Symmetry Solutions of Some Nonlinear PDE's

By

Aijaz Ahmad

A Thesis Presented to the

DEANSHIP OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the

Requirements for the Degree of

Master of Science

In

Mathematics

October 2005

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN 31261, SAUDI ARABIA

DEANSHIP OF GRADUATE STUDIES

This thesis, written by AIJAZ AHMAD under the direction of his thesis advisor and approved by his thesis committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

THESIS COMMITTEE

Dr. A. H. Bokhari (Chairman)

Dr. F. D. Zaman (Co-Chairman)

Dr. A. Boucherif (Member)

Dr. K. M. Furati (Member)

Dr. S. Messaoudi (Member)

Dr. Suliman Al-Homidan
Department Chairman

Dr. Mohammad A. Al-Ohali
Dean of Graduate Studies

٢٣/٩/١٤٢٦هـ

Date 26-10-2005

عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

To my beloved parents.

ACKNOWLEDGEMENTS

All praise and glory to Almighty Allah who gave me the courage and patience to carry out this work. Peace and Blessings of Allah be upon His last prophet Muhammad (Sallallah-Alaihe-Wasallam), who guided us to the right path.

First and the foremost acknowledgement is due to the Department of Mathematical Sciences at King Fahd University of Petroleum & Minerals, which provided a genial research environment and and a wonderful working place. I am also indebted to KFUPM for supporting my research work.

My deep appreciation and heartfelt gratitude goes to my thesis advisor Dr. A. H. Bokhari for his constant endeavor, guidance and the encouragement throughout the course of this research work. His valuable suggestions made this work interesting and knowledgeable for me. Working with him in a friendly and motivating environment was really a joyful and learning experience—not easy to forget.

I extend my deepest gratitude to my thesis co-advisor Dr. F. D. Zaman, thesis committee members Dr. A. Boucherif, Dr. K. M. Furati and Dr. S. Messaoudi for their constructive and positive criticism, extraordinary attention and thought-provoking contribution in my research. It was surely an honor and an exceptional learning to work with them. I particularly appreciate the support of Dr. A. H. Kara and Dr. F. D. Zaman for their valuable suggestions and timely guidance.

I also take this opportunity to thank my parents who always prayed for my success and without whose continuous encouragement and unwavering support I would have never made this far!

TABLE OF CONTENTS

LIST OF TABLES

THESIS ABSTRACT

Name: Aijaz Ahmad.

Title: Symmetry Solutions of Some Nonlinear PDE's.

Major Field: Mathematics.

Date of Degree: October 2005.

Finding solutions of nonlinear partial differential equations, either exact or analytical, is one of the challenging problems in applied mathematics. In particular, the case of higher-order systems of nonlinear partial differential equations poses the most difficult challenge. Lie symmetry method provides a powerful tool for the generation of transformations that can be used to transform the given differential equation to a simpler equation while preserving the invariance of the original equation. Consequently, it enjoys a widespread application and has attracted the attention of many researchers.

In this research work a complete classification of a family of nonlinear (1+2)- dimensional wave equations, in which the nonlinearity is introduced through a function representing the wave speed, has been done. All possible symmetries of this wave equation are derived and a set of reductions to ordinary differential equations under two-dimensional sub-algebras is given.

خلاصة الرسالة

الإسم : إعجاز أحمد

عنوان الرسالة : حلول التناظر لبعض المعادلات التفاضلية الجزئية اللاخطية

التخصص : الرياضيات

تاريخ التخرج : أكتوبر 2005

إن إيجاد الحلول التامة التحليلية للمعادلات التفاضلية الجزئية الغير الخطية هو واحد من المسائل التي تشكل تحديا في الرياضيات التطبيقية. و بالأخص، فإن حالات جُمل المعادلات التفاضلية الجزئية اللاخطية تضع أمامنا أصعب التحديات. إن طريقة تناظر لي توفر أداة قوية لتوليد هذه التحويلات التي يمكن استخدامها لتحويل المعادلة التفاضلية المعطاة الى معادلة أبسط مع المحافظة على اللاتغير للمعادلة الأصلية. و نتيجة لذلك فإنها تتمتع بتطبيق واسع و قد جذبت انتباه العديد من الباحثين.

في هذا العمل البحثي فإننا سوف نعمل على حساب تصنيفا كاملا لعائلة من معادلات أمواج ذات (1+2) بعد، حيث أن اللاخطية قد قدمت من خلال دالة ممثلة لسرعة الموجة. لقد تم اشتقاق كل التناظرات الممكنة لهذه المعادلة الموجية وأعطيت مجموعة من الاختزالات لمعادلات تفاضلية عادية باستخدام جبريات جزئية ذات بعدين.

درجة الماجستير في العلوم الملك فهد للبتر

جامعة ول والمعادن

أكتوبر 2005

CHAPTER 1

INTRODUCTION

Most of the time the mathematical model of any physical problem includes differential equations, that is why many techniques have been discovered so far in order to find the solutions of differential equations. Although there are many wonderful techniques available but still there exist open problems needed to be solved. While dealing with higher order ordinary differential equations or a partial differential equation, for which there is no direct method to solve, we usually look for some transformations that can transform the differential equation into a class of known type or it reduces the differential equation either in order or in number of independent variables. The problem of finding such transformations that not only reduce the order of differential equation, in case of an ordinary differential equation (ODE), or reduces the number of independent variables, in case of a partial differential equation (PDE), but the given differential equation also remains invariant under these transformations is of great interest.

Majority of cases in which exact solutions of a differential equation can be found, the underlying property is symmetry of that equation [46]. Apparently unrelated methods, such as integrating factor, reduction of order, separable, homogeneous or exact solutions, conservation laws, invariant solutions or invertible linear transformations are in fact special cases of a general integration procedure based on the invariance of the differential equation under a continuous group of symmetries [39].

This relation was first discovered by the Norwegian mathematician Sophus Lie in the 19th century. On the basis of his findings Lie developed a noteworthy theory that gives rise to a creative mechanism for solving differential equations. Lie's fundamental discovery was that, in the case of a continuous group, the complicated nonlinear conditions of invariance of the system under the group transformations could be replaced by much simpler equivalent linear conditions reflecting a form of infinitesimal invariance of the system under the generators of the group. In almost every physically important system of differential equations, these infinitesimal symmetry conditions, called defining equations of the symmetry group of the system, can be explicitly solved in closed form and thus the most general continuous symmetry groups of the system can be explicitly determined [39]. Lie's continuous groups, known as Lie groups, have a wide range of applications in many different pure and applied areas and disciplines of mathematics and physics including algebraic topology, differential geometry, bifurcation theory and numerical analysis.

In the last two decades an enormous amount of research has been done in the field, both in the application to concrete physical system as well as extension of the scope and depth of the theory itself [9, 14, 19, 22, 26, 27, 29, 31, 41, 44, 53].

Lie symmetry group of transformations depends on continuous parameters, and it maps solutions of differential equations to other solutions. In classical framework of Lie, these groups consists of geometric transformations on the space of independent and dependent variables for the system and act on solutions by transforming their graphs. Most common examples of these transformations are: groups of translations, groups of rotations and groups of scaling symmetries, but still there are huge range of possibilities. Contrary to discrete symmetries such as reflections, continuous symmetry groups have great advantage that they all can be found using explicit computational methods. Introduction of continuous groups transforms complex nonlinear conditions expressing the invariance of a differential equation, under its symmetries, into linear conditions which expresses the infinitesimal invariance of the equation under the group generators. These infinitesimal generators can be calculated directly by a straightforward algorithm which is so mechanical that several computer packages are available to perform the calculations [52]. Because of this reason these generators are of prime interest in the theory.

By Lie's fundamental theorems, the infinitesimal generator completely characterizes the structure of the Lie symmetry group and thus the corresponding Lie algebra under the commutation operator [11, 39, 46]. After being determined, a symmetry group of a differential equation has many applications. New solutions of the system can be constructed using the defining property of such a group, from known solutions and thus build up classes of equivalent solutions, where equivalence means one solution can be reached by applying a symmetry to a different solution. Even if a given differential equation cannot be solved completely through use of its Lie group of point symmetries, the Lie group can still be used to determine what are known as invariant solutions, also known as similarity solutions or group invariant solutions. Invariant solutions are those solutions that are invariant under a particular symmetry or a subgroup of the Lie group and have proved to be exceptionally important in the area of symmetry analysis, particularly for PDEs. On occasion, it is often prudent to search for particular types of solutions to a given PDE, such as travelling waves or separable solutions, in fact, such approaches are precisely the same as looking for solutions that are invariant under a particular group of transformations. The symmetry group thus provides a means of classifying different symmetry classes of solutions, where two solutions are deemed to be equivalent if one can be transformed into the other by some group element. Symmetry groups can also be used to effect a classification of families of differential equations depending on arbitrary parameters or functions. Often there are good physical or mathematical reasons for preferring these equations with as high a degree of symmetry as possible. Types of differential equations that admit a prescribed group of symmetries can also be determined by infinitesimal methods using the theory of differential invariants.

In case of ODEs invariance under a one-parameter Lie group of transformations means that the order of the equation can be reduced by one and a single quadrature recovers the solutions of the original equation from those of the reduced equation, for first order ODE this is equivalent to determining the solutions explicitly [46]. Any ODE that has Lie symmetries equivalent to translations and a change of coordinates can be integrated directly, such coordinates are known as canonical coordinates. For higher order ODE multi-parameter symmetry groups beget further reductions in order via quadratures, but it requires group to satisfy an additional solvability requirement, solutions of the original equation may not recover from the solutions of the reduced equation by the quadratures only.

The invariance of partial differential equations under Lie groups of transformations is not quite straightforward like ODEs. Invariance under a one parameter Lie group of transformations reduces a PDE with two independent variables to an ODE; if the number of variables in a particular PDE is more than two, say n, then invariance under an m-parameter Lie group causes reduction in the number of independent variables in PDE by m. Symmetry groups are helpful in determining explicitly special types of solutions which are themselves invariant under some subgroup of the full symmetry group of the system. These group invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables then the original system [11,39]. These general group-invariant solutions include the classical similarity solutions coming from groups of scaling symmetries and travelling wave solutions reflecting some form of translational invariance in the system, as well as many other explicit solutions of direct physical or mathematical importance. For many nonlinear systems, there are only explicit exact solutions available. These solutions play an important role in both mathematical analysis and physical applications of the systems.

A lot of research is being done in the classification of symmetries [19,23,25,31,32, 34,53], linearizing transformations and invariant solutions [18,20,35–38,41,43,44]. A reference book containing symmetries of many PDEs was authored by Ibragimov [24]. Lie classical symmetries have many applications to differential equations and their solutions. There are many extensions to the classical symmetry method that expands the uses of symmetry analysis as a whole [4–6, 13, 42, 45, 49, 50]. Transformations that act as diffeomorphism (differentiable + homeomorphism) on the subset of the jet space, by a process known as prolongation, are known as generalized symmetries. The infinitesimal generators of these transformations depend on derivatives of dependent and independent variables up to a finite order. If this dependence is only up to first order derivatives, then these symmetries are called contact symmetries. These symmetries are also referred to as dynamical symmetries, internal symmetries, Lie-Backlund transformations and higher-order symmetries. There is a class of point transformations that are not symmetries at all, but can lead to exact solutions of PDEs; these symmetries are called non-classical symmetries. Notion of non-classical symmetries is given by Bluman and Cole [11, 12]; in the literature these symmetries are also referred to as conditional symmetries because the solutions obtained by these are not achievable through classical method. Symmetries generated by infinitesimal transformations are local symmetries, local due to the reason that the infinitesimals are well defined at any point if the solution of differential equation is sufficiently smooth in the neighborhood of this point. A symmetry is non-local if it depends upon integrals of dependent variables, Bluman and Kumi [11] gives the concept of potential symmetries build on the definition of non-local symmetries.

The symmetry analysis of (1+1)-dimensional nonlinear wave equation has been done by many authors [8, 13, 18, 21, 32, 34, 36, 48]. The two-dimensional (1+2) wave equation with constant coefficients has been studied with an equivalent vigor [17, 51]. However, the group theoretic approach to the equation with non-constant coefficients and the non-linear case have only been studied in specific cases and, here too, complete results have either not been attained or not presented so that very few exact solutions invariant under symmetry are known [7]. Recently Gandarias et.al. [21] has discussed (1+1)-dimensional nonlinear wave equation using Lie symmetry method. Complete group classification is presented and achieved optimal system of a nonlinear wave equation. Observing that a complete classification gives more elaborative insight of a differential equation, an in-depth study of a family of nonlinear (1+2)-dimensional wave equation has been done. Group classification and reduction to an ODE under two dimensional closed Lie algebras are obtained.

Organization of the thesis is as follows: in the first chapter a brief description of partial differential equations is given, chapter two is about the basic concepts, definitions and theorems required to find the Lie symmetries. In chapter three a detailed step by step procedure to find Lie symmetries of a PDE is given with examples. The last chapter is solely about the work done during the research, a simple case of a class of nonlinear wave equations is given following with two more general cases.

CHAPTER 2

PARTIAL DIFFERENTIAL EQUATIONS

2.1 Introduction

Partial differential equations (PDEs) are one of the fundamental areas of interest in applied analysis. The applications arise almost in all areas of science and engineering. Most of the Physical processes cannot be modeled mathematically by ordinary differential equations (ODEs) because the parameters defining the system depend on more than one parameter. For example, the temperature 'u' in a bar of length 'l' depends on the location 'x' in the bar and the time 't' from when the initial conditions were applied.

A PDE is an identity that relates more than one independent variables $x,\ y,\ z, ...$, a dependent variable $u(x,\ y,\ z, ...)$ (the number of dependent variables can be more than one) and the partial derivatives of u. Derivatives are usually denoted by subscripts i.e. $\partial u/\partial x = u_x$, and similarly for higher order derivatives. For PDEs the distinction between dependent and independent variables is always kept unlike to ODEs where the relation of dependent and independent variables can be interchanged for instance in order to solve the differential equation. One of the main aims of the studies in this regard is to find exact solutions of these PDE's. While a reasonably comprehensive theory exists for linear PDE's, the nonlinear PDE's still needs a lot to be done as there is no unified theory applicable to a wider class of nonlinear PDE's. This aspect will be on focus of attention in this thesis. In particular, we shall be considering nonlinear wave equations.

Definition 2.1 *A second order PDE in two independent variables x (known as spatial or position coordinate) and t (known as time coordinate) and one dependent variable $u(x,t)$ is an equation of the form,*

$$H(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0. \tag{1}$$

A solution $u = u(x,t)$ of a PDE is a function, which is twice continuously differentiable and that reduces (1) *to an identity for (x,t) in $\mathcal{D}$; the domain of definition for PDE [33].*

The condition of being twice continuously differentiability is due to the second order derivatives involved in (1). Graphically the solution is a smooth surface in three dimensional xyt-space, over the domain $\mathcal{D}$ in xt-plane. The domain $\mathcal{D}$ of the problem is a space-time domain. Problems that include time as an independent variable are called **evolution problems**. When two spatial coordinates, say x & y, are independent we refer to the problem as an **equilibrium** or **steady state problem** [33]. Similar to the general solution of an ODE, a PDE of type (1) has infinitely many solutions as the general solution of a PDE depend on arbitrary functions.

Example 1 *Consider a PDE,*

$$u_{tx} = tx,$$

integration with respect to x gives,

$$u_t = \frac{1}{2}tx^2 + f(t),$$

where f is an arbitrary function. Again integration by t gives the general solution,

$$u = \frac{1}{4}t^2x^2 + g(t) + h(x),$$

where $h(x)$ and $g(t) = \int f(t)dt$ are arbitrary functions. Thus the general solution depends on two arbitrary functions.

2.1.1 Initial and boundary value problems

The general solution of a PDE contains unknown functions similar to unknown constants in case of ODE's. In order to find the exact solution of a PDE we need initial or boundary conditions. An initial condition prescribes the unknown function at a fixed time $t = t_0$ whereas a boundary conditions gives its value on a curve or a surface. A condition given along any other curve in the xt-plane is called *boundary condition.* PDEs with auxiliary conditions are called *boundary value problems.* A general solution of a PDE has arbitrary functions involved in its expression. A boundary value problem consists of a PDE and corresponding initial or/and boundary conditions.

Definition 2.2 *A boundary value problem is said to be well posed if,*

1. *It has a solution.*
2. *This solution is uniquely determined.*
3. *The solution is stable, i.e. a small change in the boundary data induces only a small change in the solution.*

2.1.2 Linear and Nonlinear PDE

A PDE in the form of an operator L is given as,

$$L\ u(x,t) = f(x,t), \qquad\qquad (x,t) \in \mathcal{D} \tag{2}$$

where L is a partial differential operator. The equation (2) is a linear equation if for any functions u , v and constants α , β , it satisfies the condition,

$$L\ (\alpha\ u + \beta\ v) = \alpha\ L\ u + \beta\ L\ v. \tag{3}$$

If equation (2) does not satisfy the above condition, then it is called a nonlinear PDE.

Example 2 *Consider the heat equation* $u_t - ku_{xx} = 0$. *For this equation the differential operator is,*

$$L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\ .$$

It is easily seen that the above equation is linear because it satisfies the linearity criterion (3) *as shown below,*

$$\begin{aligned} L\,(\alpha\ u + \beta\ v) &= (\alpha\ u + \beta\ v)_t - k(\alpha\ u + \beta\ v)_{xx}\ , \\ &= \alpha\ u_t + \beta\ v_t - k\ \alpha\ u_{xx} - k\ \beta\ v_{xx}\ , \\ &= \alpha\ \ (u_t - k\ u_{xx}) + \beta\ \ (v_t - k\ v_{xx})\ \ , \\ &= \alpha\ Lu + \beta\ Lv. \end{aligned}$$

Example 3 *The PDE* $Lu = uu_t + 2txu = 0$ *is a nonlinear equation because,*

$$L(u + w) \neq Lu + Lw.$$

2.1.3 Homogeneous PDEs

A PDE is called homogeneous if in equation (2) the known function $f(x,t) = 0$ on the domain $\mathcal{D}$. Examples of some homogeneous equations are:

1. $u_{xx} + u_{yy} = 0$ (Laplace's equation)

2. $u_{tt} - c^2 u_{xx} = 0$ (Wave equation)

3. $u_t - K u_{xx} = 0$ (Heat equation)

4. $u_{tt} - c^2 u_{xx} + 2\beta u_t + \alpha u = 0$ (Telegraph equation)

5. $u_t + u_{xxx} + u u_x = 0$ (Korteweg-de Varies equation)

If the known function $f(x,t)$ in (2) is not identically zero, then it is called a ***non-homogeneous*** equation. Some examples of non-homogeneous PDEs are:

1. $u_{xx} + u_{yy} = G.$ (Poisson's equation)

2. $u u_t + 2txu = sin(tx).$

2.1.4 Superposition Principle

It is not always possible to write the general solution of a PDE in a closed form, therefore the method of combining known solutions is very important. For homogeneous equations the rule for combining the known solutions is called the superposition principle.

Theorem 2.1 *Let $u_1,\ u_2, ..., u_n$ be solutions of the homogeneous linear PDE equation $Lu = 0$, then due to the linearity,*

$$L\Big(\sum_{i=1}^{n} u_i\Big) = \sum_{i=1}^{n} L(u_i),$$

and,

$$L(c_i u_i) = c_i L(u_i),$$

where c_i *are constants* $\forall \quad i = 1, 2, ..., n,$ *we have,*

$$L\Big(\sum_{i=1}^{n} c_i u_i\Big) = 0,$$

thus the linear combination of solutions is also a solution of the PDE; this is called the superposition principle.

Superposition principle can be extended to infinite sums. If the differential operator can be shifted inside the integral sign then the applicability of superposition principle can be extended to continuous cases for example let $u(x, t, \alpha)$ be solution of (2) $\forall \ \alpha \in \mathcal{A} \subset \mathbb{R}$, and

$$u(x, t) = \int_{\mathcal{A}} c(\alpha) \ u(x, t, \alpha) d\alpha$$

where $c(\alpha)$ is bounded and continuous function $\forall \ \alpha \in \mathcal{A}$. Then $Lu(x, t) = 0$ i.e. $u(x, t)$ is also a solution of (2).

2.1.5 Subtraction Principle

Superposition principle is only applicable to the homogeneous PDE's. A principle that relate nonhomogeneous equations to homogeneous equations is known as *subtraction principle* [40].

Theorem 2.2 *If* u_1 *and* u_2 *are solutions of a nonhomogeneous linear equation* $Lu = f(x, t)$, *then* $u_1 - u_2$ *is a solution of the associated homogeneous linear equation* $Lu = 0$.

Once we know the particular solution of a nonhomogeneous PDE and a general solution of the associated homogeneous PDE, we can find the general solution of the nonhomogeneous equation.

2.1.6 Types of PDEs

The general form of a linear second order PDE in two variables is,

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{4}$$

where $A,\ B,\ C,\ D,\ E,\ F,\ G$ are functions of (x, y).

Depending on the second order coefficients $A,\ B$ and C, second order linear PDE's are classified in three fundamental types known as parabolic, elliptic and hyperbolic equations [47], as follows,

1. If $AC = B^2$, then PDE is called *Parabolic equation.*
2. If $AC > B^2$, then PDE is called *Elliptic equation.*
3. If $AC < B^2$, then PDE is called *Hyperbolic equation.*

Remark 2.1 *Parabolic equations govern diffusion processes, elliptic equations model processes in equilibrium processes and hyperbolic equations govern wave propagation. The examples of elliptic equations are Poison and Laplace equations whereas wave and telegraph equations are hyperbolic and the heat equation is a parabolic equation.*

2.2 Wave Equation

Waves are defined as disturbances which are periodic in time and space. The most common examples are water waves, sound waves, stress waves in solids and electro-magnetic waves. The convection of mater itself with wave is not necessary, energy is carried by the disturbance that propagate with wave. Mathematical model of an undistorted wave travelling with a constant velocity 'c' in two independent coordinates space 'x' and time 't' and one dependent coordinate 'u' for disturbance is given as a function,

$$u(x,t) = f(x - ct). \tag{5}$$

Initially at $t = 0$ the wave equation is $u = f(x)$, for time $t > 0$ the wave moves 'ct' units to right. The simplest partial differential equation that governs the equation of type (5) is,

$$u_t + cu_x = 0. \tag{6}$$

This equation is called the *advection equation* with general solution (5). Similarly periodic or sinusoidal travelling waves are given by,

$$u = A\cos(kx - \omega t), \tag{7}$$

here A is the amplitude, k is the wave number and ω is the angular frequency of the wave. Wavelength and time period are respectively given by $\lambda = 2\pi/k$ and $T = 2\pi/\omega$. Equation (7) can be written as,

$$u = A\ cosk\Big(x - \frac{\omega}{k}t\Big),$$

that represents a travelling wave moving to the right with velocity $c = \omega/k$, known as phase velocity. Undistorted waves are linear waves.

There are many waves that distort or break with time. These are all nonlinear waves. The examples of such waves are surface waves and stress waves propagating in solids or gases. Transmission of signals in a material increases with pressure causing disturbance, since disturbances travel faster when the pressure is higher therefore the wave steepens as time passes until it propagates as a discontinuous disturbance or shock wave. The same phenomenon causes the formation of release waves or rarefaction waves that lower the pressure. Apart from these two trends there is one third known as dispersion, in which the propagation speed depends on the wavelength of the particular wave therefore longer waves can travel faster than waves of shorter wavelength. Dispersive wave arises from both linear and nonlinear equations.

2.2.1 Linear Waves

A simple (1+1) second order linear wave equation is given as,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ . \tag{8}$$

Theorem 2.3 *The general solution of the equation* (8) *is,*

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are two arbitrary functions.

Proof *Let $u(x,t)$ be a solution of (8), therefore $u(x,t)$ is a twice differentiable smooth function i.e. $u \in \mathcal{C}^2(\mathcal{R}^2)$ where $\mathcal{R}^2$ denotes the xt-plane. Introducing new variable ξ, τ and v such that,*

$$\xi = x + ct, \qquad \tau = x - ct \qquad \textit{and} \qquad v = v(\xi,\tau) = u\left(\frac{\xi+\tau}{2}, \frac{\xi-\tau}{2c}\right),$$

we have

$$\frac{\partial^2 v}{\partial \xi \partial \tau} = 0. \tag{9}$$

Integration with respect to ξ gives,

$$\frac{\partial v}{\partial \tau} = g(\tau), \qquad \textit{or} \qquad \frac{\partial}{\partial \tau}(v - G) = 0,$$

where $g \in \mathcal{C}^1(\mathcal{R})$ and $G \in \mathcal{C}^2(\mathcal{R})$ such that $\frac{\partial G}{\partial \tau} = g(\tau)$. Further integration of the above equation with respect to τ gives,

$$v(\xi,\tau) = G(\tau) + F(\xi), \qquad \textit{where} \qquad F \in \mathcal{C}^2(R).$$

By transforming the last equation back into coordinates x, t *and* u *we obtain the solution of equation* (8) *given as,*

$$u(x,t) = F(x+ct) + G(x-ct).$$

Example 4 (Initial Value Problem) *Corresponding to the simplest PDE governing wave equation*

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < +\infty, \tag{10}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{11}$$

where u_0 *and* u_1 *are any functions in* $\mathcal{C}^2(R)$ *and* $\mathcal{C}^1(R)$ *respectively.*

Solution

The general solution of the wave equation is given as $u(x,t) = F(x+ct)+G(x-ct)$. *Therefore the initial conditions becomes,*

$$F(x) + G(x) = u_0(x), \tag{12}$$

$$cF'(x) - cG'(x) = u_1(x). \tag{13}$$

Solving (12) *and* (13) *for* $F(x)$ *and* $G(x)$ *we get,*

$$F(x) = \frac{1}{2}\left\{u_0(x) + \frac{1}{c}\int_0^x u_1(\xi)d\xi + k_1\right\}, \tag{14}$$

$$G(x) = \frac{1}{2}\left\{u_0(x) - \frac{1}{c}\int_0^x u_1(\xi)d\xi - k_2\right\}. \tag{15}$$

Thus the solution of the initial value problem is,

$$u(x,t) = \frac{1}{2}\big\{u_0(x+ct) + u_0(x-ct)\big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)d\xi \ . \tag{16}$$

The formula (16) *is called* d'Alembert formula.

2.2.2 Nonlinear Waves

Corresponding to the simplest PDE (6), governing wave equation, we have the nonlinear case,

$$u_t + f(u)u_x = 0, \qquad x \in \mathbb{R}^1, \qquad t > 0, \tag{17}$$

$$u(x,0) = g(x), \qquad x \in \mathbb{R}^1 \tag{18}$$

where $f'(u) > 0$. The characteristic equation of this PDE is given by,

$$\frac{dx}{dt} = f(u), \tag{19}$$

along these characteristic curves u is constant, because,

$$\frac{du}{dt} = u_x \ f(u) + u_t = 0.$$

Also, from (19) it is clear that,

$$\frac{d^2x}{dt^2} = \frac{df(u)}{dt} = f'(u)\frac{du}{dt} = 0,$$

therefore characteristic curves are straight lines.

Now from equation (19), by using the initial condition (18) we obtain

$$x = f(g(\xi)) \ t + \xi. \tag{20}$$

This is the equation of characteristic curves. The solution $u(x,t)$ of the initial value problem (17) is given by,

$$u(x,t) = g(\xi), \tag{21}$$

where ξ is given implicitly by (20).

Example 5 (Damped waves) *Consider the equation,*

$$u_t + u\ u_x + a\ u = 0, \qquad a \ \textit{ is a positive constant}, \qquad t > 0, \tag{22}$$

$$u(x,0) = -\frac{x}{2}\ . \tag{23}$$

In characteristic form the problem is equivalent to,

$$\frac{du}{dt} = -au, \qquad \frac{dx}{dt} = u, \tag{24}$$

with initial condition,

$$x = \xi, \qquad u = -\frac{\xi}{2}, \qquad at \qquad t = 0.$$

The general solution is given as,

$$u = c_1 e^{-at}, \qquad x = -\frac{c_1}{a}\ e^{-at} + c_2\ .$$

By applying initial conditions we get,

$$u = \ -\frac{\xi}{2}\ e^{-at}, \qquad x = \ \frac{\xi}{2a}\ e^{-at} + \xi\left(\frac{2a-1}{2a}\right),$$

which gives,

$$\xi = \ \frac{2\ a\ x}{2a - 1 + e^{-at}},$$

thus the solution of (22) *is,*

$$u(x,t) = \ \frac{a\ x\ e^{-at}}{1 - 2a - e^{-at}}\ .$$

2.2.2.1 Burgers' Equation

Burgers' equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as modelling of gas dynamics and traffic flow.

Definition 2.3 *The general form of Burgers' equation is,*

$$u_t + uu_x = \mu u_{xx}, \tag{25}$$

here $\mu > 0$ *is a viscosity coefficient and the term* $\mu\ u_{xx}$ *is called the diffusion term. When* $\mu = 0$, *Burgers' equation becomes the inviscid Burgers' equation,*

$$u_t + uu_x = 0. \tag{26}$$

It is similar in form to the advection equation. This is similar to nonlinear equation described in example (5).

Example 6 (Burgers' Equation without Diffusive Term) *Substituting* $f(u) = u$ *and* $g(x) = u(x, 0) = 2 - x$ *in equation* (17), *will give inviscid Burgers' equation (or Burgers' equation without diffusive term) with an initial condition. From* (20) *we find the equation of characteristic curves given as,*

$$x = f(2 - \xi)t + \xi = (2 - \xi)t + \xi,$$

solving this equation for ξ *we obtain,*

$$\xi = \frac{x - 2t}{1 - t}\ .$$

Therefore from equation (21), *the solution of viscid Burgers' equation is,*

$$u(x, t) = g(\xi) = 2 - \frac{x - 2t}{1 - t} = \frac{2 - x}{1 - t}\ .$$

The expression for the solution indicates the breaking time $t = 1$.

Example 7 (Burgers' Equation with Diffusive Term) *Consider the Burger Equation with diffusive term* μu_{xx} *where* $\mu > 0$, *given as,*

$$u_t + uu_x = \mu u_{xx}, \tag{27}$$

we are interested in the travelling wave solution of this equation.

Let $u(x,t) = f(x-ct)$ *be the solution of* (27), *we need to find the expressions for* f *and* c. *Let us denote* $\omega = x - ct$ *for simplicity. Substituting* $f(\omega)$ *in* (27) *we get,*

$$-cf' + ff' - \mu f'' = 0,$$

where,

$$f' = \frac{df(\omega)}{d\omega}\ , \qquad \text{and} \qquad f'' = \frac{d^2 f(\omega)}{d\omega^2}\ .$$

Integration with respect to ω *yields,*

$$-\,cf + \frac{1}{2}f^2 - \mu f' = \alpha,$$

(α *being constant of integration)*

$$\frac{df}{d\omega} = \frac{1}{2\mu}f^2 - \frac{c}{\mu}f - \frac{\alpha}{\mu}\ ,$$

$$f' = \frac{(f-f_1)(f-f_2)}{2\mu}\ , \tag{28}$$

where $f_1 = c - \sqrt{c^2+2\alpha}$ *and* $f_2 = c + \sqrt{c^2+2\alpha}$ *are the roots of quadratic equation* $f^2 - 2cf - 2\alpha$. *Assuming that* $c^2 + 2\alpha > 0$ *and integrating* (28) *we get,*

$$\frac{\omega}{2\mu} = \int \frac{df}{(f-f_1)(f-f_2)}\ ,$$

$$= \frac{1}{f_2 - f_1}\ \ln\left\{\frac{f_2 - f}{f - f_1}\right\}\ ,$$

$$\ln\left\{\frac{f_2 - f}{f - f_1}\right\} = \frac{(f_2 - f_1)}{2\mu}\ \omega\ ,$$

$$f(\omega) = \frac{f_2 + f_1\ e^{\beta\omega}}{1 + e^{\beta\omega}}\ , \tag{29}$$

where,

$$\beta = \frac{(f_2 - f_1)}{2\mu} > 0.$$

Also $f \sim f_1$ *and* $f \sim f_2$ *for* $\omega >> 0$ *and* $\omega << 0$ *respectively. The value of* f *at* $\omega = 0$ *is* $f(0) = \frac{(f_1+f_2)}{2}$ *. Therefore the solution of* (27)*, using* (7)*, is given as,*

$$u(x,t) = f(x - ct) = \frac{f_2 + f_1 \ e^{\beta(x-ct)}}{1 + e^{\beta(x-ct)}} \ ,$$

and the value of c *can be calculated from* f_1 *and* f_2 *using the formula,*

$$c = \frac{f_1 + f_2}{2} \ .$$

From the above two examples it is clear that when the diffusion term $\mu \ u_{xx}$ is absent in Burgers' equation, the solution would shock up or break. In the presence of diffusion term the effect of shock or breaking trend in the solution will reduce. The shock effect is inversely proportional to the diffusion term involved.

2.2.2.2 Korteweg-de Vries Equation

In Burgers' equation the nonlinearity comes with diffusion. The nonlinear term $u \, u_x$, that causes the shocking-up effect, is balanced by diffusive term $\mu \ u_{xx}$. In many physical problems related to wave motion, the resulting equations involve nonlinearity with dispersion, for example $u_t + u \ u_x + u_{xxx} = 0$. In this equation the shocking effect caused by $u \ u_x$ is balanced with dispersive term u_{xxx}. Nonlinearity steepens wavefronts whereas dispersion spread them out. Equation governing this type of problem is called Korteweg-de Vries (KdV) equation.

The KdV equation also models long waves in shallow water [16]. It expresses the balancing of the nonlinear steepening of shallow water waves by the effect of linear dispersion. The general form of the equation is given as,

$$u_t + \frac{\alpha\beta}{\gamma} u u_x + \frac{\beta}{\gamma^3} u_{xxx} = 0. \tag{30}$$

Example 8 (Solitary waves) *Solitary-wave solution of the KdV equation is a travelling wave of permanent form. It is a special solution of the governing equation as it does not change the shape and propagates at constant speed. Consider the KdV equation,*

$$u_t - 6uu_x + u_{xxx} = 0, \tag{31}$$

this is same as (30) *with constant terms* $\frac{\alpha\beta}{\gamma}$ *and* $\frac{\beta}{\gamma^3}$ *replaced by* -6 *and* 1 *for simplification.*

Let $u(x,t) = f(\xi)$ *where* $\xi = x - ct$, *'c' being a constant, be the travelling wave solution of this equation. Substitution in equation* (31) *will give,*

$$-cf' - 6ff' + f''' = 0,$$

which after integration gives,

$$-cf - 3f^2 + f'' = \alpha,$$

where α *is constant of integration. Using* f' *as integrating factor and integrating second time, yields,*

$$\frac{(f')^2}{2} = f^3 + \frac{cf^2}{2} + \alpha f + \beta,$$

where β *is the second constant of integration. Applying the boundary conditions* $f, f', f'' \to 0$ *as* $\xi \to \pm\infty$ *describing solitary wave, we obtain,*

$$(f')^2 = f^2(2f + c). \tag{32}$$

It will have a real solution if $2f > -c$. *Now by integrating equation* (32), *we have,*

$$\int \frac{df}{f(2f+c)^{\frac{1}{2}}} = \pm \int d\xi,$$

by using the substitution $f = -\frac{1}{2}\ c\ sech^2\theta,\quad (c \geq 0)$ *we get the solution,*

$$f(x-ct) = -\frac{1}{2}\ c\ sech^2\left\{-\frac{1}{2}\ c^{\frac{1}{2}}(x-ct-x_0)\right\}, \tag{33}$$

where x_0 *is constant of integration. The expression* (33) *is called the solitary wave solution.*

CHAPTER 3

LIE SYMMETRIES

3.1 Introduction

Lie symmetry method is a powerful technique that relates seemingly different methods for finding the solution of ODEs like, integrating factor, separable equation, homogeneous equation, reduction of order and methods of undetermined coefficients etc. Lie group depends on continuous parameters and consists of point transformations acting on the space of independent variables, dependent variables and derivatives of dependent variables. These continuous group of point transformations can be determined by an explicit computational algorithm.

Common examples of Lie groups include translations, rotations and scalings. By the application of one-parameter Lie group of point transformations, under which differential equation remain invariant, order of an ODE reduces by one and in case of PDE the number of independent variables reduces by one. Lie groups are completely characterized by infinitesimal generators which can be further prolonged to the space of independent variables, dependent variables and the derivatives of the dependent variables up to any finite order. Thus nonlinear conditions of group invariance of a given system of differential equations reduce to linear homogeneous system determining the infinitesimal generator of the group.

Similarity solutions can be found for an invariant system of partial differential equations under a Lie group of point transformations. Solutions thus found are invariant under a subgroup of the complete group. A subgroup of order r reduces the number of independent variables by r in the given PDE.

3.2 Lie Groups

Lie groups are important in mathematical analysis, physics and geometry because they serve to describe the symmetry of analytical structures. Lie groups arise as groups of symmetries of some object, or more precisely, as local groups of transformations acting on some manifolds.

3.2.1 Groups

Definition 3.1 *A group is a set G together with a binary operation '$*$' called group operation, satisfying following properties:*

1. **Closure**

   For any two elements α and β of group G there exist an element $\gamma \in G$ such that,

   $$\alpha * \beta = \gamma.$$

2. **Associativity**

   For any three elements α, β and γ in G,

   $$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma.$$

3. **Identity Element**

   There exists a unique element e in G such that,

   $$\alpha * e = e * \alpha = \alpha, \qquad \forall \ \alpha \in G.$$

4. **Inverse Element**

   For each element α *in* G *there exists a unique element* α^{-1} *in* G *such that,*

$$\alpha * \alpha^{-1} = \alpha^{-1} * \alpha = e.$$

Definition 3.2 (Abelian Group) *A group* G *is said to be Abelian if in addition to above properties it satisfies the property:*

$$\alpha * \beta = \beta * \alpha, \qquad\qquad \forall \ \ \alpha, \beta \in G.$$

Definition 3.3 (Subgroup) *Let* H *be a subset of* G*. Then* H *is said to be a subgroup of* G *if it satisfies all the conditions of the group* $(G, *)$ *under the same binary operation '$*$'.*

Example 9

1. *The set* $\mathbb{Z}$*, of integers is a group under group operation '$+$'. The identity element of the group* $(\mathbb{Z}, +)$ *is* 0 *and the inverse of each element* $\alpha \in \mathbb{Z}$ *is* $-\alpha$*. It is also an abelian group.*

2. *Another example of abelian group is the group* $(\mathbb{R}, +)$*, where* $\mathbb{R}$ *is set of real numbers, with identity element* 0 *and the inverse of each element* α *is* $-\alpha$*. Since* $\mathbb{Z} \subset \mathbb{R}$ *therefore the group* $(\mathbb{Z}, +)$ *is a subgroup of* $(\mathbb{R}, +)$*.*

3. *Similarly* $(\mathbb{R}\backslash\{0\}, \times)$ *is a group having identity element* 1 *and the inverse of each element* α *is* $1/\alpha$*.*

3.2.2 Groups of Transformations

Definition 3.4 *Let* G *be a set of transformations and* $G_i \in G$ *such that,*

$$G_i : \alpha \longrightarrow \tilde{\alpha}(\alpha; \epsilon),$$

where α *and* $\tilde{\alpha}$ *both belong to the set* $\mathcal{S} \subset \mathbb{R}^n$ *and the parameter* $\epsilon \in \mathcal{A} \subset \mathbb{R}$ *with composition law* $\psi(\epsilon, \delta)$ *for all* $\epsilon,\ \delta \in \mathcal{A}$, *satisfying the conditions:*

1. G_i *is a one-to-one transformation for each* i *and for all* $\epsilon \in \mathcal{A}$.

2. $(\mathcal{A}, \psi)$ *is a group.*

3. *For the identity element* e *of the group* $(\mathcal{A}, \psi)$, $\tilde{\alpha} = \alpha$ *i.e.*

$$G_i(\alpha; e) = \alpha, \qquad\qquad \forall\ i.$$

4. *Let* $\tilde{\alpha} = G_i(\alpha; \epsilon)$ *then,*

$$\tilde{\tilde{\alpha}} = G_i(\tilde{\alpha}; \delta) = G_i(\alpha; \psi(\epsilon, \delta)).$$

3.3 Lie Groups of Transformations

Definition 3.5 *A transformation group* G *with composition law* ψ *is said to be a Lie group of transformations of one-parameter if:*

1. *The parameter* ϵ *is continuous. i.e. the set* $\mathcal{A}$ *is an interval in* $\mathbb{R}$.

2. *Each element* G_i *of the group* G *is an infinitely differentiable function of* $\alpha \in \mathcal{S} \subset \mathbb{R}^n$.

3. *The composition function* $\psi(\epsilon, \delta)$ *is an analytic function.*

Equivalently, a group of infinitesimal point transformations of one parameter ϵ is a transformation group that is invertible and has an identity transformation. Being an invertible means that repeated application of the transformation leads to the transformation of the same family. Mathematically, this statement can be recast as:

Let $\tilde{x} = x(x, y, ; \epsilon)$ and $\tilde{y} = y(x, y, ; \epsilon)$ be a transformation group such that

1. $\tilde{\tilde{x}} = \tilde{\tilde{x}}(\tilde{x}, \tilde{y}; \tilde{\epsilon}) = \tilde{\tilde{x}}(x, y; \tilde{\tilde{\epsilon}})$ for some $\tilde{\tilde{\epsilon}} = \tilde{\tilde{\epsilon}}(\tilde{\epsilon}, \epsilon)$.

2. There exist ϵ_0 such that,

$$\tilde{x}(x, y; \epsilon_0) = x, \qquad \tilde{y}(x, y; \epsilon_0) = y,$$

then it is called an one-parameter group of point transformations.

Example 10 *The transformation defined by,*

$$G_i(x, y) \longrightarrow (\tilde{x}, \tilde{y}),$$

such that,

$$\tilde{x} = x \ \cos\epsilon - y \ \sin\epsilon, \qquad \tilde{y} = x \ \sin\epsilon + y \ \cos\epsilon,$$

where ϵ *is an infinitesimal parameter forms the group of* **rotations** *transformations. Since,*

$$\tilde{\tilde{x}} = \tilde{x} \ \cos\delta - \tilde{y} \ \sin\delta = x\cos(\epsilon + \delta) - y\sin(\epsilon + \delta),$$

$$\tilde{\tilde{y}} = \tilde{x} \ \sin\delta - \tilde{y} \ \cos\delta = x\sin(\epsilon + \delta) - y\cos(\epsilon + \delta).$$

Also for $\epsilon = 0$ *we have*

$$\tilde{x} = x, \qquad \textit{and} \qquad \tilde{y} = y.$$

Therefore, the above transformations constitutes a one-parameter group of Lie point transformations, where,

$$\psi(\epsilon, \delta) = \epsilon + \delta.$$

Example 11 *A Group of* **translations** *in the plane is defined as,*

$$\tilde{x} = x + \epsilon, \qquad \textit{and} \qquad \tilde{y} = y + \epsilon.$$

In this case,

$$\tilde{\tilde{x}} = \tilde{x} + \delta = x + \epsilon + \delta, \qquad \textit{and} \qquad \tilde{\tilde{y}} = \tilde{y} + \delta = y + \epsilon + \delta,$$

with composition law and identity element given respectively as,

$$\psi(\epsilon, \delta) = \epsilon + \delta, \qquad \textit{and} \qquad \epsilon_0 = 0.$$

Therefore the group of translations is a Lie group.

Example 12 *The group of* **scaling** *transformations is defined as,*

$$\tilde{x} = e^{\epsilon} x, \qquad \textit{and} \qquad \tilde{y} = e^{\epsilon} y.$$

In this case, the invertibility condition gives,

$$\tilde{\tilde{x}} = \tilde{x}\ e^{\delta} = x\ e^{\epsilon}\ e^{\delta} = x\ e^{\epsilon+\delta}, \qquad \textit{similarly} \qquad \tilde{\tilde{y}} = \tilde{y}\ e^{\delta} = y\ e^{\epsilon}\ e^{\delta} = y\ e^{\epsilon+\delta},$$

with identity element,

$$\epsilon_0 = 0,$$

and the composition law,

$$\psi(\epsilon, \delta) = \epsilon + \delta,$$

Thus the group of scaling transformations is a Lie group of transformation.

Example 13 *Consider the* **reflection** *transformations defined as,*

$$\tilde{x} = -x, \qquad \textit{and} \qquad \tilde{y} = -y.$$

Since,

$$\tilde{\tilde{x}} = -\tilde{x} = -(-x) = x, \qquad \textit{and} \qquad \tilde{\tilde{y}} = -\tilde{y} = -(-y) = y,$$

which shows that it is not invertible hence does not form a Lie group of transformation.

3.4 Infinitesimal Transformations

Consider one parameter (ϵ) Lie group of transformation with identity $\epsilon_0 = 0$ and composition law ψ defined as,

$$\tilde{\alpha} = G_i(\alpha; \epsilon). \tag{34}$$

Taylor expansion of the transformation (34) about $\epsilon_0 = 0$ is given as,

$$\begin{aligned} \tilde{\alpha} &= G_i(\alpha; \epsilon_0) + (\epsilon - \epsilon_0) \frac{\partial G_i(\alpha; \epsilon)}{\partial \epsilon} \Big|_{\epsilon=\epsilon_0} + O(\epsilon^2) \\ &= \alpha + \epsilon \frac{\partial \tilde{\alpha}}{\partial \epsilon} \Big|_{\epsilon=0} + O(\epsilon^2), \end{aligned} \tag{35}$$

where $\frac{\partial \tilde{\alpha}}{\partial \epsilon} \Big|_{\epsilon=0} = \xi^{\alpha}(\alpha)$.

In particular for $(x, y) \in \mathbb{R}^2$ the Taylor expansion of transformation G_1 such that,

$$G_1 : (x, y) \longrightarrow (\tilde{x}, \tilde{y}),$$

is given as,

$$\begin{aligned} \tilde{x} &= x + \epsilon \frac{\partial \tilde{x}}{\partial \epsilon} \Big|_{\epsilon=0} + \ldots, \\ \tilde{y} &= y + \epsilon \frac{\partial \tilde{y}}{\partial \epsilon} \Big|_{\epsilon=0} + \ldots. \end{aligned} \tag{36}$$

Substituting,

$$\frac{\partial \tilde{x}}{\partial \epsilon}\Big|_{\epsilon=0} = \xi(x,y), \qquad \text{and} \qquad \frac{\partial \tilde{y}}{\partial \epsilon}\Big|_{\epsilon=0} = \eta(x,y),$$

in (36) reduces it to,

$$\tilde{x} = x + \epsilon\ \xi(x,y) + \ldots,$$

$$\tilde{y} = y + \epsilon\ \eta(x,y) + \ldots. \tag{37}$$

This is called the *Infinitesimal Transformation* and the components $\xi(x,y)$ and $\eta(x,y)$ are called *infinitesimals* of the transformation. Transformation (34) can be found from the component $\xi(\alpha)$ by integrating,

$$\frac{\partial \tilde{\alpha}}{\partial \epsilon} = \xi(\tilde{\alpha}), \tag{38}$$

with initial condition $\tilde{\alpha}\ |_{\epsilon=0} = G_i\ |_{\epsilon=0} = \alpha$.

Theorem 3.1 (First Fundamental Theorem of Lie [11]) *There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations $\tilde{\alpha} = G_i(\alpha;\epsilon)$ is equivalent to the solution of the initial value problem for the system of first order differential equations,*

$$\frac{d\tilde{x}}{d\tau} = \xi(\tilde{x}), \tag{39}$$

with,

$$\tilde{x} = x \qquad \textit{when} \qquad \tau = 0. \tag{40}$$

3.4.1 Infinitesimal Generator

Consider the transformation,

$$\tilde{\alpha} = G_i(\alpha; \epsilon), \tag{41}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathbb{R}^n$. Then the operator defined by,

$$\chi = \xi(\alpha) \cdot \nabla = \sum_{k=1}^{n} \xi^k(\alpha) \frac{\partial}{\partial \alpha_k}, \tag{42}$$

is called an ***infinitesimal generator*** of the one parameter group of transformation (41) where $\xi^k = \frac{\partial \tilde{\alpha}_k}{\partial \epsilon} |_{\epsilon=0}$ give the components of the tangent vector χ_α.

Consider an arbitrary point $(x, y) \in \mathbb{R}^2$ and the transformation given in (37), the symmetry generator corresponding to this transformation is,

$$\chi = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y},$$

where $\xi(x, y) = \frac{\partial \tilde{x}}{\partial \epsilon} |_{\epsilon=0}$ and $\eta(x, y) = \frac{\partial \tilde{y}}{\partial \epsilon} |_{\epsilon=0}$.

Any transformation (41) can be determined completely with the help of infinitesimal generator χ by integrating,

$$\xi^k(\tilde{\alpha}) = \frac{\partial \tilde{\alpha}_k}{\partial \epsilon} \ , \tag{43}$$

with initial condition $\tilde{\alpha}_k |_{\epsilon=0} = \alpha_k$.

Theorem 3.2 *The one-parameter Lie group of transformations* $\tilde{\alpha} = G_i(\alpha; \epsilon)$ *is* *equivalent to:*

$$\begin{aligned}\tilde{\alpha} &= e^{\epsilon\chi}\alpha \\ &= \alpha + \epsilon\ \chi\ \alpha + \frac{\epsilon^2}{2}\ \chi^2\ \alpha + \dots \\ &= [1 + \epsilon\ \chi + \frac{\epsilon^2}{2}\ \chi^2 + \dots]\ \alpha \\ &= \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\ \chi^k \alpha, \end{aligned} \tag{44}$$

where the operator χ *is defined by* (42).

Example 14 *Consider the group of rotations defined as,*

$$\tilde{x} = x\ \cos\epsilon - y\ \sin\epsilon, \qquad \tilde{y} = x\ \sin\epsilon + y\ \cos\epsilon.$$

The components of symmetry generator are,

$$\xi(x,y) = \frac{\partial\tilde{x}}{\partial\epsilon}\Big|_{\epsilon=0} = -y, \qquad \textit{and} \qquad \eta(x,y) = \frac{\partial\tilde{y}}{\partial\epsilon}\Big|_{\epsilon=0} = x.$$

Therefore the symmetry generator is given as,

$$\chi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\ . \tag{45}$$

The inverse problem is to find the corresponding transformation of a symmetry generator. There are two different ways to do this:

a) *Lie series corresponding to the generator* (45) *is,*

$$\tilde{x} = e^{\epsilon\chi} x = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\ \chi^k x$$

$$= \chi^0 x + \frac{\epsilon^1}{1!}\ \chi^1 x + \frac{\epsilon^2}{2!}\ \chi^2 x + \frac{\epsilon^3}{3!}\ \chi^3 x + \frac{\epsilon^4}{4!}\ \chi^4 x + \frac{\epsilon^5}{5!}\ \chi^5 x + \dots$$

$$= x - \frac{\epsilon^1}{1!}\ y - \frac{\epsilon^2}{2!}\ x + \frac{\epsilon^3}{3!}\ y + \frac{\epsilon^4}{4!}\ x - \frac{\epsilon^5}{5!}\ y + \dots$$

$$= \left(1 - \frac{\epsilon^2}{2!} + \frac{\epsilon^4}{4!} + \dots\right)\ x - \ \left(\epsilon - \frac{\epsilon^3}{3!} + \frac{\epsilon^5}{5!} - \dots\right)\ y$$

$$= x\ \ \cos\epsilon - y\ \ \sin\epsilon.$$

Similarly,

$$\tilde{y} = e^{\epsilon\chi}\ y = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\ \chi^k\ y = x\ \ \sin\epsilon + y\ \ \cos\epsilon.$$

b) *For explanation of the second method consider the symmetry generator,*

$$\chi = \ x\ \frac{\partial}{\partial x} + \ y\ \frac{\partial}{\partial y}\ .$$

Using formula (43) *we have following relations,*

$$\xi(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{x}}{\partial \epsilon} = \tilde{x}, \qquad \textit{and} \qquad \eta(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{y}}{\partial \epsilon} = \tilde{y},$$

integrating these two equations and applying the initial conditions $\tilde{x}(0) = x$ *and* $\tilde{y}(0) = y$, *we obtain,*

$$\tilde{x} = e^{\epsilon} x, \qquad \textit{and} \qquad \tilde{y} = e^{\epsilon} y.$$

This is the group of scaling transformations.

3.4.2 Transformations of Generators

The infinitesimal symmetry generator $\chi_j = \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial}{\partial \alpha_i}$, in variables α_i, can be transformed in to the new variables α_i' using transformation law. This law gives the corresponding change in the components $\xi^i(\alpha)$ with the change of independent variables α_i.

Suppose,

$$\alpha_i' = \alpha_i'(\alpha_i), \qquad \text{such that} \qquad \mid \partial\alpha_i'/\partial\alpha_i \mid \ \neq 0.$$

By chain rule for derivatives,

$$\frac{\partial}{\partial \alpha_i} = \frac{\partial}{\partial \alpha_i'} \, \frac{\partial \alpha_i'}{\partial \alpha_i} \ .$$

In the light of above χ_j becomes,

$$\begin{aligned}
\chi_j &= \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial}{\partial \alpha_i} \\
&= \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial}{\partial \alpha_i'} \, \frac{\partial \alpha_i'}{\partial \alpha_i} \\
&= \sum_{i=1}^{n} \xi^{i'}(\alpha) \frac{\partial}{\partial \alpha_i'} \ ,
\end{aligned}$$

where,

$$\xi^{i'}(\alpha) = \xi^i(\alpha) \ \frac{\partial \alpha_i'}{\partial \alpha_i} \ .$$

Since,

$$\chi_j \ \alpha_k = \ \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial \alpha_k}{\partial \alpha_i} = \ \xi^k(\alpha),$$

and

$$\chi_j \ \alpha_k' = \ \sum_{i=1}^{n} \xi^{i'}(\alpha) \frac{\partial \alpha_k'}{\partial \alpha_i'} = \ \xi^{k'}(\alpha), \qquad \text{for} \quad 1 \leq k \leq n,$$

accordingly the infinitesimal symmetry generator can be written as,

$$\chi_j = \sum_{i=1}^{n} (\chi_j \ \alpha_i) \frac{\partial}{\partial \alpha_i} = \sum_{i=1}^{n} (\chi_j \ \alpha_i') \frac{\partial}{\partial \alpha_i'} \ . \tag{46}$$

Thus given infinitesimal symmetry generator χ_j in coordinates α_i can be transformed to new coordinates α_i' by the application of infinitesimal symmetry generator to coordinates α_i'.

Example 15 *Consider the infinitesimal symmetry generator defined as,*

$$\chi = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \ ,$$

In order to transform it in new variables u and v given by,

$$u = y/x, \qquad v = xy.$$

Application of χ on u and v yields,

$$\chi \ u = x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0,$$

$$\chi \ v = x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 2xy$$

$$= 2v.$$

Therefore the infinitesimal generator χ in new variables is given as,

$$\chi = 2v \frac{\partial}{\partial v} \ .$$

3.4.3 Normal Form

A result [46] in the theory of partial differential equations states that, for a system of equations,

$$\chi\ \gamma = \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial \gamma}{\partial \alpha_i} = 1,$$

$$\chi\ \alpha'_k = \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial \alpha'_k}{\partial \alpha_i} = 0, \tag{47}$$

where $i = 1, 2, \ldots, n$ and $k = 2, 3, \ldots, n$, there always exist a nontrivial solution $\{\gamma(\alpha_i), \alpha'_k(\alpha_i)\}$.

This result ensures that there always exist coordinates in which the infinitesimal symmetry generator can be maximally simplified. Therefore the symmetry generator $\chi = \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial}{\partial \alpha_i}$ can be reduced to,

$$\chi = \frac{\partial}{\partial \gamma}\ . \tag{48}$$

Equation (48) is called the *normal form* of the generator χ.

Example 16 *Consider the rotational transformations defined as,*

$$\tilde{x} = x\ \cos\epsilon - y\ \sin\epsilon, \qquad\qquad \tilde{y} = x\ \sin\epsilon + y\ \cos\epsilon.$$

Corresponding symmetry generator is given by,

$$\chi = -y\ \frac{\partial}{\partial x} + x\ \frac{\partial}{\partial y}\ .$$

Transforming the generator χ *in polar coordinates,*

$$r = (x^2 + y^2)^{1/2}, \qquad \textit{and} \qquad \phi = \arctan y/x,$$

we have,

$$\chi\ r = -\ y\ \frac{\partial r}{\partial x} + x\ \frac{\partial r}{\partial y}$$

$$= 0.$$

$$\chi\ \phi = -\ y\ \frac{\partial \phi}{\partial x} + x\ \frac{\partial \phi}{\partial y}$$

$$=1.$$

Thus in new coordinates the generator is given as,

$$\chi\ =\ \frac{\partial}{\partial \phi}\ ,$$

which is the corresponding normal form.

3.5 Invariance

Lie group of transformations can have invariant functions, surfaces, curves and invariant points. This is the most powerful observation of Lie group theory because due to the invariance property, complicated nonlinear conditions can be transformed into simple linear conditions under the corresponding infinitesimal generator of the symmetry group. A solution of a system of equations is a point that satisfies the system. Symmetry group of the system transforms its solutions to other solutions giving new invariant solutions of the system.

3.5.1 Invariance of a function

Let f be an infinitely differentiable function and let $\tilde{\alpha} = G_i(\alpha;\epsilon)$ be the Lie group of transformations of one parameter ϵ.

The function f is said to be an *invariant function* if and only if,

$$f(\tilde{\alpha}) \equiv f(\alpha). \tag{49}$$

Theorem 3.3 *A function* f *is invariant under Lie group of transformation* $\tilde{\alpha} = G_i(\alpha;\epsilon)$ *if,*

$$\chi\, f(\alpha) \equiv 0, \tag{50}$$

where χ *is the infinitesimal generator of the symmetry transformation and conversely.*

Theorem 3.4 *Given Lie group of transformation* $\tilde{\alpha} = G_i(\alpha;\epsilon)$ *with symmetry generator* χ*, the identity,*

$$f(\tilde{\alpha}) \equiv f(\alpha) + \epsilon, \tag{51}$$

holds if,

$$\chi\, f(\alpha) \equiv 1, \tag{52}$$

and conversely.

3.5.2 Invariance of a surface

Let $f(\alpha) = 0$ be a smooth surface and let $\tilde{\alpha} = G_i(\alpha;\epsilon)$ be the Lie group of symmetry transformation of one parameter ϵ. The surface $f(\alpha) = 0$ is said to be an invariant surface under the symmetry transformation if and only if $f(\tilde{\alpha}) = 0$ whenever $f(\alpha) = 0$.

3.5.3 Invariance of a curve

Consider a curve $f(\alpha) = 0$ in an n-dimensional space $\mathbb{R}^n$ and let one parameter Lie group of transformations in space $\mathbb{R}^n$ be given as,

$$\tilde{\alpha}_i = \alpha_i + \epsilon\, \xi_i(\alpha) + O(\epsilon^2), \qquad \forall\, i = 1, 2, \ldots, n. \tag{53}$$

The infinitesimal generator corresponding to this transformation is,

$$\chi \; = \sum_{i=1}^{n} \xi^i(\alpha) \frac{\partial}{\partial \alpha_i} \; . \tag{54}$$

Then the curve $f(\alpha) = 0$ is said to be an invariant curve if $f(\tilde{\alpha}) = 0$ whenever $f(\alpha) = 0$ and conversely.

3.5.4 Invariance of a point

A point $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ is called an invariant point for one-parameter Lie group of transformation $\tilde{\alpha} = G_i(\alpha; \epsilon)$ if and only if $\tilde{\alpha} = \alpha$ under this transformation.

3.6 Multi-Parameter Lie Transformations

Lie groups of transformations can depend on more than one parameter ϵ . Let $G(\alpha; \epsilon_k)$ for $k = 1, 2, \ldots, r$ be an r-parameter Lie group of transformations. Corresponding to each parameter ϵ_j there exist an infinitesimal symmetry generator χ_j that belongs to an r-dimensional linear vector space with the *commutator* structure. This vector space is know as r-dimensional Lie Algebra [11,39]. A one-parameter Lie group of transformations is a subgroup of the r-parameter Lie group of transformations.

Definition 3.6 *An r-parameter group of transformations* $\tilde{\alpha} = G(\alpha; \epsilon_k)$ *with composition law* ψ *is said to be r-parameters Lie group of transformations if:*

1. *The parameters* ϵ_k *are continuous.*

2. *Each element* G_{r_i} *for parameter* ϵ_i *of the group* G *is an infinitely differentiable function of* $\alpha \in \mathbb{R}^n$ *.*

3. The composition law for parameters $\psi(\epsilon_i, \delta_j)$ *is analytic.*

The composition law for parameters, denoted by,

$$\psi(\epsilon_i, \delta_j) = (\psi_1(\epsilon_i, \delta_j), \psi_2(\epsilon_i, \delta_j), \ldots, \psi_r(\epsilon_i, \delta_j)) \,,$$

satisfies the group axioms with $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r) = 0$ *corresponds to identity transformation.*

Equivalently, the transformation $\tilde{\alpha} = G(\alpha; \epsilon_k)$ where $k = 1, 2, \ldots, r$ is an r-parameter Lie group of transformations if:

1. Each parameter ϵ_k is independent of other parameters.
2. There exist an identity transformation.
3. Transformations are invertible and include their repeated application with possibly different parameter ϵ_j.

3.6.1 Infinitesimal Generators

Consider a Lie group of r-parameter transformations given as,

$$\tilde{\alpha} = G(\alpha; \epsilon_k), \tag{55}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ and $k = 1, 2, \ldots, r$.

The infinitesimal symmetry generator for this group is same as that of one-parameter group of transformations, but for each parameter ϵ_i there exist a corresponding symmetry generator χ_i , given by,

$$\chi_i \;=\; \sum_{k=1}^{n} \xi_i^k(\alpha) \, \frac{\partial}{\partial \alpha_k} \;. \tag{56}$$

The components $\xi_i^k(\alpha)$ of symmetry generator χ_i can be derived as,

$$\xi_i^k(\alpha) = \frac{\partial \tilde{\alpha}_k}{\partial \epsilon_i}\Big|_{\epsilon_j=0} , \tag{57}$$

Rescaling of parameter ϵ_i rescales the corresponding infinitesimal symmetry generator χ_i by a constant factor. Let,

$$\epsilon_i = \epsilon_i(\hat{\epsilon}_j), \qquad \text{with} \qquad \epsilon_i(0) = 0. \tag{58}$$

Therefore by the definition (57) of the components for infinitesimal symmetry generator we have,

$$\begin{aligned}
\hat{\xi}_i^k &= \frac{\partial \tilde{\alpha}_k}{\partial \hat{\epsilon}_i} \Big|_{\epsilon_j=0} \\
&= \frac{\partial \tilde{\alpha}_k}{\partial \epsilon_l} \frac{\partial \epsilon_l}{\partial \hat{\epsilon}_i}\Big|_{\epsilon_j=0} \\
&= \xi_l^k \frac{\partial \epsilon_l}{\partial \hat{\epsilon}_i}\Big|_{\epsilon_j=0} \\
&= C_i^l \, \xi_l^k,
\end{aligned}$$

where $C_i^l = \frac{\partial \epsilon_l}{\partial \hat{\epsilon}_i} |_{\epsilon_j=0}$ is a constant. Because of linearity, transformed infinitesimal symmetry generator is given as,

$$\hat{\chi}_i \; = \; C_i^l \, \chi_l \; . \tag{59}$$

If the parameters ϵ_i are not independent then in order to find a specific transformation, the relation between different parameters is required. That is, we have to define all ϵ_i's in terms of a single parameter ϵ. In this case,

$$\xi = \frac{\partial \tilde{\alpha}_k}{\partial \epsilon}\Big|_{\epsilon=0}$$

$$= \frac{\partial \tilde{\alpha}_k}{\partial \epsilon_i} \frac{\partial \epsilon_i}{\partial \epsilon}\Big|_{\epsilon=0}$$

$$= D^i \xi_i^k \ ,$$

where D^i is a constant. Therefore the infinitesimal symmetry generator is,

$$\chi \ = \ D^i \ \chi_i \ . \tag{60}$$

The difference between one-parameter transformations and multi-parameter transformations is that the multi-parameter transformations contains some constants linearly. Therefore they constitute all the properties of one-parameter transformations.

Example 17 *Consider two-parameter group of transformations in two-dimensional space defined as,*

$$\tilde{x} = \ x \ e^{\theta} \ \cos\phi - y \ e^{\theta} \ \sin\phi, \qquad \textit{and} \qquad \tilde{y} \ = \ x \ e^{\theta} \ \sin\phi + y \ e^{\theta} \ \cos\phi.$$

Then the generators corresponding to the parameters θ *and* ϕ *are respectively given as,*

$$\chi_\theta \ = \ x \ \frac{\partial}{\partial x} + y \ \frac{\partial}{\partial y} \ , \qquad \textit{and} \qquad \chi_\phi \ = \ - y \ \frac{\partial}{\partial x} + x \ \frac{\partial}{\partial y} \ ,$$

whereas a specific infinitesimal symmetry generator corresponding to the parameter γ *such that* $\theta = 2\gamma$ *and* $\phi = 3\gamma$ *will be,*

$$\chi_\gamma \ = D^\theta \ \chi_\theta + D^\phi \ \chi_\phi$$

$$= (2x - 3y)\frac{\partial}{\partial x} + (2y + 3x)\frac{\partial}{\partial y},$$

where $D^\theta = \frac{\partial \theta}{\partial \gamma} \mid_{\gamma=0} = 2$ *and* $D^\phi = \frac{\partial \phi}{\partial \gamma} \mid_{\gamma=0} = 3.$

3.7 Lie Algebra

Infinitesimal symmetry generator χ_j corresponding to the parameter ϵ_j belongs to an r-dimensional linear vector space known as r-dimensional Lie Algebra with the *commutator* structure.

Definition 3.7 (Commutator Operator) *Consider an r-parameter Lie group of transformations given by* (55) *with infinitesimal symmetry generators χ_i corresponding to each parameter ϵ_i defined in* (56)*. Then the commutator operator [,] for any two symmetry generators χ_i and χ_j is defined as [39],*

$$[\ \chi_i\ ,\ \chi_j\ ]\ =\ \chi_i\ \chi_j - \chi_j\ \chi_i\ . \tag{61}$$

Since,

$$\chi_i\ \chi_j = \sum_{k=1}^{n} \xi_i^k(\alpha) \frac{\partial}{\partial \alpha_k} \left\{ \sum_{l=1}^{n} \xi_j^l(\alpha; \epsilon_i) \frac{\partial}{\partial \alpha_l} \right\}$$

$$= \sum_{l,k=1}^{n} \xi_i^k(\alpha) \frac{\partial}{\partial \alpha_k} \left\{ \xi_j^l(\alpha; \epsilon_i) \frac{\partial}{\partial \alpha_l} \right\}\ .$$

Then,

$$\begin{aligned}[\ \chi_i\ ,\ \chi_j\ ]\ &=\ \sum_{l,k=1}^{n} \left[\xi_i^k(\alpha) \frac{\partial}{\partial \alpha_k} \left\{ \xi_j^l(\alpha) \frac{\partial}{\partial \alpha_l} \right\} - \xi_j^l(\alpha) \frac{\partial}{\partial \alpha_l} \left\{ \xi_i^k(\alpha) \frac{\partial}{\partial \alpha_k} \right\} \right] \\ &= \sum_{i=1}^{n} \eta_i(\alpha) \frac{\partial}{\partial \alpha_i}\ , \end{aligned} \tag{62}$$

where,

$$\eta_i(\alpha) = \sum_{l,k=1}^{n} \left(\xi_i^k \ \frac{\partial \xi_j^l}{\partial \alpha_k} + \xi_i^k \ \xi_j^l \ \frac{\partial^2}{\partial \alpha_k \partial \alpha_l} - \xi_j^l \ \xi_i^k \ \frac{\partial^2}{\partial \alpha_l \partial \alpha_k} - \xi_j^l \ \frac{\partial \xi_i^k}{\partial \alpha_k} \right)$$

$$= \sum_{l,k=1}^{n} \left(\xi_i^k \ \frac{\partial \xi_j^l}{\partial \alpha_k} - \xi_j^l \ \frac{\partial \xi_i^k}{\partial \alpha_k} \right) \ . \tag{63}$$

Equation (62) implies that the commutator of any two generators is again an infinitesimal symmetry generator. From equation (62) and (63) it is obvious that the commutator operator also known as Lie Bracket is skew symmetric and bilinear i.e.

$$[\ \chi_i\ ,\ \chi_j\ ] = \ -[\ \chi_j\ ,\ \chi_i\ ], \tag{64}$$

$$[\ c\ \chi_i\ +\ c'\ \chi_j\ ,\ \chi_k\ ] = c\ [\ \chi_i\ ,\ \chi_k\ ]\ +\ c'\ [\ \chi_j\ ,\ \chi_k\ ], \tag{65}$$

$$[\ \chi_i\ ,\ c\ \chi_j\ +\ c'\ \chi_k\ ] = c\ [\ \chi_i\ ,\ \chi_j\ ]\ +\ c'\ [\ \chi_i\ ,\ \chi_k\ ]. \tag{66}$$

where c and c' are constants.

Any three infinitesimal symmetry generators χ_i , χ_j and χ_k satisfies the *Jacobi's identity* defined as,

$$[\ \chi_i\ ,\ [\ \chi_j\ ,\ \chi_k\ ]\ ]\ +\ [\ \chi_k\ ,\ [\ \chi_i\ ,\ \chi_j\ ]\ ]\ +\ [\ \chi_j\ ,\ [\ \chi_k\ ,\ \chi_i\ ]\ ] = 0. \tag{67}$$

Definition 3.8 (Lie Algebra) *Let G be an r-parameter Lie group of transformations with basis $\{\chi_1, \chi_2, \ldots, \chi_r\}$ where χ_i is an infinitesimal symmetry generator corresponding to the parameter ϵ_i . The Lie group G of transformations, form an r-dimensional Lie algebra $\mathcal{G}^r$ over the field $\mathcal{F} = \mathbb{R}$ with respect to commutation law [11].*

Lie algebra is a vector space '$\mathcal{G}$' together with commutator operator that is Bi-linear, skew symmetric *and satisfies* jacobi identity.

Definition 3.9 (Subalgebra) *A subset $\mathcal{H}$ of Lie algebra $\mathcal{G}$ is called a subalgebra of $\mathcal{G}$ if it is closed under the commutation operator i.e. for all χ_i , $\chi_j \in \mathcal{H}$,*

$$[\ \chi_i\ ,\ \chi_j\ ] \in \mathcal{H}.$$

3.7.1 Solvable Lie Algebras

The order of an *nth* order ordinary differential equation can be reduced constructively by two if it admits a Lie algebra of transformations of two parameters. But for an r-parameter Lie algebra ($r \geq 3$) the order of the differential equation can be reduced constructively by p, if there exist a p-dimensional solvable subalgebra.

Definition 3.10 *Let $\mathcal{H}$ be a subalgebra of the Lie algebra $\mathcal{G}$. If,*

$$[\ g\ ,\ h\ ] \in \mathcal{H}, \qquad\qquad \forall \quad h \in \mathcal{H} \quad and \quad \forall \quad g \in \mathcal{G},$$

then $\mathcal{H}$ is called an ideal *or* normal subalgebra *of $\mathcal{G}$.*

Definition 3.11 *$\mathcal{H}^p$ is a p-dimensional solvable Lie algebra if there exists a chain of subalgebras,*

$$\mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^{p-1} \subset \mathcal{H}^p,$$

such that $\mathcal{H}^{i-1}$ is an ideal of $\mathcal{H}^i$ $\quad \forall \quad i = 2, 3, \ldots, p.$

Definition 3.12 *An algebra* $\mathcal{G}$ *is called an abelian Lie algebra if* $[\ \chi_i\ ,\ \chi_j\ ] = 0$ *for all* $\chi_i\ ,\ \chi_j \in \mathcal{G}$.

Theorem 3.5 *Every Abelian Lie algebra and every two-dimensional Lie algebra is a solvable Lie algebra.*

3.7.2 Structure Constants

Theorem 3.6 (Second Fundamental Theorem of Lie [11]) *The commutator of any two infinitesimal generators of an* r*-parameter Lie group of transformations is again an infinitesimal symmetry generator.*

Definition 3.13 *Lie bracket [39] of any two basis vectors must again lie in* $\mathcal{G}^r$ *i.e.*

$$[\ \chi_i\ ,\ \chi_j\ ] \ = \ \sum_{k=1}^{r} C_{ij}^k\ \chi_k \in \mathcal{G}, \qquad \forall \ \ i,j = 1,2,\ldots,r. \tag{68}$$

The constants C_{ij}^k *are called* structure constants *of the Lie algebra* $\mathcal{G}^r$.

Definition 3.14 (Commutation Relations) *For an* r*-parameter Lie group of transformations with basis* $\{\chi_1, \chi_2, \ldots, \chi_r\}$ *the relations defined by equation* (68) *are called* commutation relations.

Theorem 3.7 (Third Fundamental Theorem of Lie [11]) *The structure constants, defined by commutation relations* (68), *satisfy the relations:*

1. $C_{ij}^k \ = \ -\, C_{ji}^k$ ***(skew symmetry)***.

2. $C_{ij}^k\ C_{kl}^m \ + \ C_{jl}^k\ C_{ki}^m \ + \ C_{li}^k\ C_{kj}^m \ = \ 0$ ***(Jacobi identity)***.

3.8 Prolongation

In order to apply transformations (34) to an *nth* order differential equation, the corresponding infinitesimal symmetry generator (42) needs to be extended or prolonged to *nth* order.

3.8.1 Case I: (One dependent and one independent variable)

Consider a differential equation,

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0, \tag{69}$$

of order n, with one independent variable x, and one dependent variable y. The one-parameter (ϵ) Lie group of infinitesimal transformations are given by,

$$\left.\begin{aligned} \tilde{x} &= G_x(x, y; \epsilon) = x + \epsilon\, \xi(x, y) + O(\epsilon^2) \\ \tilde{y} &= G_y(x, y; \epsilon) = y + \epsilon\, \eta(x, y) + O(\epsilon^2) \end{aligned}\right\} \tag{70}$$

Infinitesimal components of symmetry generator χ are,

$$\xi(x, y) = \frac{\partial \tilde{x}}{\partial \epsilon}\Big|_{\epsilon=0}, \qquad \eta(x, y) = \frac{\partial \tilde{y}}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Then the corresponding infinitesimal symmetry generator is,

$$\chi \;=\; \xi(x, y)\, \frac{\partial}{\partial x} \;+\; \eta(x, y)\, \frac{\partial}{\partial y}\ . \tag{71}$$

For an *nth* order differential equation we need to prolong the infinitesimal symmetry generator to *nth* order. Thus (71) becomes,

$$\chi^{(n)} \;=\; \chi + \eta' \frac{\partial}{\partial y'} + \ldots\cdots + \eta^{(n)} \frac{\partial}{\partial y^{(n)}}\ . \tag{72}$$

We need to find expressions of the infinitesimal components $\eta^{(i)}$ for $i = 1, 2, \ldots, n$. Extension of equation (70) up to order n is given as,

$$\left.\begin{aligned}
\tilde{y}' &= y' + \epsilon\ \eta'(x, y, y') + O(\epsilon^2) \\
\tilde{y}'' &= y'' + \epsilon\ \eta''(x, y) + O(\epsilon^2) \\
&\ \vdots \\
\tilde{y}^{(n)} &= y^{(n)} + \epsilon\ \eta^{(n)}(x, y) + O(\epsilon^2)
\end{aligned}\right\} \tag{73}$$

where,

$$\eta^{(n)} = \left.\frac{\partial \tilde{y}^{(n)}}{\partial \epsilon}\right|_{\epsilon=0} .$$

The expressions for $\tilde{y}', \ldots, \tilde{y}^{(n)}$ are also given as,

$$\begin{aligned}
\tilde{y}' &= \frac{d\tilde{y}(x, y; \epsilon)}{d\tilde{x}(x, y; \epsilon)} = \frac{y'\,(\partial\tilde{y}/\partial y) + (\partial\tilde{y}/\partial x)}{y'\,(\partial\tilde{x}/\partial y) + (\partial\tilde{x}/\partial x)} = \tilde{y}'(x, y, y'; \epsilon), \\
\tilde{y}'' &= \frac{d\tilde{y}'(x, y, y'; \epsilon)}{d\tilde{x}(x, y; \epsilon)} = \frac{y''\,(\partial\tilde{y}'/\partial y') + y'\,(\partial\tilde{y}'/\partial y) + (\partial\tilde{y}'/\partial x)}{y'\,(\partial\tilde{x}/\partial y) + (\partial\tilde{x}/\partial x)} = \tilde{y}''(x, y, y', y''; \epsilon), \\
&\ \vdots \\
&\ \vdots \\
\tilde{y}^{(n)} &= \frac{d\tilde{y}^{(n-1)}}{d\tilde{x}} = \tilde{y}^{(n)}(x, y, y', \ldots, y^{(n)}; \epsilon).
\end{aligned} \tag{74}$$

Also,

$$\begin{aligned}
\tilde{y}' &= y' + \epsilon\, \eta' + \dots \\
&= \frac{d\tilde{y}}{d\tilde{x}} = \frac{dy + \epsilon\, d\eta + \dots}{dx + \epsilon\, d\xi + \dots} \\
&= \frac{y' + \epsilon\,(d\eta/dx) + \dots}{1 + \epsilon\,(d\xi/dx) + \dots} \\
&= y' + \epsilon\left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right) + \dots, \\
&\;\vdots \\
&\;\vdots
\end{aligned} \tag{75}$$

$$\begin{aligned}
\tilde{y}^{(n)} &= y^{(n)} + \epsilon\, \eta^{(n)} + \dots = \frac{d\tilde{y}^{(n-1)}}{d\tilde{x}} \\
&= y^{(n)} + \epsilon\left(\frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}\right) + \dots .
\end{aligned} \tag{76}$$

From these equations following expressions for $\eta', \dots, \eta^{(n)}$ can be read off,

$$\begin{aligned}
\eta' &= \frac{d\eta}{dx} - y'\frac{d\xi}{dx} \\
&= \frac{\partial\eta}{\partial x} + y'\left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right) - y'^2\frac{\partial\xi}{\partial y}\ , \\
&\;\vdots \\
\eta^{(n)} &= \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}\ .
\end{aligned}$$

3.8.2 Case II: (One dependent and p independent variables)

Consider an nth order differential equation with one dependent variable 'u' and p independent variables 'x_i' for $i = 1, 2, \ldots, p$ given as,

$$F(x, u, u^{(n)}) = 0, \tag{77}$$

where $x = (x_1, x_2, \ldots, x_p)$ and $u^{(n)}$ denotes the set of all derivatives of u of order less than or equal to n. Let $J = (j_1, j_2, \ldots, j_k)$ be the k tuple of integers (number of tuples in J are equal to the order of derivative). In index notation the derivative of order m is denoted by,

$$u_J = \frac{\partial^m u}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_m}} \ , \tag{78}$$

where $J = (j_1, j_2, \ldots, j_m)$ with $1 \leq j_i \leq p$ for all $i = 1, 2, \ldots, m$.

Lie group of transformations of one parameter ϵ are given as,

$$\left.\begin{aligned} \tilde{x}_k = G_{x_k}(x_i, u; \epsilon) = x_k + \epsilon\ \xi^k(x_i, u) + O(\epsilon^2) \\ \tilde{u} = G_u(x_i, u; \epsilon) = u + \epsilon\ \phi(x_i, u) + O(\epsilon^2) \end{aligned}\right\} \tag{79}$$

where $i = 1, 2, \ldots, p$ and $x_k \in \{x_1, x_2, \ldots, x_p\}$. Infinitesimal components of symmetry generator χ are given as,

$$\xi^k(x_i, u) = \left.\frac{\partial \tilde{x}_k}{\partial \epsilon}\right|_{\epsilon=0}, \qquad \& \qquad \phi(x_i, u) = \left.\frac{\partial \tilde{u}}{\partial \epsilon}\right|_{\epsilon=0}. \tag{80}$$

Thus the corresponding infinitesimal symmetry generator of transformations (79) is,

$$\chi \ = \sum_{k=1}^{p} \xi^k(x_i, u) \frac{\partial}{\partial x_k} + \phi(x_i, u) \frac{\partial}{\partial u} \ . \tag{81}$$

In order to apply the infinitesimal symmetry generator to nth order differential equation we need to extend it to nth order. The general prolongation formula can be given in simple form by using the total derivative operator D, as given in the definition below:

Definition 3.15 *Let* $f(x, u^{(n)})$ *be a continuously differentiable function with* p *independent variables* $x = (x_1, x_2, \ldots, x_p)$*, one dependent variable* u *and the derivatives of* u *up to order* n*. The general form of* i*th* **total derivative** *of* f *is,*

$$D_i f = \frac{\partial f}{\partial x_i} + \sum_J u_{J,i} \frac{\partial f}{\partial u_J} \ , \tag{82}$$

where $J = (j_1, \ldots, j_k)$*,* $1 \le j_i \le p$ *for all* $i = 1, 2, \ldots, k$ *and,*

$$u_{J,i} = \frac{\partial u_J}{\partial x_i} = \frac{\partial^{k+1} u}{\partial x_i \partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}} \ . \tag{83}$$

The sum in (82) *is over all* J*'s of order* $0 \le \#J \le n$*.*

3.8.2.1 Derivation of extended infinitesimal symmetry generator

From equation (79) we have,

$$\tilde{u}_{x_k} = \ u_{x_k} + \epsilon \phi^{x_k} + \ldots, \tag{84}$$

where $\phi^{x_k} = \frac{\partial u_{x_k}}{\partial \epsilon}\Big|_{\epsilon=0}$.

Also from (79) we have,

$$\begin{aligned} d\tilde{u} &= du + \epsilon d\phi + O(\epsilon^2) \\ &= \left\{ \frac{\partial u}{\partial x_i} + \epsilon \left(\frac{\partial \phi}{\partial x_i} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x_i} \right) \right\} dx_i + \ldots \\ &= \left(\frac{\partial u}{\partial x_i} + \epsilon D_i \phi \right) dx_i + \ldots . \end{aligned} \tag{85}$$

Like above,

$$
\begin{aligned}
d\tilde{x}_k &= dx_k + \epsilon d\phi + O(\epsilon^2) \\
&= \left\{ \frac{\partial x_k}{\partial x_j} + \epsilon \left(\frac{\partial \xi^k}{\partial x_j} + \frac{\partial \xi^k}{\partial u} \frac{\partial u}{\partial x_j} \right) \right\} dx_j + \ldots \\
&= \left\{ \delta^k_j + \epsilon \left(\frac{\partial \xi^k}{\partial x_j} + \frac{\partial \xi^k}{\partial u} \frac{\partial u}{\partial x_j} \right) \right\} dx_j + \ldots \\
&= \left(\delta^k_j + \epsilon D_j \xi^k \right) dx_j + \ldots . \qquad (86)
\end{aligned}
$$

Using (85) and (86) we can construct the following relation,

$$
\begin{aligned}
\tilde{u}_{x_k} &= \frac{d\tilde{u}}{d\tilde{x}_k} \\
&= \frac{\left(\frac{\partial u}{\partial x_i} + \epsilon D_i \phi \right) dx_i + \ldots}{\left(\delta^k_j + \epsilon D_j \xi^k \right) dx_j + \ldots} \\
&= \frac{\left(\frac{\partial u}{\partial x_i} + \epsilon D_i \phi \right) + \ldots}{\left(\delta^k_j + \epsilon D_j \xi^k \right) + \ldots} \; \delta^i_j \; , \\
&= \left(\frac{\partial u}{\partial x_i} + \epsilon D_i \phi + \ldots \right) \; \delta^i_j \; \left\{ \left(\delta^k_j \right)^{-1} - \epsilon \left(\delta^k_j \right)^{-2} D_j \xi^k + \ldots \right\} \\
&= \left(\frac{\partial u}{\partial x_i} + \epsilon D_i \phi + \ldots \right) \left(\delta^i_k - \epsilon D_k \xi^i + \ldots \right) \\
&= u_{x_k} + \epsilon D_k \phi - \epsilon u_{x_i} D_k \xi^i + \ldots \\
&= u_{x_k} + \epsilon \left(D_k \phi - u_{x_i} D_k \xi^i \right) + \ldots \qquad (87)
\end{aligned}
$$

Comparing the coefficients of ϵ in equations (84) and (87), we find the expression for ϕ^{x_k} given as,

$$\begin{aligned} \phi^{x_k} &= D_k\phi - u_{x_i}D_k\xi^i \\ &= D_k\left(\phi - u_{x_i}\xi^i\right) + \xi^i u_{x_i x_k}. \end{aligned} \tag{88}$$

Similarly the expression for $\phi^J = \phi^{x_{j_1}x_{j_2}\cdots x_{j_k}}$, where $J = (j_1, j_2, \ldots, j_k)$ and $1 \le j_i \le p$ for all $i = 1, 2, \ldots, k$, can be found using the general formula,

$$\phi^J\left(x, u^{(n)}\right) = D_J\left(\phi - \sum_{k=1}^{p}\xi^k u_k\right) + \sum_{k=1}^{p}\xi^k\ u_{J,k}\ . \tag{89}$$

Thus the nth prolongation of symmetry generator (81) is,

$$\chi^{(n)} = \chi + \sum_J \phi^J\left(x, u^{(n)}\right)\frac{\partial}{\partial u_J}, \tag{90}$$

where the summation is over all multi-indices $J = (j_1, j_2, \ldots, j_k)$, with $1 \le j_k \le p$ and $1 \le k \le n$. Whereas ϕ^J and u_J are defined in (89) and (78) respectively.

3.8.3 Case III: (q dependent and p independent variables)

Let,

$$F(x, u, u^{(n)}) = 0,$$

be an nth order partial differential equation with p-independent variables $x = (x_1, x_2, \ldots, x_p)$, q-dependent variables $u = (u^1, u^2, \ldots, u^q)$ and the derivatives of dependent variables with respect to independent variables up to the order n. The infinitesimal symmetry generator be given as,

$$\chi = \sum_{k=1}^{p}\xi^k(x, u)\frac{\partial}{\partial x_k} + \sum_{\alpha=1}^{q}\phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}\ . \tag{91}$$

Then the *nth* prolongation of the generator (91) will be,

$$\chi^{(n)} = \chi + \sum_{\alpha=1}^{q} \sum_{J} \phi_{\alpha}^{J}(x, u, u^{(n)}) \frac{\partial}{\partial u_J^{\alpha}} \ , \tag{92}$$

where ϕ_α corresponds to u^α.

CHAPTER 4

LIE SYMMETRIES AND PDE'S

A symmetry group for the system of differential equations is a group of transformations acting on dependent and independent variables in the system such that the system remain invariant under these transformations and it transforms solutions of the system to other solutions. Lie group of point transformations lead to invariant solutions also called similarity solutions obtained from the solution of PDE's with fewer independent variables than the given PDE's.

4.1 Invariance of a PDE

Consider a system of PDEs of order n with p-independent and q-dependent variables represented as,

$$F_{\mu}(x, u, u^{(n)}) = 0, \qquad \mu = 1, 2, \ldots, k, \tag{93}$$

where $x = (x_1, x_2, x_3, \ldots, x_p)$ denotes independent variables, $u = (u^1, u^2, \ldots, u^q)$ denotes dependent variables and $u^{(n)}$ represents the set of all derivatives of order less and equal to n. We denote the derivative of order m is denoted as,

$$u_J^{\alpha} = \frac{\partial^m u^{\alpha}}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_m}} \ , \tag{94}$$

where $1 \leq j_i \leq p$ for all $i = 1, 2, \ldots, m$ and the order of m-tuple of integers $J = (j_1, j_2, \ldots, j_m)$ indicates the order of the derivative to be taken.

Definition 4.1 (Maximal Rank of Jacobian) *The system* (93) *is of maximal rank* *if the corresponding* $k \times \left(p + qp^{(n)}\right)$ *Jacobian matrix,*

$$J(x, u, u^{(n)}) = \left(\frac{\partial F_\mu}{\partial x_j}, \frac{\partial F_\mu}{\partial u_J^\alpha}\right) \ , \tag{95}$$

is of rank k *whenever* $F_\mu(x, u, u^{(n)}) = 0$.

Example 18 *The Burgers' equation*

$$F = u_t - u_{xx} - u_x^2 = 0,$$

is of maximal rank, since,

$$\begin{aligned} J = & \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial t}; \frac{\partial F}{\partial u}; \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_y}, \frac{\partial F}{\partial u_t}; \frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xy}}, \frac{\partial F}{\partial u_{xt}}, \frac{\partial F}{\partial u_{yy}}, \frac{\partial F}{\partial u_{yt}}, \frac{\partial F}{\partial u_{tt}}\right) \\ = & \ (0, 0, 0; 0; -2\ u_x, 0, 1; -1, 0, 0, 0, 0, 0) \ , \end{aligned}$$

which is of rank one everywhere.

Example 19 *The equation*

$$F = (u_t - u_{xx})^2 = 0$$

is not of maximal rank, since,

$$\begin{aligned} J = & \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t}; \frac{\partial F}{\partial u}; \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_t}; \frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xt}}, \frac{\partial F}{\partial u_{tt}}\right) \\ = & \ (0, 0; 0; 0, 2(u_t - u_{xx}); -2(u_t - u_{xx}), 0, 0) \\ = & \ 0 \ , \end{aligned}$$

whenever $(u_t - u_{xx})^2 = 0$.

Theorem 4.1 (Infinitesimal criterion for the invariance of PDE) *Let the system,*

$$F_\mu(x, u, u^{(n)}) = 0, \qquad \mu = 1, 2, \ldots, k$$

of k differential equations be of maximal rank. If G *is a group of transformations and,*

$$\chi^{(n)}\left\{F_\mu(x, u, u^{(n)})\right\} = 0, \qquad \mu = 1, 2, \ldots, k \qquad \textit{whenever} \qquad F_\mu(x, u, u^{(n)}) = 0\ , \tag{96}$$

for every infinitesimal symmetry generator χ *of the group* G *, then* G *is a symmetry group of the system.*

4.2 Procedure to calculate symmetries

Lie group of infinitesimal transformations and infinitesimal symmetry generators of a partial differential equation can be calculated by a systematic computational procedure in the light of Theorem 4.1 and using the prolongation formula.

The first step of the procedure is to find the hypothetical one-parameter symmetry generator χ. The coefficients $\xi^i(x, u)$ and $\phi_\alpha(x, u)$ of symmetry generator χ will be the functions of x and u. Using prolongation formula symmetry generator χ needs to be prolonged to the order n equivalent to the order of differential equation. The coefficients ϕ_α^J of the prolonged infinitesimal symmetry generator $\chi^{(n)}$ involve the partial derivatives of the coefficients ξ^i and ϕ_α with respect to both x and u.

Application of prolonged symmetry generator on the differential equation using theorem of infinitesimal criterion for the invariance of PDE gives a general equation that involves x , u and the derivatives of u with respect to x, as well as $\xi^i(x, u)$, $\phi_\alpha(x, u)$ and their partial derivatives with respect to x and u. Since (96) holds only on solutions of the system, therefore dependence among the derivatives of u caused by the system itself required to be removed. By comparing the coefficients of the partial derivatives of u, a large number of coupled partial differential equations are obtained. This system of equations will be solved for the coefficients functions ξ^i and ϕ_α of the infinitesimal symmetry generator. These equations are called *defining equations* of the symmetry group of the given system. The general solution of this system of defining equations determines the most general expressions for ξ^i and ϕ_α, thus giving the general infinitesimal symmetry generator χ. Following are some examples illustrating the procedure.

4.3 The Heat Equation

The equation governing the heat conduction in one dimensional rod is given as,

$$u_t = u_{xx} \tag{97}$$

Rank of Jacobian

Jacobian of the heat equation $F(x,t,u) = u_t - u_{xx} = 0$ equation is,

$$\begin{aligned} J &= \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t}; \frac{\partial F}{\partial u}; \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_t}; \frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xt}}, \frac{\partial F}{\partial u_{tt}}\right) \\ &= (0,0;0;0,1;-1,0,0)\,. \end{aligned}$$

Thus the rank of Jacobian is always one for heat equation.

Symmetry Generator

Since there are two independent variables x and t and one dependent variable u, therefore the infinitesimal symmetry generator will be of the form,

$$\chi = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}\ . \tag{98}$$

Prolongation

For a second order differential equation we need to prolong the infinitesimal symmetry generator (98) to second order. The prolongation is given as,

$$\chi^{(2)} = \chi + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}\ . \tag{99}$$

Symmetry Criterion

Now applying the symmetry criterion (96) we get the following relation,

$$\phi^{t} - \phi^{xx} = 0. \tag{100}$$

Substituting expressions for ϕ^t and ϕ^{xx} in (100) and replacing u_t by u_{xx} the general relation for the symmetry criterion becomes,

$$\begin{aligned}
&\phi_t + (\phi_u - \tau_t)\,u_{xx} - \xi_t u_x - \xi_u u_x u_{xx} - \tau_u u_{xx}^2 - \phi_{xx} \\
&- (2\phi_{xu} - \xi_{xx})\,u_x + \tau_{xx}u_{xx} - (\phi_{uu} - 2\xi_{ux})\,u_x^2 + 2 \\
&\tau_{xu}u_x u_{xx} + \xi_{uu}u_x^3 + \tau_{uu}u_x^2 u_{xx} - (\phi_u - 2\xi_x)\,u_{xx} + 2\tau_x \\
&u_{tx} + 3\xi_u u_x u_{xx} + \tau_u u_{xx}^2 + 2\tau_u u_x u_{xt} = 0.
\end{aligned}$$

Defining Equations

By comparing the coefficients of monomials we obtain the following ten coupled defining equations,

$$\begin{aligned}
&u_t u_{xt} &&: &&\tau_u = 0 \\
&u_{xt} &&: &&\tau_x = 0
\end{aligned}$$

$$
\begin{array}{lcl}
u_{xx}^2 & : & \tau_u = \tau_u \\
u_x^2 u_{xx} & : & \tau_{uu} = 0 \\
u_x u_{xx} & : & \xi_u = 2\tau_{xu} + 3\xi_u \\
u_{xx} & : & \tau_t - \phi_u = \tau_{xx} - \phi_u + 2\xi_x \\
u_x^3 & : & \xi_{uu} = 0 \\
u_x^2 & : & \phi_{uu} = 2\xi_{xu} \\
u_x & : & \xi_t = \xi_{xx} - 2\phi_{xu} \\
1 & : & \phi_t = \phi_{xx}
\end{array}
$$

Solving the above equations simultaneously, we obtain the general expression for ξ, τ and ϕ given by,

$$
\begin{aligned}
\tau &= c_1 t^2 + c_2 t + c_3, \\
\xi &= c_1 tx + 1/2 c_2 x + c_4 t + c_5, \\
\phi &= \left(-1/4 c_1 x^2 - 1/2 c_4 x - 1/2 c_1 t + c_6\right) u + \alpha.
\end{aligned}
$$

Symmetry Generators

Substituting $c_i = 1$ and $c_j = 0 \; \forall \; i \neq j$ for $i = 1, \ldots, 6$ we obtain seven infinitesimal symmetry generators including one infinite-dimensional symmetry generator as follows;

$$\chi_1 = \; u \, \frac{\partial}{\partial u} \; , \qquad \chi_2 = \; \frac{\partial}{\partial x} \; , \qquad \chi_3 = \; \frac{\partial}{\partial t} \; ,$$

$$\chi_4 = \; t \, \frac{\partial}{\partial x} - \frac{1}{2} \, x \, u \, \frac{\partial}{\partial \phi} \; , \qquad \chi_5 = \; \frac{1}{2} \; x \, \frac{\partial}{\partial x} + t \, \frac{\partial}{\partial t} \; ,$$

$$\chi_6 = \; t \, x \, \frac{\partial}{\partial x} + t^2 \, \frac{\partial}{\partial t} \; - \left(\frac{x^2 u}{4} + \frac{t \, u}{2} \right) \frac{\partial}{\partial u} \; ,$$

$$\chi_\alpha = \; \alpha \, \frac{\partial}{\partial u} \; .$$

Commutator Table

The commutation relations for all of these generators are given below in Table 1.

$[\chi_i, \chi_j]$	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
χ_1	0	0	0	χ_1	$-\chi_3$	$2\chi_5$
χ_2	0	0	0	$2\chi_2$	$2\chi_1$	$4\chi_4 - 2\chi_3$
χ_3	0	0	0	0	0	0
χ_4	$-\chi_1$	$-2\chi_2$	0	0	χ_5	$2\chi_6$
χ_5	χ_3	$-2\chi_1$	0	$-\chi_5$	0	0
χ_6	$-2\chi_5$	$2\chi_3 - 4\chi_4$	0	$-2\chi_6$	0	0

Table 1: Commutation Relations

Transformation Groups

Group of transformations corresponding to each infinitesimal symmetry generator χ_i can be calculated using the general formula,

$$\xi^i(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{\alpha}_i}{\partial \epsilon} \; , \tag{101}$$

with initial condition $\tilde{\alpha}_i|_{\epsilon=0} = \alpha_i$.

Consider the infinitesimal symmetry generator $\chi_1 = u\frac{\partial}{\partial u}$ we obtain following expressions:

1.
$$\frac{\partial \tilde{x}}{\partial \epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) = 0 \ ,$$
$$\tilde{x} = c,$$
applying initial condition $\tilde{x}|_{\epsilon=0} = x$ we obtain,
$$\tilde{x} = x.$$

2.
$$\frac{\partial \tilde{t}}{\partial \epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) = 0 \ ,$$
$$\tilde{t} = c,$$
applying initial condition $\tilde{t}|_{\epsilon=0} = t$ we obtain,
$$\tilde{t} = t.$$

3.
$$\frac{\partial \tilde{u}}{\partial \epsilon} = \phi(\tilde{x}, \tilde{t}, \tilde{u}) = \tilde{u} \ ,$$
$$\ln \tilde{u} = \epsilon + c,$$
applying initial condition $\tilde{u}|_{\epsilon=0} = u$ we obtain,
$$\tilde{u} = ue^{\epsilon}.$$

Therefore the transformation group G_1 generated by infinitesimal symmetry generator χ_1 is given as,

$$G_1: \qquad (\tilde{x}, \tilde{t}, \tilde{u}) = (x, t, ue^{\epsilon}).$$

Similarly for remaining infinitesimal symmetry generators we have following the transformation groups:

$$\begin{aligned}
G_2: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= (x+\epsilon, t, u),\\
G_3: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= (x, t+\epsilon, u),\\
G_4: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= (x+\epsilon t, t, ue^{-(\epsilon x+\epsilon^2 t)/2}),\\
G_5: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= (xe^{\epsilon/2}, te^{\epsilon}, u),\\
G_6: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= \left(\frac{x}{1-\epsilon t}, \frac{t}{1-\epsilon t}, \frac{u}{1-\epsilon t}\, e^{x^2\epsilon/4(1-\epsilon t)}\right),\\
G_\alpha: &\qquad (\tilde{x}, \tilde{t}, \tilde{u}) &&= (x, t, u+\epsilon\ \alpha(x,t)).
\end{aligned}$$

4.4 The KdV Equation

Consider the KdV equation,

$$u_t + u_{xxx} + uu_x = 0 \tag{102}$$

Rank of Jacobian

For the KdV equation $F(x,t,u) = u_t + u_{xxx} + uu_x = 0$ Jacobian is always one since,

$$J = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial t}; \frac{\partial F}{\partial u}; \frac{\partial F}{\partial u_x}, \frac{\partial F}{\partial u_t}; \frac{\partial F}{\partial u_{xx}}, \frac{\partial F}{\partial u_{xt}}, \frac{\partial F}{\partial u_{tt}}; \frac{\partial F}{\partial u_{xxx}}, \frac{\partial F}{\partial u_{xxt}}, \frac{\partial F}{\partial u_{xtt}}, \frac{\partial F}{\partial u_{ttt}}\right)$$

$$= (0,1; u_x; u, 1; 0,0,0; 1,0,0,0)\,.$$

Symmetry Generator

The infinitesimal symmetry generator for this equation is a vector field on a three dimensional space, as there are two independent and one dependent variable. Therefore,

$$\chi = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}\ . \tag{103}$$

Prolongation

We need to prolong the symmetry generator (103) to third order. Thus the prolongation of the infinitesimal symmetry generator is as follows,

$$\chi^{(3)} = \chi + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}$$

$$+ \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxt}\frac{\partial}{\partial u_{xxt}} + \phi^{xtt}\frac{\partial}{\partial u_{xtt}} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}}\ . \tag{104}$$

Symmetry Criterion

Symmetry criterion (96) for partial differential equations gives the relation,

$$\phi^{t} + \phi^{xxx} + u\phi^{x} + u_x\phi = 0. \tag{105}$$

Substitution of expressions for ϕ^t, ϕ^{xxx} and ϕ^x in (105) and replacement of u_t by $-u_{xxx} - uu_x$ yields,

$$\phi_t + (-\phi_u + \tau_t + \tau_{xxx} + \phi_u - 3\xi_x)\, u_{xxx} + (\tau_t - \phi_u + \tau_{xxx} + \phi_u$$

$$-\xi_x)\, uu_x + (-\xi_t + 3\phi_{xxu} - \xi_{xxx} + \phi)\, u_x + (\xi_u - 4\xi_u + 3\tau_{xxu})$$

$$u_x u_{xxx} + (\xi_u + 3\tau_{xxu} - \xi_u)\, uu_x^2 + (-\tau_u + \tau_u)\, u_{xxx} + (-2\tau_u + 2$$

$$\tau_u)\, uu_x u_{xxx} + (-\tau_u + \tau_u)\, u^2 u_x^2 + \phi_{xxx} + (3\phi_{xuu} - 3\xi_{xxu})\, u_x^2\, (3$$

$$\phi_{xu} - 3\xi_{xx})\, u_{xx} + (\phi_{uuu} - 3\xi_{uux})\, u_x^3 + (3\phi_{uu} - 9\xi_{xu})\, u_x u_{xx} -$$

$$\xi_{uuu} u_x^4 - 6\xi_{uu} u_x^2 u_{xx} - 3\xi_u u_{xx}^2 + 3\tau_{uux} u_x^2 u_{xxx} + 3\tau_{uux} uu_x^3 + 3\tau_{xu}$$

$$u_{xxx} u_{xx} + 3\tau_{xu} uu_x u_{xx} - 6\tau_{xu} u_x u_{tx} - 3\tau_{xx} u_{tx} - 3\tau_x u_{txx} + \tau_{uuu}$$

$$u_x^3 u_{xxx} + \tau_{uuu} uu_x^4 + 3\tau_{uu} u_x u_{xx} u_{xxx} + 3\tau_{uu} uu_x^2 u_{xx} - 3\tau_{uu} u_x^2 u_{tx}$$

$$- 3\tau_u u_{xx} u_{xt} - 3\tau_u u_x u_{txx} + \phi_x u + \tau_x uu_{xxx} + \tau_x u^2 u_x = 0.$$

Defining Equations

Comparison of coefficients of the monomials as in the case of heat equation (97) gives the following defining equations: Which can be solved for ξ, τ and ϕ to give,

$$\xi = c_1 + c_2 x + c_3 t,$$

$$\tau = c_4 + 3c_2 t,$$

$$\phi = c_3 - 2c_2 u.$$

Symmetry Generators

Substituting $c_i = 1$ and $c_j = 0\ \forall\ i \neq j$ for $i = 1, \ldots, 6$ we obtain following seven infinitesimal symmetry generators, including one infinite-dimensional symmetry generator,

$$\chi_1 = \frac{\partial}{\partial x}, \qquad \chi_2 = x\frac{\partial}{\partial x} + 3\, t\frac{\partial}{\partial t} - 2\, u\frac{\partial}{\partial u},$$

$$\chi_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad \chi_4 = \frac{\partial}{\partial t}.$$

Commutator Table

Commutation relations for these generators are given below in Table 2.

$[\chi_i, \chi_j]$	χ_1	χ_2	χ_3	χ_4
χ_1	0	0	0	χ_1
χ_2	0	0	χ_1	$3\chi_2$
χ_3	0	$-\chi_1$	0	$-2\chi_3$
χ_4	$-\chi_1$	$-3\chi_2$	$2\chi_3$	0

Table 2: Commutation Relations

Transformation Groups

Transformation groups corresponding to each infinitesimal symmetry generator χ_i can be calculated using the general formula,

$$\xi^i(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{\alpha}_i}{\partial \epsilon}\ , \tag{106}$$

with initial condition $\tilde{\alpha}_i|_{\epsilon=0} = \alpha_i$.

Consider the infinitesimal symmetry generator $\chi_2 = \ x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}$ we obtain following expressions:

1.

$$\frac{\partial \tilde{x}}{\partial \epsilon} = \ \xi(\tilde{x}, \tilde{t}, \tilde{u}) = \tilde{x},$$

$$\ln \tilde{x} = \epsilon + c,$$

applying initial condition $\tilde{x}|_{\epsilon=0} = x$ we obtain,

$$\tilde{x} = xe^{\epsilon}.$$

2.

$$\frac{\partial \tilde{t}}{\partial \epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) = 3\tilde{t},$$

$$\ln \tilde{t} = \epsilon + c,$$

applying initial condition $\tilde{t}|_{\epsilon=0} = t$ we obtain,

$$\tilde{t} = te^{3\epsilon}.$$

3.

$$\frac{\partial \tilde{u}}{\partial \epsilon} = \phi(\tilde{x}, \tilde{t}, \tilde{u}) = -2\tilde{u},$$

$$\ln \tilde{u} = -2\epsilon + c,$$

applying initial condition $\tilde{u}|_{\epsilon=0} = u$ we obtain,

$$\tilde{u} = ue^{-2\epsilon}.$$

Therefore the transformation group G_1 generated by infinitesimal symmetry generator χ_1 is given as,

$$G_2 : \qquad (\tilde{x}, \tilde{t}, \tilde{u}) = (xe^{\epsilon}, te^{3\epsilon}, ue^{-2\epsilon}).$$

Similarly for remaining infinitesimal symmetry generators we have following the transformation groups:

$$G_1 : \qquad (\tilde{x}, \tilde{t}, \tilde{u}) = (x + \epsilon, t, u),$$

$$G_3 : \qquad (\tilde{x}, \tilde{t}, \tilde{u}) = (t\epsilon + x, t, u + \epsilon),$$

$$G_4 : \qquad (\tilde{x}, \tilde{t}, \tilde{u}) = (x, t + \epsilon, u).$$

CHAPTER 5

NONLINEAR WAVE EQUATION

A large amount of literature is available on Lie symmetry analysis of (1+1)-dimensional nonlinear wave equation [1, 2, 10]. More recently Magda and Lahno considered the classification problem of wave equation [32, 34]. Also study has been made about invariance properties and invariance groups [30]. Invariance of solutions under infinitesimal Lie group of transformations for various (1+1)-dimensional nonlinear wave equations has been worked out [21, 36]. Moreover it is shown that with the use of conservation laws non-local (potential) symmetries lead to new solutions for a large class of (1+1) wave equations with variables speeds [11]. This formulation has led to a variety of interesting applications such as equations with perturbed terms [28] and conservation laws associated with potential symmetries [3, 15].

Consequently the two-dimensional (1+2) wave equation with constant coefficients has been studied with an equivalent vigor [17, 51, 52]. However, the group theoretic approach to the equation with non-constant coefficients and the non-linear case have only been studied in specific cases and complete results have either not been obtained or not presented because of which very few exact solutions invariant under symmetry are known [7].

Whereas all these studies have focused on providing some exact invariant solutions, none gives a complete classification of these invariant solutions. With a view that a complete classification of the solutions may add to a further understanding we undertake this research to conduct a detailed symmetry analysis of a family of non-linear (1+2)-dimensional wave equation. While providing this complete classification we have obtained some new interesting solutions of this nonlinear wave equation.

The details and method presented here sets the scene for further interesting studies regarding the non-linear n-dimensional wave equation which may even include dissipative terms that arise in practice as in the telegraph equation.

In the next section we present reduction of a wave equation in which the nonlinearity is due to the velocity term involved.

5.1 The Equation $u_{tt} = u(u_{xx} + u_{yy})$

Lie group of point transformations of one parameter ϵ under which the given equation remains invariant are given as [46]:

$$\tilde{x} = x + \epsilon\, \xi(x, y, t, u) + O(\epsilon^2),$$

$$\tilde{y} = y + \epsilon\, \eta(x, y, t, u) + O(\epsilon^2),$$

$$\tilde{t} = t + \epsilon\, \tau(x, y, t, u) + O(\epsilon^2),$$

$$\tilde{u} = u + \epsilon\, \phi(x, y, t, u) + O(\epsilon^2).$$

Using above transformations, we now construct the **symmetry generator**. The symmetry generator is a vector field that generates the symmetry group under which the given nonlinear wave equation remain invariant [46]. This generator is given by [39],

$$\chi = \xi(x, y, t, u)\frac{\partial}{\partial x} + \eta(x, y, t, u)\frac{\partial}{\partial y} + \tau(x, y, t, u)\frac{\partial}{\partial t} + \phi(x, y, t, u)\frac{\partial}{\partial u}\ . \tag{107}$$

Since we are dealing with a second order PDE, the above generator needs to be **prolonged** to include second order derivatives [11]. The prolonged generator can be found using prolongation formula [39],

$$\chi^{(2)} = \chi + \sum_J \phi^J(x,u)\frac{\partial}{\partial u_J}\ , \tag{108}$$

where the summation is taken over all multi-indices $J = (j_1, j_2)$.

Having the prolongation of the infinitesimal symmetry generator, the next step is to satisfy the **infinitesimal criterion** [46], that requires;

$$\chi^{(2)}\left\{u_{tt} - u(u_{xx} + u_{yy})\right\}\Big|_{u_{tt}-u(u_{xx}+u_{yy})=0} = 0\ . \tag{109}$$

The above equation can be easily recast in the form,

$$\phi^{tt} - (u_{xx} + u_{yy})\phi - u\left(\phi^{xx} + \phi^{yy}\right) = 0\ . \tag{110}$$

At this stage we need to evaluate the expressions for ϕ^{tt} , ϕ^{xx} and ϕ^{yy} using the formula, [39],

$$\phi^J\left(x, u^{(n)}\right) = D_J\left(\phi - \sum_{k=1}^{p}\xi^k u_k\right) + \sum_{k=1}^{p}\xi^k\ u_{J,k}\ , \tag{111}$$

where $J = (j_1, j_2, \ldots, j_k)$ and $1 \le j_i \le p$ for all $i = 1, 2, \ldots, k$. Using (111) it is easy to show that,

$$\begin{aligned}\phi^x =\ & D_x\left(\phi - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx}\ ,\\ =&\phi_x + \phi_u u_x - \xi_x u_x - \xi_u u_x^2 - \eta_x u_y - \eta_u u_x u_y\\ &- \tau_x u_t - \tau_u u_x u_t\ .\end{aligned}$$

$$\phi^{xx} = D_x^2\left(\phi - \xi u_x - \eta u_y - \tau u_t\right) + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx} \ ,$$

$$= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 + (\phi_u - 2\xi_x)u_{xx} -$$

$$\xi_{uu}u_x^3 - 3\xi_u u_x u_{xx} - 2\eta_x u_{xy} - 2\eta_u u_{xy} u_x - \eta_{xx} u_y - 2\eta_{xu}$$

$$u_y u_x - \eta_{uu} u_y u_x^2 - \eta_u u_y u_{xx} - 2\tau_x u_{tx} - 2\tau_u u_x u_{tx} - \tau_{xx} u_t -$$

$$2\tau_{xu} u_x u_t - \tau_{uu} u_t u_x^2 - \tau_u u_{xx} u_t \ .$$

Similarly, the expressions for ϕ^{yy} and ϕ^{tt} can be calculated. These expressions are given as;

$$\phi^{yy} = \phi_{yy} + (2\phi_{yu} - \eta_{yy})u_y + (\phi_{uu} - 2\eta_{yu})u_y^2 + (\phi_u - 2\eta_y)u_{yy} -$$

$$\eta_{uu}u_y^3 - 3\eta_u u_y u_{yy} - 2\xi_y u_{xy} - 2\xi_u u_{xy} u_y - \xi_{yy} u_x - 2\xi_{yu}$$

$$u_y u_x - \xi_{uu} u_x u_y^2 - \xi_u u_x u_{yy} - 2\tau_y u_{ty} - 2\tau_u u_y u_{ty} - \tau_{yy} u_t -$$

$$2\tau_{yu} u_y u_t - \tau_{uu} u_t u_y^2 - \tau_u u_{yy} u_t \ .$$

$$\phi^{tt} = \phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t + (\phi_{uu} - 2\tau_{tu})u_t^2 + (\phi_u - 2\tau_t)u_{tt} -$$

$$\tau_{uu}u_t^3 - 3\tau_u u_t u_{tt} - 2\eta_t u_{ty} - 2\eta_u u_{ty} u_t - \eta_{tt} u_y - 2\eta_{tu} u_y u_t$$

$$- \eta_{uu} u_y u_t^2 - \eta_u u_y u_{tt} - 2\xi_t u_{tx} - 2\xi_u u_t u_{tx} - \xi_{tt} u_x - 2\xi_{tu}$$

$$u_x u_t - \xi_{uu} u_x u_t^2 - \xi_u u_{tt} u_x \ .$$

Now replacing u_{tt} with $u(u_{xx}+u_{yy})$ and substituting the above expressions (ϕ^{tt} , ϕ^{xx} , ϕ^{yy}) in (110) gives,

$$\begin{aligned}
&\phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t + (\phi_{uu} - 2\tau_{tu})u_t^2 + (\phi_u - 2\tau_t)u \\
&(u_{xx} + u_{yy}) - \tau_{uu}u_t^3 - 3\tau_u u_t u(u_{xx} + u_{yy}) - 2\eta_t u_{ty} \\
&- 2\eta_u u_{ty} u_t - \eta_{tt} u_y - 2\eta_{tu} u_t u_y - \eta_{uu} u_y u_t^2 - \eta_u u_y u \\
&(u_{xx} + u_{yy}) - 2\xi_t u_{tx} - 2\xi_u u_t u_{tx} - \xi_{tt} u_x - 2\xi_{tu} u_x u_t - \\
&\xi_{uu} u_x u_t^2 - \xi_u u_x u(u_{xx} + u_{yy}) - (u_{yy} + u_{xx})\phi - u\,\{\phi_{xx} \\
&+ (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 + (\phi_u - 2\xi_x)u_{xx} \\
&- \xi_{uu} u_x^3 - 3\xi_u u_x u_{xx} - 2\eta_x u_{xy} - 2\eta_u u_{xy} u_x - \eta_{xx} u_y - \\
&2\eta_{xu} u_y u_x - \eta_{uu} u_y u_x^2 - \eta_u u_y u_{xx} - 2\tau_x u_{tx} - 2\tau_u u_x u_{tx} \\
&- \tau_{xx} u_t - 2\tau_{xu} u_x u_t - \tau_{uu} u_t u_x^2 - \tau_u u_{xx} u_t\} - u\{\phi_{yy} \\
&+ (2\phi_{yu} - \eta_{yy})u_y + (\phi_{uu} - 2\eta_{uy})u_y^2 + (\phi_u - 2\eta_y)u_{yy} \\
&- \eta_{uu} u_y^3 - 3\eta_u u_y u_{yy} - 2\xi_y u_{xy} - 2\xi_u u_{xy} u_y - \xi_{yy} u_x - \\
&2\xi_{yu} u_x u_y - \xi_{uu} u_x u_y^2 - \xi_u u_x u_{yy} - 2\tau_y u_{ty} - 2\tau_u u_y u_{ty} \\
&- \tau_{yy} u_t - 2\tau_{uy} u_y u_t - \tau_{uu} u_t u_y^2 - \tau_u u_{yy} u_t\} = 0\ .
\end{aligned}$$

To Find the most general expression for the infinitesimal symmetry generator we need to find the general expressions for the components ξ , η , τ and ϕ . For this we treat above equation as an algebraic equation and compare the coefficients of like terms. The equations which arise as a consequence of this comparison are called **defining equations** [11].

To start with we compare the coefficients of u_{xx} to get,

$$u(\phi_u - 2\tau_t) - 3u\tau_u u_t - u\eta_u u_y - u\xi_u u_x - \phi -$$
$$(\phi_u - 2\xi_x)u + 3\xi_u u u_x + \eta_u u u_y + \tau_u u u_t = 0 \ . \tag{112}$$

Differentiating (112) with respect to u_t and u_x respectively, we obtain

$$\tau_u = 0, \qquad \text{and} \qquad \xi_u = 0 \ .$$

In the light of above, equation (112) reduces to,

$$2u(\xi_x - \tau_t) = \phi \ .$$

Now comparison of coefficients of u_{yy} yields,

$$u(\phi_u - 2\tau_t) - 3u\tau_u u_t - u\eta_u u_y - u\xi_u u_x - \phi -$$
$$(\phi_u - 2\eta_y)u + 3\eta_u u u_y + \xi_u u u_x + \tau_u u u_t = 0 \ , \tag{113}$$

which on differentiation with respect to u_y gives,

$$\eta_u = 0 \ .$$

As before, we substitute the above result back in (113), this reduces it to,

$$2u(\eta_y - \tau_t) = \phi \ .$$

Also the coefficients of mixed second order derivatives u_{xy}, u_{xt} and u_{ty} respectively give the following equations,

$$\eta_x = -\xi_y \ , \tag{114}$$

$$\xi_t = u\tau_x \ , \tag{115}$$

$$\eta_t = u\tau_y \ . \tag{116}$$

Similarly comparing the coefficients of remaining monomials and simplification gives the following set of ten coupled equations

$$\tau_u = \xi_u = \eta_u = 0 \ , \tag{117}$$

$$\phi = 2u(\xi_x - \tau_t) = 2u(\eta_y - \tau_t) \ , \tag{118}$$

$$\eta_x = -\xi_y \ , \tag{119}$$

$$\xi_t = u\tau_x \ , \tag{120}$$

$$\eta_t = u\tau_y \ , \tag{121}$$

$$\phi = \alpha(x, y, t)u + \beta(x, y, t) \ , \tag{122}$$

$$\xi_{tt} + u(2\phi_{xu} - \xi_{xx} - \xi_{yy}) = 0 \ , \tag{123}$$

$$\eta_{tt} + u(2\phi_{yu} - \eta_{xx} - \eta_{yy}) = 0 \ , \tag{124}$$

$$\tau_{tt} - 2\phi_{tu} - u(\tau_{xx} + \tau_{yy}) = 0 \ , \tag{125}$$

$$\phi_{tt} - u(\phi_{xx} + \phi_{yy}) = 0 \ . \tag{126}$$

At this stage we solve the above coupled system (117)-(126) for the components of infinitesimal symmetry generator. Solving the above equations iteratively and requiring consistency criterion by substituting the resulting equations back and forth into each other, the solution of the above system takes the form,

$$\xi = c_0 + c_1 x + c_2 y \ , \tag{127}$$

$$\eta = c_3 - c_2 x + c_1 y \ , \tag{128}$$

$$\tau = \frac{2}{5}(2c_1 - c_4)t + c_5 \ , \tag{129}$$

$$\phi = \frac{2}{5}(c_1 + 2c_4)u + \beta(x, y, t) \ , \tag{130}$$

where $\beta(x, y, t)$ satisfies the equation

$$\beta_{tt} - u(\beta_{xx} + \beta_{yy}) = 0.$$

Corresponding to every one parameter there exists an infinitesimal symmetry generator [39]. Therefore, by substituting $c_i = 1$ and $c_j = 0 \ \forall \ i \neq j$ for $i = 0, \ldots, 5$ we obtain seven infinitesimal symmetry generators given by;

$$\chi_0 = \frac{\partial}{\partial x}\ , \qquad \chi_1 = x\,\frac{\partial}{\partial x} + y\,\frac{\partial}{\partial y} + \frac{4}{5}\,t\,\frac{\partial}{\partial t} + \frac{2}{5}\,u\,\frac{\partial}{\partial u}\ ,$$

$$\chi_2 = y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}\ , \qquad \chi_3 = \frac{\partial}{\partial y}\ ,$$

$$\chi_4 = -\frac{2}{5}\,t\,\frac{\partial}{\partial t} + \frac{4}{5}\,u\,\frac{\partial}{\partial u}\ , \qquad \chi_5 = \frac{\partial}{\partial t}\ ,$$

$$\chi_\beta = \beta\,\frac{\partial}{\partial u}\ .$$

Commutation relations (c.f. Definition 3.7.7) for these generators are given in the form of table 3. The **commutator table** describes the structure of associated Lie algebra in a convenient way [39].

Corresponding to each infinitesimal symmetry generator we can find the **transformation groups** [39] using the formula

$$\xi^i(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{\alpha}_i}{\partial \epsilon}\ , \tag{131}$$

with initial condition $\tilde{\alpha}_i|_{\epsilon=0} = \alpha_i$.

$[\chi_i, \chi_j]$	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
χ_0	0	χ_0	$-\chi_3$	0	0	0
χ_1	$-\chi_0$	0	0	$-\chi_3$	0	$-\frac{4}{5}\chi_5$
χ_2	0	0	0	$-\chi_0$	0	0
χ_3	0	χ_3	χ_0	0	0	0
χ_4	0	0	0	0	0	$\frac{2}{5}\chi_5$
χ_5	0	$\frac{4}{5}\chi_5$	0	0	$-\frac{2}{5}\chi_5$	0

Table 3: Algebra of commutators

Considering the generator $\chi_1 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{4}{5} t \frac{\partial}{\partial t} + \frac{2}{5} u \frac{\partial}{\partial u}$, we have,

1.

$$\frac{\partial \tilde{x}}{\partial \epsilon} = \xi(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \tilde{x},$$

$$\ln \tilde{x} = \epsilon + \ln c,$$

applying initial condition $\tilde{x}|_{\epsilon=0} = x$ we obtain,

$$\tilde{x} = x\, e^{\epsilon}.$$

2.

$$\frac{\partial \tilde{y}}{\partial \epsilon} = \eta(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \tilde{y},$$

$$\ln \tilde{y} = \epsilon + \ln c,$$

applying initial condition $\tilde{y}|_{\epsilon=0} = y$ we obtain,

$$\tilde{y} = \; y \; e^{\epsilon}.$$

3.

$$\frac{\partial \tilde{t}}{\partial \epsilon} = \; \tau(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \; \frac{4}{5} \; \tilde{t},$$

$$\ln \tilde{t} = \; \frac{4}{5} \; \epsilon + c,$$

applying initial condition $\tilde{t}|_{\epsilon=0} = t$ we obtain,

$$\tilde{t} = \; t \; e^{\frac{4\epsilon}{5}}.$$

4.

$$\frac{\partial \tilde{u}}{\partial \epsilon} = \; \phi(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \frac{2}{5} \; \tilde{u},$$

$$\ln \tilde{u} = \frac{2}{5} \; \epsilon + c,$$

applying initial condition $\tilde{u}|_{\epsilon=0} = u$ we obtain,

$$\tilde{u} = \; u \; e^{\frac{2\epsilon}{5}}.$$

Therefore the transformation group G_1 generated by infinitesimal symmetry generator χ_1 is given as,

$$G_1 : \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = (\; x \; e^{\epsilon}, \; y \; e^{\epsilon}, \; t \; e^{\frac{4\epsilon}{5}}, \; u \; e^{\frac{2\epsilon}{5}}) \; .$$

Similarly the transformation groups for the remaining infinitesimal symmetry generator are,

$$G_0 : \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) \;\; = \;\; (x + \epsilon, y, t, u) \; ,$$

$$G_2: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \left(\frac{x + y\epsilon}{1+\epsilon^2}, \frac{y - x\epsilon}{1+\epsilon^2}, t, u\right) ,$$

$$G_3: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = (x, y+\epsilon, t, u) ,$$

$$G_4: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \left(x, y, te^{-\frac{2}{5}\epsilon}, ue^{\frac{4}{5}\epsilon}\right) ,$$

$$G_5: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = (x, y, t+\epsilon, u) ,$$

$$G_\beta: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = (x, y, t, u + \epsilon\ \beta(x,y,t)) .$$

5.1.1 Reduction under infinitesimal symmetry generators

Infinitesimal symmetry generator reduces the number of independent variables by one in the partial differential equation [11, 39]. In this section we find reduction of the given wave equation under each infinitesimal symmetry generator. The detailed calculations for the reduction under χ_1 are given.

Consider the generator,

$$\chi_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{4}{5}t\frac{\partial}{\partial t} + \frac{2}{5}u\frac{\partial}{\partial u} . \tag{132}$$

The characteristic equation for this generator is

$$\frac{dx}{x} = \frac{dy}{y} = \frac{5\ dt}{4t} = \frac{5\ du}{2u} . \tag{133}$$

We now find the similarity variables for the above generator, (132)

1.

$$\frac{dx}{x} = \frac{dy}{y} ,$$

$$\ln x = \ln y + \ln r ,$$

$$r = \frac{x}{y} \ .$$

2.

$$\frac{dx}{x} = \frac{5\ dt}{4t} \ ,$$

$$\ln x = \frac{5}{4} \ \ln t + \ln s \ ,$$

$$x = s\ t^{\frac{5}{4}} \ .$$

3.

$$\frac{5\ dt}{4t} = \frac{5\ du}{2u} \ ,$$

$$\frac{1}{2} \ \ln t + \ln w = \ln u \ ,$$

$$u = w\sqrt{t} \ .$$

The given wave equation can be transformed into these new similarity variables r, s and w as follows,

$$u_{tt} \ = \ -\frac{1}{4}\ t^{-\frac{3}{2}}\ w \ + \ \frac{25}{16}\ s\ t^{-\frac{3}{2}}\ w_s \ + \ \frac{25}{16}\ s^2\ t^{-\frac{3}{2}}\ w_{ss} \ ,$$

$$u_{xx} \ = \ w_{ss}\ t^{-2} \ + \ 2\ w_{rs}\ \frac{t^{-\frac{3}{4}}}{y} \ + \ w_{rr}\ \frac{t^{\frac{1}{2}}}{y^2} \ ,$$

$$u_{yy} \ = \ 2\ \sqrt{t}\ w_r\ \frac{r}{y^2} \ + \ \sqrt{t}\ w_{rr}\ \frac{r^2}{y^2} \ .$$

Thus the wave equation $u_{tt} + u(u_{xx} + u_{yy})$ reduces to a PDE with two independent and one dependent variable,

$$-4\,w\,s^2 + 25\,w_s\,s^3 + 25\,w_{ss}\,s^4 \ = \ 16\,w\,(s^2\,w_{ss} + 2\,w_{sr}\,r\,s + w_{rr}\,r^2 + 2\,w_r\,r^3 + w_{rr}\,r^4) \, .$$

A complete table of reductions under each generator is given in Table 8, Appendix-A.

5.1.2 Reduction under two dimensional subalgebra

A two dimensional subalgebra reduces the number of independent variables by two [39]. In this section we will show the reduction of given nonlinear wave equation to an ODE under each two dimensional subalgebra.

Consider the algebra $[\, \chi_1 \, , \, \chi_2 \,] = 0$ the reduction can be started with any one of the generators, but starting with χ_1 will lead to a cumbersome expression $-4\, w\, s^2 + 25\, w_s\, s^3 + 25\, w_{ss}\, s^4 \;=\; 16\, w\, (s^2\, w_{ss} + 2\, w_{sr}\, r\, s + w_{rr}\, r^2 + 2\, w_r\, r^3 + w_{rr}\, r^4)$. Therefore, instead of χ_1 , we will start with χ_2 that gives rather simpler expression. Similarity variables for χ_2 are $r \;=\; x^2 \;+\; y^2,\; s \;=\; t,\; w \;=\; u$ and the reduced differential equation is,

$$w_{ss} = \; 4\, w\; (\; w_r \;+\; r\, w_{rr}) \; . \tag{134}$$

To proceed further we need to transform χ_1 in new variables r, s and w thus,

$$\widetilde{\chi}_1 \;=\; 2\, r\, \frac{\partial}{\partial r} \;+\; \frac{4}{5}\, s\, \frac{\partial}{\partial s} \;+\; \frac{2}{5}\, w\, \frac{\partial}{\partial w} \; .$$

The characteristic equation corresponding to the transformed infinitesimal symmetry generator is,

$$\frac{dr}{2\, r} \;=\; \frac{5\; ds}{4s} \;=\; \frac{5\; dw}{2\; w} \; . \tag{135}$$

From this equation we find the following new similarity variables,

1. $\alpha \;=\; r\; s^{-\frac{5}{2}},$

2. $\sqrt{s}\; \beta(\alpha) \;=\; w.$

Transformation of (134) in new similarity variables α and β leads to,

$$w_{rr} \;=\; s^{-\frac{9}{2}}\; \beta_{\alpha\alpha} \; ,$$
$$w_{ss} \;=\; -\frac{1}{4}\, s^{-\frac{3}{2}}\, \beta \;+\; \frac{25}{4}\, \alpha\, \beta_\alpha\, s^{-\frac{3}{2}} \;+\; \frac{25}{4}\, \alpha^2\, \beta_{\alpha\alpha}\, s^{-\frac{3}{2}} \; .$$

Thus $w_{ss} \; = \; 4 \, w \, (w_r \; + \; r \; w_{rr})$ reduces to,

$$-\frac{1}{4}\,\beta \; + \; \frac{25}{4}\,\alpha\,\beta_\alpha \; + \; \frac{25}{4}\,\alpha^2\,\beta_{\alpha\alpha} \; = \; 4\,\beta\,(\beta_\alpha \; + \; \alpha\,\beta_{\alpha\alpha}) \; ,$$

which is an ODE of order two.

Since the generators χ_1 and χ_2 forms an abelian subalgebra so we can start the reduction with any one of them, but, this is not always true. If the coefficient of a commutator forming a closed algebra is nonzero then we have to start with the generator that comes as a result of the commutation. Consider, for example, the algebra $[\,\chi_5 \; , \; \chi_1\,] = \frac{4}{5}\,\chi_5$, in this case we have to start reduction with χ_5 that reduces the given wave equation to,

$$w_{rr} \; + \; w_{ss} \; = \; 0 \; , \tag{136}$$

with similarity variables $r = x, \; s = y$ and $w = u$. Transformation of χ_1 in similarity variables is given as,

$$\widetilde{\chi}_1 \; = \; r\,\frac{\partial}{\partial r} \; + \; s\,\frac{\partial}{\partial s} \; + \; \frac{2}{5}\,w\,\frac{\partial}{\partial w} \; .$$

Characteristic equation corresponding to the infinitesimal symmetry generator $\widetilde{\chi}_1$ is,

$$\frac{dr}{r} \; = \; \frac{ds}{s} \; = \; \frac{5\;dw}{w} \; .$$

Therefore new similarity variables are,

1. $\alpha \; = \; r/s$,

2. $s^{\frac{2}{5}}\;\beta(\alpha) \; = \; w$.

In these variables the differential equation (136) reduces to,

$$\beta_{\alpha\alpha} \; - \; \frac{6}{25}\,\beta \; + \; \frac{10}{5}\,\alpha\,\beta_\alpha \; + \; \alpha^2\,\beta_{\alpha\alpha} = 0 \; ,$$

which is again an ODE of order two.

Reductions under the remaining two dimensional subalgebras are given in Table 9, Appendix-A.

5.2 The Equation $u_{tt} = u^n(u_{xx} + u_{yy})$

In this section we find the symmetry reductions and possible solutions using classical Lie symmetry method for (1+2)-dimensional nonlinear wave equation,

$$u_{tt} = u^n(u_{xx} + u_{yy}). \tag{137}$$

This is the general case of the equation solved in previous section. Lie group of point transformations of one parameter ϵ under which equation (137) remain invariant are given as [46],

$$\widetilde{\alpha_i} = \alpha_i + \epsilon\ \xi^i(\alpha) + O\ (\epsilon^2)\ , \tag{138}$$

where α_i represents the variables $x,\ y,\ t,\ u$ and ξ^i represents $\xi(x, y, t, u)$, $\eta(x, y, t, u)$, $\tau(x, y, t, u)$, $\phi(x, y, t, u)$ for $i = 1,\ 2,\ 3,\ 4$. Also the kth-order derivative (of the transformed 'dependent' variable with respect to the transformed 'independent' variables) [46] is given as,

$$\widetilde{u_J} = u_J + \epsilon\ \phi^J(x, y, t, u) + O\ (\epsilon^2), \tag{139}$$

where $u_J = \frac{\partial u}{\partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}$, $J = J(j_1, \ldots, j_k)$ and $1 \leq j_k \leq 4$ for all k .

In order to find solution of (137), we begin with **symmetry generator** corresponding to the variables x, y, t and u, given by the formula [39],

$$\begin{aligned}\chi = \ & \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t} \\ & + \phi(x,y,t,u)\frac{\partial}{\partial u} \ .\end{aligned} \tag{140}$$

Infinitesimal symmetry generator (140) is a vector field of tangent vectors and components ξ, η, τ and ϕ, of vector field are arbitrary, real valued smooth functions defined in some subspace of the space of the independent variables x, y, t and the dependent variable u.

Since the differential equation (137) is of second order therefore **prolonging** the generator up to second order [11] using the general formula (108), which gives,

$$\begin{aligned}\chi^{(2)} = \ & \chi + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{y}\frac{\partial}{\partial u_y} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} \\ & + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \ .\end{aligned} \tag{141}$$

Lie symmetry criterion for PDEs requires that $\chi^2(H) = 0$ subject to $H = 0$ [11], which is equivalent to the requirement,

$$\chi^{(2)}\big\{u_{tt} - u^n(u_{xx} + u_{yy})\big\}\Big|_{u_{tt}-u^n(u_{xx}+u_{yy})=0} = 0 \ . \tag{142}$$

From (142), we easily obtain the relation,

$$\phi^{tt} - nu^{(n-1)}(u_{xx} + u_{yy})\phi - u^n(\phi^{xx} + \phi^{yy}) = 0 \ . \tag{143}$$

Substitution of the expressions for ϕ^{tt}, ϕ^{xx}, ϕ^{yy} in (143) and u^n $(u_{xx} + u_{yy})$ for

u_{tt} yields the following general equation,

$$
\begin{aligned}
&\phi_{tt} + (2\phi_{tu} - \tau_{tt})u_t + (\phi_{uu} - 2\tau_{tu})u_t^2 + (\phi_u - 2\tau_t)u^n \\
&(u_{xx} + u_{yy}) - \tau_{uu}u_t^3 - 3\tau_u u_t u^n(u_{xx} + u_{yy}) - 2\eta_t u_{ty} \\
&- 2\eta_u u_{ty} u_t - \eta_{tt} u_y - 2\eta_{tu} u_t u_y - \eta_{uu} u_y u_t^2 - \eta_u u_y u^n \\
&(u_{xx} + u_{yy}) - 2\xi_t u_{tx} - 2\xi_u u_t u_{tx} - \xi_{tt} u_x - 2\xi_{tu} u_x u_t - \\
&\xi_{uu} u_x u_t^2 - \xi_u u_x u^n(u_{xx} + u_{yy}) - nu^{n-1}(u_{yy} + u_{xx})\phi \\
&- u^n\{\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + (\phi_{uu} - 2\xi_{xu})u_x^2 + (\phi_u \\
&- 2\xi_x)u_{xx} - \xi_{uu}u_x^3 - 3\xi_u u_x u_{xx} - 2\eta_x u_{xy} - 2\eta_u u_{xy} u_x \\
&- \eta_{xx} u_y - 2\eta_{xu} u_y u_x - \eta_{uu} u_y u_x^2 - \eta_u u_y u_{xx} - 2\tau_x u_{tx} \\
&- 2\tau_u u_x u_{tx} - \tau_{xx} u_t - 2\tau_{xu} u_x u_t - \tau_{uu} u_t u_x^2 - \tau_u u_{xx} u_t\} \\
&- u^n\{\phi_{yy} + (2\phi_{yu} - \eta_{yy})u_y + (\phi_{uu} - 2\eta_{uy})u_y^2 + (\phi_u - \\
&2\eta_y)u_{yy} - \eta_{uu}u_y^3 - 3\eta_u u_y u_{yy} - 2\xi_y u_{xy} - 2\xi_u u_{xy} u_y - \\
&\xi_{yy} u_x - 2\xi_{yu} u_x u_y - \xi_{uu} u_x u_y^2 - \xi_u u_x u_{yy} - 2\tau_y u_{ty} - 2 \\
&\tau_u u_y u_{ty} - \tau_{yy} u_t - 2\tau_{uy} u_y u_t - \tau_{uu} u_t u_y^2 - \tau_u u_{yy} u_t\} = 0 \ .
\end{aligned}
$$

Comparison of the coefficients of monomials and the coefficients of terms without any monomial gives following set of nine **defining equations**,

$$2(\xi_x - \tau_t)u^n - 2\tau_u u_t u^n - nu^{n-1}\phi + 2u^n \xi_u u_x = 0 \ , \tag{144}$$

$$2(\eta_y - \tau_t)u^n - 2\tau_u u_t u^n - nu^{n-1}\phi + 2u^n \eta_u u_y = 0 \ , \tag{145}$$

$$u^n \eta_x + u^n \eta_u u_x + u^n \xi_y + u^n \xi_u u_y = 0 \ , \tag{146}$$

$$\xi_t - u^n \tau_x = 0 \ , \tag{147}$$

$$\eta_t - u^n \tau_y = 0 \ , \tag{148}$$

$$2\phi_{tu} - \tau_{tt} + \phi_{uu}u_t + u^n(\tau_{xx} + \tau_{yy}) = 0 \ , \tag{149}$$

$$\eta_{tt} + 2u^n\phi_{uy} + \eta_{xx} + \eta_{yy} = 0 \ , \tag{150}$$

$$\xi_{tt} + 2u^n\phi_{ux} + \xi_{xx} + \xi_{yy} = 0 \ , \tag{151}$$

$$\phi_{tt} - u^n(\phi_{xx} + \phi_{yy}) = 0 \ . \tag{152}$$

We now solve these equations simultaneously for the components ξ, η, τ and ϕ of infinitesimal symmetry generator. Coefficients of u_x and u_y in equation (144) requires,

$$\tau_u = 0 = \xi_u \ , \tag{153}$$

which implies that τ and ξ are functions of x, y and t only. Substitution of (153) in equation (144) gives,

$$2(\xi_x - \tau_t) = nu^{-1}\phi \ . \tag{154}$$

Similarly by comparing coefficients of u_y in equation (145) we find that $\eta_u = 0$ and by substituting back this value and using (153) we get,

$$2(\eta_y - \tau_t) = nu^{-1}\phi \ . \tag{155}$$

By using equation (153) in (146) and from (147) and (148) we obtain following results,

$$\eta_x = -\xi_y, \qquad \xi_t = u^n\tau_x \qquad \text{and} \qquad \eta_t = u^n\tau_y \ . \tag{156}$$

The coefficients of u_t and the terms without any monomials in equation (149) respectively requires,

$$\phi = \alpha(x,y,t)u + \beta(x,y,t) \qquad \text{and} \qquad 2\alpha_t - \tau_{tt} + u^n(\tau_{xx} + \tau_{yy}) = 0 \ . \tag{157}$$

By using (157) in (151) and (150) we obtain,

$$-\eta_{tt} - 2u^n\alpha_y - \eta_{xx} - \eta_{yy} \qquad \text{and} \qquad -\xi_{tt} - 2u^n\alpha_x - \xi_{xx} - \xi_{yy} = 0 \ . \tag{158}$$

The remaining terms not involving any derivatives of u are,

$$\phi_{tt} - u^n(\phi_{xx} + \phi_{yy}) = 0 \ . \tag{159}$$

Now solving (153) to (159) iteratively and requiring consistency criterion by substituting the resulting equations into each other, we obtain following expressions for $\xi,\ \eta,\ \tau$ and ϕ,

$$\xi = a_0 - a_1 y + a_2 x \ ,$$

$$\eta = a_3 + a_1 x + a_2 y \ ,$$

$$\tau = \left(\frac{4a_2}{n+4} - \frac{2na_4}{n+4}\right) t + a_5 \ ,$$

$$\phi = \left(\frac{2a_2}{n+4} + \frac{4a_4}{n+4}\right) u + \beta(x,y,t) \ ,$$

where $a_i' s$ are arbitrary constants. Now substituting $a_i = 1$ and $a_j = 0$ for $j \neq i$, we have following **infinitesimal generators**,

$$\chi_0 = \frac{\partial}{\partial x} \ , \qquad \chi_1 = -y\ \frac{\partial}{\partial x} + x\ \frac{\partial}{\partial y} \ ,$$

$$\chi_2 = x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y} + \left(\frac{4t}{n+4}\right)\frac{\partial}{\partial t} + \left(\frac{2u}{n+4}\right)\frac{\partial}{\partial u} \ ,$$

$$\chi_3 = \frac{\partial}{\partial y} \ , \qquad \chi_4 = -\left(\frac{2\ n\ t}{n+4}\right)\frac{\partial}{\partial t} + \left(\frac{4u}{n+4}\right)\frac{\partial}{\partial u} \ ,$$

$$\chi_5 = \frac{\partial}{\partial t} \ , \qquad \chi_\beta = \beta\ \frac{\partial}{\partial u} \ ,$$

where from equation (159),

$$\beta_{tt} - u^n(\beta_{xx} + \beta_{yy}) = 0,$$

which implies that $\beta(x, y, t)$ is any solution of the nonlinear wave equation (137) and χ_β is an infinite dimensional subalgebra. The **commutation relations** for these generators are given in Table 4.

$[\ \chi_i,\ \chi_j\ ]$	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
χ_0	0	χ_3	χ_0	0	0	0
χ_1	$-\chi_3$	0	0	χ_0	0	0
χ_2	$-\chi_0$	0	0	$-\chi_3$	0	$\left(\frac{-4}{n+4}\right)\chi_5$
χ_3	0	$-\chi_0$	χ_3	0	0	0
χ_4	0	0	0	0	0	$\left(\frac{-2n}{n+4}\right)\chi_5$
χ_5	0	0	0	$\left(\frac{4}{n+4}\right)\chi_5$	$\left(\frac{-2n}{n+4}\right)\chi_5$	0

Table 4: Commutator Algebra for Symmetry Generators

Lie **transformation groups** for an infinitesimal symmetry generator can be found using the formula,

$$\xi^i(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{\alpha}_i}{\partial \epsilon} \ , \tag{160}$$

with initial condition $\tilde{\alpha}_i|_{\epsilon=0} = \alpha_i$.

Considering the generator,

$$\chi_2 = x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y} + \left(\frac{4\ t}{n+4}\right)\ \frac{\partial}{\partial t} + \left(\frac{2\ u}{n+4}\right)\frac{\partial}{\partial u}\ ,$$

thus the components of transformation group are, 1.

$$\frac{\partial \tilde{x}}{\partial \epsilon} = \xi(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \tilde{x},$$

$$\ln \tilde{x} = \epsilon + \ln c,$$

applying initial condition $\tilde{x}|_{\epsilon=0} = x$ we obtain,

$$\tilde{x} = x \ e^{\epsilon}.$$

2.

$$\frac{\partial \tilde{y}}{\partial \epsilon} = \eta(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \tilde{y},$$

$$\ln \tilde{y} = \epsilon + \ln c,$$

applying initial condition $\tilde{y}|_{\epsilon=0} = y$ we obtain,

$$\tilde{y} = y \ e^{\epsilon}.$$

3.

$$\frac{\partial \tilde{t}}{\partial \epsilon} = \tau(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \frac{4 \ \tilde{t}}{n+4},$$

$$\ln \tilde{t} = \frac{4}{n+4} \ \epsilon + c,$$

applying initial condition $\tilde{t}|_{\epsilon=0} = t$ we obtain,

$$\tilde{t} = t \ e^{\frac{4\epsilon}{n+4}}.$$

4.

$$\frac{\partial \tilde{u}}{\partial \epsilon} = \phi(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = \frac{2 \ \tilde{u}}{n+4},$$

$$\ln \tilde{u} = \frac{2}{n+4} \ \epsilon + c,$$

applying initial condition $\tilde{u}|_{\epsilon=0} = u$ we obtain,

$$\tilde{u} = \ u\ e^{\frac{2\epsilon}{n+4}}.$$

Thus the transformation group G_2 generated by infinitesimal symmetry generator χ_2 is given as,

$$G_1: \qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) = (\ x\ e^{\epsilon},\ \ y\ e^{\epsilon},\ \ t\ e^{\frac{4\epsilon}{n+4}},\ \ u\ e^{\frac{2\epsilon}{n+4}}).$$

Similarly the transformation groups for the remaining infinitesimal symmetry generator are,

$$\begin{aligned}
G_0: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \ (x+\epsilon, y, t, u),\\
G_2: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \left(\frac{x - y\epsilon}{1+\epsilon^2}, \frac{y + x\epsilon}{1+\epsilon^2}, t, u\right),\\
G_3: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \ (x, y+\epsilon, t, u),\\
G_4: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \left(x, y, te^{-\frac{-2n\epsilon}{n+4}}, ue^{\frac{4\epsilon}{n+4}}\right),\\
G_5: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \ (x, y, t+\epsilon, u),\\
G_\beta: &\qquad (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}) &=& \ (x, y, t, u + \epsilon\ \beta(x, y, t)).
\end{aligned}$$

5.2.1 Reduction under infinitesimal symmetry generators

The number of independent variables in (137) can be reduced by one with each symmetry generator. Detailed calculations to find the similarity variables r, s and w and also the reduction of (137) are given for the symmetry generator,

$$\chi_4 = -\Big(\frac{2\ n\ t}{n+4}\Big)\frac{\partial}{\partial t} + \Big(\frac{4\ u}{n+4}\Big)\frac{\partial}{\partial u}\ .$$

Characteristic equation for this generator is,

$$\frac{dx}{0}=\frac{dy}{0}=\frac{-(n+4)dt}{2nt}=\frac{(n+4)du}{4u}\ .$$

Relation $\frac{dx}{0}=\frac{-(n+4)dt}{2nt}$ gives the first similarity variable $s=x$, similarly $\frac{dy}{0}=\frac{-(n+4)dt}{2nt}$ gives second similarity independent variable $r=y$ and finally the relation $\frac{-(n+4)dt}{2nt}=\frac{(n+4)du}{4u}$ gives similarity dependent variable $u=t^{\frac{-2}{n}}e^{w}$. In these similarity variables differential equation (137) reduces to,

$$\frac{2}{n}\Big(\frac{2}{n}+1\Big)=e^{nw}\big(w_{rr}+w_{ss}\big).$$

Reductions under remaining infinitesimal symmetry generators with their corresponding similarity variables are given in Appendix-B, Table 10.

5.2.2 Reduction under two dimensional subalgebra

Two dimensional subalgebra reduces the PDE (137) to an ODE of second order, the solution of this ODE gives the solution of PDE (137), by substituting back the variables. Since $[\ \chi_3,\ \chi_5\ ]=0$, therefore χ_3 and χ_5' form a closed subalgebra, we can begin with any one of them, starting with $\chi_3=\frac{\partial}{\partial y}$ we can reduce the equation (137) to,

$$w_{rr}=w^{n}w_{ss}\ , \tag{161}$$

with similarity variables $s=x,\ r=t$ and $w(r,s)=u,$ also transformation of χ_5 in these new variables is,

$$\widetilde{\chi}_5=0\frac{\partial}{\partial s}+\frac{\partial}{\partial r}+0\frac{\partial}{\partial w}\ ,$$

that reduces (161) to $\beta''=0$ and the similarity variables in this case are $\alpha=s$ and $\beta(\alpha)=\ w$.

Similarly $[\ \chi_0,\ \chi_2\ ] =\ \chi_0$ forms two dimensional closed algebra, we start with χ_0 obtaining,

$$w_{rr} = w^n w_{ss}\ , \tag{162}$$

as a reduction of (137) where $s = y,\ r = t$ and $w(r,s) =\ u$ so that χ_2 is transformed to $\widetilde{\chi}_2 =\ s\,\frac{\partial}{\partial s} + \left(\frac{4r}{n+4}\right)\frac{\partial}{\partial r} + \left(\frac{2w}{n+4}\right)\frac{\partial}{\partial w}$ as a Lie symmetry generator in new variables. The invariants of $\widetilde{\chi}_2$ are $\alpha = \frac{r^{\frac{n}{4}+1}}{s}$ and $\sqrt{r}e^{\beta(\alpha)} =\ w$ and the reduction of (162) under these variables to an ODE is,

$$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) =\ \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')\ .$$

Reduction under remaining two dimensional subalgebras is given in Appendix-B, Table 11.

5.3 A general form of a nonlinear wave equation

In this section we perform the symmetry classification of a more general nonlinear one-two wave equation,

$$u_{tt} - f(u)(u_{xx} + u_{yy}) = 0, \tag{163}$$

where $f(u)$ is an arbitrary function of the variable u, has been given as well as various commutator tables. New symmetries are obtained for large classes of the equations; exact solutions invariant under two-dimensional sub-algebras are obtained.

We use the classical Lie symmetry method to obtain exact solutions of the above equation for all possibilities in $f(u)$. The one parameter Lie point transformations which leave (163) invariant are given by,

$$\widetilde{\alpha_i} = \alpha_i + \epsilon\ \xi^i(\alpha) + O\ (\epsilon^2), \tag{164}$$

where α_i represents the variables $x,\ y,\ t,\ u$ and ξ^i represents $\xi(x,y,t,u)$, $\eta(x,y,t,u)$, $\tau(x,y,t,u)$, $\phi(x,y,t,u)$ for $i = 1,\ 2,\ 3,\ 4$. Corresponding to transformations (164), the expressions for the kth-order derivatives (of the transformed 'dependent' variable with respect to the transformed 'independent' variables) is given as,

$$\widetilde{u_J} = u_J + \epsilon\ \phi^J(x,y,t,u) + O\ (\epsilon^2), \tag{165}$$

where $u_J = \frac{\partial u}{\partial x^{j_1}\partial x^{j_2}\ldots\partial x^{j_k}}$, $J = J(j_1,\ldots,j_k)$ and $1 \le j_k \le 4$ for all k .

In order to find solution of (163), we begin by writing symmetry generator corresponding to the variables $x,\ y,\ t$ and u,

$$\begin{aligned}\chi = \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \tau(x,y,t,u)\frac{\partial}{\partial t}\\ + \phi(x,y,t,u)\frac{\partial}{\partial u},\end{aligned} \tag{166}$$

where $\xi,\ \eta,\ \tau$ and ϕ are the components of the tangent vector. We then proceed to prolong the above generator up to second order,

$$\begin{aligned}\chi^{(2)} =\ &\chi + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}}\\ &+ \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}\ .\end{aligned} \tag{167}$$

To write $\chi^{(2)}$ explicitly, we evaluate the values for ϕ^x, ϕ^y, ϕ^t, ϕ^{xx}, ϕ^{xy}, ϕ^{xt}, ϕ^{yy}, ϕ^{yt} and ϕ^{tt} using,

$$\phi^J(\alpha^i,u) = D_J\left(\phi - \sum_{i=1}^{3}\xi^i u_i\right) + \sum_{i=1}^{3}\xi^i u_{J,i} \tag{168}$$

where,

$$D_i\phi = \frac{\partial\phi}{\partial x^i} + \sum_J u_{J,i}\frac{\partial\phi}{\partial u_J}\ . \tag{169}$$

At this stage we apply the Lie point symmetry criterion $\chi^2(H)\ \mid_{H=0} =\ 0$ for partial differential equations to the wave equation,to obtain,

$$\phi^{tt} - f_u(u_{xx}+u_{yy})\phi - f(u)(\phi^{xx}+\phi^{yy}) = 0. \tag{170}$$

The general solution of (170) determines the expressions of the components of the infinitesimal symmetry generator 'χ'. These expressions can be obtained by substituting the values of ϕ^{tt}, ϕ^{xx} and ϕ^{yy} and replacing u_{tt} by $f(u)(u_{xx}+u_{yy})$ in (170), that gives,

$$
\begin{aligned}
&\phi_{tt}+(2\phi_{tu}-\tau_{tt})u_t+(\phi_{uu}-2\tau_{tu})u_t^2+(\phi_u-2\tau_t)f(u)\\
&(u_{xx}+u_{yy})-\tau_{uu}u_t^3-3\tau_u u_t f(u)(u_{xx}+u_{yy})-2\eta_t u_{ty}\\
&-2\eta_u u_{ty}u_t-\eta_{tt}u_y-2\eta_{tu}u_t u_y-\eta_{uu}u_y u_t^2-\eta_u u_y f(u)\\
&(u_{xx}+u_{yy})-2\xi_t u_{tx}-2\xi_u u_t u_{tx}-\xi_{tt}u_x-2\xi_{tu}u_x u_t-\\
&\xi_{uu}u_x u_t^2-\xi_u u_x f(u)(u_{xx}+u_{yy})-f_u(u_{yy}+u_{xx})\phi-f\\
&\{\phi_{xx}+(2\phi_{xu}-\xi_{xx})u_x+(\phi_{uu}-2\xi_{xu})u_x^2+(\phi_u-2\xi_x)\\
&u_{xx}-\xi_{uu}u_x^3-3\xi_u u_x u_{xx}-2\eta_x u_{xy}-2\eta_u u_{xy}u_x-\eta_{xx}u_y\\
&-2\eta_{xu}u_y u_x-\eta_{uu}u_y u_x^2-\eta_u u_y u_{xx}-2\tau_x u_{tx}-2\tau_u u_x u_{tx}\\
&-\tau_{xx}u_t-2\tau_{xu}u_x u_t-\tau_{uu}u_t u_x^2-\tau_u u_{xx}u_t\}-f(u)\{\phi_{yy}\\
&+(2\phi_{yu}-\eta_{yy})u_y+(\phi_{uu}-2\eta_{uy})u_y^2+(\phi_u-2\eta_y)u_{yy}-\\
&\eta_{uu}u_y^3-3\eta_u u_y u_{yy}-2\xi_y u_{xy}-2\xi_u u_{xy}u_y-\xi_{yy}u_x-2\xi_{yu}\\
&u_x u_y-\xi_{uu}u_x u_y^2-\xi_u u_x u_{yy}-2\tau_y u_{ty}-2\tau_u u_y u_{ty}-\tau_{yy}u_t\\
&-2\tau_{uy}u_y u_t-\tau_{uu}u_t u_y^2-\tau_u u_{yy}u_t\}=0.
\end{aligned}
\tag{171}
$$

From the above equation we now compare the coefficients of like terms in derivatives of 'u' and terms without monomials. This comparison of the terms gives rise to the following system of 'nine' coupled PDEs to be solved for classification of symmetries,

$$2(\xi_x-\tau_t)f(u)-2\tau_u u_t f(u)-f_u\phi+2f(u)\xi_u u_x=0, \tag{172}$$

$$2(\eta_y - \tau_t)f(u) - 2\tau_u u_t f(u) - f_u\phi + 2f(u)\eta_u u_y = 0, \tag{173}$$

$$f(u)\eta_x + f(u)\eta_u u_x + f(u)\xi_y + f(u)\xi_u u_y = 0, \tag{174}$$

$$\xi_t - f(u)\tau_x = 0, \tag{175}$$

$$\eta_t - f(u)\tau_y = 0, \tag{176}$$

$$2\phi_{tu} - \tau_{tt} + \phi_{uu}u_t + f(u)(\tau_{xx} + \tau_{yy}) = 0, \tag{177}$$

$$\eta_{tt} + 2f(u)\phi_{uy} + \eta_{xx} + \eta_{yy} = 0, \tag{178}$$

$$\xi_{tt} + 2f(u)\phi_{ux} + \xi_{xx} + \xi_{yy} = 0, \tag{179}$$

$$\phi_{tt} - f(u)(\phi_{xx} + \phi_{yy}) = 0. \tag{180}$$

At this stage we solve the above system to find the components of the symmetry generator 'χ'. To do so we begin by first considering equation (172). Differentiating this equation with respect to 'u_t' first and then 'u_x' respectively gives,

$$\tau_u = 0 = \xi_u \ . \tag{181}$$

Substituting above expressions in (172) reduces it to,

$$2f(u)(\xi_u - \tau_u) = f_u \ \phi \ . \tag{182}$$

Now differentiating (173) with respect to 'u_y' gives $\eta_u = 0$. Differentiating (175) and (176) with respect to 'u' and substituting (181) with $\eta_u = 0$ in the resultant expressions gives,

$$\tau_x = 0 = \tau_y \ . \tag{183}$$

Substituting this result and (181) in (173)-(176) respectively, we obtain,

$$2f(u)(\eta_y - \tau_t) = f_u\phi \ , \tag{184}$$

$$\eta_x + \xi_y = 0 \ , \tag{185}$$

$$\xi_t = 0 = \eta_t \ . \tag{186}$$

As above, we first compare the coefficients of 'u_t' from (177) to get,

$$\phi = \alpha(x, y, t)u + \beta(x, y, t) \ , \tag{187}$$

and then substitute the resulting expression in (177)-(179) to respectively obtain,

$$\tau_{tt} = 2\alpha_t \ , \tag{188}$$

$$2f(u)\alpha_y + \eta_{xx} + \eta_{yy} = 0 \ , \tag{189}$$

$$2f(u)\alpha_x + \xi_{xx} + \xi_{yy} = 0 \ . \tag{190}$$

To find a complete solution of the above coupled system we start from (184) by writing it in the form,

$$\phi = 2 \ \frac{f}{f_u} \ (\eta_y - \tau_t) \ , \tag{191}$$

and considering the possible cases. This is done in the following section.

5.3.1 Classification of Symmetries

In this section we give a complete classification of the symmetries of the nonlinear wave equation (163). This requires solving the above coupled system (172)-(190) of PDEs to include all possibilities of $f(u)$. To obtain this classification we begin our procedure by first considering (191). From this equation it can be easily noticed that following *two* cases arise, namely,

$$I \qquad \frac{f}{f_u} = \widetilde{A} \quad \text{(some constant)}, \tag{192}$$

$$II \qquad \frac{f}{f_u} = g(u). \tag{193}$$

We consider these possibilities one by one.

Case I

To determine $f(u)$ in this case, we integrate $\frac{f}{f_u} = \widetilde{A}$ over 'u' to get,

$$f(u) = Ke^{Au},$$

where 'K' is a constant of integration and $A = \tilde{A}^{-1}$.

Now differentiating (192) with respect to 'u' and then inserting the resulting expression in (187) sets $\alpha = 0$. Using this value of α in (187) instantly yields $\phi = \beta$. In the light of these results, (191) simplifies to,

$$\phi = \beta = 2A(\eta_y - \tau_t). \tag{194}$$

Substituting above expressions in (188)-(190) respectively reduces to $\tau_{tt} = 0 = \eta_{xx} + \eta_{yy} = \xi_{xx} + \xi_{yy}$. These expressions with (194) give following relations for $\xi,\ \eta,\ \tau$ and ϕ,

$$\begin{aligned}
\xi &= c_0 - c_1 y + c_2 x + 2c_3 xy + c_4(x^2 - y^2)\ , \\
\eta &= c_5 + c_1 x + c_2 y + 2c_4 xy + c_3(y^2 - x^2)\ , \\
\tau &= c_6 t + c_7 \\
\phi &= 2Ac_2 - 2Ac_6 + 4Ac_4 x + 4Ac_3 y\ .
\end{aligned} \tag{195}$$

At this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators, given by,

$$\begin{aligned}
&\chi_0 = \frac{\partial}{\partial x}\ , \quad \chi_1 = -y\ \frac{\partial}{\partial x} + x\ \frac{\partial}{\partial y}\ ,\\
&\chi_2 = x\ \frac{\partial}{\partial x} + y\ \frac{\partial}{\partial y} + 2\ A\ \frac{\partial}{\partial u}\ ,\\
&\chi_3 = 2\ x\ y\ \frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial}{\partial y} + 4\ A\ y\ \frac{\partial}{\partial u}\ ,\\
&\chi_4 = (x^2 - y^2)\frac{\partial}{\partial x} + 2\ x\ y\ \frac{\partial}{\partial y} + 4\ A\ x\ \frac{\partial}{\partial u}\ ,\\
&\chi_5 = \frac{\partial}{\partial y}\ , \quad \chi_6 = t\ \frac{\partial}{\partial t} - 2\ A\ \frac{\partial}{\partial u}\ , \quad \chi_7 = \frac{\partial}{\partial t}\ .
\end{aligned} \tag{196}$$

The Lie algebra satisfied by the above generators can be constructed by solving the Lie bracket operation for each one of these generators. It turns out that they all form a closed Lie algebra which is given in the form of Table 5.

5.3.2 Reduction under infinitesimal symmetry generators

Reduction by each infinitesimal symmetry generator for this case is given in Table 12, Appendix-C.

5.3.3 Reduction under two dimensional subalgebra

As an exact solution invariant under $\{\ \chi_3,\ \chi_7\ \}$ is not given elsewhere, we present some of the details involved. Since $[\ \chi_3,\ \chi_7\ ] = 0$, form a subalgebra and the equation (163) with this choice of $f(u)$ is reducible to an ODE, the reduction may begin with either of χ_3 or χ_7. If we begin with χ_3, the reduced partial differential equation in two independent variables is obtainable from $s = \frac{x}{x^2+y^2}$, $r = t$ and

$[\chi_i,\ \chi_j]$	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
χ_0	0	χ_5	χ_0	$-2\chi_1$	$2\chi_2$	0	0	0
χ_1	$-\chi_5$	0	0	χ_4	$-\chi_3$	χ_0	0	0
χ_2	$-\chi_0$	0	0	χ_3	χ_4	$-\chi_5$	0	0
χ_3	$2\chi_1$	$-\chi_4$	$-\chi_3$	0	0	$-2\chi_2$	0	0
χ_4	$-2\chi_2$	χ_3	$-\chi_4$	0	0	$-2\chi_1$	0	0
χ_5	0	$-\chi_0$	χ_5	$2\chi_2$	$2\chi_1$	0	0	0
χ_6	0	0	0	0	0	0	0	$-\chi_7$
χ_7	0	0	0	0	0	0	χ_7	0

Table 5: Algebra of generators for case I

$u =\ 2A\ln(xw)$, where w is a function of s and r. However, obtaining the reduced PDE is a messy task. Instead, we start with χ_7. It is easy to note that this symmetry trivially leads to $s = x,\ r = y$ and $w = u$ and the reduced PDE is the Laplace equation,

$$w_{rr} + w_{ss} = 0, \tag{197}$$

which has χ_3 in the new variables as a Lie symmetry, viz.,

$$\begin{aligned} \widetilde{\chi}_3 &= \chi_3(s)\frac{\partial}{\partial s} + \chi_3(r)\frac{\partial}{\partial r} + \chi_3(w)\frac{\partial}{\partial w} \\ &= 2\ s\ r\frac{\partial}{\partial s} + (r^2 - s^2)\frac{\partial}{\partial r} + 4\ A\ r\frac{\partial}{\partial w}\ . \end{aligned} \tag{198}$$

The invariants of $\widetilde{\chi}_3$ are $\alpha = \frac{r^2+s^2}{s}$ and $\beta = w - 2A\ln s$ with β being function of α. In these variables, the Laplace equation (197) reduces to the ODE,

$$\alpha^2\beta'' - 2\ \alpha\ \beta' + 2\ A = 0, \tag{199}$$

which, after substituting back leads to the solution,

$$u = 2A\ln x + c_2 + \frac{c_1}{3}\left(\frac{x^2+y^2}{y}\right)^3 + \frac{2}{3}A\ln\left(\frac{x^2+y^2}{y}\right)\ . \tag{200}$$

We note here that since χ_7 form a two dimensional subalgebra with any of the other $\chi_i' s$, the reduction via χ_7 will lead to $\widetilde{\chi}_i$ being a symmetry of the Laplace equation.

Similarly, a solution invariant under the subalgebra $\{\ \chi_2,\ \chi_3\ \}$ can be attained but the reduction has to begin with χ_3 as $[\ \chi_2,\ \chi_3\ ] = \chi_3$. Here, from χ_3, transformed form of (163) for $A = 1$ and $A = -1$ is given as,

$$w_{rr} = Ke^{w}(-2 + s^2 w_{ss} + 2sw_s), \tag{201}$$

$$w_{rr} = Ke^{-w}(2 + s^2 w_{ss} + 2sw_s), \tag{202}$$

where $s = \frac{x^2+y^2}{x}$, $r = t$ and $u = w + 2A\ln x$, and K is a constant. Then,

$$\widetilde{\chi_2} = s\frac{\partial}{\partial s}\ , \tag{203}$$

is a symmetry of this reduced equation. The invariants $\alpha = r$ and $\beta(\alpha) = w$ leads to the ODE,

$$\beta'' = -2Ke^{\beta}, \tag{204}$$

$$\beta'' = 2Ke^{-\beta}, \tag{205}$$

corresponding to $A = 1$ and $A = -1$, respectively. The first case, e.g., leads to,

$$4Ke^{\beta} = c\ \sec h^2\left\{\frac{1}{2}\sqrt{c(\alpha+k)^2}\right\}, \tag{206}$$

where c and k are constants. Thus, we have a solution,

$$e^u = \frac{c}{4kx^2} \sec h^2 \left\{ \frac{1}{2}\sqrt{c}(t+k) \right\} \ . \tag{207}$$

The second case yields,

$$(\alpha + k)^2 + \frac{4}{k} \ln^2 \left\{ 2 \left(ce^{\frac{1}{2}\beta} + \sqrt{4k + ce^{\beta}} \right) \right\} = 0, \tag{208}$$

and the solution to (163) is obtainable from resetting $\alpha = t$ and $\beta(\alpha) = u - 2\ln x$.

Furthermore, a solution invariant under the subalgebra $\{ \chi_2, \chi_1 \}$ can be obtained by the reduction with either χ_1 or χ_2 first as $[\chi_2, \chi_1] = 0$. Here, from χ_1, we get (163) to be (for $A = 1$ and $A = -1$) ,

$$w_{rr} = 4Ke^{Aw}(sw_{ss} + w_s), \tag{209}$$

where $s = x^2 + y^2,\ r = t$ and $w(r, s) = u$ and K is a constant. Then,

$$\widetilde{\chi_2} = 2s\frac{\partial}{\partial s} + 2A\frac{\partial}{\partial w}, \tag{210}$$

is a symmetry of this reduced equation. The invariants $\alpha = r$ and $\beta(\alpha) + A\ln s = w$ leads to the ODE,

$$\beta'' = 0 \ . \tag{211}$$

Having given the reduction of the wave equation in three cases $(\{ \chi_3, \chi_7 \}, \{ \chi_2, \chi_3 \}$ and $\{ \chi_2, \chi_1 \})$, we now consider case (II), while reductions in the remaining cases through generators forming subalgebra are given in the form of Table 13, Appendix-C.

Case II

In this case we give classification of the solutions of the wave equation by considering the second possibility (193) which is equivalent to,

$$\left(\frac{f}{f_u}\right)_u \neq 0,$$

arising from (191). To classify solutions here, we consider equations (181)-(190). Using the procedure followed in first case, we can solve these equations to find that the components, $\xi,\ \eta,\ \tau$ and ϕ of infinitesimal symmetry generator χ given by (166) take the form,

$$\xi = c_0 + c_1 x - c_2 y \ ,$$

$$\eta = c_3 + c_2 x + c_1 y \ ,$$

$$\tau = \frac{4\big(f_u^2 - f f_{uu}\big)c_1 t - 2c_4 f_u^2 t}{5f_u^2 - 4f f_{uu}} + c_5 \ ,$$

$$\phi = \left\{\frac{2\big(f_u^2 - f f_{uu}\big)c_1 - c_4 f_u^2}{5f_u^2 - 4f f_{uu}} + c_4\right\}u + \beta(x,y,t) \ ,$$

where $\beta_{tt} - f(u)\Big(\beta_{xx} + \beta_{yy}\Big) = 0$.

The expression for τ includes the function $f(u)$ and its derivatives of first and second order. However, from (181) we have a constraint on 'τ' that $\tau_u = 0$. This condition on τ requires that,

$$\Big(f_u^3 f_{uu} - 2f f_u f_{uu}^2 + f f_u^2 f_{uuu}\Big)(c_1 + 2c_4) = 0. \tag{212}$$

Writing $G(u) = f_u^3 f_{uu} - 2f f_u f_{uu}^2 + f f_u^2 f_{uuu}$, we can write the above equation as,

$$G(u)(c_1 + 2c_4) = 0. \tag{213}$$

Equation (213) gives rise to further two cases which are as follows:

$$II(a): \qquad G(u) = 0 \quad \text{and} \quad c_1 + 2\,c_4 \neq 0,$$

$$II(b): \qquad G(u) \neq 0 \quad \text{and} \quad c_1 + 2\,c_4 = 0.$$

In these two cases we have different sets of expressions for $\xi,\ \eta,\ \tau$ and ϕ which we will consider one by one and find the symmetry generators in each case, with the corresponding Lie algebra and reduction under each closed algebra.

Symmetry generators for case II(a):

The condition $G(u) = 0$ requires that $f_u^3 f_{uu} - 2ff_u f_{uu}^2 + ff_u^2 f_{uuu} = 0$ or $f_u(f_u^2 f_{uu} - 2ff_{uu}^2 + ff_u f_{uuu}) = 0$, which implies that either $f_u = 0$ or $f_u^2 f_{uu} - 2ff_{uu}^2 + ff_u f_{uuu} = 0$. Here $f_u = 0$ corresponds to the case of linear wave equation of the type $u_{tt} = \lambda(u_{xx} + u_{yy})$. Since we are interested in nonlinear wave equation, we will not consider this case. On the other hand if $f_u^2 f_{uu} - 2ff_{uu}^2 + ff_u f_{uuu} = 0$, then the most general infinitesimal symmetry of the wave equation that satisfies this condition has the following expressions for $\xi,\ \eta,\ \tau$ and ϕ,

$$\xi = c_0 + c_1 x - c_2 y\ ,$$

$$\eta = c_3 + c_2 x + c_1 y\ ,$$

$$\tau = \frac{4\big(f_u^2 - ff_{uu}\big)c_1 t - 2c_4 f_u^2 t}{5f_u^2 - 4ff_{uu}} + c_5\ ,$$

$$\phi = \left\{\frac{2\big(f_u^2 - ff_{uu}\big)c_1 - c_4 f_u^2}{5f_u^2 - 4ff_{uu}} + c_4\right\}u + \beta(x,y,t)\ .$$

Therefore we have following group of seven symmetry generators including one infinite dimensional case χ_β ,

$$\chi_0 = \frac{\partial}{\partial x} \; ,$$

$$\chi_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \left\{\frac{4\,(f_u^2 - ff_{uu})}{5f_u^2 - 4ff_{uu}}\right\} t\frac{\partial}{\partial t} + \left\{\frac{2\,(f_u^2 - ff_{uu})}{5f_u^2 - 4ff_{uu}}\right\} u\frac{\partial}{\partial u} \; ,$$

$$\chi_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \; ,$$

$$\chi_3 = \frac{\partial}{\partial y} \; ,$$

$$\chi_4 = \left\{\frac{-2f_u^2}{5f_u^2 - 4ff_{uu}}\right\} t\frac{\partial}{\partial t} + \left\{\frac{4\,(f_u^2 - ff_{uu})}{5f_u^2 - 4ff_{uu}}\right\} u\frac{\partial}{\partial u} \; ,$$

$$\chi_5 = \frac{\partial}{\partial t} \; , \qquad \chi_\beta = \beta\frac{\partial}{\partial u} \; .$$

Corresponding algebra of commutators for these generators is given in Table 6.

5.3.4 Reduction under infinitesimal symmetry generators

Reduction of equation (163) by each infinitesimal symmetry generator is given in Table 14, Appendix-D.

5.3.5 Reduction under two dimensional subalgebra

There are eight subalgebras in this case, each subalgebra reduces the PDE (163) into an ODE whose solution, on back substitution gives the solution of (163). We give the complete table of reductions to an ODE for all the eight subalgebras in Table 15, Appendix-D. Reduction to an ODE in some cases is given below.

Consider the algebra given by χ_1 and χ_2. Since $[\chi_1, \chi_2] = 0$ we can start with

$[\,\chi_i,\ \chi_j\,]$	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
χ_0	0	χ_0	χ_3	0	0	0
χ_1	$-\chi_0$	0	0	$-\chi_3$	0	$-2\lambda\chi_5$
χ_2	$-\chi_3$	0	0	χ_0	0	0
χ_3	0	χ_3	$-\chi_0$	0	0	0
χ_4	0	0	0	0	0	$-(4\lambda-2)\chi_5$
χ_5	0	$2\lambda\chi_5$	0	0	$(4\lambda-2)\chi_5$	0

Table 6: Algebra of generators for case II(a)
where $\lambda = \frac{2\left(f_u^2 - f f_{uu}\right)}{5f_u^2 - 4f f_{uu}}$ should be a constant

either χ_1 or χ_2, but the reduction under χ_1 would be a cumbersome task therefore we begin with $\chi_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. The similarity variables for this generator include two independent variables $r = x^2 + y^2$ and $s = t$, and one dependent variable $w(r, s) = u$. Using these transformations, equation (163) reduces to a PDE with two independent and one dependent variable given as,

$$w_{ss} = 4f(w)(rw_{rr} + w_r)\ . \tag{214}$$

In order to get the second reduction to an ODE of second order we need to transform χ_1 in new variables $r,\ s$ and $w(r, s)$. Thus we have,

$$\widetilde{\chi_1} = 2r\frac{\partial}{\partial r} + 2\lambda s\frac{\partial}{\partial s} + \lambda w\frac{\partial}{\partial w}\ .$$

From the requirement on $\lambda = \frac{2\left(f_u^2 - f f_{uu}\right)}{5f_u^2 - 4f f_{uu}}$, being a constant, the general expression for $f(u)$ is given by $f(u) = u^{\frac{2-4\lambda}{\lambda}}$. Let $\lambda = 2$ so that $f(u) = u^{-3}$. In this case the similarity variables for $\widetilde{\chi_1}$ are $\alpha = sr^{-2}$ and $w = \beta(\alpha)r$, under these invariants (214) reduces to $\beta''(\beta^3 - 16\alpha^2) - 4\beta = 0$, an ODE of second order. Similarly For $\lambda = 4$ we have $f(u) = u^{\frac{-7}{2}}$, now the similarity variables are $\alpha = sr^{-4}$ and $w = \beta(\alpha)r^2$. Therefore (214) reduces to $\beta''(\beta^{\frac{7}{2}} - 16\alpha^2) - 4\beta = 0$ which is again an ODE of second order. Other constant values of $\lambda \neq 0$ can similarly be considered.

Similarly, a reduction can be obtained using the subalgebra $[\chi_1, \chi_3] = \chi_3$. Here we need to start with $\chi_3 = \frac{\partial}{\partial y}$, that reduces equation (163) to a PDE,

$$w_{rr} = f(w)w_{ss} \ ,$$

with two independent variables $s = x$ and $r = t$, and one dependent variable $w(r,s) = u$. Substituting the expression $f(u) = u^{\frac{2-4\lambda}{\lambda}}$ for $f(u)$ in this equation, it can be written as,

$$w_{rr} = w^{\frac{2-4\lambda}{\lambda}} w_{ss} \ . \tag{215}$$

Now transforming χ_1 in these new variables for the second reduction we get,

$$\widetilde{\chi_1} = s\frac{\partial}{\partial s} + 2\lambda r\frac{\partial}{\partial r} + \lambda w\frac{\partial}{\partial w} \ .$$

The invariants of $\widetilde{\chi_1}$ are $\alpha = \frac{r}{s^{2\lambda}}$ and $w = \sqrt{r}\beta(\alpha)$. In these variables, equation (215) reduces to a second order ODE given by,

$$-\frac{1}{4}\beta + \frac{1}{2}\alpha\beta' + \alpha^2\beta'' = \alpha^{\frac{1}{\lambda}}\beta^{\frac{2-4\lambda}{\lambda}}\{(2\lambda)(2\lambda+1)\alpha\beta' + 4\lambda^2\alpha^2\beta''\}.$$

Symmetry generators for case II(b):

In this case the components $\xi,\ \eta,\ \tau$ and ϕ of symmetry generator (166) can easily be found as,

$$\xi = c_0 - 2c_4 x - c_2 y \ ,$$

$$\eta = c_3 + c_2 x - 2c_4 y \ ,$$

$$\tau = -2c_4 t + c_5 \ ,$$

$$\phi = \beta(x, y, t) \ .$$

Corresponding to each constant c_i we have the following six symmetry generators, where χ_β is an infinite dimensional subalgebra,

$$\chi_0 = \frac{\partial}{\partial x} \ , \ \chi_1 = -y \ \frac{\partial}{\partial x} + x \ \frac{\partial}{\partial y}, \ \chi_2 = \frac{\partial}{\partial y} \ ,$$

$$\chi_3 = -2 \ x \ \frac{\partial}{\partial x} - 2 \ y \ \frac{\partial}{\partial y} - 2t\frac{\partial}{\partial t} \ ,$$

$$\chi_4 = \frac{\partial}{\partial t} \ , \ \chi_\beta = \beta\frac{\partial}{\partial u} \ .$$

Commutation relations for these generators are given in Table 7. From this table we find that there are eight two dimensional subalgebras.

5.3.6 Reduction under infinitesimal symmetry generators

Reduction of equation (163) by each infinitesimal symmetry generator is given in Table 16, Appendix-E.

5.3.7 Reduction under two dimensional subalgebra

Considering the two dimensional algebra $[\chi_0, \chi_2] = 0$ as an example, we start with χ_0, which reduces (163) to a PDE involving less independent variables given by,

$$w_{rr} = f(w) w_{ss} \ , \tag{216}$$

$[\chi_i, \chi_j]$	χ_0	χ_1	χ_2	χ_3	χ_4
χ_0	0	χ_2	0	$-2\chi_0$	0
χ_1	$-\chi_2$	0	χ_0	$-4\chi_1$	0
χ_2	0	$-\chi_0$	0	$-2\chi_2$	0
χ_3	$2\chi_0$	$4\chi_1$	$2\chi_2$	0	$2\chi_4$
χ_4	0	0	0	$-2\chi_4$	0

Table 7: Algebra of generators for case II(b)

where $s = y,\ r = t$ and $w(r, s) = u$ are the invariants of χ_0, as before the transformation of generator χ_2 in new variables is,

$$\widetilde{\chi}_2 = \frac{\partial}{\partial s} + 0\frac{\partial}{\partial r} + 0\frac{\partial}{\partial w}\ ,$$

which has similarity variables given by $\alpha = r$ and $\beta(\alpha) = w$. Using these new variables, (216) reduces to a second order ODE,

$$\beta'' = 0\ .$$

Furthermore the reduction under the algebra $[\chi_3, \chi_1] = 4\chi_1$ can be obtained by starting with $\chi_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ for first reduction to a PDE and then by $\widetilde{\chi}_3$ for a second reduction to an ODE. Therefore equation (163) reduces to $4\alpha\beta'' + 2\beta' = 4f(\beta)\Big(\beta''\alpha^2 + \beta'\alpha\Big)$ where $\alpha = \frac{s^2}{r},\ \beta(\alpha) = w$ and $s = t,\ r = x^2 + y^2,\ w(r, s) = u$ are the similarity variables for χ_1 and $\widetilde{\chi}_3$ respectively. Reduction for the remaining subalgebras in this case are given in Appendix-E, Table 17.

A.1 Appendix-A

Reduction table for $u_{tt} = u(u_{xx} + u_{yy})$

Table 8: First order reduction

Generator	**Reduction & Similarity Variables**
$\chi_0 = \frac{\partial}{\partial x}$	$w_{ss} = w w_{rr}$ where $r = y,\ s = t,\ w = u$
$\chi_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{4}{5}\ t\ \frac{\partial}{\partial t} + \frac{2}{5}u\frac{\partial}{\partial u}$	$-4\ w\ s^2 + 25\ w_s\ s^3 + 25\ w_{ss}\ s^4 = 16\ w\ (s^2\ w_{ss} + 2\ w_{sr}\ r\ s + w_{rr}\ r^2 + 2\ w_r\ r^3 + w_{rr}\ r^4)$ where $r = x/y,\ s = x\ t^{-\frac{5}{4}},\ w = u\ t^{-\frac{1}{2}}$
$\chi_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$	$w_{rr} = 4\ w\ (w_r + r\ w_{rr})$ where $r = x^2 + y^2,\ s = t,\ w = u$
$\chi_3 = \frac{\partial}{\partial y}$	$w_{ss} = w\ w_{rr}$ where $r = x,\ s = t,\ w = u$

Generator	Reduction & Similarity Variables
$\chi_4 \; = -\frac{2}{5}\; t\; \frac{\partial}{\partial t} + -\frac{4}{5}\; u\; \frac{\partial}{\partial u}$	$w_{rr} \; + \; w_{ss} \; = \; 6$ where $r \; = \; x, \; s \; = \; y, \; w \; = \; u\; t^2$
$\chi_5 \; = \frac{\partial}{\partial t}$	$w_{rr} \; + \; w_{ss} = 0$ where $r \; = \; x, \; s \; = \; y, \; w \; = \; u$

Table 9: Reductions under two dimensional subalgebra

Algebra	Reduction
$[\; \chi_0 \; , \; \chi_1 \;] \; = \; \chi_0$	$-\frac{1}{4}\; \beta \; + \; \frac{25}{16}\; \alpha\; \beta_\alpha \; + \; \frac{25}{16}\; \alpha^2\; \beta_{\alpha\alpha} \; = \; \beta\; \beta_{\alpha\alpha}$
$[\; \chi_0 \; , \; \chi_3 \;] \; = 0$	$\beta_{\alpha\alpha} = 0$
$[\; \chi_0 \; , \; \chi_4 \;] \; = 0$	$6\; \beta \; = \; \beta\; \beta_{\alpha\alpha}$
$[\; \chi_0 \; , \; \chi_5 \;] \; = \; \chi_0$	$\beta_{\alpha\alpha} = 0$
$[\; \chi_1 \; , \; \chi_2 \;] \; = 0$	$-\frac{1}{4}\; \beta \; + \; \frac{25}{4}\; \alpha\; \beta_\alpha \; + \; \frac{25}{4}\; \alpha^2\; \beta_{\alpha\alpha} \; = \; 4\; \beta\; (\beta_\alpha \; + \; \alpha\; \beta_{\alpha\alpha})$

Algebra	Reduction
$[\, \chi_3 \,,\, \chi_1 \,] \;=\; \chi_3$	$-\frac{1}{4}\, \beta \;+\; \frac{25}{16}\, \alpha\, \beta_\alpha \;+\; \frac{25}{16}\, \alpha^2\, \beta_{\alpha\alpha} \;=\; \beta\, \beta_{\alpha\alpha}$
$[\, \chi_1 \,,\, \chi_4 \,] \;=\; 0$	$\beta_{\alpha\alpha} \;+\; \frac{14}{25}\, \beta \;+\; \frac{10}{5}\, \alpha\, \beta_\alpha \;+\; \alpha^2\, \beta_{\alpha\alpha} \;=\; 6$
$[\, \chi_5 \,,\, \chi_1 \,] \;=\; \frac{4}{5}\, \chi_5$	$\beta_{\alpha\alpha} \;-\; \frac{6}{25}\, \beta \;+\; \frac{10}{5}\, \alpha\, \beta_\alpha \;+\; \alpha^2\, \beta_{\alpha\alpha} = 0$
$[\, \chi_2 \,,\, \chi_4 \,] \;=\; 0$	$\beta_\alpha \;+\; \alpha\, \beta_{\alpha\alpha} \;=\; \frac{3}{2}$
$[\, \chi_2 \,,\, \chi_5 \,] \;=\; 0$	$\beta_\alpha \;+\; \alpha\, \beta_{\alpha\alpha} = 0$
$[\, \chi_3 \,,\, \chi_4 \,] \;=\; 0$	$\beta_{\alpha\alpha} \;=\; 6$
$[\, \chi_3 \,,\, \chi_5 \,] \;=\; 0$	$\beta\, \beta_{\alpha\alpha} \;= 0$

A.2 Appendix-B

Table 10: Reduction under symmetry generators

Generator	Reduction and similarity variables
$\chi_0 = \frac{\partial}{\partial x}$	$w_{rr} = w^n w_{ss}$

Generator	Reduction and similarity variables
	where $s=y,\ r=t$ and $w(r,s)=u$
$\chi_1=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$	$w_{rr}=4w^n(w_s+sw_{ss})$ where $s=x^2+y^2,\ r=t$ and $w(r,s)=u$
$\chi_2=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+$ $\left(\frac{4t}{n+4}\right)\frac{\partial}{\partial t}+\left(\frac{2u}{n+4}\right)\frac{\partial}{\partial u}$	$-\frac{1}{4}+\left(\frac{n}{4}+1\right)^2r^2w_r^2+\left(\frac{n}{4}+1\right)w_r+\left(\frac{n}{4}+1\right)^2r^2w_{rr}$ $=r^2e^{nw}\left\{r^2w_r^2+(s^2+1)w_s^2+2rsw_sw_r+\right.$ $\left.r^2w_{rr}+2srw_{sr}+2rw_r+(s^2+1)w_{ss}+2sw_s\right.$ where $s=\frac{y}{x},\ r=\frac{t^{\frac{n}{4}+1}}{x}$ and $\sqrt{t}w^{w(r,s)}=u$
$\chi_3=\frac{\partial}{\partial y}$	$w_{rr}=w^nw_{ss}$ where $s=x,\ r=t$ and $w(r,s)=u$
$\chi_4=-\left(\frac{2nt}{n+4}\right)\frac{\partial}{\partial t}+\left(\frac{4u}{n+4}\right)\frac{\partial}{\partial u}$	$\frac{2}{n}\left(\frac{2}{n}+1\right)=e^{nw}(w_{ss}+w_{rr})$ where $s=x,\ r=y$ and $u=t^{-\frac{2}{n}}e^{w(r,s)}$
$\chi_5=\frac{\partial}{\partial t}$	$w_{ss}+w_{rr}=0$ where $s=x,\ r=y$ and $w(r,s)=u$

Table 11: Reductions under two dimensional algebra

Algebra	Reductions and Similarity variables
$[\chi_0, \chi_3] = 0$	$\beta'' = 0$ where $\alpha = r\ , \beta(\alpha) = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[\chi_0, \chi_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = e^{n\beta}(\beta'^2 + \beta'')$ where $\alpha = s,\ e^{\beta(\alpha)} r^{-\frac{2}{n}} = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[\chi_0, \chi_5] = 0$	$\beta'' = 0$ where $\alpha = s,\ \beta(\alpha) = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[\chi_1, \chi_2] = 0$	$-\frac{1}{2} + \left(\frac{n}{2}+2\right)^2(\alpha\beta' + \alpha^2\beta' + \alpha^2\beta'') = 4\alpha e^{n\beta}(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{2}+2}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = x^2 + y^2,\ r = t,\ w(r,s) = u$
$[\chi_1, \chi_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = 4e^{n\beta}\{\beta' + \alpha(\beta'^2 + \beta'')\}$ where $\alpha = s,\ e^{\beta(\alpha)} r^{-\frac{2}{n}} = w$ and $s = x^2 + y^2,\ r = t,\ w(r,s) = u$

Algebra	Reductions and Similarity variables
$[\chi_1, \chi_5] = 0$	$\beta' + \alpha\beta'' = 0$ where $\alpha = s^2 + r^2,\ \beta(\alpha) = w$ and $s = x,\ r = y,\ w(r,s) = u$
$[\chi_2, \chi_4] = 0$	$\frac{2}{n}\left(\frac{2}{n} + 1\right) = e^{n\beta}\left(\beta'' - \frac{2}{n} + \alpha^2\beta'' + 2\alpha\beta'\right)$ where $\alpha = \frac{s}{r},\ \frac{2}{n}\ln r + \beta(\alpha) = w$ and $s = x,\ r = y,\ w(r,s) = \ln\left(ut^{\frac{2}{n}}\right)$
$[\chi_3, \chi_4] = 0$	$\frac{2}{n}\left(\frac{2}{n} + 1\right) = e^{n\beta}(\beta'' + \beta'^2)$ where $\alpha = s,\ e^{\beta(\alpha)}r^{-\frac{2}{n}} = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[\chi_3, \chi_5] = 0$	$\beta'' = 0$ where $\alpha = s,\ \beta(\alpha) = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[\chi_0, \chi_2] = \chi_0$	$-\frac{1}{4} + \left(\frac{n}{4} + 1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) = \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{4}+1}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = y,\ r = t,\ w(r,s) = u$

Algebra	Reductions and Similarity variables
$[\chi_3, \chi_2] = \chi_3$	$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) = \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{4}+1}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[\chi_5, \chi_2] = c_1\chi_5$	$\frac{2(n+2)}{(n+4)^2} = \left(\frac{2n+12}{n+4}\right)\alpha\beta' + \beta'^2 + \beta'' + \alpha^2\beta'^2 + \alpha^2\beta''$ where $\alpha = \frac{r}{s},\ s^{\frac{2}{n+4}}e^{\beta(\alpha)} = w$ and $s = x,\ r = y,\ w(r,s) = u$

A.3 Appendix-C

Table 12: Reduction under generators for Case I

Generator	Reduction and similarity variables
$\chi_0 = \frac{\partial}{\partial x}$	$w_{rr} = K\ e^{Aw}\ w_{ss}$ where $s = y,\ r = t$ and $w(r,s) = u$
$\chi_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$w_{rr} = 4\ K\ e^{Aw}\ (w_s + s w_{ss})$

Generator	Reduction and similarity variables
	where $s = x^2 + y^2,\ r = t$ and $w(r,s) = u$
$\chi_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $+2A\frac{\partial}{\partial u}$	$w_{rr} = Ke^{w}\left\{-2 + \left(s^2+1\right)w_{ss} + 2sw_s\right\}$ for $A = 1$ $w_{rr} = Ke^{-w}\left\{2 + \left(s^2+1\right)w_{ss} + 2sw_s\right\}$ for $A = -1$ where $s = \frac{y}{x},\ r = t$ and $w(r,s) = u - 2A\ln x$
$\chi_3 = 2xy\frac{\partial}{\partial x} + \left(y^2 - x^2\right)\frac{\partial}{\partial y}$ $4Ay\frac{\partial}{\partial u}$	$w_{rr} = Ke^{w}\left\{-2 + s^2w_{ss} + 2sw_s\right\}$ for $A = 1$ $w_{rr} = Ke^{-w}\left\{2 + s^2w_{ss} + 2sw_s\right\}$ for $A = -1$ where $s = \frac{x^2+y^2}{x},\ r = t$ and $w(r,s) = u - 2A\ln x$
$\chi_4 = \left(x^2 - y^2\right)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$ $4Ax\frac{\partial}{\partial u}$	$w_{rr} = Ke^{w}\left\{-2 + s^2w_{ss} + 2sw_s\right\}$ for $A = 1$ $w_{rr} = Ke^{-w}\left\{2 + s^2w_{ss} + 2sw_s\right\}$ for $A = -1$ where $s = \frac{x^2+y^2}{y},\ r = t$ and $w(r,s) = u - 2A\ln y$
$\chi_5 = \frac{\partial}{\partial y}$	$w_{rr} = ke^{Aw}w_{ss}$ where $s = x,\ r = t$ and $w(r,s) = u$
$\chi_6 = t\frac{\partial}{\partial t} - 2A\frac{\partial}{\partial u}$	$Ke^{w}\left(w_{ss} + w_{rr}\right) = 2$ for $A = 1$

Generator	Reduction and similarity variables
	$Ke^{-w}(w_{ss}+w_{rr})=-2 \quad \text{for} \quad A=-1$ where $s=x,\ r=y$ and $w(r,s)=u+2A\ln t$
$\chi_7=\frac{\partial}{\partial t}$	$Ke^{Aw}(w_{rr}+w_{ss})=0$ where $s=x,\ r=y$ and $w(r,s)=u$

Table 13: Reductions for case I

Algebra	Reduction
$[\chi_0,\chi_5]=0$	$\beta''=0$
$[\chi_0,\chi_3]=0$	$\beta''=0$
$[\chi_0,\chi_6]=0$	$\beta''Ke^{\beta}=2 \quad \text{for} \quad A=1$ $\beta''Ke^{-\beta}=-2 \quad \text{for} \quad A=-1$
$[\chi_0,\chi_7]=0$	$\beta''=0$

Algebra	Reduction
$[\chi_1, \chi_2] = 0$	$\beta'' = 0$
$[\chi_1, \chi_6] = 0$	$2\beta'' K e^{\beta} = 1 \quad \text{for} \quad A = 1$
	$2\beta'' K e^{-\beta} = -1 \quad \text{for} \quad A = -1$
$[\chi_1, \chi_7] = 0$	$\alpha\beta'' + \beta' = 0$
$[\chi_2, \chi_6] = 0$	$K e^{\beta} \alpha^2 \Big\{\beta''(\alpha^2+1) + 2\beta'\alpha - \frac{2}{\alpha^2}\Big\} = 2 \quad \text{for} \quad A = 1$
	$K e^{-\beta} \alpha^2 \Big\{\beta''(\alpha^2+1) + 2\beta'\alpha + \frac{2}{\alpha^2}\Big\} = -2 \quad \text{for} \quad A = -1$
$[\chi_2, \chi_7] = 0$	$\beta''(\alpha^2+1) + 2\beta'\alpha - \frac{2A}{\alpha^2} = 0$
$[\chi_3, \chi_4] = 0$	$\beta'' = 0$
$[\chi_3, \chi_6] = 0$	$K e^{\beta}(-2 + \alpha^2\beta'' + 2\alpha\beta') = 2 \quad \text{for} \quad A = 1$
	$K e^{-\beta}(2 + \alpha^2\beta'' + 2\alpha\beta') = -2 \quad \text{for} \quad A = -1$

Algebra	**Reduction**
$[\chi_3, \chi_7] = 0$	$\alpha^2\beta'' + 2\alpha\beta' - 2A = 0$
$[\chi_4, \chi_6] = 0$	$Ke^{\beta}(-2 + \alpha^2\beta'' + 2\alpha\beta') = 2 \quad \text{for} \quad A = 1$
	$Ke^{-\beta}(2 + \alpha^2\beta'' + 2\alpha\beta') = -2 \quad \text{for} \quad A = -1$
$[\chi_4, \chi_7] = 0$	$\alpha^2\beta'' + 2\alpha\beta' - 2A = 0$
$[\chi_5, \chi_6] = 0$	$\beta'' Ke^{\beta} = 2 \quad \text{for} \quad A = 1$
	$\beta'' Ke^{-\beta} = -2 \quad \text{for} \quad A = -1$
$[\chi_5, \chi_7] = 0$	$\beta'' = 0$
$[\chi_0, \chi_2] = \chi_0$	$\beta'' = 2Ke^{\beta} \quad \text{for} \quad A = 1$
	$\beta'' = -2Ke^{-\beta} \quad \text{for} \quad A = -1$

Algebra	Reduction
$[\chi_2, \chi_3] = \chi_3$	$\beta'' = -2Ke^{\beta} \quad \text{for} \quad A = 1$
	$\beta'' = 2Ke^{-\beta} \quad \text{for} \quad A = -1$
$[\chi_2, \chi_4] = \chi_4$	$\beta'' = -2Ke^{\beta} \quad \text{for} \quad A = 1$
	$\beta'' = 2Ke^{-\beta} \quad \text{for} \quad A = -1$
$[\chi_2, \chi_5] = \chi_5$	$\beta'' = -2Ke^{\beta} \quad \text{for} \quad A = 1$
	$\beta'' = 2Ke^{-\beta} \quad \text{for} \quad A = -1$

A.4 Appendix-D

Table 14: Reduction under generators for Case II(a)

Generator	Reduction and similarity variables
$\chi_0 = \frac{\partial}{\partial x}$	$w_{rr} = f(w)\ w_{ss}$

Generator	Reduction and similarity variables
	where $s = y,\ r = t$ and $w(r,s) = u$
$\chi_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} +$ $2\lambda t\frac{\partial}{\partial t} + \lambda u\frac{\partial}{\partial u}$	$-\frac{1}{4}w + rw_r + r^2 w_{rr} = w^{\frac{2-4\lambda}{\lambda}} r^{\frac{1}{\lambda}} \{w_{ss}(1+s^2) +$ $4\lambda s r w_{sr} + 2sw_s + 4\lambda^2 r^2 w_{rr} + 2\lambda(2\lambda+1) r w_r\}$ where $s = \frac{y}{x},\ r = tx^{-2\lambda}$ and $w(r,s) = ut^{-\frac{1}{2}}$
$\chi_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$w_{rr} = 4f(w)(w_s + sw_{ss})$ where $s = x^2 + y^2,\ r = t$ and $w(r,s) = u$
$\chi_3 = \frac{\partial}{\partial y}$	$w_{rr} = f(w)w_{ss}$ where $s = x,\ r = t$ and $w(r,s) = u$
$\chi_4 = (4\lambda - 2)\, t\frac{\partial}{\partial t} + 2\lambda u\frac{\partial}{\partial u}$	$\frac{\lambda(1-\lambda)}{(2\lambda-1)^2} = w^{\frac{2-4\lambda}{\lambda}}$ where $r = x,\ s = y$ and $w(r,s) = ut^{\frac{\lambda}{1-2\lambda}}$
$\chi_5 = \frac{\partial}{\partial t}$	$w_{rr} + w_{ss} = 0$ where $r = x,\ y = s$ and $w(r,s) = u$

Table 15: Reductions for case II(a)

Algebra	**Reduction**
$[\chi_0, \chi_3] = 0$	$\beta'' = 0$
$[\chi_0, \chi_4] = 0$	$\frac{\lambda(1-\lambda)}{(2\lambda-1)^2} = \beta^{\frac{2-5\lambda}{\lambda}}\beta''$
$[\chi_0, \chi_5] = 0$	$\beta'' = 0$
$[\chi_1, \chi_2] = 0$	$-\frac{1}{4}\beta + \frac{1}{2}\alpha\beta' + \alpha^2\beta'' = 4\lambda^2\beta^{\frac{2-4\lambda}{\lambda}}\alpha^{\frac{1}{\lambda}}(\alpha\beta' + \alpha^2\beta'')$
$[\chi_1, \chi_4] = 0$	$\frac{\lambda(1-\lambda)}{(2\lambda-1)^2} = \beta^{\frac{2-5\lambda}{\lambda}}\left\{(1+\alpha^2)\beta'' + 2\left(\frac{3\lambda-1}{2\lambda-1}\right)\alpha\beta' + \frac{\lambda(\lambda-1)}{(2\lambda-1)^2}\beta\right\}$
$[\chi_2, \chi_4] = 0$	$\frac{\lambda(1-\lambda)}{(2\lambda-1)^2} = 4\beta^{\frac{2-5\lambda}{\lambda}}(\beta' + \beta'')$
$[\chi_2, \chi_5] = 0$	$\beta' + \alpha\beta'' = 0$
$[\chi_3, \chi_4] = 0$	$\frac{\lambda(1-\lambda)}{(2\lambda-1)^2} = \beta^{\frac{2-5\lambda}{\lambda}}\beta''$
$[\chi_3, \chi_5] = 0$	$\beta'' = 0$

Algebra	Reduction
$[\chi_0, \chi_1] = \chi_0$	$-\frac{1}{4}\beta + \alpha\beta' + \alpha^2\beta'' = 2\lambda\beta^{\frac{2-4\lambda}{\lambda}}\alpha^{\frac{1}{\lambda}}\left\{(2\lambda+1)\alpha\beta' + 2\lambda\alpha^2\beta''\right\}$
$[\chi_3, \chi_1] = \chi_3$	$-\frac{1}{4}\beta + \frac{1}{2}\alpha\beta' + \alpha^2\beta'' = \alpha^{\frac{1}{\lambda}}\beta^{\frac{2-4\lambda}{\lambda}}\{(2\lambda)(2\lambda+1)\alpha\beta' + 4\lambda^2\alpha^2\beta''\}$
$[\chi_5, \chi_1] = 2\lambda\chi_5$	$\lambda(\lambda-1) + 2\alpha\beta'(1-\lambda) + \beta'^2(\alpha^2+1) + \beta''(\alpha^2+1) = 0$

A.5 Appendix-E

Table 16: Reduction under generators for Case II(b)

Generator	Reduction and similarity variables
$\chi_0 = \frac{\partial}{\partial x}$	$w_{rr} = f(w)\ w_{ss}$ where $s = y,\ r = t$ and $w(r,s) = u$
$\chi_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$w_{ss} = 4f(w)(w_r + rw_{rr})$ where $r = x^2 + y^2,\ s = t$ and $w(r,s) = u$

Generator	Reduction and similarity variables
$\chi_2 = \frac{\partial}{\partial y}$	$w_{ss} = f(w) w_{rr}$ where $r = x,\ s = t$ and $w(r,s) = u$
$\chi_3 = -2x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$ $-2t\frac{\partial}{\partial t}$	$2s^3 w_s + s^4 w_{ss} = f(w)\left\{2r^3 w_r + r^2(r^2+1) w_{rr} + s^2 w_{ss}\right\}$ where $r = \frac{y}{x},\ s = \frac{y}{t}$ and $w(r,s) = u$
$\chi_4 = \frac{\partial}{\partial t}$	$w_{rr} + w_{ss} = 0$ where $r = x,\ s = y$ and $w(r,s) = u$

Table 17: Reductions for case II(b)

Algebra	Reduction
$[\chi_3, \chi_5] = 0$	$\beta'' = 0$
$[\chi_0, \chi_4] = 0$	$\beta'' = 0$
$[\chi_1, \chi_4] = 0$	$\alpha\beta'' + \beta' = 0$

Algebra	Reduction
$[\chi_2, \chi_4] = 0$	$\beta'' = 0$
$[\chi_3, \chi_0] = 2\chi_0$	$\alpha^2\beta'' + 2\alpha\beta' = 0$
$[\chi_3, \chi_1] = 4\chi_1$	$4\alpha\beta'' + 2\beta' = 4f(\beta)(\alpha^2\beta'' + \alpha\beta')$
$[\chi_3, \chi_2] = 2\chi_2$	$\alpha^2\beta'' + 2\alpha\beta' = f(\beta)\beta''$
$[\chi_3, \chi_4] = 2\chi_4$	$\beta''(\alpha^2 + 1) + 2\alpha\beta' = 0$

REFERENCES

[1] W. F. Ames, *Nonlinear partial differential equations in engineering*, Academic Press **2** (1972), 87–142.

[2] W. F. Ames, E. Adams, and R. J. Lohner, *Group properties of* $u_{tt} = [f(u)u_x]_x$, Int. J. of Nonlinear Mech. **16** (1981), 439–447.

[3] S. C. Anco and G. Bluman, *Nonlocal symmetries and nonlocal conservation laws of maxwell's equations*, J. Math. Phys. **38** (1997), no. 7, 3508–3532.

[4] S. C. Anco and Dennis The, *Symmetries, conservation laws, and cohomology of maxwells equations using potentials*, arXiv:math-ph/0501052 (2005), 1–56.

[5] D. J. Arrigo and J. R. Beckham, *Nonclassical symmetries of evolutionary partial differential equations and compatibility*, Journal of Mathematical Analysis and Applications **289** (2004), 55–65.

[6] D. J. Arrigo and F. Hickling, *On the determining equations for the nonclassical reductions of the heat and burgers' equation*, Journal of Mathematical Analysis and Applications **270** (2002), 582–589.

[7] V. Baikov, R. Gazizov, and N. Ibragimov, *Crc handbook of lie group analysis of differential equations*, CRC, Boca Raton.

[8] A. Barannyk and I. Yuryk, *On some exact solutions of nonlinear wave equations*, Symmetry in Nonlinear Mathematical Physics **1** (1997), 98–107.

[9] A. F. Barannyk and Y. D. Moskalenko, *Conditional symmetry and exact solutions of the multidimensional nonlinear dalembert equation*, Nonlinear Mathematical Physics **3** (1996), 336–340.

[10] A. Barone, C. Esposito, and A. C. Scott, *Theory and application of sine-gordon equation*, Rivista Nuovo Cimento **1** (1971), no. 2, 227–267.

[11] G. Bluman and S. Kumei, *Symmetries and differential equations*, Springer,New York, 1989.

[12] G. W. Bluman and J. D. Cole, *The general similarity solution of the heat equation*, J. Math. Mech. **18** (1969), 1025–1042.

[13] G. I. Burde, *Potential symmetries of the nonlinear wave equation $u_{tt} = (uu_x)_x$ and related exact and approximate solutions*, Journal of Physics A: Mathematical and General **34** (2001), 5355–5371.

[14] R. Cherniha and M. Serov, *Lie and non-lie symmetries of nonlinear diffusion equations with convection term*, Symmetry in Nonlinear Mathematical Physics **2** (1997), 444–449.

[15] A. H. Davison and A. H. Kara, *Potential symmetries and associated conservation laws–a generalization*, Nonlinear Dynamics **36** (2003), no. 4, 369–377.

[16] P. G. Drazin, *Solitons*, Cambridge University Press.

[17] P. G. Estevez and S. Leble, *A wave equation in 2+1: Painleve analysis and solutions*, Inverse Problems **11** (1995), 925–937.

[18] M. V. Foursov and E. M. Vorob'ev, *Solutions of the nonlinear wave equation* $u_{tt} = \left(uu_x\right)_x$ *invariant under conditional symmetries*, J. Phy. A: Math Gen **29** (1996), 6363–6373.

[19] W. I. Fushchych and V. M. Boyko, *Symmetry classification of the one-dimensional second order equation of hydrodynamical type*, W. I. Fushchych, Scientific Works **5** (2003), 371–380.

[20] L. Gagnon and P. Winternitz, *Lie symmetries of a generalised non-linear schrodinger equation: I. the symmetry group and its subgroups*, J . Phys. A: Math. Gen. **21** (1988), 1493–1511.

[21] M. L. Gandarias, M. Torrisi, and A. Valenti, *Symmetry classification and optimal systems of a non-linear wave equation*, International Journal of Non-Linear Mechanics **39** (2004), 389–398.

[22] C. Geronimi, P. G. L. Leach, and M. R. Feix, *First integrals and parametric solutions for equations integrable through lie symmetries*, Journal of Nonlinear Mathematical Physics **8** (2001), no. 2, 211–216.

[23] P. B. Horwath and V. Lahno, *Group classiffication of nonlinear partial differential equations: a new approach to resolving the problem*, Proceedings of Institute of Mathematics of NAS of Ukraine **43** (2002), no. 1, 86–92.

[24] N. H. Ibragimov, *Introduction to modern group analysis*, 2000.

[25] N. H. Ibragimov, M. Torrisi, and A. Valenti, *Preliminary group classification of equations* $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, J. Math Phys. **32** (1991), 2988–2995.

[26] F. S. Ibrahim, M. A. Mansour, and M. A. A. Hamad, *Lie-group analysis of radiative and magnetic field effects on free convection and mass transfer flow past a semi-infinite vertical flat plate*, Electronic Journal of Differential Equations **2005** (2005), no. 39, 1–17.

[27] Z. Jiang, *Lie symmetries and preliminary classification of* $u_n^{(k)}(t) = f_n(t, u_{n+a}, ..., u_{n+b})$, Symmetry in Nonlinear Mathematical Physics **1** (1997), 89–97.

[28] A. H. Kara, C. Qu, and F. M. Mahomed, *Nonlocal symmetries and associated conservation laws for wave equations with variable speed*, Int Journal of Theoretical Physics **39** (2000), no. 10, 2503–2512.

[29] A. Karasu and H. Yildrim, *On the lie symmetries of keplerermakov systems*, Journal of Nonlinear Mathematical Physics **9** (2002), no. 4, 475–482.

[30] S. Kumei, *Invariance transformation, invariance group transformations and invariance groups of sine-gordon equations*, J. Math. Phys. **16** (1975), 2461–2472.

[31] S. Lafortune, S. Tremblay, and P. Winternitz, *Symmetry classification of diatomic molecular chains*, Journal of Mathematical Physics **42** (2001), 5341–5357.

[32] V. Lahno and O. Magda, *The group classification of one class of nonlinear wave equations*, arXiv:math-ph/0310049 **2** (2003), 1–8.

[33] J. D. Logan, *Applied mathematics*, John Wiley & Sons, INC., 1997.

[34] O. Magda, *The group classification of nonlinear wave equations invariant under two-dimensional lie algebras*, Proceedings of Institute of Mathematics of NAS of Ukraine **30** (2000), 165–169.

[35] ______, *Invariance of quasilinear equations of hyperbolic type with respect to three-dimensional lie algebras*, Proceedings of Institute of Mathematics of NAS of Ukraine **43** (2002), 167–170.

[36] G. H. Maluleke and D. P. Mason, *Optimal system and group invariant solutions for a nonlinear wave equation*, Communications in Nonlinear Science and Numerical Simulation **9** (2004), 93–104.

[37] V. Marchenko, *On symmetry reduction of nonlinear generalization of the heat equation*, Nonlinear Mathematical Physics **3** (1996), 432–434.

[38] P. Nattermann and H. D. Doebner, *Gauge classification, lie symmetries and integrability of a family of nonlinear schrodinger equations*, Nonlinear Mathematical Physics **3** (1996), 302–310.

[39] P. J. Olver, *Applications of lie groups to differential equations*, Springer-Verlag, New York.

[40] M. A. Pinsky, *Introduction to partial differential equations with applications*, McGraw-Hill Book Company, 1984.

[41] R. O. Popovych and N. M. Ivanova, *New results on group classification of nonlinear diffusion-convection equations*, J. Phys. A: Math. Gen. **37** (2004), 7547–7565.

[42] V. Repeta, *Conditional and lie symmetry of nonlinear wave equation*, Nonlinear Mathematical Physics **3** (1996), 414–416.

[43] E. A. Saied, R. G. Abd El-Rahman, and M. I. Ghonamy, *On the exact solution of (2+1)-dimensional cubic nonlinear schrodinger (nls) equation*, J. Phys. A: Math. Gen. **36** (2003), 6751–6770.

[44] F. Schwarz, *Canonical form transformation of second order differential equations with lie symmetries*, Journal of Lie Theory **10** (2000), 411–433.

[45] C. Sophocleous, *On cyclic symmetries of n-dimensional nonlinear wave equations*, J. Phys. A: Math. Gen. **33** (2000), 8319–8330.

[46] H. Stephani, *Differential equations their solution using symmetries*, Cambridge University Press, 1989.

[47] W. A. Strauss, *Partial differential equations an introduction*, John Wiley & Sons, INC., 1992.

[48] I. Tsyfra, *Conditional symmetry reduction and invariant solutions of nonlinear wave equations*, Proceedings of Institute of Mathematics of NAS of Ukraine **43** (2002), no. 1, 229–233.

[49] ______, *Symmetry reduction of nonlinear differential equations*, Proceedings of Institute of Mathematics of NAS of Ukraine **50** (2004), no. 1, 266–270.

[50] V. A. Tychynin, *Nonlocal symmetry of nonlinear wave equations*, Nonlinear Mathematical Physics **3** (1996), 409–413.

[51] M. S. Velan and M. Lakshmanan, *Invariance analysis of the (2+1) dimensional long dispersive wave equation*, Nonlinear Mathematical Physics **4** (1997), no. 34, 251–260.

[52] E. M. Vorob'ev, *Symmetry analysis of nonlinear pde with a mathematica program symman*, Nonlinear Mathematical Physics **3** (1996), 85–89.

[53] R. Z. Zhdanov and V. I. Lahno, *Group classification of heat conductivity equations with a nonlinear source*, J. Phy. A: Math Gen **32** (1999), 7405–7418.

VITAE

Aijaz Ahmad was born on 20th April, 1976, in Karachi, Pakistan. He Graduated from the Punjab University in 1996 and achieved a Master's degree in Mathematics from University of Peshawar, Pakistan in 1998. He joined KFUPM as a research assistant in 2003 and obtained a second Master's degree in Mathematics in 2005.

Aijaz Ahmad has been actively involved in education and research from the very beginning of his career. He has held the positions of lecturer at PIPS, ICB and COMSATS. His areas of interest include Differential Equations, Lie Symmetries and Numerical Methods.