e+001 \\ -3.4803e-003 & -3.0010e-003 \\ -1.4600e-003 & 1.6267e-003 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 2.9603e+000 & -3.9690e+000 \\ -3.9399e+000 & -2.9379e+000 \\ -8.9808e+000 & -6.6341e+000 \\ -6.6282e+000 & 8.9737e+000 \\ 5.6545e-002 & -6.6836e-002 \\ -1.2732e-001 & -9.7200e-002 \\ 2.2025e+000 & 1.6586e+000 \\ 1.3792e+000 & -1.8046e+000 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -4.0688e+001 & 5.4639e+001 & -2.4384e+001 & -1.8020e+001 \\ 5.4635e+001 & 4.0691e+001 & -1.8012e+001 & 2.4394e+001 \\ 5.7500e+006 & -7.2030e-006 & -3.2723e-005 & -2.5258e-005 \\ -7.4254e-006 & -5.3128e-006 & -2.3847e-005 & 3.3487e-005 \end{matrix}$$

$$\begin{matrix} -4.6115e+001 & 5.2621e+001 & 2.1224e+000 & 1.3966e+000 \\ 5.2621e+001 & 4.6115e+001 & 1.7232e+000 & -1.7348e+000 \\ -3.4744e-006 & 1.7983e-006 & 4.4768e-002 & -6.9921e-002 \\ 3.7053e-006 & 2.8927e-006 & & \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -2.0865e-002 & 2.7970e-002 & 5.9153e-002 & 4.3751e-002 \\ 2.8018e-002 & 2.0830e-002 & 4.3690e-002 & -5.9233e-002 \end{matrix}$$

$$\begin{matrix} -2.7865e-003 & 3.1738e-003 & 2.8153e-001 & 2.6499e-001 \\ 3.1776e-003 & 2.7840e-003 & 2.1387e-001 & -3.4805e-001 \end{bmatrix}$$

$$\left[\begin{array}{c|c} D_{11} & D_{12} \\ \hline D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 6.0000e-002 & 0 & 0 & 0 \\ 0 & 6.0000e-002 & 0 & 0 \\ 0 & 0 & 4.1120e-006 & 5.2801e-006 \\ \hline 0 & 0 & 4.0306e-006 & 5.7135e-006 \\ 1.0000e+000 & 1.0000e+000 & 0 & 0 \end{array}\right]$$

The state space realization of the $H_\infty$ compensator for the large space structure is given by:

$$A_k = \begin{bmatrix} -1.1041e+010 & 1.7844e+009 & -8.1706e+008 & -4.9356e+009 \\ 3.2592e+009 & -8.1632e+008 & 3.7065e+008 & 1.4565e+009 \\ 7.5163e+009 & -1.8726e+009 & 8.5034e+008 & 3.3589e+009 \\ 2.4841e+010 & -4.0119e+009 & 1.8370e+009 & 1.1105e+010 \\ 1.9840e+008 & 3.2383e+007 & -1.4824e+007 & -8.8692e+007 \\ 1.0218e+008 & -2.5951e+007 & 1.1780e+007 & 4.5660e+007 \\ -1.7993e+009 & 4.5325e+008 & -2.0578e+008 & -8.0406e+008 \end{matrix}$$

$$\begin{matrix} -8.8697e+009 & 7.1782e+008 & 2.2587e+007 & 1.4752e+008 \\ 2.6366e+009 & -4.4154e+008 & -1.1868e+007 & -4.3723e+007 \\ 6.0798e+009 & -1.0104e+009 & -2.7192e+007 & -1.0083e+008 \\ 1.9956e+010 & 1.612e+009 & -5.0765e+007 & -3.3191e+008 \\ -1.5941e+008 & 1.3150e+007 & 4.1156e+005 & 2.6511e+006 \\ 8.2679e+007 & -1.4127e+007 & -3.7852e+005 & -1.3709e+006 \\ -1.4557e+009 & 2.4581e+008 & 6.5983e+006 & 2.4139e+007 \\ -4.0827e+009 & 3.3165e+008 & 1.0425e+007 & 6.7902e+007 \end{bmatrix}$$

$$B_k = \begin{bmatrix} 4.0581e+001 & 5.4490e+001 \\ 5.4497e+001 & 4.0586e+001 \\ -2.2671e+001 & -1.6746e+001 \\ -1.6756e+001 & 2.2684e+001 \\ 4.6115e+001 & -5.2621e+001 \\ -5.1621e+001 & -4.6114e+001 \\ -3.4803e-003 & -3.0010e-003 \\ -1.4600e-003 & 1.6267e-003 \end{bmatrix}$$

$$C_k = \left[\begin{matrix} 1.8645e+009 & 3.4857e+008 & -1.5909e+008 & -8.3341e+008 \\ 1.3911e+009 & -1.8960e+008 & 8.7195e+007 & 6.2192e+008 \end{matrix}\right.$$

$$\left.\begin{matrix} -1.5008e+009 & 1.5868e+008 & 4.6626e+006 & 2.4941e+007 \\ 1.1153e+008 & -6.2504e+007 & 2.2131e+006 & -1.8565e+007 \end{matrix}\right]$$