Generalization of Formal Concept Analysis to Fuzzy Contexts: Application for Knowledge Extraction

by

Faisal Alvi

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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This thesis, written by **FAISAL ALVI** under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN COMPUTER SCIENCE**

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Dedicated to my grandfather

Shaikh Abdul Waheed

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All Praise is due to ALLAH to whom belongs the dominion of The Heavens and The Earth. Peace and Mercy Be Upon His Prophet. I thank ALLAH for giving me the courage, patience and knowledge to carry out this work.

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THESIS ABSTRACT

Name: FAISAL ALVI

Title: Generalization of Formal Concept Analysis to Fuzzy Contexts: Application for Knowledge Extraction

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Formal Concept Analysis is a sound methodology for information structuring and knowledge extraction. At the theoretical level, several mathematical properties have been proved and some efficient algorithms for building the lattice structure and knowledge database derivation from crisp data have been proposed and developed. At the practical level, we can find several data mining tools, which have been applied to various real data.

Starting from the bibliographical database on formal concept analysis and from fuzzy set theory we have extended formal concept analysis to the case of fuzzy binary relations. At the theoretical level new theoretical definitions and properties have been be developed. Specifically, we have defined fuzzy Galois connection and fuzzy knowledge extraction. Algorithms have also been developed for fuzzy Galois lattice generation. We have also extracted a minimal and complete set of implications from fuzzy relations, as well as provided an interpretation of the extracted implications. At the practical level we have built a prototype of fuzzy rule extractions from data. Several directions for continuation of this work in future have been proposed.

Keywords: Formal Concept Analysis, Fuzzy Set Theory, Fuzzy Context, Fuzzy Concept, Fuzzy Implications, Fuzzy Extent, Fuzzy Intent, Objects, Atributes, Complexity

King Fahd University of Petroleum and Minerals, Dhahran.
July 2000

خلاصة الرسالة

اسم الطالب: فيصل علوى

عنوان الرسالة: تعميم التحليل المفهوم ذوالاساس الرياضى للاحوال غيرالمحدودة:

تطبيق فى استنباط المعرفة

التخصص: الحاسب الآلى

تاريخ التخرج: يوليو ٢٠٠٠م

التحليل المفهوم ذوالاساس الرياضي هي طريقة سليمة لترتيب المعلومات واستنباط المعرفة. على المستوى النظري لعديد من الخصائص الرياضية تم اثباتها و بعض طرق الحل ذات الكفائة لبناء تنظيم مشبك و قاعدة بيانات للمعرفة من بيانات محددة تم ارضها و تنفيذها. على المستوى العملي يوجد الكثير من ادوات استخراج البيانات التيتم تطبيقها على بيانات حقيقية مختلفة.

بداية من قاعدة البيانات الببليوغرافيا الخاصة بالتحليل المفهوم ذوالاساس الرياضى و بداية من نظرية المجموعة غيرالمحددة لقد قمنا بمد التحليل المفهوم ذوالاساس الرياضى الى حالة العلاقات الثنائية غيرالمحددة. على المستوى النظري تعاريف و خصائص نظرية جديدة تم استنتاجها. بالتحديد، لقد قمنا بتعريف تربيط جيلونس غيرالمحدد و استنباط المعرفة غيرالمحددة. لقد تم بناء طرق الحل لتوليد تشبيك جيلونس غيرالمحدد. لقد قمنا ايضا باستنباط اصغر مجموعة كاملة من القواعد من علاقات غيرالمحددة. لقد قمنا ايضا بعرض تفسير للقواعد المستنبطة. على المستوى العملي قمنا ببناء نموذج اولى لاستخراج القواعد من البيانات. ايضاً تم عرض من التوجيهات لتكملة هذا العمل فى المستقبل.

المصطلحات: التحليل المفهوم ذوالاساس الرياضي، نظرية المجموعة غيرالمحددة، حالة غيرمحدد، مفهوم غيرمحدد، قواعد غيرمحدد، المدى غيرمحدد، نية غيرمحدد، اشياء، خصائص، درجة التعقيد.

جامعة الملك فهد للبترول والمعادن

الظهران - المملكة العربية السعودية

يوليو ٢٠٠٠م

Chapter 1

INTRODUCTION

1.1 Formal Concept Analysis

Formal Concept Analysis is a theory of data analysis which identifies conceptual structures among data sets. It is based upon the mathematization of *concept* and *conceptual thinking* .

Formal Concept Analysis is mainly used for the analysis of data, i.e., for investigating and processing explicitly given information. Such data will be structured into units which are formal abstractions of *concepts* of human thought allowing meaningful and comprehensible interpretation [4].

Since its inception, formal concept analysis has found several applications. Major areas of application [8, 13, 4] include:

- Knowledge Extraction

- Information Restructuring and Classification
- Data Mining
- Data Visualization
- Decision Making.

1.2 Definitions

Before stating the proposed objectives, it is worthwhile to review definitions of basic terms used in Formal Concept Analysis. For a detailed description refer to [4]:

1.2.1 Context

A *formal context* or *crisp binary relation* is a two-valued relation between a set of *objects* and a set of *attributes*. In case of two-valued context, this relation is a binary relation and it simply states the truth value of the proposition: *Is a particular object from the domain set related to a particular attribute from the range set?*, for all objects and attributes.

As an example, a formal context R may be used to map the relation "is a divisor of" for the set of divisors of 30. Both the domain and the range sets are the same in this case, which is the set of divisors of the 30, i.e., $\{1, 2, 3, 5, 6, 10, 15, 30\}$. This relation is presented in Table 1.1.

	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	0	1	0	0	1	1	0	1
3	0	0	1	0	1	0	1	1
5	0	0	0	1	0	1	1	1
6	0	0	0	0	1	0	0	1
10	0	0	0	0	0	1	0	1
15	0	0	0	0	0	0	1	1
30	0	0	0	0	0	0	0	1

Table 1.1: Formal Context R

1.2.2 Concept

A *formal concept* or a *maximal rectangle* of a context R is an ordered pair of two sets (A, B) such that whenever $A \times B \subseteq A' \times B' \subseteq R$ then $A = A'$ and $B = B'$ [4].

More simply, in a formal context, if a maximal set of objects A shares a maximal set of attributes B, then the ordered pair (A, B) is called a formal concept. Here A is called the *extent* and B is called the *intent* of the concept.

Consider the context R shown in Table 1.1. The set of objects $A = \{1, 2, 5\}$ shares a maximal set of attributes $B = \{10, 30\}$. Hence the ordered pair (A, B) is a concept. The significance of this concept is that the numbers 1, 2 and 5 divide *only* the numbers 10 and 30 in the set of divisors of 30. In other words, a concept indicates, for any given object set, the maximal set of attributes related to all the objects in the object set.

1.2.3 Galois Connection

Let A be any arbitrary set of objects and B be any arbitrary set of attributes from a given context R. Then, the operators f and h defined by:

$$f(A) = A^R = \{d \mid \forall g, g \in A \Rightarrow (g,d) \in R\} \tag{1.1}$$

$$h(B) = B^Q = \{g \mid \forall d, d \in B \Rightarrow (g,d) \in R\} \tag{1.2}$$

define a *Galois connection*

Galois Connections are used in order to discover a concept corresponding to any given set of objects. The operator f when applied to a set of objects results in the set of attributes shared by those objects. The operator h is then applied to this resulting set of attributes to complete the initial set of objects, i.e., to find any additional objects that may share the same set of attributes.

As an example, consider the context given in Table 1.1. Consider the set of objects $A = \{3, 5\}$. The set of attributes shared by these objects is obtained by: $\Rightarrow B = f(A) = A^R = \{15, 30\}$.

This means that 3 and 5 divide the numbers 15 and 30. To find out any additional numbers which may divide both 15 and 30, we apply the h operator on B, i.e.,

$\Rightarrow B^Q = h(A^R) = A^{RQ} = \{1, 3, 5\}$.

The ordered pair $(B^Q, A^R) = (A^{RQ}, A^R) = (\{1, 3, 5\}, \{15, 30\})$ is a *concept.*

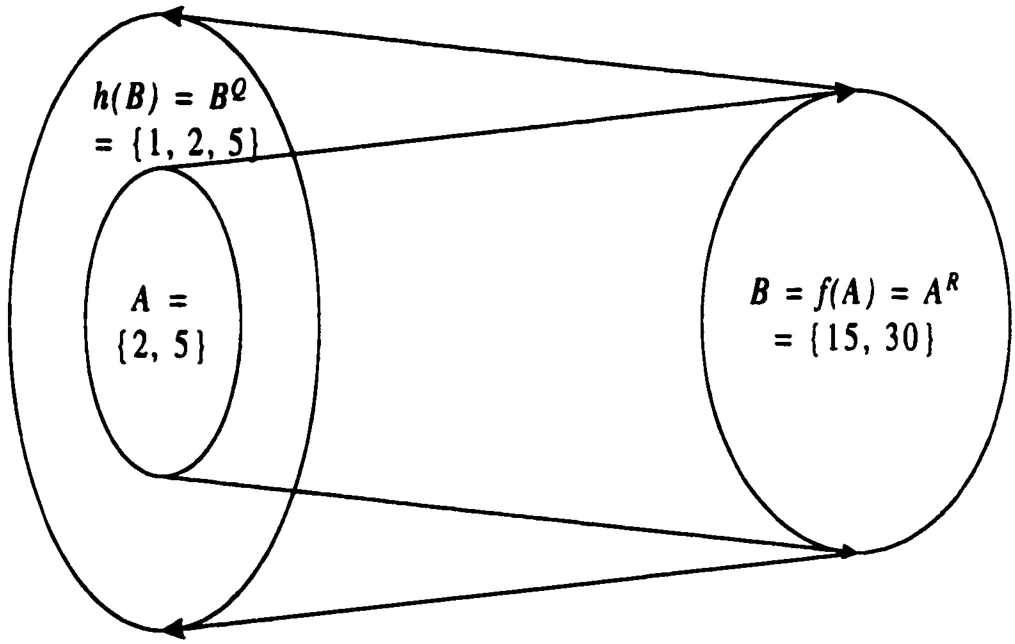

Figure 1.1: Discovering Concepts using Galois Connections

1.2.4 Galois Lattice

For a given context, Galois connections can be used to generate the set of all possible concepts. These concepts are naturally ordered by a subconcept-superconcept relation ($\ll$). This relation is defined by:

$$(A_1, B_1) \ll (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \text{ and } B_2 \subseteq B_1. \tag{1.3}$$

It can be shown that $\ll$ is a partial order relation i.e., $\ll$ is reflexive, antisymmetrical and transitive. The set of all concepts of a given context can therefore be organized as an ordered set.

The ordered set of all concepts of a context, also consisting of an upper bound as well as a lower bound, is known as a *Galois lattice* of the context. Galois lattices can be represented as a line diagram in which there is an arrow from (A_1, B_1) to (A_2, B_2) if (A_1, B_1) $\ll$ (A_2, B_2). (A_2, B_2) is referred to as the *predeccesor* and (A_1, B_1) is called the *successor*. The lower-bound of the Galois lattice is the concept with minimal number of objects and maximal number of attributes. As we follow the ordering of the lattice (shown by arrow-heads) we find that the object set increases incrementally, while the attribute set decreases incrementally. This change agrees with the observation that "The more the number of objects is, the less common they will be". The upper-bound of the lattice is the concept with the maximum number of objects and minimum number of attributes.

Figure 1.2 shows the Galois lattice corresponding to the relation R of Table 1.1.

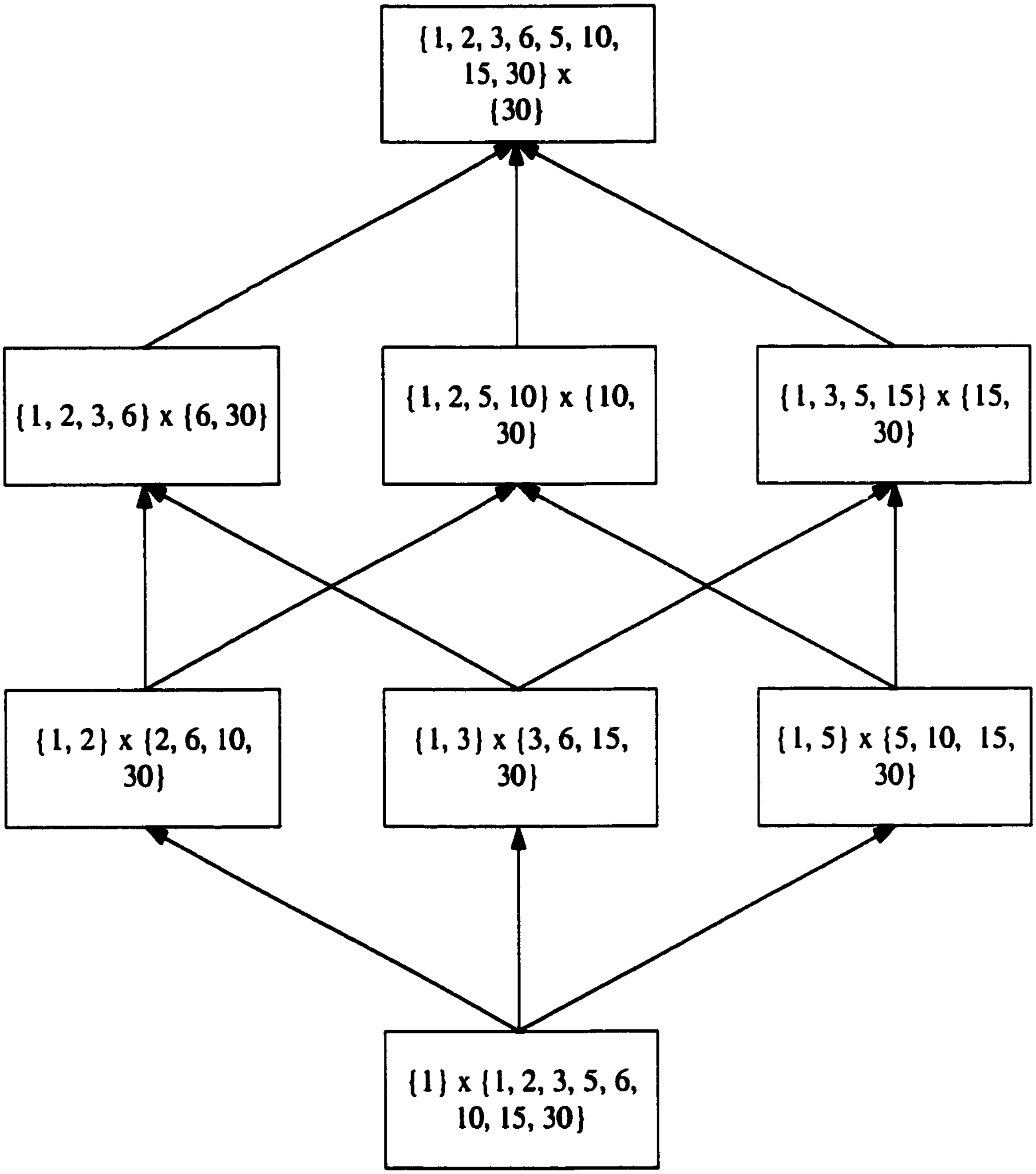

Figure 1.2: Galois Lattice Structure for *R*

Galois Lattice structures provide several advantages over tabular representation of data. A Galois lattice structure exhibits the following properties [4]:

- Information classification
- Reduced redundancy by organizing objects with similar attributes into concepts
- Inheritance of attributes thereby creating a class hierarchy of data.
- Semantic information from unorganized data
- Graphical representation of data, thus giving a view of the inherent structures hidden within the data.
- Associative rules extraction, providing knowledge from information.

1.2.5 Inheritance of objects and attributes

We can observe that the graphical ordering of the concepts in a Galois lattice exhibits inheritance of attributes and objects. This is the case by the very definition of the order relation ($\ll$) defined in the previous section. As we move from the lower bound to the upper bound, we find the object set increasing and the attribute set decreasing. So attributes are inherited from the predeccessor to the successor concept, while objects are inherited from the successor to the predeccessor concept.

Inheritance plays a vital role in interpreting the Galois lattice structure. For example, in the Galois lattice shown in figure 1.2, we can see that each concept's predeccessor concept consists of all the objects in the successor concept plus any additional objects. So, the concept with the object set {1, 2, 3, 6} has successors with the object sets {1, 2} and {1, 3}. *Thus each concept's successors consist of the set of divisors of the additional elements in the predeccessor concept.* For example, {1, 2, 3, 6} inherits from {1, 2} and {1, 3}. The additional element here is 6 and we find that this whole object set forms the set of divisors of 6.

A further advantage of inheritance is reduced redundancy in representation. For example, we can eliminate the inherited objects and attributes to give a more readable representation.

Figure 1.3 shows the reduced Galois lattice structure for *R*. In this figure both the reduced attribute set and the reduced object set for each concept are identical. An important semantic meaning can be extracted from this figure: *The set of divisors of each element are given by its all successors.*

Inheritance also aids in creating a class hierarchy of data as will be shown in the coming chapters.

1.2.6 Knowledge Extraction

By definition, knowledge extraction deals with extracting knowledge from unorganized data. In knowledge extraction *association rules* or *implications* of the form

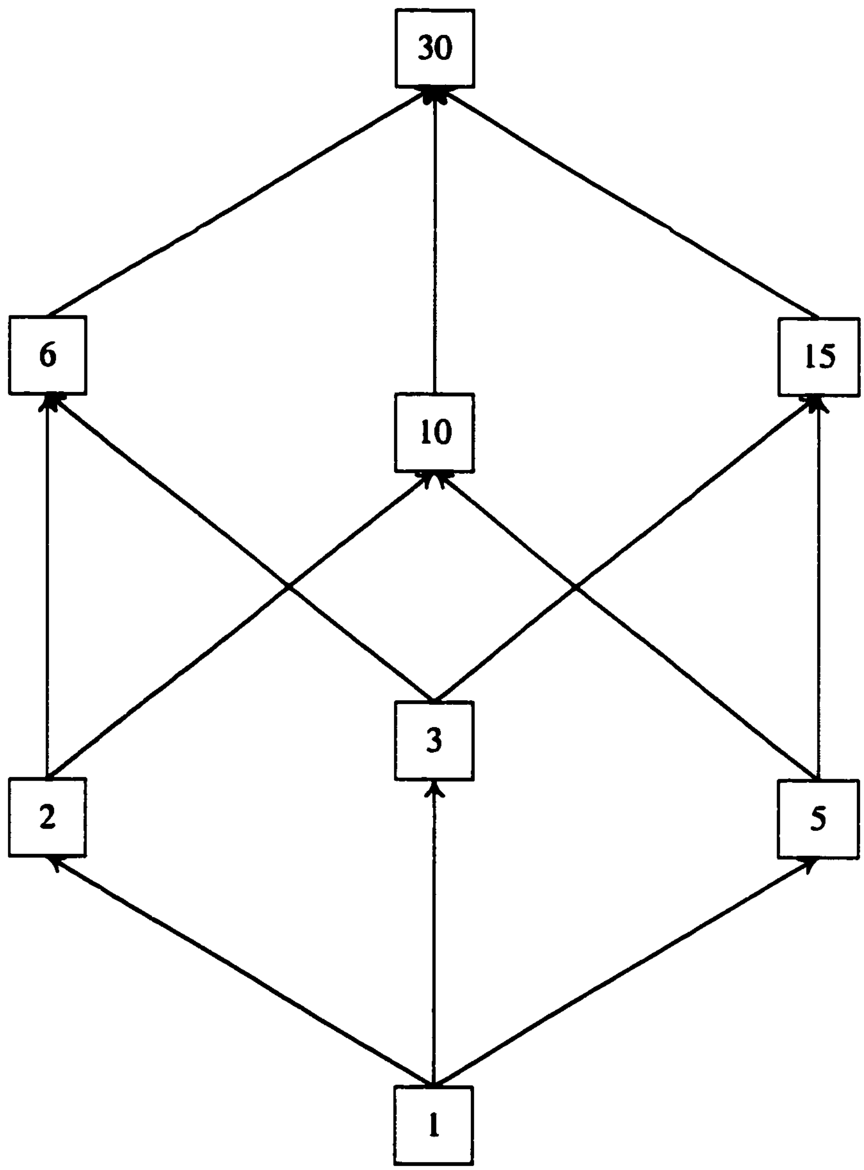

Figure 1.3: Reduced Galois Lattice Structure for R

$P \Rightarrow C$ are extracted from a context, where P is the premise and C is the conclusion. Both sets P and C are subsets of the range or attribute set of the given context. The implication $P \Rightarrow C$ is *valid* if the following is true for a given context R:

For every object $g \in R$: if every attribute from the premise P applies to the object g then every attribute from the conclusion C also applies to g.

As an example, suppose that we are interested in finding the conclusion C corresponding to the following premise from the context given in table 1.1: *What is the conclusion we can draw about all numbers divisible by 2 and 3?*. We find the following association rule:

$$\{2, 3\} \Rightarrow \{6\}$$

This means that any number divisible by 2 and 3 within the context R is also divisible by 6. Although the above association rule is true in general for any set of numbers, it may not be the case with all association rules. Association rules are only valid for the context from which they have been extracted.

Knowledge extraction is thus useful for discovering trends or tendencies from empirical data. For example, association rules may be discovered regarding the performance of students in a course and their age.

1.3 Fuzzy Set Theory

Traditional tools for formal modeling, reasoning and computing are crisp and deterministic. In conventional logic, for example, a statement can either be true or false - nothing in between. In set theory, an element can either belong to a set or not - there is no notion of partial membership [15, 16].

Fuzzy set theory extends these models by allowing an element to partially belong to a set. A fuzzy set consists of a set of members along with their degree of membership denoted by μ. For an element $\mu = 0$ may indicate no membership, $\mu = 0.2$ may indicate weak membership, $\mu = 0.8$ may indicate strong membership and $\mu = 1.0$ may infer complete membership.

Fuzziness introduces into a system the characteristics of fuzziness and uncertainty which may be used to model data in which attribute membership is related to objects by degrees of possibility.

1.3.1 Fuzzy Context

A *fuzzy context* or a *fuzzy binary relation* is a set of objects and a set of attributes related to each other by fuzzy values indicating that a particular object possesses a particular attribute partially. Table 1.2 models a data situation in which attributes cannot be possessed by objects with a true/false value. For such situations, fuzzy contexts are best suited.

Employee	Age	Education	Salary	Benefits
Instructor A	0.2	0.4	0.5	0.1
Instructor B	0.5	0.9	0.7	0.8
Instructor C	0.3	0.2	0.4	0.4
Instructor D	0.7	0.9	1.0	0.8

Table 1.2: Fuzzy Context $\widetilde{R}$

1.4 Proposed Work

1.4.1 Motivation

Formal Concept Analysis is a useful tool for data analysis within crisp data. Our fundamental objective in undertaking this work is to extend formal concept analysis to fuzzy contexts. The primary motivation behind this objective is to develop the ability to extract knowledge from fuzzy data. Doing so we shall be able to extend formal concept analysis to the case of fuzziness where each attribute has a degree of possibility rather than the conventional true/false possession of attributes.

This work would involve extending the definitions of the terms and notions used in formal concept analysis for the crisp case. We would have to define the terms *fuzzy concept*, *fuzzy Galois connection*, *fuzzy Galois lattice* and *fuzzy knowledge extraction*.

1.4.2 Objectives

Our objectives in undertaking this research are the following:

1. To extend formal concept analysis to the case of fuzzy contexts

2. To define and develop the notion of a fuzzy concept.

3. To define fuzzy Galois connection, fuzzy Galois lattice and fuzzy knowledge extraction.

4. To extract theoretical properties of fuzzy Galois lattice.

5. To develop and analyze algorithms for fuzzy Galois lattice development and fuzzy knowledge extraction.

6. To develop a prototype of fuzzy rule extractions from data.

7. To carry out experimentation on some real data set and to interpret the results.

1.4.3 Organization

This thesis report is organized as follows: Chapter 1 provides a brief introduction to FCA. Chapter 2 gives a brief overview of attempts to extend well-defined notions in FCA to fuzzy relations. Chapter 3 provides an extension of Galois Connection and Galois Lattice to fuzzy relations. Chapter 4 provides an extension of Knowledge Extraction to fuzzy relations. Both Chapters 3 and 4 begin with sections on background that is pertinent to the proofs and extensions within the framework of extensions. Chapter 5 provides an experimentation on a real data set. Chapter 6 concludes this work with some proposals for extension of this work.

Chapter 2

RELATED WORK

This chapter provides a survey of some of the attempts to exten formal concept analysis to fuzzy relations. Then motivation for our work which came from fuzzy regular relations is discussed briefly.

2.1 Literature Survey

Attempts to extend formal concept analysis to fuzzy contexts have a short but rich history. Below some of the related works pertinent to this research are outlined:

2.1.1 Fuzzy Difunctional Relations

Difunctional Relations have proved to play an important role in software design and information restructuring [8]. By definition, a relation R is difunctional if and only if it satisfies the following condition:

$$RR^{-1}R \subseteq R \tag{2.1}$$

Ounalli et. al [9] extended the notion of difunctional relations to fuzzy relations and characterized fuzzy difunctional relations for the first time. Relational and theoretical properties of fuzzy difunctional relations were investigated. It was found out that fuzzy difunctional relations have properties analogous to those of crisp difunctional relations. A key result of this work was: A fuzzy relation $\widetilde{R}$ is fuzzy difunctional if and only if it satisfies the following condition:

$$\widetilde{R}\widetilde{R}^{-1}\widetilde{R} = \widetilde{R} \tag{2.2}$$

The similarity between the defining condition of difunctionality for the crisp and the fuzzy case can be readily observed.

2.1.2 Fuzzy Difunctional Dependencies

Ounalli et. al [10] proposed an extension of difunctional dependencies in the framework of fuzzy relational database in which every fuzzy relation is a set of weighted tuples. The concept of fuzzy difunctionality was defined and characterized and it was shown that inference rules applied in the classical case remain valid in the case of fuzzy relations. A hierarchical decomposition approach for the decomposition of fuzzy difunctional relations was also proposed which allowed better data clustering and reduced storage space by its particular inheritance mechanism.

2.1.3 Silke's Extension of FCA to Many-Valued Contexts

Silke Pollandt's book [11] which is based on her doctoral thesis [12] introduces an extension of formal concept analysis to many-valued contexts. She uses many-valued contexts in order to model contexts where an object possesses any attribute with some grade of possiblity. A set-theoretic model for such many-valued contexts is developed by means of fuzzy sets [12].

Her work goes on further to prove that any multi-valued context with ordered scales can be naturally made to correspond to a fuzzy valued context. One of the conclusions of her doctoral thesis is: *Fuzzy valued context can be considered as a generalization of a multi-valued context.*

Her work utilises this extension of formal concept analysis to many-valued contexts by applying it to define attribute simplification from a set of implications of a fuzzy-valued context. The question of completeness of implications in fuzzy valued contexts is also addressed.

2.1.4 Wolff's Proposition

Wolff[14] proposed L-Fuzzy Scaling Theory. He observes that linguistic variables play the same role in Fuzzy Theory as conceptual scales in formal concept analysis.

Wolff started his proposition[14] by presenting the notion of L-fuzzy sets as a generalization of fuzzy sets. For each L-fuzzy set f, he introduced the notion of cut-context of f by using the α-cut notion. A cut context is mainly applied to a linguistic variable λ in order to derive a scale S_λ. Roughly speaking, the scale S_λ constitutes a crisp relation (which is equivalent to λ) derived from the fuzzy sets describing the linguistic values of λ. The definition of a formal concept remains unchanged since it becomes useless to treat directly any fuzzy relation.

Wolff proposed a representation for cut-contexts of L-fuzzy sets as follows: Let X be a set and $(L, \leq)$ an ordered set and $f \in \mathbf{F}(X, L)$. Then f can be reconstructed from the cut-context $\mathbf{K}_f(L, X, I_f)$ and $(L, \leq)$ by the formula $f(x) = \max\{\alpha \in L \mid \alpha\, I_f\, x\}$ i.e., $f(x)$ is the maximum in $(L, \leq)$ of the extent of the attribute concept of x in $(B(\mathbf{K}_f), \leq)$.

2.1.5 Belohlavek's proposition: Fuzzy Galois Connections

Belohlavek [2] proposed an extension of Galois connections between power sets from the point of view of fuzzy logic. His proposition attempts to establish a one-to-one correspondence between fuzzy Galois connections and fuzzy binary relations.

His work primarily focuses on three points:

1. Crisp Galois connections are just **L**-Galois Connections for $\mathbf{L} = 2$, where **L** is a complete residuated lattice.

2. Fuzzy Galois Connections are in one-to-one correspondence with fuzzy binary relations.

3. L-Galois Connections may be represented by special systems of 2-Galois Connections.

Behlolavek [3] not only proposes fuzzy Galois connections but also characterizes lattices of fixed points of fuzzy Galois connections. His work is mostly centered around applications for traditional logic.

2.2 Motivation for fuzzy concept: Fuzzy Regular Relations

A crisp regular relation [6] is a relation R such that $\forall\ u,\ v \in \text{domain}(R)$,

$$u.R \cap v.R \neq \phi \Rightarrow u.R = v.R \tag{2.3}$$

A relation is regular if and only if it is the union of disjoint maximal rectangles. In case of crisp relations, any crisp difunctional relations is always regular and the converse is also true, i.e., any crisp regular relation is always difunctional.

In case of fuzzy relations, we found that this may not be the case. Although a fuzzy regular relation is always difunctional. But the converse may not be true, i.e., a fuzzy difunctional relation may not be regular.

This discovery of the inequality of fuzzy difunctional relation and fuzzy regular relation led to the notion of fuzzy concept or a fuzzy rectangle which we shall outline in subsequent chapters.

Example 2.1 *Consider the following fuzzy difunctional relation:*

	p_1	p_2
o_1	0.5	0.1
o_2	0.1	0.3
o_3	0.1	0.5

Table 2.1: Fuzzy Difunctional Relation $\widetilde{S}$

Obviously $\widetilde{S}$ is difunctional (it can be readily verified that $\widetilde{S}\ \widetilde{S}^{-1}\ \widetilde{S} = \widetilde{S}$) but not regular as it cannot be decomposed into a fuzzy union of disjoint maximal rectangles.

Example 2.2 *Consider the following fuzzy regular relation:*

	p_1	p_2
o_1	0.5	0.0
o_2	0.0	0.3
o_3	0.0	0.3

Table 2.2: Fuzzy Regular Relation $\widetilde{S}$

Obviously $\widetilde{S}$ is regular as it can be decomposed into the following fuzzy maximal rectangles: $\widetilde{S} = \{o_1\} \times \{p_1/0.5\} \cup \{o_2, o_3\} \times \{p_2/0.3\}$.

$\widetilde{S}$ is also difunctional (it can be readily verified that $\widetilde{S}\ \widetilde{S}^{-1}\ \widetilde{S} = \widetilde{S}$).

The discovery that fuzzy difunctional relations may not be regular led to the notion of the importance of the smallest basic unit that may be the most important for structuring fuzzy relations. This led to the idea of a fuzzy concept which is discussed in the next chapter - it also became the basis of this research.

Chapter 3

FUZZY GALOIS CONNECTION AND LATTICE

Fuzzy Galois Connection as well as fuzzy Galois lattice are smooth extensions of their crisp counterparts. Below we outline the theoretical foundations behind crisp concepts, crisp Galois connections and crisp Galois lattices. This underlying theory is then smoothly extended to the fuzzy case. [7]

3.1 Crisp Contexts

In this section, we start by presenting some formal properties of crisp contexts. Along all this section and the following ones, J stands for any set of indices.

3.1.1 Crisp Rectangles

Below we formally state the definitions of crisp rectangle and crisp maximal rectangle. It can be seen that a maximal rectangle is the same as a formal concept.

Definition 3.1 *Let R be a binary relation. A rectangle $A \times B$ is a cartesian product of two sets (A, B) such that $A \times B \subseteq R$. A is the domain of the rectangle (A, B) and B is its range.*

Definition 3.2 *Let (A, B) be a rectangle of a crisp relation R. The rectangle (A, B) is said to be maximal if whenever $A \times B \subseteq A' \times B' \subseteq R$, then $A = A'$ and $B = B'$.*

3.1.2 Crisp Galois Connection

Let R be a binary relation. For two sets A and B such that $A \subseteq domain(R)$, $B \subseteq range(R)$, we define the operators $f(A) = A^R$ and $h(B) = B^Q$ as follows:

$$f(A) \quad = A^R = \{d \mid \forall g, g \in A \Rightarrow (g, d) \in R\} \tag{3.1}$$

$$h(B) \quad = B^Q = \{g \mid \forall d, d \in B \Rightarrow (g, d) \in R\} \tag{3.2}$$

The operators ^{R} and ^{Q} define a Galois connection [4] between the ordered sets A and B by satisfying the following conditions:

$$A_i \subseteq A_j \quad \Rightarrow A_i^R \supseteq A_j^R \tag{3.3}$$

$$B_i \subseteq B_j \quad \Rightarrow B_i^Q \supseteq B_j^Q \tag{3.4}$$

$$A_i \subseteq A_i^{RQ} \quad \text{and } B_i \subseteq B_i^{QR} \tag{3.5}$$

Proposition 3.1 *[4] A pair (f, h) or $(^R, ^Q)$ of maps is called a Galois connection if and only if*

$$A \subseteq B^Q \Leftrightarrow B \subseteq A^R \tag{3.6}$$

Proposition 3.2 *[4] For every Galois connection (f, h) or $(^R, ^Q)$*

$$f = fhf \text{ and } h = hfh \tag{3.7}$$

3.1.3 Crisp Galois Lattice

The set of maximal rectangles of a binary relation R, using the Galois connection operators ^{R} and ^{Q} can be organized under a complete lattice [1].

Definition 3.3 *Let R be a binary relation and T_r the set of maximal rectangles of R ordered by the relation $\ll$. Hence, $(T_r, \ll)$ is a complete lattice where the supremum P and the infinimum H of any set of maximal rectangles of T_r are given respectively as follows:*

$$P_{j\in J}(A_j, B_j) = (\cup_{j\in J} A_j, \cap_{j\in J} B_j) \tag{3.8}$$

$$H_{j\in J}(A_j, B_j) = (\cap_{j\in J} A_j, \cup_{j\in J} B_j) \tag{3.9}$$

3.2 Mathematical Background on Fuzzy Sets

Here we review some definitions and results that will be needed in the sequel. For details we refer to [9, 16]

3.2.1 Fuzzy Sets

In this work, a fuzzy set [15] $\widetilde{F}$ which includes a member x_i, that has degree of membership $\mu(x_i)$, ($\mu(x_i) \in (0,1)$) will be represented as $\widetilde{F} = \{x_i/\mu(x_i)\}$, $1 \leq i \leq n$.

Crisp sets can be defined as special cases of fuzzy sets with the membership degrees restricted to the set $\{0,1\}$.

3.2.2 Basic Set-Theoretic operations on Fuzzy Sets

The fundamental set-theoretic operations on fuzzy sets which are relevant to us are inclusion, union and intersection.

Fuzzy Inclusion

A fuzzy set $\widetilde{A}$ is said to be included in another fuzzy set $\widetilde{B}$ if $\forall x \in \widetilde{A}\ \exists x \in \widetilde{B} \mid \mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x)$.

Fuzzy Intersection

The membership function $\mu_{\widetilde{C}}(x)$ of the intersection $\widetilde{C} = \widetilde{A} \cap \widetilde{B}$ is defined by $\mu_{\widetilde{C}}(x) = min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$.

Fuzzy Union

The membership function $\mu_{\widetilde{D}}(x)$ of the union $\widetilde{D} = \widetilde{A} \cup \widetilde{B}$ is defined by $\mu_{\widetilde{D}}(x) = max\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)\}$.

3.2.3 Cartesian Product

Since a crisp set can be defined as a fuzzy set with membership degrees restricted to {0, 1}, so, the cartesian product of a crisp set and a fuzzy set $\widetilde{R} = A \times \widetilde{B}$ is defined by:

$$\widetilde{R} = \{(x, y)/min(\mu_A(x), \mu_{\widetilde{B}}(y)) \mid \forall x \in A, y \in \widetilde{B}\} \tag{3.10}$$

Example 3.1 *Consider* $A = \{a/1, b/0\}$ *and* $\widetilde{B} = \{p/0.1, q/0.9\}$. *So* $\widetilde{R} = \{(a,p)/0.1,$ $(b,p)/0,$ $(a,q)/0.9,$ $(b,q)/0\}$ *is the cartesian product* $\widetilde{R} = A \times \widetilde{B}$

3.3 Fuzzy Contexts

In this section, we start by introducing the notions of fuzzy rectangle and fuzzy maximal rectangle and we prove some formal properties; we also define a pair of Galois connection operators by proving its conditions. Then we propose to organize the set of fuzzy rectangles under a complete and distributive lattice.

3.3.1 Fuzzy Rectangles

Below we formally state the definitions of fuzzy rectangle and fuzzy maximal rectangle. The close correspondence between the crisp case and the fuzzy case can be readily seen.

Definition 3.4 *Let* $\widetilde{R}$ *be a fuzzy binary relation defined from* E *to* $\widetilde{F}$. *A fuzzy* *rectangle of* $\widetilde{R}$ *is an ordered pair of two sets* $(A, \widetilde{B})$ *such that* $A \times \widetilde{B} \subseteq \widetilde{R}$. *i.e.,* $\mu_{A\times\widetilde{B}}(x, y) \leq \mu_{\widetilde{R}}(x, y)$, *where* $x \in E$ *and* $y \in \widetilde{F}$.

Definition 3.5 *Let* $\widetilde{R}$ *be a fuzzy binary relation defined from* E *to* $\widetilde{F}$. *The relation* $A \times \widetilde{B}$, *such that* $A \subseteq E$ *and* $\widetilde{B} \subseteq \widetilde{F}$ *is called fuzzy rectangular relation associated with the rectangle* $(A, \widetilde{B})$ *of* $\widetilde{R}$. A *is the domain of this relation and* $\widetilde{B}$ *is its range.*

Definition 3.6 *A fuzzy maximal rectangle, or fuzzy concept is defined as the cartesian product* $A \times \widetilde{B} \subseteq \widetilde{R}$, *where* A *is a crisp set and* $\widetilde{B}$ *is a fuzzy one, such that whenever* $A \times \widetilde{B} \subseteq A' \times \widetilde{B}' \subseteq R$, *then* $A = A'$ *and* $\widetilde{B} = \widetilde{B}'$.

From the above definitions, it can be seen that a fuzzy maximal rectangle thus associates a degree of membership with each object-attribute pair within it. In contrast to a crisp rectangle which only states the presence or absence of an object-attribute pair, a fuzzy maximal rectangle or a fuzzy concept also gives the "degree of association" between the object and the attribute.

3.3.2 Fuzzy Galois Connection

Let $\widetilde{R}$ be a fuzzy binary relation defined from E to $\widetilde{F}$. For two sets A and $\widetilde{B}$ such that $A \subseteq E$ and $\widetilde{B} \subseteq \widetilde{F}$ we define the operators $\widetilde{f}(A) = A^{\widetilde{R}}$ and $\widetilde{h}(\widetilde{B}) = B^{\widetilde{Q}}$ as follows:

$$\widetilde{f}(A) = A^{\widetilde{R}} = \{d/\alpha \mid \forall g, g \in A, \alpha = \min \mu_{\widetilde{R}}(g, d)\} \tag{3.11}$$

$$\widetilde{h}(\widetilde{B}) = \widetilde{B}^{\widetilde{Q}} = \{g \mid \forall d, d \in \widetilde{B}, \Rightarrow \mu_{\widetilde{R}}(g, d) \geq \mu_{\widetilde{B}}(d)\} \tag{3.12}$$

Proposition 3.3 *The operators* $^{\tilde{R}}$ *and* $^{\tilde{Q}}$ *form a fuzzy Galois connection on the sets* A *and* $\tilde{B}$ *by satisfying the following conditions:*

$$A_i \subseteq A_j \quad \Rightarrow A_i^{\tilde{R}} \supseteq A_j^{\tilde{R}} \tag{3.13}$$

$$\tilde{B}_i \subseteq \tilde{B}_j \quad \Rightarrow \tilde{B}_i^{\tilde{Q}} \supseteq \tilde{B}_j^{\tilde{Q}} \tag{3.14}$$

$$A \subseteq A^{\tilde{R}\tilde{Q}} \quad \text{and } \tilde{B} \subseteq \tilde{B}^{\tilde{Q}\tilde{R}} \tag{3.15}$$

Proof 3.1 *Let* $A, A_i, A_j \in E$ *and* $\tilde{B}, \tilde{B}_i, \tilde{B}_j \in \tilde{F}$.

- *Then*

$$A_i^{\tilde{R}} = \{d/\alpha_i \mid \forall g, g \in A_i, \alpha_i = \min \mu_{\tilde{R}}(g,d)\}$$

$$A_j^{\tilde{R}} = \{d/\alpha_j \mid \forall g, g \in A_j, \alpha_j = \min \mu_{\tilde{R}}(g,d)\}$$

If $A_i \subseteq A_j \Rightarrow \alpha_i \geq \alpha_j$. *Hence* $A_i^{\tilde{R}} \supseteq A_j^{\tilde{R}}$ *This proves (3.13).*

- *For (3.14), if* $\tilde{B}_i \subseteq \tilde{B}_j \Rightarrow \forall d \in \tilde{B}_i, \mu_{\tilde{B}_j}(d) \geq \mu_{\tilde{B}_i}(d)$. *Then*

$$\tilde{B}_i^{\tilde{Q}} = \{g_i \mid \forall d, d \in \tilde{B}_i, \Rightarrow \mu_{\tilde{R}}(g_i,d) \geq \mu_{\tilde{B}_i}(d)\}$$

$$\tilde{B}_j^{\tilde{Q}} = \{g_j \mid \forall d, d \in \tilde{B}_j, \Rightarrow \mu_{\tilde{R}}(g_j,d) \geq \mu_{\tilde{B}_j}(d)\}$$

If $g_j \in B_j^{\tilde{Q}} \Rightarrow \mu_{\tilde{R}}(g_j,d) \geq \mu_{\tilde{B}_j}(d) \geq \mu_{\tilde{B}_i}(d) \Rightarrow g_j \in B_i^{\tilde{Q}}$. *Hence* $B_i^{\tilde{Q}} \supseteq B_j^{\tilde{Q}}$. *This proves (3.14).*

- *For (3.15)*

$$A^{\tilde{R}} = \{d/\alpha \mid \forall g, g \in A, \alpha = \min \mu_{\tilde{R}}(g,d)\}$$

$$A^{\tilde{R}\tilde{Q}} = \{g \mid \forall d, d \in A^{\tilde{R}}, \Rightarrow \mu_{\tilde{R}}(g,d) \geq \min \mu_{\tilde{R}}(g,d)\}$$

Obviously, if $g \in A \Rightarrow g \in A^{\tilde{R}\tilde{Q}} \Rightarrow A \subseteq A^{\tilde{R}\tilde{Q}}$. *This proves the first part of (3.15).*

- *Similarly, let* $d \in \widetilde{B}$ *with* $\mu_{\widetilde{B}}(d)$. *Then*

$$\widetilde{B}^{\tilde{Q}} = \{g \mid \forall d,\ d \in \widetilde{B}, \Rightarrow \mu_{\tilde{R}}(g,d) \geq \mu_{\widetilde{B}}(d)\}$$

$$\widetilde{B}^{\tilde{Q}\tilde{R}} = \{d/\alpha \mid \forall g, g \in \widetilde{B}^{\tilde{R}},\ \alpha = \min \mu_{\tilde{R}}(g,d)\}$$

$\Rightarrow \alpha \geq \mu_{\widetilde{B}}(d) \Rightarrow \widetilde{B} \subseteq \widetilde{B}^{\tilde{Q}\tilde{R}}$. *This proves the other part of (3.15).*

Proposition 3.4 *A pair* $(^{\tilde{R}},^{\tilde{Q}})$ *of maps is a fuzzy Galois connection if and only if* $A \subseteq \widetilde{B}^{\tilde{R}} \Leftrightarrow \widetilde{B} \subseteq A^{\tilde{Q}}$.

Proof 3.2 *If* $A \subseteq \widetilde{B}^{\tilde{R}}$ *then by (3.13)* $\Rightarrow A^{\tilde{Q}} \supseteq \widetilde{B}^{\tilde{Q}\tilde{R}}$ *and by (3.15)* $\Rightarrow A^{\tilde{Q}} \supseteq \widetilde{B}$. *This proves that* $A \subseteq \widetilde{B}^{\tilde{R}} \Rightarrow \widetilde{B} \subseteq A^{\tilde{Q}}$ *The other direction follows symmetrically.*

Proposition 3.5 *For a fuzzy Galois connection* $(^{\tilde{R}},^{\tilde{Q}})$, $A^{\tilde{R}} = A^{\tilde{R}\tilde{Q}\tilde{R}}$ *and* $\widetilde{B}^{\tilde{Q}} = B^{\tilde{Q}\tilde{R}\tilde{Q}}$.

Proof 3.3 *With* $\widetilde{B} = A^{\tilde{R}}$ *by (3.15)*

$$\Rightarrow A^{\tilde{R}} \subseteq A^{\tilde{R}\tilde{Q}\tilde{R}} \tag{3.16}$$

and from $A \subseteq A^{\tilde{R}\tilde{Q}}$ *by (3.13)*

$$\Rightarrow A^{\tilde{R}} \supseteq A^{\tilde{R}\tilde{Q}\tilde{R}} \tag{3.17}$$

(3.16) and (3.17) $\Rightarrow A^{\tilde{R}} = A^{\tilde{R}\tilde{Q}\tilde{R}}$. *The other part can be proved similarly.*

3.3.3 Fuzzy Galois Lattice

In the previous subsection we defined the Galois connection operators for the case of fuzzy relations. Here we show that the set of fuzzy maximal rectangles of a fuzzy binary relation can be organized under a complete and distributive lattice by showing that:

- A partial order relation $\widetilde{\lll}$ exists for the set of fuzzy maximal rectangles of a fuzzy relation $\widetilde{T}_r$.
- An upperbound (supremum) and a lower bound (infinimum) exists in $\widetilde{T}_r$ for any subset of $\widetilde{T}_r$.

Proposition 3.6 *The following relation $\widetilde{\lll}$ defined on $\widetilde{R}$ is a partial order relation: $(A_1, \widetilde{B}_1) \widetilde{\lll} (A_2, \widetilde{B}_2) \Leftrightarrow A_1 \subseteq A_2$ and $\widetilde{B}_2 \subseteq \widetilde{B}_1$, where $(A_1, \widetilde{B}_1)$, $(A_2, \widetilde{B}_2) \in \widetilde{R}$.*

Proof 3.4 *According to the definition of a partial order relation, we have to prove that $\widetilde{\lll}$ is reflexive, antisymmetrical and transitive.*

- *Reflexivity*

  $\forall A_1 \subseteq E$, $\widetilde{B}_1 \subseteq \widetilde{F}$ *we have* $A_1 \subseteq A_1$ *and* $\widetilde{B}_1 \subseteq \widetilde{B}_1$ *(by reflexivity of $\subseteq$). Hence,* $(A_1, \widetilde{B}_1) \widetilde{\lll} (A_1, \widetilde{B}_1)$.

- *Antisymmetry*

  $(A_1, \widetilde{B}_1) \widetilde{\lll} (A_2, \widetilde{B}_2)$ *and* $(A_2, \widetilde{B}_2) \widetilde{\lll} (A_1, \widetilde{B}_1)$

$\Leftrightarrow A_1 \subseteq A_2$ *and* $\widetilde{B}_2 \subseteq \widetilde{B}_1$ *and* $A_2 \subseteq A_1$ *and* $\widetilde{B}_1 \subseteq \widetilde{B}_2$

$\Rightarrow A_1 = A_2$ *and* $\widetilde{B}_1 = \widetilde{B}_2$

$\Rightarrow (A_1, \widetilde{B}_1) = (A_2, \widetilde{B}_2)$.

- *Transitivity*

$(A_1, \widetilde{B}_1) \mathrel{\widetilde{\ll}} (A_2, \widetilde{B}_2)$ *and* $(A_2, \widetilde{B}_2) \mathrel{\widetilde{\ll}} (A_3, \widetilde{B}_3)$.

$\Leftrightarrow A_1 \subseteq A_2$ *and* $\widetilde{B}_2 \subseteq \widetilde{B}_1$ *and* $A_2 \subseteq A_3$ *and* $\widetilde{B}_3 \subseteq \widetilde{B}_2$

$\Rightarrow A_1 \subseteq A_3$ *and* $\widetilde{B}_3 \subseteq \widetilde{B}_1$

$\Leftrightarrow (A_1, \widetilde{B}_1) \ll (A_3, \widetilde{B}_3)$.

Hence $\widetilde{\ll}$ *is a partial order relation.*

Proposition 3.7 *Let* $\{(A_j, \widetilde{B}_j)\}$ *(with* $j \in J$*) be a set of fuzzy rectangular relations of a fuzzy binary relation* $\widetilde{R}$*. The relation* $(\cup_{j\in J} A_j) \times (\cap_{j\in J} \widetilde{B}_j)$ *is a rectangular relation of* $\widetilde{R}$*. Therefore* $(\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j)$ *is a rectangle of* $\widetilde{R}$*.*

Proof 3.5 *We have to prove that* $\forall (a,b) \in (\cup_{j\in J} A_j) \times (\cap_{j\in J} \widetilde{B}_j) \Rightarrow (a,b) \in \widetilde{R}$

$$\forall (a,b) \in (\cup_{j\in J} A_j) \times (\cap_{j\in J} \widetilde{B}_j) \quad \Rightarrow \quad a \in \cup_{j\in J} A_j \Rightarrow \exists k \in J \mid a \in A_k \qquad (3.18)$$

$$\forall (a,b) \in (\cup_{j\in J} A_j) \times (\cap_{j\in J} \widetilde{B}_j) \quad \Rightarrow \quad \mu_{\cap_{j\in J} \widetilde{B}_j}(b) = \min_{j\in J}(\mu_{\widetilde{B}_j}(b))$$

Hence, and particularly for k *we have*

$$\mu_{\cap_{j\in J} \widetilde{B}_j}(b) \leq \mu_{\widetilde{B}_k}(b) \qquad (3.19)$$

$$(3.18) \text{ and } (3.19) \Rightarrow \quad \mu_{(\cup_{j\in J} A_j)\times(\cap_{j\in J} \widetilde{B}_j)}(a,b) \leq \mu_{(A_k, B_k)}(a,b) \leq \mu_{\widetilde{R}}(a,b)$$

Since $(A_k, \widetilde{B}_k)$ *is a fuzzy rectangle of* $\widetilde{R} \Rightarrow (\cup_{j\in J} A_j) \times (\cap_{j\in J} \widetilde{B}_j)$ *is a fuzzy rectangle of* $\widetilde{R}$.

Proposition 3.8 *Let* $\{(A_j, \widetilde{B}_j)\}$ *(with* $j \in J$*) be a set of fuzzy rectangular relations of a fuzzy binary relation* $\widetilde{R}$*. The relation* $(\cap_{j\in J} A_j) \times (\cup_{j\in J} \widetilde{B}_j)$ *is a rectangular relation of* $\widetilde{R}$*. Therefore* $(\cap_{j\in J} A_j, \cup_{j\in J} \widetilde{B}_j)$ *is a fuzzy rectangle of* $\widetilde{R}$.

Proof 3.6 *Conversely follows from the proof of the previous proposition.*

Theorem 3.1 *Let* $\widetilde{R}$ *be a fuzzy binary relation defined from* E *to* $\widetilde{F}$ *and* T_r *the set of fuzzy rectangles of* $\widetilde{R}$ *ordered by the relation* $\widetilde{\lll}$*.* $(T_r, \widetilde{\lll})$ *is a complete lattice with the supremum* P *and the infimum* H *as follows:*

$$P_{j\in J}(A_j, \widetilde{B}_j) = (\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j) \tag{3.20}$$

$$H_{j\in J}(A_j, \widetilde{B}_j) = (\cap_{j\in J} A_j, \cup_{j\in J} \widetilde{B}_j) \tag{3.21}$$

Proof 3.7 *First, let us show that any set of fuzzy rectangles of* $\widetilde{R}$ *has a smallest superior boundary and a biggest inferior one which are both fuzzy rectangles of* $\widetilde{R}$.

- *Smallest Superior Boundary*

  $\forall j \in J,\ (A_j, \widetilde{B}_j) \mathrel{\widetilde{\lll}} (C, \widetilde{D})$

  $\Leftrightarrow \forall j \in J,\ (A_j \subseteq C)$ *and* $(\widetilde{D} \subseteq \widetilde{B}_j)$

  $\Leftrightarrow \cup_{j\in J} A_j \subseteq C$ *and* $\widetilde{D} \subseteq \cap_{j\in J} \widetilde{B}_j$

  $\Leftrightarrow (\cup_{j\in J} A_j, \cap_{j\in J} \widetilde{B}_j) \mathrel{\widetilde{\lll}} (C, \widetilde{D})$

The rectangle $(\cup_{j\in J}A_j, \cap_{j\in J}\widetilde{B}_j)$ *is a fuzzy rectangle of* T_r.

Infact, according to proposition 3.7, $P_{j\in J}(A_j, \widetilde{B}_j) = (\cup_{j\in J}A_j, \cap_{j\in J}\widetilde{B}_j)$, *is the smallest superior boundary.*

- *Biggest Inferior Boundary*

  We have to prove that any set of fuzzy rectangles of T_r *has a biggest inferior boundary.*

  Let $(A_j, \widetilde{B}_j)$ *be a fuzzy rectangle of* T_r :

  $\forall j \in J$, $(C, \widetilde{D}) \mathrel{\widetilde{\ll}} (A_j, \widetilde{B}_j)$

  $\Leftrightarrow \forall j \in J$, $(C \subseteq A_j)$ *and* $(\widetilde{B}_j \subseteq \widetilde{D})$

  $\Leftrightarrow C \subseteq \cap_{j\in J}A_j$ *and* $\cup_{j\in J}\widetilde{B}_j \subseteq \widetilde{D}$

  $\Leftrightarrow (C, \widetilde{D}) \mathrel{\widetilde{\ll}} (\cap_{j\in J}A_j, \cup_{j\in J}\widetilde{B}_j)$

- *The rectangle* $(\cap_{j\in J}A_j, \cup_{j\in J}\widetilde{B}_j)$ *is a fuzzy rectangle of* T_r.

  Infact, according to proposition 3.8, $H_{j\in J}(A_j, \widetilde{B}_j) = (\cap_{j\in J}A_j, \cup_{j\in J}\widetilde{B}_j)$, *is the biggest inferior boundary.*

3.4 Theoretical Properties

This section explores the theoretical properties of a fuzzy Galois lattice. Here we state two important properties of a fuzzy Galois lattice: Inheritance and Incremental BuildUp.

3.4.1 Inheritance

In the previous sections we saw that the fuzzy maximal rectangles $(A_i, \widetilde{B}_i)$ of a fuzzy binary relation $\widetilde{R}$ can be ordered by a subconcept-superconcept relation $\widetilde{\lll}$ into a complete lattice. This relationship $\widetilde{\lll}$ is defined as follows:

$$(A_1, \widetilde{B}_1) \widetilde{\lll} (A_2, \widetilde{B}_2) \text{ iff } A_1 \subseteq A_2 \text{ and } \widetilde{B}_2 \subseteq \widetilde{B}_1$$

The above relationship consists of redundant elements. For example A_1 is repeated in A_2 and $\widetilde{B}_2$ is repeated in $\widetilde{B}_1$. If we define new sets in which only the non-repeated elements are shown then we would have a more compact way of representing the above relationship. Thus for example, we could eliminate repeated elements from A_2 by representing the set $A_2 \backslash A_1$. Similar is the case with $\widetilde{B}_1$, i.e., we could as well represent it by $\widetilde{B}_1 \backslash \widetilde{B}_2$. The relationship would be still valid although the representation would become

$$(A_1, \widetilde{B}_1 \backslash \widetilde{B}_2) \widetilde{\lll} (A_2 \backslash A_1, \widetilde{B}_2)$$

The above simplification leads us to a property called inheritance. It leads to a reduced representation of a Galois lattice structure. For all ordered sets, it is understood that as we move from lower orders to higher ones, the object sets include all the lower order objects. Similarly as we move from higher orders to lower orders the attribute sets include all the higher order attributes.

Inheritance leads us to a much simpler representation of a Galois lattice. It also aids in discovering knowledge from data.

3.4.2 Incremental Buildup

Here we prove that in any incremental buildup of a fuzzy Galois lattice of a fuzzy binary relation $\widetilde{R}_n$, concepts that have not appeared in $\widetilde{R}_{n-1}$ would also not appear in $\widetilde{R}_n$. Not only this but those extents that have not appeared in $\widetilde{R}_{n-1}$, they will not appear with the new object o_n appended to them.

Proposition 3.9 *Let $\widetilde{R}_{n-1}$ be a fuzzy binary relation with $n-1$ objects, and let $\widetilde{R}_n$ be a fuzzy binary relation with n objects such that $\widetilde{R}_n = \widetilde{R}_{n-1} \cup \{o_n\} \times \{p_1/\alpha_{n1}, p_2/\alpha_{n2}, \ldots, p_m/\alpha_{nm}\}$. If $\{o_j\}$ is not an extent of $\widetilde{R}_{n-1}$ then $\{o_j, o_n\}$ is not an extent of $\widetilde{R}_n$.*

Proof 3.8 *If $\{o_j\}$ is not an extent of $\widetilde{R}_{n-1}$ this means that $\{o_j\} \times \{p_1/\alpha_{j1}, p_2/\alpha_{j2}, \ldots, p_m/\alpha_{jm}\}$ is not a concept in $\widetilde{R}_{n-1}$. Therefore the smallest concept containing o_j must have at least one more object. Let this concept have an extent $\{o_j, o_k\}$.*

The above condition implies that

$$\{o_j\}^{\widetilde{R}\widetilde{Q}} = \{o_j, o_k\} \tag{3.22}$$

Obviously, the membership degree for each attribute of o_j must be less than that for o_k, i.e., $\{o_j, o_k\}$ is the smallest extent containing o_j if and only if

$$\alpha_{ki} \geq \alpha_{ji} \tag{3.23}$$

Let us now suppose that $\{o_j, o_n\}$ is an extent of $\widetilde{R}_n$. Applying the Galois con-

nection operator to the set $\{o_j, o_n\}$ *we get, usin equation 3.23*

$$\{o_j, o_n\}^{\widetilde{R}\widetilde{Q}} = \{o_j, o_k, o_n\} \tag{3.24}$$

This means that the smallest extent containing o_j *and* o_n *in* $\widetilde{R}_n$ *also contains* o_k. *This proves proposition 3.9.*

Proposition 3.10 *If* $\{o_j, o_n\}$ *is an extent of* $\widetilde{R}_n$*, then* $\{o_j\}$ *is an extent of* $\widetilde{R}_{n-1}$.

Proof 3.9 *Conversely follows from the proof of propostion 3.9.*

The significance of the above results is that we don't need to test for all combinations of object sets in order to generate concepts for $\widetilde{R}_n$. Instead we could utilise the extents of $\widetilde{R}_{n-1}$. Furthermore, we get an upper bound on the maximum number of concepts for $\widetilde{R}_n$ which obviously cannot be more than twice the number of concepts in $\widetilde{R}_{n-1}$.

3.5 Complexity Analysis

3.5.1 Space Complexity

Let us investigate what could be the worst case space requirement for the fuzzy Galois lattice of a fuzzy binary relation.

In the subsequent discussion o_i stands for any object and p_i stands for any attribute of a fuzzy relation.

Consider an incremental development of the fuzzy Galois lattice of $\widetilde{R}_{n-1}$. Let $S(n-1)$ denote the space requirement (the number of fuzzy maximal rectangles contained in the fuzzy Galois lattice of $n-1$ objects). Let us append another object o_n with the attributes $\{p_1/\alpha_{n1}, p_2/\alpha_{n2}, \dots, p_m/\alpha_{nm}\}$. to $\widetilde{R}_{n-1}$. Let the resulting relation be called $\widetilde{R}_n$.

3.5.2 Worst Case Analysis

In the worst case, the maximum number of fuzzy maximal rectangles in $\widetilde{R}_n$ would be twice that of those in $\widetilde{R}_{n-1}$. To see this, observe that in the worst case all the fuzzy maximal rectangles in $\widetilde{R}_{n-1}$ would also be in $\widetilde{R}_n$. Furthermore, the new object o_n would be appended to the object set of each concept of $\widetilde{R}_{n-1}$ and tested if it is forms a fuzzy maximal rectangle or not. Suppose further that the newly generated fuzzy maximal rectangles are all included in the fuzzy Galois lattice and such that none of the existing rectangles is replaced.

In that case the maximum number of fuzzy maximal rectangles are given by:

Initial number of fuzzy maximal rectangles = $S(n-1)$

Maximal number of newly generated fuzzy maximal rectangles = $S(n-1)$

The recurrence relation for the number of fuzzy maximal rectangles in the worst case is:

$$S(n) = 2S(n-1) \tag{3.25}$$

As $S(0) = 1$, so the solution to Recurrence 3.25 is $S(n) = 2^n$.

This quantity represents the maximum number of fuzzy maximal rectangles that could be present in a fuzzy Galois lattice.

3.5.3 Average Case Analysis

Suppose that the maximum number of fuzzy maximal rectangles generated in the fuzzy Galois lattice of n objects is not the same as $S(n-1)$. In other words, if each time an object is added, a constant fraction α_i of the rectangles is being discarded (because they are no longer maximal), then the recurrence reduces to

$$S(n) = (2 - \alpha_{n-1})S(n-1) \tag{3.26}$$

With $S(0) = 1$, the solution to the above recurrence becomes,

$$S(n) = \prod_{i=0}^{n-1}(2 - \alpha_i) \tag{3.27}$$

Equation 3.27 represents an exponential function for $0 \le \alpha \le 1$. *The above analysis clearly reveals that the number of fuzzy maximal rectangles generated in a fuzzy Galois lattice is an exponential function of the number of objects.*

3.5.4 Best Case Analysis

In particular if $\alpha_i = 1$, $0 \le i \le n$, then $S(n) = 1$ which is a constant.

This only happens when the added object o_n has membership degree for each attribute:

- CASE 1: Either greater than or equal to the maximum membership degree for each attribute OR
- CASE 2: Either less than or equal to the minimum membership degree for each attribute.

In both the above cases, the Galois lattice structure can be built incrementally from the existing structure in constant time.

3.5.5 Lower Bound on Time Complexity

From the above analysis on space complexity is clear that generation of fuzzy Galois lattice is a mapping from $n \rightarrow 2^n$ distinct elements in the worst case. If we consider the generation of each fuzzy maximal rectangle as an operation requiring unit time, the per unit construction time in the worst case cannot be less than $2^n/n$. So that we have the lower limit as: $O(2^n/n)$.

3.6 An Incremental Algorithm

Our proposed algorithm for generating a fuzzy Galois lattice is incremental in nature. We propose to build a Galois lattice structure in two steps.

3.6.1 Generation Step

In the generation step we generate the fuzzy maximal rectangles for $\widetilde{R}_n$ by first generating the fuzzy maximal rectangles for $\widetilde{R}_1$. From it, we generate the fuzzy maximal rectangles for $\widetilde{R}_2$ and so on until the fuzzy maximal rectangles for $\widetilde{R}_n$ are generated.

Each new fuzzy maximal rectangle for $\widetilde{R}_n$ would:

- Either be already present in $\widetilde{R}_{n-1}$.
- Be a modified form of an existing fuzzy maximal rectangle $\widetilde{R}_{n-1}$. This modification occurs due to the addition of a new object o_n to $\widetilde{R}_{n-1}$. From proposition 3.10 we can conclude that every rectangle with o_n in $\widetilde{R}_n$ must have a counterpart without o_n in $\widetilde{R}_{n-1}$.

  Hence the fuzzy Galois lattice contains only the above-mentioned fuzzy maximal rectangles.

The recurrence for this step is the same as that for the average case presented in the previous section. Therefore,

$$T_{generation} = \prod_{i=0}^{n-1}(2 - \alpha_i) \tag{3.28}$$

3.6.2 Ordering Step

Having generated the set of fuzzy maximal rectangles, we now have to order them into an ordered set according to the subconcept-superconcept relation ($\widetilde{\lll}$). This ordering step would take time proportional to the number of elements of the newly obtained lattice elements.

The problem of ordering of fuzzy maximal rectangles is quite similar to the problem of sorting of L elements. Each element (fuzzy maximal rectangle) must be compared with all the other elements present to determine its rank. Therefore the number of comparisons would be:

Number of Comparisons $\leq \underbrace{L + L + ... + L}_{L elements} = L^2$

This ordering step would require time of the order of $O(L^2)$ where L is the number of fuzzy maximal rectangles of fuzzy Galois lattice of n elements.

This ordering step is carried out only after all the generated fuzzy maximal rectangles are obtained.

3.6.3 Overall Time Complexity

Figure 3.1 shows the flowchart corresponding to the above algorithm.

The overall time complexity would be:

$T(n) = \prod_{i=0}^{n-1}(2 - \alpha_i) + L^2$

Obviously $T(n)$ is an exponential function.

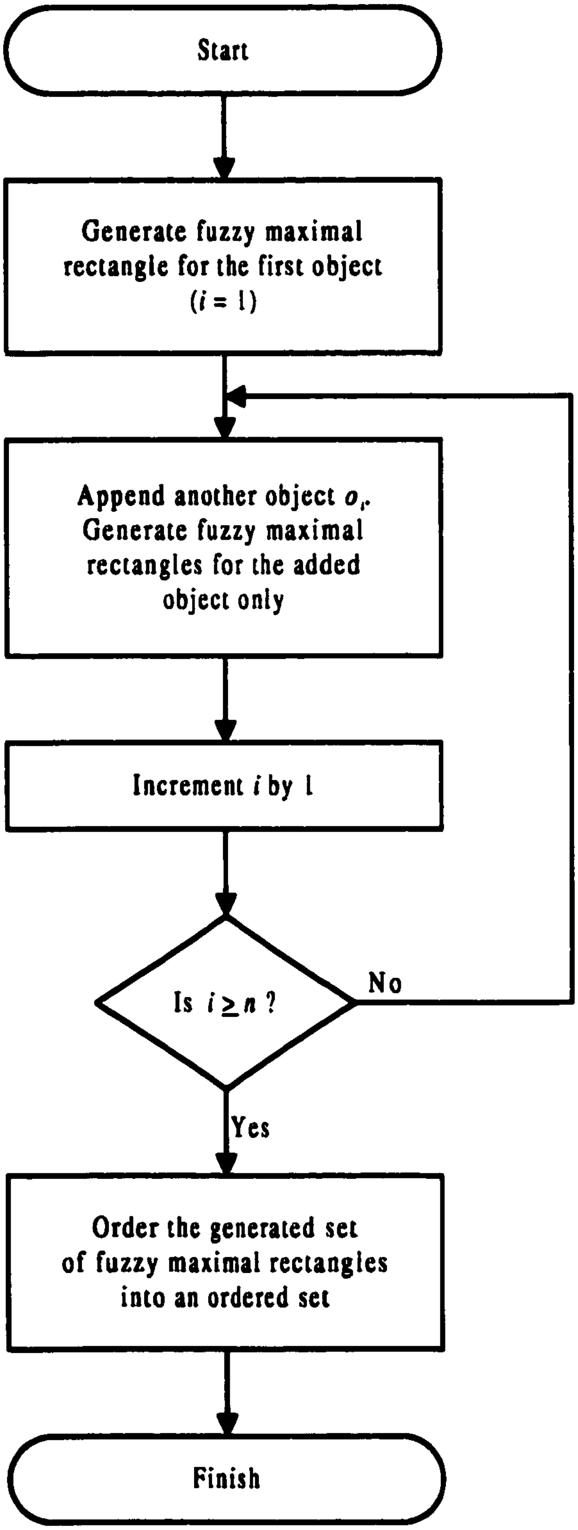

Figure 3.1: Incremental Algorithm for Fuzzy Galois Lattice Generation

Example 3.2 *We'll use the incremental approach to build a fuzzy Galois lattice from the following fuzzy relation:*

	p_1	p_2	p_3	p_4	*Class*
o_1	0.5	1	0.7	0.5	*C1*
o_2	0.6	0.7	1	0.5	*C2*
o_3	1	0.9	1	0.1	*C3*
o_4	1	0.9	0.9	0.1	*C3*

Table 3.1: The fuzzy binary relation $\widetilde{W}$

Note that $\widetilde{W}$ has four objects as well as four attributes.

GENERATION STEP:

Suppose that we have the set of fuzzy maximal rectangles of the context corresponding to o_2, o_3 and o_4, which has already been obtained incrementally. Table 3.2 shows the set of fuzzy concepts for the context with three objects o_2, o_3, o_4.

Fuzzy Concept
$\emptyset \times \{p_1/1.0, p_2/0.9, p_3/1.0, p_4/0.5\}$
$\{o_2\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.5\}$
$\{o_3\} \times \{p_1/1.0, p_2/0.9, p_3/1.0, p_4/0.1\}$
$\{o_2, o_3\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.1\}$
$\{o_2, o_4\} \times \{p_1/0.6, p_2/0.7, p_3/0.9, p_4/0.1\}$
$\{o_3, o_4\} \times \{p_1/1.0, p_2/0.9, p_3/0.9, p_4/0.1\}$
$\{o_2, o_3, o_4\} \times \{p_1/0.6, p_2/0.7, p_3/0.9, p_4/0.1\}$

Table 3.2: The fuzzy concepts in $\widetilde{W}$ with three objects

Observe that the object set $\{o_4\}$ is not present in the Table 3.2 as it does not form a fuzzy maximal rectangle.

Now we append o_1 to our object set in order to generate the context $\widetilde{W}$.

Instead of generating ($2^4 = 16$) maximal rectangles, we can only generate 7 new rectangles and utilise 7 others listed in Table 3.2. The newly generated 7 rectangles would be generated by appending o_1 to the rectangles shown in Table 3.2 and tested to see if they form a concept. (The only exception to this rule is the rectangle with the empty object set). The new set of maximal rectangles is generated and redundant rectangles which are no longer maximal are removed (for example, after appending o_1 to the rectangle $\{o_2, o_4\}$, the newly generated rectangle with the object set $\{o_1, o_2, o_4\}$ is no longer maximal as the rectangle with the object set $\{o_1, o_2, o_3, o_4\}$ has superseded it).

The final list of fuzzy maximal rectangles is shown in the Table 3.2. Observe that there are L = 10 maximal rectangles, hence we discarded 4 existing/newly generated rectangles (7+7-10 = 4). Hence for this step $\alpha = 4/7 = 57\%$.

Label	*Fuzzy Concept*
FC_0	$\emptyset \times \{p_1/1.0, p_2/1.0, p_3/1.0, p_4/0.5\}$
FC_1	$\{o_1\} \times \{p_1/0.5, p_2/1.0, p_3/0.7, p_4/0.5\}$
FC_2	$\{o_2\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.5\}$
FC_3	$\{o_3\} \times \{p_1/1.0, p_2/0.9, p_3/1.0, p_4/0.1\}$
FC_4	$\{o_1, o_2\} \times \{p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.5\}$
FC_5	$\{o_2, o_3\} \times \{p_1/0.6, p_2/0.7, p_3/1.0, p_4/0.1\}$
FC_6	$\{o_3, o_4\} \times \{p_1/1.0, p_2/0.9, p_3/0.9, p_4/0.1\}$
FC_7	$\{o_1, o_3, o_4\} \times \{p_1/0.5, p_2/0.9, p_3/0.7, p_4/0.1\}$
FC_8	$\{o_2, o_3, o_4\} \times \{p_1/0.6, p_2/0.7, p_3/0.9, p_4/0.1\}$
FC_9	$\{o_1, o_2, o_3, o_4\} \times \{p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.1\}$

Table 3.3: The fuzzy concepts in $\widetilde{W}$

ORDERING STEP:

The ordering step basically consists of determining which pairs of fuzzy concepts share the subconcept-superconcept relation $\widetilde{\ll}$. *Obviously, in the worst case, it would take* $O(L^2)$ *steps.*

The fuzzy Galois lattice for the context $\widetilde{W}$ *is shown in figure 3.2.*

It can easily be verified that if now a new object o_5 *with attributes* $\{p_1/1.0, p_2/1.0, p_3/1.0, p_4/0.5+\}$ *(having attribute values greater than all existing attribute values) is appended then the fuzzy Galois lattice can be generated in constant time.*

Similar is the case if we append o_6 *with attribute values less than or equal to existing attribute values.*

INHERITANCE:

Figure 3.2 shows the reduced Galois lattice structure for $\widetilde{W}$. *We can immediately observe that a few association rules can be read directly from it.*

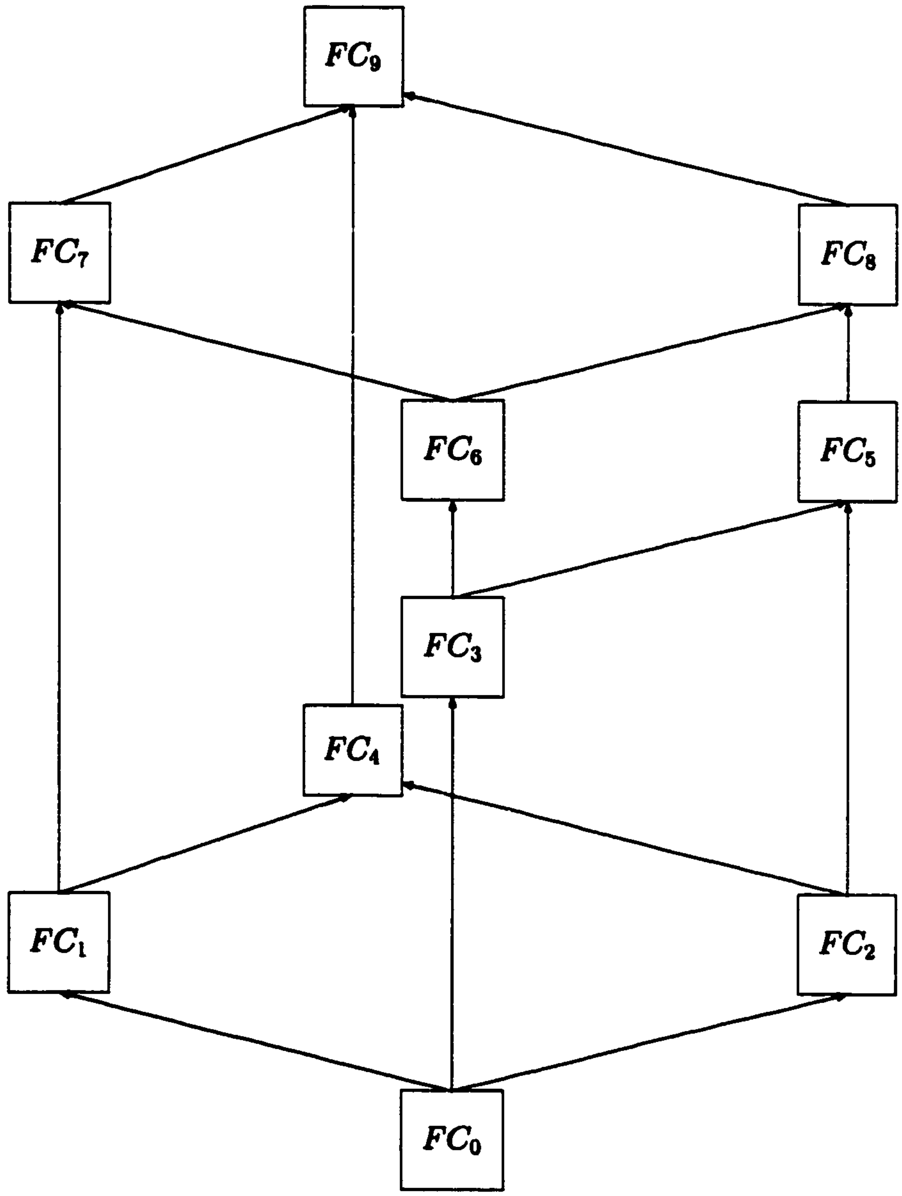

Figure 3.2: Fuzzy Galois lattice of $\widetilde{W}$

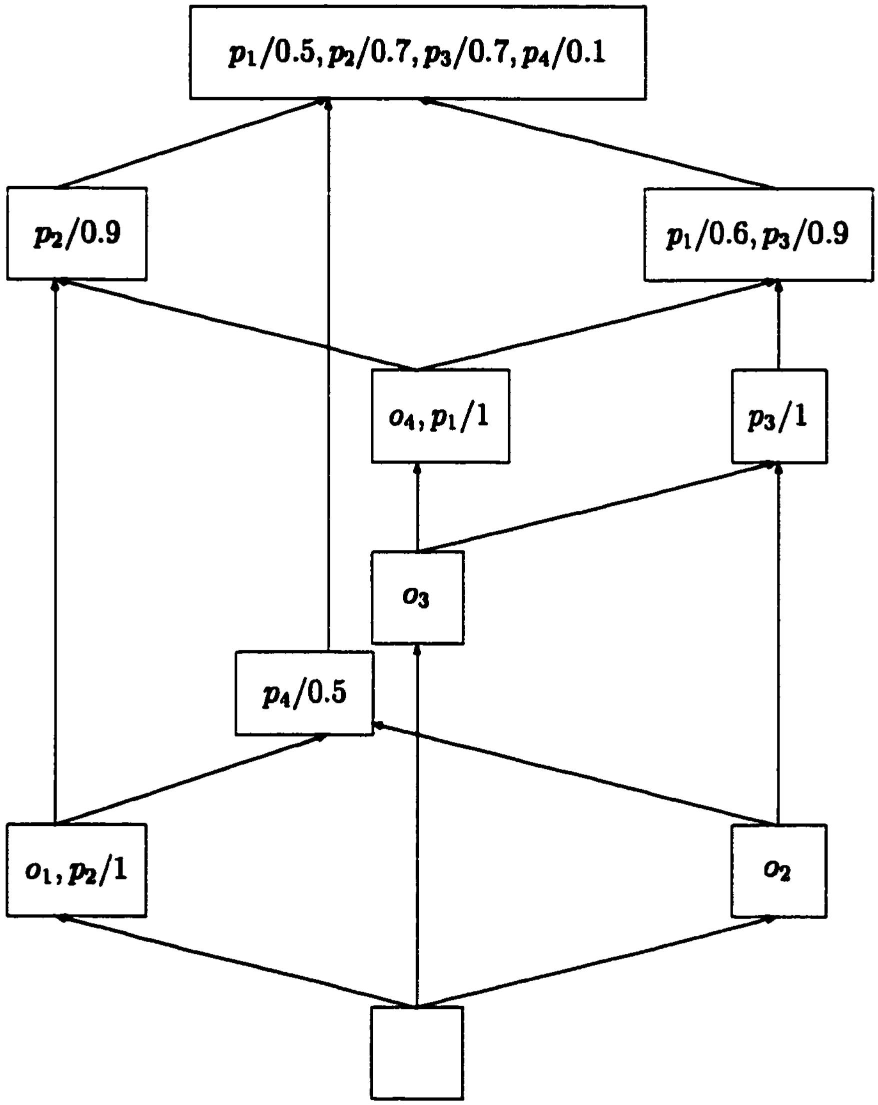

Figure 3.3: Reduced fuzzy Galois lattice of $\widetilde{W}$

Chapter 4

KNOWLEDGE EXTRACTION

Knowledge Extraction deals with extracting implications from initial data. In knowledge extraction *association rules* or *implications* of the form $P \to C$ are extracted from a context, where P is the premise and C is the conclusion. Both sets P and C are subsets of the range or attribute set of the given context. This chapter provides an extension of knowledge extraction to fuzzy contexts.

In the following section we give an explanation of knowledge extraction in the crisp case.

4.1 Knowledge Extraction from Crisp Contexts

Below we state the basic terms and definitions of knowledge extraction as applied to crisp contexts [4]:

4.1.1 Implications

A crisp implication of the form $A \rightarrow B$ for a crisp context R indicates the statement that *"Every object of R having attributes given by attribute set A will also have attributes given by attribute set B."* Here A and B are subsets of the $range(R) = M$ which is the complete attribute set of R. The set A is called the **premise** and B is known as the **conclusion.**

A subset of attributes $T \subseteq M$ **respects** an implication $A \rightarrow B$ if $A \not\subseteq T$ or $B \subseteq T$. This definition is the same as used in formal logic.

We get a simpler explanation of the above definition if we take the contrapositive: A subset of attributes T **does not respect** an implication $A \rightarrow B$ if $A \subseteq T$ and $B \not\subseteq T$. In simpler words it means that any set of attributes respects a given implication $A \rightarrow B$ except for the case when the premise is true but the conclusion is false. T **respects a set of implications** $\mathfrak{R}$ if T respects every single implication in $\mathfrak{R}$. An implication $A \rightarrow B$ **follows (semantically)** from a set $\mathfrak{R}$ of implications between attributes if each subset of the attribute set M respecting $\mathfrak{R}$ also respects $A \rightarrow B$.

$A \rightarrow B$ **holds in a set** $\{T_1, T_2, \ldots\}$ of subsets if each of the subsets T_i respects the implication $A \rightarrow B$. $A \rightarrow B$ **holds in a context** R if it holds in the system of object intents. This means that the attribute sets of all the concepts in R must respect $A \rightarrow B$. In this case, we also say that $A \rightarrow B$ is **an implication of the context** R.

Proposition 4.1 *[4] An implication $A \to B$ holds in a crisp context R if and only if $B \subseteq A^{RQ}$.*

4.1.2 Properties of Implication Sets

A family of implications $\Re$ is called **closed** if every implication following from $\Re$ is already contained in $\Re$. A set of implications $\Re$ of a context R is called **complete** if every implication from R follows from $\Re$.

For the present work, we will be concerned with complete implication sets as they are applied to extract knowledge from crisp contexts.

A set of implications $\Re$ is called **non-redundant** if none of the implications follows from others. Guiges and Duquenne [5] have shown that there is a natural complete and non-redundant set of implications for every context with a finite attribute set M.

4.1.3 Pseudo-Intents

Proposition 4.1 provides us with a first yet inefficient method of extracting implications from a crisp context R. The resulting implication set although complete, has redundant implications.

The notion of a **pseudo-intent** [4] helps in finding out a complete non-redundant set of implications for a given crisp context R. To put iT simply, a pseudo-intent is any attribute set P such that $P \neq P^{RQ}$. The set of pseudo-intents is also an ordered set. Below we provide a recursive definition of a pseudo-intent.

Definition 4.1 *$P \subseteq M$ is a* **pseudo-intent** *of R if and only if $P \neq P^{RQ}$ and for every pseudo-intent $Q \subset P$, $Q^{RQ} \subseteq P$.*

Proposition 4.2 *The set of implications:*

$$\Re = \{P \rightarrow P^{RQ} \mid P \text{ is a pseudo-intent of } R\} \tag{4.1}$$

is non-redundant and complete.

In practice the implications are not atated in the form $P \rightarrow P^{RQ}$ but in the form $P \rightarrow (P^{RQ} \setminus P)$. This form of stating a complete non-redundant set of implications between attributes is called **Duquenne-Guiges-Basis** or simply the **stem base** of the attribute implications.

4.2 Knowledge Extraction in fuzzy contexts

Below we state an extension of basic terms and definitions of knowledge extraction as applied to fuzzy countexts.

4.2.1 Fuzzy Implications

A fuzzy implication of the form $p_1/\alpha_1 \rightarrow p_2/\alpha_2$ indicates the statement that *"Every object with attribute p_1 having membership degree α_1 will also have attribute p_2 with membership degree α_2."*

Formally, a fuzzy implication between attributes is a pair of subsets of the attribute set $\widetilde{M}$. It is denoted by $\widetilde{A} \rightarrow \widetilde{B}$. It indicates the statement that *"Every object of $\widetilde{R}$ having attributes given by attribute set $\widetilde{A}$ will also have attributes given by attribute set $\widetilde{B}$."* The set A is called the **premise** and B is known as the **conclusion.**

A fuzzy subset of attributes $\widetilde{T} \subseteq \widetilde{M}$ **respects** a fuzzy implication $\widetilde{A} \rightarrow \widetilde{B}$ if $\widetilde{A} \not\subseteq \widetilde{T}$ or $\widetilde{B} \subseteq \widetilde{T}$.

We get a simpler explanation of the above definition if we take the contrapositive: A subset of attributes $\widetilde{T}$ **does not respect** an implication $\widetilde{A} \rightarrow \widetilde{B}$ if $\widetilde{A} \subseteq \widetilde{T}$ and $\widetilde{B} \not\subseteq \widetilde{T}$. In simpler words it means that any set of attributes respects a given implication $\widetilde{A} \rightarrow \widetilde{B}$ except for the case when the premise is true but the conclusion is false. $\widetilde{T}$ **respects a set of implications** $\widetilde{\mathfrak{R}}$ if $\widetilde{T}$ respects every single implication in $\widetilde{\mathfrak{R}}$.

$\widetilde{T}$ **respects a set** $\widetilde{\mathfrak{R}}$ of implications if $\widetilde{T}$ respects every single implication in $\widetilde{\mathfrak{R}}$. $\widetilde{A} \rightarrow \widetilde{B}$ **holds** in a set $\{\widetilde{T}_1, \widetilde{T}_2, \ldots\}$ of subsets if each of the subsets $\widetilde{T}_i$ respects the implication $\widetilde{A} \rightarrow \widetilde{B}$. $\widetilde{A} \rightarrow \widetilde{B}$ **holds in a fuzzy context** if it holds in the system of object intents. In this case we also say that $\widetilde{A} \rightarrow \widetilde{B}$ is an **implication of the fuzzy context** or equivalently that within the fuzzy context $\widetilde{A}$ is **a premise** of $\widetilde{B}$.

Proposition 4.3 *An implication $\widetilde{A} \rightarrow \widetilde{B}$ holds in a fuzzy context $\widetilde{R}$ if and only if $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.*

Proof 4.1 *Here we have to prove that $\widetilde{A} \rightarrow \widetilde{B} \Leftrightarrow \widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.*

Let's first prove that $\widetilde{A} \rightarrow \widetilde{B} \Rightarrow \widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.

If $\widetilde{A} \rightarrow \widetilde{B}$ *holds in a fuzzy context, this means that all object intents given by* $\{\widetilde{T}_1, \widetilde{T}_2, \ldots\}$ *respect the implication* $\widetilde{A} \rightarrow \widetilde{B}$. *Therfore,* $0 \leq i \leq n$,

$$\widetilde{A} \not\subseteq \widetilde{T}_i \text{ or } \widetilde{B} \subseteq \widetilde{T}_i \tag{4.2}$$

CASE 1: Suppose that $\widetilde{A} \not\subseteq \widetilde{T}_i$, *then* $\widetilde{A}$ *is not a subset of any object intent, or in other words, it is not a subset of any attribute set of a concept. This can be possible in two ways: either* $\widetilde{A} \supset \widetilde{T}_i$, *or* $\widetilde{A}$ *and* $\widetilde{T}_i$ *are disjoint. In either of these cases,*

$$\Rightarrow \widetilde{A}^{\widetilde{R}} = \phi \tag{4.3}$$

$$\Rightarrow \widetilde{A}^{\widetilde{R}\widetilde{Q}} = \widetilde{M} \tag{4.4}$$

From the above results, we can conclude that $\widetilde{B} \subseteq \widetilde{M} = \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.

Hence $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.

CASE 2: Suppose now that the other condition specified in equation 4.2 is true, i.e., $\widetilde{B} \subseteq \widetilde{T}_i$. *In this case again we have two sub-cases. The first case is that* $\widetilde{A} \not\subseteq \widetilde{T}_i$ *which we have already covered in CASE 1. The second sub-case would be when* $\widetilde{A} \subseteq \widetilde{T}_i$. *In this case* $\widetilde{A}^{\widetilde{R}\widetilde{Q}} = \widetilde{T}_i$ *which is an object intent.*

But since $\widetilde{B} \subseteq \widetilde{T}_i$, *and* $\widetilde{A}^{\widetilde{R}\widetilde{Q}} = \widetilde{T}_i$ *which implies that*

$$\Rightarrow \widetilde{B} \subseteq \widetilde{T}_i = \widetilde{A}^{\widetilde{R}\widetilde{Q}} \tag{4.5}$$

Hence $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.

This proves the first direction.

Now we have to prove that if $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$, then $\widetilde{A} \to \widetilde{B}$ holds in a fuzzy context.

The proof of this part is trivial. Observe that if $\widetilde{A}$ is any attribute set, then $\widetilde{A}^{\widetilde{R}\widetilde{Q}}$ must be an object intent. So $\widetilde{A}^{\widetilde{R}\widetilde{Q}} = \widetilde{T}_i$. Therefore, $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}} = \widetilde{T}_i$, hence $\widetilde{B} \subseteq \widetilde{T}_i$ which means that $\widetilde{T}_i$ respects $\widetilde{A} \to \widetilde{B}$. This proves the other direction.

The above proposition provides us with a first yet trivial way of extracting knowledge from a context. We know that $\widetilde{A} \to \widetilde{B}$ holds in a fuzzy context if and only if $\widetilde{B} \subseteq \widetilde{A}^{\widetilde{R}\widetilde{Q}}$.

Hence for any given context we can find out all the possible implications by finding all subsets of object intents.

4.2.2 Properties of fuzzy implication sets

A fuzzy implication $\widetilde{A} \to \widetilde{B}$ **follows (semantically)** from a set $\widetilde{\mathfrak{R}}$ of implications between fuzzy attributes if each subset of $\widetilde{M}$ respecting $\widetilde{\mathfrak{R}}$ also respects $\widetilde{\mathfrak{R}}$. A set $\widetilde{\mathfrak{R}}$ of implications of a fuzzy context $\widetilde{R}$ is called **complete** if every implication of $\widetilde{R}$ follows from $\widetilde{\mathfrak{R}}$.

A set $\widetilde{\mathfrak{R}}$ of fuzzy implications is called **non-redundant** if none of the implications follows from others.

4.2.3 Fuzzy Pseudo-intents

In order to determine a complete and non-redundant set of implications for a fuzzy context $\widetilde{R}$ we attempt to extend the notion of pseudo-intents to fuzzy contexts.

Let us define $\widetilde{P} \subseteq \widetilde{M}$ as the **fuzzy pseudo-intent** of $\widetilde{R}$ if $\widetilde{P} \neq \widetilde{P}^{\widetilde{R}\widetilde{Q}}$ and $\widetilde{Q}^{\widetilde{R}\widetilde{Q}} \subseteq \widetilde{P}$ holds for every fuzzy pseudo-intent $\widetilde{Q} \subseteq \widetilde{P}$, $\widetilde{Q} \neq \widetilde{P}$.

Proposition 4.4 *The set of fuzzy implications:*

$$\widetilde{\Re} = \{\widetilde{P} \rightarrow \widetilde{P}^{\widetilde{R}\widetilde{Q}} \mid \widetilde{P} \text{ is a pseudo-intent of } \widetilde{R}\} \tag{4.6}$$

is non-redundant and complete.

Proof 4.2 *In order to show that $\widetilde{\Re}$ is complete we have to show that every set $\widetilde{T}$ respecting $\widetilde{\Re}$ is an intent. Let $\widetilde{Q}$ be a fuzzy pseudo-intent.*

$\widetilde{T}$ respects all implications of the form $\widetilde{Q} \rightarrow \widetilde{Q}^{\widetilde{R}\widetilde{Q}}$ when $\widetilde{Q} \subseteq \widetilde{T}$. Let us suppose that $\widetilde{T}$ is not an intent, that is, $\widetilde{T} \neq \widetilde{T}^{\widetilde{R}\widetilde{Q}}$. Therefore, $\widetilde{T}$ is a fuzzy pseudo-intent according to the definition of a fuzzy pseudo-intent. For the sake of simplicity let us substitute $\widetilde{S}$ in place of $\widetilde{T}$ for stating the implication. Hence,

$$\widetilde{S} = \widetilde{T} \tag{4.7}$$

As $\widetilde{T}$ is a fuzzy pseudo-intent, so

$$\widetilde{T} \neq \widetilde{T}^{\widetilde{R}\widetilde{Q}} \tag{4.8}$$

$$\Rightarrow \quad \widetilde{T} \subset \widetilde{T}^{\widetilde{R}\widetilde{Q}} \tag{4.9}$$

$$\Rightarrow \quad \widetilde{T}^{\widetilde{R}\widetilde{Q}} \not\subseteq \widetilde{T} \tag{4.10}$$

We can see that $\widetilde{T}$ does not respect the implication $\widetilde{S} \rightarrow \widetilde{S}^{\widetilde{R}\widetilde{Q}}$ because the premise is true and the conclusion is false. To see this, we observe that $\widetilde{T} \subseteq \widetilde{S}$ but $\widetilde{T} \nsubseteq \widetilde{S}^{\widetilde{R}\widetilde{Q}}$ from equations 4.7 and 4.10.

This result is a contradiction that $\widetilde{T}$ does not respect $\widetilde{S} \rightarrow \widetilde{S}^{\widetilde{R}\widetilde{Q}}$ which is the same as $\widetilde{T} \rightarrow \widetilde{T}^{\widetilde{R}\widetilde{Q}}$. Hence the underlying assumption must be false, i.e., $\widetilde{T}$ is a fuzzy pseudo-intent. Therefore $\widetilde{T}$ is an intent and hence every attribute subset $\widetilde{T}$ respecting $\widetilde{\Re}$ is an intent.

In order to show that $\widetilde{\Re}$ is non-redundant, we consider an arbitrary pseudo-intent $\widetilde{P}$ and show that $\widetilde{P}$ respects the set $\widetilde{\Re} \setminus \{\widetilde{P} \rightarrow \widetilde{P}^{\widetilde{R}\widetilde{Q}}$. Infact, if $\widetilde{Q} \rightarrow \widetilde{Q}^{\widetilde{R}\widetilde{Q}}$ is an implication in $\widetilde{\Re} \setminus \{\widetilde{P} \rightarrow \widetilde{P}^{\widetilde{R}\widetilde{Q}}$, with $\widetilde{Q} \subseteq \widetilde{P}$ then $\widetilde{Q}^{\widetilde{R}\widetilde{Q}} \subseteq \widetilde{P}$ must hold since $\widetilde{P}$ is a pseudo-intent.

In the above proposition we have provided an extension to the crisp Duquenne Guiges Basis to fuzzy contexts. Henceforth, we define Fuzzy Duquenne-Guiges Basis as the set of implications, written in the form $\widetilde{\Re} \setminus \{ \widetilde{P} \rightarrow \widetilde{P}^{\widetilde{R}\widetilde{Q}}\}$. It can also be defined as the fuzzy stem base.

We now propose an algorithm to compute this complete and non-redundant set of implications.

4.3 Algorithm for Computing Fuzzy Stem Base

In this section, we propose an algortihm for computing the fuzzy stem base or the Fuzzy Duquenne-Guiges Basis of a given fuzzy context $\widetilde{R}$. From the defintion of the pseudo-intent, the algorithm is relatively straightforward. The steps of this algorithm are:

1. The initial fuzzy pseudo-intent is the empty attribute set $\widetilde{\phi}$. $\widetilde{\phi}^{\widetilde{R}\widetilde{Q}}$ is basically the set of attributes shared by all the objects.

2. Here we generate the next fuzzy attribute set in lectic order. By lectic order we mean that if the initial fuzzy pseudo-intent is $\{p_1/\alpha_1, p_2/\beta_1, p_3/\gamma_1, ... \}$ then in the lectic order, we generate a new fuzzy attribute set by considering the next attribute value of p_1. In this case the next attribute set in lectic order would be $\{p_1/\alpha_2, p_2/\beta_1, p_3/\gamma_1, ... \}$. After we're done with generating attribute sets by varying attribute degrees of p_1, we would then vary attribute degrees of p_2. So the next attribute set in lectic order would be $\{p_1/\alpha_1, p_2/\beta_2, p_3/\gamma_1, ... \}$

3. For each attribute set $\widetilde{K}$ generated in Step 2, determine if it is a fuzzy pseudo-intent by applying the following tests.

   - $\widetilde{K} \neq \widetilde{K}^{\widetilde{R}\widetilde{Q}}$

   - For each fuzzy pseudo-intent $\widetilde{L}$ determined previously, if $\widetilde{L} \subset \widetilde{K}$, then

$\widetilde{L}^{\widetilde{R}\widetilde{Q}} \subseteq \widetilde{K}$.

4. If $\widetilde{K}$ is a pseudo-intent, add it to the list of pseudo-intents.

5. Add the implication $\widetilde{K} \rightarrow \widetilde{K}^{\widetilde{R}\widetilde{Q}}$ to the list of implications.

6. Stop when there are no more attribute sets in the lectic order.

4.3.1 Time Complexity

The time complexity of the proposed algorithm chiefly depends upon Step 2.

Suppose that attribute p_1 has n_1 distinct values of α, p_2 has n_2 distinct values of β and so on upto p_m which has n_m distinct values. Then the total number of times fuzzy attribute sets are generated is given by:

$$T(n) = n_1 n_2 ... n_i ... n_m \tag{4.11}$$

For the particular case when $n_i = n =$ number of objects, we have $T(n) = n^m$ which is clearly exponential.

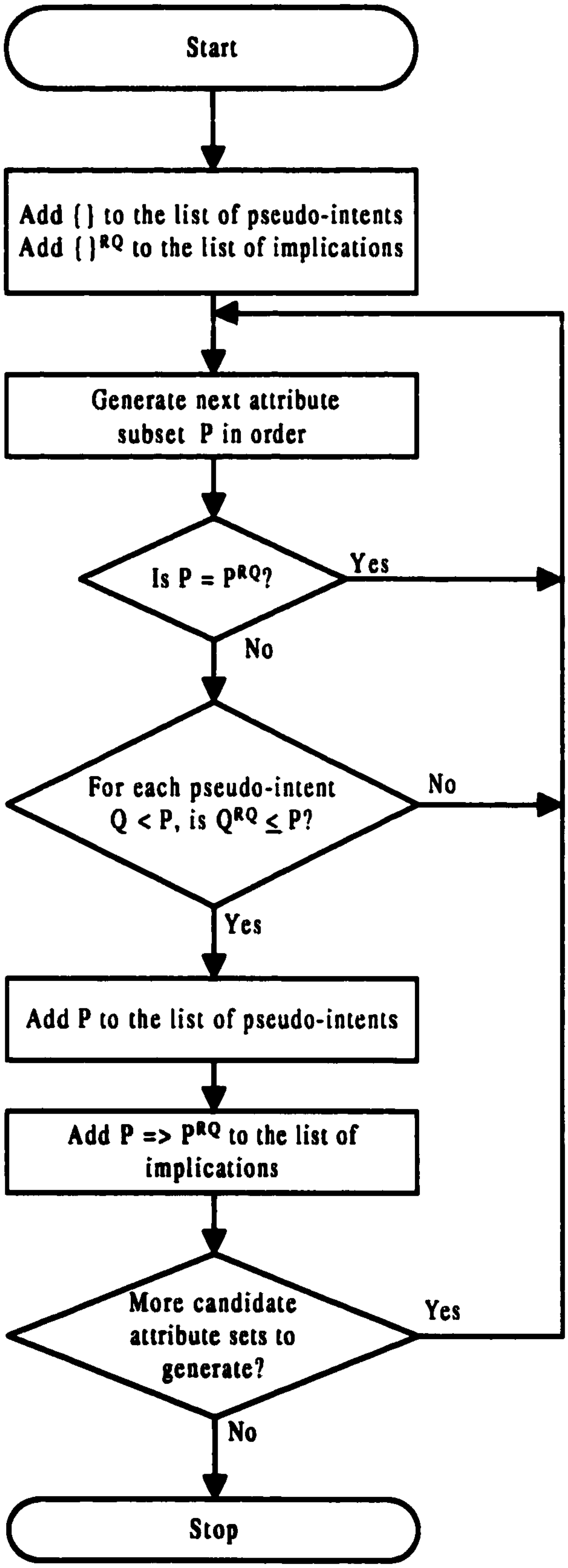

Figure 4.1: Algorithm for Computation of Fuzzy Galois Stem Base

Example 4.1 *For the fuzzy context given in Table 3.2, (reproduced here), we find out the Fuzzy Duquenne-Guiges Basis using the proposed algorithm.*

	p_1	p_2	p_3	p_4
o_1	0.5	1	0.7	0.5
o_2	0.6	0.7	1	0.5
o_3	1	0.9	1	0.1
o_4	1	0.9	0.9	0.1

Table 4.1: The fuzzy binary relation $\widetilde{W}$

The first fuzzy pseudo-intent is the $\widetilde{P_0} = \tilde{\phi}$.

Obviously, $\tilde{\phi}^{\widetilde{R}\widetilde{Q}} = \{p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.1\}$.

Hence below is our first implication.

$$\rightarrow p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.1 \tag{4.12}$$

Our first candidate attribute set would be the set of minimal attribute values given by $\tilde{\phi}^{\widetilde{R}\widetilde{Q}}$. *Obviously* $\tilde{\phi}^{\widetilde{R}\widetilde{Q}}$ *does not qualify as a pseudo-intent since it is an an intent.*

Lectically, the next candidate attribute set would be $\widetilde{P_1} = \{p_1/0.6,\ p_2/0.7,\ p_3/0.7,\ p_4/0.1\}$. *We see that* $\widetilde{P_1}^{\widetilde{R}\widetilde{Q}} = \{p_1/0.6,\ p_2/0.7,\ p_3/0.9,\ p_4/0.1\}$. *Hence* $\widetilde{P_1} \neq \widetilde{P_1}^{\widetilde{R}\widetilde{Q}}$. *Also, we note that for the previous pseudo-intent* $\widetilde{P_0}$, $\widetilde{P_0} \subset \widetilde{P_1}$ *and* $\widetilde{P_0}^{\widetilde{R}\widetilde{Q}} \subseteq \widetilde{P_1}$. *Hence* $\widetilde{P_1}$ *is a fuzzy pseudo-intent.*

While stating the implication, we don't need to include attributes that are trivially derivable. Hence we shall exclude $p_2/0.7$, $p_3/0.7$ *and* $p_4/0.1$ *from the implication as they are true for any element of this context by implication 4.12.*

Therefore, we now state our second implication

$$p_1/0.6 \rightarrow p_3/0.9 \tag{4.13}$$

The next candidate attribute set would be $\widetilde{P_2} = \{p_1/1.0,\ p_2/0.7,\ p_3/0.7,\ p_4/0.1\}$. *We find that* $\widetilde{P_2}^{\tilde{R}\tilde{Q}} = \{p_1/1.0,\ p_2/0.9,\ p_3/0.9,\ p_4/0.1\}$. *We also verify that* $\widetilde{P_2}$ *qualifies as a pseudo-intent.*

Hence our next implication (devoid of trivially satisfying attributes is)

$$p_1/1.0 \rightarrow p_2/0.9, p_3/0.9 \tag{4.14}$$

The next candidate attribute set is $\widetilde{P_3} = \{p_1/0.5,\ p_2/0.9,\ p_3/0.7,\ p_4/0.1\}$. *This set does not qualify as a pseudo-intent because* $\widetilde{P_3}^{\tilde{R}\tilde{Q}} = \widetilde{P_3}$. *Hence we discard it.*

The next candidate attribute set is $\widetilde{P_4} = \{p_1/0.5, p_2/1.0, p_3/0.7, p_4/0.1\}$. *This set qualifies as a pseudo-intent because* $\widetilde{P_3}^{\tilde{R}\tilde{Q}} \neq \widetilde{P_3}$. *It also verifies the other condition.*

Hence our next implication is:

$$p_2/1.0 \rightarrow p_4/0.5 \tag{4.15}$$

An interesting candidate attribute set is $\widetilde{P_5} = \{p_1/1.0,\ p_2/0.9,\ p_3/0.7,\ p_4/0.1\}$. *Although it satisfies the condition* $\widetilde{P_5} \neq \widetilde{P_5}^{\tilde{R}\tilde{Q}}$, *it fails to satisfy the other condition as* $\widetilde{P_2} \subset \widetilde{P_5}$ *but* $\widetilde{P_2}^{\tilde{R}\tilde{Q}} \not\subseteq \widetilde{P_5}$. *Hence* $\widetilde{P_5}$ *is not a fuzzy pseudo-intent.*

Therefore, the final complete set of Duquenne-Guiges Basis of $\widetilde{W}$ is:

$$\rightarrow p_1/0.5, p_2/0.7, p_3/0.7, p_4/0.1(support = 4) \quad (4.16)$$

$$p_1/0.6 \rightarrow p_3/0.9(support = 3) \quad (4.17)$$

$$p_1/1.0 \rightarrow p_2/0.9, p_3/0.9(support = 2) \quad (4.18)$$

$$p_2/1.0 \rightarrow p_4/0.5(support = 1) \quad (4.19)$$

$$p_3/0.9 \rightarrow p_1/0.6(support = 3) \quad (4.20)$$

$$p_1/0.6, p_2/0.9, p_3/0.9 \rightarrow p_1/1.0(support = 2) \quad (4.21)$$

$$p_1/0.6, p_3/0.9, p_4/0.5 \rightarrow p_3/1.0 \quad (4.22)$$

$$p_2/0.9, p_4/0.5 \rightarrow p_2/1.0(support = 1) \quad (4.23)$$

4.4 Derived Implications

The method of pseudo-intents stated in the previous section provides us with a way for computing the minimal set of implications that is complete and non-redundant. In this section we state some rules that can be used to derive further implications from this initial set that can present more information. It may however be pointed out that derived implications do not provide any new information that is not present within the fuzzy stem base. They only provide with a richer representation of the same information that is present within the stem base but is not immediately evident.

4.4.1 Rules for Derived Implications

Below we state the rules for deriving implications from a set of given implications:

1. If $\tilde{X}$, $\tilde{Y}$ are fuzzy attribute sets such that $\tilde{Y} \subseteq \tilde{X}$ then $\tilde{X} \to \tilde{Y}$ is an implication.

2. For $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ being attribute sets, if $\tilde{X} \to \tilde{Y}$ is an implication, then $\tilde{X} \cup \tilde{Z} \to \tilde{Y}$ is also an implication.

3. For $\tilde{X}$, $\tilde{Y}$, $\tilde{A}$ and $\tilde{B}$ being attribute sets, if $\tilde{X} \to \tilde{Y}$ and $\tilde{A} \to \tilde{B}$ are two implications then $\tilde{X} \cup \tilde{A} \to \tilde{Y} \cup \tilde{B}$ is an implication.

4. If $\tilde{X} \to \tilde{Y}$ is an implication and $\tilde{Y} \to \tilde{Z}$ is an implication, then $\tilde{X} \to \tilde{Z}$ is an implication. This is called the transitivity property.

5. If $\tilde{X} \to \tilde{Y}$ is an implication and $\tilde{Y} \cup \tilde{Z} \to \tilde{A}$ is an implication then $\tilde{X} \cup \tilde{Z} \to \tilde{A}$ is an implication.

The above stated rules are natural extensions of the crisp case. They can simply be proven by arguments of premise and conclusion.

(1) is always true since the premise contains the conclusion. Whenever the premise is true, the conclusion already is true.

(2) is always true because if the the union of a premise ($\tilde{X}$) with another set ($\tilde{Z}$) does not nullify the already existing conclusion ($\tilde{Y}$) which holds whenever the premise is true.

(3) is always true, since each premise($\tilde{X}$ and $\tilde{A}$) produces its own conclusion ($\tilde{Y}$ and $\tilde{B}$). Hence the union of the premises at least produces the union of conclusions.

(4) is always true, since the truth of the first premise $\tilde{X}$ implies the truth of the first conclusion which becomes the second premise $\tilde{Y}$, which implies the truth of the second conclusion $\tilde{Z}$.

(5) is always true since the first premise gives a conclusion which is part of the premise for the second conclusion. The other part is $\tilde{A}$. The truth of $\tilde{Y}$ is established by the truth of $\tilde{X}$ and so the truth of both $\tilde{X}$ and $\tilde{A}$ imply the truth of $\tilde{Z}$.

Chapter 5

EXPERIMENTATION

In this chapter we first discuss briefly the use of an existing crisp context analysis tool CONIMP for fuzzy Galois lattice generation and fuzzy knowledge extraction. Then an implementation of the algorithms for fuzzy Galois lattice generation as well as fuzzy knowledge extraction discussed in Chapter 3 and Chapter 4 is stated in the form of a prototype. Finally an experimentation on a real-world data set provides an interpretation of some of the notions of FCA in the light of actual data.

5.1 Mapping

In order to utilise tools for crisp context analysis on fuzzy contexts we need to map a fuzzy context to a crisp context. The resulting crisp context can then be analyzed using tools for crisp context analysis. The results of this analysis - concepts, their ordering and implications can then be mapped back to fuzzy values.

Mapping a fuzzy context to a crisp one involves extending the number of attributes. The procedure we adopted is as follows:

1. The number of objects remains the same.

2. Each fuzzy attribute is mapped to a set of crisp attributes. This is achieved by defining a new crisp attribute for each value of α present within the fuzzy attribute. Thus for example, if attribute p_1 has three membership grades α_1, α_2 and α_3 such that $\alpha_1 \geq \alpha_2 \geq \alpha_3$ then the mapped crisp relation will have three crisp attributes p_1/α_1, p_2/α_2 and p_3/α_3.

3. Each object which possesses an attribute with membership degree α necessarily possesses that attribute with lower membership degrees also. For the above stated context, if an object possesses an attribute with membership degree α_2 it also possesses that attribute at least with degree α_3. So we place a '1' for such an object at attribute p_1/α_2 and p_1/α_3.

4. The resulting crisp context can now be analyzed.

In our case we mapped the crisp relation presented in Table 3.2 and mapped it to the crisp case. Table 5.1 shows the resulting crisp context.

The resulting crisp context was input to CONIMP - a crisp context analysis tool. We found out that the number of concepts was the same - 10 in all. But

	$p_1/0.5$	0.6	1.0	$p_2/0.7$	0.9	1.0	$p_3/0.7$	0.9	1	$p_4/0.1$	0.5
o_1	1	0	0	1	1	1	1	0	0	1	1
o_2	1	1	0	1	0	0	1	1	1	1	1
o_3	1	1	1	1	1	0	1	1	1	1	0
o_4	1	1	1	1	1	0	1	1	0	1	0

Table 5.1: Crisp Context corresponding to $\widetilde{W}$

the concepts needed refinement. For example attribute sets like $\{..., p_4/0.1, p_4/0.5\}$ were generated. Such sets can be simplified by simply retaining the highest value attribute for each p_i and removing all others.

Similarly the minimal base set of implications was also generated which were 10 in all. This number is two more than our result. In this case two implications are redundant. For example, the implication

$$p_3/1.0 \rightarrow p_1/0.6 \tag{5.1}$$

is redundant since $p_3/0.9 \rightarrow p_1/0.6$ is already included in the minimal set.

To summarise, we need a preprocessor to generate a crisp context from our initial fuzzy context. We then feed this crisp context to CONIMP. The results then need refinement using a post-processor.

The immediate drawback of this approach is the need to incorporate a preprocessing as well as a postprocessing step. For the pre-processing case the problem is the large crisp context generated. For our case, instead of the 16 entries in the fuzzy context, the corresponding crisp context has 44 entries which is an increase of more than two times. In general a the number of entries is the sum of by the number of possible fuzzy values possessed by each attribute for each object.

Figure 5.1 states the procedure in the form of a block diagram.

5.2 Prototype for Fuzzy Context Analysis

A prototype for fuzzy context analysis was implemented in C language. The prototype implements the algorithms proposed in this work for fuzzy Galois lattice generation and fuzzy knowledge extraction.

5.2.1 Fuzzy Maximal Rectangles Generation

Fuzzy Maximal rectangles are generated using the algorithm proposed in Section 3.5. We used a linked list of nodes as our data structure with each node having an object set and an attribute set as two linear arrays. At each iteration, the linked list grows by as much as twice its previous length; redundant nodes which are no longer fuzzy maximal rectangles are then truncated to adjust the length of the list.

5.2.2 Ordering: Successor List

After the generation of the linked list, the orderig of the generated fuzzy maximal rectangles is done using a successor list. Successor lists are used to specify the ordering of concepts in a Galois lattice structure. They consist of each set with a

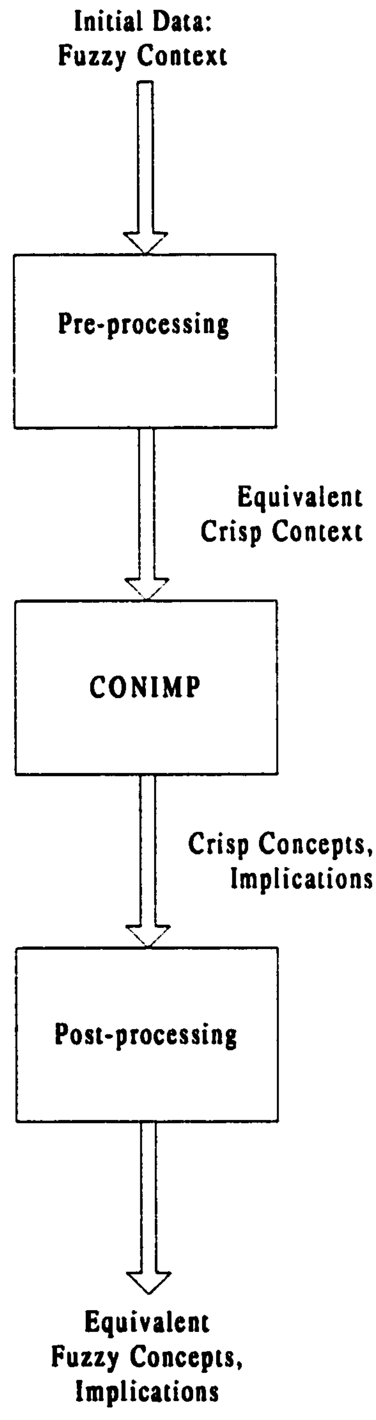

Figure 5.1: Block diagram for fuzzy context analysis using CONIMP

list of all the parents of that set. In our prototype, a linked list of nodes was used with each node having a label denoting the fuzzy maximal rectangle as well as a linear array having a list of rectangles that are its parents.

5.2.3 Knowledge Extraction

Finally, implications are extracted using the algorithm presented in Section 4.3. Again, a linked list of nodes was used as the chosen data structure.

5.3 Experimentation

We used our prototype to analyze admission statistics of 10 engineering schools in the United States ranked among the best by US News. This analysis provided several useful results as well as highlighted many important points.

To begin with, the admission statistics are intended as a guide for potential applicants to these schools. For our purposes, we adopted some approximations in order to simlify the analysis. These approximations for each attributes values will be stated in the next sub-section.

The data under consideration has 10 objects representing 10 universities: MIT, Stanford, UC Berkeley, Georgia Tech, Carnegie Mellon, Cornell, Purdue, UT Austin and USC. The attributes consisted of the following: Average Applicant GRE Scores, Total Acceptance, Acceptance Rate, Total Aid (No of applicants), Percentage of

Aid given to Assistantships. This data is intended to provide useful information for potential applicants. In the next section, we provide an explanation of each attribute.

The data in the context form is stated in Table 5.2:

Name	Avg GRE	Total Acc	Acc Rate	Total Aid	% Asst
MIT	2000	1400	30	2200	80
Stanford	2000	1800	45	2000	60
UC Berkeley	2000	1000	25	1600	75
Georgia Tech	1900	1600	25	1900	80
Univ Michigan	2000	1400	35	1300	85
Carnegie Mellon	2000	650	20	650	80
Cornell Univ	2000	1200	30	700	70
Purdue Univ	1950	1500	33	1300	85
UT Austin	1950	1100	36	1600	70
Univ South CA	1850	1700	40	600	90

Table 5.2: University Admission Data: Source (US News)

5.3.1 Fuzzification

Table 5.2 consists of real values which need to be transformed to fuzzy values in order to apply FCA for fuzzy contexts. In order to transform Table 5.2 to a fuzzy table, we need to define functions mapping these real data to fuzzy values. This transformation is based upon functions transforming real data to fuzzy data which essentially capture the "linguistic interpretation" of the stated values. A linguistic variable is used to characterize less precise description of an attribute.

Table 5.3 displays an association between linguistic values and fuzzy values which will be used throughout in our analysis.

Linguistic Value	Associated Fuzzy Value
Non-existent	0.0
Very Low	0.1-0.2
Low	0.3
Moderate	0.4-0.6
High	0.7
Very High	0.8-0.9
Perfect	1.0

Table 5.3: Linguistic Values and Associated Fuzzy Values

Below we provide a description of each attribute appearing within Table 5.2 as well as its associated mapping function.

Average Applicant GRE Scores

This attribute represents the average scores of typical applicants to a particular school. These values have been rounded to the nearest fiftieth. Table 5.2 has scores within the range of 1850-2000. Hence for our purpose, we classify scores from "low" to "perfect" within the range 1750-2100. With each particular value appearing within Table 5.2 we may associate a fuzzy value according to a linear mapping function.

Equation 5.2 provides a mapping function for transforming values from actual values to their fuzzy equivalents. The mapping function is linear within the range

Score Range	Fuzzy Range
Below 1750	0.3
1750 - 1790	0.3 - 0.38
1800 - 1890	0.4 - 0.58
1900 - 1990	0.6 - 0.78
2000 - 2090	0.8 - 0.98
2100+	1.0

Table 5.4: Fuzzy Equivalent of Average Applicant GRE Scores

1750-2100 and constant elsewhere.

$$f(x) = \begin{cases} 0.3, & x \leq 1750, \\ \dfrac{x - 1750}{500} + 0.3, & 1750 \geq x \leq 2100, \\ 1.0, & x \geq 2100 \end{cases} \tag{5.2}$$

Total Acceptance

This attribute represents the total number of admissions for a particular school. For this attribute a round-up staircase distribution is adopted.

The mapping function for this attribute is the staircase function given by the following expression:

$$f(x) = \begin{cases} 0.2\lceil \frac{x}{500} \rceil, & 0 \leq x \leq 2000, \\ 1.0 & x \geq 2000 \end{cases} \tag{5.3}$$

Acceptance Rate

Acceptance Rate is the ratio of the total number of acceptances to the total number of applications. The lower the acceptance rate, the greater will be the chances of an application being rejected. Therefore, a higher acceptance rate is preferable.

For this attribute also a round-off staircase distribution is adopted. We only state here the mapping function.

$$f(x) = 0.1\lfloor \frac{x}{10} + 0.5 \rfloor, \ 0 \leq x \leq 100 \tag{5.4}$$

For next two attributes namely *Total Aid* and *Percentage of Aid for Assistantships* the fuzzy equivalent distribution is the same a Total Acceptance and Acceptance Rate. Therefore, we only provide their mapping functions:

Total Aid

$$f(x) = \begin{cases} 0.2\lceil \frac{x}{500} \rceil, & 0 \leq x \leq 2000, \\ 1.0 & x \geq 2000 \end{cases} \tag{5.5}$$

Percentage of Aid for Assistantships

$$f(x) = 0.1\lfloor \frac{x}{10} + 0.5 \rfloor, \ 0 \leq x \leq 100 \tag{5.6}$$

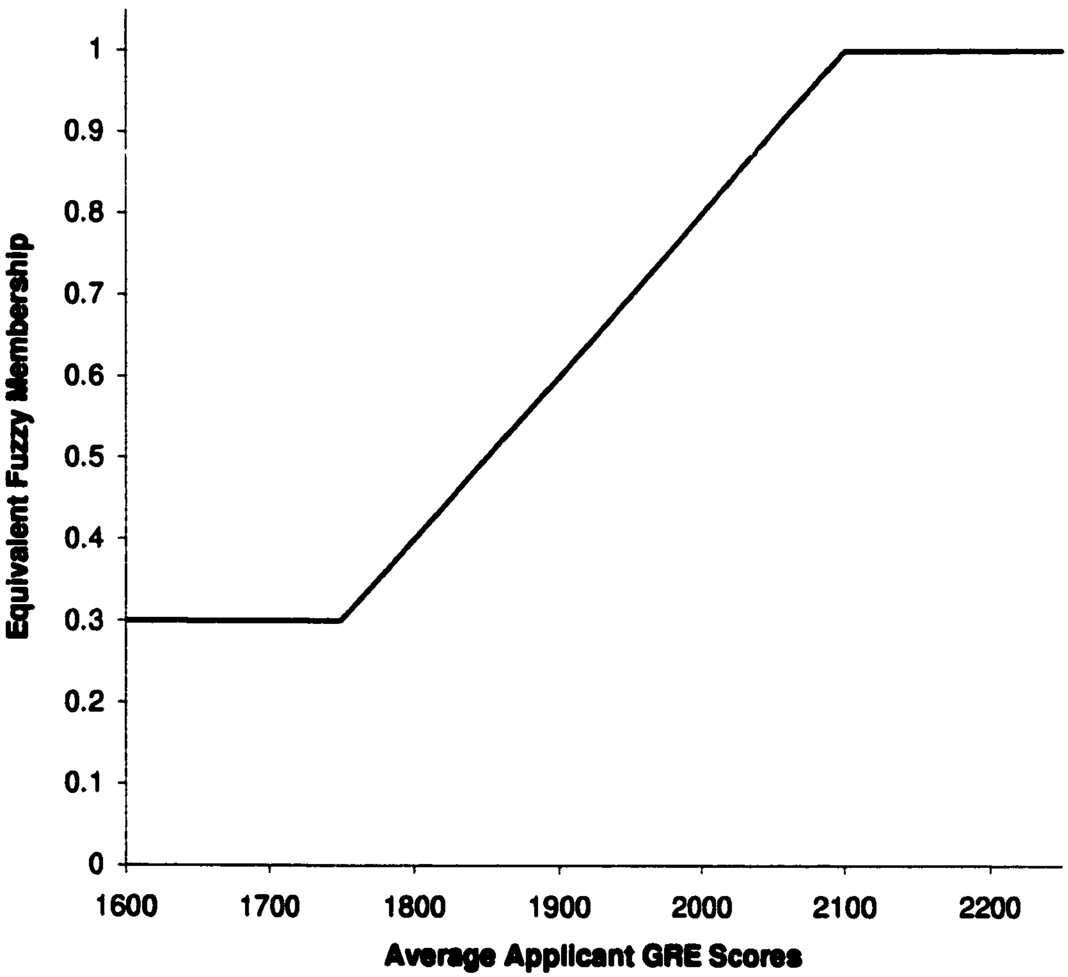

Figure 5.2: Mapping function for Average Applicant GRE Scores

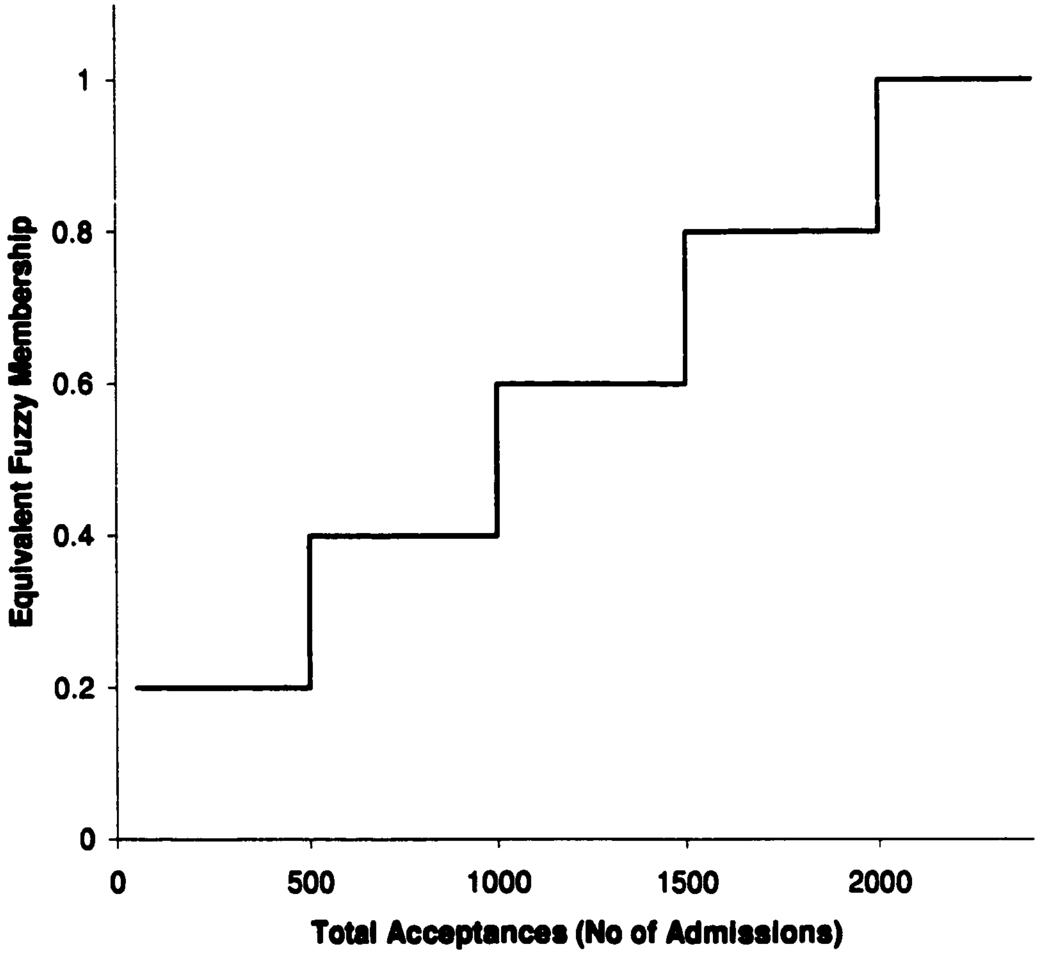

Figure 5.3: Mapping function for Total Acceptance

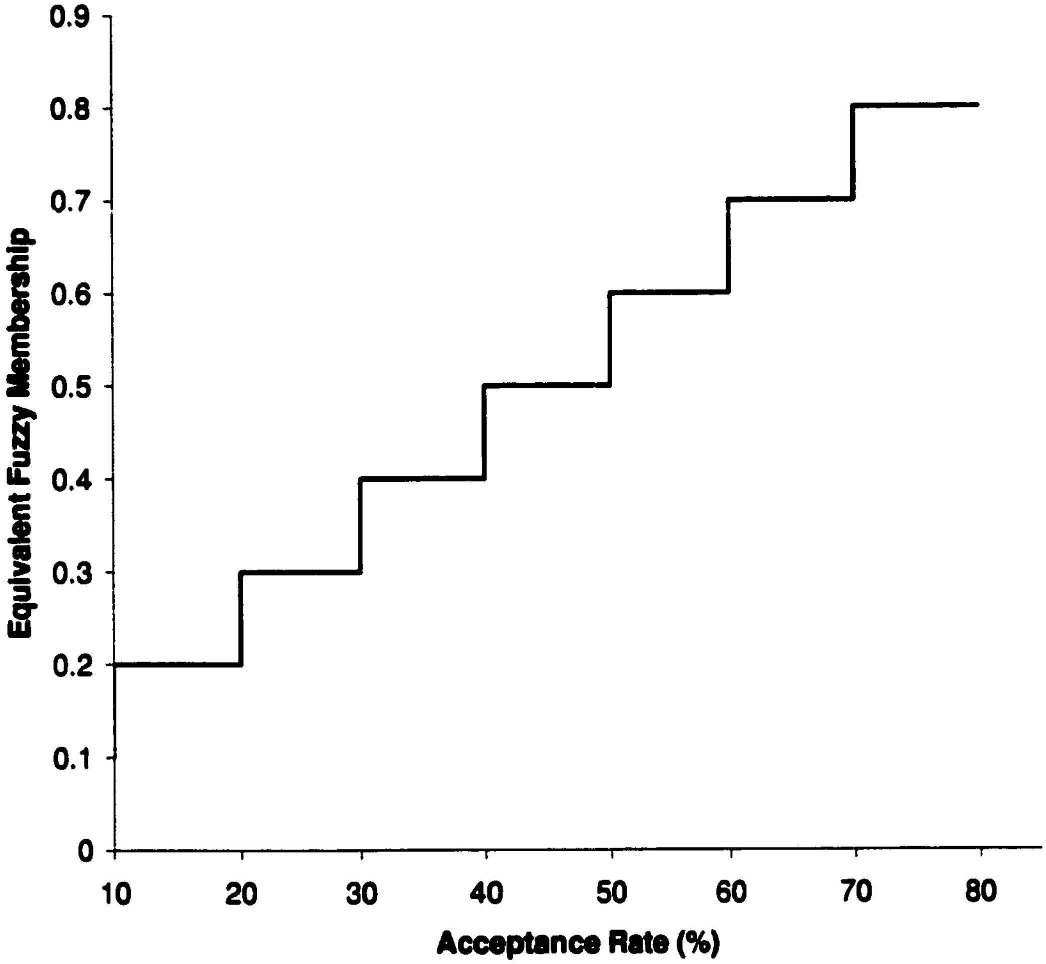

Figure 5.4: Mapping function for Acceptance Rate

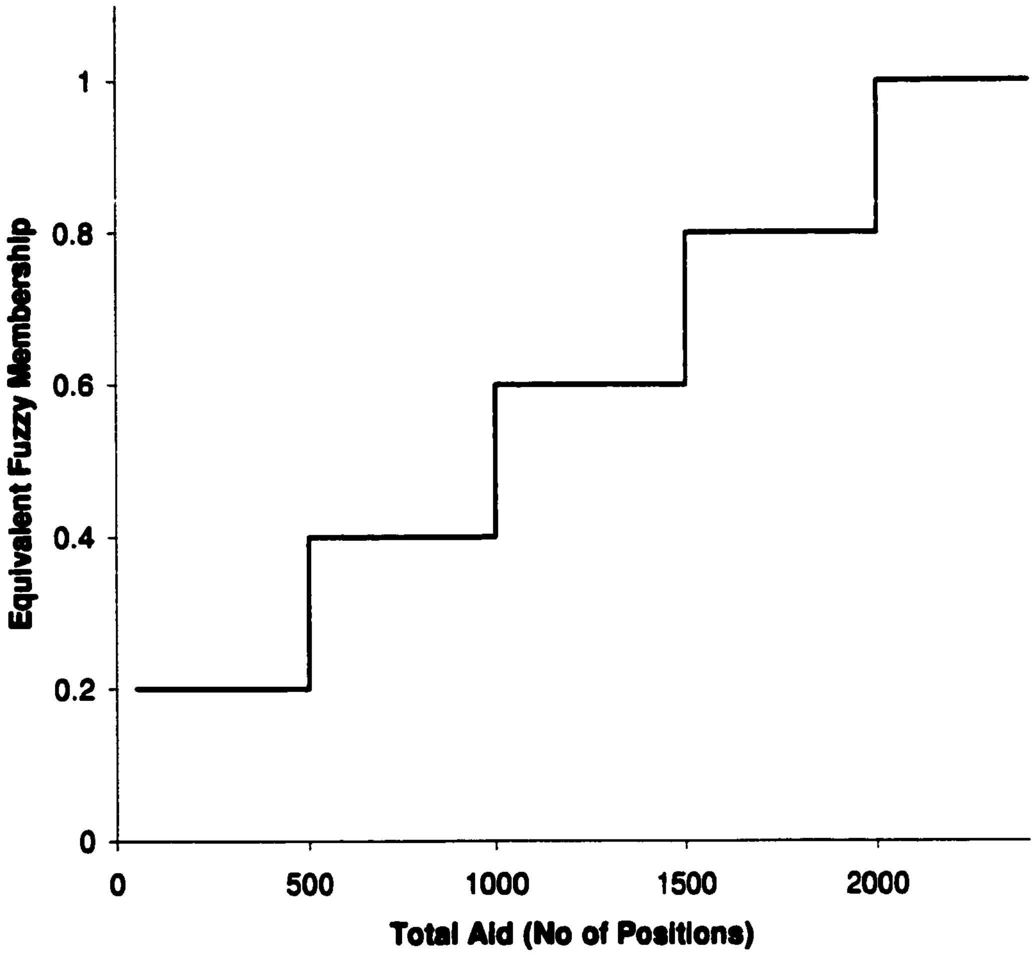

Figure 5.5: Mapping function for Total Aid

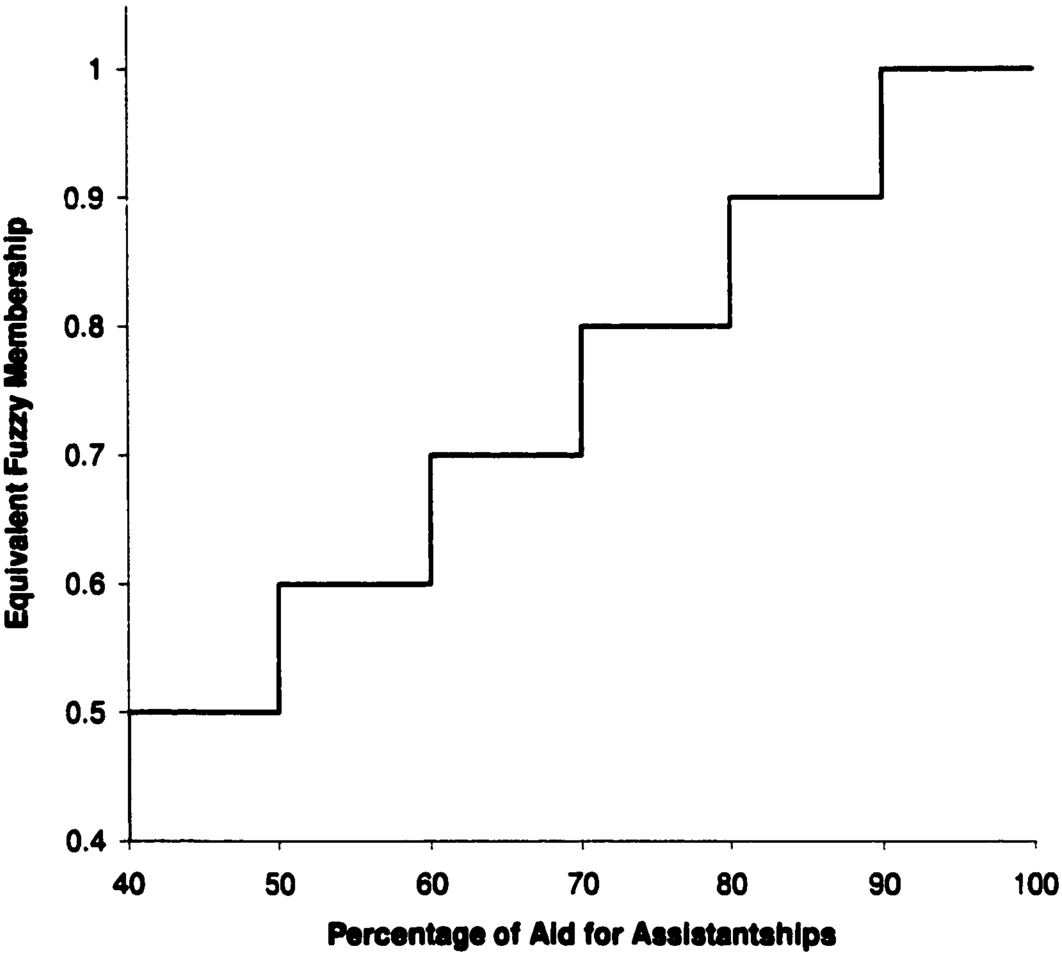

Figure 5.6: Mapping function for Percentage Aid for Assistantships

5.3.2 The fuzzy context

The context obtained after application of mapping functions stated in previous subsection result in the following context.

Name	Avg GRE (p_1)	Total Acc (p_2)	Acc Rate (p_3)	Total Aid (p_4)	% Asst (p_5)
MIT (o_1)	0.8	0.6	0.3	1	0.8
Stanford (o_2)	0.8	0.8	0.5	0.8	0.6
UC Berkeley (o_3)	0.8	0.4	0.3	0.8	0.8
Georgia Tech (o_4)	0.6	0.8	0.3	0.8	0.8
Univ Michigan (o_5)	0.8	0.6	0.4	0.6	0.9
Carnegie Mellon (o_6)	0.8	0.4	0.2	0.4	0.8
Cornell Univ (o_7)	0.8	0.6	0.3	0.4	0.7
Purdue Univ (o_8)	0.7	0.6	0.3	0.6	0.9
UT Austin (o_9)	0.7	0.6	0.4	0.8	0.7
Univ South CA (o_{10})	0.5	0.8	0.4	0.4	0.9

Table 5.5: University Admission Data: Fuzzified Context Source (US News)

5.3.3 Maximal Rectangles List

The fuzzified context is fed as input to the developed prototype. The list of fuzzy maximal rectangles or fuzzy concepts is generated. There are 79 concepts in the given context labelled from 0 to 78. They are stated in Appendix A.

5.3.4 Successor List

A successor list is also generated for the input context. The successor list contains links to successors of each concept. The generated successor list is stated in

Appendix B.

5.3.5 Implications

For the stated example, 21 implications were generated. These are listed in Appendix A.

5.4 Interpretation of the Results

The list of concepts, successor list as well as implications can be used to extract knowledge as well as information from the given context which is not apparent on the surface. Below we state several ways in which the obtained results can be applied.

5.4.1 Information Retrieval

Fuzzy Maximal Rectangles just like their crisp counterparts can be used for querying as well as for answering the question and later on completing the question. For example, if we wish to ask the question: *Which schools have very high total acceptance?*. Looking for an answer to this query, we first transform the linguistic term "very high total acceptance" to is fuzzy equivalent which is 0.8-0.9. We then search for the fuzzy maximal rectangle with the maximum number of objects that has $p_2/0.8 - 0.9$ in its attribute set. We find that concept labelled 65 fulfils this criteria. So the object set of this concept is the answer to the query which is: Stanford, Georgia Tech and USC. But searching for a concept gives us more pertinent information. The concept discovered also gives us the Average Applicant GRE Scores, Acceptance Rate, Total Aid, and Percentage of Aid for Assistantships. The values for these attributes are $p_1/0.5$, $p_3/0.3$, $p_4/0.4$, $p_5/0.6$. Hence we can state the following information: *For schools having high total acceptance, the Average Applicant GRE Score is at least moderate, the acceptance rate is at least low, the total aid is at least moderate and the percentage of aid given to assistantships is moderate.*

Hence we can now also state the complete question: *Which schools have moderate Average Applicant GRE Scores, High total acceptance, low acceptance rate, moderate total aid and moderate percentage of aid given to assistantships?.*

The above discussion clearly unveils an important point. The attribute sets of the fuzzy Galois lattice form complete questions and the object sets are the answers to these questions.

5.4.2 Successor Lists

As pointed out in Section 5.2, Successor Lists are used as a verbal description of the Galois lattice structure. For the given context, the Galois lattice structure is too complex to be visually represented. Hence we rely on a verbal description using its successor list.

5.4.3 Classification: Using Inheritance

Using the obtained results, we can classify the schools or groups of schools according to some criteria.

For example in the successor list, we find that the concept labelled 2 has concepts 3, 4, 7, 11 and 31 as its children. We can provide an interpretation to this link.

Concept 2 reprents Stanford and all its attributes with their associated membership degrees. All its children represent other objects whose attributes are at least less than or equal to those of Stanford. The attributes which are higher are not shown.

For example if we wish to find out which schools are closer in attributes to Stanford? Figure 5.7 gives us an answer. We have applied inheritance to eliminate attributes as well as objects that are already present within the parent. For example Stanford has attribute set:- Average Applicant GRE Score: Very High, Total Acceptance: Very High, Acceptance Rate: Moderate, Total Aid: Very High and % of Aid for Assistantships: Moderate. Each child concept of concept# 2 gives us the names of schools which have some particular attribute less than or equal to Stanford. So based on {Stanford}, we get all the schools that are closest to it in terms of attribute differences. For example compared to Stanford, UT Austin (Concept# 32) has Average Applicant GRE high and total acceptance moderate. The rest of the attributes are either equal or greater.

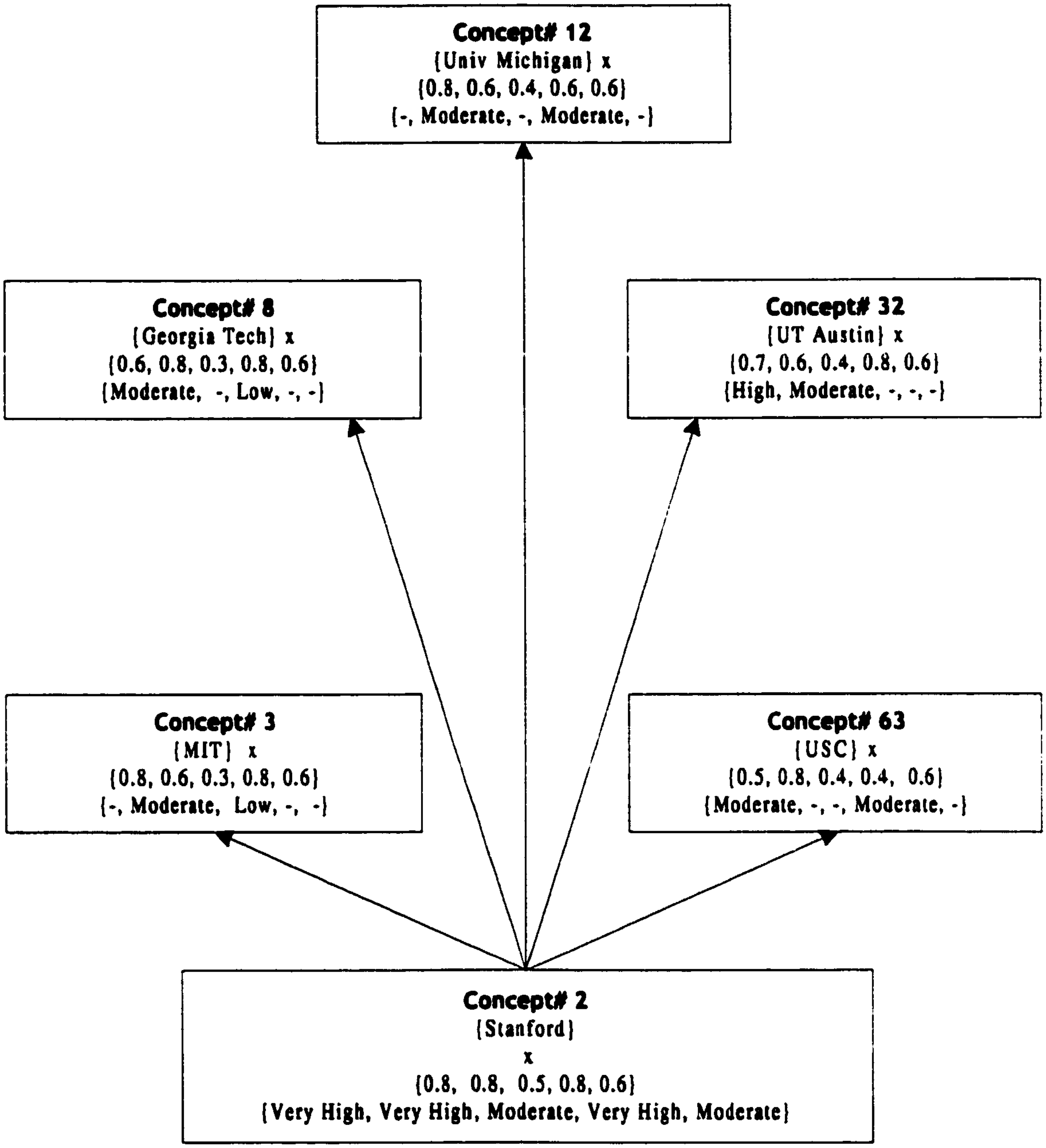

Figure 5.7: Applying Inheritance to classify Schools

Similarly Concept# 12 gives us the attribute differences between Stanford and University of Michigan.

5.4.4 Assocation Rules or Implications

The association rules stated in Appendix A provide us with an association between various attributes of the context.

The first rule

$$\Rightarrow p_1/0.5, p_2/0.4, p_3/0.2, p_4/0.4, p_5/0.6 \tag{5.7}$$

states the information that for the objects share the attribute p_1 at least with degree 0.5, p_2 at least with degree 0.4, p_3 at least with degree 0.2, p_4 at least with degree 0.4 and p_5 at least with degree 0.6. This means that for all the 10 schools have an Average Applicant GRE Score at least moderate, Total Acceptance is at least moderate, Acceptance Rate is at least low, Total Aid is at least moderate and Percentage of Aid given to Assistantships is at least moderate.

Consider the following implication:

$$p_5/0.9 \Rightarrow p_2/0.6, p_3/0.3 \tag{5.8}$$

This implication provides the information that for all schools that have a very high percentage of aid given to assistantships, their total acceptance is moderate but their acceptance rate is low.

In some implications, the premise as well as the conclusion have the same attribute with different degrees. This is useful in establishing association when we are sure about some attribute with a less degree but sure about other attributes with greater degrees. Then we can be sure about the former attributes with a greater degree. Such implications are useful in improving upon an initial value of an attribute.

For example the implication

$$p_2/0.8, p_3/0.3, p_5/0.7 \Rightarrow p_5/0.8 \quad (5.9)$$

states the information that in the given context if p_2 is possessed with degree at least 0.8, p_3 with degree at least 0.3 and p_5 with degree at least 0.7 then p_5 is being possessed with degree 0.8. This can be translated in linguistic terms as: if some schools have high total acceptance, low acceptance rate and high percentage of aid given to assistantships, then the percentage of aid given to assistantships must be very high.

5.4.5 Derived Implications

As stated in Chapter 4, we could use the minimal implication set to derive further implications. For example, we could combine implications (A-4) and (A-7). These implications are:

$$p_3/0.4 \Rightarrow p_2/0.6 \tag{5.10}$$

$$p_4/0.6 \Rightarrow p_1/0.6, p_3/0.3 \tag{5.11}$$

By combining them (using union of premises and union of conclusions, we get)

$$p_3/0.4, p_4/0.6 \Rightarrow p_1/0.6, p_2/0.6, p_3/0.3 \tag{5.12}$$

Obviously $p_3/0.3$ is redundant in the conclusion, so we remove it to get,

$$p_3/0.4, p_4/0.6 \Rightarrow p_1/0.6, p_2/0.6 \tag{5.13}$$

This implication gives us an association between the Acceptance Rate & Total Aid as the premise and Average Applicant GRE & Total Acceptance on the other. It simply states that if the Acceptance Rate & Total Aid are moderate then the Average Applicant GRE & Total Acceptance will also be moderate.

Chapter 6

Conclusions and Future Work

In this research an attempt had been made to extend formal concept analysis to fuzzy contexts. Several useful conclusions can be drawn from the extensions.

6.1 Conclusions

1. Fuzzy Galois lattice is an exponential structure just like its crisp counterpart.

2. We can substitute linguistic variables to interpret the concepts as well as the implications obtained from a fuzzy context.

3. Functions transforming real data to fuzzy data must take into account the linguistic interpretation of the data.

4. Mapping a fuzzy context to a crisp one increases the size of the context.

5. Fuzzy Galois lattice can be used for information retrieval, classification, discovering implications.

6.2 Future Work

There are several avenues for possible research which had not been explored in this work. These are:

1. A good investigation point is to analyze the effect of changing the nature of mapping functions and compare the extracted knowledge with that obtained by the choice of linear functions.

2. The exponential nature of fuzzy Galois lattice motivates the devlopment of an optimized fuzzy coverage of initial data which can cover the entire context. Several heuristics may be used for obtaining a near-optimal solution for knowledge extraction.

3. Visual Representation of the fuzzy Galois lattice as it grows can be extremely cumbersome and inappropriate. Some type of reduction scheme as well as graphical methods could be applied to develop a quick and informatice visual representation.

Appendix A

Fuzzy Concepts, Successors and Implication Lists

This appendix states the complete fuzzy concepts list, the successor list as well as the implications derived from the context presented in Table 5.5.

A.1 Fuzzy Concepts List

Below we state the complete list of fuzzy concepts derived from Table 5.5.

The format adopted is as follows: Each item within the list has two sets. The first set is the object set which consists of 1's and 0's. A 1 in a place simply means the presence of that object. A 0 implies an absence.

Similar is the case with the attribute set. The value appearing within the attribute set imply the degree to which each attribute is possessed by the object set.

For example the set:
$\{o_1, o_2, o_5, o_7\} \times \{p_1/0.8, p_2/0.6, p_3/0.3, p_4/0.4, p_5/0.6\}$
will be represented as
Object Set: 1 1 0 0 1 0 1 0 0 0
Attribute Set: 0.8 0.6 0.3 0.4 0.6

The label appearing in the list is just the serial number or an identifier assigned to each fuzzy concept.

Label	Object Set	Attribute Set
	o_1 o_2 o_3 o_4 o_5 o_6 o_7 o_8 o_9 o_{10}	p_1 p_2 p_3 p_4 p_5
0:	0 0 0 0 0 0 0 0 0 0	0.8 0.8 0.5 1.0 0.9
1:	1 0 0 0 0 0 0 0 0 0	0.8 0.6 0.3 1.0 0.8
2:	0 1 0 0 0 0 0 0 0 0	0.8 0.8 0.5 0.8 0.6
3:	1 1 0 0 0 0 0 0 0 0	0.8 0.6 0.3 0.8 0.6
4:	1 0 1 0 0 0 0 0 0 0	0.8 0.4 0.3 0.8 0.8
5:	1 1 1 0 0 0 0 0 0 0	0.8 0.4 0.3 0.8 0.6
6:	0 0 0 1 0 0 0 0 0 0	0.6 0.8 0.3 0.8 0.8
7:	1 0 0 1 0 0 0 0 0 0	0.6 0.6 0.3 0.8 0.8
8:	0 1 0 1 0 0 0 0 0 0	0.6 0.8 0.3 0.8 0.6
9:	1 0 1 1 0 0 0 0 0 0	0.6 0.4 0.3 0.8 0.8
10:	0 0 0 0 1 0 0 0 0 0	0.8 0.6 0.4 0.6 0.9
11:	1 0 0 0 1 0 0 0 0 0	0.8 0.6 0.3 0.6 0.8
12:	0 1 0 0 1 0 0 0 0 0	0.8 0.6 0.4 0.6 0.6
13:	1 1 0 0 1 0 0 0 0 0	0.8 0.6 0.3 0.6 0.6
14:	1 0 1 0 1 0 0 0 0 0	0.8 0.4 0.3 0.6 0.8
15:	1 1 1 0 1 0 0 0 0 0	0.8 0.4 0.3 0.6 0.6
16:	1 0 1 0 1 1 0 0 0 0	0.8 0.4 0.2 0.4 0.8
17:	1 0 0 0 1 0 1 0 0 0	0.8 0.6 0.3 0.4 0.7
18:	1 1 0 0 1 0 1 0 0 0	0.8 0.6 0.3 0.4 0.6
19:	1 0 1 0 1 0 1 0 0 0	0.8 0.4 0.3 0.4 0.7
20:	1 1 1 0 1 0 1 0 0 0	0.8 0.4 0.3 0.4 0.6
21:	1 0 1 0 1 1 1 0 0 0	0.8 0.4 0.2 0.4 0.7
22:	1 1 1 0 1 1 1 0 0 0	0.8 0.4 0.2 0.4 0.6
23:	0 0 0 0 1 0 0 1 0 0	0.7 0.6 0.3 0.6 0.9
24:	1 0 0 0 1 0 0 1 0 0	0.7 0.6 0.3 0.6 0.8
25:	1 0 1 0 1 0 0 1 0 0	0.7 0.4 0.3 0.6 0.8
26:	1 0 0 1 1 0 0 1 0 0	0.6 0.6 0.3 0.6 0.8
27:	1 0 1 1 1 0 0 1 0 0	0.6 0.4 0.3 0.6 0.8
28:	1 0 1 0 1 1 0 1 0 0	0.7 0.4 0.2 0.4 0.8
29:	1 0 1 1 1 1 0 1 0 0	0.6 0.4 0.2 0.4 0.8
30:	0 0 0 0 0 0 0 0 1 0	0.7 0.6 0.4 0.8 0.7

Label	Object Set	Attribute Set
	o_1 o_2 o_3 o_4 o_5 o_6 o_7 o_8 o_9 o_{10}	p_1 p_2 p_3 p_4 p_5
31:	1 0 0 0 0 0 0 0 1 0	0.7 0.6 0.3 0.8 0.7
32:	0 1 0 0 0 0 0 0 1 0	0.7 0.6 0.4 0.8 0.6
33:	1 1 0 0 0 0 0 0 1 0	0.7 0.6 0.3 0.8 0.6
34:	1 0 1 0 0 0 0 0 1 0	0.7 0.4 0.3 0.8 0.7
35:	1 1 1 0 0 0 0 0 1 0	0.7 0.4 0.3 0.8 0.6
36:	1 0 0 1 0 0 0 0 1 0	0.6 0.6 0.3 0.8 0.7
37:	1 1 0 1 0 0 0 0 1 0	0.6 0.6 0.3 0.8 0.6
38:	1 0 1 1 0 0 0 0 1 0	0.6 0.4 0.3 0.8 0.7
39:	1 1 1 1 0 0 0 0 1 0	0.6 0.4 0.3 0.8 0.6
40:	0 0 0 0 1 0 0 0 1 0	0.7 0.6 0.4 0.6 0.7
41:	0 1 0 0 1 0 0 0 1 0	0.7 0.6 0.4 0.6 0.6
42:	1 0 0 0 1 0 0 1 1 0	0.7 0.6 0.3 0.6 0.7
43:	1 1 0 0 1 0 0 1 1 0	0.7 0.6 0.3 0.6 0.6
44:	1 0 1 0 1 0 0 1 1 0	0.7 0.4 0.3 0.6 0.7
45:	1 1 1 0 1 0 0 1 1 0	0.7 0.4 0.3 0.6 0.6
46:	1 0 0 1 1 0 0 1 1 0	0.6 0.6 0.3 0.6 0.7
47:	1 1 0 1 1 0 0 1 1 0	0.6 0.6 0.3 0.6 0.6
48:	1 0 1 1 1 0 0 1 1 0	0.6 0.4 0.3 0.6 0.7
49:	1 1 1 1 1 0 0 1 1 0	0.6 0.4 0.3 0.6 0.6
50:	1 0 0 0 1 0 1 1 1 0	0.7 0.6 0.3 0.4 0.7
51:	1 1 0 0 1 0 1 1 1 0	0.7 0.6 0.3 0.4 0.6
52:	1 0 1 0 1 0 1 1 1 0	0.7 0.4 0.3 0.4 0.7
53:	1 1 1 0 1 0 1 1 1 0	0.7 0.4 0.3 0.4 0.6
54:	1 0 0 1 1 0 1 1 1 0	0.6 0.6 0.3 0.4 0.7
55:	1 1 0 1 1 0 1 1 1 0	0.6 0.6 0.3 0.4 0.6
56:	1 0 1 1 1 0 1 1 1 0	0.6 0.4 0.3 0.4 0.7
57:	1 1 1 1 1 0 1 1 1 0	0.6 0.4 0.3 0.4 0.6
58:	1 0 1 0 1 1 1 1 1 0	0.7 0.4 0.2 0.4 0.7
59:	1 1 1 0 1 1 1 1 1 0	0.7 0.4 0.2 0.4 0.6
60:	1 0 1 1 1 1 1 1 1 0	0.6 0.4 0.2 0.4 0.7

Label	Object Set	Attribute Set
	o_1 o_2 o_3 o_4 o_5 o_6 o_7 o_8 o_9 o_{10}	p_1 p_2 p_3 p_4 p_5
61:	1 1 1 1 1 1 1 1 1 0	0.6 0.4 0.2 0.4 0.6
62:	0 0 0 0 0 0 0 0 0 1	0.5 0.8 0.4 0.4 0.9
63:	0 1 0 0 0 0 0 0 0 1	0.5 0.8 0.4 0.4 0.6
64:	0 0 0 1 0 0 0 0 0 1	0.5 0.8 0.3 0.4 0.8
65:	0 1 0 1 0 0 0 0 0 1	0.5 0.8 0.3 0.4 0.6
66:	0 0 0 0 1 0 0 0 0 1	0.5 0.6 0.4 0.4 0.9
67:	0 0 0 0 1 0 0 1 0 1	0.5 0.6 0.3 0.4 0.9
68:	1 0 0 1 1 0 0 1 0 1	0.5 0.6 0.3 0.4 0.8
69:	1 0 1 1 1 0 0 1 0 1	0.5 0.4 0.3 0.4 0.8
70:	1 0 1 1 1 1 0 1 0 1	0.5 0.4 0.2 0.4 0.8
71:	0 0 0 0 1 0 0 0 1 1	0.5 0.6 0.4 0.4 0.7
72:	0 1 0 0 1 0 0 0 1 1	0.5 0.6 0.4 0.4 0.6
73:	1 0 0 1 1 0 1 1 1 1	0.5 0.6 0.3 0.4 0.7
74:	1 1 0 1 1 0 1 1 1 1	0.5 0.6 0.3 0.4 0.6
75:	1 0 1 1 1 0 1 1 1 1	0.5 0.4 0.3 0.4 0.7
76:	1 1 1 1 1 0 1 1 1 1	0.5 0.4 0.3 0.4 0.6
77:	1 0 1 1 1 1 1 1 1 1	0.5 0.4 0.2 0.4 0.7
78:	1 1 1 1 1 1 1 1 1 1	0.5 0.4 0.2 0.4 0.6

Table A.1: Fuzzy Maximal Rectangles List

A.2 Successor List

Below we state the successor list of the concepts appearing within Table A-1.

0: 1 2 6 10 30 62
1: 3 4 7 11 31
2: 3 8 12 32 63
3: 5 13 33
4: 5 9 14 34
5: 15 35
6: 7 8 64
7: 9 26 36
8: 37 65
9: 27 38
10: 11 12 23 40 66
11: 13 14 17 24

12: 13 41
13: 15 18 43
14: 15 16 19 25
15: 20 45
16: 21 28
17: 18 19 50
18: 20 51
19: 20 21 52
20: 22 53
21: 22 58
22: 59
23: 24 67
24: 25 26 42
25: 27 28 44
26: 27 46 68
27: 29 48 69
28: 29 58
29: 60 70
30: 31 32 40
31: 33 34 36 42
32: 33 41
33: 35 37 43
34: 35 38 44
35: 39 45
36: 37 38 46
37: 39 47
38: 39 48
39: 49
40: 41 42 71
41: 43 72
42: 43 44 46 50
43: 45 47 51
44: 45 48 52
45: 49 53
46: 47 48 54
47: 49 55
48: 49 56
49: 57
50: 51 52 54
51: 53 55
52: 53 56 58

53: 57 59
54: 55 56 73
55: 57 74
56: 57 60 75
57: 61 76
58: 59 60
59: 61
60: 61 77
61: 78
62: 63 64 66
63: 65 72
64: 65 68
65: 74
66: 67 71
67: 68
68: 69 73
69: 70 75
70: 77
71: 72 73
72: 74
73: 74 75
74: 76
75: 76 77
76: 78
77: 78
78:

A.3 Implications

The complete list of minimal base implications extracted from Table 5.2 is as follows:

$$\Rightarrow p_1/.5, p_2/.4, p_3/.2, p_4/.4, p_5/.6 \quad (A.1)$$
$$p_2/0.6 \Rightarrow p_3/0.3 \quad (A.2)$$
$$p_1/0.6, p_2/0.8, p_3/0.3 \Rightarrow p_4/0.8 \quad (A.3)$$
$$p_3/0.4 \Rightarrow p_2/0.6 \quad (A.4)$$
$$p_1/0.6, p_2/0.6, p_3/0.4 \Rightarrow p_1/0.7, p_4/0.6 \quad (A.5)$$
$$p_2/0.6, p_3/0.5 \Rightarrow p_1/0.8, p_2/0.8, p_4/0.8 \quad (A.6)$$
$$p_4/0.6 \Rightarrow p_1/0.6, p_3/0.3 \quad (A.7)$$
$$p_1/0.7, p_2/0.8, p_3/0.3, p_4/0.8 \Rightarrow p_1/0.8, p_3/0.5 \quad (A.8)$$
$$p_1/0.8, p_2/0.6, p_3/0.4, p_4/0.8 \Rightarrow p_2/0.8, p_3/0.5 \quad (A.9)$$
$$p_1/0.6, p_3/0.3, p_4/1.0 \Rightarrow p_1/0.8, p_2/0.6, p_5/0.8 \quad (A.10)$$
$$p_2/0.8, p_3/0.3, p_5/0.7 \Rightarrow p_5/0.8 \quad (A.11)$$
$$p_1/0.8, p_3/0.3, p_4/0.6, p_5/0.7 \Rightarrow p_5/0.8 \quad (A.12)$$
$$p_1/0.6, p_3/0.3, p_5/0.8 \Rightarrow p_4/0.6 \quad (A.13)$$
$$p_2/0.6, p_3/0.4, p_5/0.8 \Rightarrow p_5/0.9 \quad (A.14)$$
$$p_1/0.7, p_3/0.3, p_4/0.8, p_5/0.8 \Rightarrow p_1/0.8 \quad (A.15)$$
$$p_1/0.8, p_2/0.6, p_3/0.3, p_4/0.8, p_5/0.8 \Rightarrow p_4/1.0 \quad (A.16)$$
$$p_5/0.9 \Rightarrow p_2/0.6, p_3/0.3 \quad (A.17)$$
$$p_2/0.8, p_3/0.3, p_5/0.9 \Rightarrow p_3/0.4 \quad (A.18)$$
$$p_1/0.6, p_2/0.6, p_3/0.3, p_4/0.6, p_5/0.9 \Rightarrow p_1/0.7 \quad (A.19)$$
$$p_1/0.8, p_2/0.6, p_3/0.3, p_4/0.6, p_5/0.9 \Rightarrow p_3/0.4 \quad (A.20)$$
$$p_1/0.7, p_2/0.6, p_3/0.4, p_4/0.6, p_5/0.9 \Rightarrow p_1/0.8 \quad (A.21)$$

Nomenclature

English Symbols

A, B, ...	crisp set of objects or attributes
R, S, ...	crisp binary relation or context
$\widetilde{A}$, $\widetilde{B}$, ...	fuzzy set of objects or attributes
$\widetilde{R}$, $\widetilde{S}$, ...	fuzzy binary relation or context
f, h	crisp galois connection operators
M	complete crisp attribute set of a crisp relation R
$\widetilde{M}$	complete fuzzy attribute set of a fuzzy relation $\widetilde{R}$
T	any subset of M
$\widetilde{T}$	any subset of $\widetilde{M}$
$\widetilde{P}_i$	candidate attribute set for premise of an implication
o_i	a single object
p_i/α	a single attribute of a fuzzy relation with degree of membership α
$f(x)$	mapping function for a real-to-fuzzy mapping

Greek Symbols

α	membership degree of an element in a fuzzy relation
α	constant fraction of rectangles being eliminated (Section 3.5 only)
$\mu(x)$	membership degree of an element x in a fuzzy set
$\widetilde{\phi}$	The empty attribute set
$\Re$	set of implications of a crisp relation
$\widetilde{\Re}$	set of implications of a fuzzy relation

Abbreviations

FCA	Formal Concept Analysis

Subscripts

i, j	any set of indices

Superscripts

^{R}, ^{Q}	crisp galois connection operators
$^{\tilde{R}}$, $^{\tilde{Q}}$	fuzzy galois connection operators

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