A generalized theory for bending of thick isotropic rectangular plates

Ammar Khalil Hafedh Mohammed

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Abstract

Several refined theories of plates have been developed in the recent decade. All such theories have attempted to incorporate the effects of tranverse shear stresses and tranverse normal stress and strain which become important as the ratio of the plate thickness to characteristic length (h/L) increases. The theory developed in this dissertation belongs to this category, except that it differs in that generalized forms of stress are assumed initially, which leads to the formulation of a more accurate theory of bending of hick plates.

Upon comparison of the results from this present work with the exact solution and other previous refined theories, the present theory yields results closest to the exact solution for both deflection w and inplane stresses, up to a ratio of h/L as high as 3.0 for the case of cylindrical bending, and up to a ratio of h?l as high as 1.0 for the case of rectangular plates.

A Generalized Theory for Bending of Thick Isotropic Rectangular Plates

by

Ammar Khalil Hafedh Mohammed

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

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In

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This dissertation, written by AMMAR KHALIL HAFEDH MOHAMMAD under the direction of his Dissertation Advisor and approved by the Dissertation Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

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بسم الله الرحمن الرحيم

أهدي رسالة الدكتوراة هذه إلى :

والديّ العزيزين ،

وإلى

زوجتي (أم ياسر) العزيزة ،

وإلى

ابنيّ المحبوبين : ياسر ومهنّـــــد

THIS PH.D DISSERTATION IS DEDICATED TO :

MY DEAR PARENTS

MY DEAR WIFE (UM YASER)

AND

MY BELOVED SONS : YASER AND MOHANNED

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DISSERTATION ABSTRACT

FULL NAME OF STUDENT : **AMMAR KHALIL HAFEDH MOHAMMED**

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Several refined theories of plates have been developed in the recent decade. All such theories have attempted to incorporate the effects of tranverse shear stresses and tranverse normal stress and strain which become important as the ratio of the plate thickness to characteristic length (h/L) increases. The theory developed in this dissertation belongs to this category, except that it differs in that generalized forms of stresses are assumed initially, which lead to the formulation of a more accurate theory of bending of thick plates.

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خلاصة الرسالة

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لقد تطورت في السنوات الاخيرة نظريات منقحة للصفائح . حاولت كل هذه النظريات اعتبار تأثير الجهود العرضية والجهود العمودية والتمدد العمودي للصفائح والتي تزداد أهميتها بازدياد نسبة سمك الصفيحة الى طولها . تنتمي النظرية المشتقة في هذه الرسالة الى هذا القسم من النظريات الا أنها تختلف عن باقي النظريات بأنها تفترض توزيعا عاما للجهود بالبداية والتي تقود الى اشتقاق نظرية لانحناء الصفائح أكثر دقة من سابقاتها .

عند مقارنة النتائج من هذه بالنظرية الأكيدة - نظرية المرونة - وغيرها من النظريات المنقحة ، تبين أن هذه النظرية تعطي نتائج أقرب ماتكون من نتائج النظرية الأكيدة وذلك بالنسبة - لانحراف الصفيحة والجهود المستوية الى تتعرض لها حتى عندما تصل نسبة سمك الصفيحة الى طولها الى " ٣ " ، بالنسبة لانحناء الاسطواني ، والى نسبة من سمك الصفيحة الى طولها الى " ١ " بالنسبة للصفائح المستطيلة .

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Chapter **1**

INTRODUCTION

The behavior of a plate is affected greatly by its thickness. For this reason, plates can be divided into three categories [1]:

(1) thin plates with small deflections

(2) thin plates with large deflections

(3) thick plates.

In order to simplify the theory of plates, many assumptions have been made when developing a theory for thin plates with small deflections. These assumptions can be summarized as [1]:

(1) No stretching of the middle plane of the plate. This plane remains neutral during bending.

(2) Points of the plate lying initially on a normal-to-the middle plane of the plate remain on the normal to the middle surface of the plate after bending.

(3) The normal stresses in the direction transverse to the plate can be disregarded.

As a result to the above assumptions, many limitations are imposed on the classical theory of plates. As the thickness of the plate increases, the effect of transverse stresses and strains on the deflection of the plate and on the inplane stresses can not be neglected. Also, the resulting governing equation for deflection of the middle surface is of the fourth order which implies that two boundary conditions on each edge are needed for solution. This contradicts the requirement of satisfying three boundary conditions on each edge as elasticity theory states.

In order to overcome some of the limitations of thin or classical plate theory, researchers have developed a number of refined theories. Reissner [2] was the first to provide a refined theory that takes into account shear deformation. He did not include the effect of transverse normal strain. A special variational theorem was used by Reissner to develop his theory. As a result of his work, only mid-plane displacement w_o and bending moments and shear forces were modified. Stresses σ_x, σ_y, and τ_{xy} were not modified in Reissner's theory.

Some other theories [3,4,5,6] were developed to include the effects of transverse shear, transverse normal stress, and transverse normal strain. However as in all previous refined theories only the displacement "w" was corrected and the inplane stresses: σ_x, σ_y, and τ_{xy} were left as for the Kirchoff thin plate theory.

Another refined theory was developed by Kromm [7,8]. Kromm introduced more general stress distributions across the thickness of the plate. But Kromm neglected the effects of the transverse normal stress, σ_z and normal strain, ϵ_z.

Panc [9] had modified Kromm's work by deriving the governing equation for the function $f_1(z)$, used by Kromm, in a different way. Panc called this refined theory a "Generalized Theory".

In the present work, a new refined theory will be developed making use of Panc's generalized theory and a refined theory presented by Baluch et al. [10]. Figure 1.1 summarizes the state of the art and highlights characterestics of present formulation.

The effect of the transverse shear stresses, the transverse normal stress, and the transverse normal strain on the deflection "w" and on inplane stresses: σ_x, σ_y, and τ_{xy} will be considered. Also, a general stress distribution across the thickness of the plate will be assumed. Solution of problems of bending for isotropic thick rectangular plates with different boundary conditions (i.e.: simply supported, free or clamped at $y = \pm b/2$) will be considered. Also the applied load will be of general form (i.e.: concentrated, uniformly distributed or other continuous distribution).

In this present work, the importance of developing a refined theory that takes into account the effects of normal stress σ_z, and shearing stresses τ_{xz}, τ_{yz} on inplane stresses and on deflection will be illustrated explicitly. The normal stress σ_z, for example, will be shown to have values of the same order as the inplane stresses σ_x, σ_y, and τ_{xy} for plates of appreciable thickness.

A Levy type semi-inverse method will be followed to obtain the solution for bending of isotropic rectangular plates. In order to test the present theory, some problems of thick isotropic rectangular plates will be considered and compared to already existing theories and to exact solution, whenever it may exist.

FIG. 1.1 : STATE OF ART + PRESENT THEORY

1. NEGLECTS INFLUENCE OF :

τ_{xz} , τ_{yz}

ON DEFLECTION .

2. NEGLECTS INFLUENCE OF :

σ_z , ε_z

ON PLATE RESPONSE .

CLASSICAL

1. σ_z , ε_z

MISSING .

2. σ_x , σ_y , τ_{xy} , τ_{xz} ,

τ_{yz}

NOT CORRECTED .

REISSNER

FIG. 1.1 (CONTINUED)

1. ILL CONDITIONING . 2. STRESSES NOT FOUND . (IN-PLANE PROBLEM NOT SOLVED)	*BALUCH, VOYIADJIS, and AZAD*
1. INCLUDES EFFECTS OF : τ_{xz} , τ_{yz} , σ_z , and ε_z ON PLATE RESPONSE . 2. IN-PLANE PROBLEM SOLVED . 3. STRESSES FOUND 4. ILL CONDITIONING REMOVED.	*PRESENT*

Chapter 2

THEORETICAL BACKGROUND

In this chapter, basic relations in the classical theory of isotropic elastic plates will be shown. Particular simplifications are introduced into the governing equations of the mathematical theory of elasticity. These simplifications give results which do not differ significantly from those obtained from the exact equations for the range of definition of the problem.

The simplifying assumptions used in various plates theories come from using the definition of a plate as a body which has one dimension which is small and also from results of elementary beam theory.

The stress-strain relations for an isotropic body are given by [9]:

$$\varepsilon_x = \frac{1}{E} [\sigma_x - \nu (\sigma_y + \sigma_z)] \tag{2.1}$$

$$\varepsilon_y = \frac{1}{E} [\sigma_y - \nu (\sigma_x + \sigma_z)] \tag{2.2}$$

$$\varepsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x + \sigma_y)] \tag{2.3}$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} \tag{2.4}$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \tag{2.5}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \tag{2.6}$$

In the classical theory of plates, the following assumptions are adopted:

$$\sigma_z = 0 \tag{2.7.1}$$

$$\varepsilon_z = 0 \tag{2.7.2}$$

$$\gamma_{xz} = 0 \tag{2.7.3}$$

$$\gamma_{yz} = 0 \tag{2.7.4}$$

For small deflections, compared with the plate thickness h, the strain-displacement relations in rectangular coordinates are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{2.8.1}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{2.8.2}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{2.8.3}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.8.4}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2.8.5}$$

Because of the assumption in equation (2.7.2) the deflection function depends on the variables x and y, thus:

$$w = w(x,y) \tag{2.8.6}$$

Introducing equation (2.8.6) and (2.7.3), (2.7.4) into (2.8.4) and (2.8.5) yields for the displacements u and v after performing integration with respect to z:

$$u = -z\frac{\partial w}{\partial x} + u_o(x,y) \tag{2.8.7}$$

$$v = -z\frac{\partial w}{\partial y} + v_o(x,y) \tag{2.8.8}$$

<u>where</u>: u_o, v_o are functions of integration. These functions define a state of plane strain of the plate (i.e. deformations independent of z). They correspond to forces acting in the middle plane of the plate or to a uniform heating of the plate. These functions can be neglected during bending, if the only load acting on the plate is normal to its surface, and if the edges of the plate are free to move in the plane of the plate.

Introducing the simplifications (or assumptions) in (2.7.1-4), the stress-strain relations become:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{2.9.1}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{2.9.2}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \tag{2.9.3}$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \tag{2.9.4}$$

The above set of equations represent the elasticity relations used in the classical theory of isotropic plates.

Consider an element of volume dxdydz (Fig. 2.1). Then the stress components acting on this element must satisfy three conditions of equilibrium which are expressed in the absence of body forces by the equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (2.10)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (2.11)$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} = 0 \qquad (2.12)$$

The shearing stresses satisfy conditions of symmetry which result from equations of moment equilibrium

$$\tau_{xy} = \tau_{yx}$$
$$\tau_{xz} = \tau_{zx} \qquad (2.13)$$
$$\tau_{yz} = \tau_{zy}$$

The equilibrium equations in 2.10, 2.11, and 2.12 are also known as the Cauchy equations. In the solution of plate problems, the stress components are usually replaced by the corresponding resultants per unit length. These resultants are denoted by bending moments, twisting moments, and shearing forces. They are defined by:

$$M_x = \int_{-h/2}^{+h/2} \sigma_x \, z \, dz \tag{2.14.1}$$

$$M_y = \int_{-h/2}^{+h/2} \sigma_y \, z \, dz \tag{2.14.2}$$

$$M_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} \, z \, dz \tag{2.14.3}$$

$$Q_x = \int_{-h/2}^{+h/2} \tau_{xz} \, dz \tag{2.14.4}$$

$$Q_y = \int_{-h/2}^{+h/2} \tau_{yz} \, dz \tag{2.14.5}$$

Neglecting body forces, the equilibrium equations in terms of the internal forces as defined by equations (2.14) and the lateral load p(x,y) acting on an element hdxdy of a plate (Fig. 2.2) take the form:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x \tag{2.15}$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \tag{2.16}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = o \tag{2.17}$$

The relations given above represent the basis of the classical theory of elastic isotropic plates.

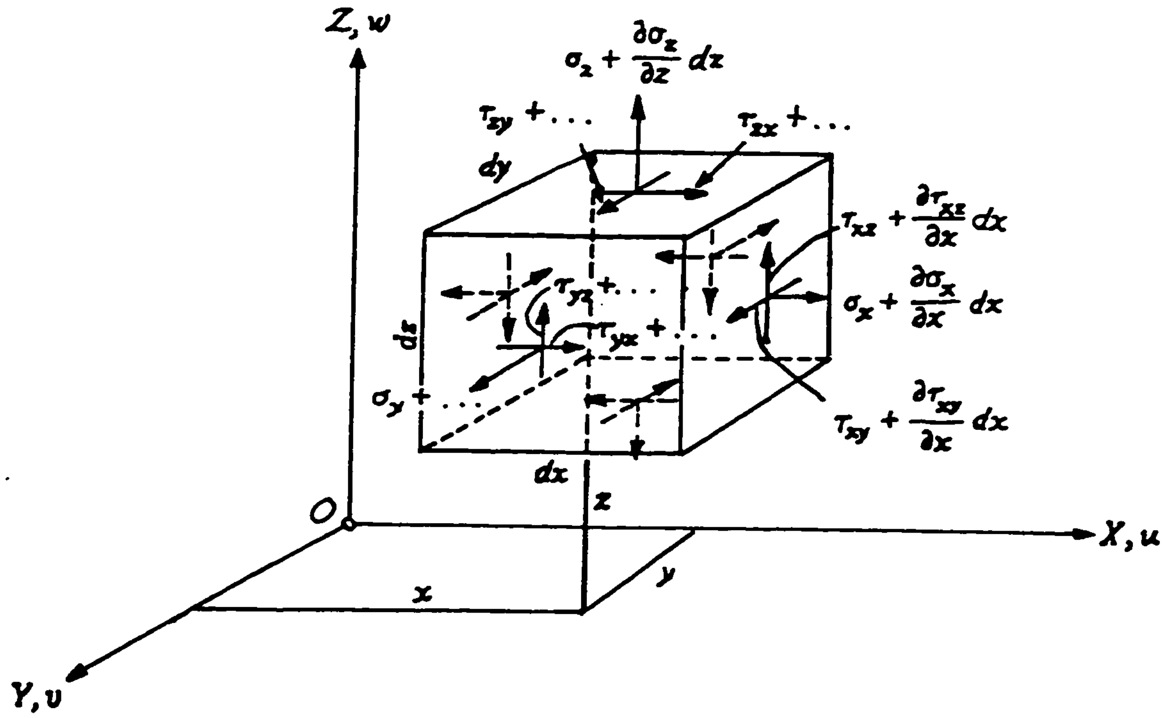


Figure 2.1 : Three-Dimensional Element (Note : +... = increment)

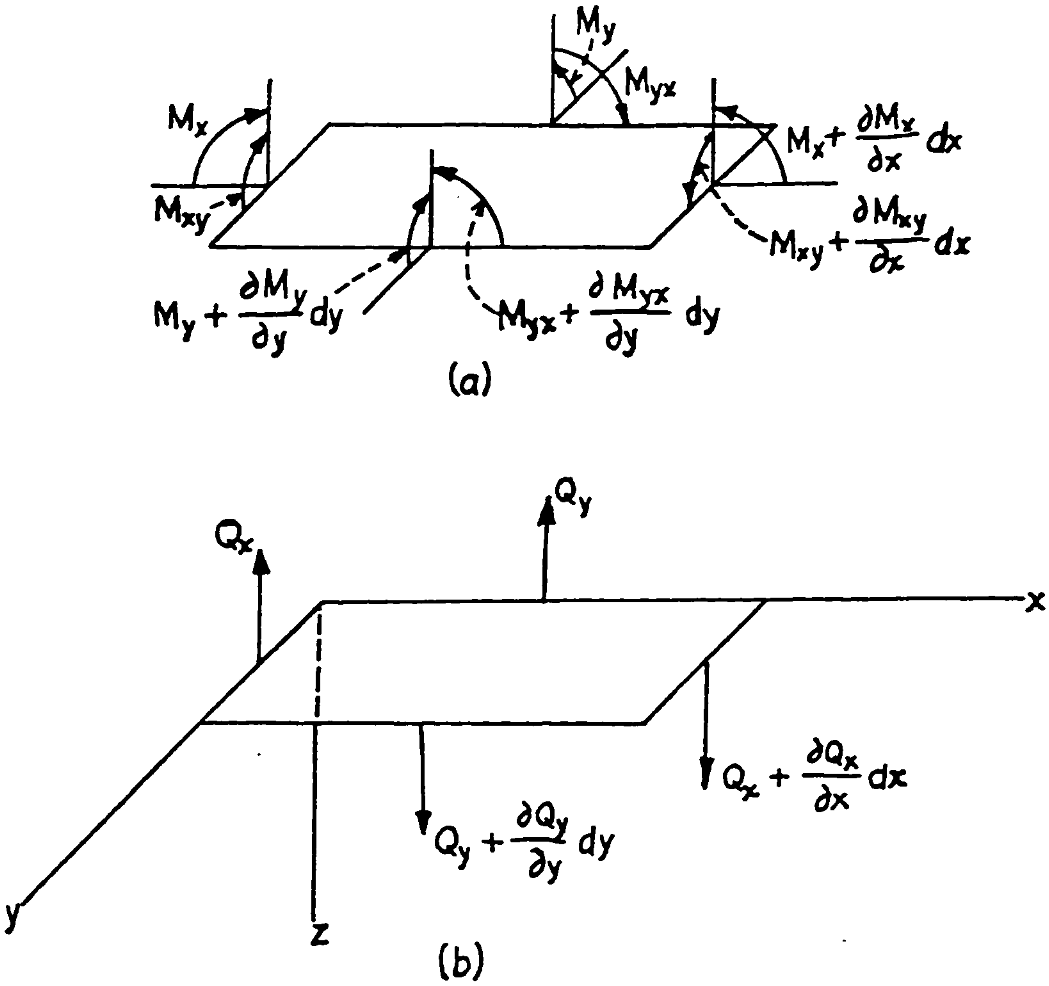


Figure 2.2 : *a) Resultant Moments.*

b) Resultant Shear Forces.

Chapter 3

FORMULATION

3.1 Governing Equations for the Bending Problem

The following generalized assumption has been introduced by Kromm [7,8] to approximate the variation of the transverse normal stress [1]

$$\sigma_z = p(x,y)\, f_1(z) \tag{3.1}$$

If the load p(x,y) acts only at the upper surface z = -h/2 of the plate, the function $f_1(z)$ must satisfy the boundary conditions:

$$f_1(-h/2) = -1, \quad f_1(+h/2) = 0 \tag{3.2}$$

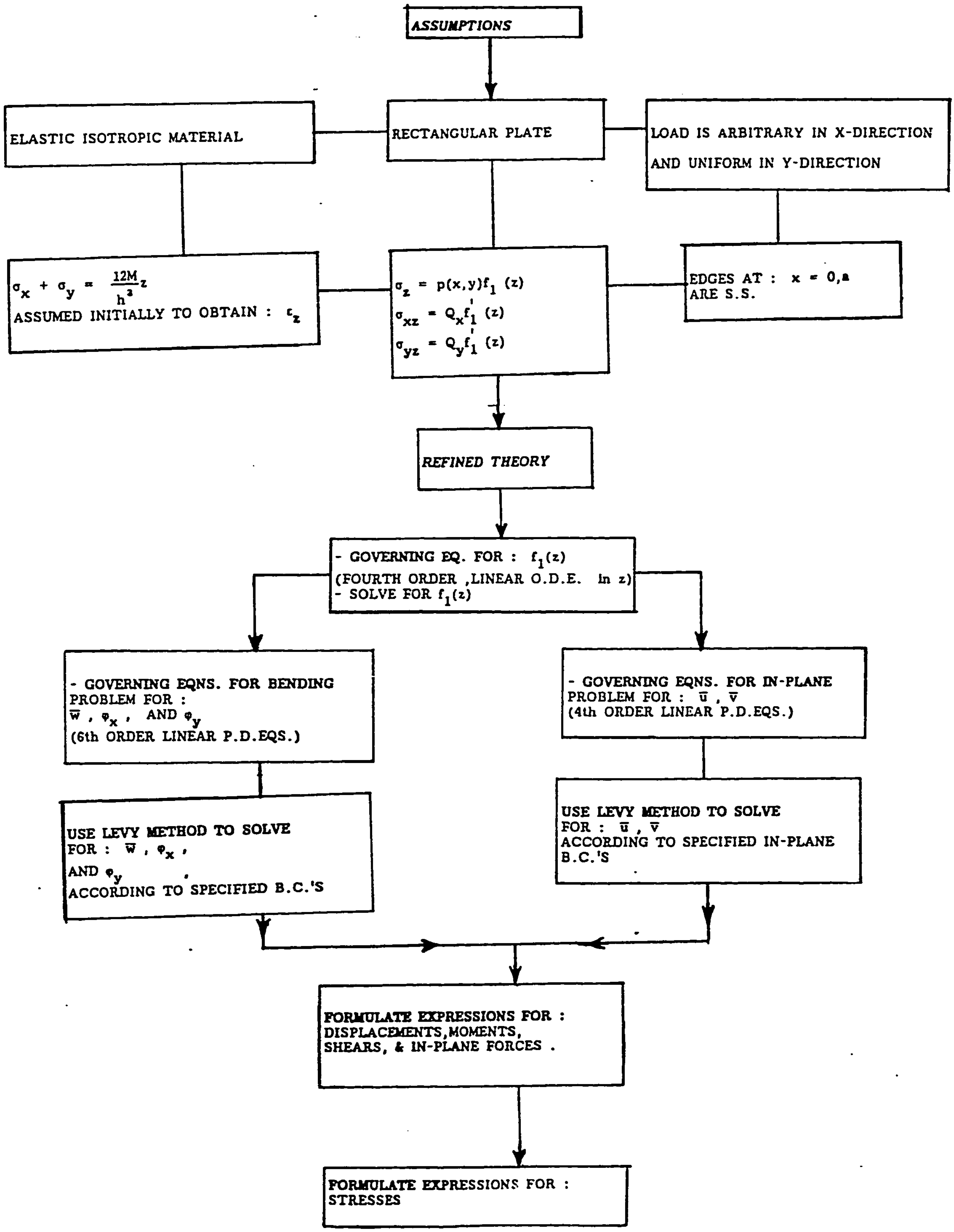
The distribution of transverse shears is assumed in the form:

$$\begin{aligned} \tau_{xz} &= Q_x(x,y)\, \bar{f}_2(z) \\ \tau_{yz} &= Q_y(x,y)\, \bar{f}_2(z) \end{aligned} \tag{3.3}$$

----

(1) See Figure 3.1 for a flowchart presentation of the theory developed.

Figure 3.1 : Flowchart For Present Theory.



where $\bar{f}_2(z)$ must satisfy the stress boundary conditions at the surface of the plate i.e.

$$\bar{f}_2(\pm h/2) = 0 \tag{3.4}$$

On substituting equations (3.1) and (3.3) into the stress differential equation of equilibrium

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{3.5}$$

one obtains

$$\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right) \bar{f}_2(z) + p(x,y)\frac{df_1(z)}{dz} = 0 \tag{3.6}$$

However

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \tag{3.7}$$

Thus for identical satisfaction of equation (3.6) one should have

$$\bar{f}_2(z) = \frac{df_1(z)}{dz} = f_1'(z) \tag{3.8}$$

Thus τ_{xz}, τ_{yz} can be written as:

$$\tau_{xz} = Q_x f_1'(z) \tag{3.9}$$

$$\tau_{yz} = Q_y f_1'(z)$$

and conditions given in equation (3.4) can be written as

$$f_1'(\pm h/2) = 0 \tag{3.10}$$

The transverse normal strain ε_z is given by:

$$\varepsilon_z = \frac{1}{E}\,[\sigma_z - \mu(\sigma_x + \sigma_y)] \tag{3.11}$$

Using equation (3.1) in (3.11)

$$\varepsilon_z = \frac{\partial w}{\partial z} = \frac{1}{E}\left(p(x,y)f_1(z)\right) - \frac{\mu}{E}\frac{(12M)z}{h^3} \tag{3.12}$$

where

$$M = M_x + M_y \tag{3.13}$$

and $\sigma_x + \sigma_y$ has been assumed to be of the form

$$\sigma_x + \sigma_y = \frac{12M}{h^3}z \tag{3.14}$$

The above linear distribution for the stresses σ_x and σ_y was used as an input stress to enable us to get an expression for ε_z, which on integration, yields a rational assumed form for the transverse displacement w.

Integrating (3.12) with respect to z yields the rational form for w as:

$$w(x,y,z) = \frac{1}{E}p(x,y)f_2(z) - \frac{6\mu M}{Eh^3}z^2 + w_o(x,y) \qquad (3.15)$$

where

$$f_2(z) = \int f_1(z)dz \qquad (3.16)$$

$w_o(x,y)$ = transverse displacement of the surface $z = 0$. (3.17)

The displacements $u(x,y,z)$ and $v(x,y,z)$ are obtained by making use of the strain-displacement relations:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad (3.18.1)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad (3.18.2)$$

Using equations (3.3) and (3.15) in (3.18.1) and integrating with respect to z gives for u

$$u = -z\frac{\partial w_o}{\partial x} + \frac{Q_x}{G} f_1(z) - \frac{1}{E}\frac{\partial p}{\partial x} f_3(z) + \frac{2\mu}{Eh^3}\frac{\partial M}{\partial x}z^3 + u_o(x,y) \qquad (3.19)$$

where

$$f_3(z) = \int f_2(z)dz \tag{3.19.1}$$

$u_o(x,y)$ = u-displacement of the mid surface (3.19.2)

Proceeding similarly, one may obtain an expression for the displacement v in the form

$$v = -z\frac{\partial w_o}{\partial y} + \frac{Q_y}{G} f_1(z) - \frac{1}{E}\frac{\partial p}{\partial y} f_3(z) + \frac{2\mu}{Eh^3}\frac{\partial M}{\partial y}z^3 + v_o(x,y) \tag{3.20}$$

where

$v_o(x,y)$ = v-displacement of the mid surface (3.20.1)

In refined theories taking into account influence of transverse shear only, u_o and v_o are taken to be identically zero.

The remaining stress-strain relations are

$$\sigma_x = \frac{E}{(1-\mu^2)}[\varepsilon_x + \mu\varepsilon_y] + \frac{\mu}{(1-\mu)}\sigma_z \tag{3.21.1}$$

$$\sigma_y = \frac{E}{(1-\mu^2)}[\varepsilon_y + \mu\varepsilon_x] + \frac{\mu}{(1-\mu)}\sigma_z \tag{3.21.2}$$

$$\tau_{xy} = G\gamma_{xy} \tag{3.21.3}$$

The strain-displacement relations are given by

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y},\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.21.4}$$

Substituting equations (3.1), (3.19), (3.20) and (3.21.4) into the set (3.21.1), (3.21.2) and (3.21.3) yields

$$\sigma_x = \frac{E}{(1-\mu^2)}\left[-z\frac{\partial^2 w_o}{\partial x^2} + \frac{f_1(z)}{G}\frac{\partial Q_x}{\partial x} - \frac{f_3(z)}{E}\frac{\partial^2 p}{\partial x^2} + \frac{2\mu}{Eh^3}\frac{\partial^2 M}{\partial x^2}z^3 + \mu\left\{-z\frac{\partial^2 w_o}{\partial y^2} + \frac{f_1(z)}{G}\frac{\partial Q_y}{\partial y} - \frac{f_3(z)}{E}\frac{\partial^2 p}{\partial y^2} + \frac{2\mu}{Eh^3}\frac{\partial^2 M}{\partial y^2}z^3\right\}\right] + \frac{E}{(1-\mu^2)}\left(\frac{\partial u_o}{\partial x} + \frac{\mu\partial v_o}{\partial y}\right) + \frac{\mu p}{(1-\mu)}f_1(z) \tag{3.22}$$

$$\sigma_y = \frac{E}{(1-\mu^2)}\left[-z\frac{\partial^2 w_o}{\partial y^2} + \frac{f_1(z)}{G}\frac{\partial Q_y}{\partial y} - \frac{f_3(z)}{E}\frac{\partial^2 p}{\partial y^2} + \frac{2\mu}{Eh^3}\frac{\partial^2 M}{\partial y^2}z^3 + \mu\left\{-z\frac{\partial^2 w_o}{\partial y^2} + \frac{f_1(z)}{G}\frac{\partial Q_x}{\partial x} - \frac{f_3(z)}{E}\frac{\partial^2 p}{\partial x^2} + \frac{2\mu}{Eh^3}\frac{\partial^2 M}{\partial x^2}z^3\right\}\right] + \frac{E}{(1-\mu^2)}\left(\frac{\partial v_o}{\partial y} + \frac{\mu\partial u_o}{\partial x}\right) + \frac{\mu p}{(1-\mu)}f_1(z) \tag{3.23}$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\left\{-2z\frac{\partial^2 w_o}{\partial x\partial y} + \frac{f_1(z)}{G}\frac{\partial Q_x}{\partial y} + \frac{f_1(z)}{G}\frac{\partial Q_y}{\partial x} - \frac{2f_3(z)}{E}\frac{\partial^2 p}{\partial x\partial y} + \frac{4\mu}{Eh^3}\frac{\partial^2 M}{\partial x\partial y}z^3\right\}$$

$$+ \frac{E}{2(1+\mu)}\left[\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right] \tag{3.24}$$

Using the definitions for the moment stress resultants

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz \quad ; \quad M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz$$

$$M_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z \, dz \tag{3.25}$$

one obtains

$$M_x = \frac{E}{(1-\mu^2)}\left[\frac{-h^3}{12}\frac{\partial^2 w_o}{\partial x^2} + \frac{h^3}{12G}F_1\frac{\partial Q_x}{\partial x} - \frac{h^3}{12E}F_3\frac{\partial^2 p}{\partial x^2} + \frac{\mu h^5}{40Eh^3}\frac{\partial^2 M}{\partial x^2} + \mu\left\{-\frac{h^3}{12}\frac{\partial^2 w_o}{\partial y^2} + \frac{h^3}{12G}F_1\frac{\partial Q_y}{\partial y} - \frac{h^3}{12E}F_3\frac{\partial^2 p}{\partial y^2} + \frac{\mu h^5}{40Eh^3}\frac{\partial^2 M}{\partial y^2}\right\}\right] + \frac{\mu h^3 p}{12(1-\mu)} F_1 \tag{3.26.1}$$

where :

$$F_1 = \frac{12}{h^3}\int_{-h/2}^{h/2} z \, f_1(z)dz \tag{3.26.2}$$

$$F_3 = \frac{12}{h^3}\int_{-h/2}^{h/2} z \, f_3(z)dz \tag{3.26.3}$$

or

$$M_x = D\left[\frac{\partial\varphi_x}{\partial x} + \mu\frac{\partial\varphi_y}{\partial y} + \frac{\mu(1+\mu)}{E}\,pF_1\right] \tag{3.27.1}$$

$$M_y = D\left[\frac{\partial\varphi_y}{\partial y} + \mu\frac{\partial\varphi_x}{\partial x} + \frac{\mu(1+\mu)}{E}\,pF_1\right] \tag{3.27.2}$$

$$M_{xy} = -\frac{D(1-\mu)}{2}\left[\frac{\partial\varphi_x}{\partial y} + \frac{\partial\varphi_y}{\partial x}\right] \tag{3.27.3}$$

where [1] :

$$\varphi_x = -\frac{\partial w_o}{\partial x} + \frac{F_1}{G}Q_x - \frac{F_3}{E}\frac{\partial p}{\partial x} + \frac{3\mu}{10Eh}\frac{\partial M}{\partial x}$$

$$= -\frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N}\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial M}{\partial x} \tag{3.27.4}$$

$$\varphi_y = -\frac{\partial w_o}{\partial y} + \frac{Q_y}{S} - \frac{1}{N}\frac{\partial p}{\partial y} + \frac{1}{R}\frac{\partial M}{\partial y} \tag{3.27.5}$$

in which

$$S = \frac{G}{F_1} \tag{3.27.6}$$

$$N = \frac{E}{F_3} \tag{3.27.7}$$

$$R = \frac{10Eh}{3\mu} \tag{3.27.8}$$

In order to obtain the governing differential equation for w_o,

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(1) See Appendix (A-4) for physical interpretation of φ_x and φ_y

one first eliminates φ_x and φ_y by using equations (3.27.4) and (3.27.5) in equation (3.27.1) resulting in

$$M_x = -D\left[\frac{\partial^2 w_o}{\partial x^2} + \mu\frac{\partial^2 w_o}{\partial y^2}\right] + \frac{h^3}{6}F_1\frac{\partial Q_x}{\partial x} - \frac{\mu h^3 p}{12(1-\mu)}F_1$$

$$- \frac{D}{N}\left(\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial^2 p}{\partial y^2}\right) + \frac{D}{R}\left[\frac{\partial^2 M}{\partial x^2} + \mu\frac{\partial^2 M}{\partial y^2}\right] \tag{3.28}$$

Similarly, one obtains for the moments M_y and M_{xy} the expressions

$$M_y = -D\left[\frac{\partial^2 w_o}{\partial y^2} + \mu\frac{\partial^2 w_o}{\partial x^2}\right] + \frac{h^3}{6}F_1\frac{\partial Q_y}{\partial y} - \frac{\mu h^3 p}{12(1-\mu)}F_1$$

$$- \frac{D}{N}\left[\frac{\partial^2 p}{\partial y^2} + \mu\frac{\partial^2 p}{\partial x^2}\right] + \frac{D}{R}\left[\frac{\partial^2 M}{\partial y^2} + \mu\frac{\partial^2 M}{\partial x^2}\right] \tag{3.29}$$

$$M_{xy} = D(1-\mu)\frac{\partial^2 w_o}{\partial x\partial y} - \frac{h^3}{12}F_1\left[\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right]$$

$$+ \frac{D(1-\mu)}{N}\frac{\partial^2 p}{\partial x\partial y} - \frac{D(1-\mu)}{R}\frac{\partial^2 M}{\partial x\partial y} \tag{3.30}$$

The remaining two equations of equilibrium are

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x \tag{3.31}$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y \tag{3.32}$$

By substituting equations (3.28) and (3.30) in equation (3.31), one obtains

$$Q_x - \frac{h^3F_1}{12}\Delta Q_x = -D\frac{\partial}{\partial x}\Delta w_o - \frac{h^3F_1}{12(1-\mu)}\frac{\partial p}{\partial x} - \frac{D}{N}\frac{\partial}{\partial x}\Delta p + \frac{D}{R}\frac{\partial}{\partial x}\Delta M \tag{3.33}$$

Similarly, substitution of equations (3.29) and (3.30) into equation (3.32) yields

$$Q_y - \frac{h^3F_1}{12}\Delta Q_y = -D\frac{\partial}{\partial y}\Delta w_o - \frac{h^3F_1}{12(1-\mu)}\frac{\partial p}{\partial y} - \frac{D}{N}\frac{\partial}{\partial y}\Delta p + \frac{D}{R}\frac{\partial}{\partial y}\Delta M \tag{3.34}$$

Finally, on substituting equations (3.33) and (3.34) in equation (3.7) yields the plate differential equation in terms of displacement w_o

$$D\Delta^2 w_o = p - \frac{h^3F_1}{6(1-\mu)}\Delta p + \frac{\mu h^3F_1}{12(1-\mu)}\Delta p - \frac{D}{N}\Delta^2 p + \frac{D}{R}\Delta^2 M \tag{3.35}$$

3.2 Governing Equations for the Inplane Problem

On substituting for σ_x, σ_y and τ_{xy} from equations (3.22), (3.23) and (3.24) into

$$N_x = \int_{-h/2}^{h/2} \sigma_x dz \; ; \; N_y = \int_{-h/2}^{h/2} \sigma_y dz \; ; \; N_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz \tag{3.36}$$

and further making use of the inplane equilibrium equation

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{3.37}$$

results in the following differential equation in terms of displacement u_o and v_o

$$\frac{\partial^2 u_o}{\partial x^2} + \frac{(1-\mu)}{2}\frac{\partial^2 u_o}{\partial y^2} + \frac{(1+\mu)}{2}\frac{\partial^2 v_o}{\partial x \partial y} = \frac{(1+\mu)}{Eh} F_2 \frac{\partial p}{\partial x}$$

$$+ \frac{F_4}{Eh}\frac{\partial}{\partial x}\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right] - \frac{(1-\mu^2)}{Eh}F_2\left[\frac{\partial^2 Q_x}{\partial x^2} + \frac{\partial^2 Q_x}{\partial y^2}\right]$$

(3.38)

where:

$$F_2 = \int_{-h/2}^{h/2} f_1(z)dz, \quad F_4 = \int_{-h/2}^{h/2} f_3(z)dz \tag{3.38.1}$$

Similarly, operating on the other inplane equilibrium equation

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{3.39}$$

yields

$$\frac{\partial^2 v_o}{\partial y^2} + \frac{(1-\mu)}{2}\frac{\partial^2 v_o}{\partial x^2} + \frac{(1+\mu)}{2}\frac{\partial^2 u_o}{\partial x \partial y}$$

$$= \frac{(1+\mu)}{Eh}F_2\frac{\partial p}{\partial y} + \frac{F_4}{Eh}\frac{\partial}{\partial y}\left[\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right]$$

$$- \frac{(1-\mu^2)}{Eh} F_2 \left[\frac{\partial^2 Q_y}{\partial x^2} + \frac{\partial^2 Q_y}{\partial y^2} \right] \tag{3.40}$$

3.3 Boundary Conditions

Physical interpretation for the terms φ_x, φ_y follows the same reasoning previously used in [10]. Thus φ_x is the rotation of a vertical element x = constant of the plate and φ_y is the rotation of a vertical element y = constant of the plate. Also, average displacement functions $\bar{u}$, $\bar{v}$ and $\bar{w}$ are used here in all boundary conditions where [11] (1)

$$\bar{w} = w_o + \frac{P}{N} - \frac{M}{R} \tag{3.41}$$

Since the order of equations in bending is six and in inplane problem is four, three boundary conditions are needed to be specified for bending and two boundary conditions for the inplane problem at each end.

Bending Problem

1. Simply Supported Edge (x = 0)

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(1) See Appendix (A-4) for physical interpretation of φ_x, φ_y, $\bar{u}$, $\bar{v}$, and $\bar{w}$.

$$\bar{w}(0,y) = 0, \ \varphi_y(0,y) = 0, \ M_x(0,y) = 0 \tag{3.42}$$

2. Clamped Edge $(x = 0)$

$$\bar{w}(0,y) = 0, \ \varphi_y(0,y) = 0, \ \varphi_x(0,y) = 0 \tag{3.43}$$

3. Free Edge $(x = 0)$

$$M_x(0,y) = 0, \ Q_x(0,y) = 0, \ M_{xy}(0,y) = 0 \tag{3.44}$$

Inplane Problem

1. Edge Clamped Against Stretching $(x = 0)$

$$\bar{u}(0,y) = 0, \ \bar{v}(0,y) = 0 \tag{3.45}$$

2. Edge Free to Stretch $(x = 0)$

$$N_x(0,y) = 0, \ \bar{v}(0,y) = 0 \tag{3.46}$$

3.4 Derivation of the Function $f_1(z)$

In order to derive the exact form of $f_1(z)$ that satisfies the four boundary conditions given by equations (3.2) and (3.10), one starts with the stress differential equations of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \tag{3.47.1}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = 0 \qquad (3.47.2)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad (3.47.3)$$

Solving for $\frac{\partial \tau_{xz}}{\partial z}$ and $\frac{\partial \tau_{yz}}{\partial z}$ from equations (3.47.1) and (3.47.2) by using expressions for σ_x, σ_y, τ_{xy} from equations (3.22), (3.23) and (3.24) and then substituting the result in the derivative of equation (47.3) with respect to z yields

$$\frac{E}{(1-\mu^2)} \left[z\Delta^2 w_o + \left(\frac{2-\mu}{2G}\right) f_1(z)\Delta p + \frac{f_3(z)}{E}\Delta^2 p - \frac{2\mu}{Eh^3} z^3 \Delta^2 M - \frac{\partial}{\partial x}\Delta u_o - \frac{\partial}{\partial y}\Delta v_o \right] + p f_1''(z) = 0 \qquad (3.48)$$

Differentiating equation (3.48) twice with respect to z and using the relation $f_3''(z) = f_1(z)$ yields the following fourth order differential equation in $f_1(z)$

$$p f_1^{(iv)}(z) + \frac{(2-\mu)}{(1-\mu)} f_1''(z)\Delta p + \frac{f_1(z)}{(1-\mu^2)}\Delta^2 p = \frac{12\mu}{Eh^3} z\Delta^2 M \qquad (3.49)$$

Expand the loading function p(x,y) in double Fourier series

$$p(x,y) = \sum_m \sum_n p_{mn} \sin \alpha_m x \sin\beta_n y \qquad (3.50.1)$$

The solution for M can be shown to be :

$$M = M_h + M_p$$

where M_h is the homogeneous part of the solution (i.e when $p = 0$) and M_p the particular solution.

Substituting for M in equation (3.49) above, one obtains for the homogeneous part of the solution corresponding to $p = 0$ the relationship that

$$\Delta^2 M_h = 0 \qquad (3.50.1a)$$

Relation (3.50.1a) indicates that it is the particular solution of M that plays a role in determination of the function $f_1(z)$.

The particular solution for M(x,y) corresponding to the loading p(x,y) given by (3.50.1) may be taken to be of the form

$$M_p(x,y) = \sum_m \sum_n M_{mn} \sin \alpha_m x \sin\beta_n y \qquad (3.50.2)$$

Substituting the expansions given by equations (3.50.1) and (3.50.2) into equation (3.49) and dividing by p_{mn} yields

$$f_1^{(iv)}(z) - \overline{A}f_1''(z) + \overline{B}f_1(z) = \overline{C}z \qquad (3.50.3)$$

where

$$\overline{A} = \frac{(2-\mu)}{(1-\mu)} (\alpha_m^2 + \beta_n^2) \qquad (3.50.4)$$

$$\overline{B} = \frac{(\alpha_m^2 + \beta_n^2)^2}{(1-\mu^2)} \qquad (3.50.5)$$

$$\overline{C} = \frac{12\mu M_{mn}}{h^3(1-\mu^2)p_{mn}} (\alpha_m^2 + \beta_n^2) \qquad (3.50.6)$$

$$\alpha_m = \frac{m\pi}{a}, \quad \beta_n = \frac{n\pi}{b} \qquad (3.50.7)$$

Equation (3.50.3) is a fourth order non-homogeneous differential equation in $f_1(z)$ whose solution is given by

$$f_1(z) = f_{1p}(z) + f_{1h}(z) = A_o + A_1 z + A_2 \cosh \overline{a}z$$
$$+ A_3 \sinh \overline{a}z + A_4 \cosh \overline{b}z + A_5 \sinh \overline{b}z \qquad (3.51)$$

where

$$\overline{a} = \sqrt{(\overline{A} + \sqrt{\overline{A}^2 - 4\overline{B}})/2} \qquad (3.51.1)$$

$$\overline{b} = \sqrt{(\overline{A} - \sqrt{\overline{A}^2 - 4\overline{B}})/2} \qquad (3.51.2)$$

and $f_{1p}(z)$ is the particular solution as given by $A_o + A_1 z$, and $f_{1h}(z)$ being the homogeneous solution. Coefficients in the particular solution are readily found to be

$$A_o = 0 \qquad (3.52)$$

$$A_1 = \frac{12\mu M_{mn}}{h^3 p_{mn}} \tag{3.53}$$

and the constants A_2 through A_5 involved in the homogeneous solution are found by using the four conditions given by equations (3.2) and (3.10).

Subsequent to obtaining $f_1(z)$, all other functions dependent on $f_1(z)$ are readily obtained and given by:

$$f_2(z) = \frac{A_1}{2} z^2 + \frac{A_2}{\bar{a}} \sinh \bar{a}z + \frac{A_3}{\bar{a}} \cosh \bar{a}z + \frac{A_4}{\bar{b}} \sinh \bar{b}z + \frac{A_5}{\bar{b}} \cosh \bar{b}z + C_1 \tag{3.54}$$

$$f_3(z) = \frac{A_1}{6} z^3 + \frac{A_2}{\bar{a}^2} \cosh \bar{a}z + \frac{A_3}{\bar{a}^2} \sinh \bar{a}z + \frac{A_4}{\bar{b}^2} \cosh \bar{b}z + \frac{A_5}{\bar{b}^2} \sinh \bar{b}z + C_1 z + C_2 \tag{3.55}$$

$$F_1 = A_1 + \frac{12}{h^3} \left(\frac{h}{\bar{a}} \cosh \frac{\bar{a}h}{2} - \frac{2}{\bar{a}^2} \sinh \frac{ah}{2} \right) A_3 + \frac{12}{h^3} \left(\frac{h}{\bar{b}} \cosh \frac{bh}{2} - \frac{2}{\bar{b}^2} \sinh \frac{\bar{b}h}{2} \right) A_5 \tag{3.56}$$

$$F_3 = A_1 + \frac{12}{h^3}\left[\frac{h}{\bar{a}}\cosh\frac{\bar{a}h}{2} - \frac{2}{\bar{a}^2}\sinh\frac{\bar{a}h}{2}\right]A_3 + \frac{12}{h^3}\left[\frac{h}{\bar{b}}\cosh\frac{\bar{b}h}{2} - \frac{2}{\bar{b}^2}\sinh\frac{\bar{b}h}{2}\right]A_5 \qquad (3.57)$$

$$F_2 = \left(\frac{2}{\bar{a}}\sinh\frac{\bar{a}h}{2}\right)A_2 + \left(\frac{2}{\bar{b}}\sinh\frac{\bar{b}h}{2}\right)A_4 \qquad (3.58)$$

$$F_4 = C_2 h + \left[\frac{2}{\bar{a}^3}\sinh\frac{\bar{a}h}{2}\right]A_2 + \left[\frac{2}{\bar{b}^3}\sinh\frac{\bar{b}h}{2}\right]A_4 \qquad (3.59)$$

The constant C_1 appearing in $f_2(z)$ is found by imposing the condition (with no loss in generality) that

$$w(x,y,o) = w_o(x,y) \qquad (3.60)$$

in equation (3.15) resulting in

$$C_1 = -\left(\frac{A_3}{\bar{a}} + \frac{A_5}{\bar{b}}\right) \qquad (3.61)$$

Similarly the constant C_2 appearing in $f_3(z)$ is found by imposing the condition that

$$u(x,y,o) = u_o(x,y) \qquad (3.62)$$

in equation (19) resulting in

$$C_2 = \frac{2(1+\mu)}{\alpha_m^2}(A_2 + A_4) - \left[\frac{A_2}{\bar{a}^2} + \frac{A_4}{\bar{b}^2}\right] \tag{3.63}$$

As an additional check on the particular solution for plate deflection w_o, one may differentiate equation (3.48) with respect to z and then set z = 0 in the resulting expression which yields

$$w_{oo} = \frac{h^3}{12}\left[\frac{p_{mn}}{(\alpha_m^2 + \beta_n^2)^2}\right]\left[AA_1 - BC_1 - A_3\left(\bar{a}^3 - \bar{a}A + \frac{B}{\bar{a}}\right) - A_5\left(\bar{b}^3 - \bar{b}A + \frac{B}{\bar{b}}\right)\right] \tag{3.64}$$

where:

$$w_o = \sum_m \sum_n w_{oo} \sin \alpha_m x \sin \beta_n y \tag{3.65}$$

and p(x,y) is as given by equation (3.50.1).

It should be noticed that equation (3.64) give particular solution for w_o which should coincide with the particular solution obtained from the differential equation derived for the plate deflection w_o i.e. equation (3.35).

Chapter 4

SOLUTION OF PROBLEM BY SEMI-INVERSE LEVY TYPE METHOD

4.1 Solution of the Bending Problem

4.1.1 Derivation of the Governing Equations

From work in Chap. 3, one has the following:

$$M_x = D\left[\frac{\partial \varphi_x}{\partial x} + \mu \frac{\partial \varphi_y}{\partial y} + \kappa p\right] \tag{4.1}$$

$$M_y = D\left[\frac{\partial \varphi_y}{\partial y} + \mu \frac{\partial \varphi_x}{\partial x} + \kappa p\right] \tag{4.2}$$

$$M_{xy} = \frac{-D(1-\mu)}{2}\left[\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right] \tag{4.3}$$

where:

$$\varphi_x = -\frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N}\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial M}{\partial x} \tag{4.4}$$

$$\varphi_y = -\frac{\partial w_o}{\partial y} + \frac{Q_x}{S} - \frac{1}{N}\frac{\partial p}{\partial y} + \frac{1}{R}\frac{\partial M}{\partial y} \qquad (4.5)$$

$$S = \frac{G}{F_1} \qquad (4.6)$$

$$N = \frac{E}{F_3} \qquad (4.7)$$

$$R = \frac{10Eh}{3\mu} \qquad (4.8)$$

$$\kappa = \frac{\mu\ (1+\mu)\ F_1}{E} \qquad (4.9)$$

$$F_1 = \frac{12}{h^3}\int_{-h/2}^{+h/2} z\ f_1(z)\ dz \qquad (4.10)$$

$$F_3 = \frac{12}{h^3}\int_{-h/2}^{+h/2} z\ f_3(z)\ dz \qquad (4.11)$$

Using equations (4.4) and (4.5) and the following equation:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \qquad (4.12)$$

one obtains alternate forms for M_x, M_y and M_{xy} as:

$$M_x = -D\left(\frac{\partial^2 w_o}{\partial x^2} + \mu\frac{\partial^2 w_o}{\partial y^2}\right) + \frac{h^3}{6}F_1\ \frac{\partial Q_x}{\partial x} - \frac{\mu h^3 F_1}{12(1-\mu)}p$$

$$\frac{-D}{N}\left[\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial^2 p}{\partial y^2}\right] + \frac{D}{R}\left[\frac{\partial^2 M}{\partial x^2} + \mu\frac{\partial^2 M}{\partial y^2}\right] \tag{4.13}$$

$$M_y = -D\left[\frac{\partial^2 w_o}{\partial y^2} + \mu\frac{\partial^2 w_o}{\partial x^2}\right] + \frac{h^3}{6}F_1\frac{\partial Q_y}{\partial y} - \frac{\mu h^3 F_1}{12(1-\mu)}p$$

$$\frac{-D}{N}\left[\frac{\partial^2 p}{\partial y^2} + \mu\frac{\partial^2 p}{\partial x^2}\right] + \frac{D}{R}\left[\frac{\partial^2 M}{\partial y^2} + \mu\frac{\partial^2 M}{\partial x^2}\right] \tag{4.14}$$

$$M_{xy} = D(1-\mu)\frac{\partial^2 w_o}{\partial x \partial y} - \frac{h^3}{12}F_1\left[\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right]$$

$$+ \frac{D(1-\mu)}{N}\frac{\partial^2 p}{\partial x \partial y} - \frac{D(1-\mu)}{R}\frac{\partial^2 M}{\partial x \partial y} \tag{4.15}$$

where:

$$M = M_x + M_y$$

Defining average transverse displacement $\overline{w}$ and average rotations φ_x, φ_y as (Appendix A-4)

$$\overline{w} = w_o + \frac{p}{N} - \frac{M}{R} \tag{4.16}$$

$$\varphi_x = -\frac{\partial \overline{w}}{\partial x} + \frac{Q_x}{S} \tag{4.17}$$

$$\varphi_y = -\frac{\partial \overline{w}}{\partial y} + \frac{Q_y}{S} \tag{4.18}$$

the set of equations (4.4), (4.5), (4.13), (4.14) and (4.15) are rewritten in the form

$$M_x = -D\left(\frac{\partial^2\overline{w}}{\partial x^2} + \mu\frac{\partial^2\overline{w}}{\partial y^2}\right) + \frac{h^3}{6}F_1\,\frac{\partial Q_x}{\partial x} - \frac{\mu h^3 F_1}{12(1-\mu)}p \tag{4.19}$$

$$M_y = -D\left(\frac{\partial^2\overline{w}}{\partial y^2} + \mu\frac{\partial^2\overline{w}}{\partial x^2}\right) + \frac{h^3}{6}F_1\,\frac{\partial Q_y}{\partial y} - \frac{\mu h^3 F_1}{12(1-\mu)}p \tag{4.20}$$

$$M_{xy} = D(1-\mu)\frac{\partial^2\overline{w}}{\partial x\partial y} - \frac{h^3}{12}F_1\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right) \tag{4.21}$$

Eliminating shears from equations (4.19), (4.20), (4.21) by using equations (4.17) and (4.18), one obtains:

$$M_x = \left(-D+\frac{h^3F_1}{6}S\right)\frac{\partial^2\overline{w}}{\partial x^2} - D\mu\frac{\partial^2\overline{w}}{\partial y^2} + \frac{h^3F_1}{6}S\frac{\partial\varphi_x}{\partial x} - \frac{h^3\mu F_1}{12(1-\mu)}p \tag{4.22}$$

$$M_y = \left(-D+\frac{h^3F_1}{6}S\right)\frac{\partial^2\overline{w}}{\partial y^2} - D\mu\frac{\partial^2\overline{w}}{\partial x^2} + \frac{h^3F_1}{6}S\frac{\partial\varphi_y}{\partial y} - \frac{h^3\mu F_1}{12(1-\mu)}p \tag{4.23}$$

$$M_{xy} = \left[D(1-\mu) - \frac{h^3F_1}{6}S\right]\frac{\partial^2\overline{w}}{\partial x\partial y} - \frac{h^3F_1}{12}S\left(\frac{\partial\varphi_x}{\partial y} + \frac{\partial\varphi_y}{\partial x}\right) \tag{4.24}$$

Using equations (4.17), (4.18) to eliminate shears in the equilibrium equations (3.31), (3.32), one obtains:

$$\left[\left(-D + \frac{h^3F_1S}{6}\right)\frac{\partial^3}{\partial x^3} - \left(D - \frac{h^3F_1S}{6}\right)\frac{\partial^3}{\partial x\partial y^2} - S\frac{\partial}{\partial x}\right]\overline{w}$$

$$+ \left[\frac{h^3F_1}{6}S\frac{\partial^2}{\partial x^2} + \frac{h^3F_1}{12}S\frac{\partial^2}{\partial y^2} - S\right]\varphi_x$$

$$+ \left[\frac{h^3F_1S}{12}\frac{\partial^2}{\partial x\partial y}\right]\varphi_y = \frac{\mu h^3F_1}{12(1-\mu)}\frac{\partial p}{\partial x} \tag{4.25}$$

$$\left[\left(-D + \frac{h^3F_1S}{6}\right)\frac{\partial^3}{\partial y^3} - \left(D - \frac{h^3F_1S}{6}\right)\frac{\partial^3}{\partial x^2\partial y} - S\frac{\partial}{\partial y}\right]\overline{w}$$

$$+ \left[\frac{h^3F_1S}{12}\frac{\partial^2}{\partial x\partial y}\right]\varphi_x + \left[\frac{h^3F_1S}{6}\frac{\partial^2}{\partial y^2} + \frac{h^3F_1S}{12}\frac{\partial^2}{\partial x^2} - S\right]\varphi_y$$

$$= \frac{\mu h^3F_1}{12(1-\mu)}\frac{\partial p}{\partial y} \tag{4.26}$$

The third equation involving $\overline{w}$, φ_x, and φ_y is obtained by substituting equations (4.17) and (4.18) into equation (4.12):

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\overline{w} + \left[\frac{\partial}{\partial x}\right]\varphi_x + \left[\frac{\partial}{\partial y}\right]\varphi_y = \frac{-p}{S} \tag{4.27}$$

The set of equations (4.25) through (4.27) represents a sixth order bending problem.

By using the set of equations (4.25), (4.26), and (4.27), the governing plate differential equation in terms of the average transverse displacement $\overline{w}$ can be obtained as[1]:

$$M'(\Delta^3\overline{w}) + N'(\Delta^2\overline{w}) = A\Delta^2p + B\Delta p + Cp \tag{4.28}$$

----

(1) See Appendix (A-1) for derivation of this equation.

<u>where</u>:

$$M' = \frac{h^3 F_1 SD}{12} \qquad (4.29.1)$$

$$N' = -\ SD \qquad (4.29.2)$$

$$A = -\ \frac{(2-\mu)}{(1-\mu)}\left(\frac{h^3 F_1}{12}\right)^2 S \qquad (4.29.3)$$

$$B = +\ \frac{(3-2\mu)}{12(1-\mu)} h^3 F_1 S \qquad (4.29.4)$$

$$C = -\ S \qquad (4.29.5)$$

$$\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \qquad (4.29.6)$$

4.1.2 Solution of Bending Problem by Semi-Inverse Levy type Method

For plates with the pair of edges at x=0, x=a being simply supported, see figure 4.1 , the solution to equation (4.28) may be expressed in the Levy form as:

$$\bar{w}(x,y) = \bar{w}_1(x) + \bar{w}_2(x,y) \tag{4.30}$$

in which the governing equations to be satisfied by $\bar{w}_1$ and $\bar{w}_2$ are given by [with Load p = p(x)]:

$$M'\frac{d^6\bar{w}_1}{dx^6} + N'\frac{d^4\bar{w}_1}{dx^4} = A\frac{d^4p}{dx^4} + B\frac{d^2p}{dx^2} + Cp \tag{4.31}$$

and:

$$M'\Delta^3\bar{w}_2 + N'\Delta^2\bar{w}_2 = 0 \tag{4.32}$$

Expanding the load in a half range sine series

$$p = \sum_{m=1}^{\infty} p_m \sin \alpha_m x \tag{4.33}$$

<u>with</u>: $\alpha_m = \frac{m\pi}{a}$ (4.33.1)

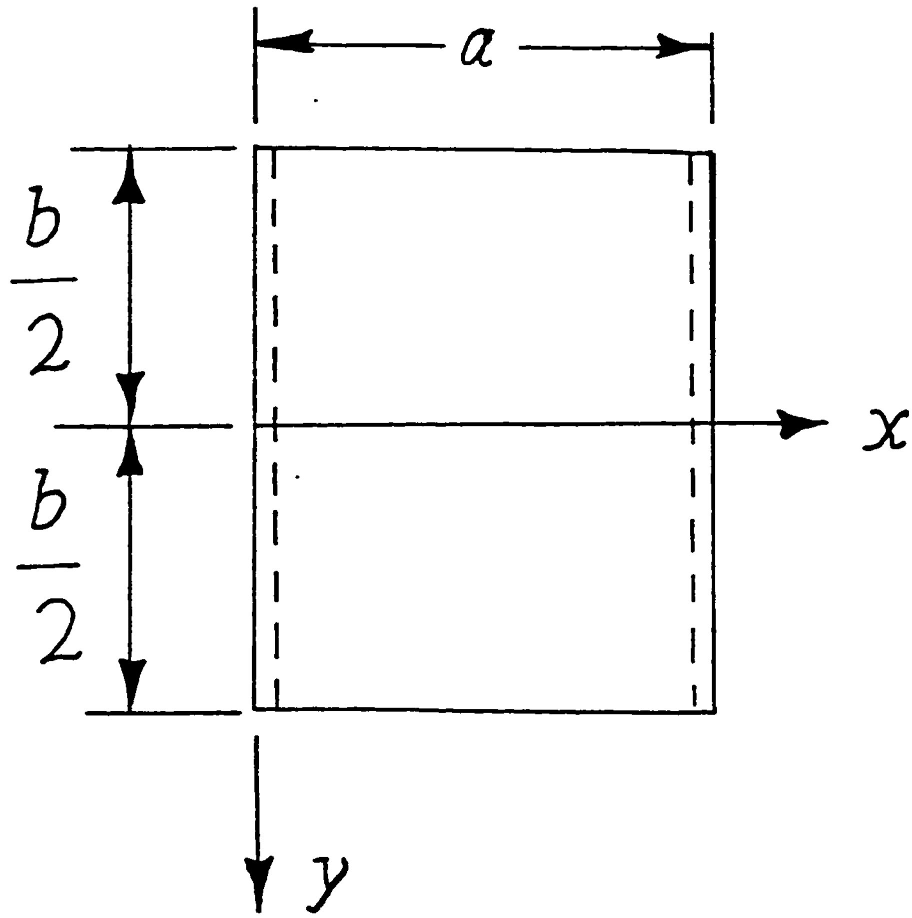


Figure 4.1 : Coordinate Axis For The Plate.

and the function $\overline{w}_1$ expressed in the form:

$$\overline{w}_1 = \sum_{m=1}^{\infty} \beta_m \sin \alpha_m x \tag{4.34}$$

The Fourier coefficients β_m are then determined from equation (4.31) to be:

$$\beta_m = \left\{ \frac{A\alpha_m^4 - B\alpha_m^2 + C}{-M'\alpha_m^6 + N'\alpha_m^4} \right\} p_m \tag{4.35}$$

The solution for $\overline{w}_2$ may be taken in the following form:

$$\overline{w}_2 = \sum_{m=1}^{\infty} Y_m(y) \sin \alpha_m x \tag{4.36}$$

in which Y_m is obtained by substituting appropriate expressions for $\overline{w}_2$ and its derivatives in equation (4.32). The function $Y_m(y)$ can be shown to be:

$$\begin{aligned} Y_m(y) = {} & A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + C_m \sinh \alpha_m y \\ & + D_m \alpha_m y \cosh \alpha_m y + E_m \cosh \gamma_m y + F_m \sinh \gamma_m y \end{aligned} \tag{4.37}$$

<u>where</u>:

$$\gamma_m^2 = \alpha_m^2 - \frac{N'}{M'} \tag{4.37-1}$$

Restricting the development to plates with loading and boundary conditions that are symmetrical with respect to the x-axis necessitates

$$C_m = D_m = F_m = 0$$

The complete solution for $\overline{w}$ becomes:

$$\overline{w} = \sum_{m=1}^{\infty} \overline{w}_m(y) \sin \alpha_m x$$

$$= \sum_{m=1}^{\infty} (A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + E_m \cosh \gamma_m y + \beta_m) \sin \alpha_m x \quad (4.38)$$

<u>where</u>:

$$\overline{w}_m(y) = A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + E_m \cosh \gamma_m y + \beta_m \quad (4.39)$$

In a similar way the same set of equations (4.25), (4.26), (4.27) can be used to obtain the governing equations for the average rotations φ_x and φ_y.

For the symmetric problem considered, the solutions are of the form:

$$\varphi_x = \sum_{m=1}^{\infty} \varphi_{xm}(y) \cos \alpha_m x \quad (4.40)$$

$$\varphi_y = \sum_{m=1}^{\infty} \varphi_{ym}(y) \sin \alpha_m x \quad (4.41)$$

where:

$$\varphi_{xm}(y) = A'_m \cosh \alpha_m y + B'_m \alpha_m y \sinh \alpha_m y + E'_m \cosh \gamma_m y + \beta'_m \tag{4.42}$$

$$\varphi_{ym}(y) = C''_m \sinh \alpha_m y + D''_m \alpha_m y \cosh \alpha_m y + F''_m \sinh \gamma_m y + \beta''_m \tag{4.43}$$

It should be noticed that due to symmetrical loading and boundary conditions with respect to the x-axis, φ_{xm} is even in "y" while φ_{ym} is odd in "y".

Relations between the constants in $\overline{w}$, φ_x and φ_y :

In view of the order of the plate problem, there exists a linear dependence among the nine constants A_m through F''_m. One way of arriving at these relationships, together with the particular solutions β'_m and β''_m, is by the following procedure:

Substituting equations (4.1), (4.2), and (4.3) into equations (3.31) and (3.32) and using equations (4.17) and (4.18) to eliminate the transverse shears yields

$$\varphi_x + \frac{\partial \overline{w}}{\partial x} = \frac{D}{S}\left[\frac{\partial^2 \varphi_x}{\partial x^2} + \frac{(1-\mu)}{2}\frac{\partial^2 \varphi_x}{\partial y^2} + \frac{(1+\mu)}{2}\frac{\partial^2 \varphi_y}{\partial x \partial y} + \kappa\frac{\partial p}{\partial x}\right] \tag{4.44}$$

$$\varphi_y + \frac{\partial \bar{w}}{\partial y} = \frac{D}{S}\left[\frac{\partial^2 \varphi_y}{\partial y^2} + \frac{(1-\mu)}{2}\frac{\partial^2 \varphi_y}{\partial x^2} + \frac{(1+\mu)}{2}\frac{\partial^2 \varphi_x}{\partial x \partial y} + \kappa\frac{\partial p}{\partial y}\right] \qquad (4.45)$$

Substituting for φ_x, φ_y, $\bar{w}$ and p from equations (4.38), (4.40), (4.41), and (4.33), respectively, into equations (4.44) and (4.45), the following coupled ordinary differential equations in $\varphi_{xm}(y)$ and $\varphi_{ym}(y)$ are obtained:

$$\left[\frac{D}{S}\frac{(1-\mu)}{2}\frac{d^2}{dy^2} - \alpha_m^2\frac{D}{S} - 1\right]\varphi_{xm} + \left[\frac{D}{S}\frac{(1+\mu)}{2}\alpha_m\frac{d}{dy}\right]\varphi_{ym}$$

$$= \quad \alpha_m\bar{w}_m - \kappa\frac{D}{S}\alpha_m p_m \qquad (4.46)$$

<u>and</u>:

$$-\left[\frac{D}{S}\frac{(1+\mu)}{2}\alpha_m\frac{d}{dy}\right]\varphi_{xm} + \left[\frac{D}{S}\frac{d^2}{dy^2} - \frac{D}{S}\frac{(1-\mu)}{2}\alpha_m^2 - 1\right]\varphi_{ym}$$

$$= \quad \frac{d\bar{w}_m}{dy} \qquad (4.47)$$

Uncoupling equations (4.46) and (4.47) for φ_{xm} and φ_{ym} results in:

$$\Bigg\{\left[\left(\frac{D}{S}\right)^2\left(\frac{1-\mu}{2}\right)\right]\frac{d^4}{dy^4} + \left[-(1-\mu)\left(\frac{D}{S}\right)^2\alpha_m^2 - \frac{(3-\mu)}{2}\frac{D}{S}\right]\frac{d^2}{dy^2}$$

$$+ \left[\left(\frac{D}{S}\right)^2\alpha_m^4\frac{(1-\mu)}{2} + \frac{D}{S}\left(\frac{3-\mu}{2}\right)\alpha_m^2 + 1\right]\Bigg\}\{\varphi_{xm}\}$$

$$= \alpha_m \left[\frac{D}{S}\frac{(1-\mu)}{2}\frac{d^2}{dy^2} - \frac{D}{S}\frac{(1-\mu)}{2}\alpha_m^2 - 1 \right] \left(\overline{w}_m - \kappa\frac{D}{S}p_m\right) \quad (4.48)$$

Similarly one obtains for φ_{ym}:

$$\left\{ \left[\left(\frac{D}{S}\right)^2 \left(\frac{1-\mu}{2}\right) \right] \frac{d^4}{dy^4} + \left[-(1-\mu)\left(\frac{D}{S}\right)^2 \alpha_m^2 - \frac{(3-\mu)}{2}\frac{D}{S} \right] \frac{d^2}{dy^2} \right.$$

$$\left. + \left[\left(\frac{D}{S}\right)^2 \alpha_m^4 \frac{(1-\mu)}{2} + \frac{D}{S}\left(\frac{3-\mu}{2}\right)\alpha_m^2 + 1 \right] \right\} \{\varphi_{ym}\}$$

$$= \left[\frac{D}{S}\frac{(1-\mu)}{2}\frac{d^3}{dy^3} - \frac{D}{S}\frac{(1-\mu)}{2}\alpha_m^2 \frac{d}{dy} - \frac{d}{dy} \right] \left(\overline{w}_m - \kappa p_m \frac{D}{S}\right) \quad (4.49)$$

The required relationships among the constants, together with solutions for $\beta_m^{'}$ and $\beta_m^{''}$ are established by substituting relations in equations (4.33), (4.39), (4.42), and (4.43) into equations (4.48) and (4.49).

Then these relationships are given by:

$$A_m^{'} = -\alpha_m A_m - \frac{2D}{S}\alpha_m^3 B_m \quad (4.50.1)$$

$$B_m^{'} = -\alpha_m B_m \quad (4.50.2)$$

$$E_m^{'} = \frac{\alpha_m}{\left[\frac{D}{S}(\gamma_m^2 - \alpha_m^2) - 1\right]} E_m \quad (4.50.3)$$

$$\beta_m^{'} = \frac{-\alpha_m\left(\beta_m - \kappa\frac{D}{S}p_m\right)}{\left(\frac{D}{S}\alpha_m^2 + 1\right)} \quad (4.50.4)$$

$$C''_m = -\alpha_m A_m - \left(\frac{2D}{S}\alpha_m^3 + \alpha_m\right)B_m \tag{4.50.5}$$

$$D''_m = -\alpha_m B_m \tag{4.50.6}$$

$$F''_m = \frac{\gamma_m}{\left[\frac{D}{S}(\gamma_m^2 - \alpha_m^2) - 1\right]}E_m \tag{4.50.7}$$

$$\beta''_m = 0 \tag{4.50.8}$$

4.1.3 Derivation of the Non-Dimensional Form of $f_1(z)$ and Related Constants:

Consider the governing differential equation for $f_1(z)$ (equation 3-49):

$$f_1^{(iv)}(z) - \overline{A}\, f_1''(z) + \overline{B}\, f_1(z) = \overline{C}z \tag{3-49}$$

<u>where</u>:

$$\overline{A} = \left[\frac{2 - \mu}{1 - \mu}\right] \alpha_m^{\;2} \tag{4-51.1}$$

$$\overline{B} = \left[\frac{\alpha_m^4}{1 - \mu^2}\right] \tag{4-51.2}$$

$$\overline{C} = \frac{12\mu\alpha_m^4}{h^3(1 - \mu^2)} \frac{M_m}{P_m} \tag{4-51.3}$$

It can be shown that:

$$\overline{C} = \frac{\alpha_m^4}{(1 - \mu^2)} A_1 \tag{4-51.4}$$

Therefore, from equations (4-51.3), and (4-51.4), we get:

$$M_m = \frac{h^3 P_m}{12\mu} A_1 \tag{4-51.5}$$

where:

The particular solution for $M = M_x + M_y$ can be written as:

$$M_p\ (x) = \sum_{m=1}^{\infty} M_m \sin \alpha_m x \qquad (4\text{-}51.6)$$

It can be shown that M_m will be given by:(1)

$$M_m = p_m \left[\frac{1 + \mu}{\alpha_m^2} + \frac{\mu h^3 F_1}{12} \right] \qquad (4\text{-}52)$$

Substituting for F_1 from equation (3-5b) and for M_m from equation (4-51.5) into the above equation results in:

$$\left[(1 - \mu^2)\right] A_1' + \frac{12\mu^2}{h^3} \left[\frac{2}{\bar{a}^2} \sinh \frac{\bar{a}h}{2} - \frac{h}{\bar{a}} \cosh \frac{\bar{a}h}{2} \right] A_3$$

$$+ \frac{12\mu^2}{h^3} \left[\frac{2}{\bar{b}^2} \sinh \frac{\bar{b}h}{2} - \frac{h}{\bar{b}} \cosh \frac{\bar{b}h}{2} \right] A_5 = 0 \qquad (4\text{-}53)$$

Equation (4-53) together with equations (3-2) and (3-10) represent the boundary conditions that the function $f_1(z)$ must satisfy.

From equation (3-5):

$$f_1(z) = A_1' z + A_2 \cosh \bar{a}z + A_3 \sinh \bar{a}z + A_4 \cosh \bar{b}z$$

$$+ A_5 \sinh \bar{b}z$$

----

(1) See Appendix (A-3) for derivation of this equation.

let :

$$A_1^{'} = \frac{A_1}{h} \tag{4-54}$$

Then $f_1(z)$ can be rewritten as:

$$f_1(z) = A_1\left(\frac{z}{h}\right) + A_2 \cosh \bar{a}z + A_3 \sinh \bar{a}z$$

$$+ A_4 \cosh \bar{b}z + A_5 \sinh \bar{b}z \tag{4-55}$$

From the boundary condition on $f_1(z)$: $f_1(-h/2) = -1$

one obtains

$$-\frac{1}{2}A_1 + \cosh \frac{\bar{a}h}{2} A_2 - \sinh \frac{\bar{a}h}{2} A_3$$

$$+ \cosh \frac{\bar{b}h}{2} A_4 - \sinh \frac{\bar{b}h}{2} A_5 = -1 \tag{4-56}$$

and the boundary condition $f_1(+h/2) = 0$ results in

$$\frac{1}{2} A_1 + \mathrm{Cosh} \frac{\bar{a}h}{2} A_2 + \sinh \frac{\bar{a}h}{2} A_3 + \cosh \frac{\bar{b}h}{2} A_4$$

$$+ \sinh \frac{\bar{b}h}{2} A_5 = 0 \tag{4-57}$$

the boundary condition $f_1^{'}(-h/2) = 0$ yields

$$A_1 - \bar{a}h \sinh \frac{\bar{a}h}{2} A_2 + \bar{a}h \cosh \frac{\bar{a}h}{2} A_3 - \bar{b}h \sinh \frac{\bar{b}h}{2} A_4$$

$$+ \bar{b}h \cosh \frac{\bar{b}h}{2} A_5 = 0 \qquad (4\text{-}58)$$

And the boundary condition $f_1'(+h/2) = 0$ results in

$$A_1 + \bar{a}h \sinh \frac{\bar{a}h}{2} A_2 + \bar{a}h \cosh \frac{\bar{a}h}{2} A_3 + \bar{b}h \sinh \frac{\bar{b}h}{2} A_4$$

$$+ \bar{b}h \cosh \frac{\bar{b}h}{2} A_5 = 0 \qquad (4\text{-}59)$$

Thus equations (4-53), (4-56), (4-57), (4-58), and (4-59) can be solved for the constants A_1 through A_5.

(Note that A_1' in equation (4-53) has to be replaced by A_1 given by equation (4-54)).

Therefore the function of $f_1(z)$ given by equation (4-55) is now completely known.

Solution for other functions and constants related to $f_1(z)$:

The expression F_1 will be rewritten in the following form:

$$F_1 = \frac{1}{h}\,\overline{F}_1 \qquad (4\text{-}59.1)$$

<u>where</u>:

$$\overline{F}_1 = A_1 + 12\left[\frac{1}{\bar{a}h}\cosh\frac{\bar{a}h}{2} - \frac{2}{(\bar{a}h)^2}\sinh\frac{\bar{a}h}{2}\right]A_3$$

$$+ 12\left[\frac{1}{\bar{b}h}\cosh\frac{\bar{b}h}{2} - \frac{2}{(\bar{b}h)^2}\sinh\frac{\bar{b}h}{2}\right]A_5 \qquad (4\text{-}59.2)$$

Similarly F_3 is rewritten as:

$$F_3 = h\overline{F}_3 \qquad (4\text{-}59.3)$$

<u>where</u>:

$$\overline{F}_3 = \frac{1}{40}A_1 + \frac{12}{(\bar{a}h)^3}\left[\cosh\frac{\bar{a}h}{2} - \frac{2}{(\bar{a}h)}\sinh\frac{\bar{a}h}{2}\right]A_3$$

$$+ \frac{12}{(\bar{b}h)^3}\left[\cosh\frac{\bar{b}h}{2} - \frac{2}{(\bar{b}h)}\sinh\frac{\bar{b}h}{2}\right]A_5 + \overline{C}_1 \qquad (4\text{-}59.4)$$

in which

$$C_1 = h\overline{C}_1 \qquad (5\text{-}59.5)$$

<u>and</u>:

$$\overline{C}_1 = \left[\frac{1}{\bar{a}h} A_3 + \frac{1}{\bar{b}h} A_5\right] \tag{4-59.6}$$

F_2 is rewritten as:

$$F_2 = h\overline{F}_2 \tag{4-59.7}$$

<u>where</u>:

$$\overline{F}_2 = \left[\frac{2}{\bar{a}h} \sinh \frac{\bar{a}h}{2}\right] A_2 + \left[\frac{2}{\bar{b}h} \sinh \frac{\bar{b}h}{2}\right] A_4 \tag{4-59.8}$$

F_4 is rewritten as:

$$F_4 = h^3\overline{F}_4 \tag{4-59.9}$$

<u>where</u>:

$$\overline{F}_4 = \left[\frac{2}{(\bar{a}h)^3} \sinh \frac{\bar{a}h}{2}\right] A_2 + \left[\frac{2}{(\bar{b}h)^3} \sinh \frac{\bar{b}h}{2}\right] A_4$$

$$+ \overline{C}_2 \tag{4-59.10}$$

in which

$$C_2 = h^2\overline{C}_2 \tag{4-59.11}$$

<u>and</u>:

$$\overline{C}_2 = \left[\frac{2(1+\mu)}{\alpha_m^2 h^2} - \frac{1}{(\bar{a}h)^2}\right] A_2$$

$$+ \left[\frac{2(1+\mu)}{\alpha_m^2 h^2} - \frac{1}{(\bar{a}h)^2}\right] A_4 \qquad (4\text{-}59.12)$$

The function $f_2(z)$ is rewritten as:

$$f_2(z) = h\bar{f}_2(z) \qquad (4\text{-}59.13)$$

<u>where</u>:

$$\bar{f}_2(z) = \left[\frac{1}{2}\left(\frac{z}{h}\right)^2\right] A_1 + \left[\frac{1}{(\bar{a}h)} \sinh \bar{a}z\right] A_2$$

$$+ \left[\frac{1}{(\bar{a}h)} \cosh \bar{a}z\right] A_3 + \left[\frac{1}{(\bar{b}h)} \sinh \bar{b}z\right] A_4$$

$$+ \left[\frac{1}{(\bar{b}h)} \cosh \bar{b}z\right] A_5 + \overline{C}_1 \qquad (4\text{-}59.14)$$

And the function $f_3(z)$ is rewritten as:

$$f_3(z) = h^2\, \bar{f}_3(z) \qquad (4\text{-}59.15)$$

<u>where</u>:

$$\bar{f}_3(z) = \left[\frac{1}{6}\left(\frac{z}{h}\right)^3\right] A_1 + \left[\frac{1}{(\bar{a}h)^2} \cosh \bar{a}z\right] A_2$$

$$+ \left[\frac{1}{(\bar{a}h)^2} \sinh \bar{a}z\right] A_3 + \left[\frac{1}{(\bar{b}h)^2} \cosh \bar{b}z\right] A_4$$

$$+ \left[\frac{1}{(\bar{b}h)^2} \sinh \bar{b}z\right] A_5 + \bar{C}_1 \left(\frac{z}{h}\right) + \bar{C}_2 \qquad (4\text{-}59.16)$$

Having all the functions and constants related to $f_1(z)$ written in a non-dimensional form, one proceeds now to write the other expressions in a non-dimensional form as follows:

The constant β_m appearing in equation (4-38) is rewritten as follows:

$$\beta_m = k_1 \frac{p_o a^4}{Eh^3} \qquad (4\text{-}59.17)$$

where:

$$k_1 = 48 \left[(2 - \mu)(1 + \mu) \frac{\bar{F}_1^2}{144} (m\pi)^4 (h/a)^4 \right.$$

$$\left. + (3 - 2\mu)(1 + \mu) \frac{\bar{F}_1}{12} (m\pi)^2 (h/a)^2 + 1 - \mu^2 \right]$$

$$/ \left\{ (m\pi)^5 \left[\frac{\bar{F}_1}{12} (m\pi)^2 (h/a)^2 + 1\right] \right\} \qquad (4\text{-}59.18)$$

The parameter γ_m appearing in equation (4-43.1) is rewritten as:

$$\gamma_m = \frac{1}{h}\sqrt{(m\pi)^2(h/a)^2 + \frac{12}{\overline{F}_1}}$$

$$= \frac{1}{a}\,\bar{\gamma}_m = \frac{1}{a}\left[\frac{a}{h}\sqrt{(m\pi)^2(h/a)^2 + \frac{12}{\overline{F}_1}}\right] \tag{4-59.19}$$

and $\gamma_m \cdot \frac{b}{2}$ (a term that will be needed later) can be written as:

$$\frac{\gamma_m \cdot b}{2} = \frac{1}{2}\left(\frac{b}{a}\right)\left(\frac{a}{h}\right)\sqrt{(m\pi)^2(h/a)^2 + \frac{12}{\overline{F}_1}} \tag{4-59.20}$$

One also has the terms:

$$\frac{D}{S}\,\alpha_m^2 + 1 = \frac{\overline{F}_1}{6(1-\mu)}\,(m\pi)^2(h/a)^2 + 1 = \frac{1}{k_{11}} \tag{4-59.21}$$

And:

$$\frac{D}{S}\,(\gamma_m^2 - \alpha_m^2) - 1 = \frac{(1+\mu)}{(1-\mu)} = \frac{1}{k_{22}} \tag{4-59.22}$$

4.1.4 Expressions For Moments and Shear Forces in the Plate

Making use of the relations in equations (4-50.1) to (4-50.8), one can write:

$$\varphi_{xm} = A_m\,(-\,\alpha_m \cosh \alpha_m y)$$

$$+ B_m\left(-\frac{2D}{S} \alpha_m^3 \cosh \alpha_m y - \alpha_m^2 y \sinh \alpha_m y \right)$$

$$+ \left(k_{22} \alpha_m \cosh \gamma_m y\right) E_m + \beta_m' \qquad (4\text{-}60)$$

$$\varphi_{ym} = \left(-\alpha_m \sinh \alpha_m y\right) A_m$$

$$+ \left[- \left(\frac{2D}{S} \alpha_m^3 + \alpha_m\right) \sinh \alpha_m y - \alpha_m^2 y \cosh \alpha_m y\right] B_m$$

$$+ \left(k_{22} \gamma_m \sinh \gamma_m y\right) E_m \qquad (4\text{-}61)$$

Substituting appropriate expressions using equations (4-60), (4-61) and (4-33) into equations (4-1), (4-2) and (4-3), results in expressions for the bending and twisting moments as:

$$M_x = \Bigg\{ \left[(1 - \mu)(m\pi)^2 \cosh \alpha_m y\right] A_m$$

$$+ \left[\frac{2\overline{F}_1 m\pi)^4 (h/a)^2}{6} \cosh \alpha_m y - 2\mu(m\pi)^2 \cosh \alpha_m y\right.$$

$$\left. + (1 - \mu)(\alpha_m y)(m\pi)^2 \sinh \alpha_m y\right] B_m$$

$$- \left[K_{22}\left((m\pi)^2 - \frac{\mu}{(h/a)^2} \left((m\pi)^2 (h/a)^2 + \frac{12}{\overline{F}_1}\right) \right) \cosh \gamma_m y\right] E_m$$

$$+ \overline{\beta}_m' + \overline{kp_m} \Bigg\} \left(\frac{p_o a^2}{12(1 - \mu^2)}\right) \sin \alpha_m x \qquad (4\text{-}62)$$

where:

$$\bar{\beta}'_m = \frac{-(m\pi)^2(k_1 - k_2)k_{11}}{12(1 - \mu^2)} \tag{4-62.1}$$

where:

$$kp_m \frac{D}{S} = k_2 \left[\frac{p_o a^4}{Eh^3}\right] \tag{4-62.2}$$

And:

$$k_2 = \frac{2\mu(1 + \mu)\,\bar{\bar{F}}_1^{\,2}}{3(1 - \mu)(m\pi)}\,(h/a)^4 \tag{4-62.3}$$

And:

$$kp_m = \overline{kp_m}\left[\frac{p_o a^2}{12(1 - \mu^2)}\right] \tag{4-62.4}$$

$$\overline{kp_m} = \frac{4\mu(1 + \mu)\,\bar{F}_1}{(m\pi)}\,(h/a)^2 \tag{4-62.5}$$

Similarly M_y can be written as:

$$M_y = \frac{p_o a^2}{12(1 - \mu)^2}\left\{ -(1 - \mu)(m\pi)^2 \cosh \alpha_m y\, A_m \right.$$

$$+ \left[\frac{-2\overline{F}_1 (m\pi)^4 (h/a)^2}{6} \cosh \alpha_m y \right.$$

$$\left. -2(m\pi)^2 \cosh \alpha_m y - \alpha_m y (m\pi)^2 (1 - \mu) \sinh \alpha_m y \right] B_m$$

$$+ k_{22} \left[\left[\frac{1}{(h/a)^2} \left((m\pi)^2 (h/a)^2 + \frac{12}{\overline{F}_1} \right) - \mu (m\pi)^2 \right] E_m \cosh \gamma_m y \right]$$

$$\left. - \mu \overline{\beta}'_m + \overline{kp_m} \right\} \sin \alpha_m x \qquad (4\text{-}63)$$

Similarly for M_{xy}:

$$M_{xy} = \frac{p_o a^2}{24(1 + \mu)} \left\{ 2(m\pi)^2 \sinh \alpha_m y \quad A_m \right.$$

$$+ \left[\frac{4\overline{F}_1 (h/a)^2 (m\pi)^4}{6(1 - \mu)} \sinh \alpha_m y \right.$$

$$\left. + 2(m\pi)^2 \sinh \alpha_m y + 2(m\pi)^2 \alpha_m y \cosh \alpha_m y \right] B_m$$

$$\left. + \left[(-2k_{22} (m\pi) \overline{\gamma}_m \sinh \gamma_m y \right] E_m \right\} \cos \alpha_m x \qquad (4\text{-}64)$$

Similarly the shear force Q_x can be written as:

$$Q_x = \frac{p_o a}{12(1 - \mu^2)} \left\{ - 2(m\pi)^3 \cosh \alpha_m y \quad B_m \right.$$

$$+ \left[\frac{12k_{22}(m\pi)}{(h/a)^2\overline{F}_1} \cosh \gamma_m y \right] E_m$$

$$+ \frac{6(1 - \mu)(m\pi)}{\overline{F}_1(h/a)^2} \left(\overline{\beta}_m + \overline{\beta}_m' \right) \Bigg\} \cos \alpha_m x \qquad (4\text{-}65)$$

<u>where</u>:

$$\beta_m D = k_1 \left[\frac{p_o a^4}{Eh^3} \right] D$$

$$= \overline{\beta}_m \left[\frac{p_o a}{12(1 - \mu^2)} \right]$$

<u>or</u>:

$$\overline{\beta}_m = k_1 \qquad (4\text{-}65.1)$$

The expression for Q_y can be written as:

$$Q_y = \left[\frac{p_o a}{12(1 - \mu^2)} \right] \Bigg\{ \left(-2(m\pi)^3 \sinh \alpha_m y \right) B_m$$

$$+ \left(\frac{12k_{22}(\overline{\gamma}_m)}{\overline{F}_1(h/a)^2} \sinh \gamma_m y \right) E_m \Bigg\} \sin \alpha_m x \qquad (4\text{-}66)$$

4.2 Solution of the Inplane Problem

4.2.1 Formulation in Terms of Average Inplane Displacements $\bar{u}$ and $\bar{v}$

To start with, expressions for average inplane displacements $\bar{u}$ and $\bar{v}$ are derived as follows:

<u>Define</u>:

$$\bar{u} = \frac{1}{h} \int_{-h/2}^{+h/2} u dz \tag{4-67}$$

<u>and</u>:

$$\bar{v} = \frac{1}{h} \int_{-h/2}^{+h/2} v dz \tag{4-68}$$

Substituting for u from equation (3-19) into equation (4-67), one obtains

$$\bar{u} = u_o + \frac{F_2}{Gh} Q_x - \frac{F_4}{Eh} \frac{\partial p}{\partial x} \tag{4-69}$$

Similarly substituting for v from equation (3-20) into equation (4-68), yields

$$\bar{v} = v_o + \frac{F_2}{Gh} Q_x - \frac{F_4}{Eh} \frac{\partial p}{\partial y} \tag{4-70}$$

Noting that:

$$M = M_x + M_y$$

Then from equations (4-1), (4-2) and the above equation, one obtains:

$$M = D\left[(1+\mu)\left(\frac{\partial\varphi_x}{\partial x} + \frac{\partial\varphi_y}{\partial y}\right) + 2Kp\right] \qquad (4\text{-}71.1)$$

Thus:

$$\frac{\partial^2 M}{\partial x^2} = D\left[(1+\mu)\left(\frac{\partial^3\varphi_x}{\partial x^3} + \frac{\partial^3\varphi_y}{\partial x^2 y}\right) + 2K\frac{\partial^2 p}{\partial x^2}\right] \qquad (4\text{-}71.2)$$

$$\frac{\partial^2 M}{\partial y^2} = D\left[(1+\mu)\left(\frac{\partial^3\varphi_x}{\partial x\partial y^2} + \frac{\partial^3\varphi_y}{\partial y^3}\right) + 2K\frac{\partial^2 p}{\partial y^2}\right] \qquad (4\text{-}71.3)$$

Similarly using equations (4-69), (4-70), one has:

$$\frac{\partial u_o}{\partial x} + \mu\frac{\partial v_o}{\partial y} = \left(\frac{\partial\bar{u}}{\partial x} + \mu\frac{\partial\bar{v}}{\partial y}\right)$$

$$+ \frac{F_4}{Eh}\left(\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial^2 p}{\partial y^2}\right)$$

$$- \frac{F_2}{Gh}\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right). \qquad (4\text{-}71.4)$$

Also:

$$\frac{1}{G}\left(\frac{\partial Q_x}{\partial x} + \mu\,\frac{\partial Q_y}{\partial y}\right) = \frac{1}{F_1}\left[\left(\frac{\partial \varphi_x}{\partial x} + \mu\,\frac{\partial \varphi_y}{\partial y}\right) + \left(\frac{\partial^2 \bar{w}}{\partial x^2} + \mu\frac{\partial^2 \bar{w}}{\partial y^2}\right)\right] \tag{4-71.5}$$

Using the previous expressions and equation (3-22), the stress σ_x can be written in terms of average displacements $\bar{w}$, $\bar{u}$, $\bar{v}$ and average rotations φ_x and φ_y as follows:

$$\sigma_x = \frac{E}{(1 - \mu^2)}\Bigg[\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \mu\frac{\partial^2 \bar{w}}{\partial y^2}\right)\left[-z + \left(f_1(z) - \frac{F_2}{h}\right)\frac{1}{F_1}\right]$$

$$+ \left(\frac{\partial^3 \varphi_x}{\partial x^3} + \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \mu\frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \mu\frac{\partial^3 \varphi_y}{\partial y^3}\right)$$

$$\left[D(1 + \mu)\left(-\frac{z}{R} + \frac{2\mu z^3}{Eh^3}\right)\right]$$

$$+ \left[\frac{\partial \varphi_x}{\partial x} + \mu\,\frac{\partial \varphi_y}{\partial y}\right]\left(\left(f_1(z) - \frac{F_2}{h}\right)\frac{1}{F_1}\right)$$

$$+ \left(\frac{\partial^2 p}{\partial x^2} + \mu\frac{\partial^2 p}{\partial y^2}\right)\left[\frac{z}{N} - \frac{f_3(z)}{E} + \frac{F_4}{Eh}\right.$$

$$\left.+ 2KD\left(-\frac{z}{R} + \frac{2\mu z^3}{Eh^3}\right)\right]\Bigg]$$

$$+ \left[\frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \right] \Bigg]$$

$$+ \frac{\mu p}{(1 - \mu)} f_1(z) \qquad (4\text{-}72)$$

$$\sigma_y = \frac{E}{(1 - \mu^2)} \Bigg[\left(\frac{\partial^2 \bar{w}}{\partial y^2} + \mu \frac{\partial^2 \bar{w}}{\partial x^2} \right) \left[-z + \frac{1}{F_1} \left(f_1(z) - \frac{F_2}{h} \right) \right]$$

$$+ \left(\frac{\partial^3 \varphi_y}{\partial y^3} + \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \mu \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \mu \frac{\partial^3 \varphi_x}{\partial x^3} \right)$$

$$\left[D(1 + \mu) \left(- \frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right) \right]$$

$$+ \left[\frac{\partial \varphi_y}{\partial y} + \mu \frac{\partial \varphi_x}{\partial x} \right] \left(\frac{1}{F_1} \left(f_1(z) - \frac{F_2}{h} \right) \right)$$

$$+ \left(\frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \right) \left[\frac{z}{N} - \frac{f_3(z)}{E} + \frac{F_4}{Eh} \right.$$

$$+ 2KD \left(-\frac{z}{R} + \frac{2\mu z^3}{Eh^3} \right) \Bigg]$$

$$+ \left[\frac{\partial \bar{v}}{\partial y} + \mu \frac{\partial \bar{u}}{\partial x} \right] \Bigg]$$

$$+ \frac{\mu p}{(1 - \mu)} f_1(z) \qquad (4\text{-}73)$$

Noting that:

$$\frac{\partial u_o}{\partial y} + \frac{\partial vo}{\partial x} = \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) + \frac{2F_4}{Eh}\frac{\partial^2 p}{\partial x \partial y} - \frac{F_2}{Gh}\left(\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x}\right) \tag{4-73.1}$$

$$\frac{\partial Q_x}{\partial y} + \frac{\partial Q_y}{\partial x} = \frac{G}{F_1}\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} + 2\frac{\partial^2 \bar{w}}{\partial x \partial y}\right) \tag{4-73.2}$$

$$\frac{\partial^2 M}{\partial x \partial y} = D\left[(1+\mu)\left(\frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + \frac{\partial^3 \varphi_y}{\partial x \partial y^2}\right) + 2K\frac{\partial^2 p}{\partial x \partial y}\right] \tag{4-73.3}$$

and

$$\frac{\partial^2 w_o}{\partial x \partial y} = \frac{\partial^2 \bar{w}}{\partial x \partial y} - \frac{1}{N}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{R}\frac{\partial^2 M}{\partial x \partial y}$$

Using the above relations into equation (3-24), we get for τ_{xy} :

$$\tau_{xy} = G\left[\frac{\partial^2 \bar{w}}{\partial x \partial y}\left[-2z + \frac{2}{F_1}\left(f_1(z) - \frac{F_2}{h}\right)\right]\right.$$

$$+ \left[\frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + \frac{\partial^3 \varphi_y}{\partial x \partial y^2}\right]\left[(1+\mu)D\left(\frac{-2z}{R} + \frac{4\mu z^3}{Eh^3}\right)\right]$$

$$+ \left[\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right]\left(\frac{1}{F_1}\left(f_1(z) - \frac{F_2}{h}\right)\right)$$

$$+ \left(\frac{\partial^2 p}{\partial x \partial y}\right)\left[\frac{2z}{N} - \frac{2f_3(z)}{E} + 2\frac{F_4}{Eh}\right.$$

$$\left. + 2KD \left(\frac{-2z}{R} + \frac{4\mu z^3}{Eh^3}\right)\right]$$

$$+ \left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right) \Bigg] \tag{4-74}$$

Using equations (3-36) yields expressions for inplane stress resultant N_x:

$$N_x = \frac{E}{(1 - \mu^2)}\left[h \left(\frac{\partial \bar{u}}{\partial x} + \mu\frac{\partial \bar{v}}{\partial y}\right)\right] + \frac{\mu p}{(1 - \mu)} F_2 \tag{4-75}$$

Similarly using equation (4-73) into second of equations (3-3b) yields:

$$N_y = \frac{E}{(1 - \mu^2)}\left[h \left(\mu\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right)\right] + \frac{\mu p}{(1 - \mu)} F_2 \tag{4-76}$$

The expression for N_{xy} is given by

$$N_{xy} = Gh\left[\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x}\right] \tag{4-77}$$

Using equations (4-75), (4-76), and (4-77) into the inplane equilibrium equations (3-37) and (3-39), yields the inplane governing equations in terms of average inplane displacements $\bar{u}$, $\bar{v}$:

$$\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{(1+\mu)}{2}\frac{\partial^2 \bar{v}}{\partial x \partial y} + \frac{(1-\mu)}{2}\frac{\partial^2 \bar{u}}{\partial y^2}$$

$$= \frac{-\mu(1+\mu)\,F_2}{Eh}\frac{\partial p}{\partial x} \qquad (4\text{-}78)$$

<u>And</u>:

$$\frac{\partial^2 \bar{v}}{\partial y^2} + \frac{(1+\mu)}{2}\frac{\partial^2 \bar{u}}{\partial x \partial y} + \frac{(1-\mu)}{2}\frac{\partial^2 \bar{v}}{\partial x^2}$$

$$= \frac{-\mu(1+\mu)\,F_2}{Eh}\frac{\partial p}{\partial y} \qquad (4\text{-}79)$$

4.2.2 Solution for $\bar{u}$ and $\bar{v}$:

It can be shown that the inplane governing equations (4-78) and (4-79) can be uncoupled for $\bar{u}$ and $\bar{v}$ to give:

$$\Delta^2\,\{\bar{u}\} = k_3\,\frac{\partial}{\partial x}\,\{\Delta p\} \qquad (4\text{-}80)$$

<u>And</u>:

$$\Delta^2\,\{\bar{v}\} = k_3\,\frac{\partial}{\partial y}\,\{\Delta p\} \qquad (4\text{-}81)$$

<u>where</u>:

$$k_3 = \frac{-\mu(1+\mu)F_2}{Eh} \qquad (4\text{-}81.1)$$

Since from equation (4-33):

$$p = \sum p_m \sin \alpha_m x$$

thus

$$\frac{\partial}{\partial x} \Delta p = \sum - \alpha^3{}_m \, p_m \cos \alpha_m x \qquad (4\text{-}81\text{-}2)$$

Assume that $\bar{u}$ will be of the following form:

$$\bar{u} = \sum \bar{u}_m(y) \cos \alpha_m x \qquad (4\text{-}82)$$

Then: $\Delta^2\bar{u}$ from equation (4-82) is:

$$\Delta^2\bar{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 \bar{u}$$

$$= \sum \left[\alpha_m^4 \, \bar{u}_m - 2\alpha_m^2 \frac{d^2\bar{u}_m}{dy^2} + \frac{d^4\bar{u}_m}{dy^4} + \right] \cos \alpha_m x \qquad (4\text{-}82.1)$$

Substituting equations (4-82.1), (4-81.2) into equation (4-80) yields the governing equation for $\bar{u}$ as

$$\frac{d^4\bar{u}_m}{dy^4} - 2\alpha^2 m \frac{d^2\bar{u}_m}{dy^2} + \alpha^4{}_m \, \bar{u}_m = -\alpha^3{}_m \, p_m \, k_3 \qquad (4\text{-}82.2)$$

The solution of the above linear differential equation is given by

$$\bar{u}_m = \bar{u}_p + \bar{u}_h \tag{4-82.3}$$

From equation (4-82.2) the particular solution for $\bar{u}$ may be shown to be

$$\bar{u}_p = - k_3 \frac{p_m}{\alpha_m} \tag{4-82.4}$$

It can be shown that $\bar{u}_h$ will be of the following form:

$$\bar{u}_h = C_1 \cosh \alpha_m y + C_2 \alpha_m y \sinh \alpha_m y + C_3 \alpha_m y \cosh \alpha_m y + C_4 \sinh \alpha_m y \tag{4-82.5}$$

Asuming that $\bar{u}$ will be symmetric with respect to the x-axis, then:

$$C_3 = C_4 = 0$$

and equation (4-82.3) yields for $\bar{u}_m$ the expression

$$\bar{u}_m = \left[C_1 \cosh \alpha_m y + C_2 \alpha_m y \sinh \alpha_m y + \bar{u}_p\right] \tag{4-83}$$

<u>Similarly</u>:

$$\bar{v}_m = \left[C'_3 \sinh \alpha_m y + C'_4 \alpha_m y \cosh \alpha_m y\right] \tag{4-84}$$

Note that $\bar{v}$ is antisymmetric with respect to the x-axis.

To find relations between the constants in $\bar{u}$ and those in $\bar{v}$, appropriate expressions using equations (4-83) and (4-84) are substituted into equation (4-78). This results in:

$$C'_3 = C_1 - k_4 C_2 \qquad (4\text{-}84\text{-}1)$$

$$C'_4 = C_2$$

<u>where</u>:

$$k_4 = \frac{1 + k_1}{k_2} \qquad (4\text{-}84\text{-}2)$$

<u>and</u>:

$$k_1 = \frac{1 - \mu}{2} \qquad (4\text{-}84\text{-}3)$$

$$k_2 = \frac{1 + \mu}{2} \qquad (4\text{-}84\text{-}4)$$

Thus equation (4-84) can be rewritten for $\bar{v}_m$ as:

$$\bar{v}_m = \left[C_1 (\sinh \alpha_m y) + C_2 (\alpha_m y \cosh \alpha_m y - k_4 \sinh \alpha_m y) \right] \qquad (4\text{-}85)$$

4.3 Boundary Conditions for the Bending Problem

The plate will be always simply supported along the edges at x = 0 and x = a. The edges at y = ± b/2 can be simply supported, clamped, or free.

Case I:* *A Plate Uniformly Loaded and Simply Supported at y = ± b/2.

For a simply supported edge at y = ± b/2, the boundary conditions that need to be satisfied are:

$$\bar{w}\,(x,\ \pm\ b/2)=0 \tag{4-86.1}$$

$$\varphi_x\,(x,\ \pm\ b/2)=0 \tag{4-86.2}$$

$$M_y\,(x,\ \pm\ b/2)=0 \tag{4-86.3}$$

Using equation (4-38) for $\bar{w}$, boundary condition in equation (4-86.1) gives:

$$A_m\left(\cosh\frac{\alpha_m b}{2}\right) + B_m\left(\frac{\alpha_m b}{2}\sinh\frac{\alpha_m b}{2}\right)$$

$$+ E_m\left(\cosh\frac{\gamma_m b}{2}\right) = -\beta_m \tag{4-87}$$

From equation (4-17), one has:

$$\varphi_x = -\frac{\partial\bar{w}}{\partial x} + \frac{Q_x}{S}$$

but :

$$\left.\frac{\partial \bar{w}}{\partial x}\right|_{y = \pm b/2} = 0 \qquad (\text{ since } \bar{w}(x, \pm\ b/2) = 0)$$

Thus $\varphi_x(x,\ \pm\ b/2) = 0$ implies that $\frac{Q_x}{S}(x,\ \pm\ b/2) = 0$ *

From equation (4-65), one has:

$$\left[2(m\pi)^3 \cosh \frac{\alpha_m b}{2}\right] B_m - \left[\frac{12k_{22}(m\pi)}{(h/a)^2\ \bar{F}_1} \cosh \frac{\gamma_m b}{2}\right] E_m$$

$$= \frac{6(1 - \mu)m\pi}{\bar{F}_1 (h/a)^2} (\bar{\beta}_m + \bar{\beta}'_m) \tag{4-88}$$

For the boundary condition in equation (4-86.3), one gets from equation (4-1):

$$M_y\Big|_{y = \pm b/2} = D\left[\frac{\partial \varphi_y}{\partial y} + Kp_m\right] \tag{4-88.1}$$

The term $\left.\frac{\partial \varphi_x}{\partial x}\right|_{y = \pm b/2} = 0$ is missing in equation (4-88.1) *

since $\varphi_x(x,\ \pm\ b/2) = 0$ which implies that

----

* Such modifications in boundary conditions are necessary to avoid effects of ill conditioning

$$\left.\frac{\partial \varphi_x}{\partial x}\right|_{y = \pm b/2} = 0$$

Thus expansion of (4-88.1) results in

$$\left[-(m\pi)^2 \cosh \frac{\alpha_m b}{2}\right] A_m$$

$$- \left[2(m\pi)^2 \cosh \frac{\alpha_m b}{2} + \frac{\alpha_m b}{2}(m\pi)^2 \sinh \frac{\alpha_m b}{2}\right] B_m$$

$$- \left[a^2 \gamma^2_m \cosh \frac{\gamma_m b}{2}\right] E_m + (1 - \frac{2}{\mu}) \overline{Kp_m} = 0 \qquad (4\text{-}89)$$

Case II: Plate Uniformity Loaded and Clamped at $y = \pm b/2$

$$\overline{w} (x, \pm b/2)=0 \qquad (4\text{-}87)$$

$$\varphi_x (x, \pm b/2)=0 \qquad (4\text{-}88)$$

$$\varphi_y (x, \pm b/2)=0 \qquad (4\text{-}89.1)$$

from equation (4-61) and boundary condition in equation (4-89.1), one has:

$$\left[\alpha_m h \sinh \frac{\alpha_m b}{2}\right] A_m + \left[\left(\frac{2(m\pi)^3 (h/a)^3 F_1}{6(1 - \mu)} \right.\right.$$

$$\left. + (m\pi)(h/a) \right) \sinh \frac{\alpha_m b}{2} + \frac{\alpha_m b}{2}(m\pi)(h/a)\cosh \frac{\alpha_m b}{2} \left.\right] B_m$$

$$- \left[k_{22}(\gamma_m h) \sinh \frac{\gamma_m b}{s} \right] E_m = 0 \qquad (4\text{-}90)$$

Case III: Plate Uniformity Loaded and Free at $y = \pm b/2$

Boundary conditions for this case are:

$$M_y (x, \pm b/2) = 0 \qquad (4\text{-}91.1)$$

$$Q_y (x, \pm b/2) = 0 \qquad (4\text{-}91.2)$$

$$M_{xy} (x, \pm b/2) = 0 \qquad (4\text{-}91.3)$$

Once again, ill conditioning of the non-modified system led to numerical problems. The following equivalent set of equations were used instead:

$$M_y (x, \pm b/2) = 0 \qquad (4\text{-}91.4)$$

$$Q_y - \frac{\partial M_{xy}}{\partial x} = 0 \qquad (4\text{-}91.5)$$

$$Q_y = 0 \qquad (4\text{-}91.6)$$

Note: If $Q_y(x, \pm b/2) = 0$ in equation (4-91.6) then equation (4-91.5) implies that:

$\left.\frac{\partial M_{xy}}{\partial x}\right|_{y = \pm b/2} = 0$ or $M_{xy} (x, \pm b/2) = 0$ (which is equation 4-91.3)

Also from equation (4-65) for Q_x:

$$\frac{\partial Q_x}{\partial y} = \frac{p_o a}{12(1-\mu^2)}\left\{\left[-\ 2(m\pi)^3\alpha_m \sinh \alpha_m y\right] B_m\right.$$

$$\left. + \left[\frac{12k_{22}(m\pi)\ \gamma_m}{\bar{F}_1(h/a)^2} \sinh \gamma_m y\right] E_m\right\}\cos \alpha_m x$$

$$= \alpha_m\ Y(y)\ \cos\ \alpha_m\ x \tag{4-91.7}$$

where

$$Y(y) = \frac{p_o a}{12(1-\mu^2)}\left\{\left[-\ 2(m\pi)^3 \sinh \alpha_m y\right] B_m\right.$$

$$\left. + \left[\frac{12k_{22}\ \bar{\gamma}_m}{\bar{F}_1(h/a)^2} \sinh \gamma_m y\right] E_m\right\} \tag{4-91.8}$$

Also from previous work

$$Q_y = Y(y)\ \sin\ \alpha_m\ x \tag{4-66}$$

Thus the boundary condition that $Q_y(x, \pm\ b/2) = 0$ implies that $Y(\pm\ b/2) = 0$. (from equation (4-66))

Thus equation (4-91.7) yields that

$$\left.\frac{\partial Q_x}{\partial y}\right|_{y = \pm b/2} = 0$$

From the above (and from equation (4.21) for M_{xy}) it is seen that:

$$\left.\frac{\partial M_{xy}}{\partial x}\right|_{y = \pm b/2} = D(1 - \mu) \left.\frac{\partial^3 \overline{w}}{\partial x^2 \partial y}\right|_{y = \pm b/2}$$

$$\frac{\partial M_{xy}}{\partial x} = \frac{p_o a^4}{12(1 - \mu^2)}\Big[-\Big((1 - \mu)\, \alpha_m^3 \sinh \alpha_m y \Big) A_m$$

$$-B_m\Big[(1 - \mu)\, \alpha_m^3 \sinh \alpha_m y + (1 - \mu)\alpha_m^4 y \cosh \alpha_m y \Big]$$

$$- \Big[(1 - \mu)\alpha^2 m\, \gamma_m \sinh \gamma_m y \Big] E_m \Big] \sin \alpha_m x \qquad (4\text{-}91.9)$$

Substituting for Q_y from equation (4-66) and for $\frac{\partial M_{xy}}{\partial x}$ from equation (4-91.9) into boundary condition in equation (4-91.5) yields

$$\left[(1 - \mu)(m\pi)^3 \sinh\frac{\alpha_m b}{2}\right] A_m + B_m \left[- (1 + \mu)(m\pi)^3 \sinh\frac{\alpha_m b}{2}\right.$$

$$\left. + (1 - \mu)\frac{\alpha_m b}{2} (m\pi)^3 \cosh \frac{\alpha_m b}{2}\right]$$

$$+ E_m \left[12\frac{k_{22}}{\overline{F}_1 (h/a)^2} + (1 - \mu) (m\pi)^2\right] a\gamma_m \sinh \frac{\gamma_m b}{2} = 0 \qquad (4\text{-}92)$$

Consider boundary condition as given by equation (4-91.6):

$$Q_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = 0 \qquad (4\text{-}93.1)$$

Since $\frac{\partial^2 \overline{w}}{\partial x \partial y} \Big|_{y = \pm b/2} = 0$, it can be shown that

$$\frac{\partial M_y}{\partial y} = - D \left[\frac{\partial^3 \overline{w}}{\partial y^3}\right] + \frac{h^2 \overline{F}_1}{6} \frac{\partial^2 Q_y}{\partial y^2} \qquad (4\text{-}93.2)$$

Also it can be shown that:

$$\frac{\partial^3 \overline{w}}{\partial y^3} = \left[\left(2\alpha_m^3 \sinh \alpha_m y\right) B_m + \left(\frac{12}{\overline{F}_1 (h/a)^2} \gamma_m \sinh \gamma_m y\right) E_m\right] \sin \alpha_m x \qquad (4\text{-}93.3)$$

<u>and</u>:

$$\frac{\partial^2 Q_y}{\partial y^2} = \frac{p_o a^4}{12(1 - \mu^2)} \left\{\left(\frac{144\ k_{22}}{\overline{F}_2^2 (h/a)^4} \gamma_m \sinh \gamma_m y\right) E_m\right\} \sin \alpha_m x \qquad (4\text{-}93.4)$$

<u>Substituting</u> <u>from</u> <u>equations</u> (<u>4-93</u>.<u>3</u>), (<u>4-93</u>.<u>4</u>) <u>into</u> (<u>4-93</u>.<u>2</u>) and then into (4-93.1), we get:

$$Q_y = \Big|_{y = \pm b/2} = 0$$

$$\left[(1 - \mu)(m\pi)^3 \sinh\frac{\alpha_m b}{2}\right] A_m + B_m \left(- (1 + \mu)(m\pi)^3 \sinh\frac{\alpha_m b}{2}\right.$$

$$+ (1 - \mu)\frac{\alpha_m b}{2} (m\pi)^3 \cosh\frac{\alpha_m b}{2} \Bigg]$$

$$+ E_m \left[\frac{-12(1 - 2k_{22})}{\overline{F}_1 (h/a)^2} + (1 - \mu)(m\pi)^2\right] a\, \gamma_m \sinh \frac{\gamma_m b}{2} = 0 \qquad (4\text{-}94)$$

4.4 Boundary Conditions for the Inplane Problem

One notes that due to the form of $\overline{v}$ which is due to the method of obtaining solution by Levy method that:

$$\overline{v}(0,y) = \overline{v}(a,y) = 0 \qquad (4\text{-}95)$$

So due to the use of the Levy method for solution , the edges at x = 0 and at x = a are always free to stretch in the x-direction.Thus N_x will vanish at the edges at x = 0 and at x = a. For this reason boundary conditions on inplane displacements can be specified on the edges at y = ± b/2. We have two cases:

Case I- Edges at y = ± b/2 clamped against stretching:

In this case the following boundary conditions apply:

$$\overline{u}\,(x, \pm\, b/2)=0 \qquad (4\text{-}96.1)$$

$$\overline{v}(x, \pm\, b/2)=0 \qquad (4\text{-}96.2)$$

Substituting from equations (4-83) and (4-85) into the above boundary conditions yields

$$\left[\cosh \frac{\gamma_m b}{2}\right] C_1 + \left[\frac{\alpha_m b}{2} \sinh \frac{\gamma_m b}{2}\right] C_2 = -\bar{u}_p \tag{4-96.3}$$

and

$$\left[\sinh \frac{\gamma_m b}{2}\right] C_1 + \left[\frac{\alpha_m b}{2} \cosh \frac{\alpha_m b}{2} - k_4 \sinh \frac{\gamma_m b}{2}\right] C_2 = 0 \tag{4-96.4}$$

Case II- Edges at y = ± b/2 are free to stretch in the y-direction only:

In this case the following boundary conditions apply:

$$N_y = (x, \pm b/2)=0 \tag{4-96.5}$$

$$\bar{u}(x, \pm b/2)=0 \tag{4-96.6}$$

From boundary condition as given by equation (4-96.5), and making use of equation (4-76) yields

$$C_1 (1 - \mu) \alpha_m \cosh \alpha_m y + C_2\Big(1 - k_4) \alpha_m \cosh \alpha_m y$$

$$+ (1 - \mu) \alpha_m^2 y \sinh \alpha_m y\Big) = - \frac{k_6}{k_5} p_m + \mu \alpha_m \bar{u}_p \tag{4-96.7}$$

where:

$$k_5 = \frac{Eh}{(1 - \mu^2)} \tag{4-96.8}$$

and:

$$k_6 = \frac{\mu F_2}{(1 - \mu)} \tag{4-96.9}$$

4.5 Expressions for Stresses in a Non-dimensional Form

The stress σ_x can be written as:

$$\sigma_x = \bar{\sigma}_x\left(\frac{p_o}{(h/a)^2}\right) \tag{4-97.1}$$

Similarly other stresses can be written as:

$$\sigma_y = \bar{\sigma}_y\left(\frac{p_o}{(h/a)^2}\right) \tag{4-97.2}$$

$$\tau_{xy} = \bar{\tau}_{xy}\left(\frac{p_o}{(h/a)^2}\right) \tag{4-97.3}$$

$$\tau_{xz} = \bar{\tau}_{xz}\left(\frac{p_o}{(h/a)}\right) \tag{4-97.4}$$

$$\tau_{yz} = \bar{\tau}_{yz}\left(\frac{p_o}{(h/a)}\right) \tag{4-97.5}$$

where:

$$\bar{\sigma}_x = \left\{\frac{1}{(1 - \mu^2)}\left[\bar{I}_1(y)\bar{g}_1(z) + \frac{\mu}{12(1 - \mu)}\bar{I}_2(y)\bar{g}_2(z)\right.\right.$$

$$+ \bar{I}_3(y)\bar{g}_3(z) + \bar{I}_4(y)\bar{g}_4(z) + \bar{I}_7(y)\bar{g}_2(z)$$

$$\left. + \bar{I}_5(y)\Big] + \bar{I}_6\, f_1(z) \right\} \sin \alpha_m x \qquad (4\text{-}98)$$

$$\bar{\sigma}_y = \left\{ \frac{1}{(1-\mu^2)} \Big[\bar{J}_1(y)\bar{g}_1(z) + \frac{\mu}{12(1-\mu)} \bar{J}_2(y)\bar{g}_2(z) \right.$$

$$+ \bar{J}_3(y)\bar{g}_3(z) + \bar{J}_4(y)\bar{g}_4(z) + \bar{J}_7(y)\bar{g}_2(z)$$

$$\left. + \bar{J}_5(y)\Big] + \bar{J}_6\, f_1(z) \right\} \sin \alpha_m x \qquad (4\text{-}99)$$

$$\bar{\tau}_{xy} = \frac{1}{(1+\mu)} \Big[\bar{L}_1(y)\bar{g}_1(z) + I_2(y)\bar{g}_2(z)\frac{\mu}{12(1-\mu)}$$

$$+ \frac{1}{2} \bar{L}_3(y)\bar{g}_3(z) + \bar{L}_4(y)\bar{g}_4(z)$$

$$+ \bar{L}_6(y)\bar{g}_2(z) + \bar{L}_5(y) \Big] \cos \alpha_m x \qquad (4\text{-}100)$$

<u>And</u>:

$$I_1(y) = \frac{\partial^2 \bar{w}}{\partial x^2} + \mu \frac{\partial^2 \bar{w}}{\partial y^2} \qquad (4\text{-}101.01)$$

$$I_2(y) = \frac{\partial^3 \varphi_x}{\partial x^3} + \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \mu \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \frac{\partial^3 \varphi_y}{\partial y^3} \qquad (4\text{-}101.02)$$

$$I_3(y) = \frac{\partial \varphi_x}{\partial x} + \mu \frac{\partial \varphi_y}{\partial y} \qquad (4\text{-}101.03)$$

$$I_4(y) = \frac{\partial^2 p}{\partial x^2} + \mu \frac{\partial^2 p}{\partial y^2} \tag{4-101.04}$$

$$I_5(y) = \frac{\partial \bar{u}}{\partial x} + \mu \frac{\partial \bar{v}}{\partial y} \tag{4-101.05}$$

$$I_6(y) = \frac{4\mu(h/a)^2}{(1 - \mu)(m\pi)} \tag{4-101.06}$$

$$I_7(y) = \frac{-\ 4\mu^2(m\pi)\ \bar{F}_1}{6(1 - \mu)} (h/a)^4 \tag{4-101.07}$$

$$\bar{I}_1(y) = a^2 I_1(y) \tag{4-101.08}$$

$$\bar{I}_2(y) = a^2 h^2 I_2(y) \tag{4-101.09}$$

$$\bar{I}_3(y) = a^2 I_3(y) \tag{4-101.10}$$

$$\bar{I}_4(y) = h^2 I_4(y) \tag{4-101.11}$$

$$\bar{I}_5(y) = A \left(\frac{h^2}{a^2}\right) I_5(y) \tag{4-101.12}$$

And:

$$\bar{J}_1(y) = a^2 J_1(y) \tag{4-101.13}$$

$$\bar{J}_2(y) = a^2 h^2 J_2(y) \tag{4-101.14}$$

$$\bar{J}_3(y) = a^2 J_3(y) \quad (4\text{-}101.15)$$

$$\bar{J}_4(y) = h^2 J_4(y) \quad (4\text{-}101.16)$$

$$\bar{J}_5(y) = h \left(\frac{h^2}{a^2}\right) J_5(y) \quad (4\text{-}101.17)$$

$$\bar{J}_6(y) = \bar{I}_6(y) \quad (4\text{-}101.18)$$

$$\bar{J}_7(y) = \bar{I}_7(y) \quad (4\text{-}101.19)$$

<u>where</u>:

$$J_1(y) = \frac{\partial^2 \overline{w}}{\partial y^2} + \mu \frac{\partial^2 \overline{w}}{\partial x^2} \quad (4\text{-}101.20)$$

$$J_2(y) = \frac{\partial^3 \varphi_y}{\partial y^3} + \frac{\partial^3 \varphi_x}{\partial x \partial y^2} + \mu \frac{\partial^3 \varphi_y}{\partial x^2 \partial y} + \frac{\partial^3 \varphi_x}{\partial x^3} \quad (4\text{-}101.21)$$

$$J_3(y) = \frac{\partial \varphi_y}{\partial y} + \mu \frac{\partial \varphi_x}{\partial x} \quad (4\text{-}101.22)$$

$$J_4(y) = \frac{\partial^2 p}{\partial y^2} + \mu \frac{\partial^2 p}{\partial x^2} \quad (4\text{-}101.23)$$

$$J_5(y) = \frac{\partial \overline{v}}{\partial y} + \mu \frac{\partial \overline{u}}{\partial x} \quad (4\text{-}101.24)$$

<u>Also</u>:

$$L_1(y) = \frac{\partial^2 \overline{w}}{\partial x \partial y} \tag{4-102.01}$$

$$L_2(y) = \frac{\partial^3 \varphi_x}{\partial x^2 \partial y} + \frac{\partial^3 \varphi_y}{\partial x \partial y^2} \tag{4-102.02}$$

$$L_3(y) = \frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} \tag{4-102.03}$$

$$L_4(y) = \frac{\partial^2 p}{\partial x \partial y} \tag{4-102.04}$$

$$L_5(y) = \frac{1}{2}\left(\frac{\partial \overline{u}}{\partial y} + \frac{\partial \overline{v}}{\partial x}\right) \tag{4-102.05}$$

$$L_6(y) = L_4(y) \tag{4-102.06}$$

<u>And</u>:

$$\overline{L}_1(y) = a^2 L_1(y) \tag{4-102.07}$$

$$\overline{L}_2(y) = a^2 h^2 L_2(y) \tag{4-102.08}$$

$$\overline{L}_3(y) = a^2 L_3(y)$$

$$\overline{L}_4(y) = 0 \quad (\text{since } L_4(y) = 0\,) \tag{4-102.09}$$

$$\overline{L}_5(y) = a\left(\frac{h^2}{a^2}\right) L_5(y) \tag{4-102.10}$$

$$\bar{L}_6(y) = \bar{L}_4(y) = 0 \qquad (4\text{-}102.11)$$

Also;

$$g_1(z) = h\bar{g}_1(z) \qquad (4\text{-}103.1)$$

$$\bar{g}_1(z) = \left[\frac{1}{\bar{F}_1}\left(f_1(z) - \bar{F}_2\right) - (z/h)\right] \qquad (4\text{-}103.2)$$

$$\bar{g}_2(z) = \frac{\mu}{E}\left[\ 2\left(\frac{z}{h}\right)^3 - \frac{3}{10}\left(\frac{z}{h}\right)\right] \qquad (4\text{-}103.3)$$

$$g_3(z) = \frac{1}{F_1}\left(\ f_1(z) - \frac{F_2}{h}\right) \qquad (4\text{-}103.4)$$

$$\bar{g}_3(z) = \frac{1}{\bar{F}_1}\left[\ f_1(z) - \bar{F}_2\right] \qquad (4\text{-}103.5)$$

$$g_4(z) = \frac{h^2}{E}\ \bar{g}_4(z) \qquad (4\text{-}103.6)$$

$$\bar{g}_4(z) = \left[\ -\ \bar{f}_3(z) + \bar{F}_3\left(\frac{z}{h}\right) + \bar{F}_4\ \right] \qquad (4\text{-}103.7)$$

Chapter 5

APPLICATIONS

5.1 Cylindrical Bending

Two problems are considered to test the validity of the present formulation.

Example 5.1.1

An infinite plate strip of thickness "h" subjected to the stress field:

$$\sigma_z(x,y,-h/2) = -q_o \sin\frac{\pi x}{L} \tag{5.1}$$

is considered first.

An exact elasticity solution exists for this problem [15]. Also this case was used in [10] to evaluate a higher order plate theory.

The dependent variables may be assumed to be in the form:

$$w_o = w_{oo} \sin \frac{\pi x}{L}$$

$$u_o = u_{oo} \cos \frac{\pi x}{L}$$

$$
\begin{aligned}
v_o &= v_{oo} \sin \frac{\pi x}{L} \\
Q_x &= Q_{ox} \cos \frac{\pi x}{L} \\
Q_y &= Q_{oy} \cos \frac{\pi x}{L} \\
\varphi_x &= \varphi_{ox} \cos \frac{\pi x}{L} \\
\varphi_y &= \varphi_{oy} \cos \frac{\pi x}{L} \\
M_x &= M_{ox} \sin \frac{\pi x}{L} \\
M_y &= M_{oy} \sin \frac{\pi x}{L} \\
M_{xy} &= M_{oxy} \sin \frac{\pi x}{L}
\end{aligned}
\tag{5.2}
$$

The boundary conditions are as given by equations (3.42) and (3.46).
Substituting equations (3.66) and (3.67) into equations (3.7), (3.31), (3.32), (3.27.4), (3.27.5), (3.28), (3.29) and (3.30), one may solve for the unknown coefficients in the set of equations (3.67).

The solution for the transverse deflection w_o is given by:

$$
w_o = \frac{p_m}{\alpha_m^4 D} [1 + \frac{(2-\mu)h^3}{12(1-\mu)} \alpha_m^2 F_1 \; - \alpha_m^4 \; D/N
$$

$$+ \frac{\mu h^2 \alpha_m^2}{40(1-\mu)} - \frac{\mu^2 h^5 \alpha_m^4 F_1}{480(1-\mu)^2}$$

$$+ \frac{\mu^2 h^5 \alpha_m^4}{240(1-\mu)^2(1+\mu)} F_1] \sin\alpha_m x \tag{5.3}$$

where

$$\alpha_m = \alpha_1 = \frac{\pi}{L}, \; p_m = p_1 = q_o \text{ (for } m = 1) \tag{5.3.1}$$

Solving for the stress α_x, we get:

$$\sigma_x = \{E\alpha_m^2 w_{oo} \frac{z}{(1-\mu^2)} - \frac{(2-\mu)}{(1-\mu)} p_m f_1(z) + \frac{p_m \alpha_m^2}{(1-\mu^2)} f_3(z)$$

$$- \frac{2\mu\alpha_m^2 M_o}{h^3(1-\mu^2)} z^3 - \frac{p_m}{h(1-\mu^2)} [(\mu^2 - \mu - 2)F_2 + \alpha_m^2 F_4]\} \sin \alpha_m x \tag{5.4}$$

where

$$M_o = M_{ox} + M_{oy} \tag{5.4.1}$$

If one solves the same problem using the shear deformation generalized theory of Panc [9], the expression for σ_x may be shown to be given by

$$\sigma_x = \frac{E\alpha_m^2}{(1-\mu^2)} w_{mo} - \frac{2p_m}{(1-\mu)} \left[f_{1m}(z) + \frac{1}{2} \right] \tag{5.5}$$

where

$$w_{mo} = \frac{p_m}{k_m \alpha_m^4}$$

$$k_m = \frac{2E}{(1-\mu^2)\lambda_m^3} \left[\frac{\lambda_m h}{2} - \tanh \frac{\lambda_m h}{2}\right] \tag{5.6}$$

$$\lambda_m^2 = \frac{2}{(1-\mu)} \alpha_m^2$$

$$f_{1m}(z) = -\frac{1}{2}\left[1 - \frac{\lambda_m z ch(\lambda_m h/2) - sh(\lambda_m z)}{(\lambda_m h/2) ch(\lambda_m h/2) - sh(\lambda_m h/2)}\right]$$

Figure 5.1 shows results for w_o and Figures 5.2 to 5.9 show results for σ_x, as given by the exact solution [15], Panc [9], Baluch [10], and the present work.

The effect of normal strain on w_o becomes very clear for h/L > 1.0 as shown in Figure 5.1 . As h/L increases, the present work gives results which are closest to the exact solution.

The present work, as shown in Figures 5.2 to 5.9, gives the best results for stress σ_x as compared to the exact solution. For h/L > 1.0, previous work by Baluch [10] and Panc [9] failed to give good results for stresses. The present work yields almost exact results even up to h/L = 3.0, which is representative of an extremely thick plate. Figs. 5.4 through 5.9 show that σ_x from the present theory is almost superposed on the exact solution for h/L upto 3.0, whereas the other refined theories yield diverging solutions and which are thus not plotted.

Example 5.1.2

An infinite plate strip of thickness "h" subjected to a uniformly distributed load "p" at $z = -h/2$. For this case, the previous expressions derived for w_o and σ_x in example (5.1.1) are still valid except that for this case:

$$\alpha_m = \frac{m\pi}{L}, p_m = \frac{4p}{m\pi} \qquad m = 1,3,5,7,.., \qquad (5.7)$$

Figure 5.10 shows results for w_o and Figures 5.11 to 5.18 show results for σ_x, as given by the exact solution [15], Panc [9], and the present work.

The effect of normal strain on w_o is again apparent for $h/L > 1.0$ as shown in Figure 5.10. The present work yields w_o which is close to the exact solution as h/L is increased.

The σ_x stresses from the present theory yield results initially indistinguishable from the exact theory for h/L upto as high as 3.0 (Figs.: 5.11 to 5.18).

Figures 5.19 to 5.21 depict the variation of the transverse normal stress σ_z with the ratio h/L. As with the case of σ_x stresses, the present formulation yields results for σ_z almost identical to the exact solution. It is also of interest to note that as the plate becomes thicker, the maximum magnitude of the bending stress σ_x becomes of the same order as that of the transverse normal stress σ_z.

5.2 Examples for Rectangular Plates

A rectangular plate of sides a (along x-axis) and b (along y-axis) loaded uniformly and with the edges at $x=0$, $x=a$ being simply supported was considered. The following cases were chosen to give examples for such isotropic rectangular plates (in all cases considered, Poisson's ratio μ was taken to be 0.3).

NOTE :

In the figures that follow the notation

BC.h/a-I(OR II)

is used to indicate :

BC : Indicates the type of boundary condition

SS : indicates a simply supported edge.

SC : indicates a clamped edge.

SF : indicates a Free edge.

h/a : is the value of (thickness to span) ratio.

I OR II : indicates whether the edges at

$y = \pm b/2$

are not allowed to stretch in the y-direction (I)

OR are allowed to do so (II).

5.2.1 A Square Plate Uniformly Loaded with All Edges Simply Supported (SS) :

The boundary conditions that need to be satisfied for the bending problem for this case are given by equations (4.87), (4.88), and (4.89).

The boundary conditions that need to be satisfied for the inplane problem are given by equations (4.96.3), (4.96.4) for edges at $y = \pm b/2$ not allowed to stretch in the y-direction (Case I) and by equations (4.96-3) and (4.96.7) for edges at $y = \pm b/2$ allowed to stretch in the y-direction only (Case II). Table 5.1 shows the results for deflection $\bar{w}$ obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR) [12], refined theory in reference [11] RTB, and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.2 for M_x and Table 5.3 for M_y).

Also the stress σ_x is obtained and results are compared with results from other theories for Case I in Figures 5.22 to 5.30 and results are shown in Figures 5.31 to 5.43 for Case II.

The variation of the transverse shear stress τ_{xz} is shown in Figures 5.44 to 5.47. The results are in qualitative agreement with the elasticity solution for bending of thick curved bar by force at end [14].

The results shown demonstrate clearly the effect of including the influence of tranverse stresses and strains and normal stress and strain on the deflection and on the resultant moments. This effect becomes very clear as h/a for the plate increases up to as high as h/a = 1.0 .

The graphs for the stresses show the non-linearity in the stresses as h/a ratio increases. Also it is shown clearly in the graphs that the neutral plane is shifted and it does not coincide any more with the mid-plane as CPT and RTR predicts. The magnitude of the inplane stresses σ_x, σ_y, σ_{xy} decreases, as the ratio h/a of the plate increases, to an order of magnitude similar to that of the normal stress σ_z and thus σ_z cannot be neglected for thick plates.

5.2.2 *A Square Plate Uniformly Loaded with Clamped Edges at y =* ±b/2 (SC) :

Table 5.4 shows the results for deflection $\bar{w}$ obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR), refined theory in reference [11] RTB , and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.5 for M_x and Table 5.6 for M_y).

Also the stress σ_x is obtained and results are compared with results from other theories for Case I in Figures 5.48 to 5.53 and results are shown in Figures 5.54 to 5.59 for Case II.

Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made based on the above results for this case (i.e : simple/clamped plate).

5.2.3 A Square Plate Uniformly Loaded with Free Edges at $y = \pm b/2$ (SF) :

Table 5.7 shows the results for deflection $\bar{w}$ obtained by present work RTP and compared with results given by Classical plate theory (CPT) [1], Reissner's plate theory (RTR) [12], refined theory in reference [11] RTB , and FEM in reference [13].

The moments resultants are obtained and results are compared with results given by other theories (Table 5.8 for M_x and Table 5.9 for M_y).

Also the stress σ_x is obtained and results are compared with results from other theories for Case I in Figures 5.60 to 5.66 and results are shown in Figures 5.67 to 5.72 for Case II.

Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made based on the above results for this case (i.e : simple/free plate).

5.2.4 A Square Plate Simply Supported All Around and Loaded With A Line Load At $x = a/2$ (See Figure 5-A) :

Assuming that the plate (simply supported all around) is subjected to a line load at : $x = x_1$,in this case p_m can be shown to be given by :

$$p_m = \frac{2p_o}{a} \sin \frac{m\pi x}{a} \qquad (5.2.4\text{-}1)$$

Table 5.10 shows the results of deflection at center of the plate for this case of loading.

Table 5.11 shows the results of the resultant moment M_x at the center of the plate.

Table 5.12 shows the results of the resultant moment M_y at the center of the plate. The results were compared with results from CPT. Results from both RTR and RTB were not available. The importance of using a refined theory such as the one presented here is clear from the results shown in these tables. For a ratio of h/a as high as 1.0 , the deflection obtaned from this theory is almost 7 times the one obtained by CPT.
Stresses are not shown for this case since the load does not converge when expanded in single Fourier series but rather it's integral converges.

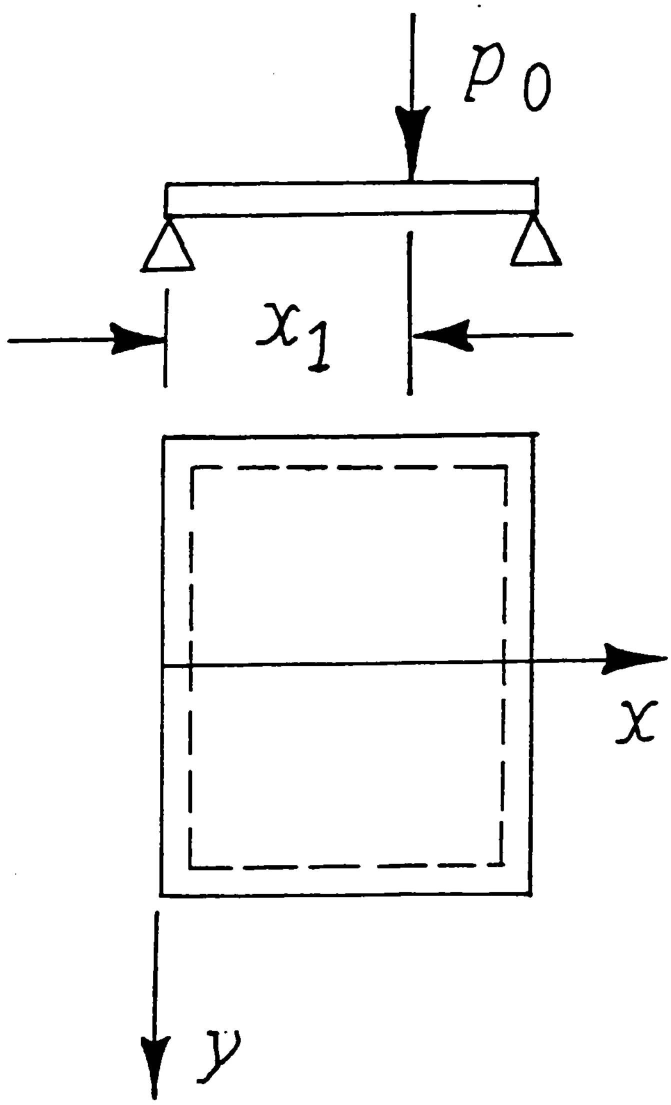


Figure 5-A : Line Load P_o At $x = x_1$

5.2.5 *A Square Plate Simply Supported All Around and Loaded With A Strip Load :*

Assuming that the plate (simply supported all around) is subjected to a strip load of width = u and centered at x = ξ, in this case p_m can be shown to be given by :

$$p_m = \frac{4p_o}{m\pi} \sin \frac{m\pi\xi}{a} \sin \frac{m\pi u}{a} \qquad (5.2.5\text{-}1)$$

Table 5.13 shows the results of deflection at center of the plate for this case of loading.

Table 5.14 shows the results of the resultant moment M_x at the center of the plate.

Table 5.15 shows the results of the resultant moment M_y at the center of the plate. The results were compared with results from CPT. Results from both RTR and RTB were not available. The importance of using a refined theory such as the one presented here is clear from the results shown in these tables. For a ratio of h/a as high as 1.0 , the deflection obtaned from this theory is almost 7 times the one obtained by CPT.

Also the stress σ_x is obtained and results are compared with results from other theories for Case II in Figures 5.73 to 5.77 .
Observations similar to those made for the case of simply supported plate for deflection, resultant moments, and stresses can be made

based on the above results for this case.
Also it may be noted that this case of loading represents a general case of strip loading since the width and center of the strip load can be varied to obtain any case of strip loading including the case of uniformly loaded plate.

For the case of distributed loading on both the top and bottom surfaces of the plate, the problem can be solved by superposition. The problem will be divided into two problems. The first will be a plate loaded at top; and this will be solved as shown in the previous sections on the type of loading (i.e. : a line load, a strip load, or a uniform load). The second problem will be for a plate loaded at the bottom only; and this can be solved by reversing the z-axis (i.e. positive z-axis will be upward). Thus this second problem will be equivalent to the first problem with the z-axis being reversed. The solution for the whole problem will be obtained by superposing solutions from the first and second problems.

5.2.6 A Plot Of w(x,y,z) Across The Plate :

Substituting for $w_o(x,y)$ from equation (3-41) in equation (3-15), the expression for w(x,y,z) can be rewritten as follows:

$$w(x,y,z) = \frac{p(x)}{E} f_2(z) - \frac{6\mu M(x,y) z^2}{Eh^3} + \bar{w}(x,y) - \frac{p(x)}{N} + \frac{M(x,y)}{R} \qquad (5.2.6\text{-}1)$$

Substiuting for N and R from equations (4.7) and (4.8), respectively, in equation (5.2.5-1) and rearranging results in

$$w(x,y,z) = \frac{p(x)}{E}\left\{f_2(z) - F_3\right\} + \frac{M(x,y)}{E}\left\{\frac{3\mu}{10h} - \frac{6\mu z^2}{h^3}\right\} + \bar{w}(x,y)$$

Noting that

$$F_3 = h\bar{F}_3 \qquad (4\text{-}59.3)$$

and

$$f_2(z) = h\bar{f}_2(z) \qquad (4\text{-}59.13)$$

the expression for w(x,y,z) can be rewritten as follows:

$$w(x,y,z) = \frac{1}{E}\left\{ p(x)[\, h\bar{f}_2(z) - h\bar{F}_3 \,] + \frac{3\mu M(x,y)}{h}[\, \frac{1}{10} - 2(\frac{z}{h})^2 \,] \right\}$$

$$+ \bar{w}(x,y)$$

Making use of equation (4.33) for p(x) and noting that

$$\bar{w}(x,y) = \sum_{m=1}^{\infty} \bar{w}_m(y) \sin \alpha_m x \qquad (4.38)$$

$$M_m(y) = M_{xm}(y) + M_{ym}(y) \qquad (3.13)$$

and

$$M_x(x,y) = \sum_{m=1}^{\infty} M_{xm}(y) \sin \alpha_m x \qquad (4\text{-}62)$$

$$M_y(x,y) = \sum_{m=1}^{\infty} M_{ym}(y) \sin \alpha_m y \tag{4-63}$$

the expression for deflection w(x,y,z) can be rewritten in the following form :

$$w(x,y,z) = \sum_{m=1}^{\infty} \frac{p_o a^4}{Eh^3} \Big\{ p_m \left(\frac{h}{a}\right)^4 \left[\bar{f}_2 (z) - \bar{F}_3 \right]$$

$$+ 3 \mu \left(\frac{h}{a}\right)^2 M_m(y) \left[\frac{1}{10} - 2\left(\frac{z}{h}\right)^2 \right]$$

$$+ \overline{W}_m(y) \Big\} \sin \alpha_m x \tag{5.2.6-2}$$

where

$$\overline{W}_m(y) = \frac{Eh^3}{p_o a^4} \overline{w}_m(y) \tag{5.2.6-3}$$

Figures 5.78,79,80 show deflection of TOP surface of the plate given by RTR and RTP for h/a = 0.1, 0.5 and 1.0,respectively .
Figures 5.81,82,83 show deflection of middle surface of the plate given by RTR and RTP for h/a = 0.1, 0.5 and 1.0,respectively .
Figures 5.84,82,83 show deflection of bottom surface of the plate given by RTR and RTP for h/a = 0.1, 0.5 and 1.0,respectively .
Figures 5.87,88,89 show deflection of top, middle, and bottom surfaces of the plate given by RTR and RTP for h/a = 0.1, 0.5 and 1.0, respectively .

From the graphs the effect of including the normal strain on deflection is very clear. Also, the present work can give the deflection as a function of z whereas RTR is giving " average deflection " across the depth of the plate. The present theory is predicting deflection at top to be much more than deflection at bottom of the plate as the ratio h/a of the plate increases. This result is expected; since as the plate thickness increases the load will be taken mostly by the top layers and the bottom layers will hardly feel the load.

5.2.7 Verifying Equilibrium Of The Plate In The Vertical Direction :

Edge reactions at edges of the plate should balance the applied load:

$$I = \int_0^a [\, Q_y\, (x,+b/2) - Q_y\, (x,-b/2)\,]\, dx$$

$$+ \int_{\frac{-b}{2}}^{\frac{b}{2}} [\, Q_x\, (a,y) - Q_x\, (0,y)\,]\, dy \qquad (5.2.7\text{-}1)$$

After performing the integrations in the above equation, it can be shown that :

$$I = \frac{p_o ab}{12(1-\mu^2)} \left\{ \frac{24k_{22}(\cos(m\pi)-1)}{F_1} \left(\frac{h}{a}\right)^2 \left\{ \frac{m\pi}{\gamma_m b} - \frac{\gamma_m}{\alpha_m b} \right\} \sinh\left(\frac{\gamma_m b}{2}\right) E_m + \frac{6(1-\mu)(m\pi)}{F_1\left(\frac{h}{a}\right)^2} [\, \beta_m + \bar{\beta}'_m\,][\, \cos(m\pi) - 1\,] \right\} \qquad (5.2.7\text{-}2)$$

Table 5.16 shows that total reaction of the edges of the plate is equal to the uniformly applied loads for different types of support at $y = \pm b/2$. The results are satisfactory compared with classical theory since the latter gives unbalanced concentrated reaction of about 26 % wheras there is no evidence of such unbalanced reaction in this work.

5.2.8 *Effect of inplane stretching on inplane stresses* :

To study the effect of inplane stretching on inplane stresses, σ_y was evaluated at the center of a simply supported plate for the two cases :

when edges at $y = \pm b/2$ are allowed to stretch in the y-direction (case-I)

and when edges at $y = \pm b/2$ are not allowed to stretch in the y-direction (case-II).

The results are shown in Figures 5.90 to 5.92 .

From the results it is noticed that the in-plane compressive stresses increase by 10-15 % for case-I over those for case-II. Also it is noticed that the in-plane tensile stresses decrease by 10-15 % for case-I over those for case-II. For thin plates the in-plane stresses were the same for both cases since the effect of the in-plane forces for thin plates is extremely small.

5.3 Computer Program

A computer program (DISS2) is developed to get the solution for any rectangular plate that is simply supported at x=0,a and can have any boundary condition on edges at $y = \pm b/2$. A flowchart is given in Fig. 5-B to show the structure of this program. A program listing is included in the Appendix A-5-1.

It should be noted that this program can handle solutions according to RTB or RTP by the use of the parameter IBALCH. (See program listing for more details).

A similar program DISS4 is developed for the case of plate strips (i.e for the case of Cylindrical Bending). The plate strip can have any boundary condition at x=0,x=l (i.e at edges of the plate strip). A program listing for DISS4 is included in the Appendix A-5-2.

FIG. 5-B : Flowchart For The Computer Program DISS2

DATA : b/a , h/a, v ,etc.

SUBROUTINE DISS1 :

Get : f1(z),f2(z),f3(z),

F1,F2,F3,F4,etc.

SUBROUTINE BOUND :

-Get Boundary Conditions

For Bending Problem.

SUBROUTINE BENDING :

-Solve For : A_m, B_m, E_m.

SUBROUTINE XPLANE :

-Get Boundary Conditions

For Inplane Problem

-Solve For : C_1, C_2

Figure 5-B (Continued) : Flowchart For The Computer Program DISS2

SUBROUTINE FORCE :

-Get Resultant Moments :

M_x, M_y, M_{xy}

-Get Resultant Inplane Forces :

N_x, N_y, N_{xy}

-Get Shear Forces :

Q_x, Q_y

SUBROUTINE STRES :

-Get Stresses :

σ_x, σ_y, σ_z,

τ_{xy}, τ_{xz}, τ_{yz}

PRINT RESULTS For :

DEFLECTION, MOMENTS,

SHEARS AND STRESSES

5.4 Conclusions

1. It may be concluded that the use of generalized distribution of transverse normal and shear stresses (as originally presented by Kromm [7,8] in the development of a new refined thick plate theory (along the lines of earlier presentation [10,11] yields a formulation that captures all essential characteristics of the exact three dimensional elasticity problem. This is reflected in that results for stresses obtained from the present formulation are almost identical to the exact solution up to ratios of h/a = 3.0 (for the case of cylindrical bending). This ratio characterizes a significantly thick plate, and all previously known refined theories breakdown at this level of plate thickness.

   For the case of rectangular plates , the results are satisfactory up to h/a = 1.0

2. Based on comparison of resultant moments and forces : M_x , M_y , M_{xy} , Q_x , Q_y from classical thin plate theory and refined theories , a plate is considered to be thick for a ratio of $h/a \geq 0.1$. Thus for plates for which $h/a \geq 0.1$ a refined theory - such as the one presented in this work - should be used to analyze the behavior of such plates completely.

3. It is shown in the results that as h/a increases (from 0.1 and above) , inplane bending and twisting shear stresses decrease to a level where they are of equivalent order as σ_z and therefore σ_z cannot be neglected.

4. This theory allows for in-plane movement of the plate, yielding new type of boundary conditions in the form of loosely or rigidly supported simple or clamped edges.

   The case of rigidly supported edges yields in-plane compression forces not present in any of the previous refined theories .

   The effect of these forces is accentuated as h/a increases. In-plane compressive normal stress σ_y increases by 10-15 % if the edges at $y = \pm b/2$ are not allowed to stretch.

5. $f_1(z)$ is the function that is responsible for yielding 3-Dimensional type behavior (in terms of stresses) from an essentially 2-Dimensional analysis for stress resultants and displacements .

6. Present theory (RTP) corrects stresses as h/a becomes large whereas Reissner's theory (RTR) predicts always linear distribution for the stresses : σ_x , σ_y , σ_{xy} , and parabolic distribution for the stresses : τ_{xz} , τ_{yz} , and assumes that : $\sigma_z = 0$.

   Present theory gives non-linear distribution similar to exact solution from theory of elasticity for deep beam type members.(For all stresses: σ_x , σ_y , σ_z , σ_{xy} , τ_{xz} ,and τ_{yz})

7. Present theory captures ' transition from " beam bending problem" to " column type problem " as plate gets thicker ' better than Reissner's theory.

8. Present work solves the numerical problem of ill-conditioning which occurs in the previous companion refined theory [10,11]. The ill-conditioning in the previous formulation was a serious shortcoming as some of the results presented in References [10,11] are in discrepancy with those presented by the most well known of refined theories i.e. Reissner theory [12].

9. The variation of the transverse shear stress τ_{xz} agrees qualitatively with the elasticity solution for bending of thick curved bar by force at end.

10. The results for vertical equilibrium of the plate are satisfactory compared with the classical theory of plates since the latter gives unbalanced concentrated reaction of about 26% wheras there is no evidence of such unbalanced reaction in this work.

11. The tranverse normal stress σ_z of previous theory [11] (RTB) is not a function of thickness of the plate, whereas present one is a function of thickness. This reflects clearly the role of $f_1(z)$ on plate behavior.

FIG.5.1 : DEFLECTION COEFFICIENT K VS H/L

(K = MAX. WO (REFINED THEORY) / WO (CLASSICAL THEORY))
(PO = SIN(PI*X/L))

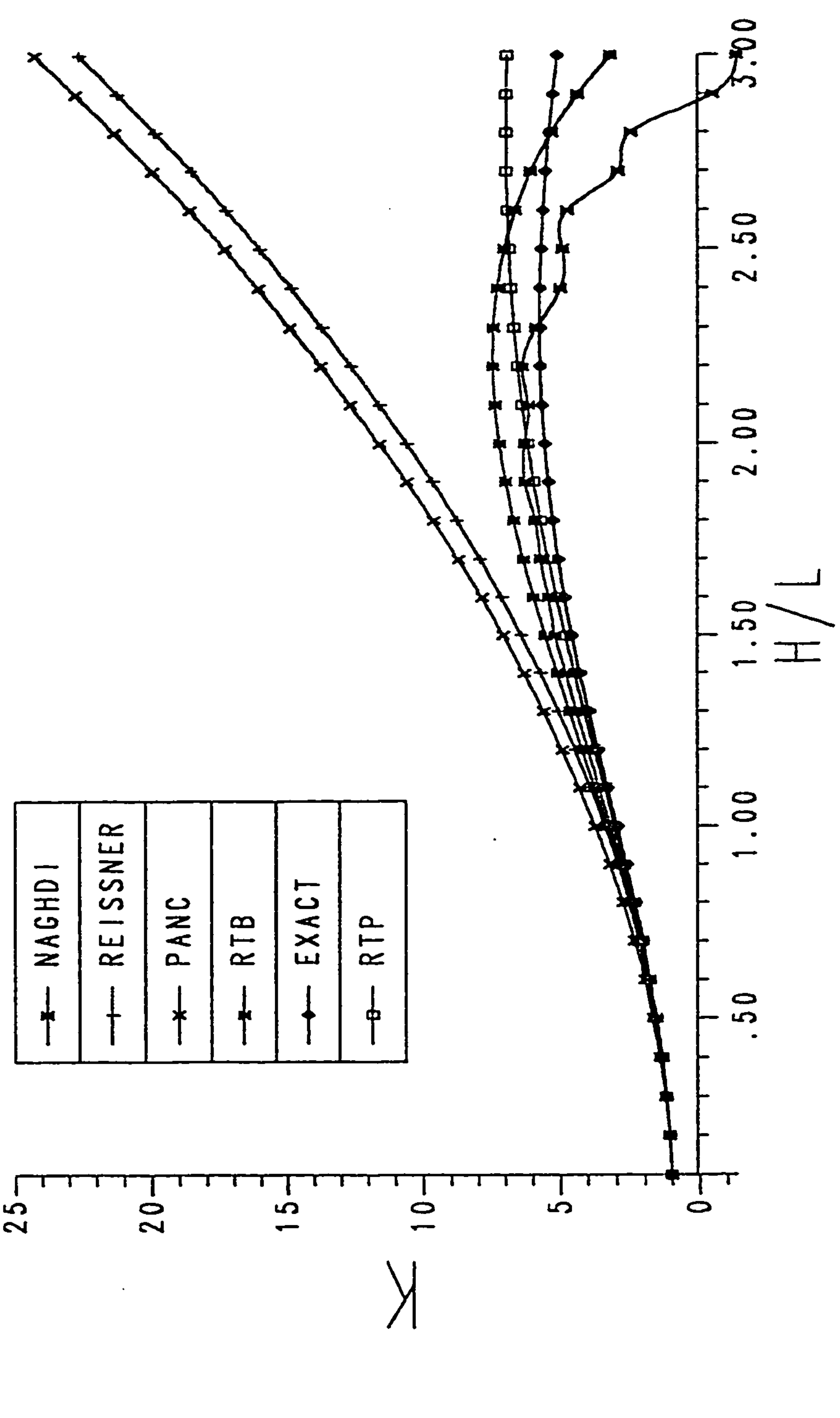


FIG.5.2 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.1,P=P0*SIN(PI*X/L))

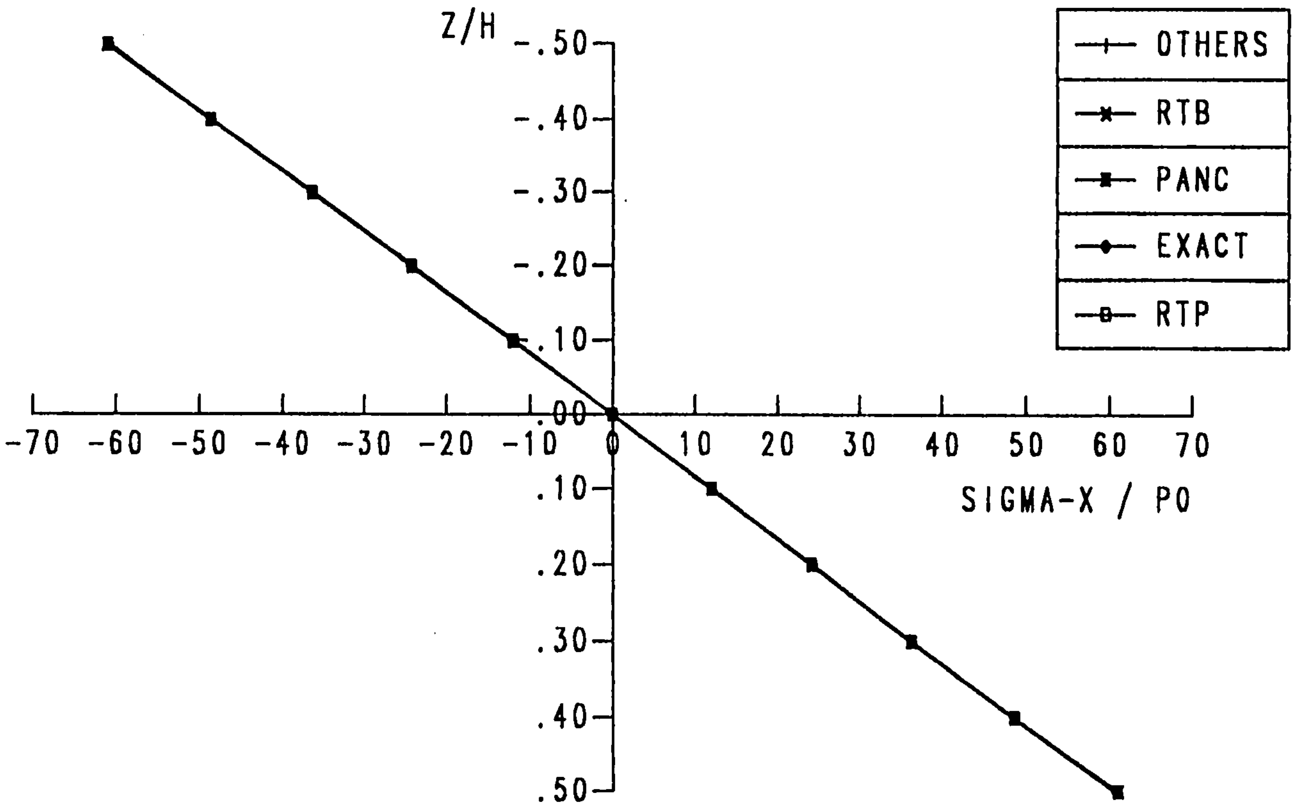


FIG.5.3 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.3,P=P0*SIN(PI*X/L))

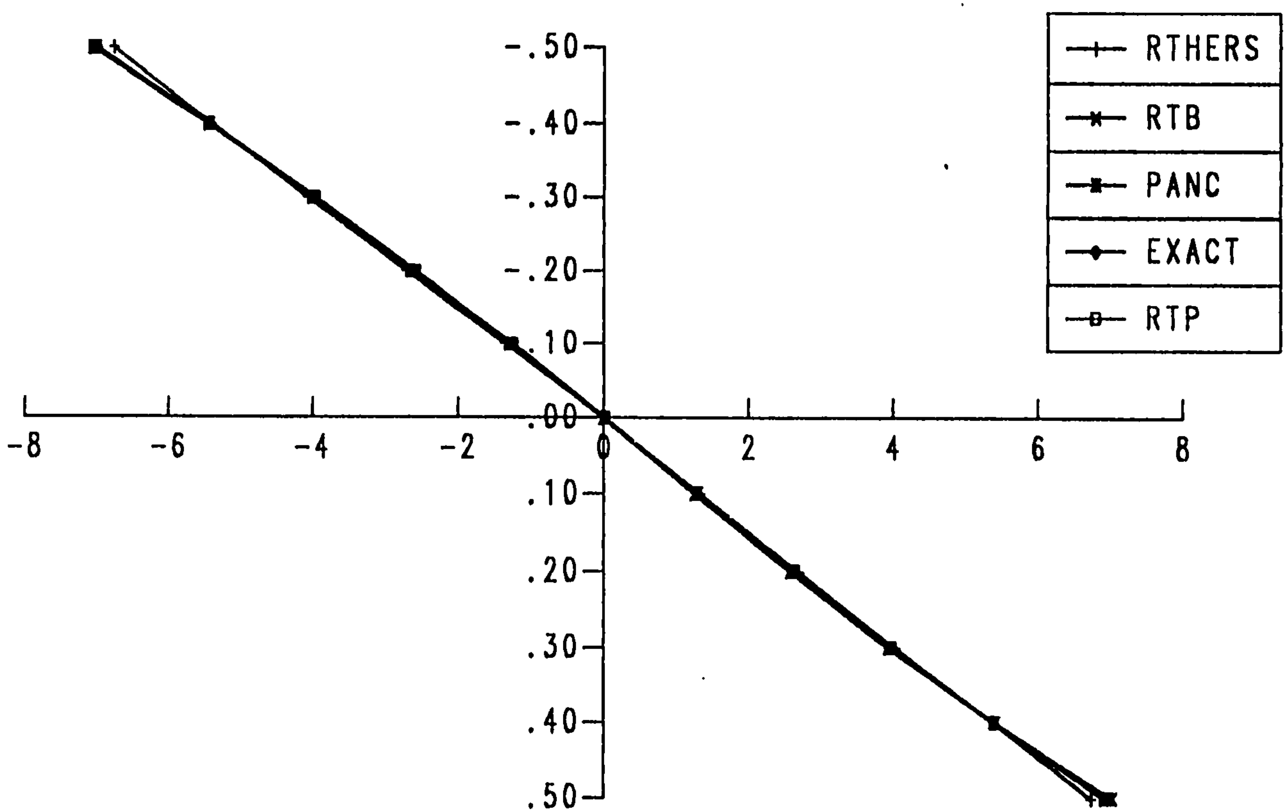


FIG.5.4 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.5,P=PO*SIN(PI*X/L))

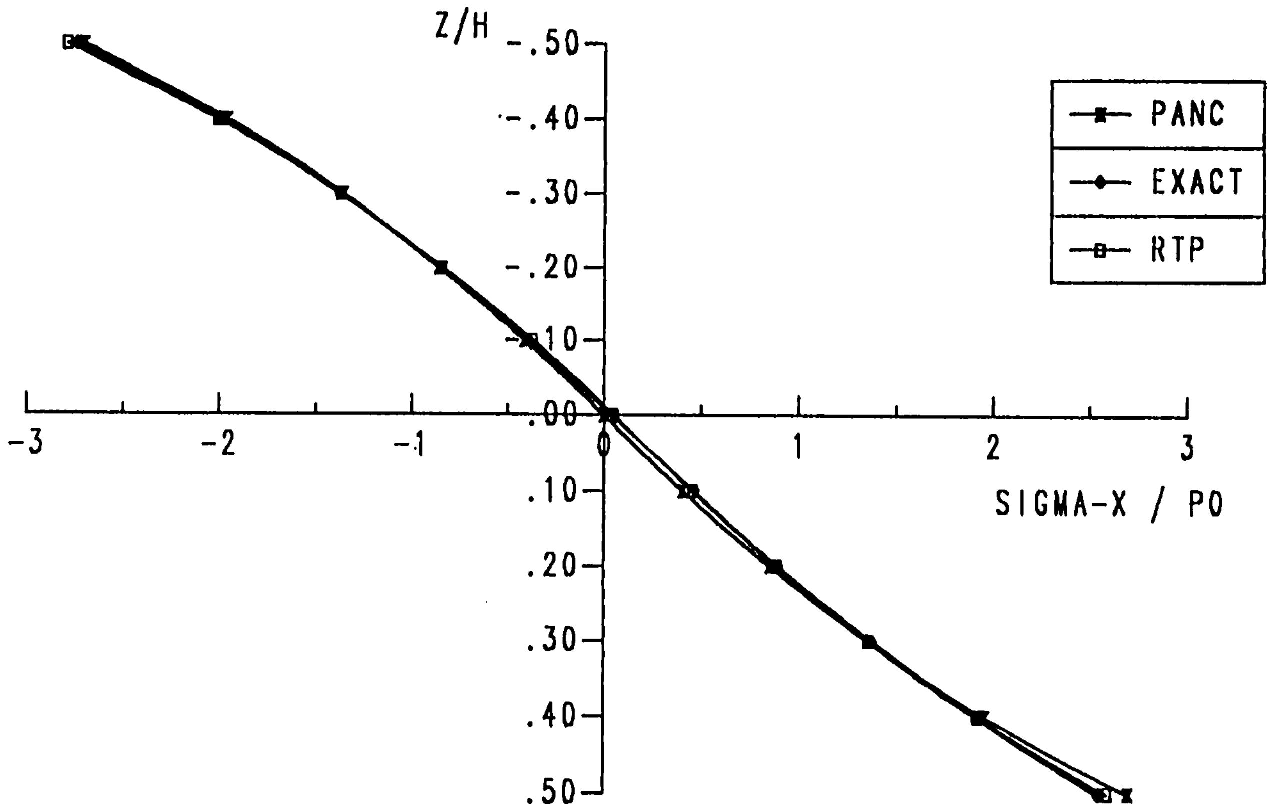


FIG.5.5 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.0,P=PO*SIN(PI*X/L))

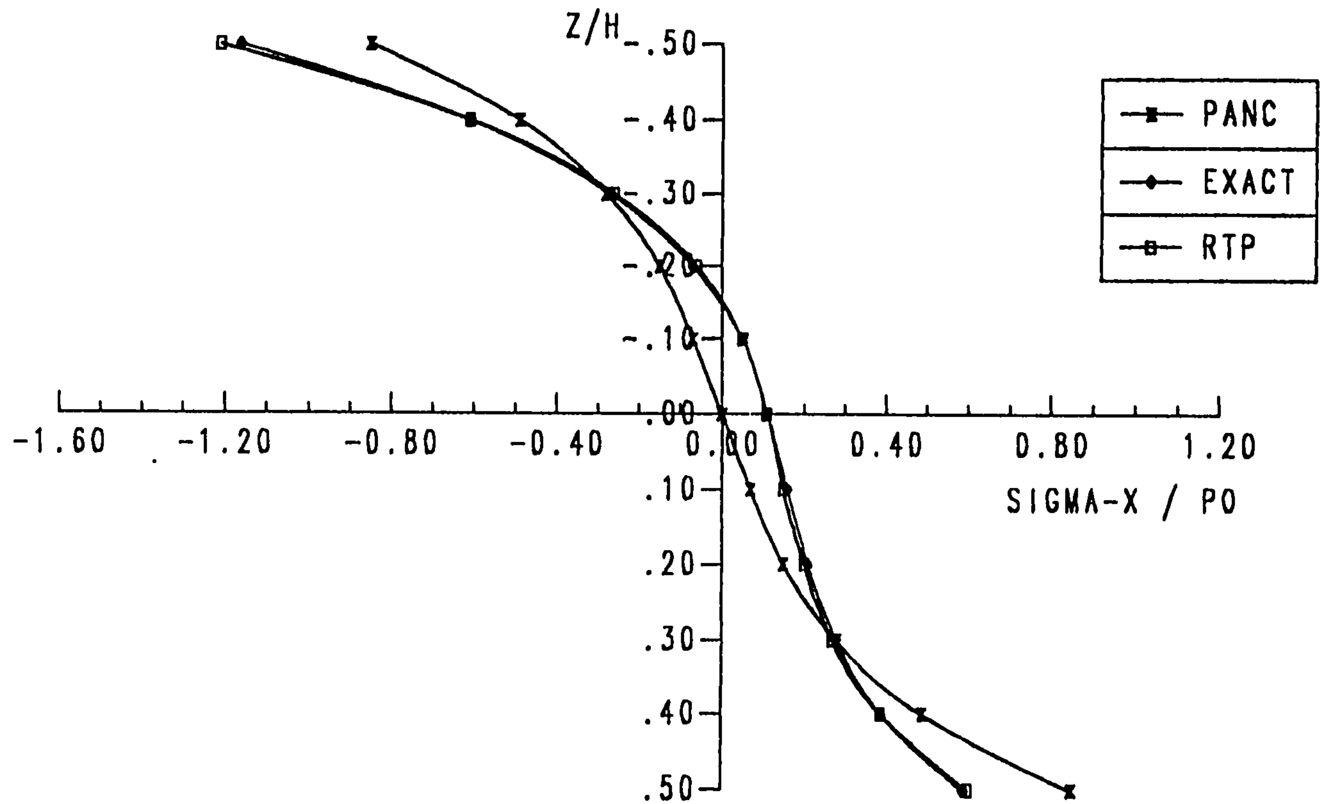


FIG.5.6 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.5,P=P0*SIN(PI*X/L))

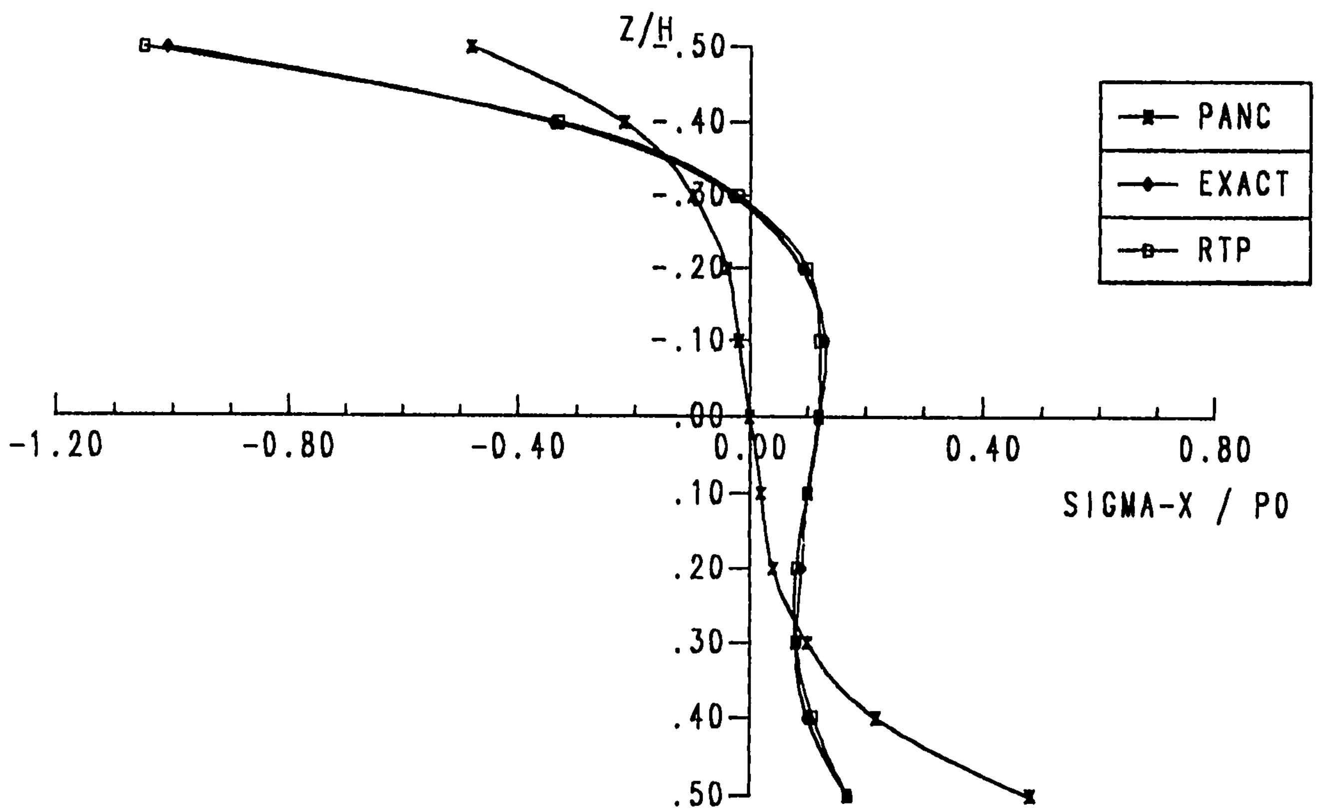


FIG.5.7 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.0,P=PO*SIN(PI*X/L))

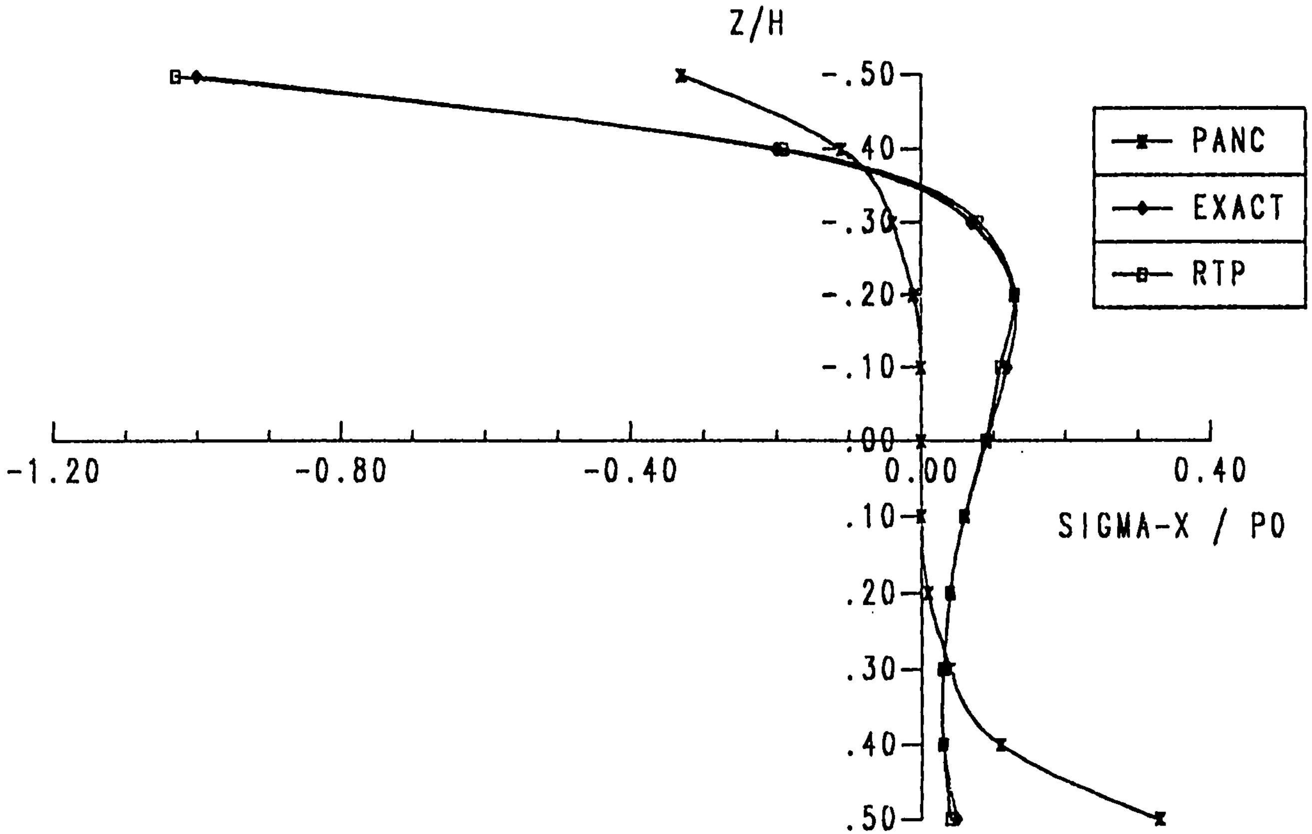


FIG.5.8 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.5,P=PO*SIN(PI*X/L))

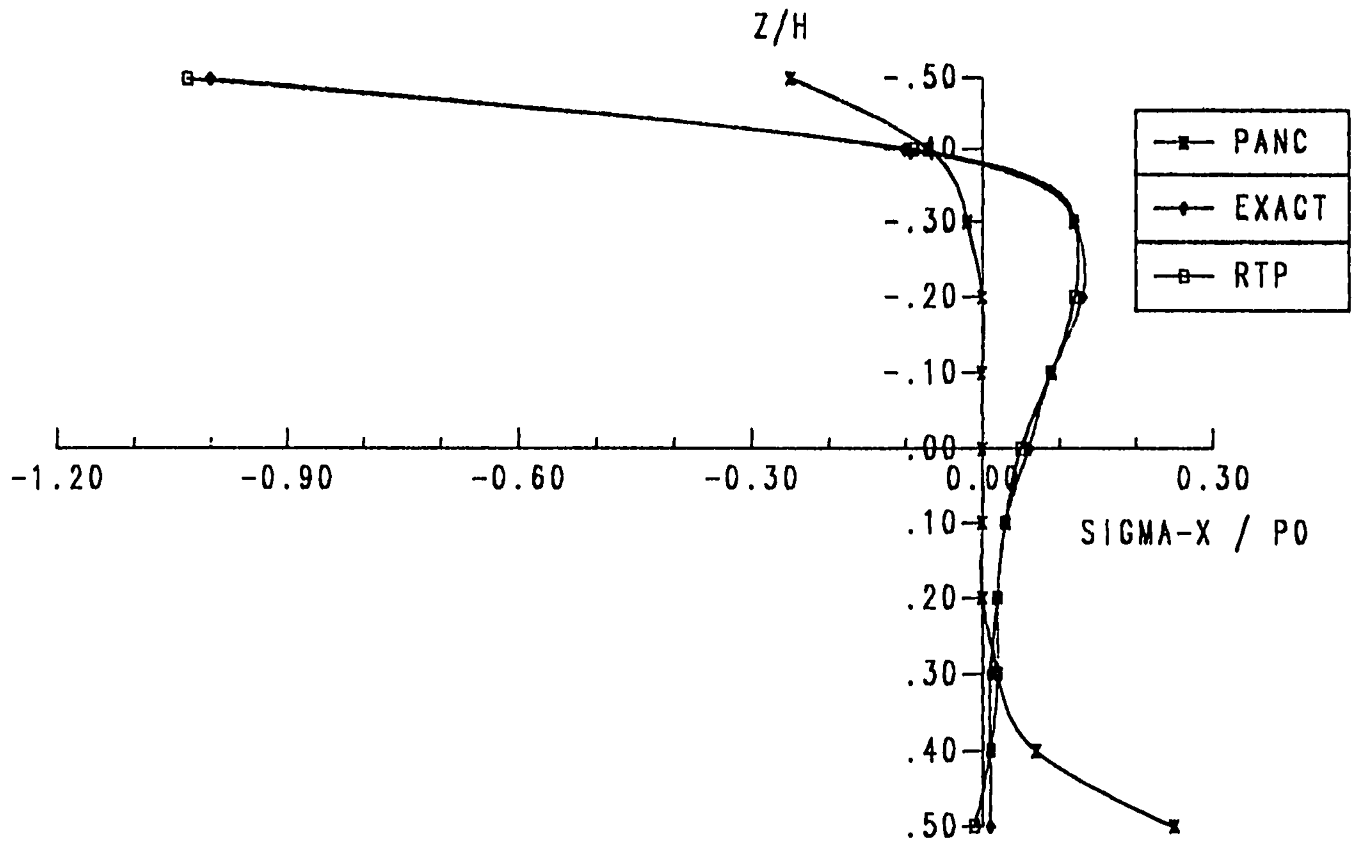


FIG.5.9 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=3.0,P=P0*SIN(PI*X/L))

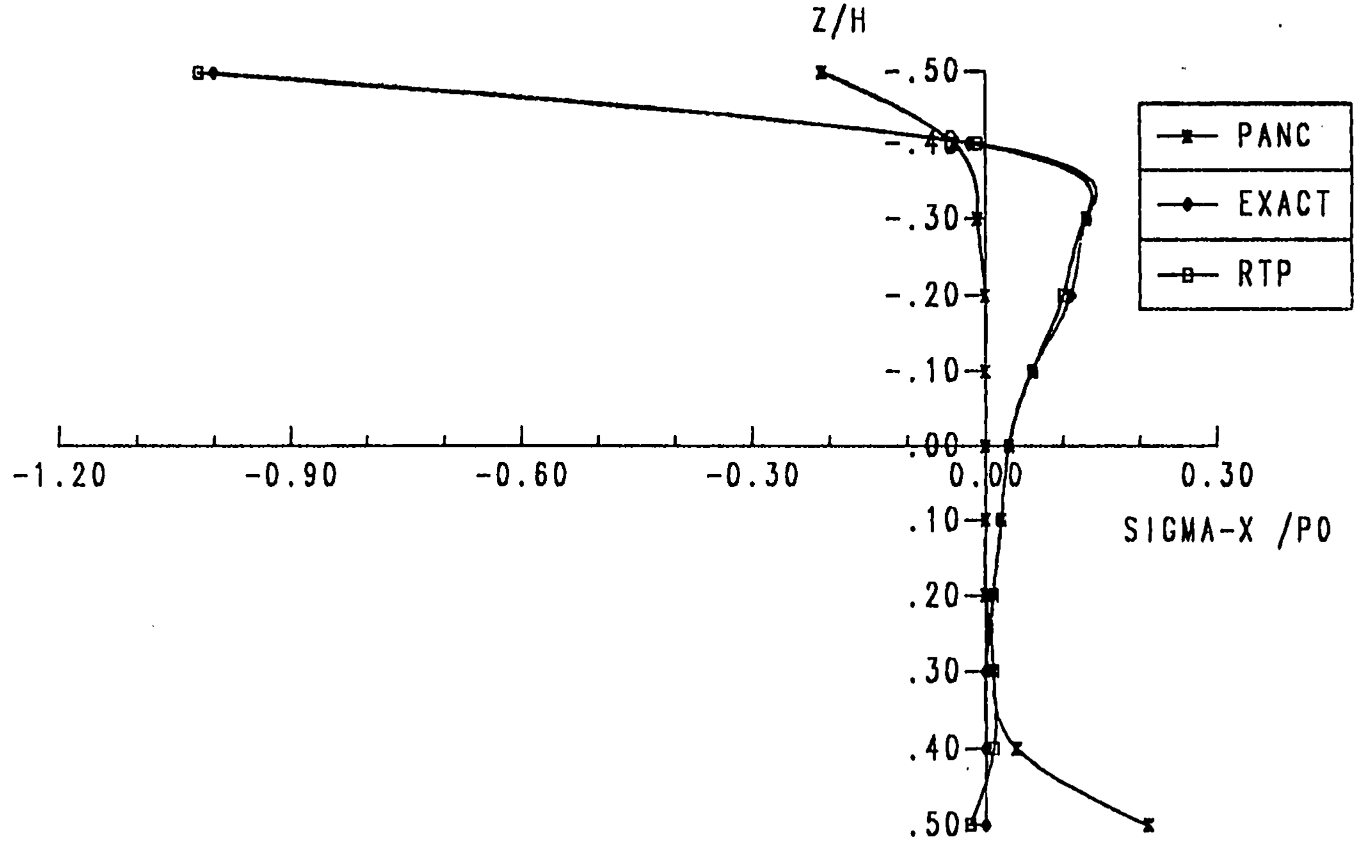


FIG.5.10 : DEFLECTION COEFFICIENT K VS H/L

(K = MAX. WO (REFINED THEORY)/MAX. WO (CLASSICAL THEORY))
(P = UNIFORM LOAD)

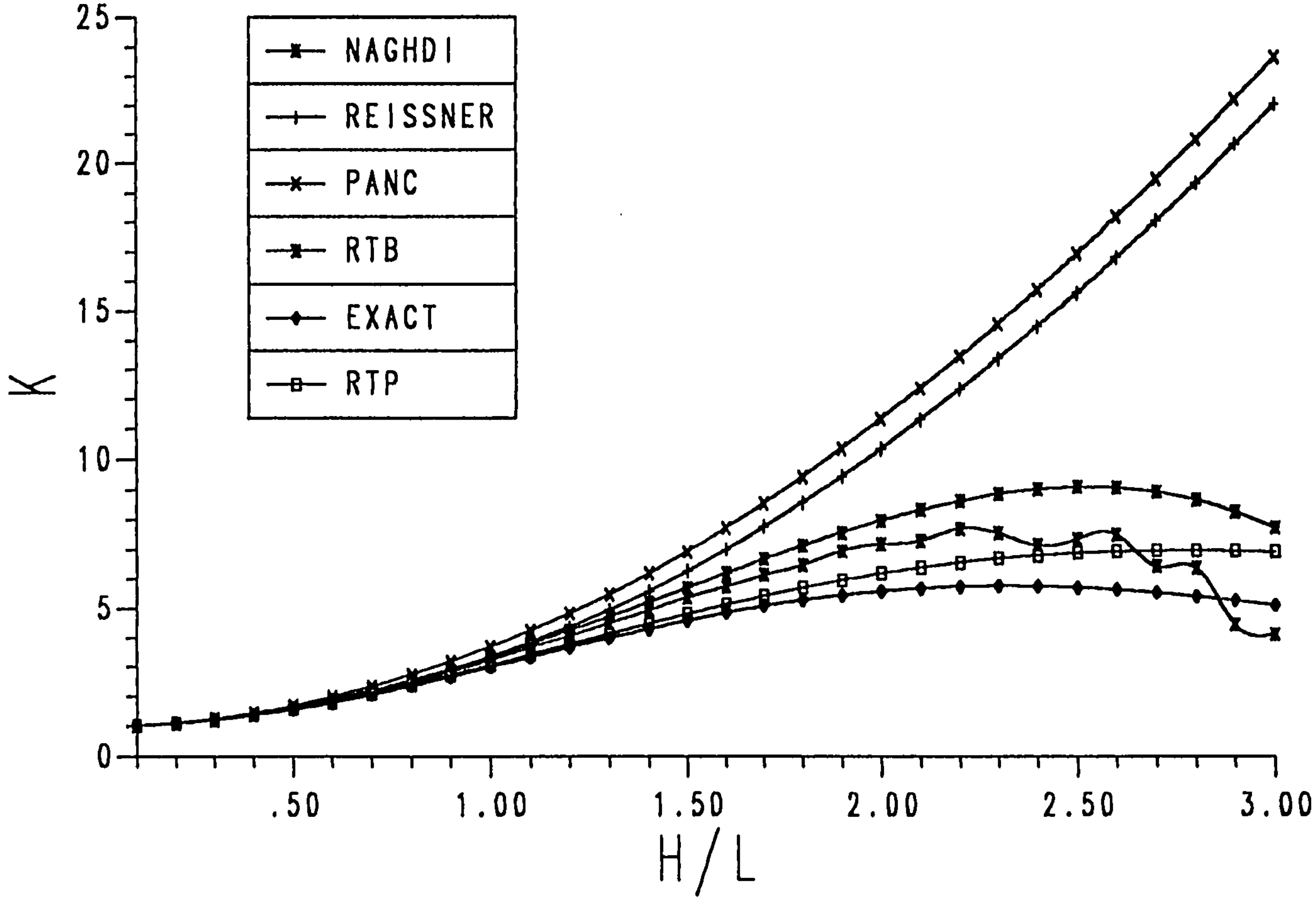


FIG.5.11: MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.1,UNIFORM LOAD)

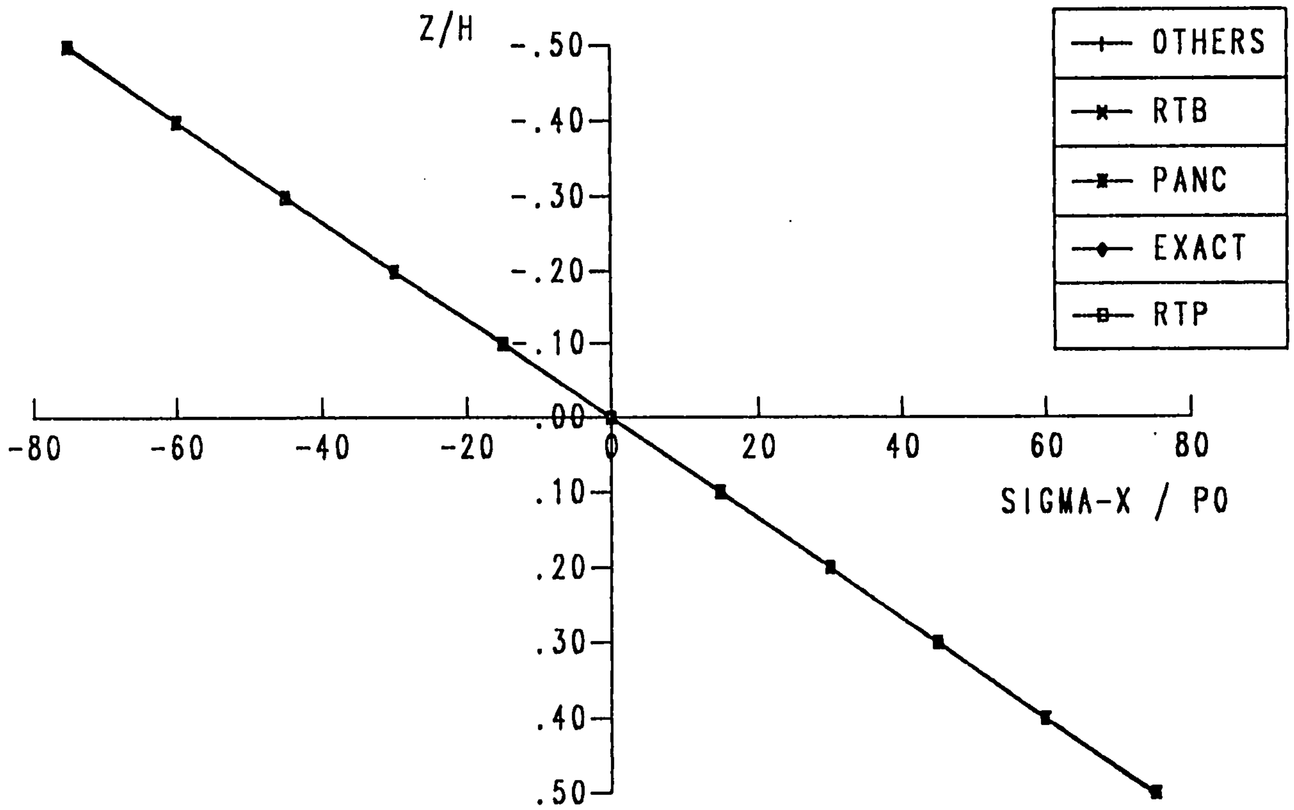


FIG.5.12 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.3,UNIFORM LOAD)

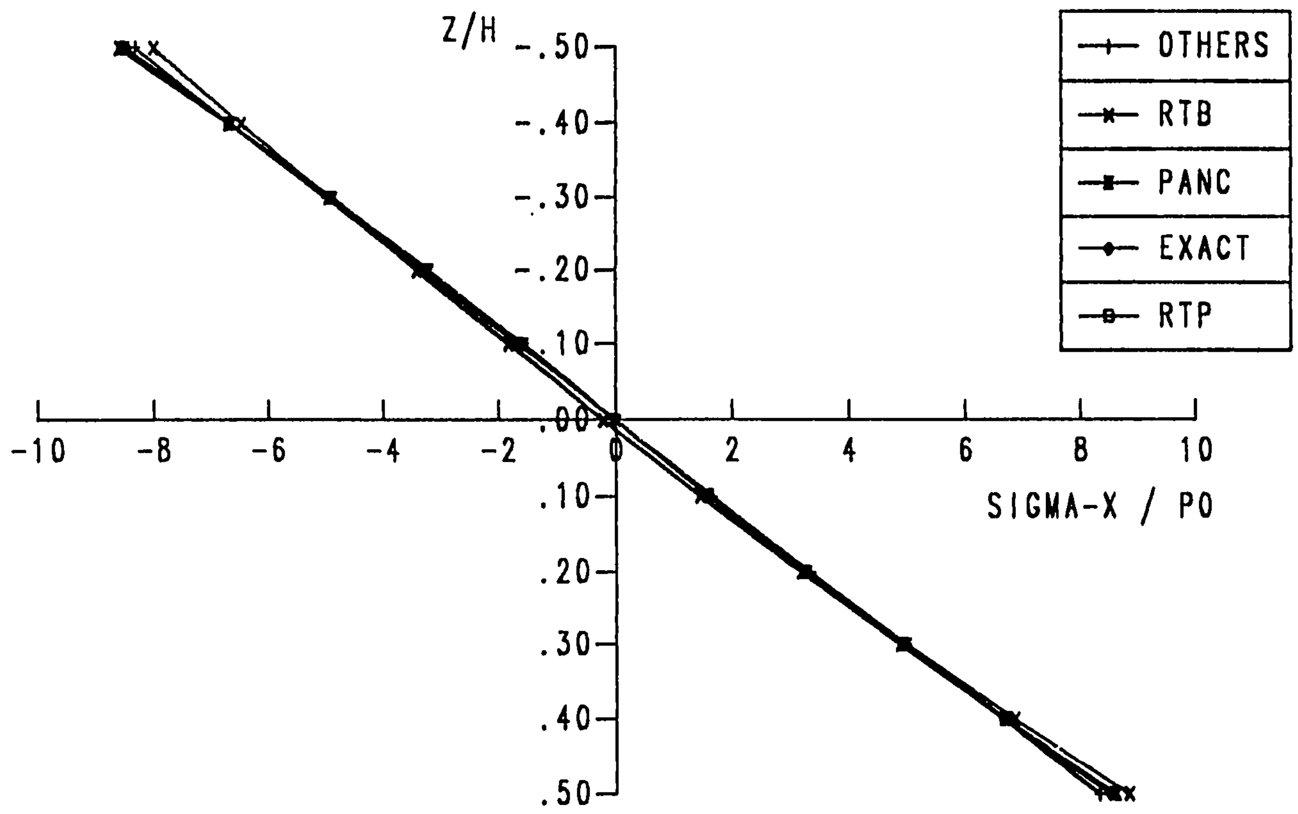


FIG.5.13 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=.5,UNIFORM LOAD)

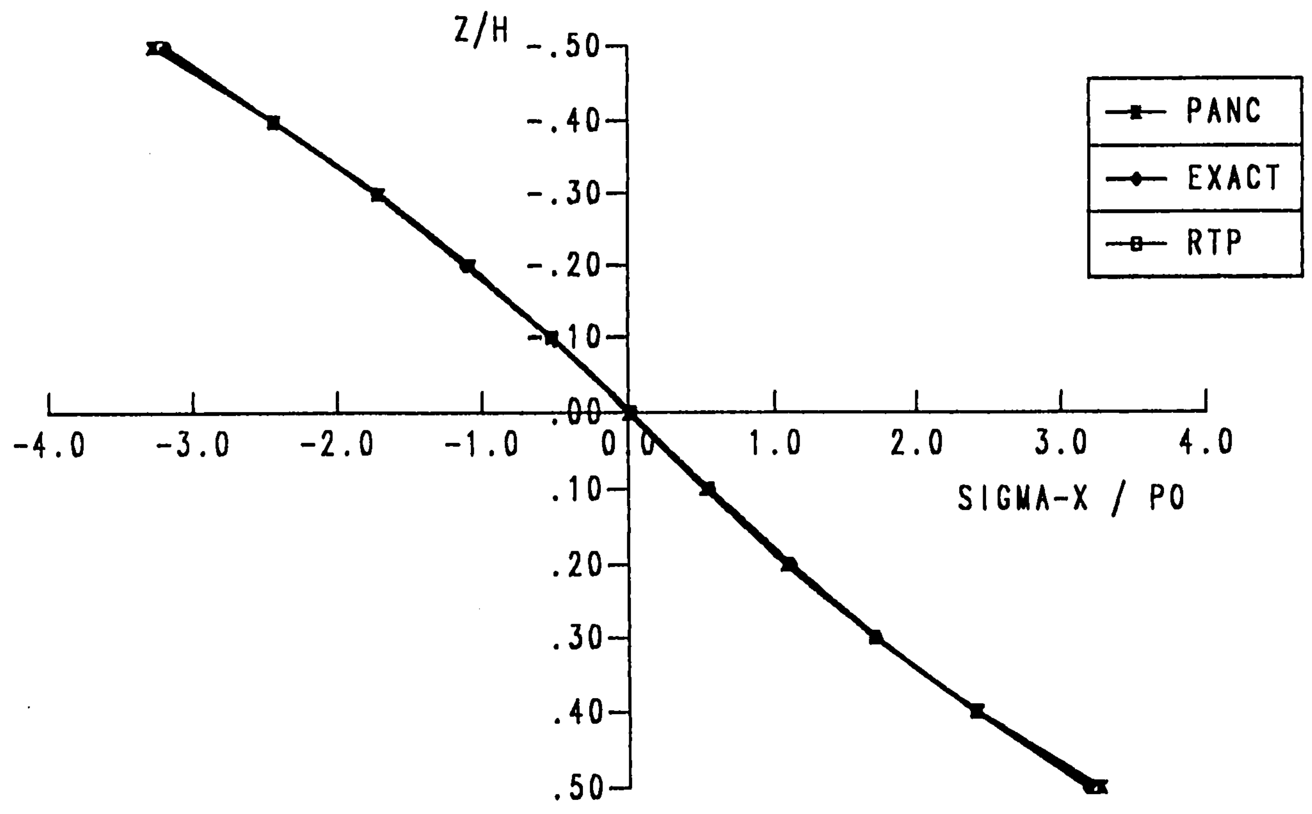


FIG. 5.14 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.0,UNIFORM LOAD)

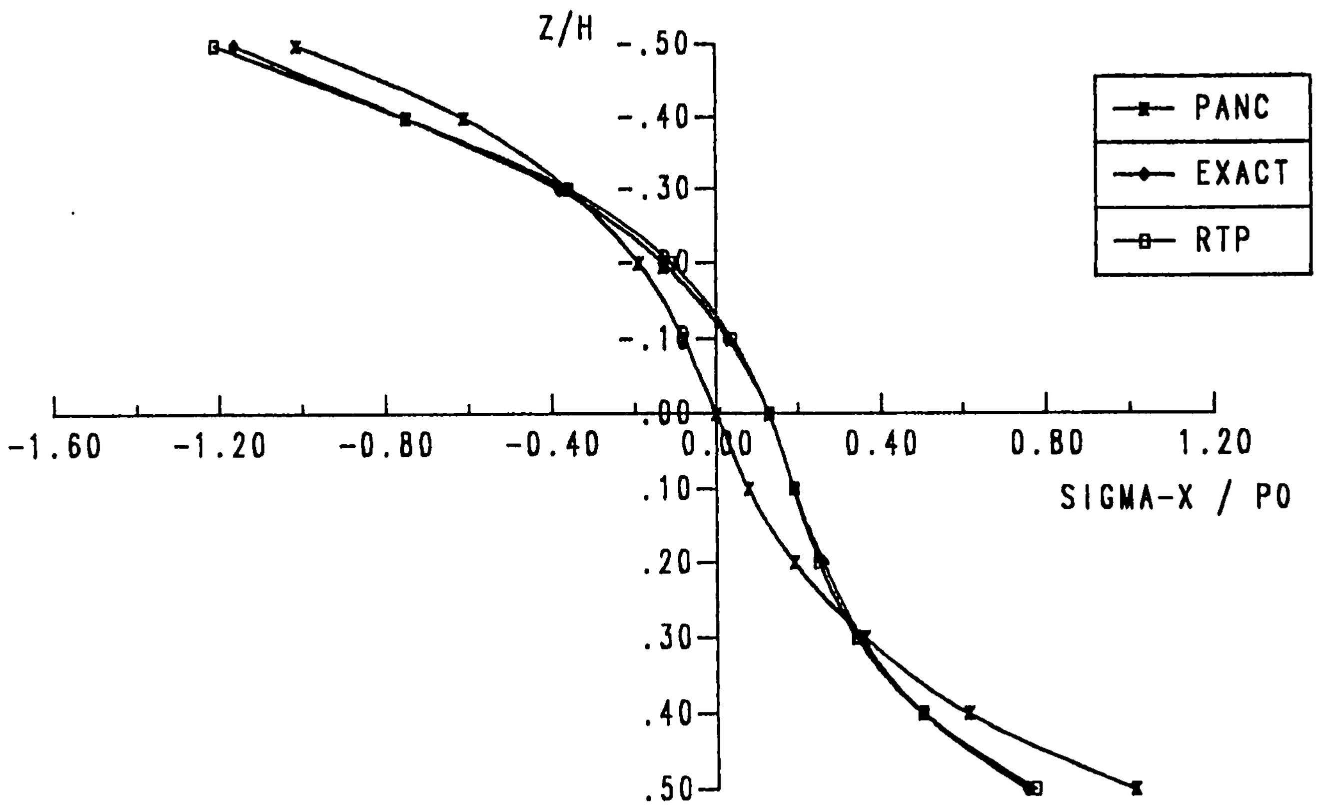


FIG. 5.15 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=1.5,UNIFORM LOAD)

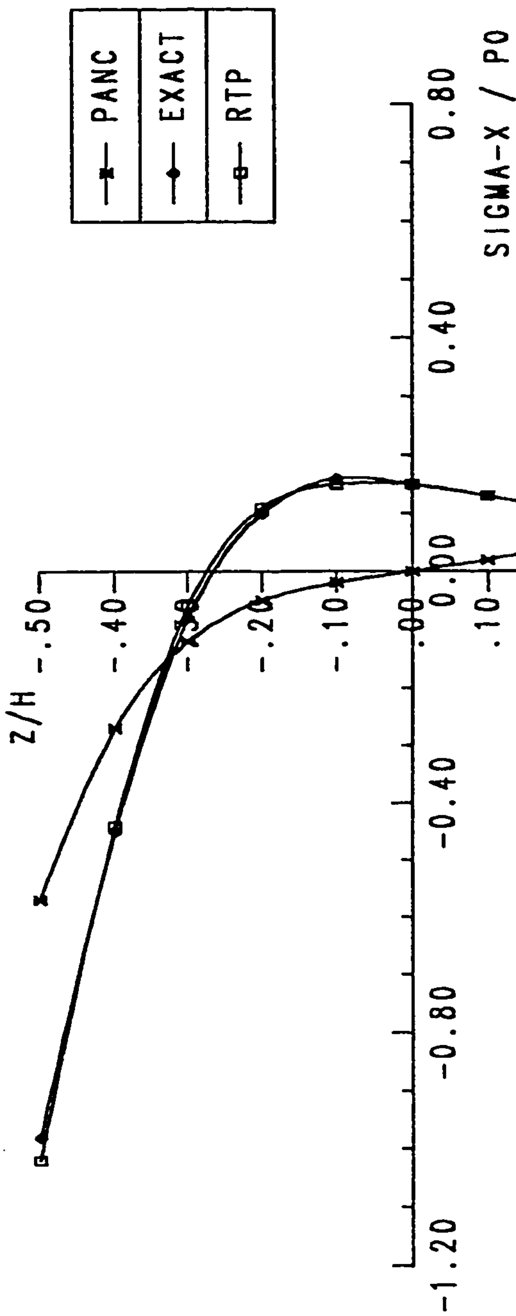
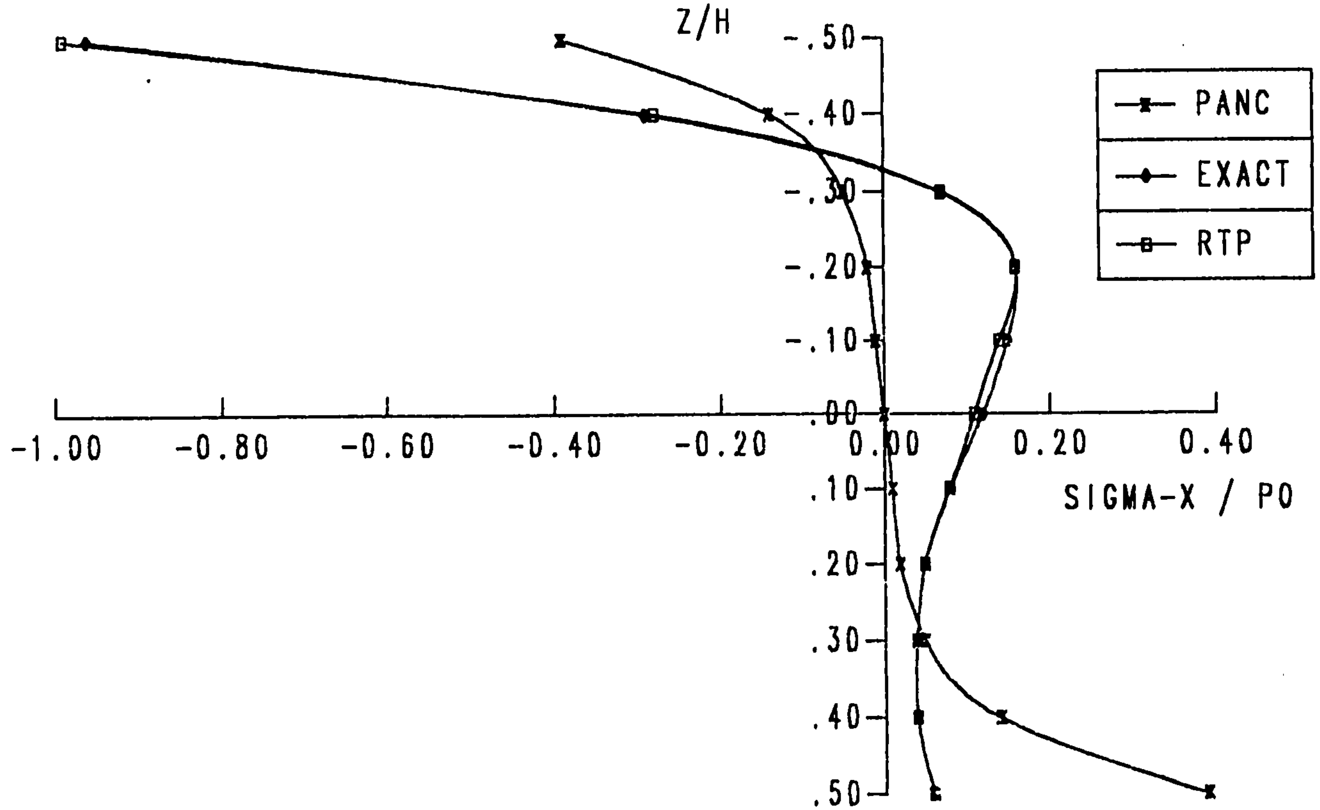


FIG. 5.16 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.0,UNIFORM LOAD)



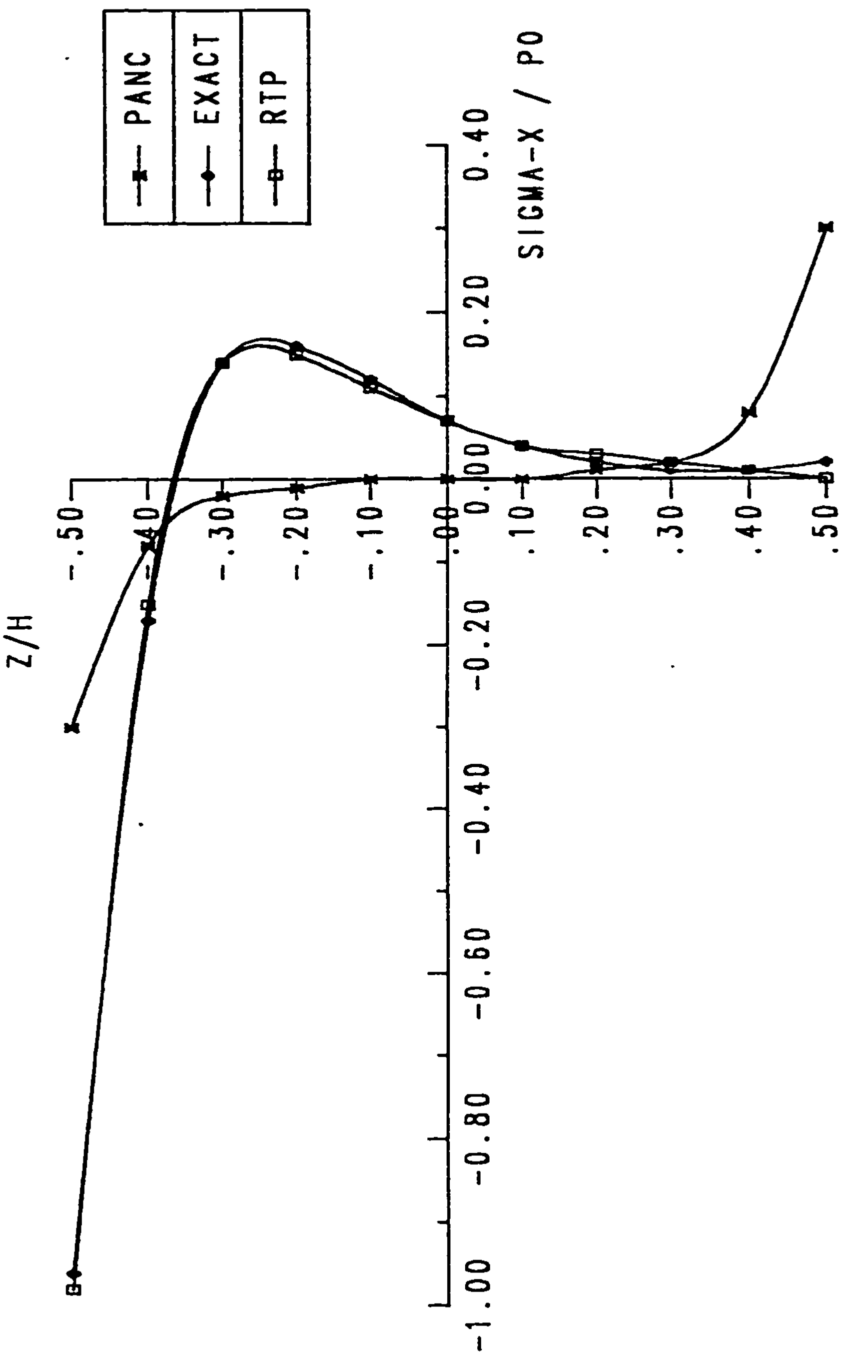


FIG. 5.17: MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=2.5,UNIFORM LOAD)

FIG. 5.18 : MAX. NORMAL STRESS SIGMA-X VS Z/H (H/L=3.0,UNIFORM LOAD)

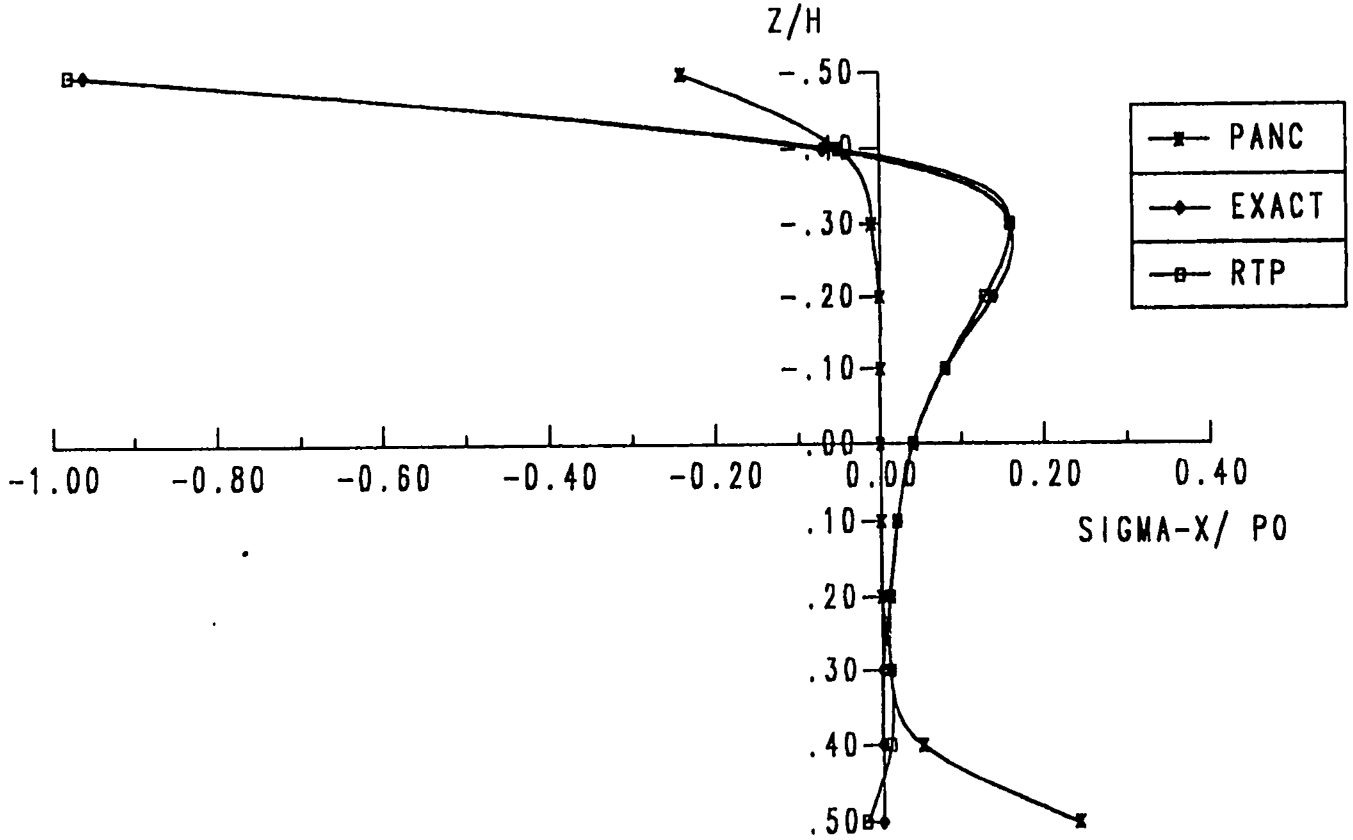


FIG. 5.19 : Max. Normal Stress Sigmaz Vs Z/H (H/L=1.0,Uniform Load)

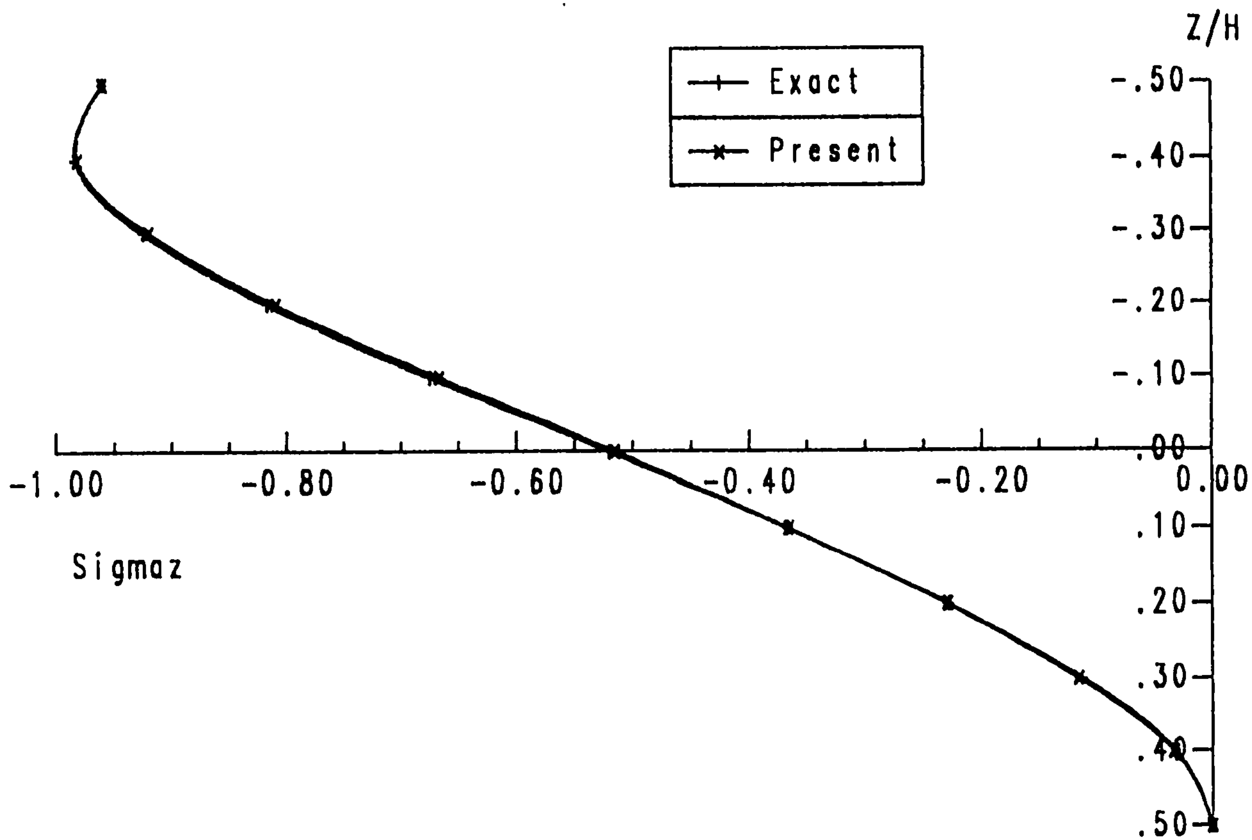


FIG. 5.20 : Max. Normal Stress Sigmaz Vs Z/H (H/L=2.0,Uniform Load)

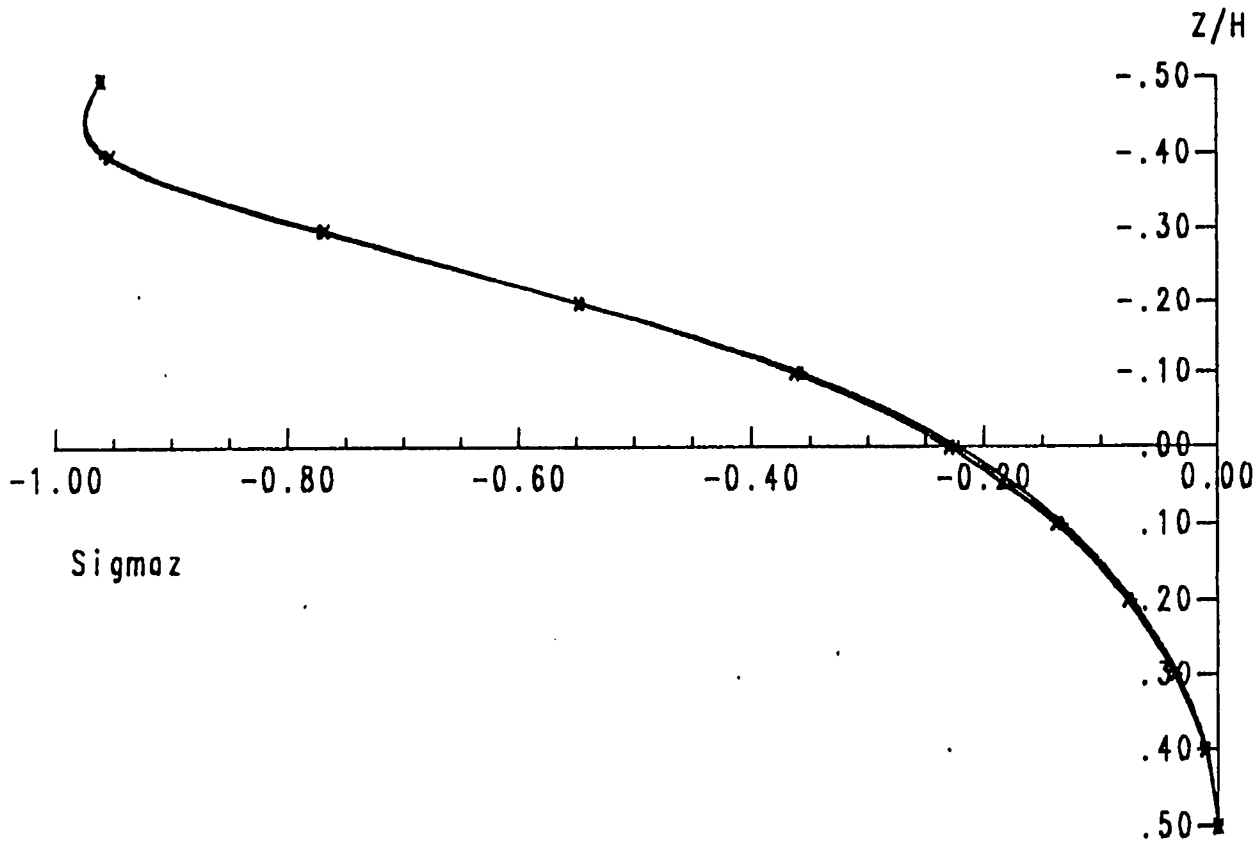


FIG. 5.21 : Max. Normal Stress Sigmaz Vs Z/H (H/L=3.0,Uniform Load)

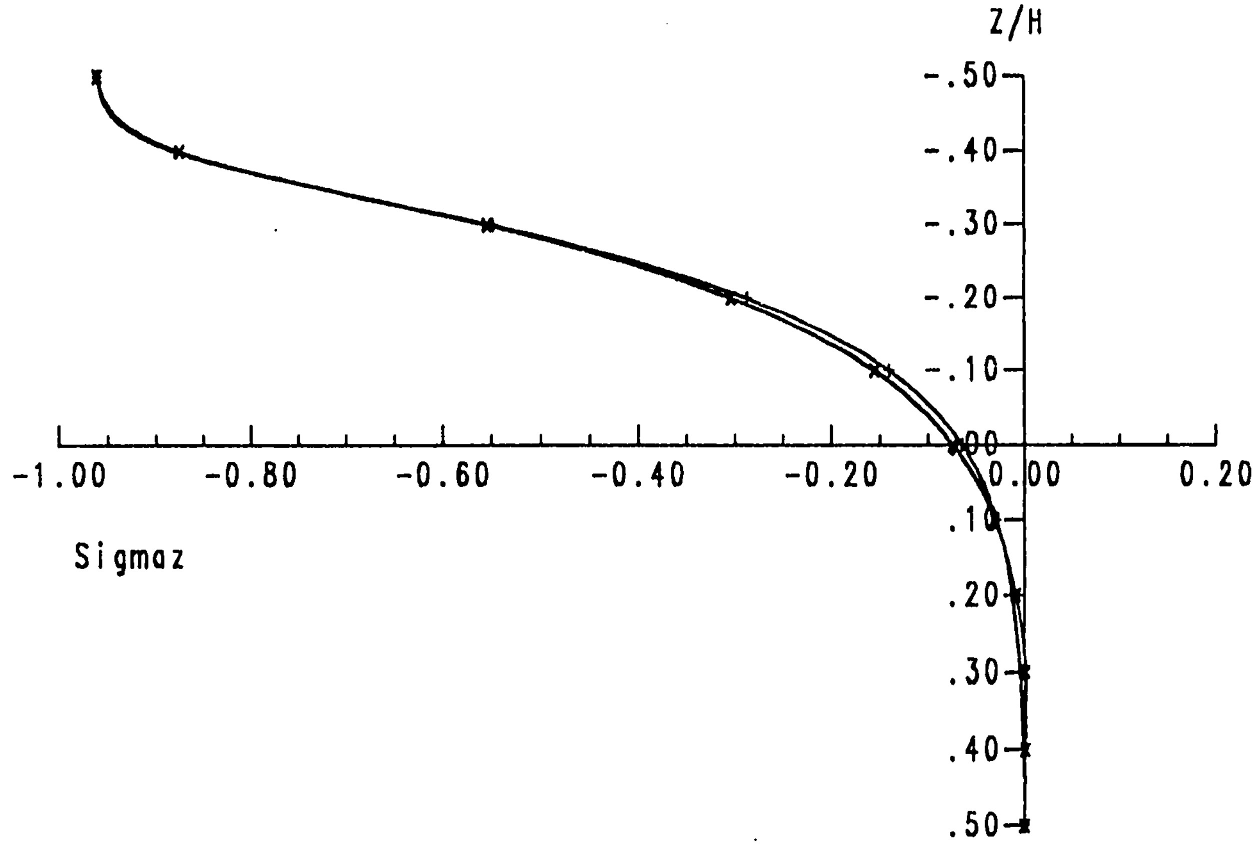


FIG. 5.22 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.005-I)

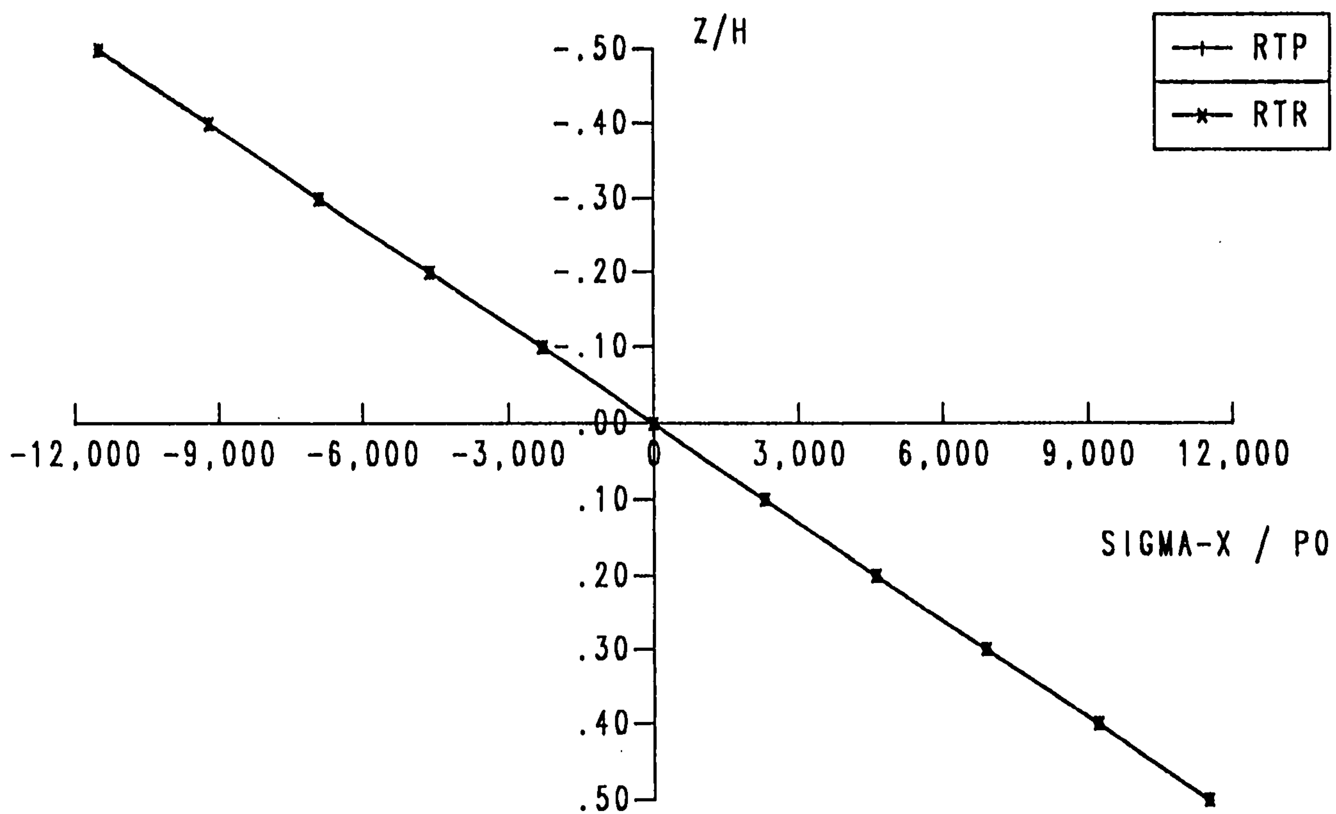


FIG. 5.23 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.01-I)

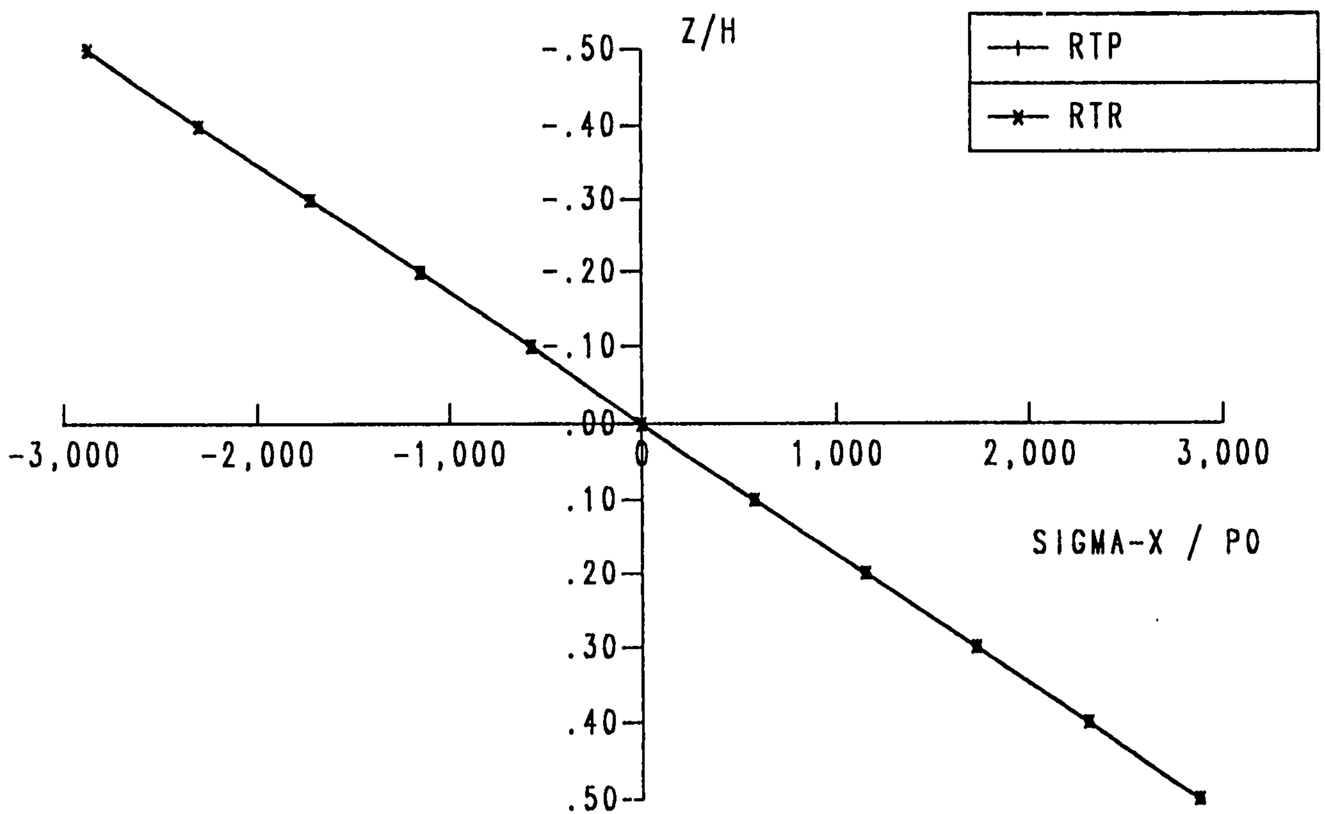


FIG. 5.24 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.05-I)

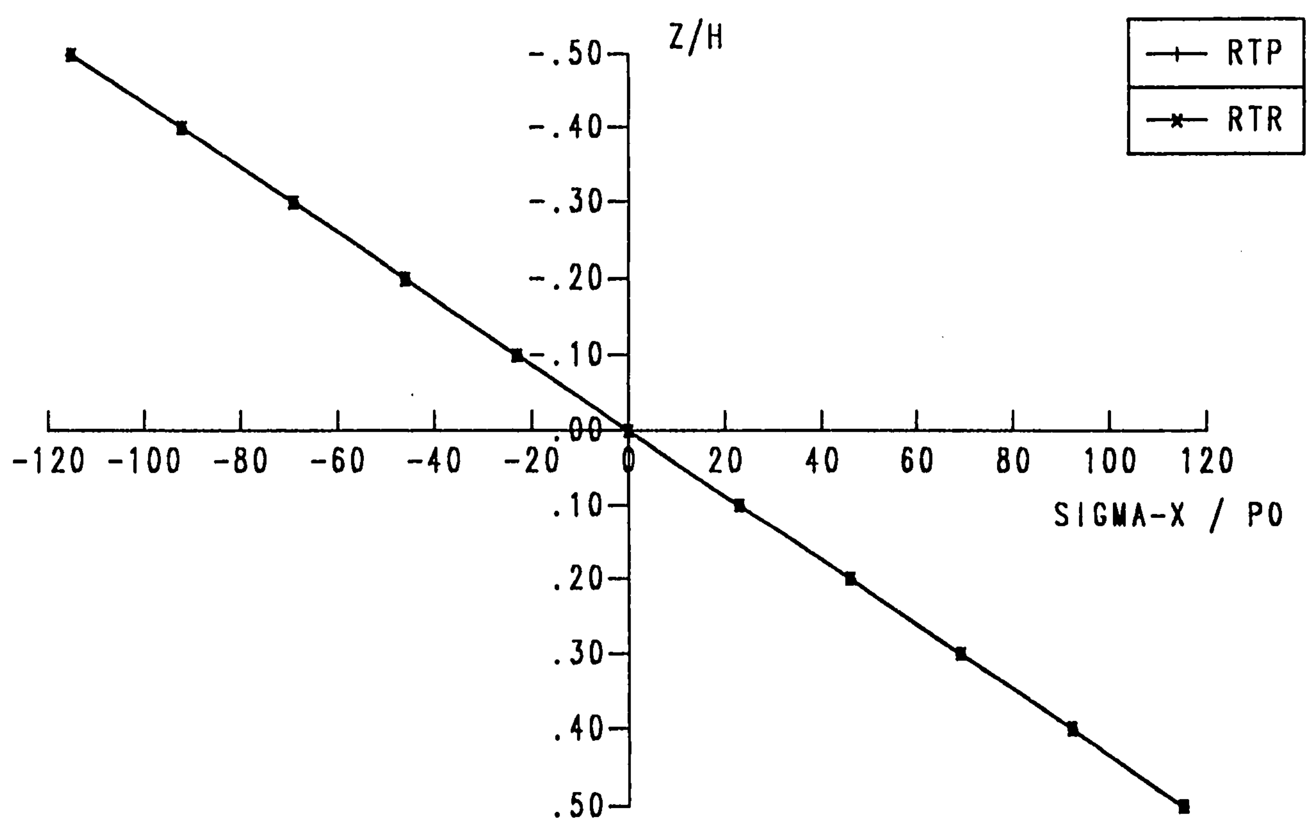


FIG. 5.25 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.1-I)

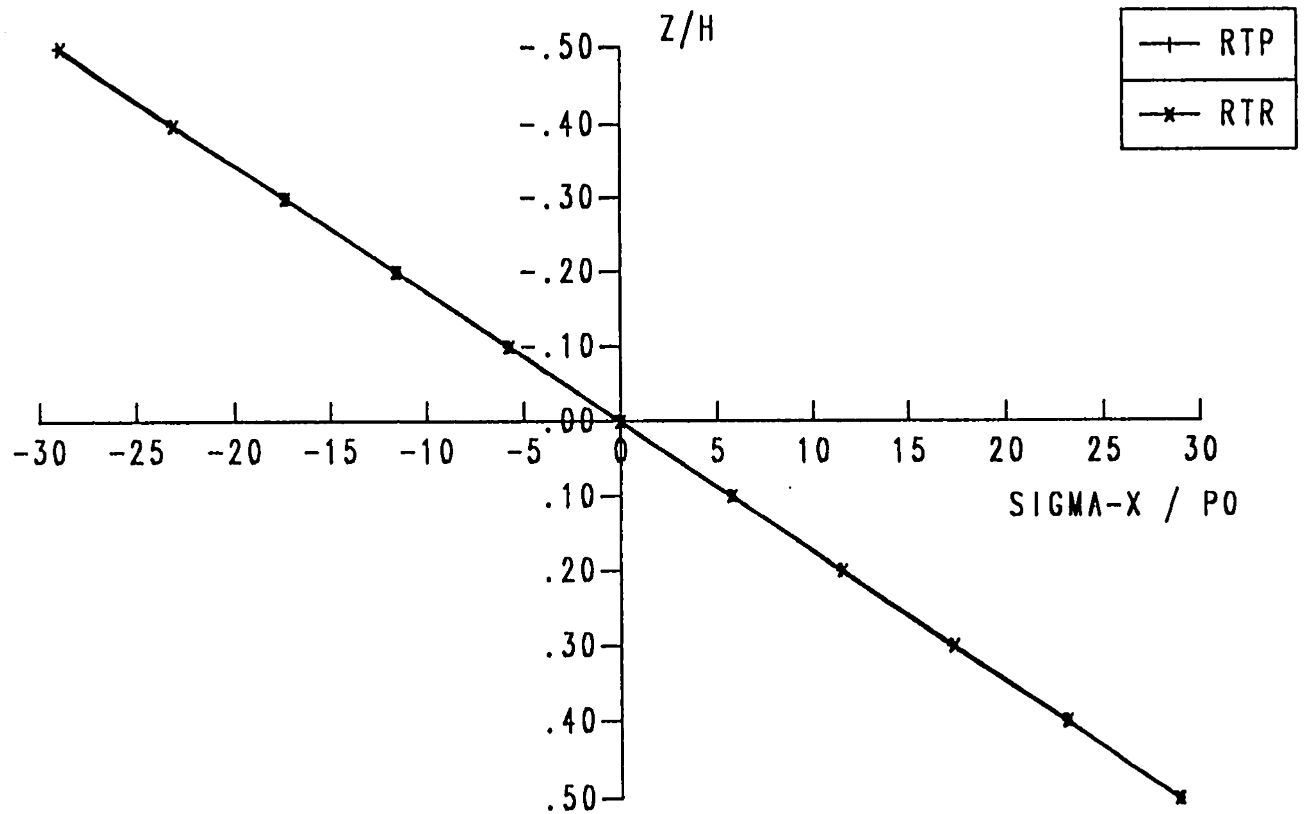


FIG. 5.26 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.2-I)

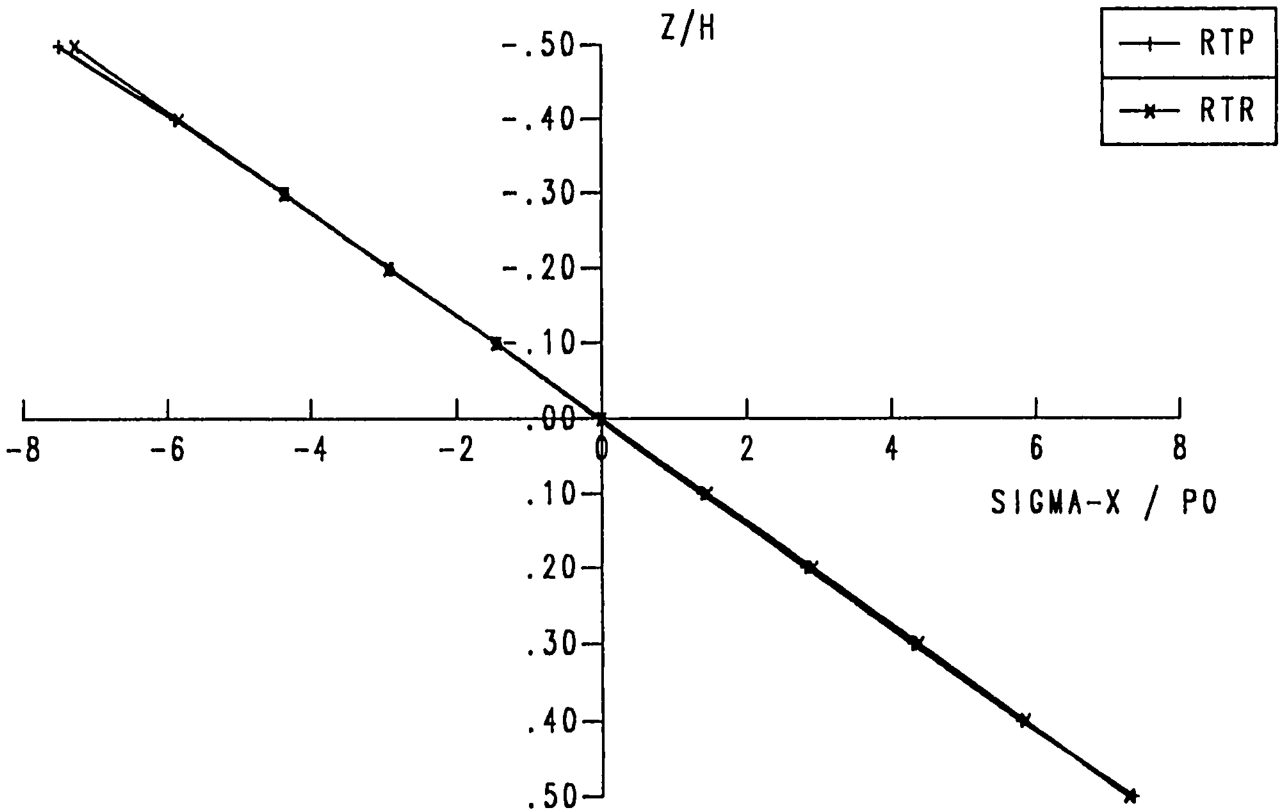


FIG. 5.27 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.3-I)

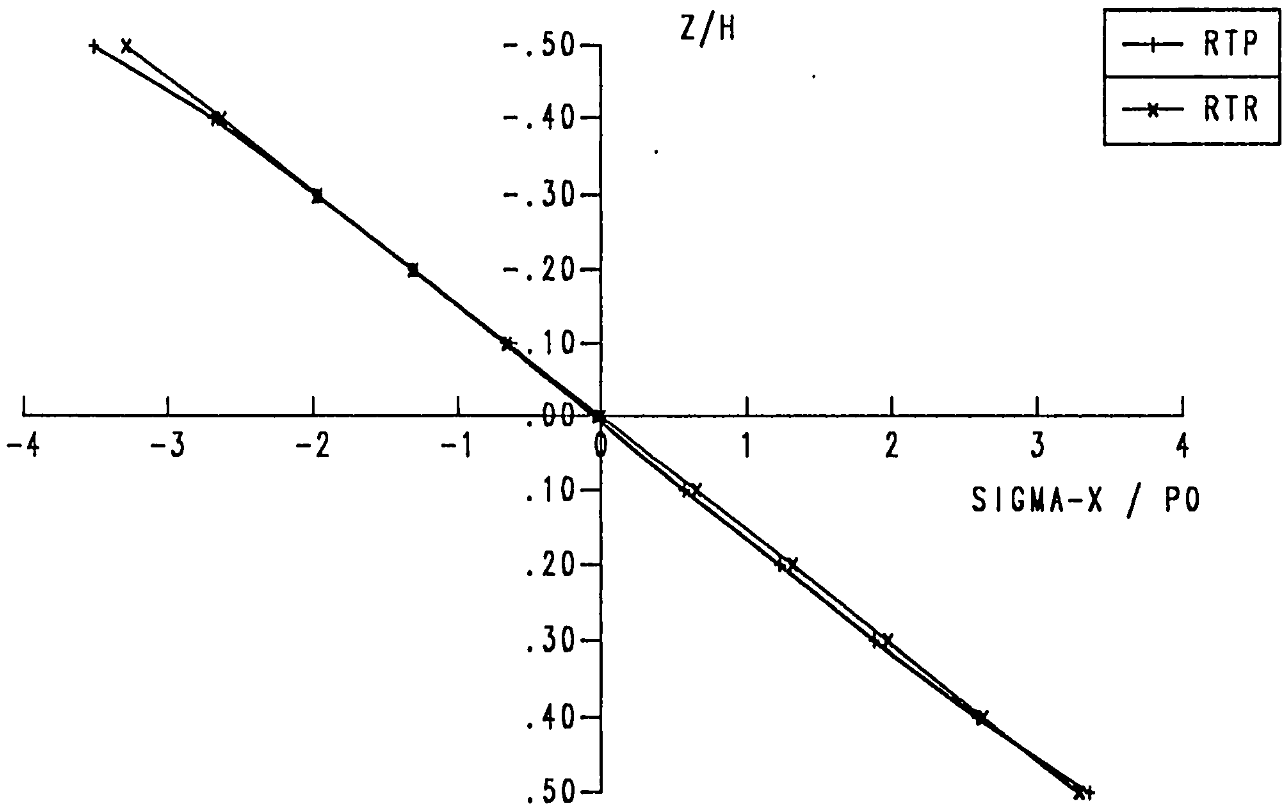


FIG. 5.28 : MAX. NORMAL STRESSSIGMA-X VS Z/H (SS.5-I)

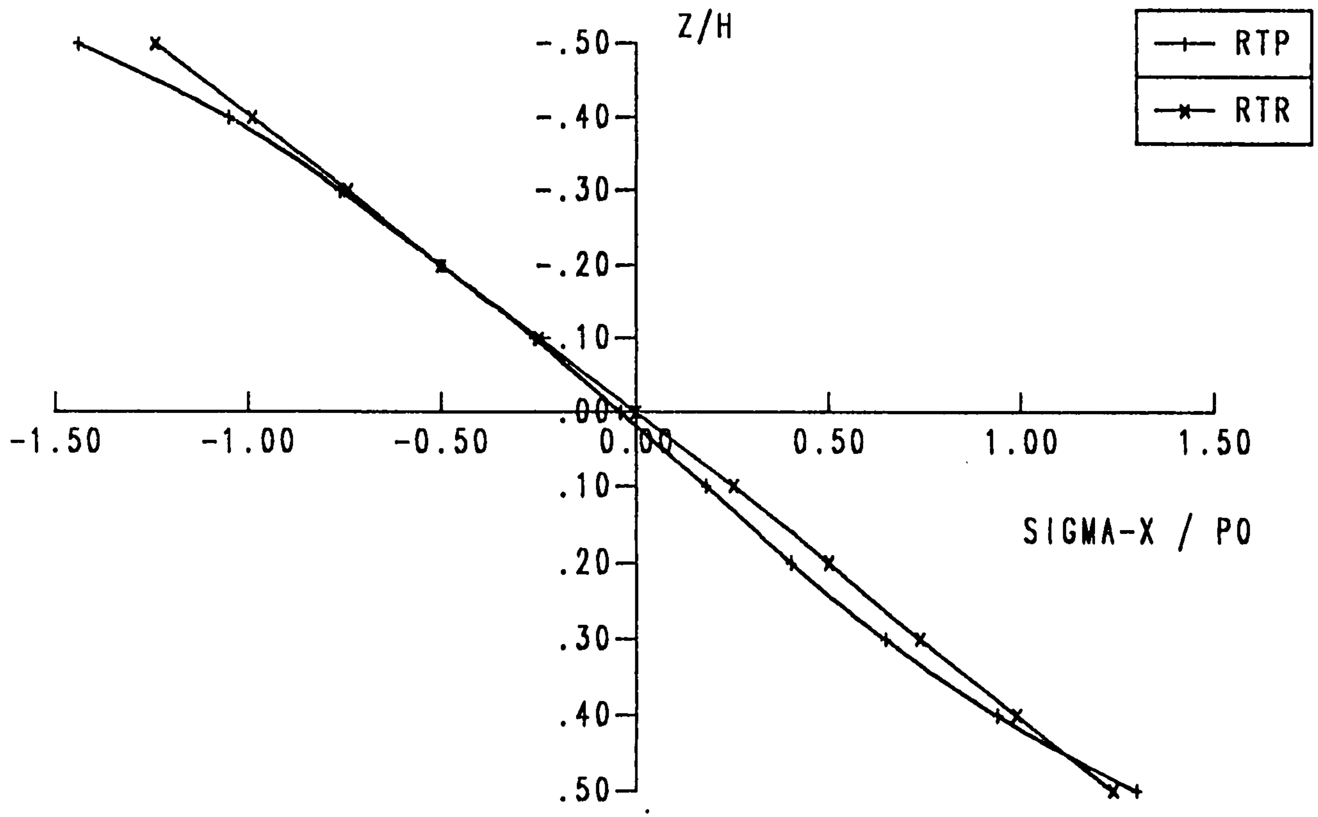


FIG. 5.29 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.7-1)

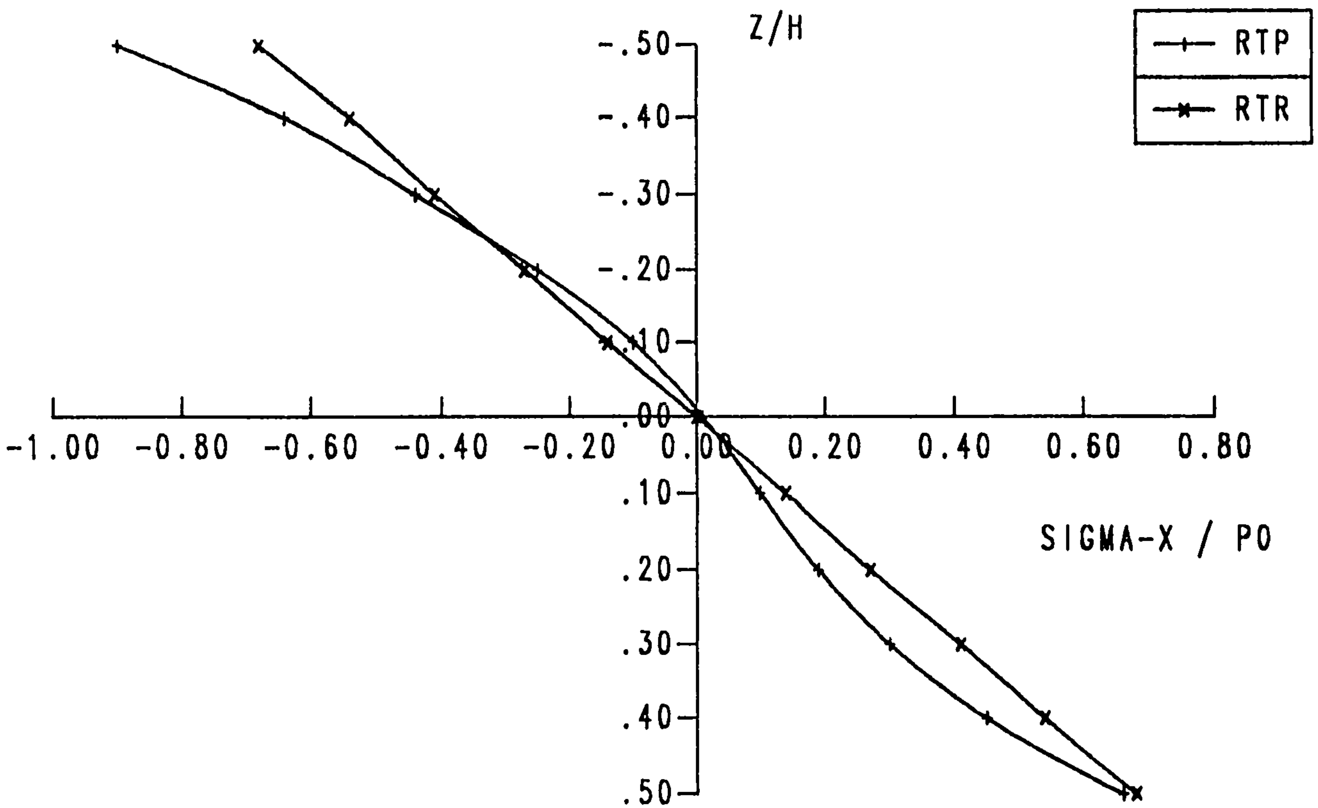


FIG. 5.30 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS1.-I)

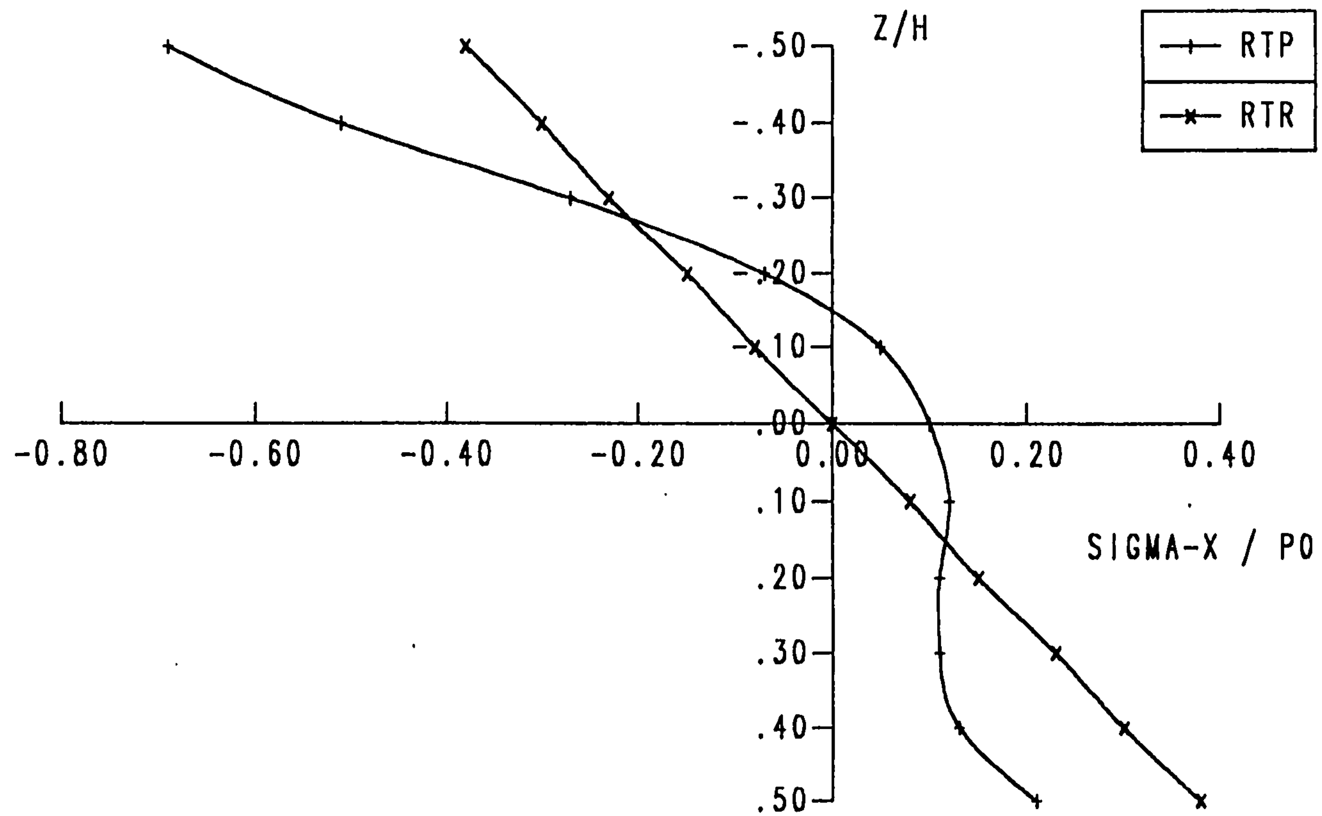


FIG. 5.31 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.005-II)

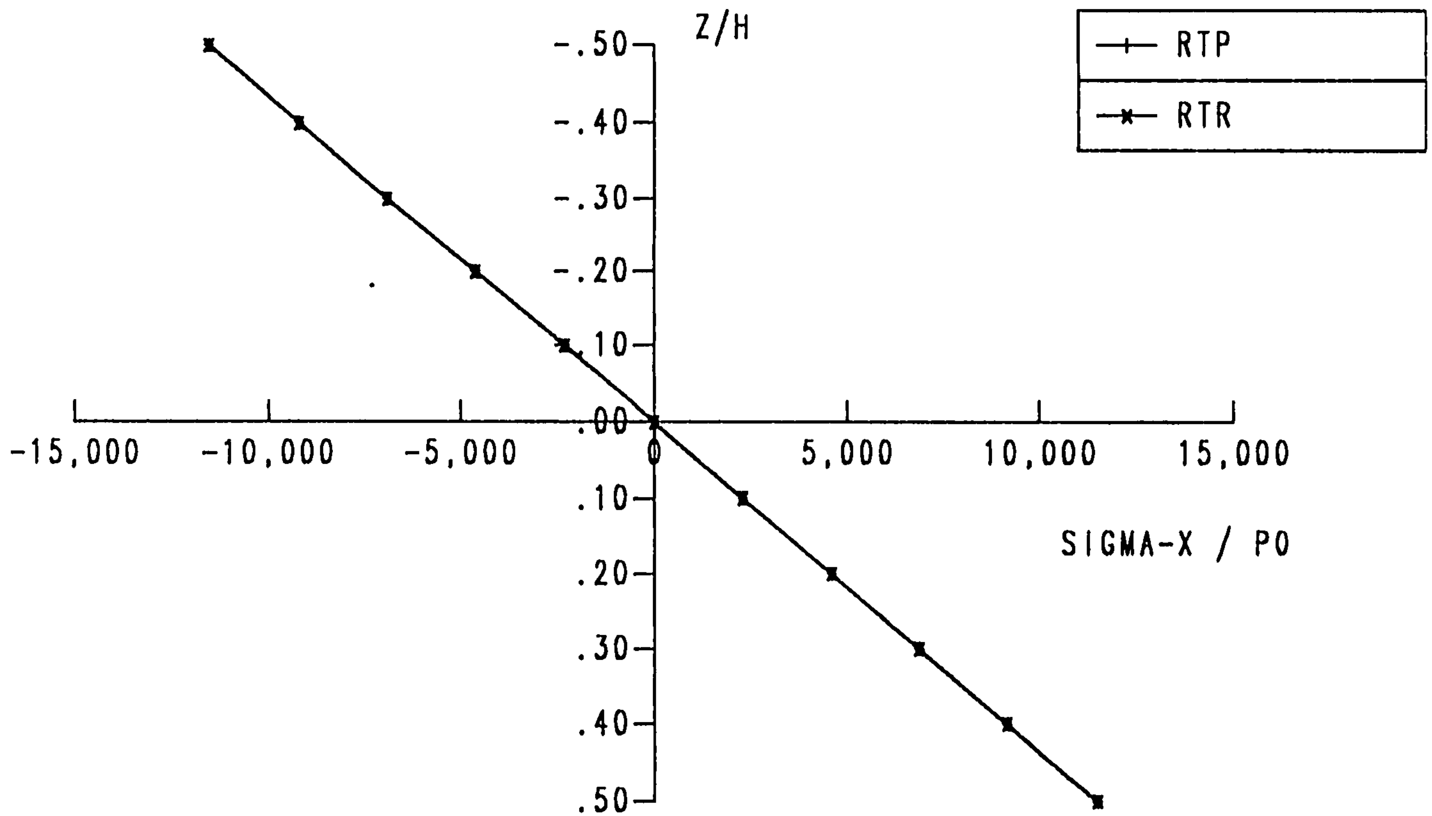


FIG. 5.32 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.01-II)

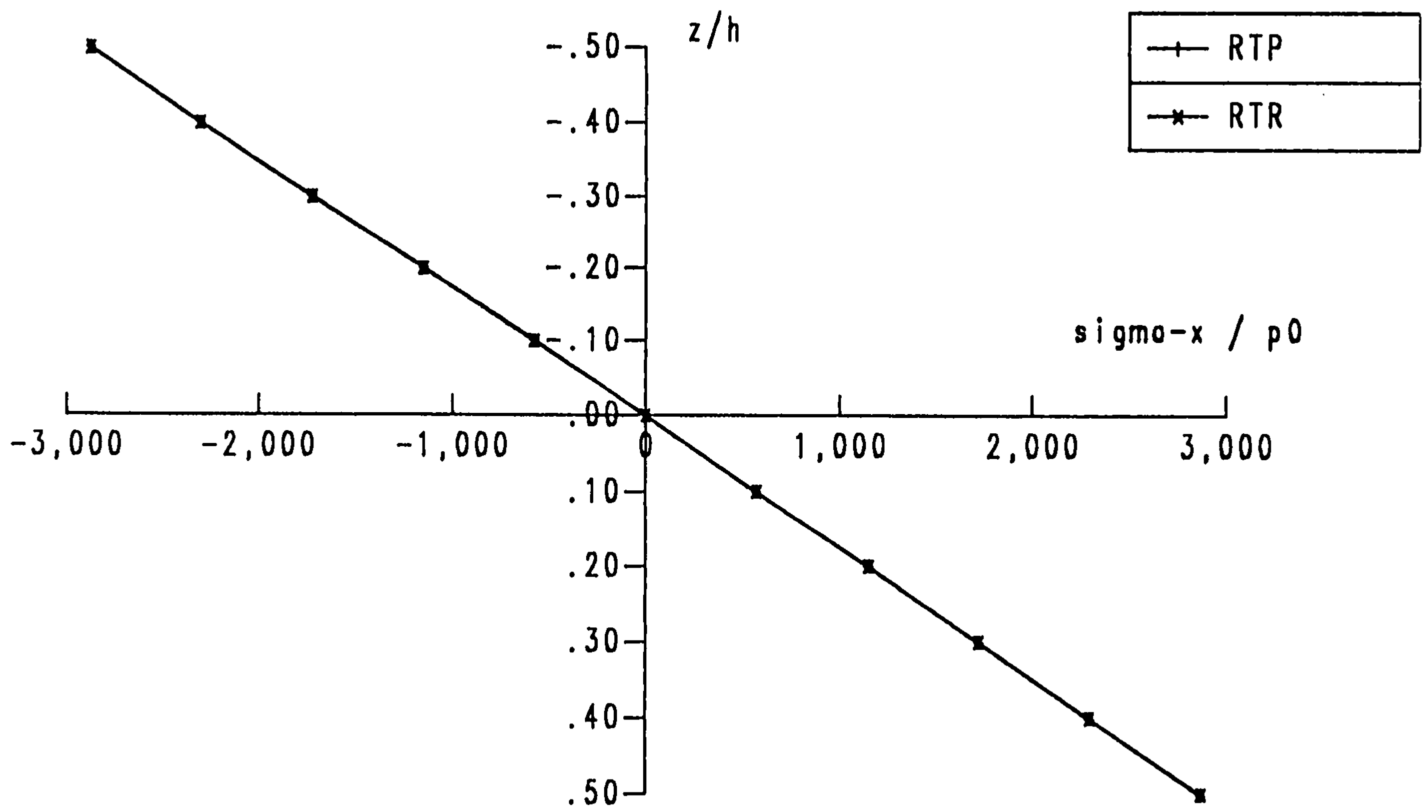


FIG. 5.33 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.05-II)

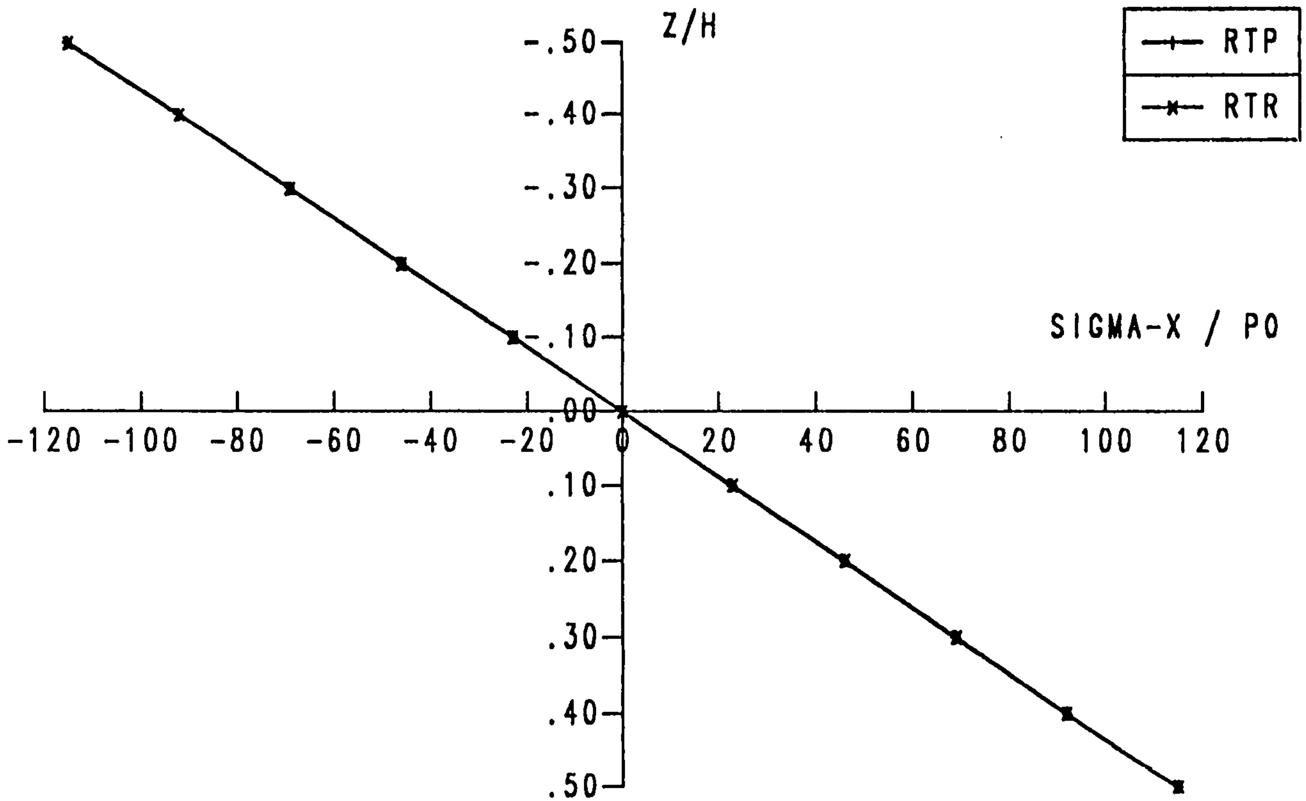


FIG. 5.34 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.1-II)

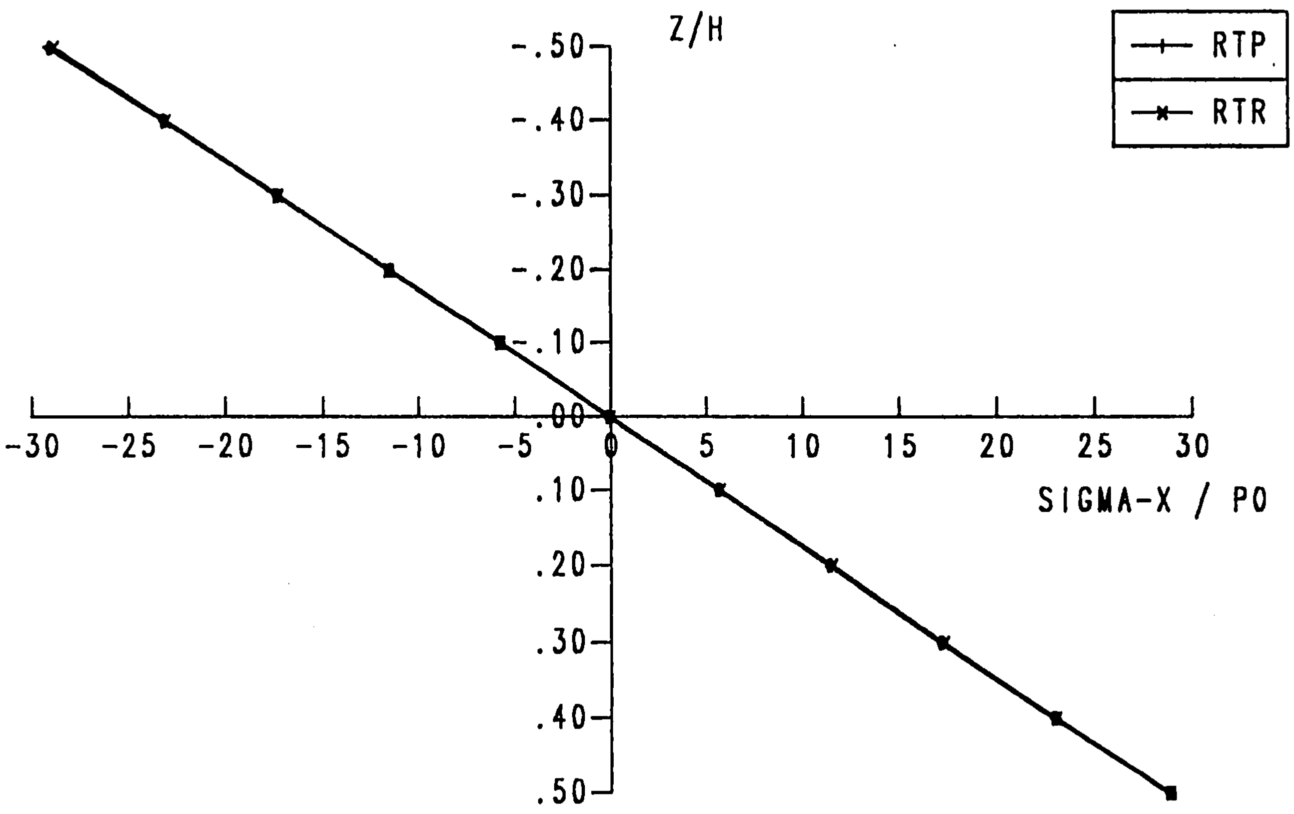


FIG. 5.35 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.2-II)

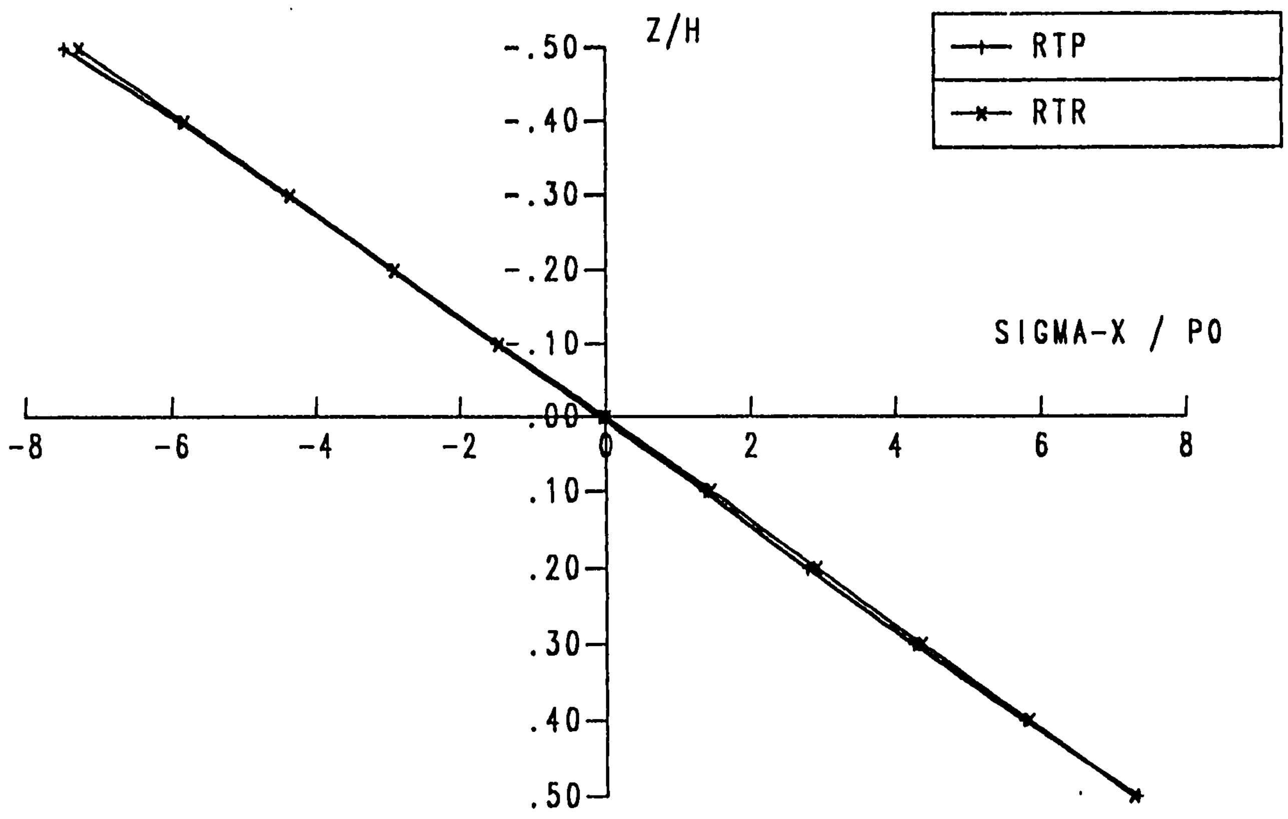


FIG. 5.36 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.3-II)

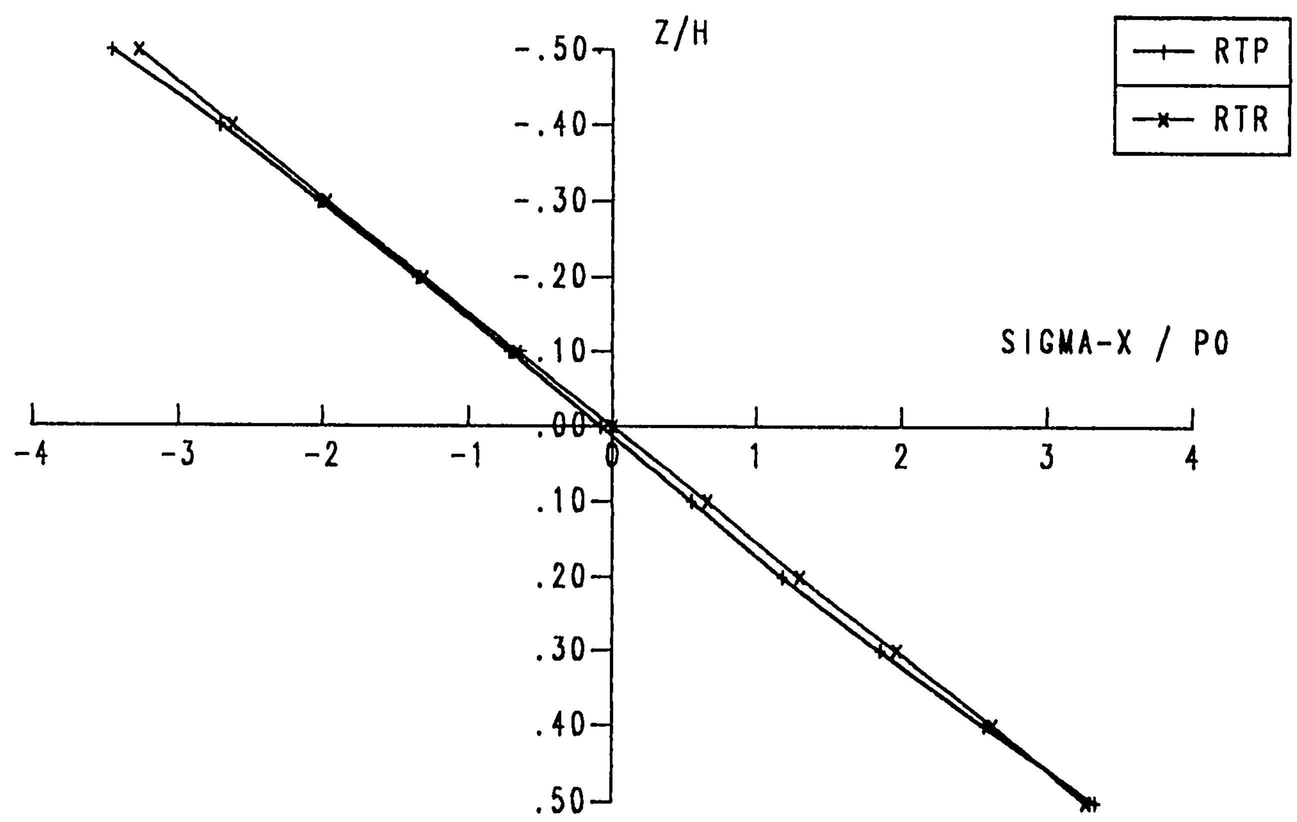


FIG. 5.37 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.4-IL)

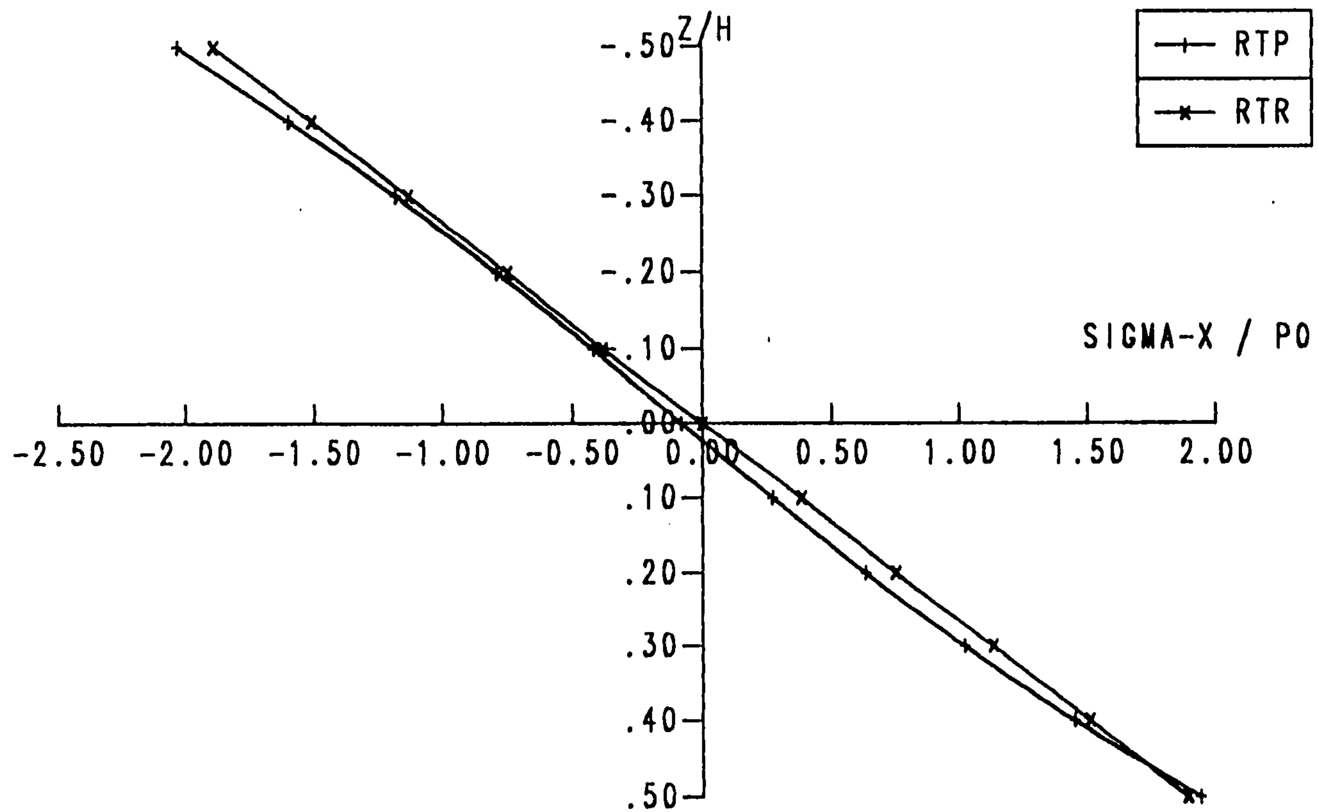


FIG. 5.38 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.5-11)

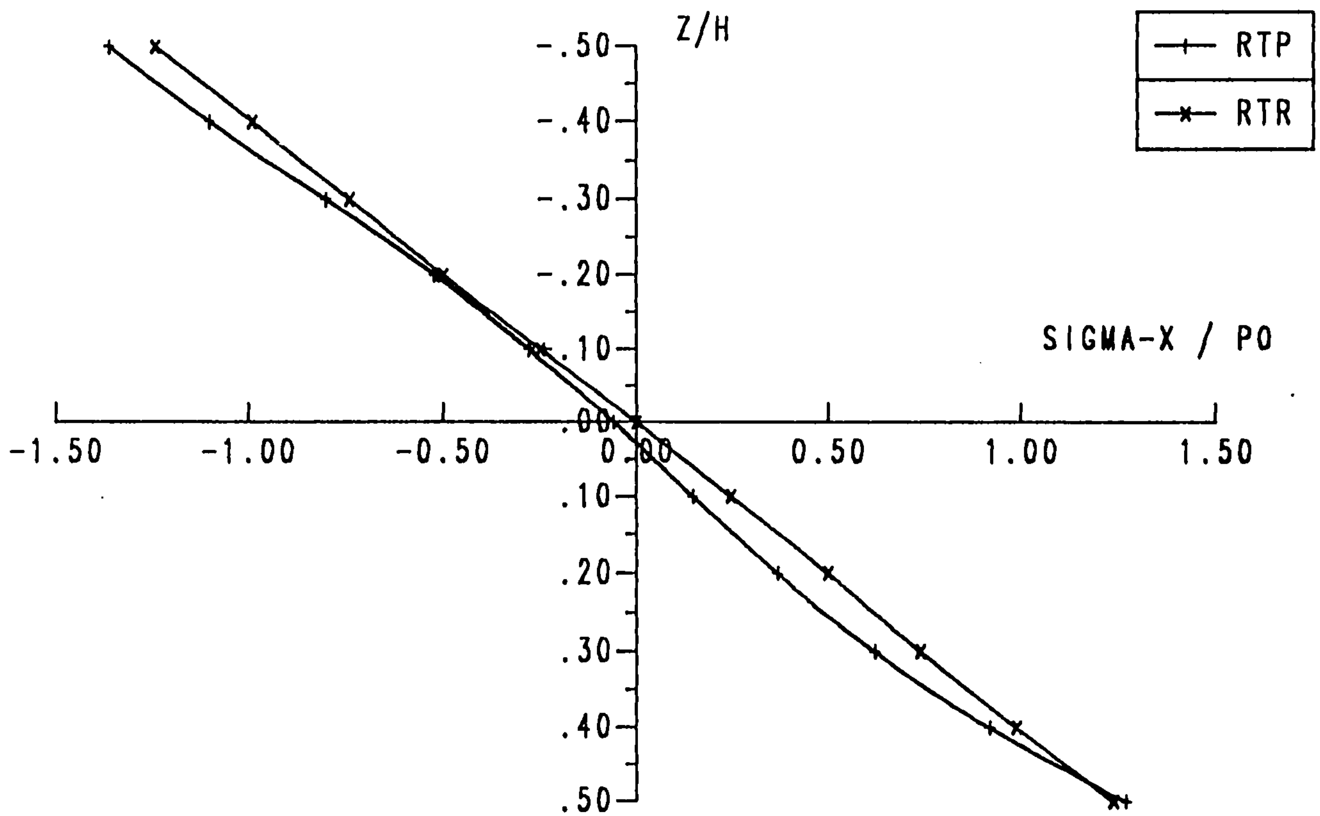


FIG. 5.39 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.6-II)

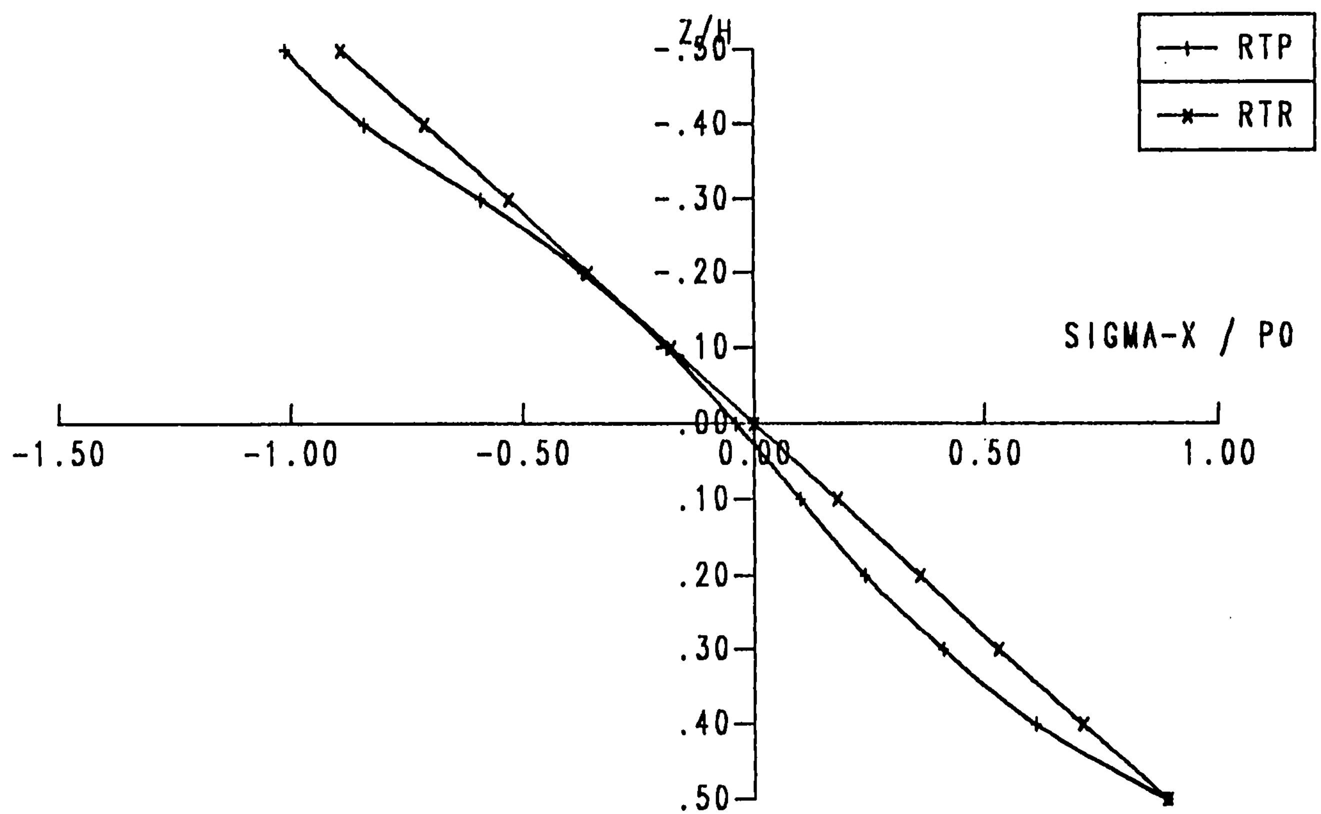


FIG. 5.40 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.7-11)

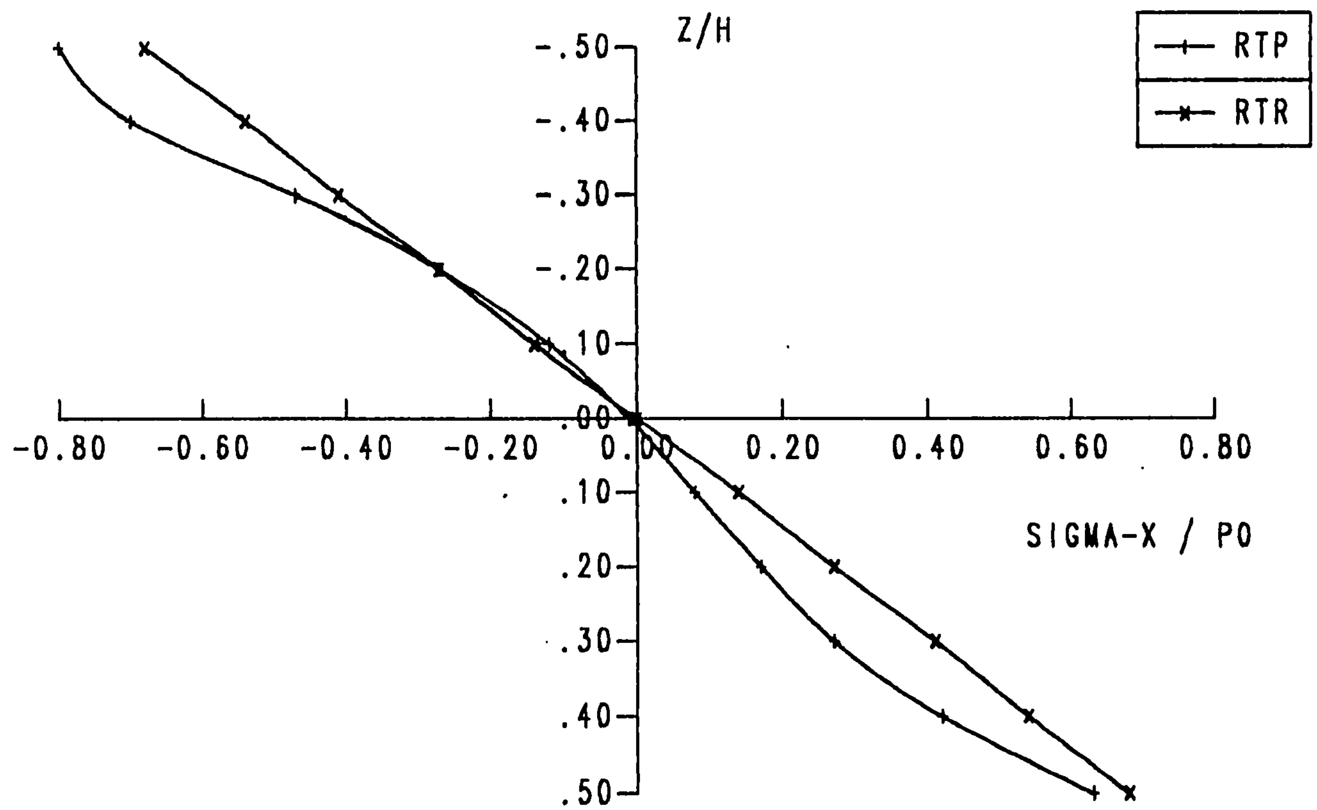


FIG. 5.41 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.8-II)

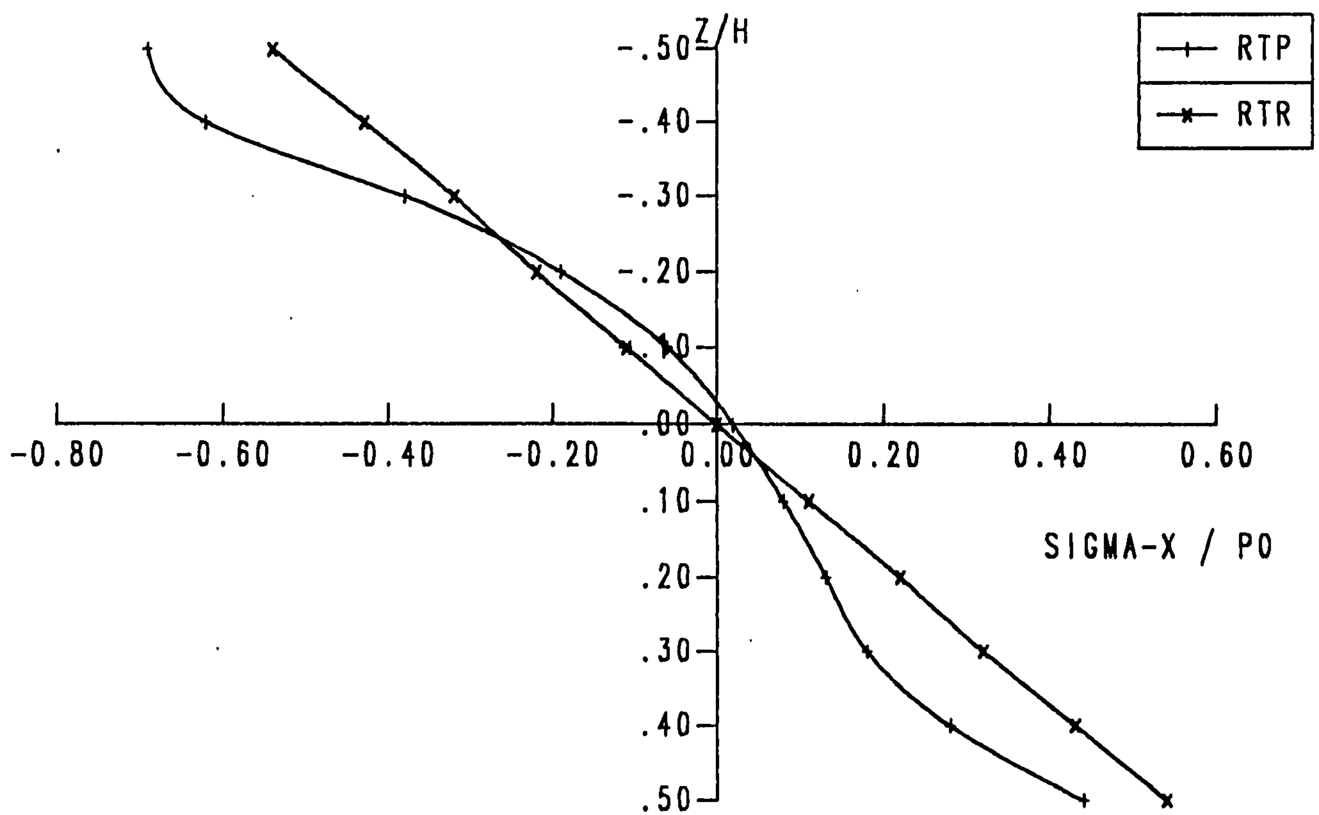
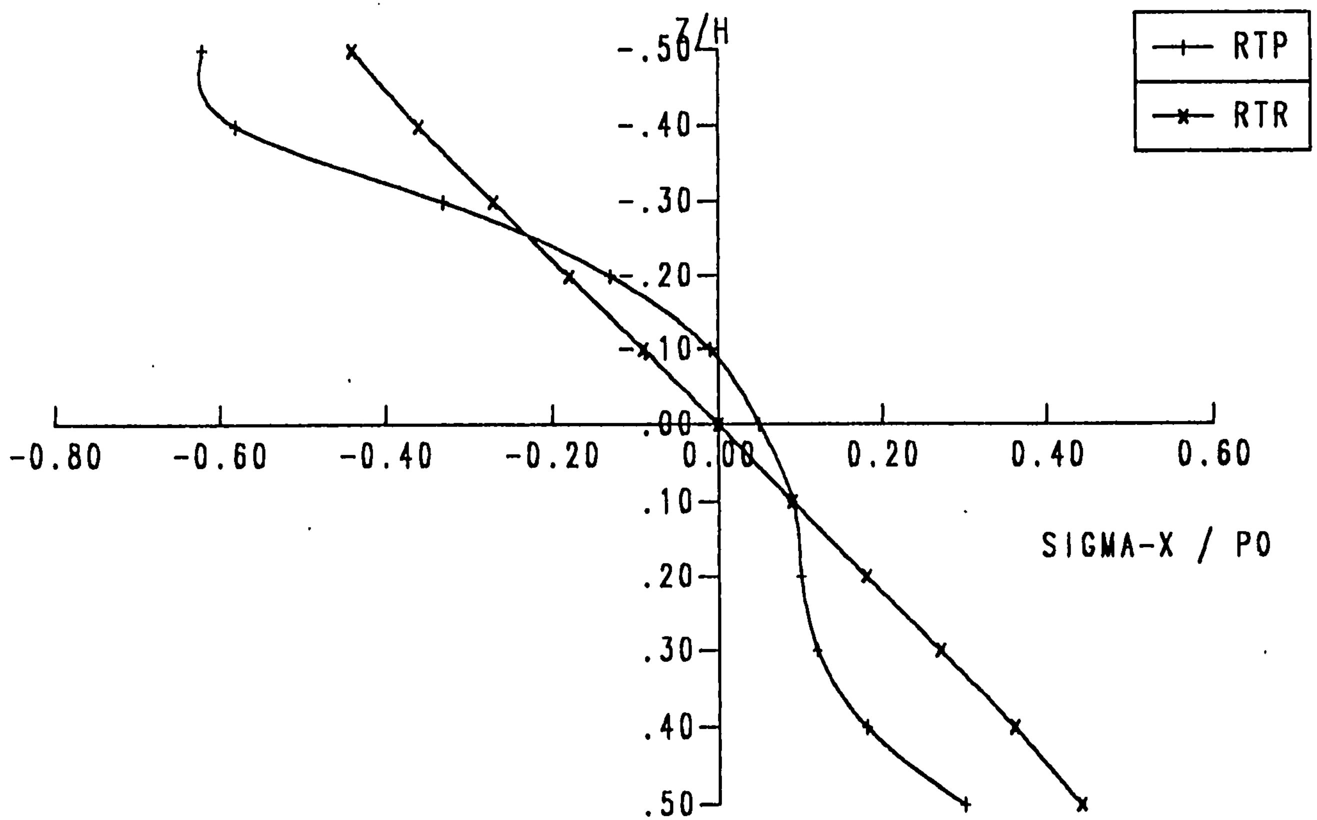


FIG. 5.42 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS.9-II)



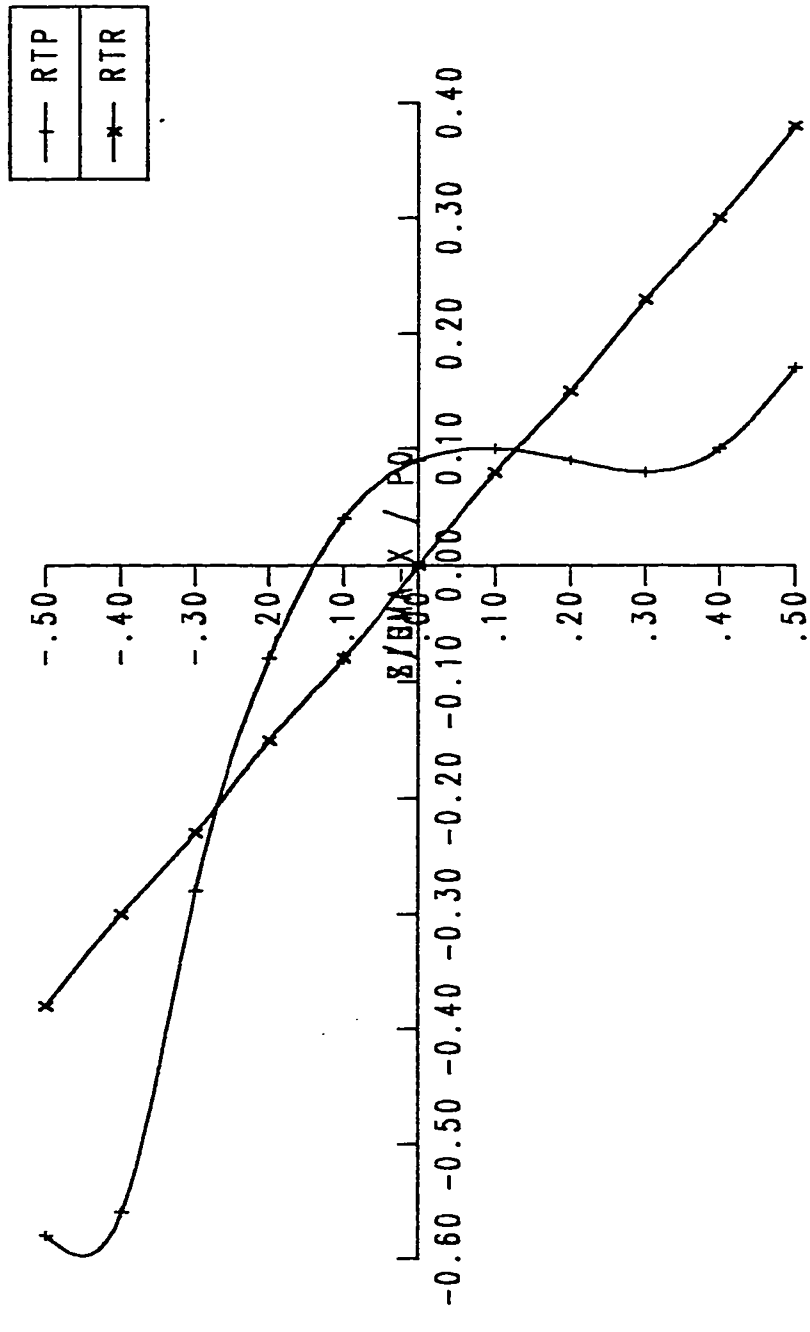


FIG. 5.43 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SS1.-II)

FIG. 5.44 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.1-II)

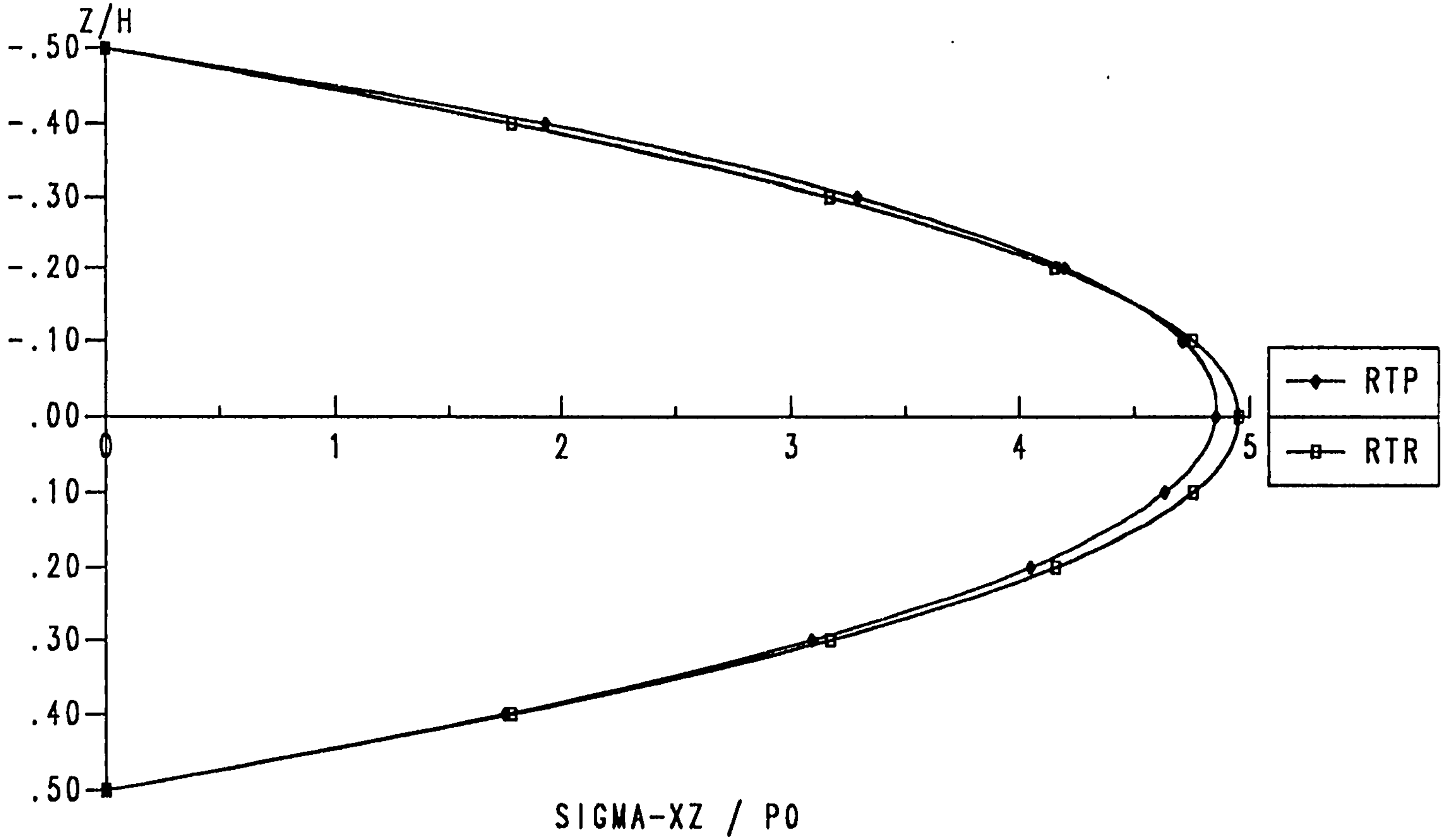


FIG. 5.45 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.3-II)

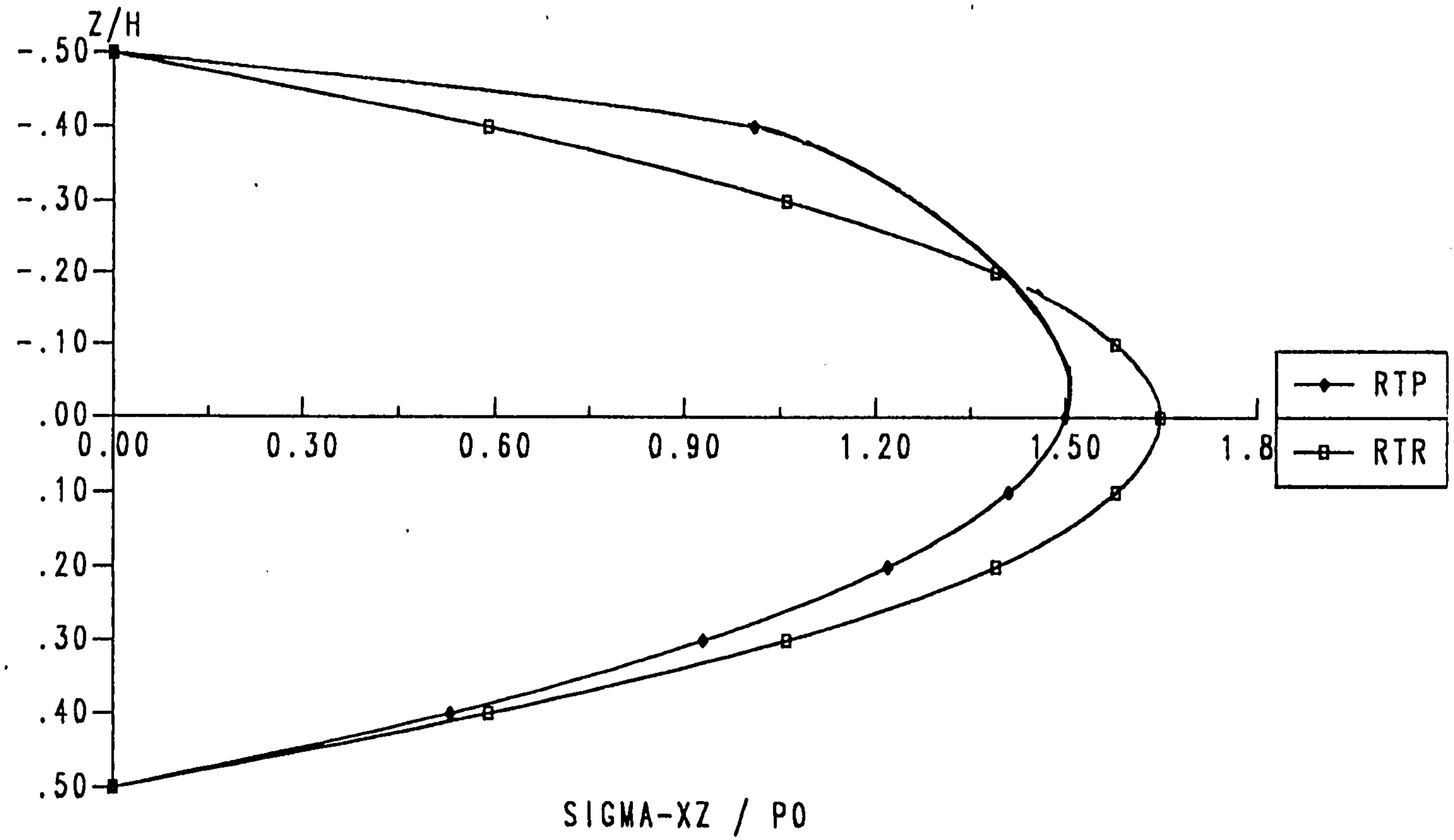


FIG. 5.46 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS.5-II)

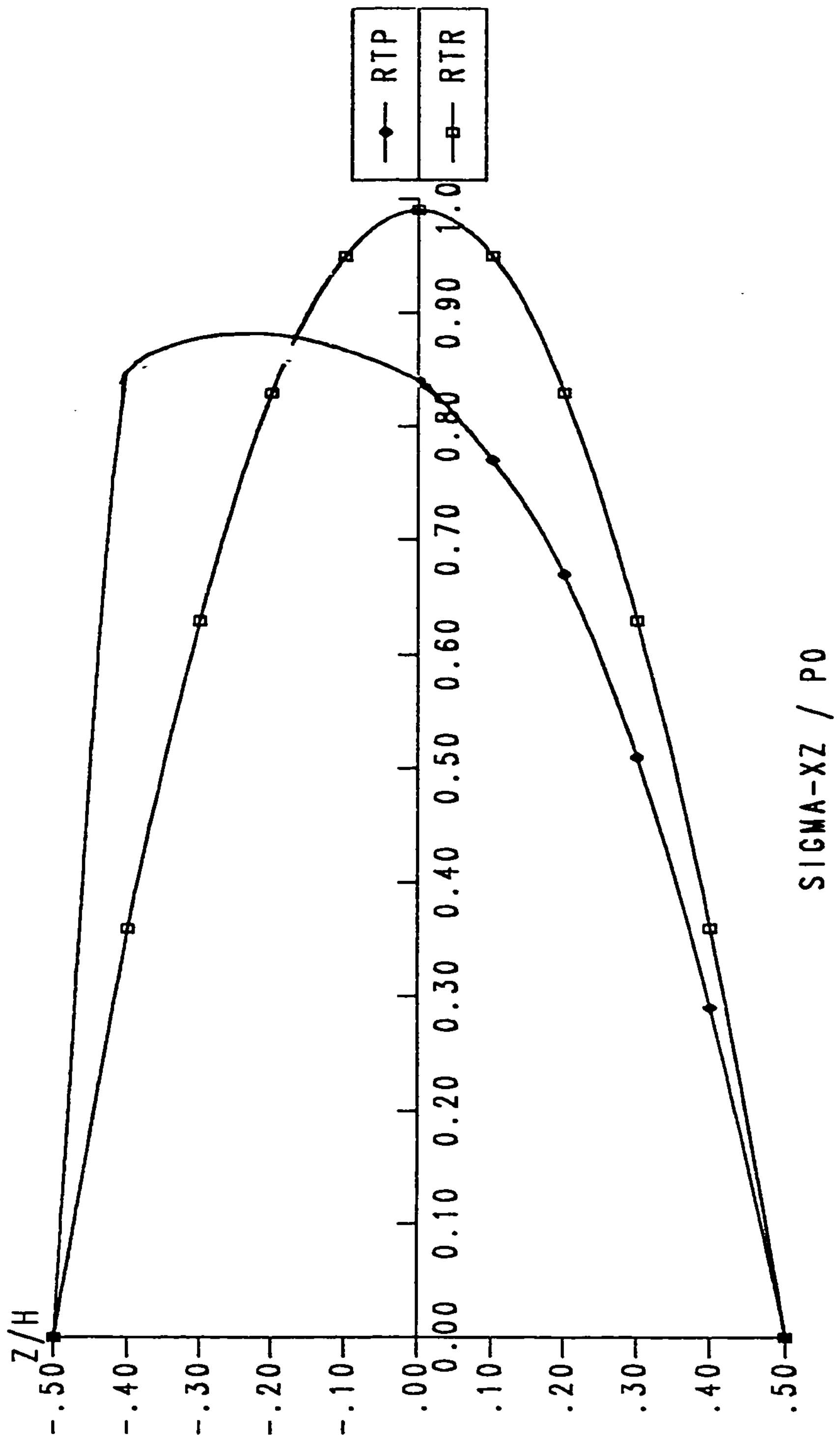


FIG. 5.47 : SIGMA-XZ AT (0,0,Z) VS Z/H (SS1.-II)

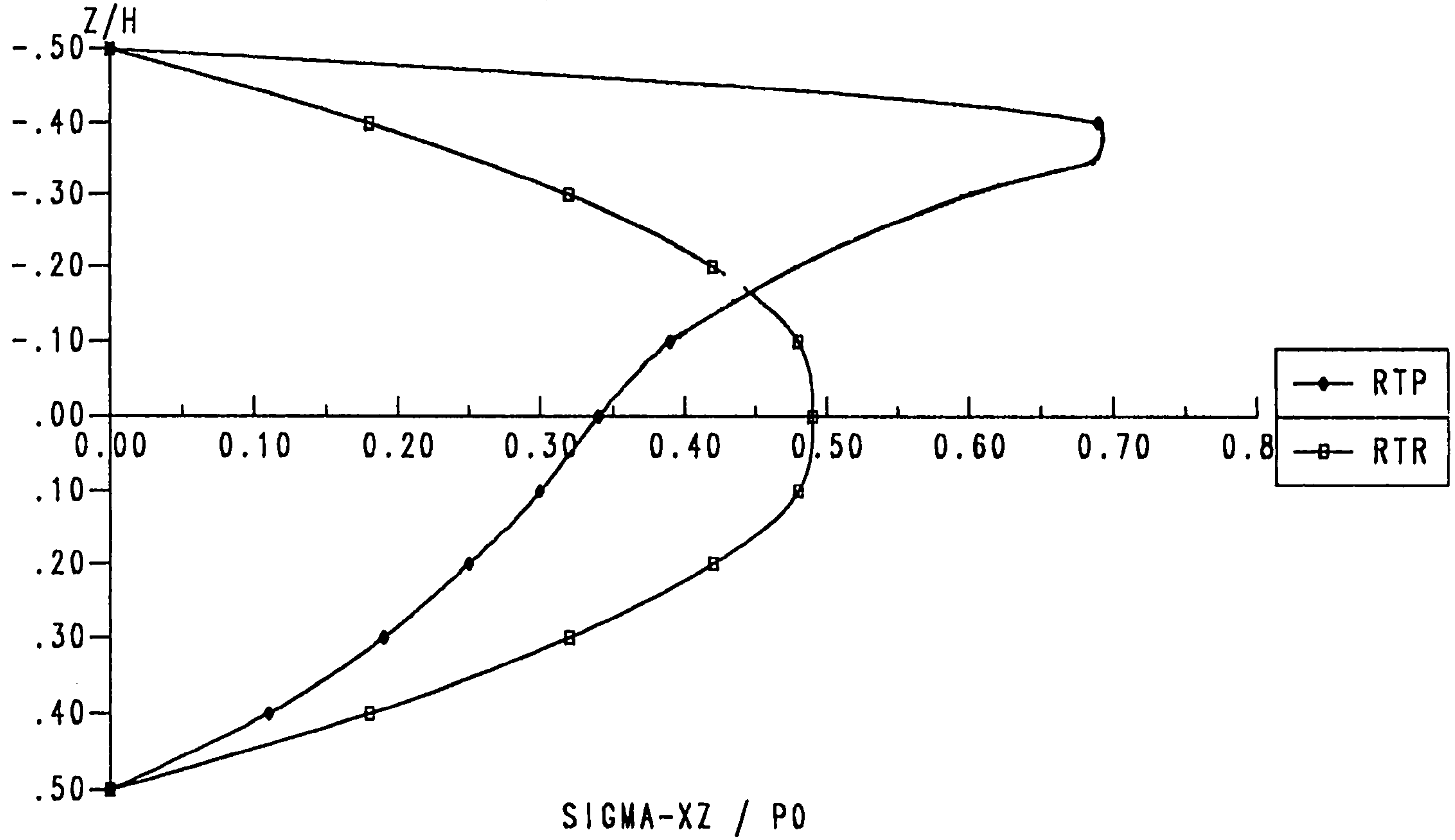


FIG. 5.48 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.005-I)

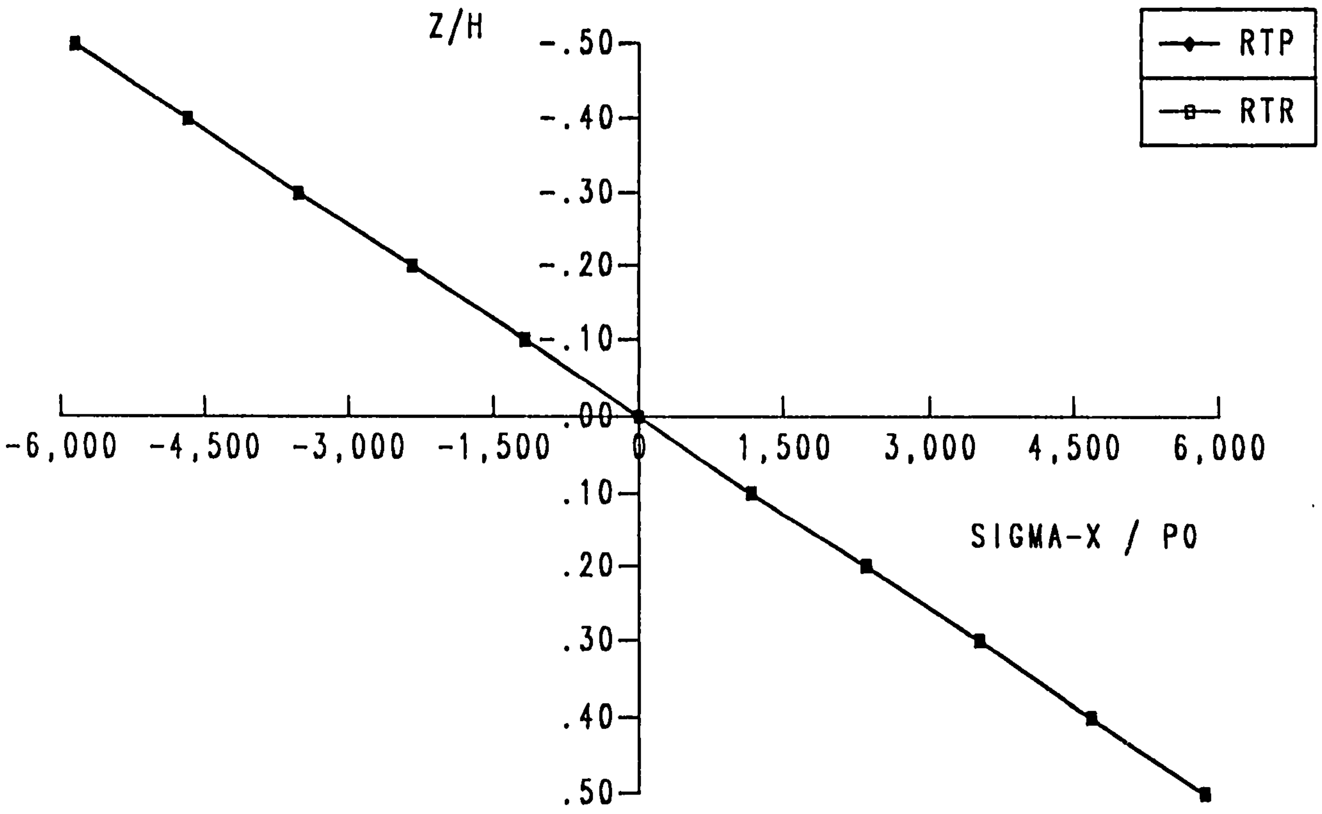


FIG. 5.49 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.1-I)

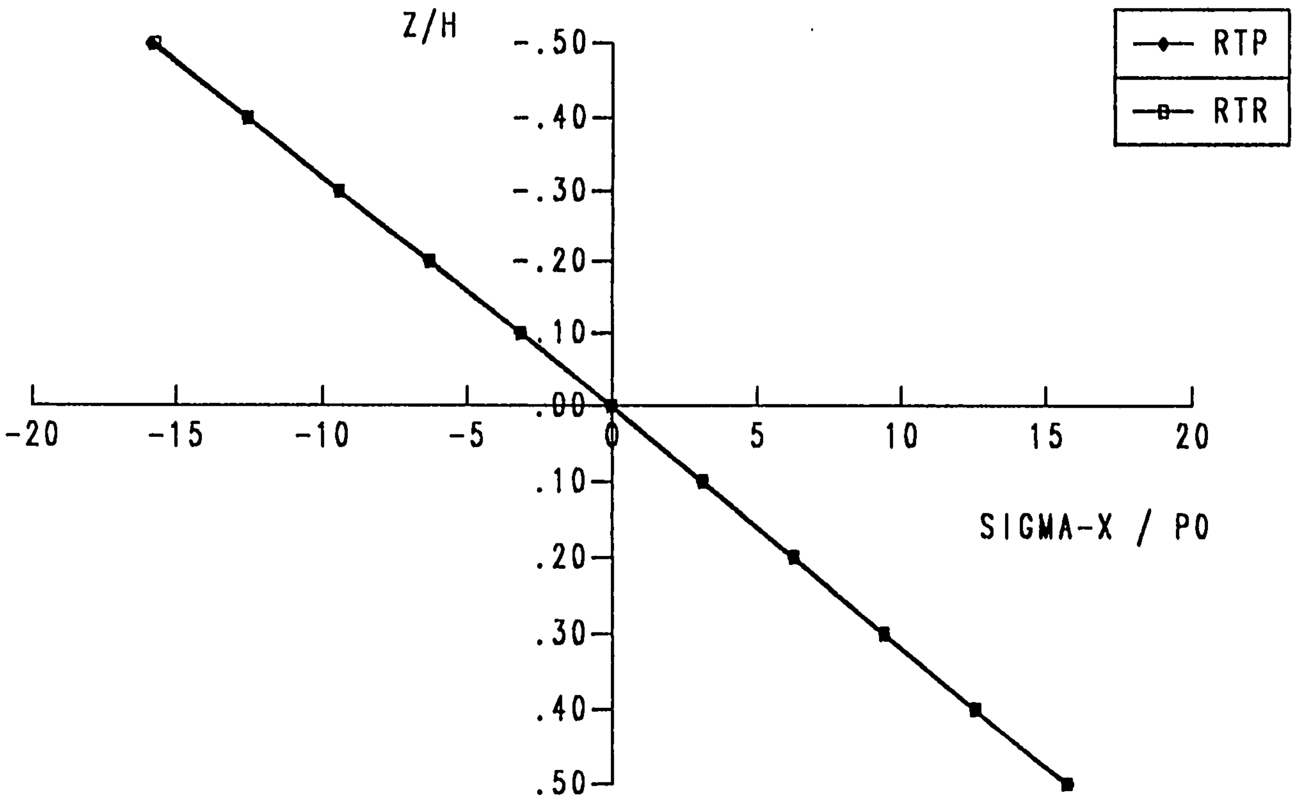


FIG. 5.50 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.3-I)

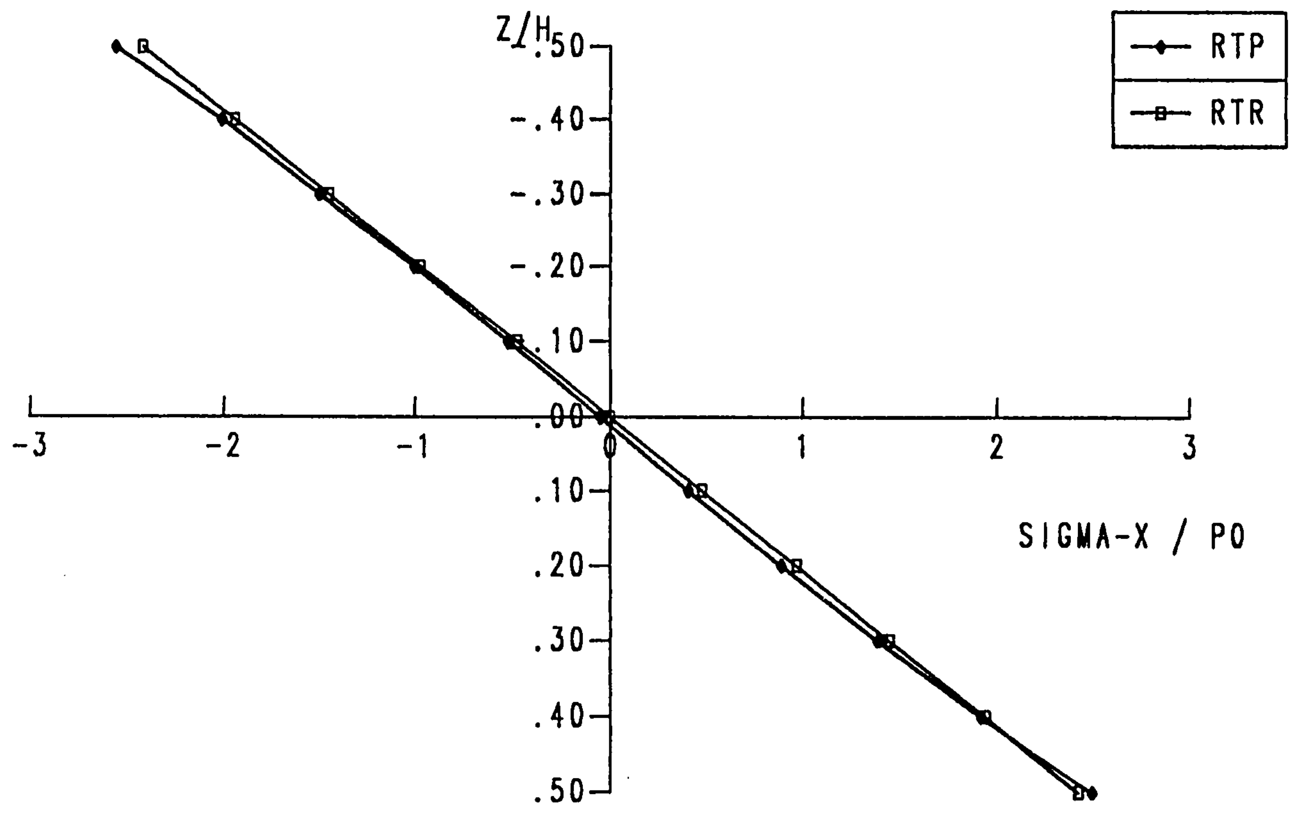


FIG. 5.51 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.5-1)

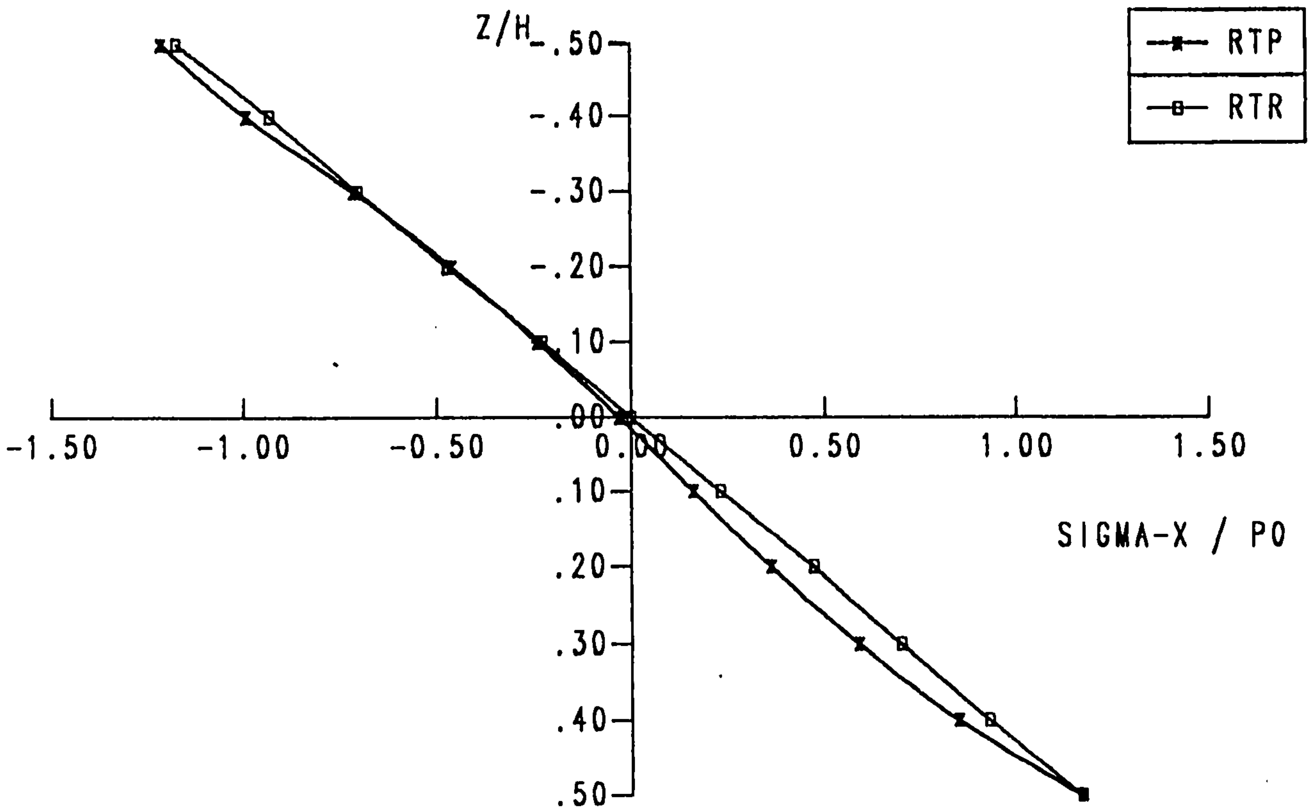


FIG. 5.52 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.7-I)

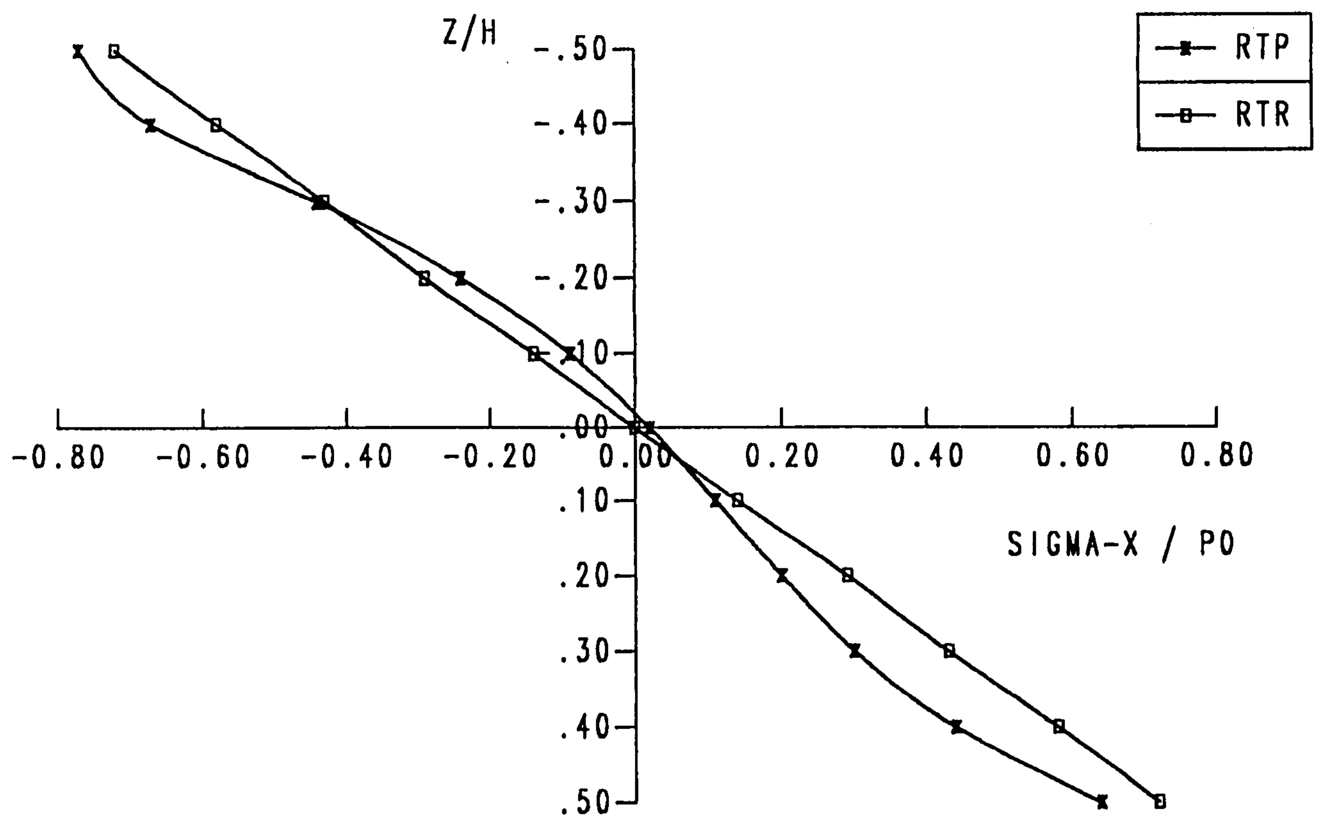


FIG. 5.53 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC1.-I)

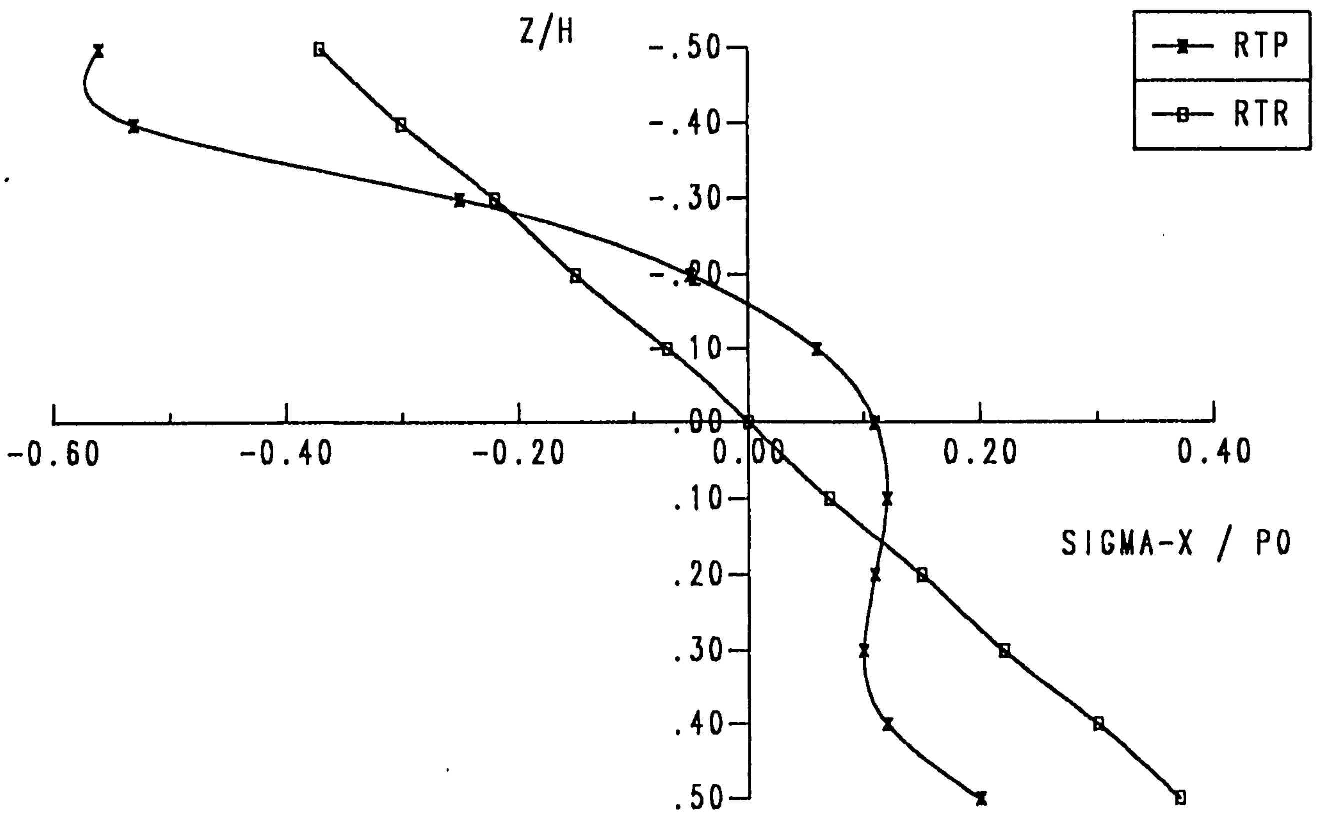


FIG. 5.54 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.005-II)

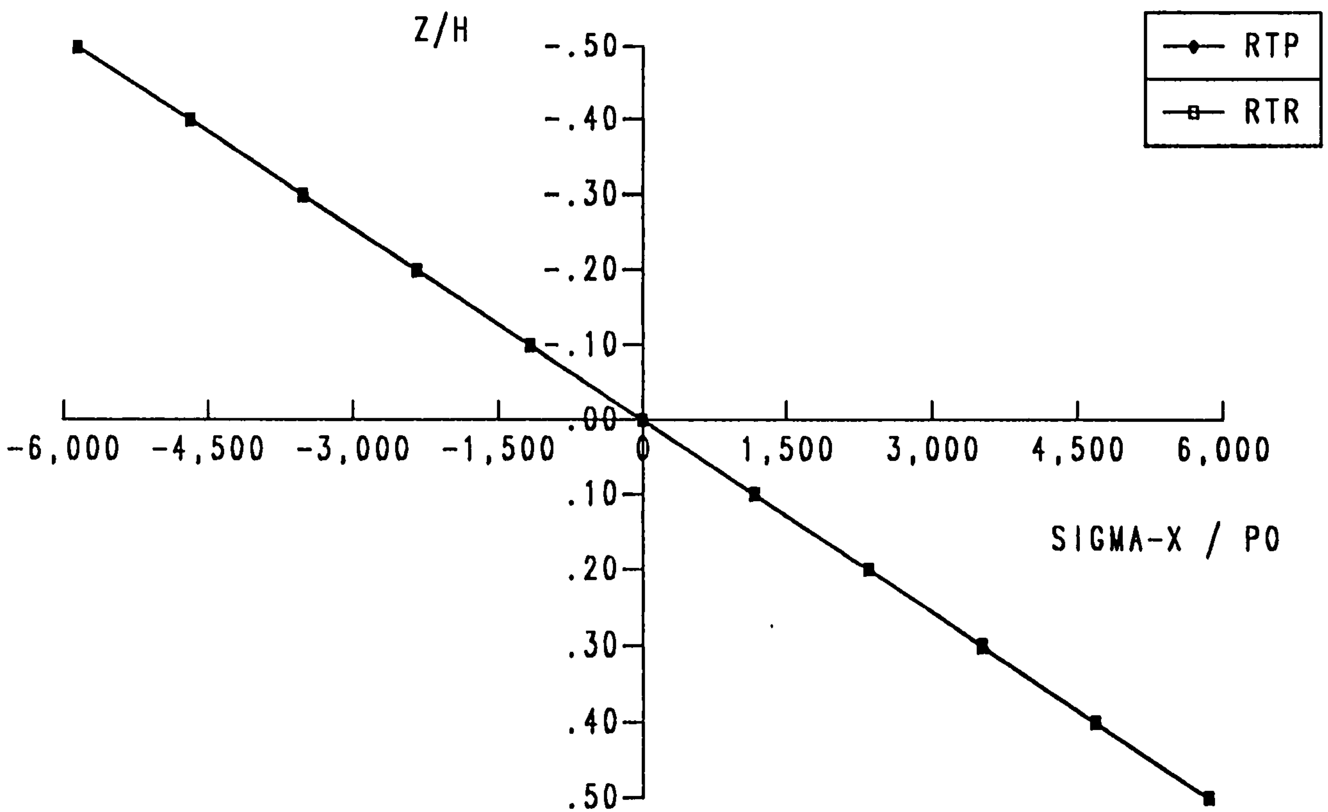


FIG. 5.55 : MAX. NORMALSTRESS SIGMA-X VS Z/H (SC.1-II)

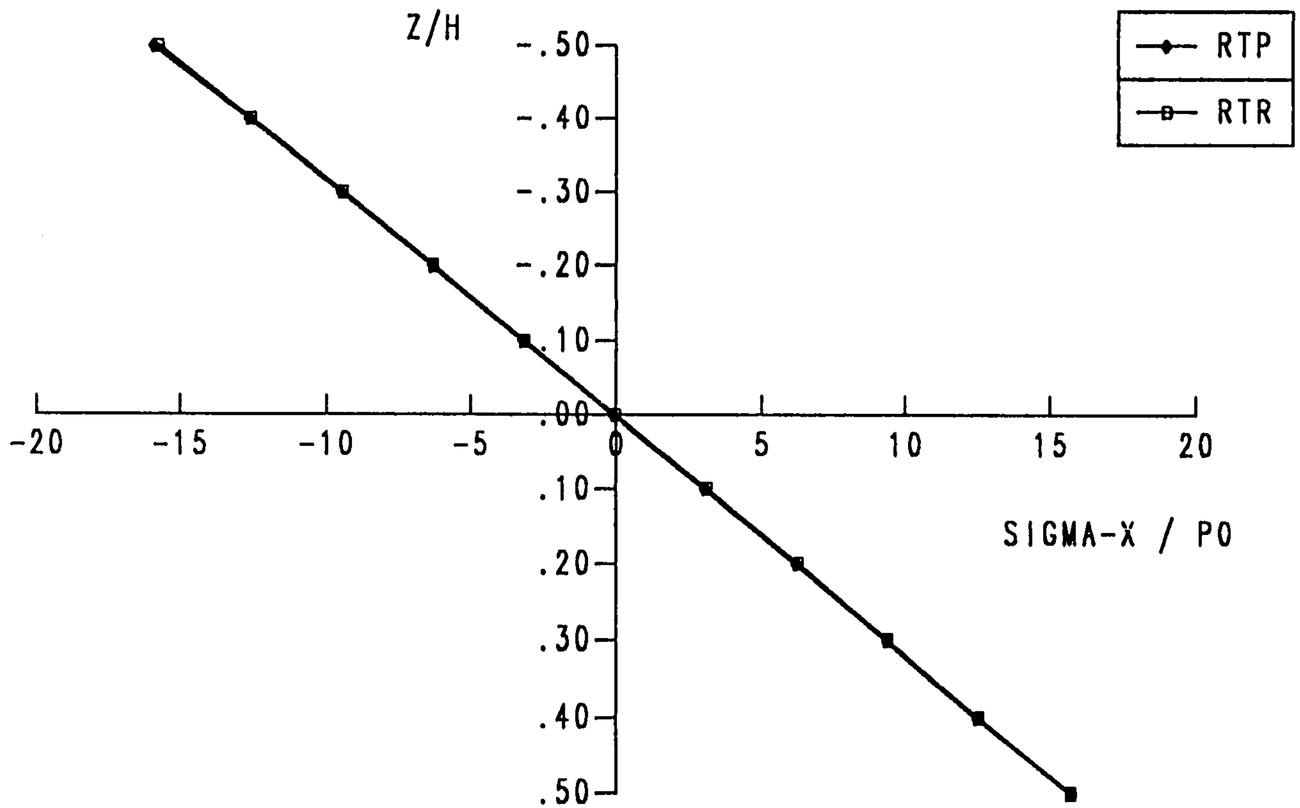


FIG. 5.56 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.3-II)

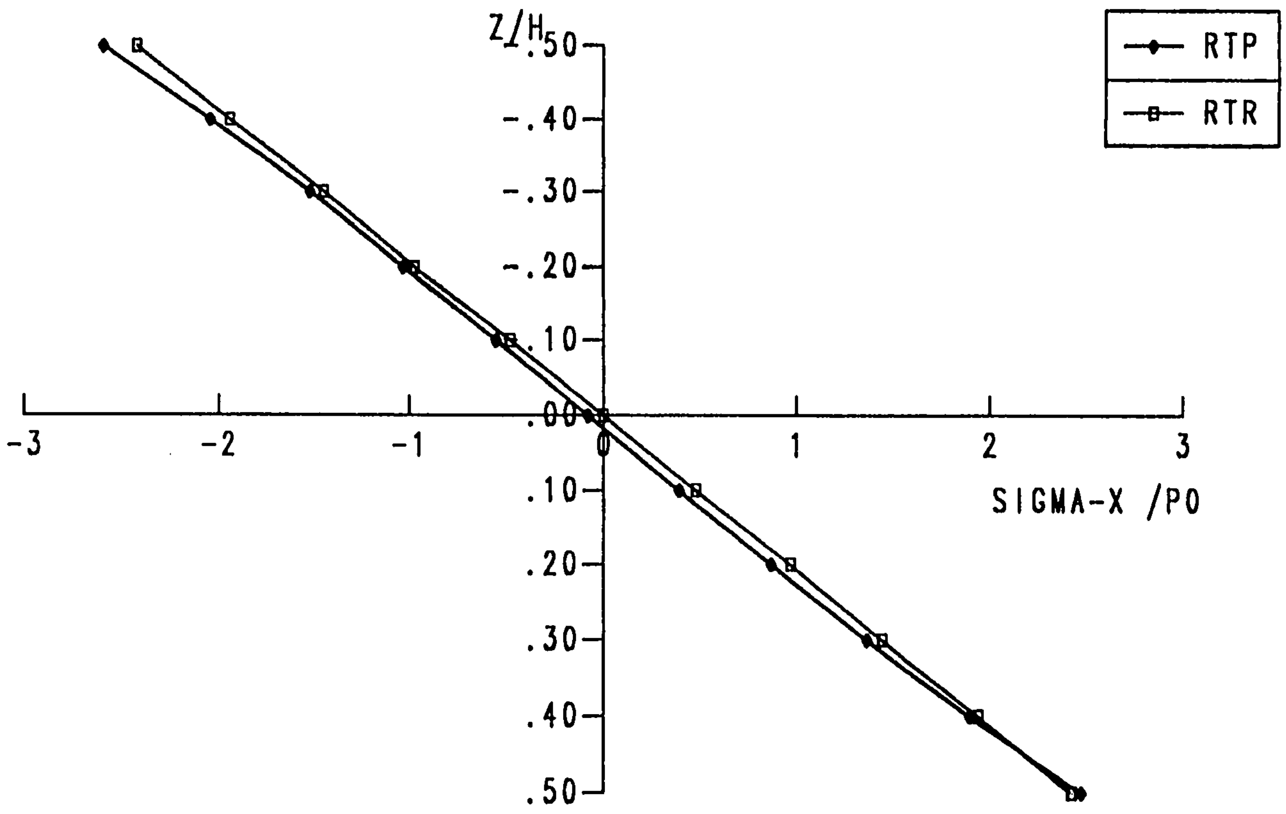


FIG. 5.57 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.5-11)

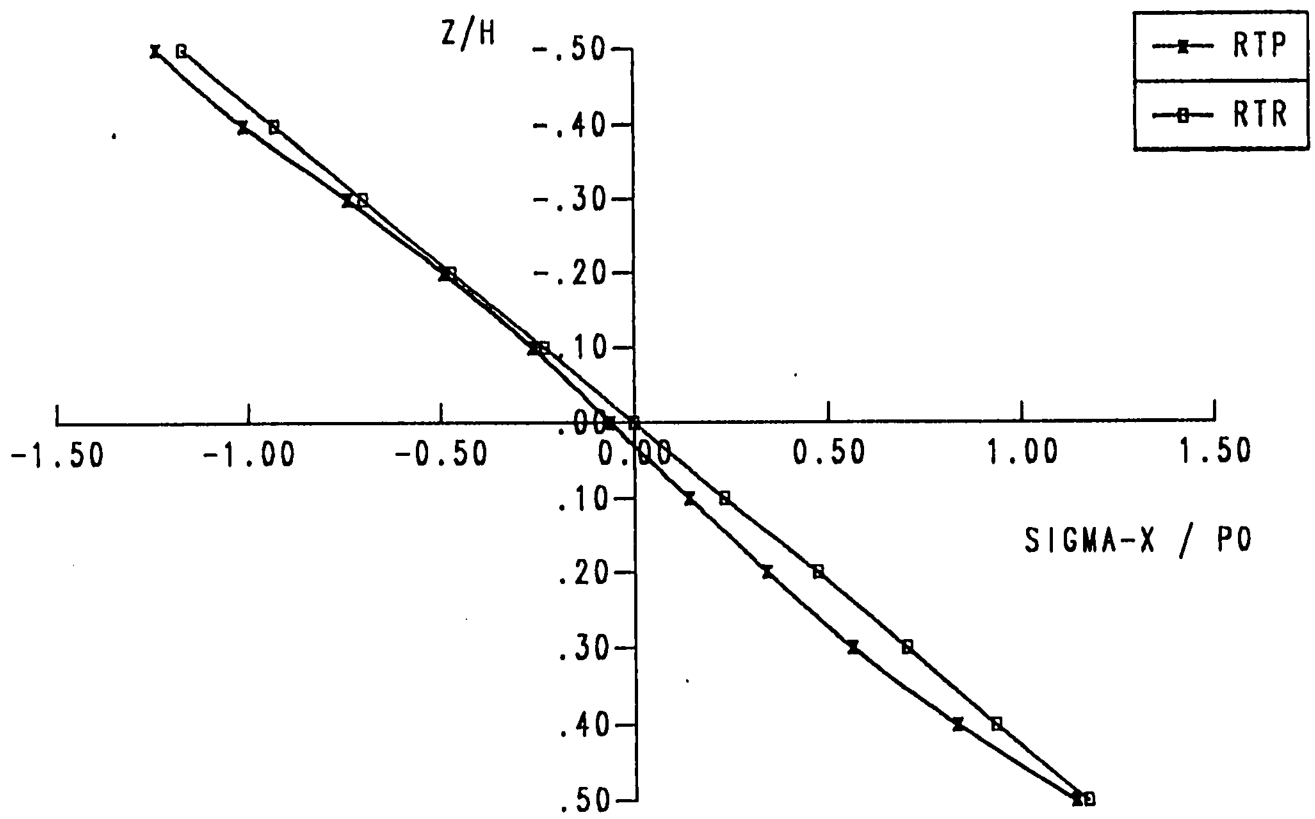


FIG. 5.58 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC.7-II)

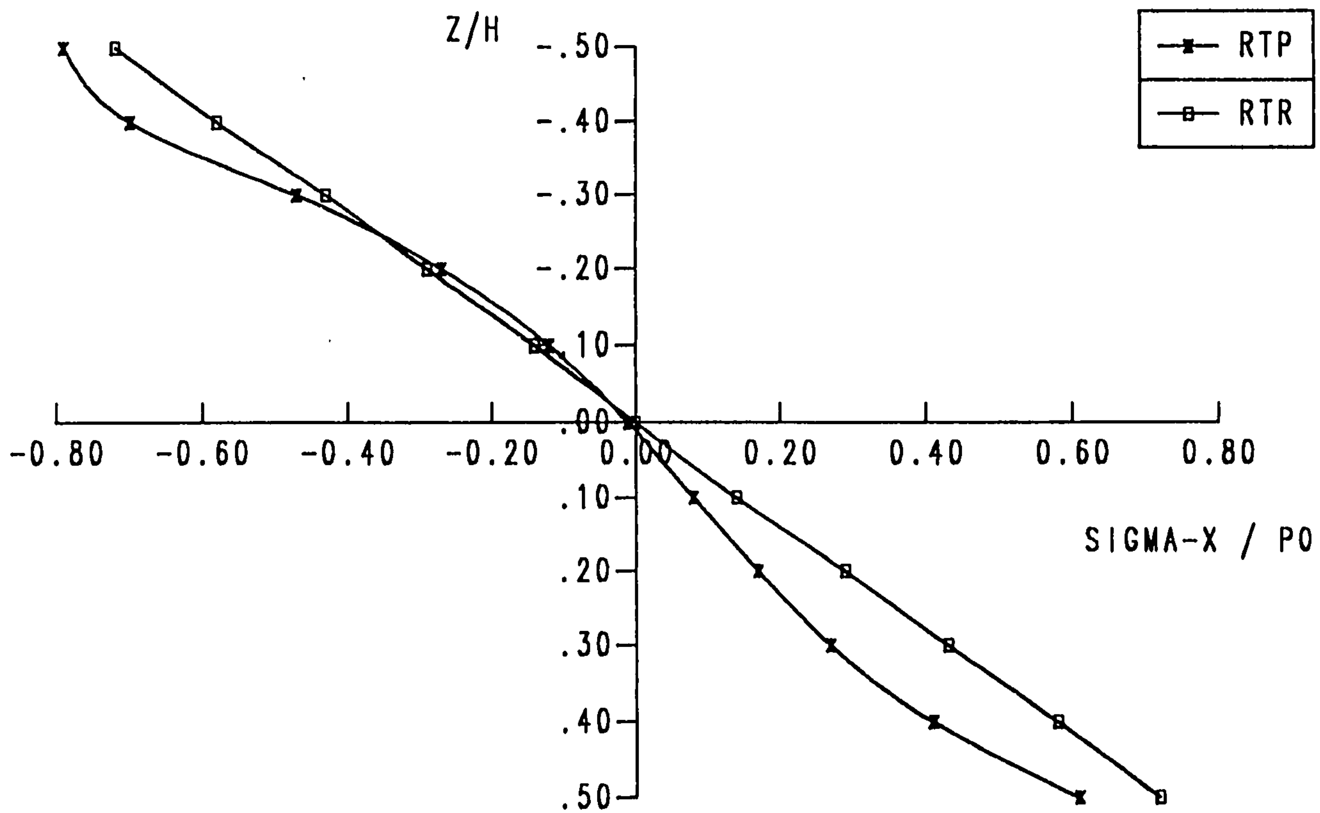


FIG. 5.59 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SC1.-II)

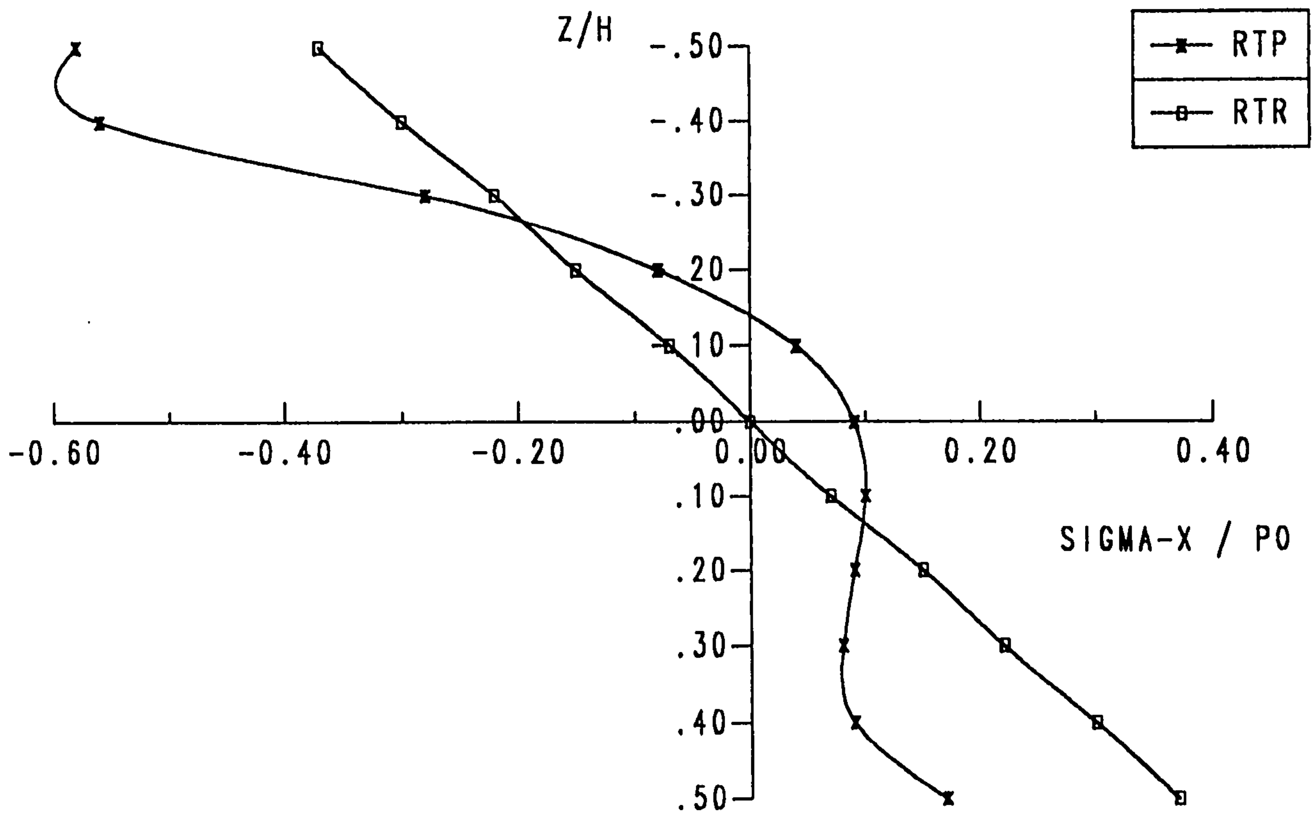


FIG. 5.60 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.005-I)

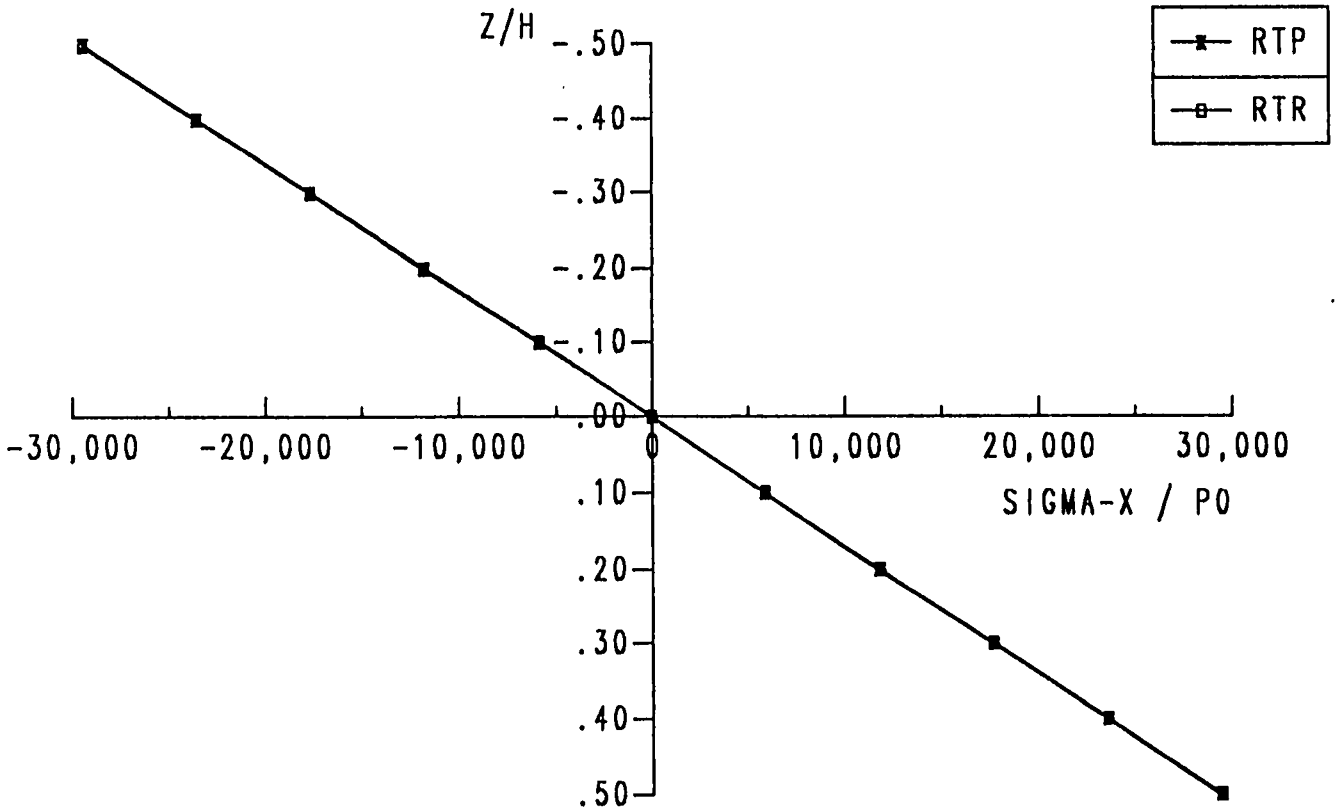


FIG. 5.61 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.1-I)

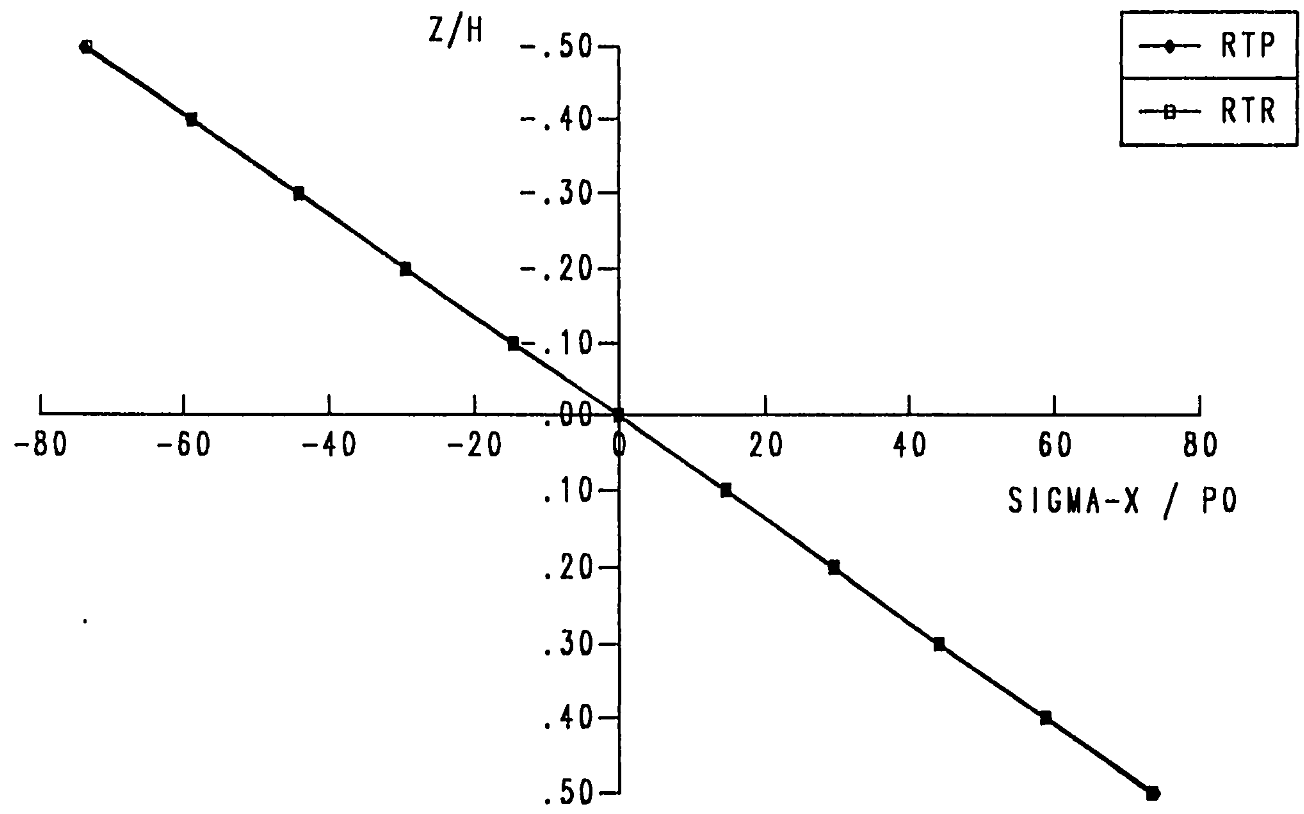


FIG. 5.62 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.3-I)

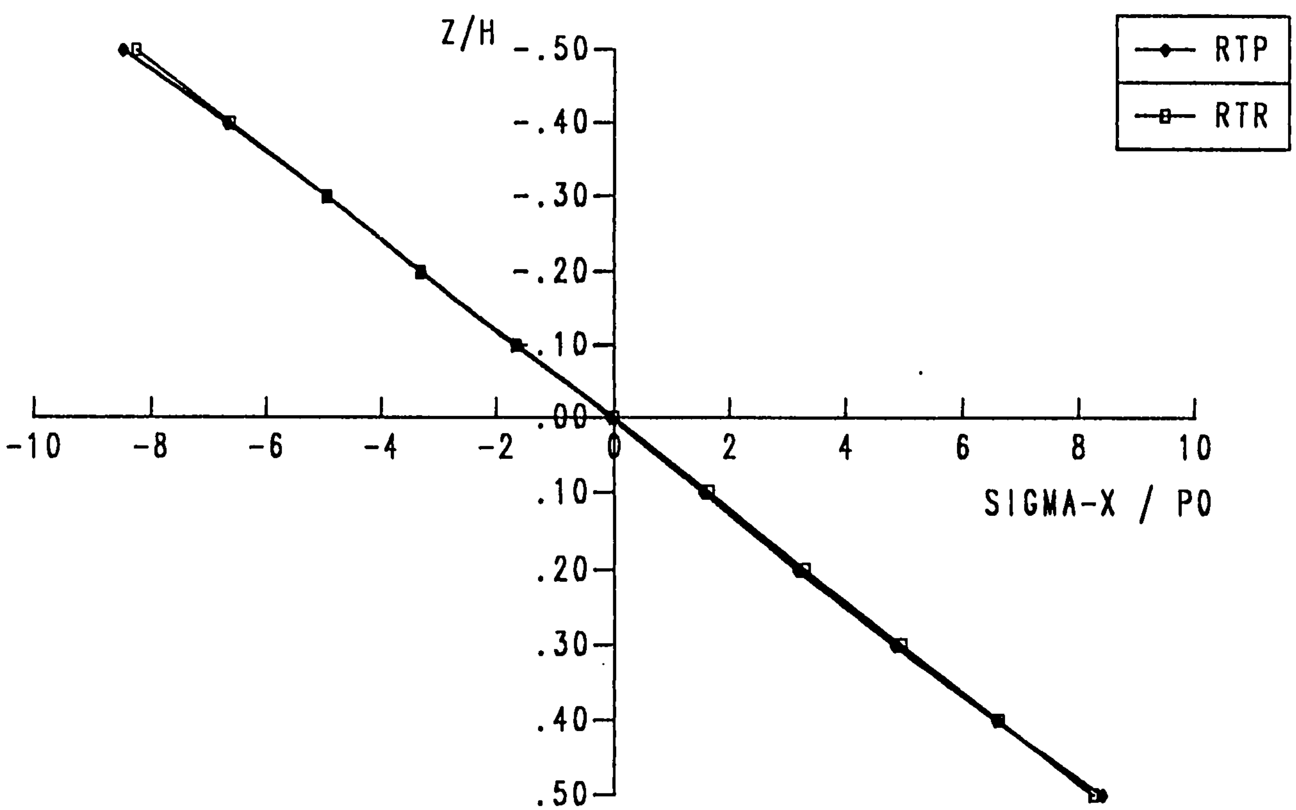


FIG. 5.63 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.5-I)

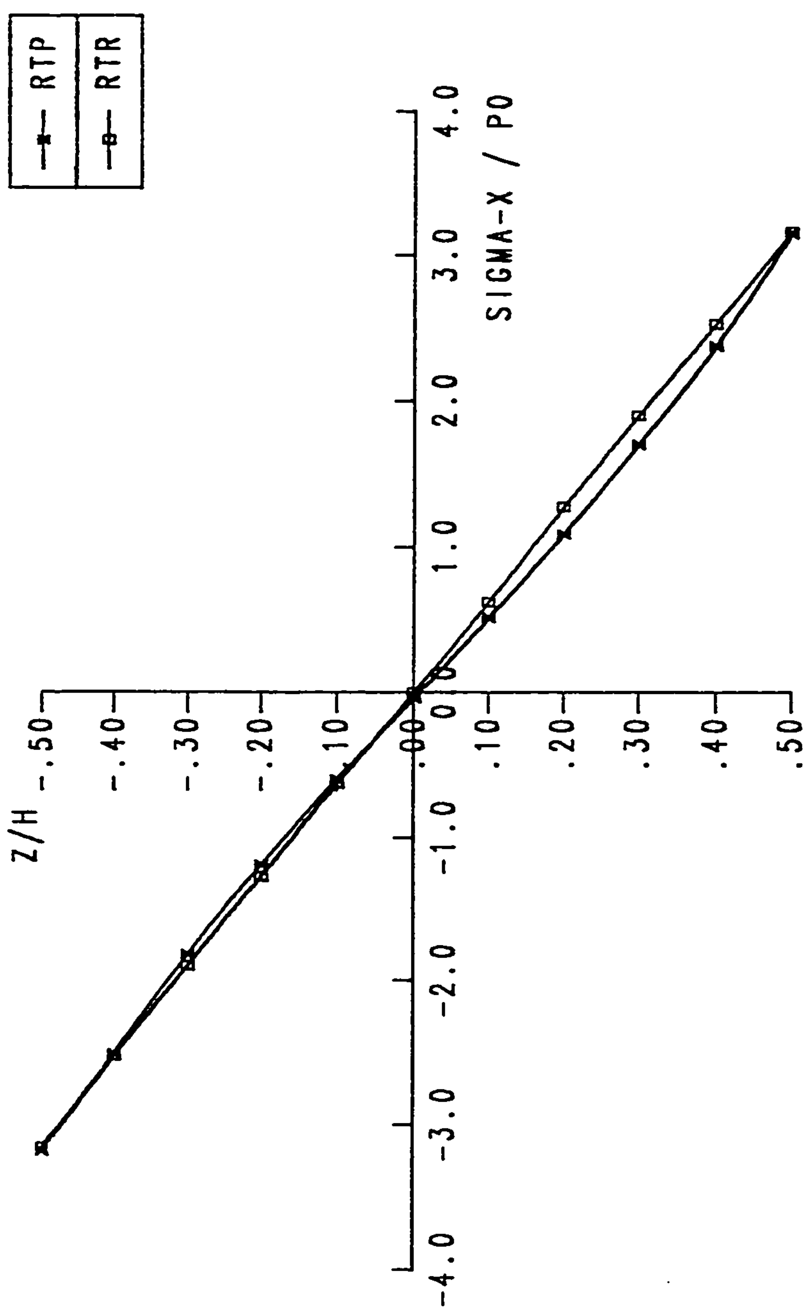


FIG. 5.64 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.7-I)

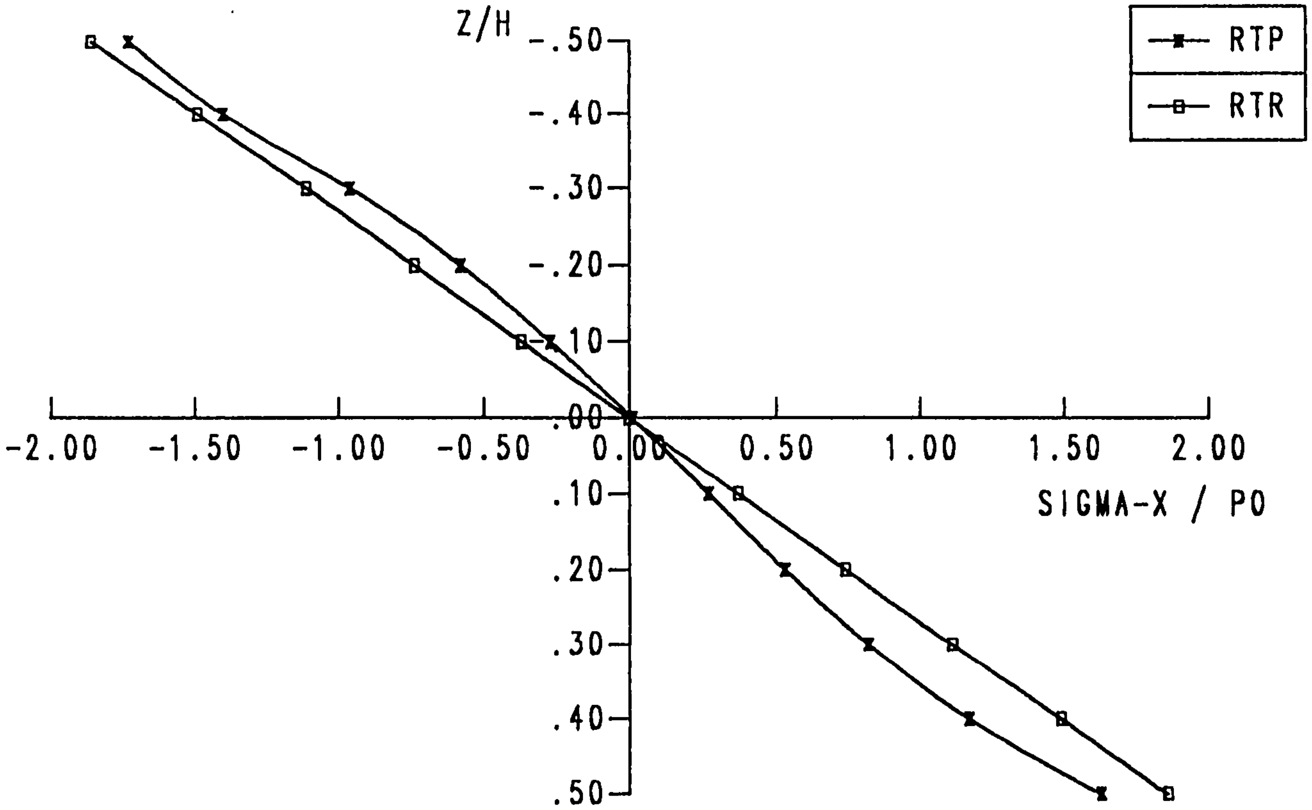


FIG. 5.65 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF1.-I)

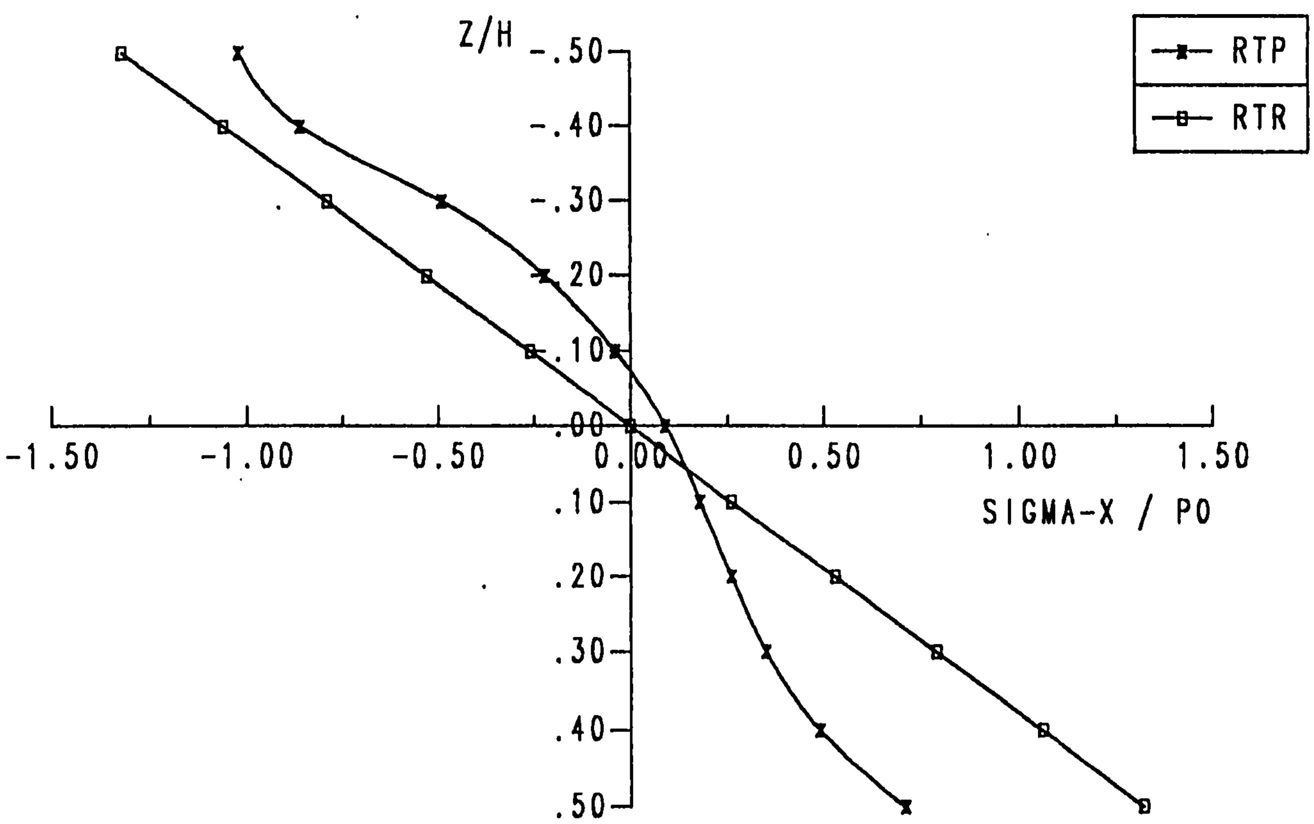


FIG. 5.66 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.005-II)

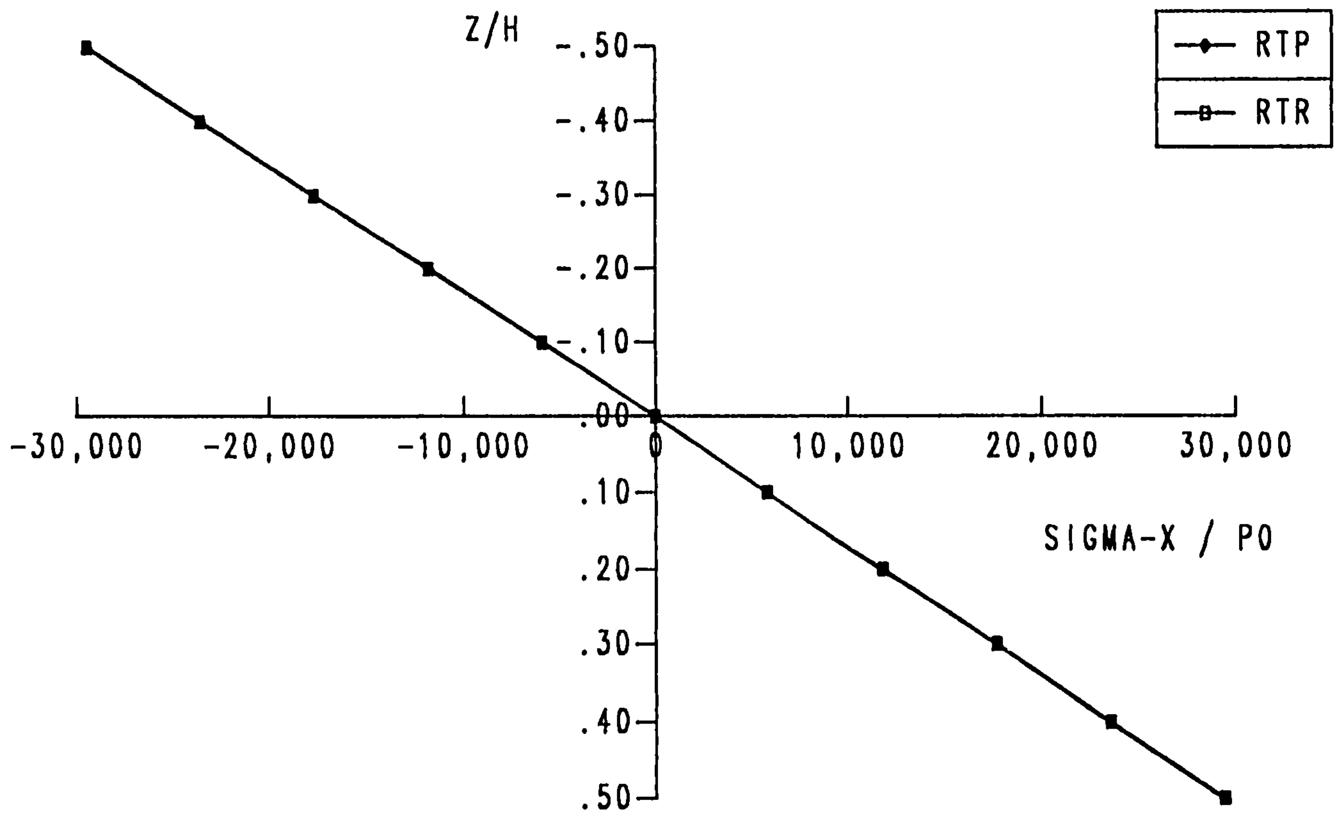


FIG. 5.67 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.1-II)

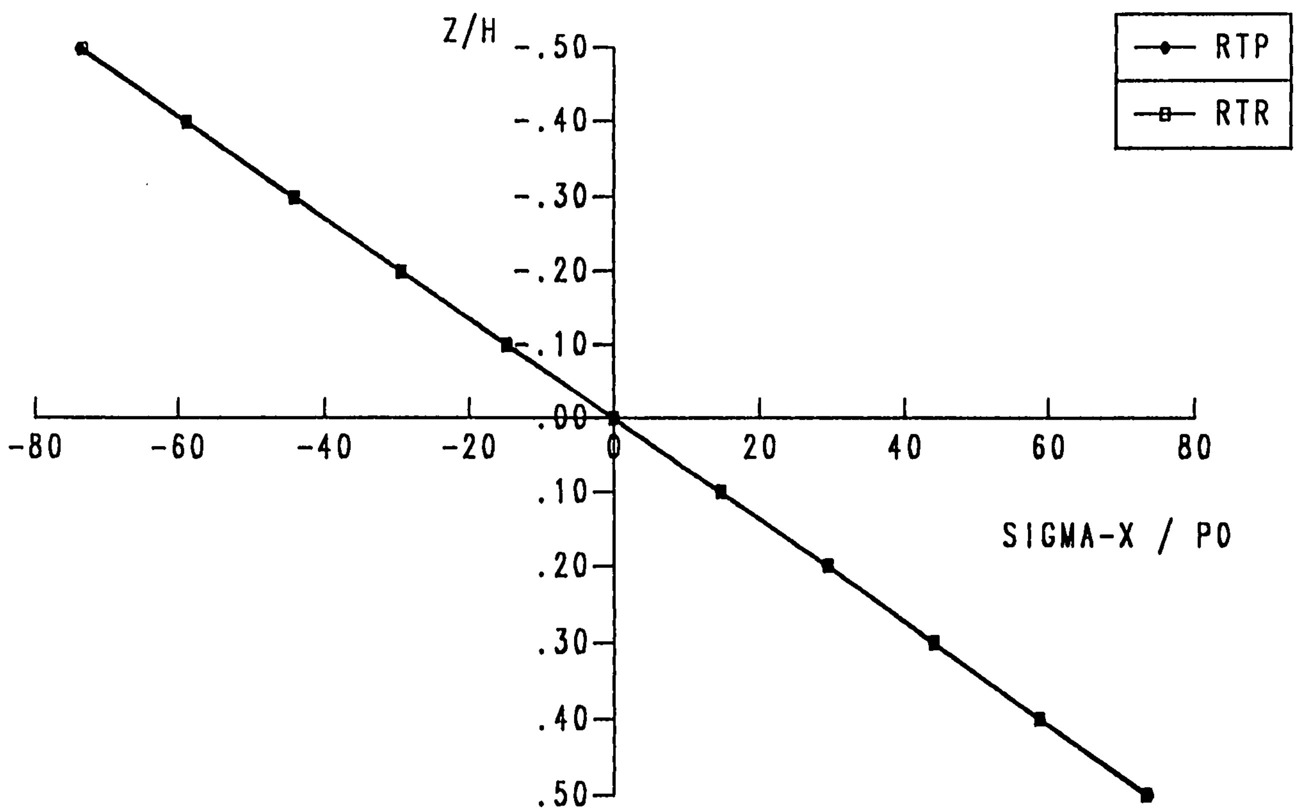


FIG. 5.68 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.3-II)

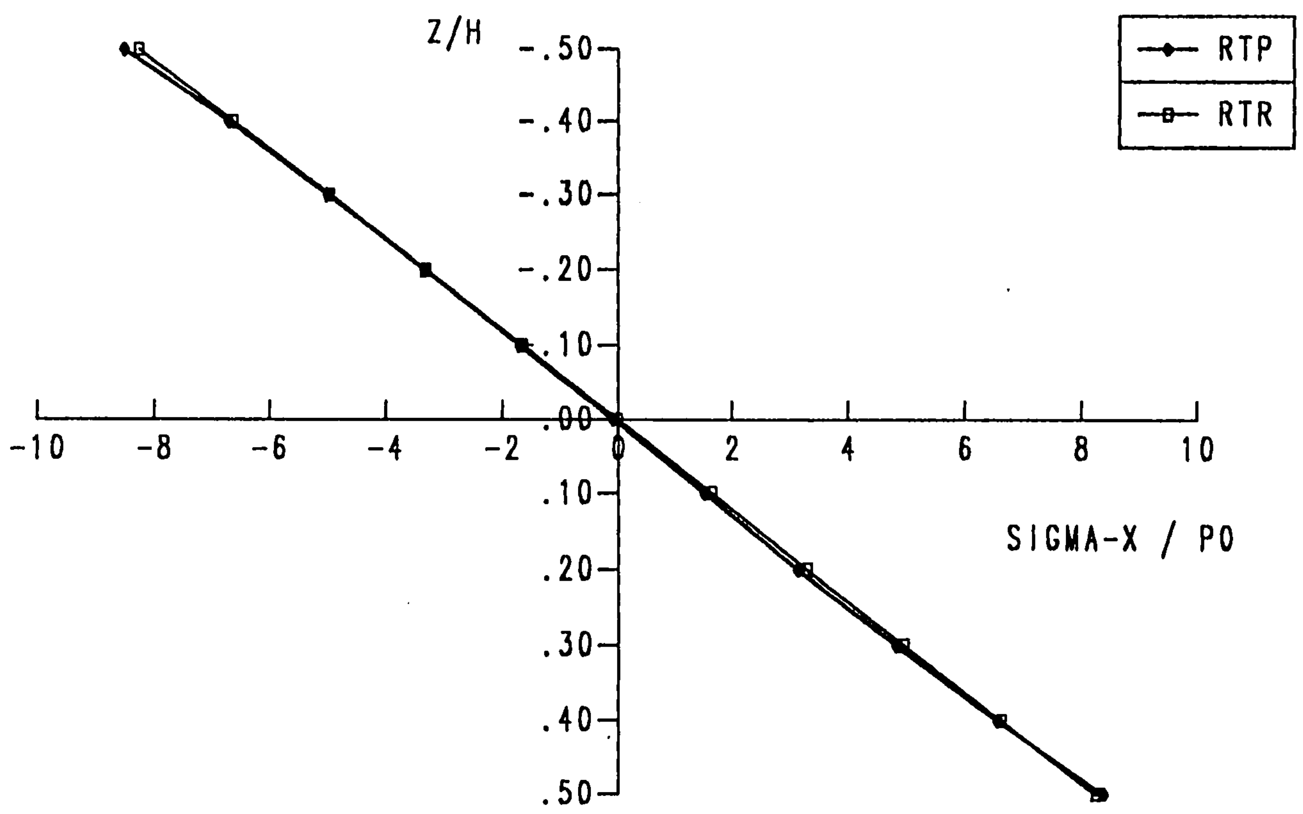


FIG. 5.69 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.5-11)

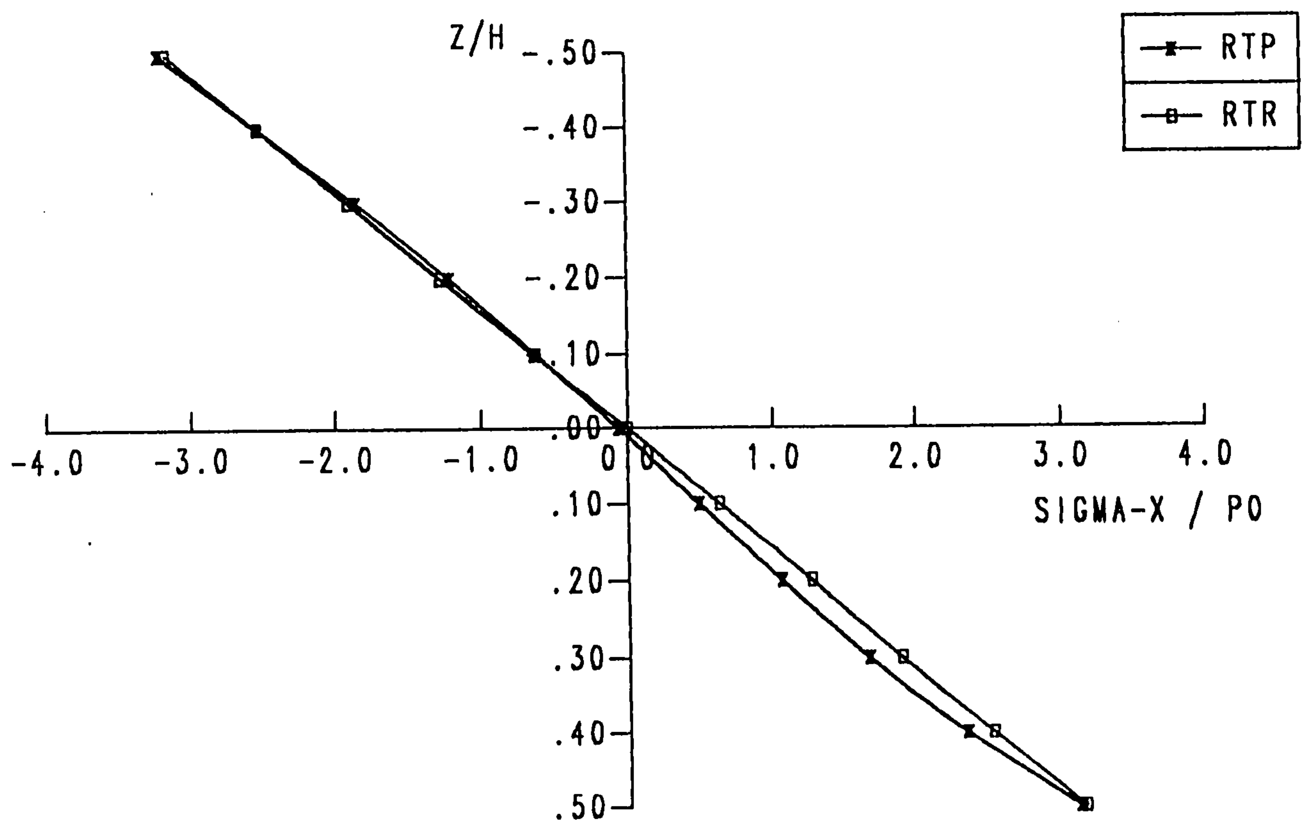


FIG. 5.70 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF.7-II)

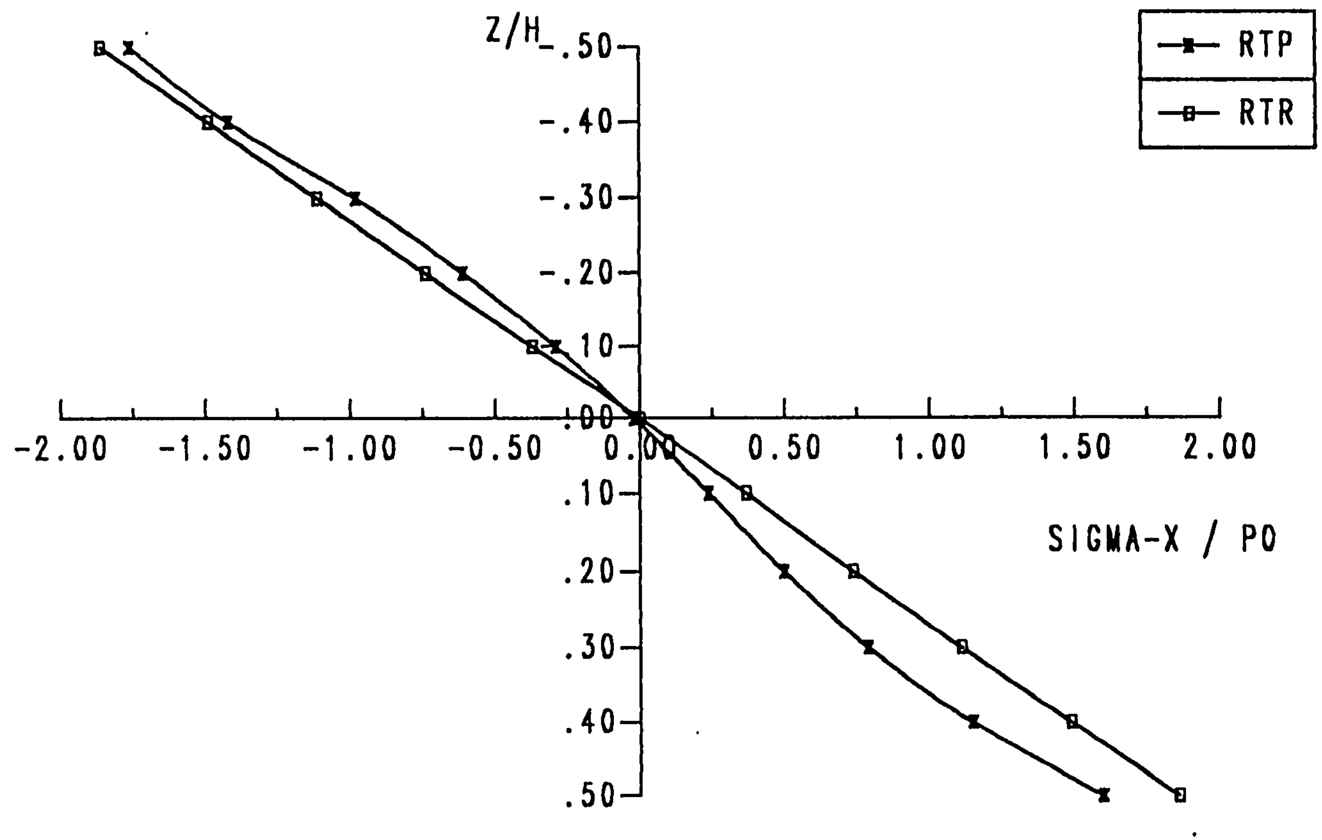


FIG. 5.71 : MAX. NORMAL STRESS SIGMA-X VS Z/H (SF1.-II)

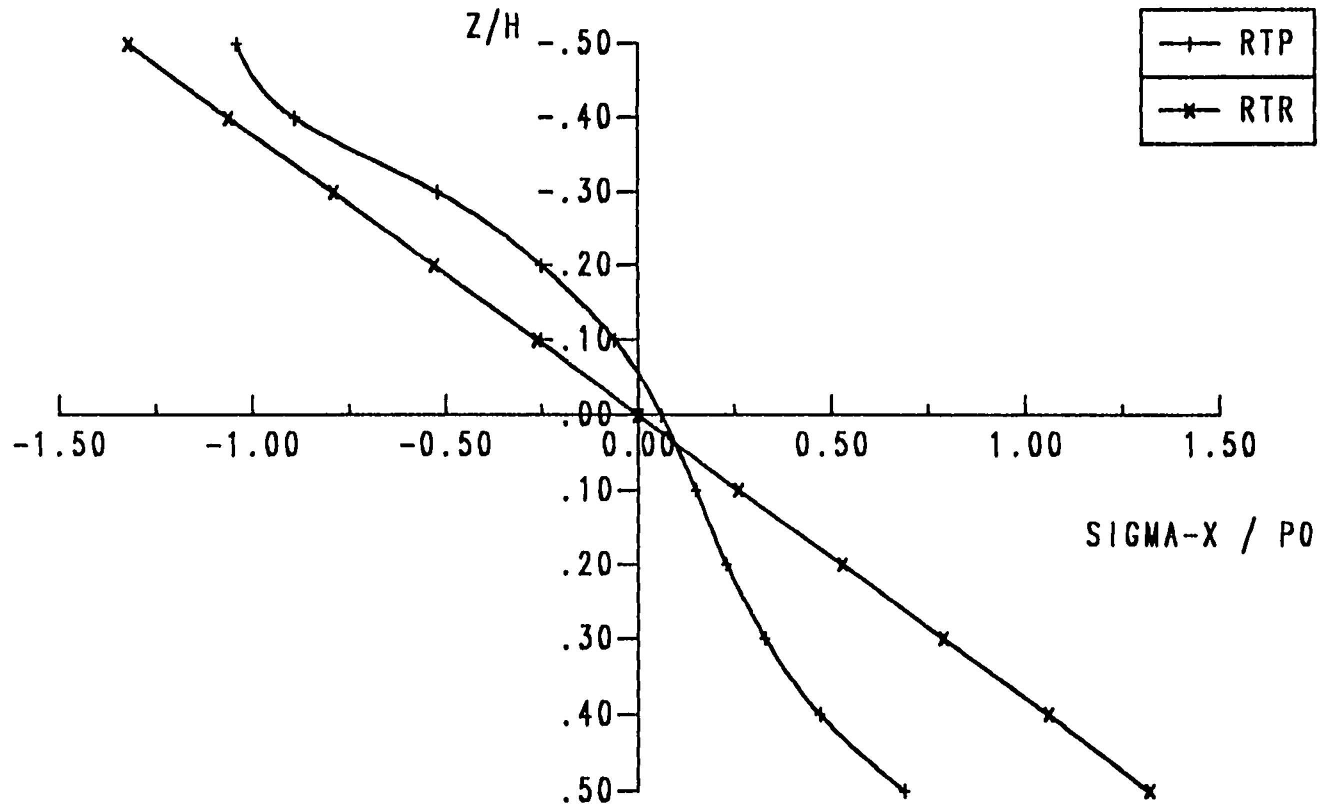


FIG. 72 : Max. Normal Stress Sigma-X VS Z/H (SS.1-11, Strip Load, Width = 0.2 a)

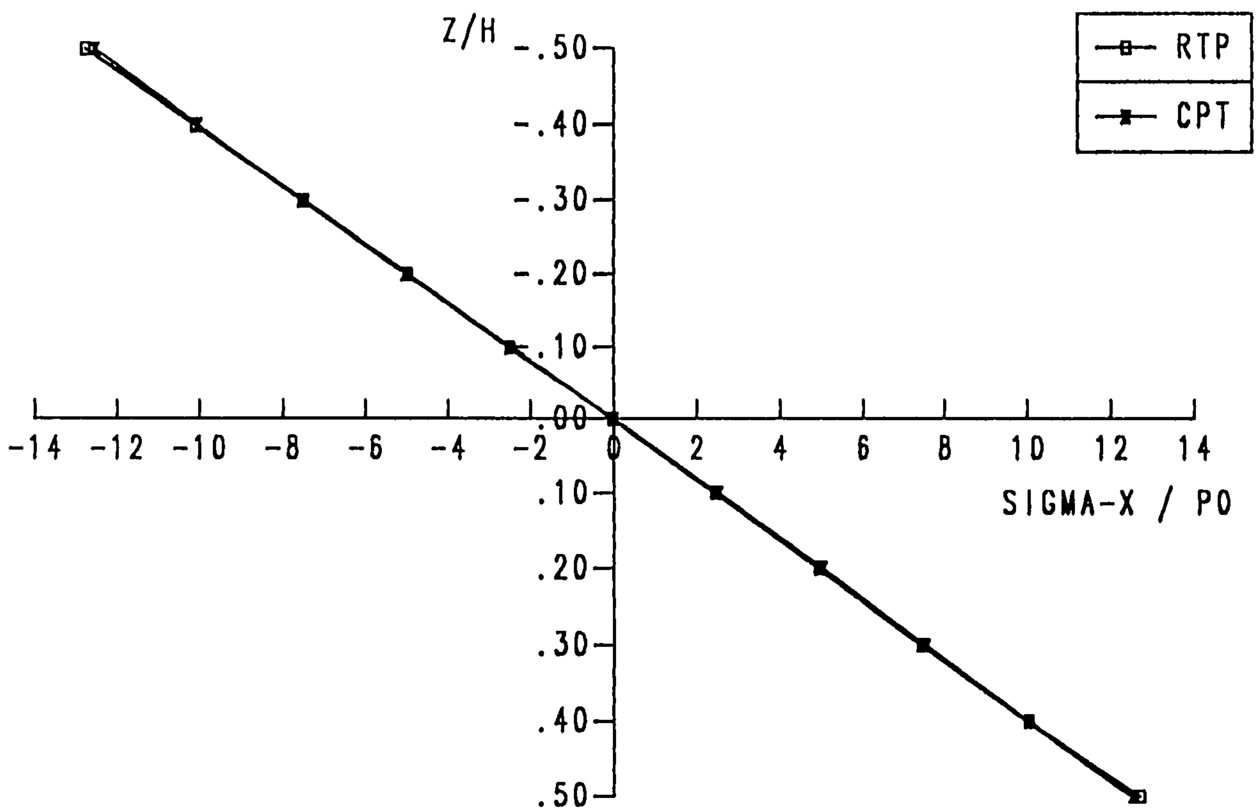


FIG. 73 : Max. Normal Stress Sigma-X VS Z/H (SS.3-II, Strip Load, Width = 0.2 a)

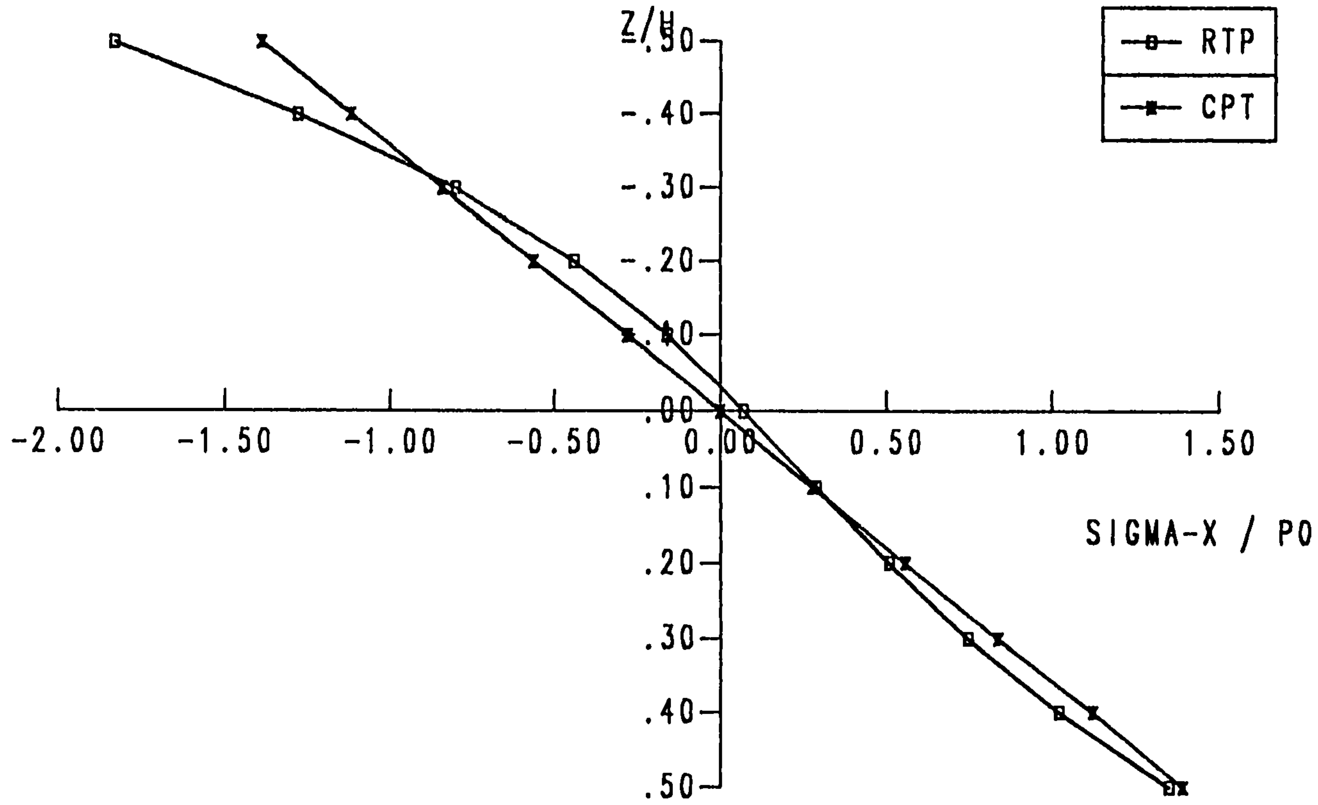


FIG. 74 : Max. Normal Stress Sigma-X VS Z/H (SS.5-II, Strip Load, Width = 0.2 a)

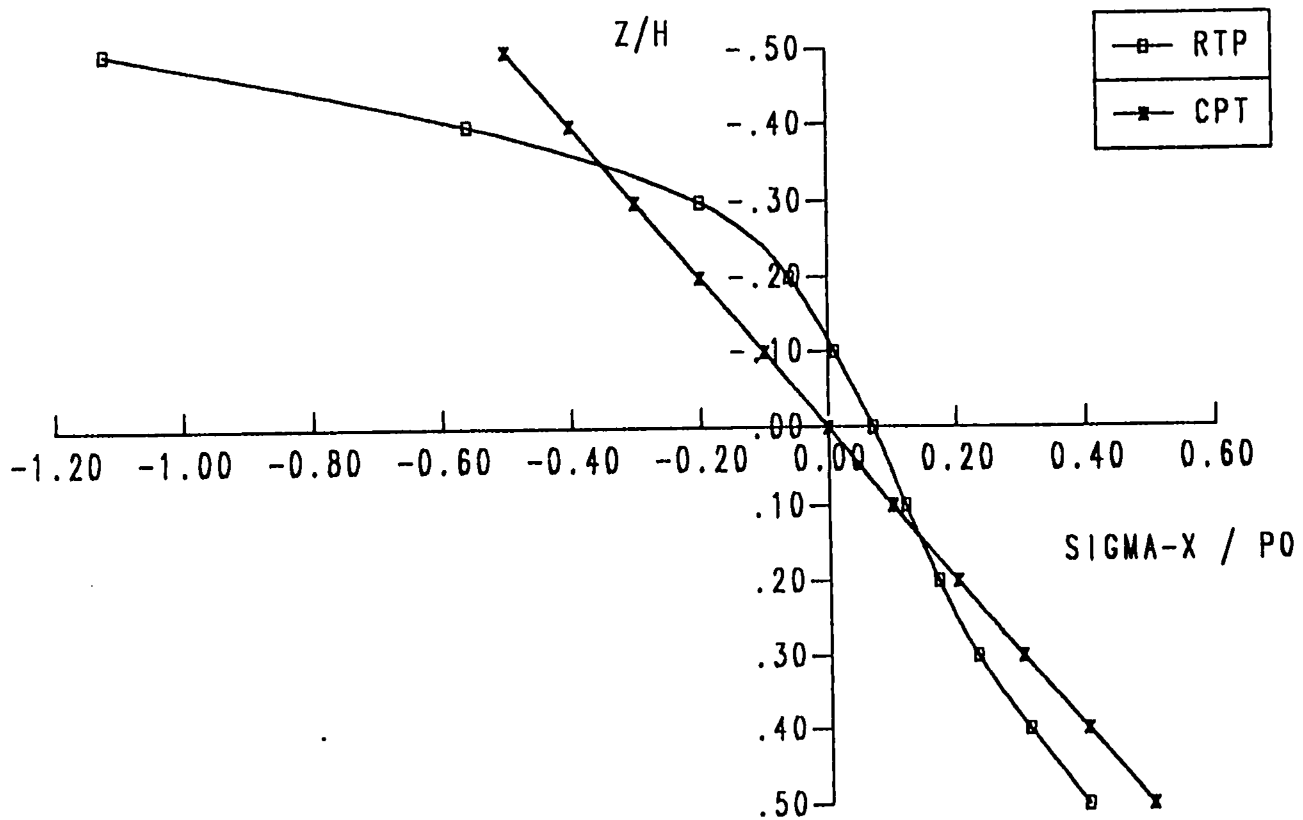


FIG. 75 : Max. Normal Stress Sigma-X VS Z/H (SS.7-II, Strip Load, Width = 0.2 a)

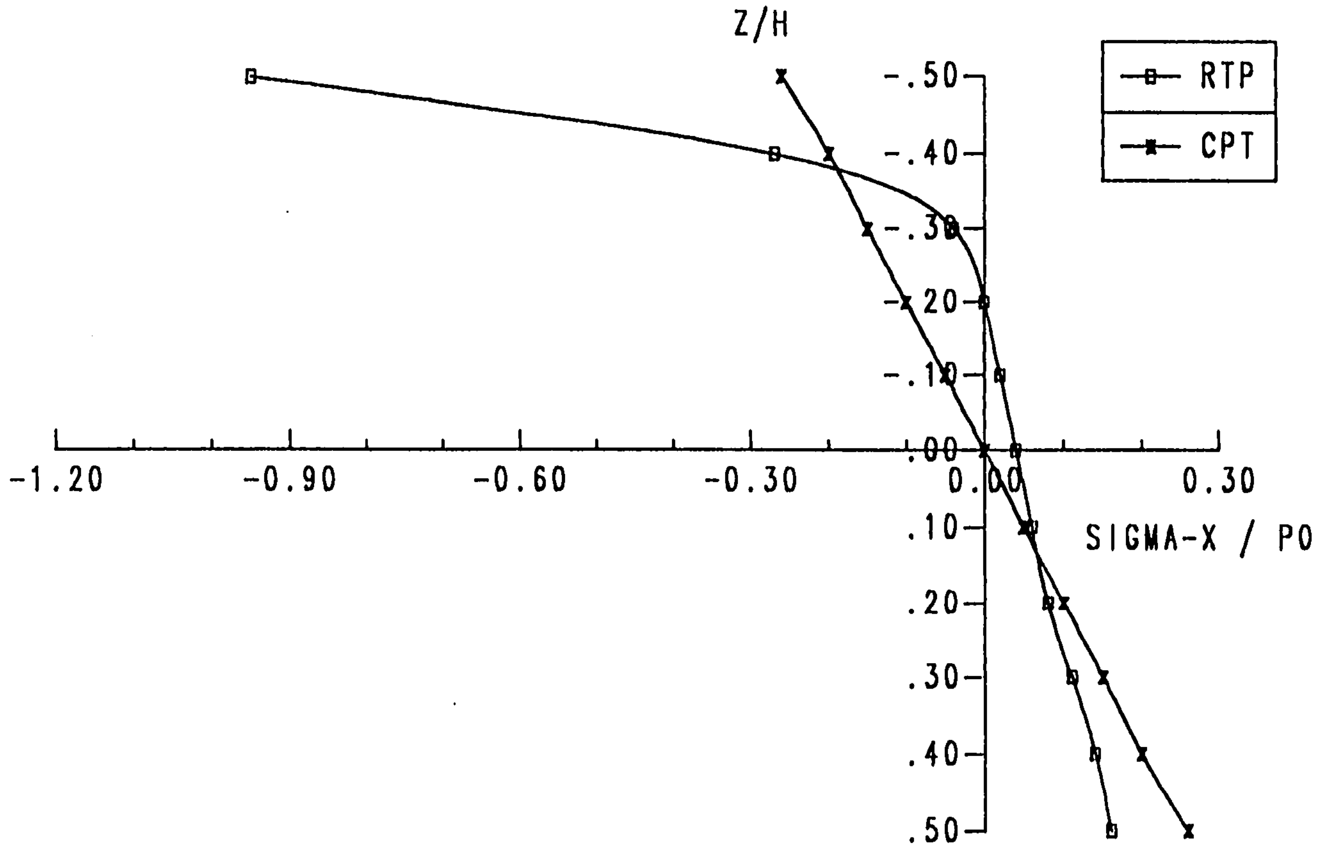


FIG. 76 : Max. Normal Stress Sigma-X VS Z/H (SS1.-II, Strip Load, Width = 0.2 a)

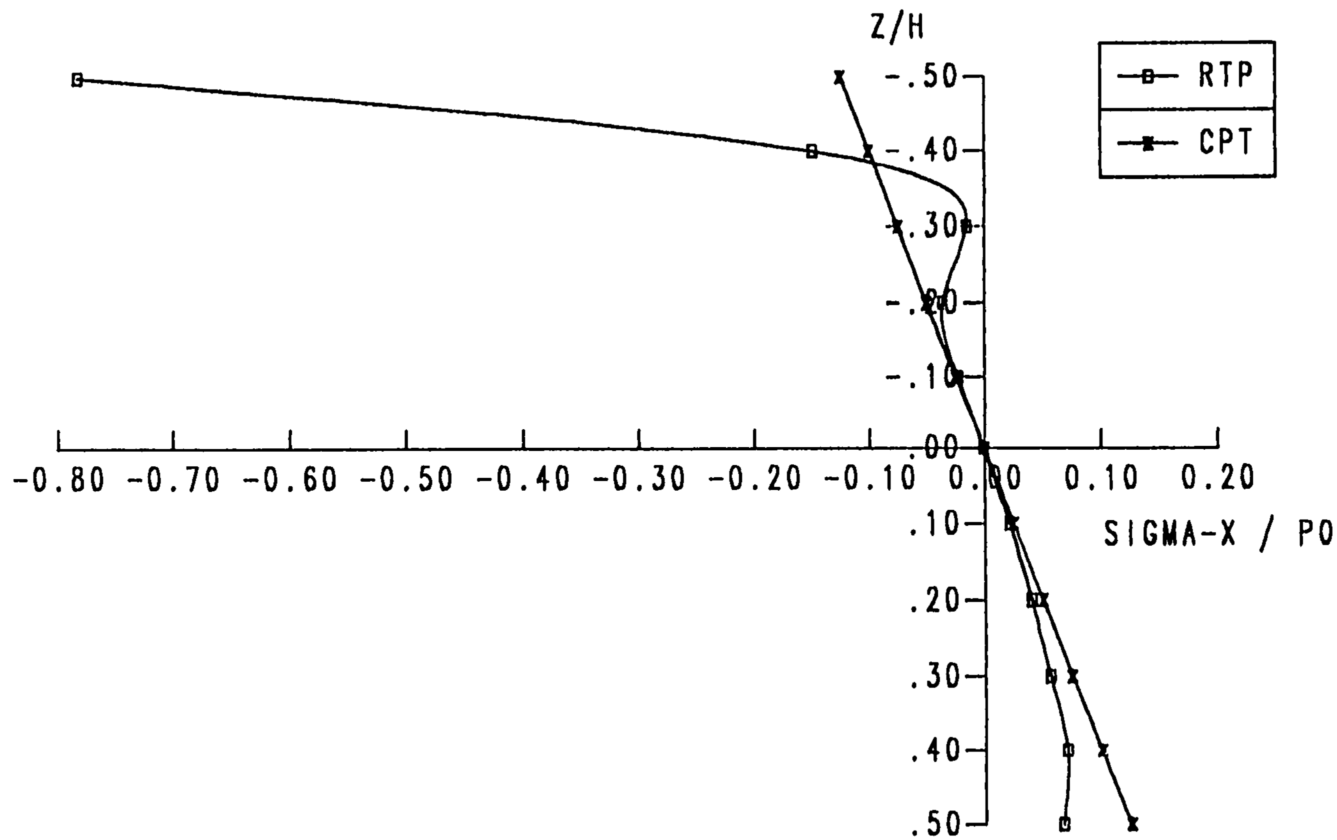


FIG. 5.77 : DEFLECTION OF TOP SURFACE OF PLATE AT Y=0.0 (SS.1-11)

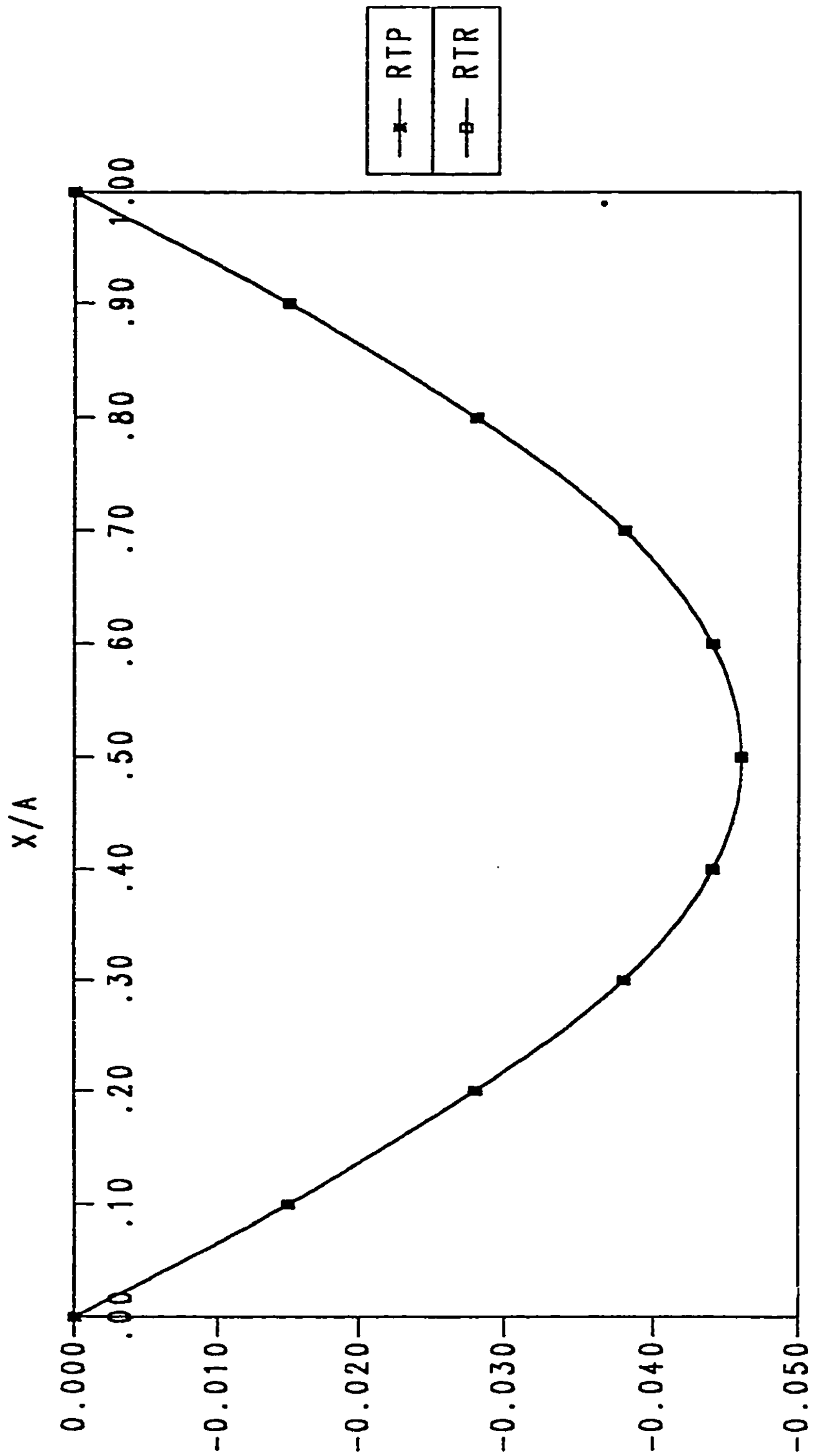


FIG. 5.78 : DEFLECTION OF TOP SURFACE OF PLATE AT Y=0.0 (SS.5-II)

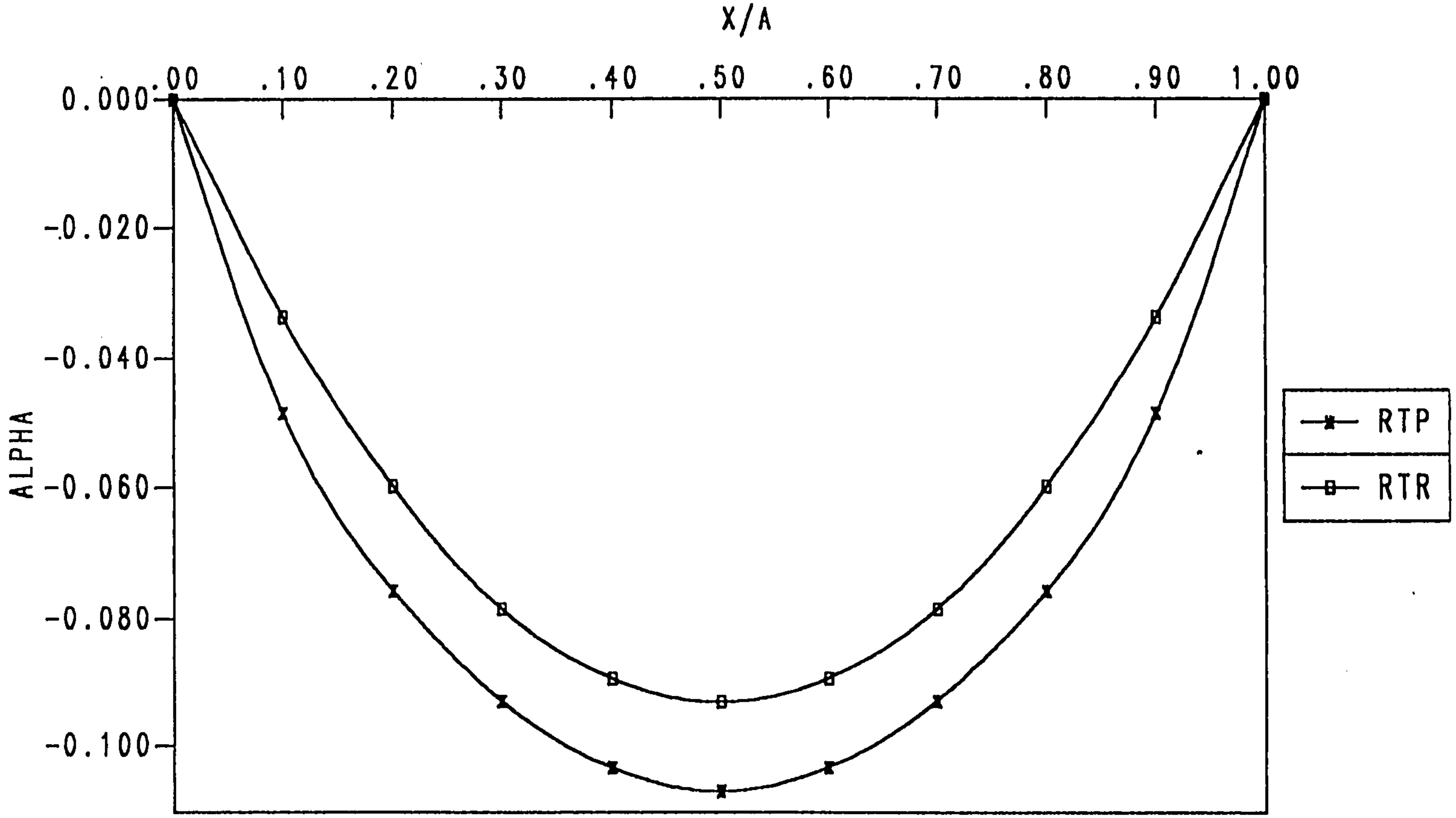


FIG. 79 : DEFLECTION OF TOP SURFACE OF PLATE AT Y=0.0 (SS1.-II)

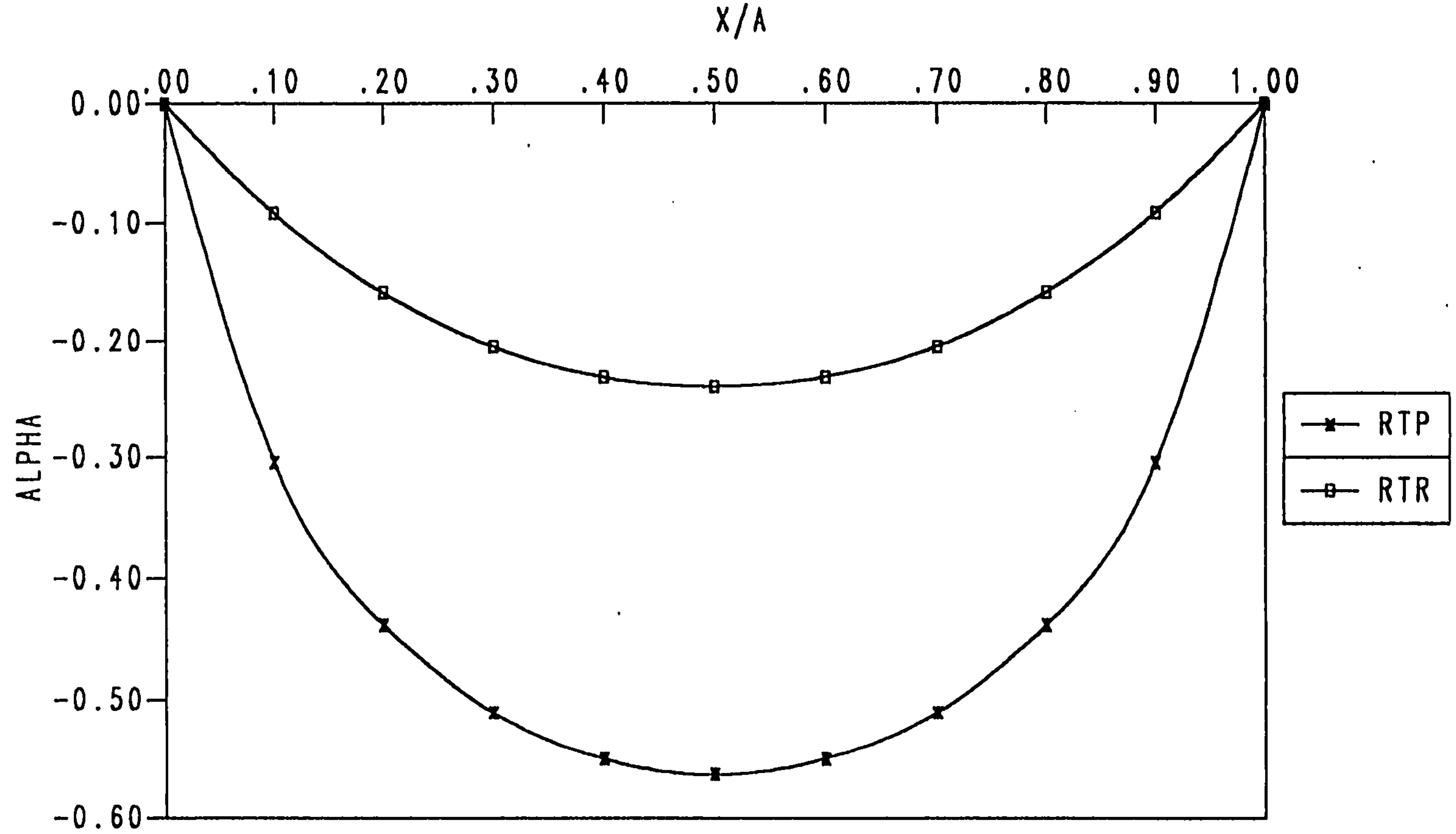


FIG. 5.80 : DEFLECTION OF MID SURFACE OF PLATE AT Y=0.0 (SS.1-II)

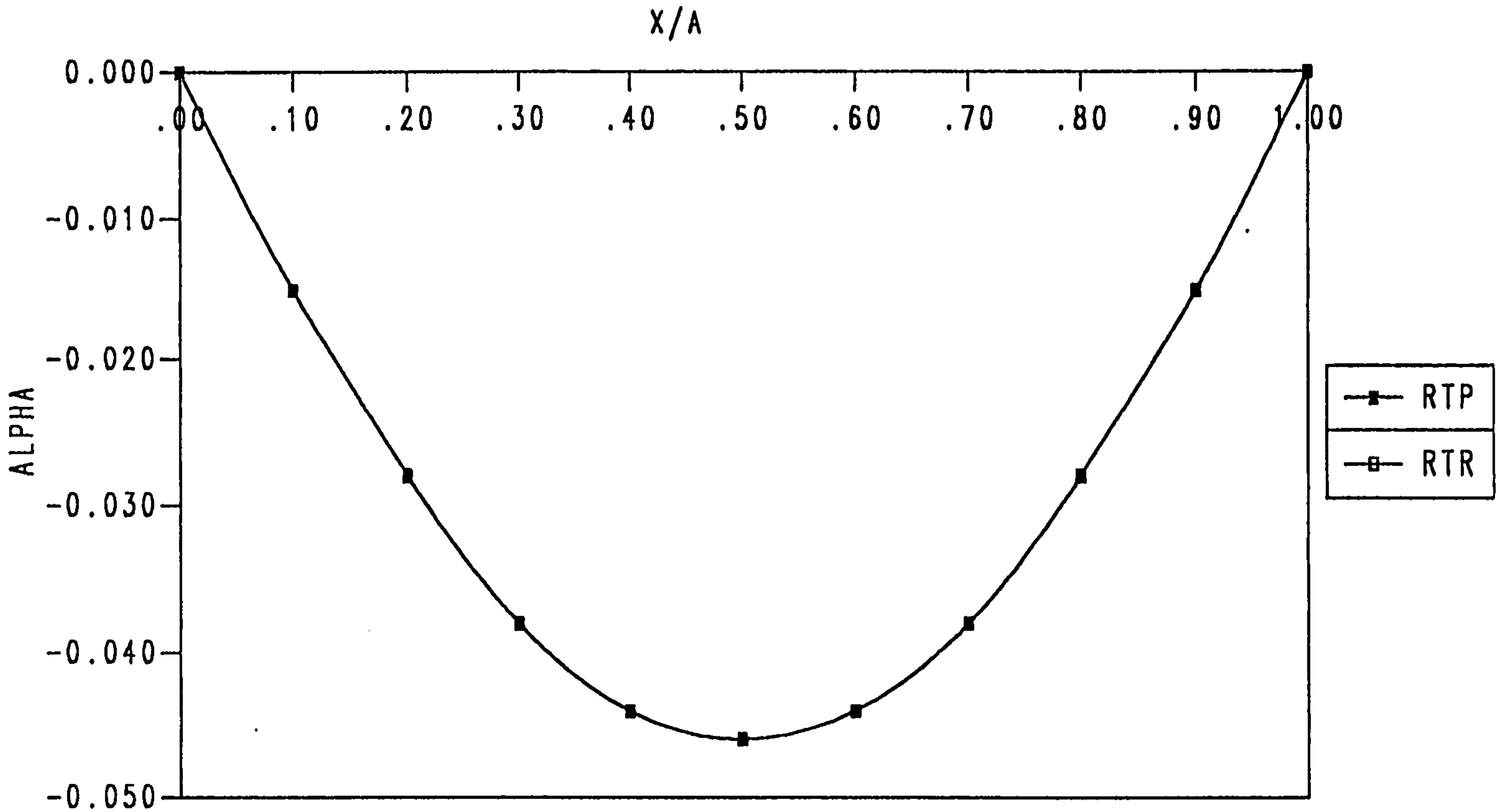


FIG. 5.81 : DEFLECTION OF MID. SURFACE OF PLATE AT Y=0.0 (SS.5-II)

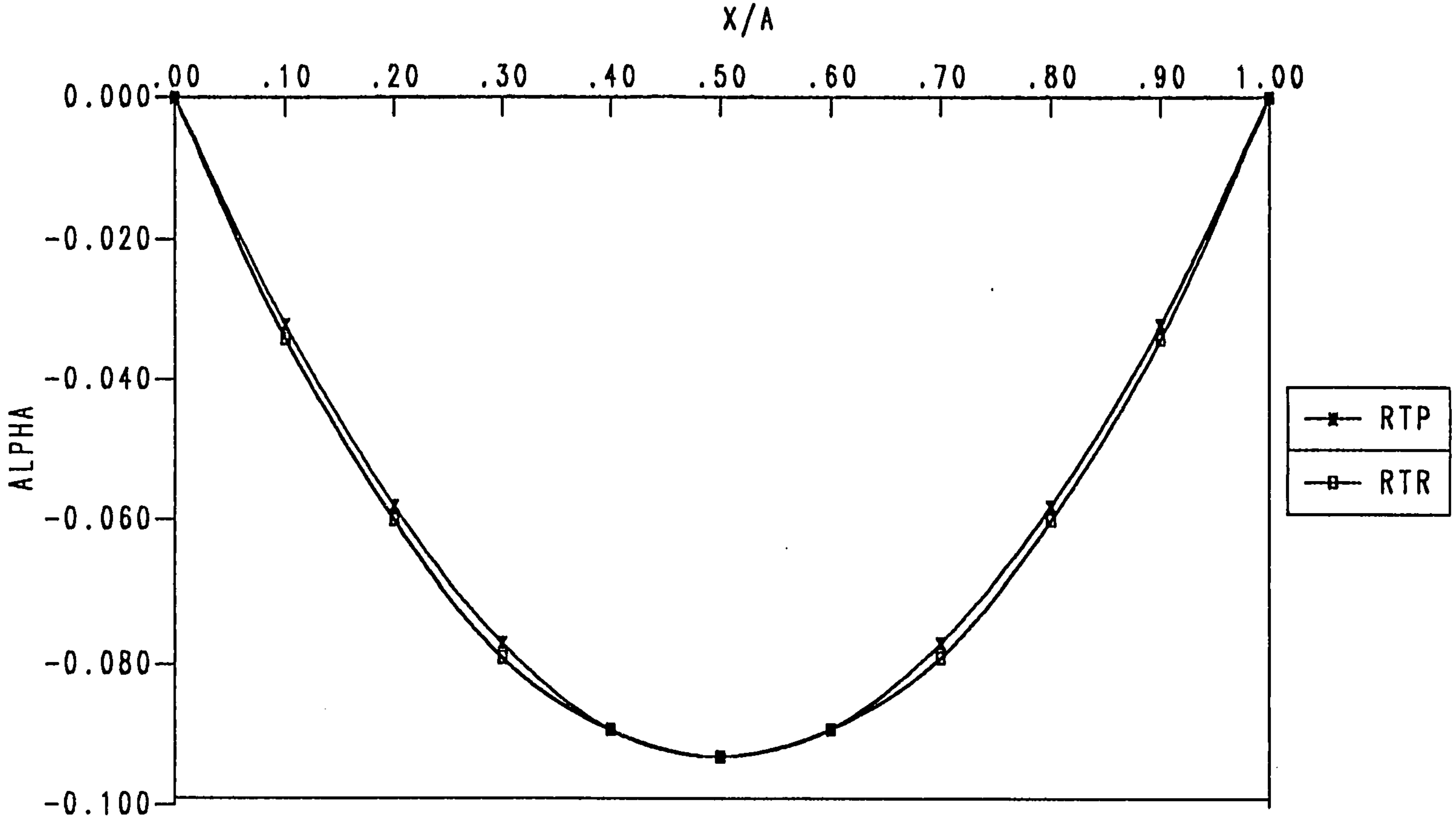


FIG. 5.82 : DEFLECTION OF MID. SURFACE OF PLATE AT Y=0.0 (SS1.-II)

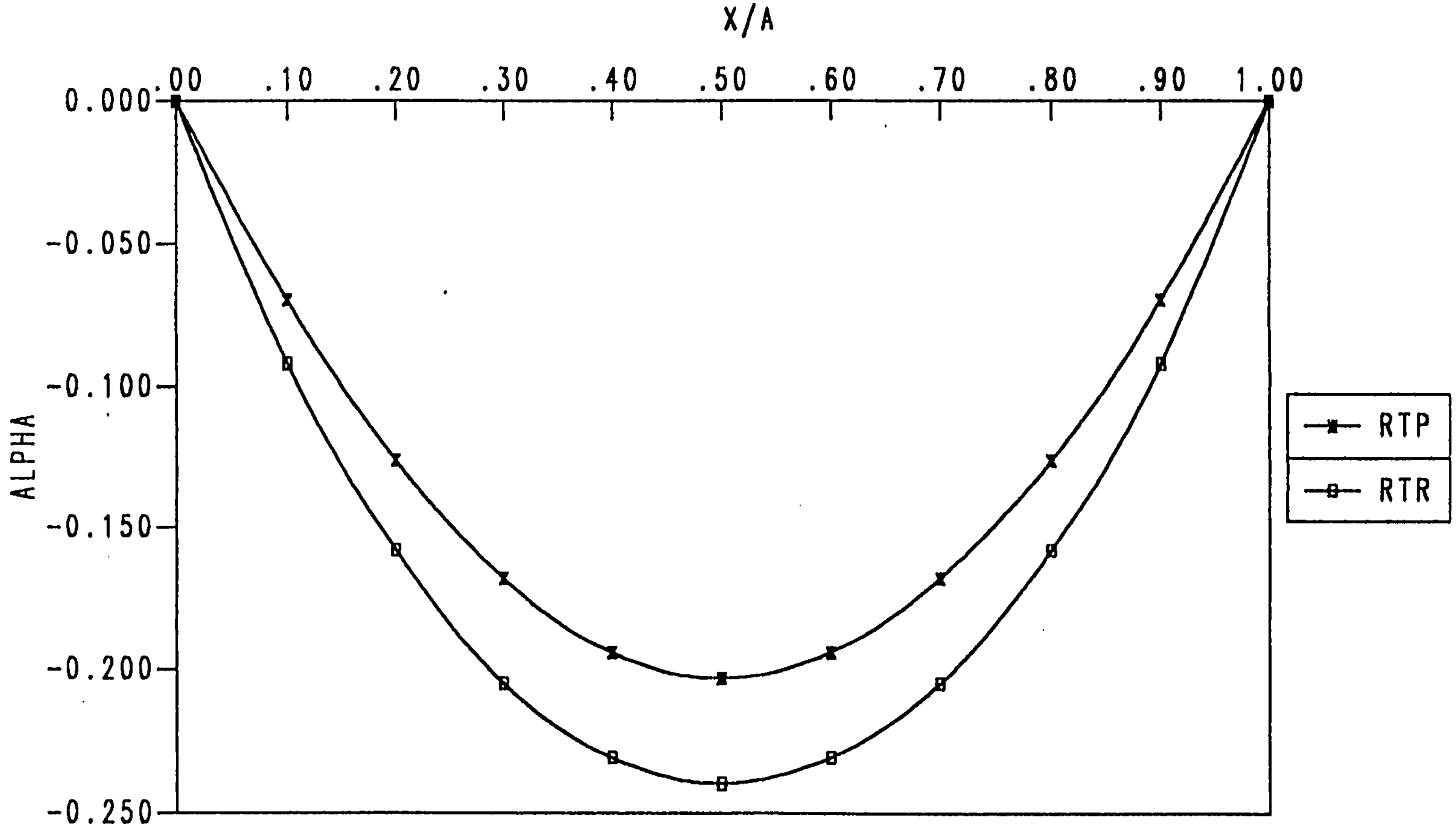
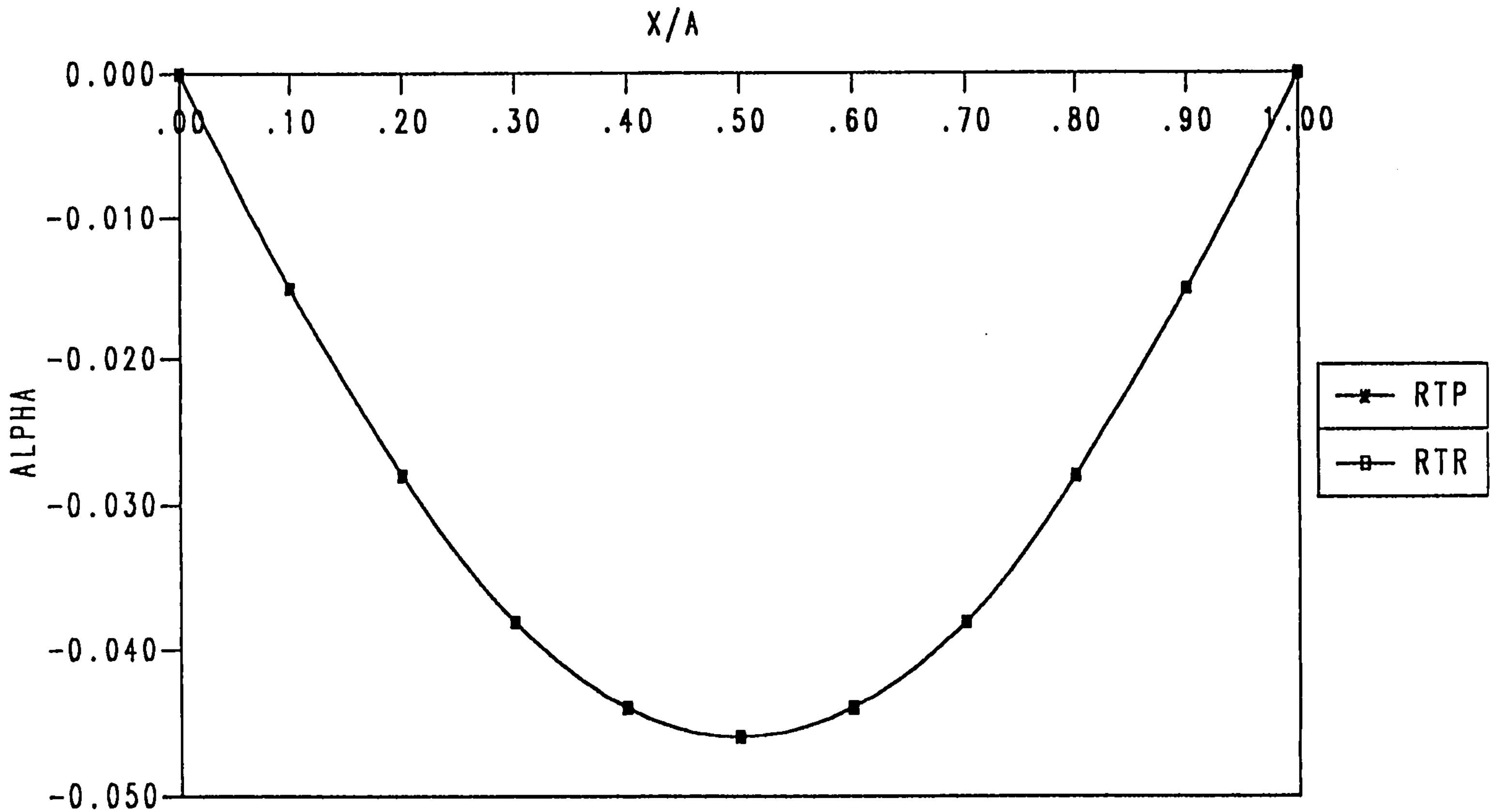


FIG. 5.83 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT Y=0.0 (SS.1-II)



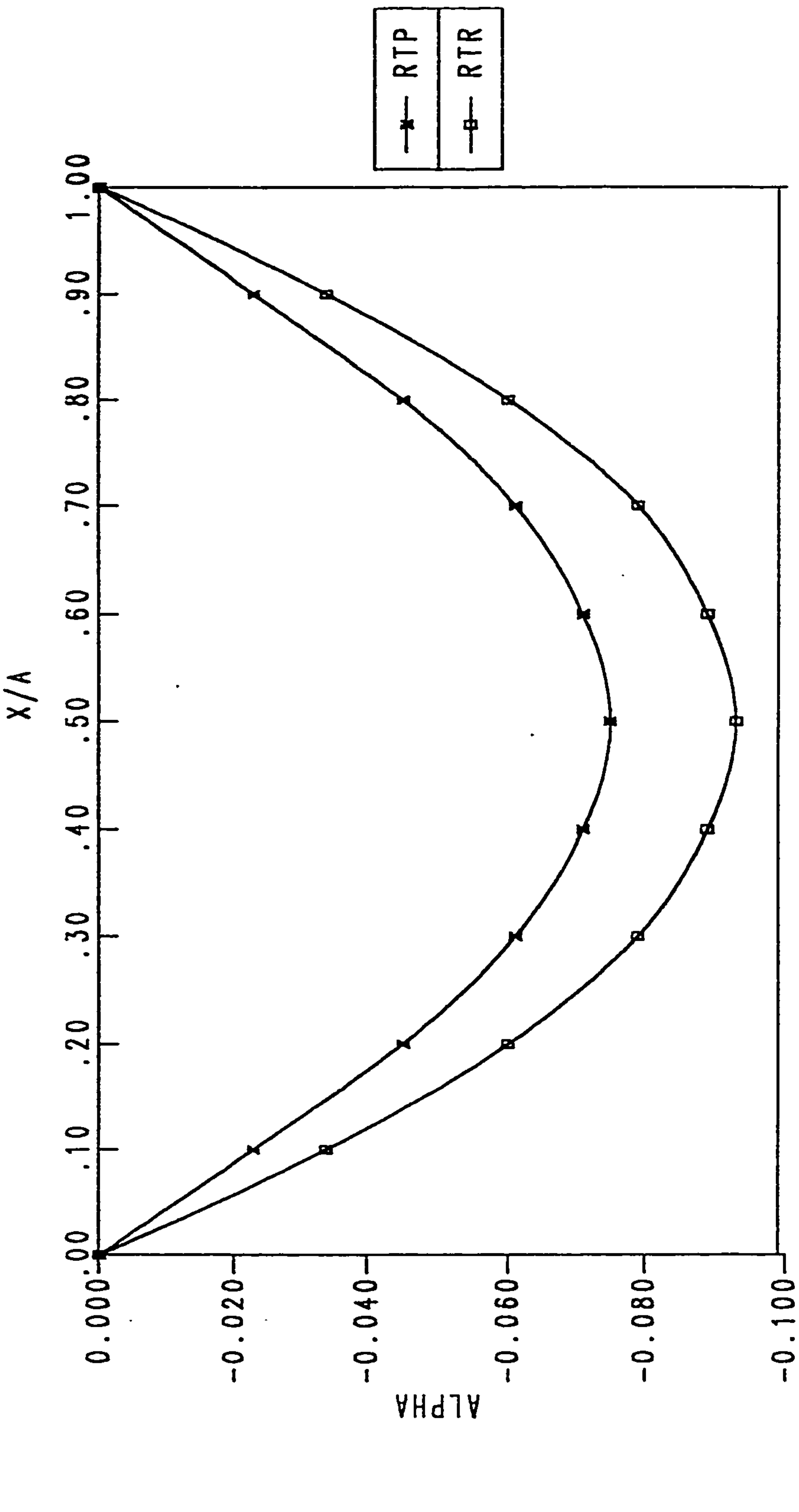


FIG. 5.84 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT Y=0.0 (SS.5-11)

FIG. 5.85 : DEFLECTION OF BOTTOM SURFACE OF PLATE AT Y=0.0 (SS1.-II)

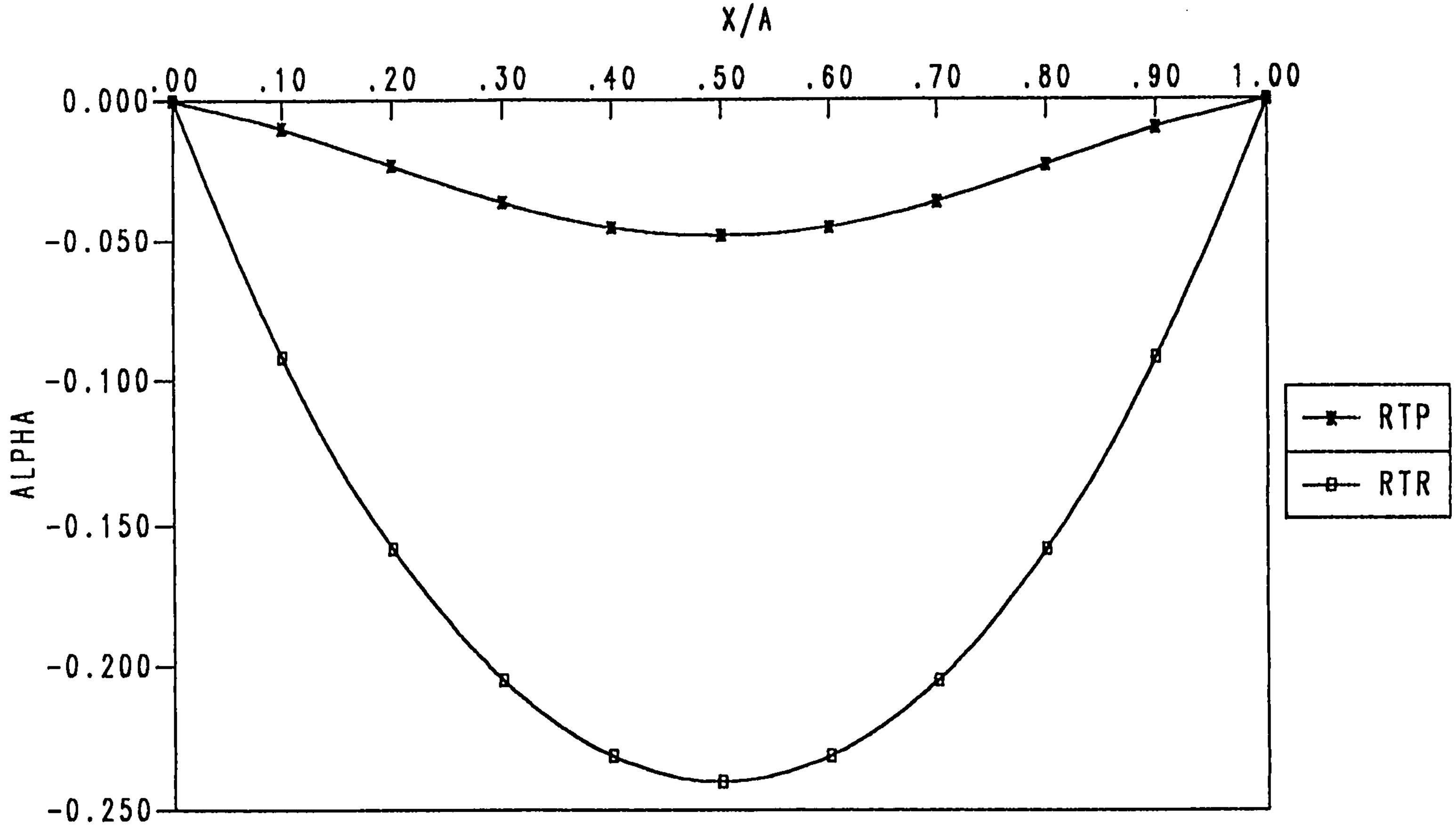


FIG. 86 : Deflection Of TOP, MID., & BOTTOM SURFACES At Y = 0.0 (SS.1-II)

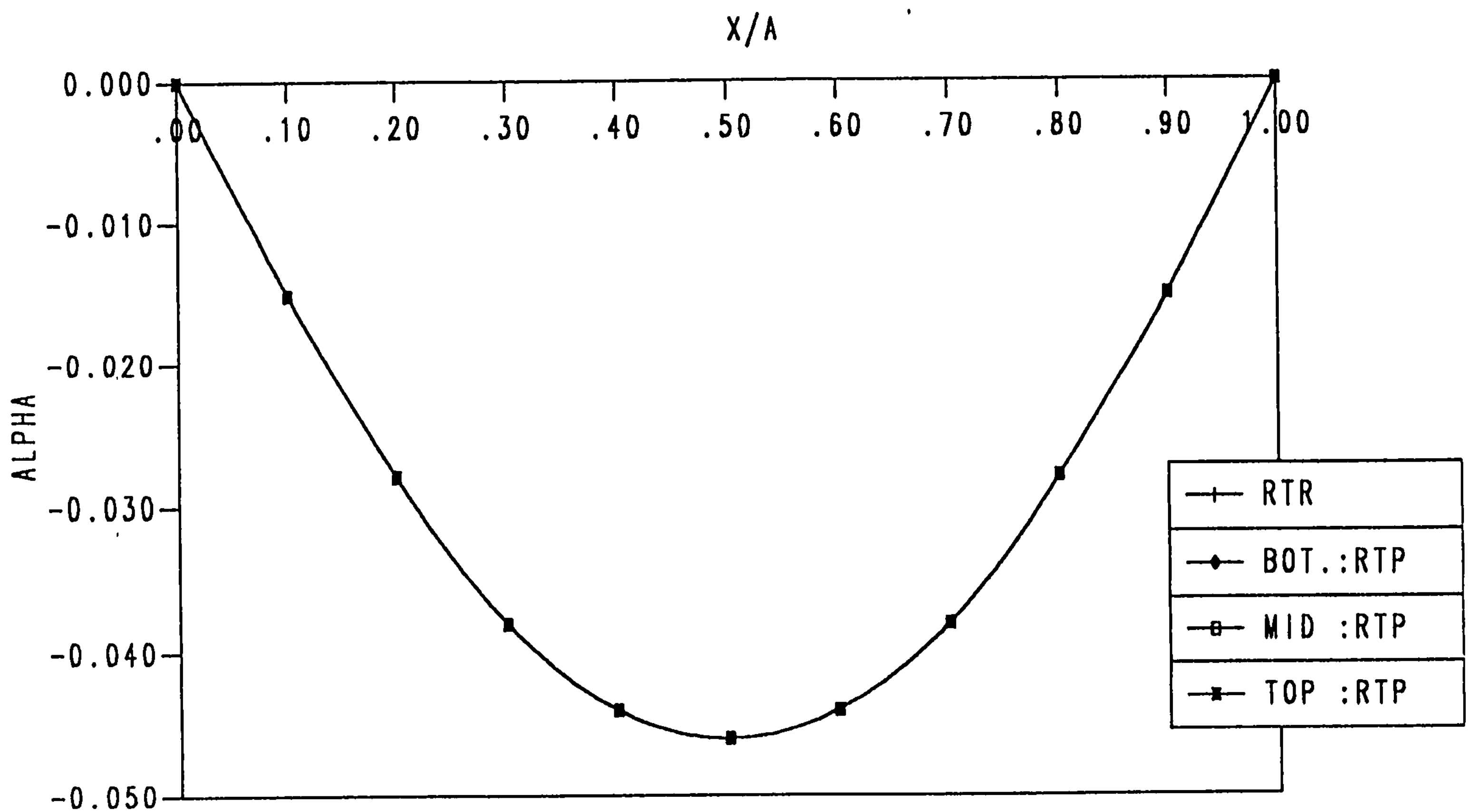


FIG. 87 : DEFLECTION OF TOP, MID, & BOT. SURFACES OF PLATE AT Y=0.0 (SS.5-II)

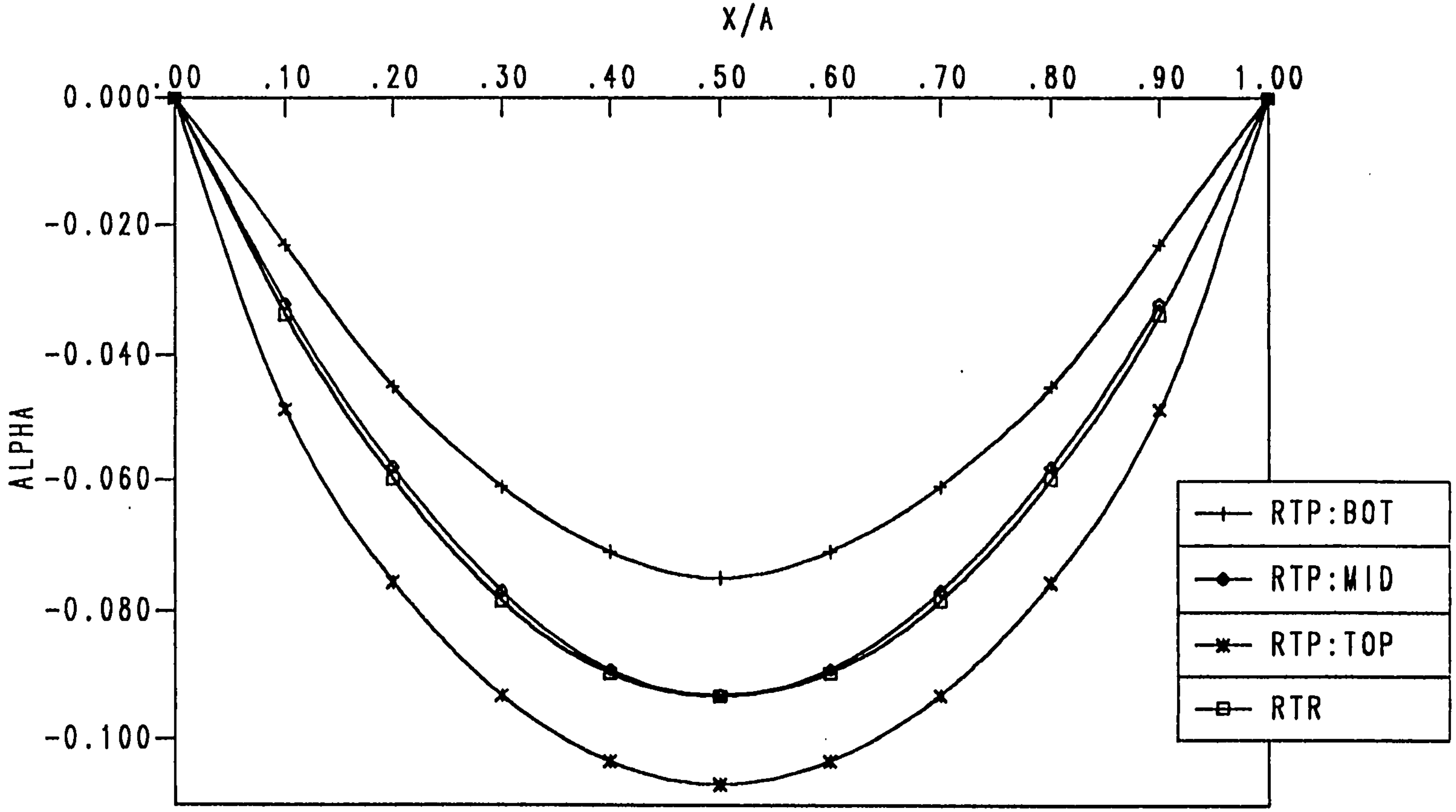


FIG. 88 : DEFLECTION OF TOP, MID, & BOT SURFACES OF PLATE AT Y=0.0 (SS1.-II)

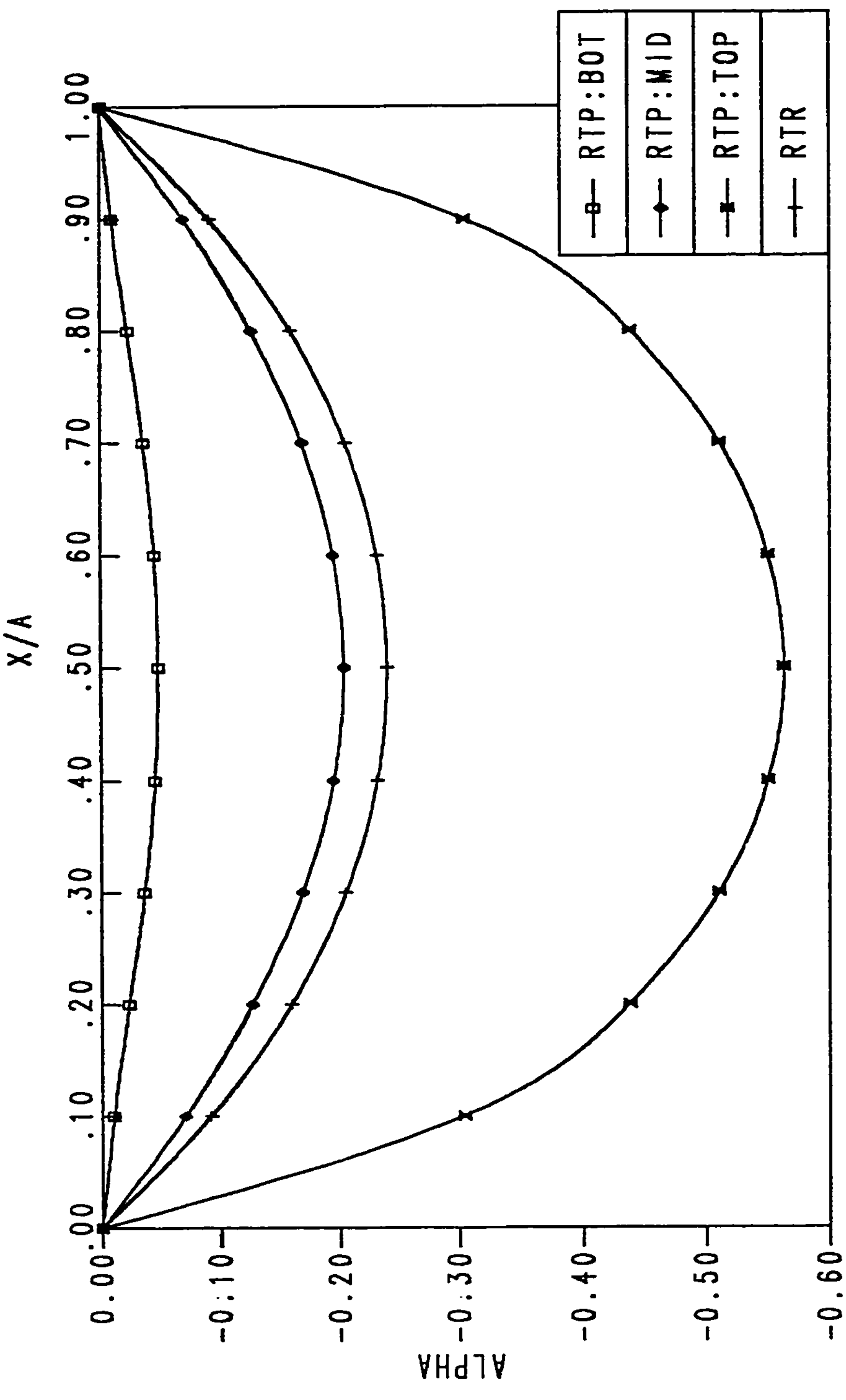


FIG. 5.89 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS.1)

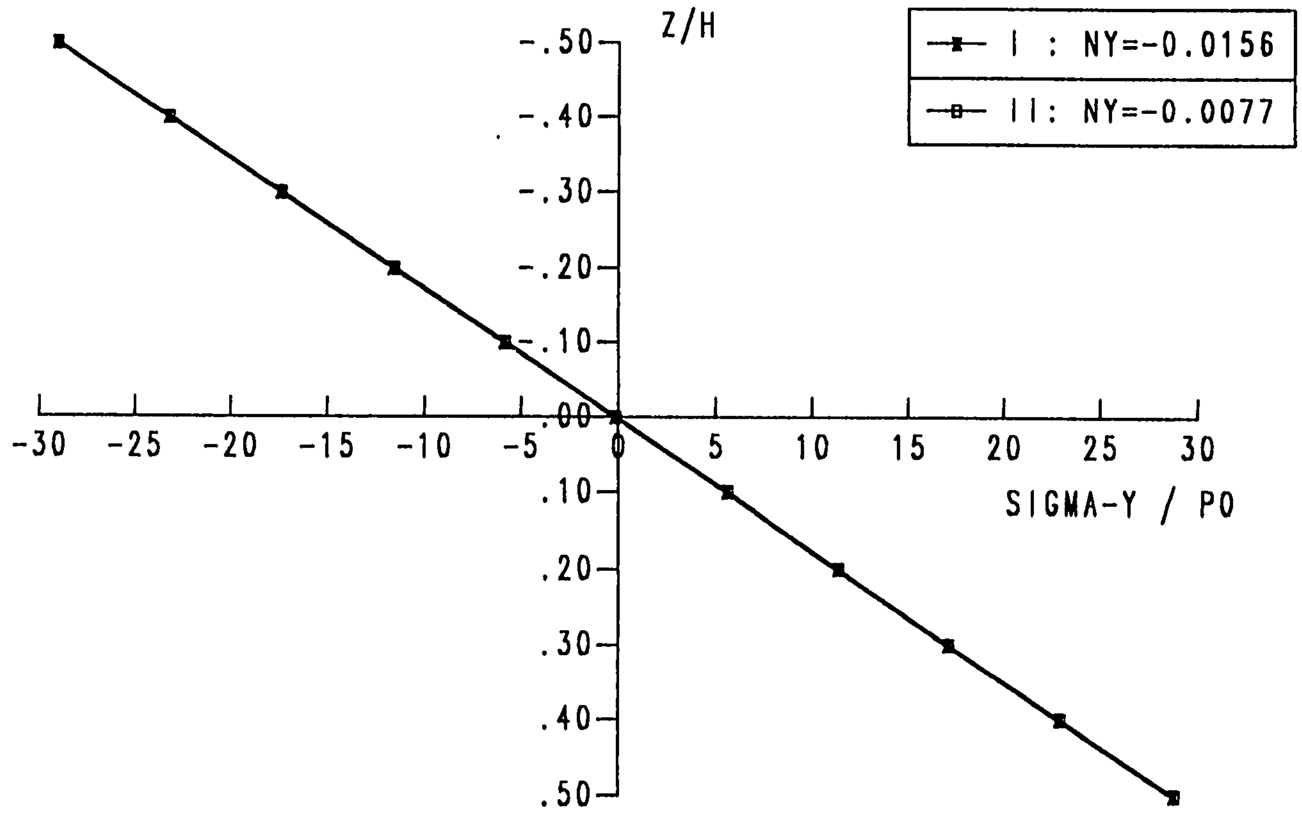


FIG. 5.90 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS.5)

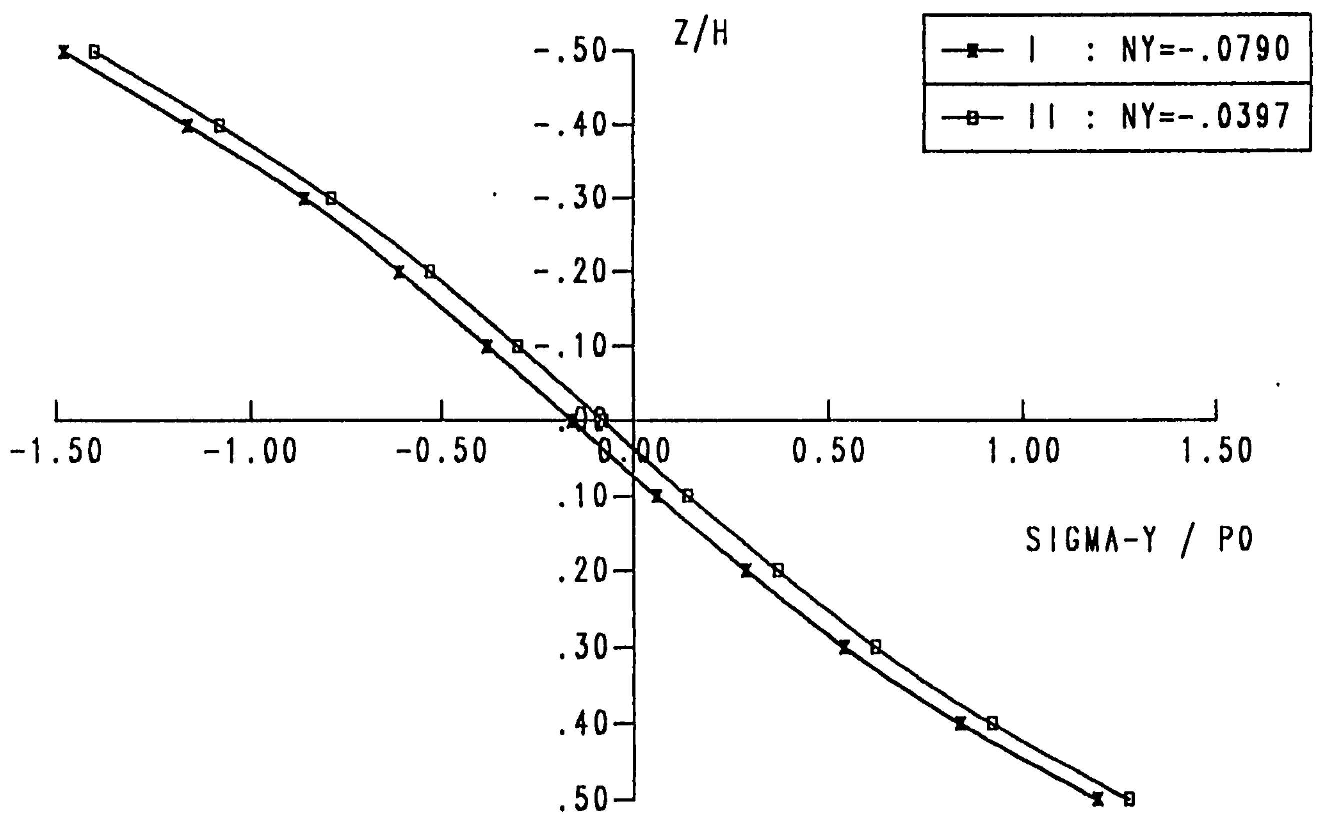


FIG. 5.91 : MAX. NORMAL STRESS SIGMA-Y VS Z/H (SS1.0)

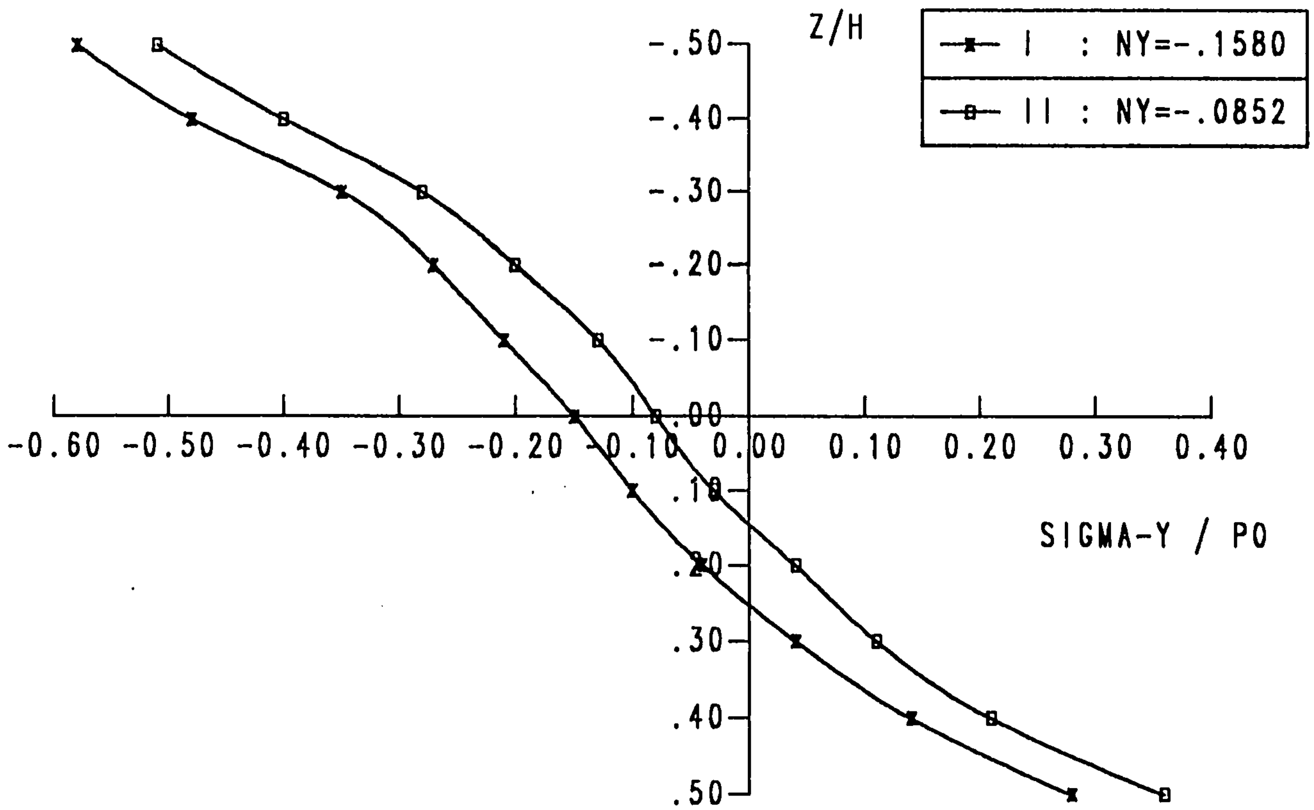


Table 5.1 Coefficient α for the Center Deflection of a Uniformly Loaded Simply Supported Square Plate

h/a	α_1	α_2	α_3	α_4
0.005	0.044009	0.044366	0.04433	0.044366
0.01	0.044149	0.044380	0.04434	0.044380
0.05	0.044789	0.044849	0.04481	0.044849
0.1	0.046294	0.046315	0.04625	0.046314
0.2	0.052171	0.052176	0.05194	0.052157
0.3	0.061946	0.061946	–	0.061867
0.4	0.075619	0.075623	0.07474	0.075312
0.5	0.093229	0.093207	–	0.092448
0.6	0.11463	0.11470	0.10853	0.11314
0.7	0.14008	0.14010	–	0.13717
0.8	0.16941	0.16941	0.15682	0.16426
0.9	0.20220	0.20262	–	0.19428
1.0	0.24024	0.23975	0.21982	0.22679

NOTE : $\alpha = 0.04433$
By CPT : Classical Plate Theory (For All h/a Ratios)

α_1 = FEM : Goma'a and Baluch
α_2 = RTR : Refined Theory (Reissner)
α_3 = RTB : Refined Theory (Voyiadjis and Baluch)
α_4 = RTP : Refined Theory (Present)

$w = \alpha(pa^4/Eh^3)$, $\mu = 0.3$

Table 5.2 Coefficient β for the Center Resultant Moment M_x of a Uniformly Loaded Simply Supported Square Plate

h/a	β_1	β_2	β_3	β_4
0.005	0.047477	0.047890	0.0479	0.047890
0.01	0.047659	0.047892	0.0479	0.047892
0.05	0.048072	0.047928	0.0492	0.047927
0.1	0.048285	0.048040	0.0512	0.048042
0.2	0.048776	0.048490	0.0534	0.048509
0.3	0.049549	0.049240	–	0.049339
0.4	0.050623	0.050290	0.0559	0.050284
0.5	0.052003	0.051640	–	0.051500
0.6	0.053689	0.053290	0.0640	0.052949
0.7	0.055682	0.055240	–	0.054611
0.8	0.057980	0.057490	0.0776	0.056460
0.9	0.063496	0.060040	–	0.058593
1.0	0.063496	0.062890	0.0964	0.060833

NOTE : $\beta = 0.0479$
By CPT : Classical Plate Theory (For All h/a Ratios)

β_1 = FEM : Goma'a and Baluch
β_2 = RTR : Refined Theory (Reissner)
β_3 = RTB : Refined Theory (Voyiadjis and Baluch)
β_4 = RTP : Refined Theory (Present)

$M_x = \beta p a^2$, $\mu = 0.3$

Table 5.3 Coefficient γ for the Center Resultant Moment M_y of a Uniformly Loaded Simply Supported Square Plate

h/a	γ_1	γ_2	γ_3	γ_4
0.005	0.047477	0.047888	0.0479	0.047888
0.01	0.047659	0.047889	0.0479	0.047889
0.05	0.048072	0.047927	0.0492	0.047927
0.1	0.048285	0.048045	0.0512	0.048043
0.2	0.048776	0.048517	0.0534	0.048498
0.3	0.049549	0.049303	–	0.049203
0.4	0.050623	0.050405	0.0559	0.050179
0.5	0.052003	0.051821	–	0.051418
0.6	0.053689	0.053552	0.0640	0.052952
0.7	0.055682	0.055597	–	0.054787
0.8	0.057980	0.057957	0.0776	0.056923
0.9	0.063496	0.060632	–	0.059369
1.0	0.063496	0.063621	0.0964	0.062159

NOTE : γ = 0.0479
By CPT : Classical Plate Theory (For All h/a Ratios)

γ_1 = FEM : Goma'a and Baluch

γ_2 = RTR : Refined Theory (Reissner)

γ_3 = RTB : Refined Theory (Voyiadjis and Baluch)

γ_4 = RTP : Refined Theory (Present)

$M_y = \gamma p a^2$, $\mu = 0.3$

Table 5.4 Coefficient α for the Center Deflection of a Uniformly Loaded Simple/Clamped Square Plate

h/a	α_1	α_2	α_3	α_4
0.005	0.0018120	0.0019179	0.00190	0.0019179
0.01	0.0018369	0.0019201	0.00188	0.0019201
0.05	0.0019672	0.0019901	0.00176	0.0019908
0.1	0.002194	0.002201	0.00166	0.002206
0.2	0.002980	0.002982	0.00158	0.003005
0.3	0.004163	0.004165	–	0.004197
0.4	0.005696	0.005697	0.00166	0.005703
0.5	0.007562	0.007565	–	0.007499
0.6	0.009763	0.009772	0.00182	0.009583
0.7	0.012314	0.012323	–	0.011966
0.8	0.015206	0.015227	0.00203	0.014653
0.9	0.018517	0.018490	–	0.017647
1.0	0.022100	0.022116	0.00231	0.020946

NOTE : α = 0.0192
By CPT : Classical Plate Theory (For All h/a Ratios)

α_1 = FEM : Goma'a and Baluch

α_2 = RTR : Refined Theory (Reissner)

α_3 = RTB : Refined Theory (Voyiadjis and Baluch)

α_4 = RTP : Refined Theory (Present)

$w = \alpha p a^4/D$, $\mu = 0.3$

Table 5.5 Coefficient β for the Center Resultant Moment M_x
of a Uniformly Loaded Simple/Clamped Square Plate

h/a	β_1	β_2	β_3	β_4
0.005	0.023429	0.024396	0.0242	0.024396
0.01	0.023643	0.024410	0.0241	0.024410
0.05	0.024784	0.024864	0.0261	0.024871
0.1	0.034170	0.026196	0.0243	0.026250
0.2	0.035011	0.030675	0.0216	0.030959
0.3	0.036073	0.036367	–	0.036721
0.4	0.037652	0.042456	0.0210	0.042240
0.5	0.040033	0.048551	–	0.046993
0.6	0.043359	0.054290	0.0279	0.050899
0.7	0.047662	0.059191	–	0.054050
0.8	0.052928	0.062634	0.0411	0.056579
0.9	0.059129	0.063861	–	0.058617
1.0	0.066235	0.061985	0.0596	0.060273

NOTE : $\beta = 0.0244$
By CPT : Classical Plate Theory (For All h/a Ratios)

β_1 = FEM : Goma'a and Baluch

β_2 = RTR : Refined Theory (Reissner)

β_3 = RTB : Refined Theory (Voyiadjis and Baluch)

β_4 = RTP : Refined Theory (Present)

$M_x = \beta p a^2$, $\mu = 0.3$

Table 5.6 Coefficient γ for the Center Resultant Moment M_y of a Uniformly Loaded Simple/Clamped Square Plate

h/a	γ_1	γ_2	γ_3	γ_4
0.005	0.031950	0.033247	0.0331	0.033247
0.01	0.032372	0.033250	0.0330	0.033250
0.05	0.033628	0.033345	0.0334	0.033350
0.1	0.02631	0.033045	0.0321	0.033639
0.2	0.03089	0.034373	0.0295	0.034647
0.3	0.03652	0.035469	–	0.036160
0.4	0.04206	0.037119	0.0269	0.038228
0.5	0.04699	0.039583	–	0.040927
0.6	0.05121	0.042990	0.0322	0.044288
0.7	0.05484	0.047370	–	0.048312
0.8	0.05803	0.052712	0.0444	0.052987
0.9	0.06090	0.058996	–	0.058297
1.0	0.06357	0.066206	0.0623	0.064220

NOTE : $\gamma = 0.0332$
By CPT : Classical Plate Theory (For All h/a Ratios)

γ_1 = FEM : Goma'a and Baluch

γ_2 = RTR : Refined Theory (Reissner)

γ_3 = RTB : Refined Theory (Voyiadjis and Baluch)

γ_4 = RTP : Refined Theory (Present)

$M_y = \gamma p a^2$, $\mu = 0.3$

Table 5.7 Coefficient α for the Center Deflection of a Simple/Free Square Plate

h/a	α_1	α_2	α_3	α_4
0.005	0.013127	0.013095	0.01309	0.013094
0.010	0.013294	0.013098	0.01309	0.013097
0.050	0.013956	0.013174	0.01310	0.013169
0.1	0.013495	0.013407	0.01312	0.013397
0.2	0.014469	0.014328	0.01326	0.014299
0.3	0.016016	0.015859	–	0.015786
0.4	0.018163	0.017999	0.01352	0.017830
0.5	0.020913	0.020748	–	0.020406
0.6	0.024278	0.024105	0.01395	0.023487
0.7	0.028229	0.028072	–	0.027053
0.8	0.032819	0.032648	0.01457	0.031090
0.9	0.037981	0.037834	–	0.035588
1.0	0.043800	0.043629	0.01527	0.040542

NOTE : $\alpha = 0.01377$
By CPT : Classical Plate Theory (For All h/a Ratios)

α_1 = FEM : Goma'a and Baluch
α_2 = RTR : Refined Theory (Reissner)
α_3 = RTB : Refined Theory (Voyiadjis and Baluch)
α_4 = RTP : Refined Theory (Present)

$w = \alpha p a^4/D$, $\mu = 0.3$

Table 5.8 Coefficient β for the Center Resultant Moment M_x of a Uniformly Loaded Simple/Free Square Plate

h/a	β_1	β_2	β_3	β_4
0.005	0.12002	0.12274	0.1225	0.12255
0.01	0.12027	0.12294	0.1225	0.12255
0.05	0.12320	0.12465	0.1228	0.12260
0.1	0.12442	0.12246	0.1240	0.12275
0.2	0.12547	0.12287	0.1252	0.12332
0.3	0.12645	0.12411	–	0.12414
0.4	0.12765	0.12683	0.1270	0.12506
0.5	0.12901	0.13180	–	0.12601
0.6	0.13048	0.13980	0.1313	0.12704
0.7	0.13202	0.15165	–	0.12823
0.8	0.13364	0.16826	0.1386	0.12964
0.9	0.13534	0.19066	–	0.13134
1.0	0.13713	0.21999	0.1489	0.13338

NOTE : $\beta = 0.1235$
By CPT : Classical Plate Theory (For All h/a Ratios)

β_1 = FEM : Goma'a and Baluch

β_2 = RTR : Refined Theory (Reissner)

β_3 = RTB : Refined Theory (Voyiadjis and Baluch)

β_4 = RTP : Refined Theory (Present)

$M_x = \beta p a^2$, $\mu = 0.3$

Table 5.9 Coefficient γ for the Center Resultant Moment M_y of a Uniformly Loaded Simple/Free Square Plate

h/a	γ_1	γ_2	γ_3	γ_4
0.005	0.026176	0.027227	0.0271	0.027080
0.01	0.026190	0.027376	0.0272	0.027081
0.05	0.026586	0.028660	0.0275	0.027115
0.1	0.026193	0.025831	0.0283	0.027222
0.2	0.024942	0.024414	0.0299	0.027644
0.3	0.023540	0.022757	–	0.028323
0.4	0.022057	0.021013	0.0324	0.029241
0.5	0.020622	0.019373	–	0.030409
0.6	0.019316	0.017927	0.0358	0.031861
0.7	0.018163	0.016687	–	0.033635
0.8	0.017150	0.015628	0.0399	0.035764
0.9	0.016253	0.014707	–	0.038268
1.0	0.015445	0.013882	0.0478	0.041153

NOTE : $\gamma = 0.0102$
By CPT : Classical Plate Theory (For All h/a Ratios)

γ_1 = FEM : Goma'a and Baluch
γ_2 = RTR : Refined Theory (Reissner)
γ_3 = RTB : Refined Theory (Voyiadjis and Baluch)
γ_4 = RTP : Refined Theory (Present)
$M_y = \gamma p a^2$, $\mu = 0.3$

Table 5.10 Coefficient α for the Center Deflection of a Simply Supported Square Plate with a Line Load at x = a/2

h/a	α_1	α_2
0.005	0.073601	0.073620
0.01	0.073601	0.073653
0.05	0.073601	0.074700
0.1	0.073601	0.077939
0.2	0.073601	0.090682
0.3	0.073601	0.11144
0.4	0.073601	0.14031
0.5	0.073601	0.17695
0.6	0.073601	0.22124
0.7	0.073601	0.13717
0.8	0.073601	0.33156
0.9	0.073601	0.39619
1.0	0.073601	0.46787

α_1 = CPT : Classical Plate Theory
α_2 = RTP : Refined Theory (Present)
$w = \alpha(pa^3/Eh^3)$, $\mu = 0.3$

Table 5.11 Coefficient β for the Center Resultant Moment M_x of a Simply Supported Square Plate with a Line Load at x = a/2

h/a	β_1	β_2
0.005	0.127	0.12405
0.01	0.127	0.12405
0.05	0.127	0.12386
0.1	0.127	0.12378
0.2	0.127	0.12505
0.3	0.127	0.12758
0.4	0.127	0.13200
0.5	0.127	0.13737
0.6	0.127	0.14366
0.7	0.127	0.15071
0.8	0.127	0.15843
0.9	0.127	0.16630
1.0	0.127	0.17515

β_1 = CPT : Classical Plate Theory
β_2 = RTP : Refined Theory (Present)
$M_x = \beta pa$, $\mu = 0.3$

Table 5.12 Coefficient γ for the center Resultant Moment M_y of a Simply Supported Square Plate With A Line Load at $x = a/2$

h/a	γ_1	γ_2
0.005	0.092	0.091064
0.01	0.092	0.091129
0.05	0.092	0.093099
0.1	0.092	0.098766
0.2	0.092	0.11682
0.3	0.092	0.14017
0.4	0.092	0.16671
0.5	0.092	0.19565
0.6	0.092	0.22639
0.7	0.092	0.25854
0.8	0.092	0.29179
0.9	0.092	0.32599
1.0	0.092	0.36103

γ_1 = CPT : Classical Plate Theory
γ_2 = RTP : Refined Theory (Present)
$M_y = \gamma pa$, $\mu = 0.3$

Table 5.13 Coefficient α for the Center Deflection of a Simply Supported Square Plate with a Strip Load (Width = 0.2 a) Centered at x = a/2

h/a	α_1	α_2
0.005	0.014368	0.014368
0.01	0.014368	0.014373
0.05	0.014368	0.014558
0.1	0.014368	0.015132
0.2	0.014368	0.017402
0.3	0.014368	0.021106
0.4	0.014368	0.026252
0.5	0.014368	0.032779
0.6	0.014368	0.040661
0.7	0.014368	0.049839
0.8	0.014368	0.060240
0.9	0.014368	0.071676
1.0	0.014368	0.084327

α_1 = CPT : Classical Plate Theory

α_2 = RTP : Refined Theory (Present)

$w = \alpha(P_o a^4/Eh^3)$, $\mu = 0.3$

Table 5.14 Coefficient β for the center Resultant Moment M_x of a Simply Supported Square Plate With A Strip Load (Width = 0.2a) Centered At x = a/2

h/a	β_1	β_2
0.005	0.020914	0.020914
0.01	0.020914	0.020915
0.05	0.020914	0.020925
0.1	0.020914	0.020940
0.2	0.020914	0.020969
0.3	0.020914	0.020999
0.4	0.020914	0.021387
0.5	0.020914	0.021848
0.6	0.020914	0.022456
0.7	0.020914	0.023192
0.8	0.020914	0.024043
0.9	0.020914	0.024925
1.0	0.020914	0.025983

β_1 = CPT : Classical Plate Theory

β_2 = RTP : Refined Theory (Present)

$M_x = \beta P_o a^2$, $\mu = 0.3$

Table 5.15 Coefficient β for the center Resultant Moment M_y of a Simply Supported Square Plate With A Strip Load (Width = 0.2a) Centered At x = a/2

h/a	β_1	β_2
0.005	0.016841	0.016841
0.01	0.016841	0.016843
0.05	0.016841	0.016904
0.1	0.016841	0.017087
0.2	0.016841	0.017807
0.3	0.016841	0.019017
0.4	0.016841	0.020633
0.5	0.016841	0.022608
0.6	0.016841	0.024875
0.7	0.016841	0.027378
0.8	0.016841	0.030075
0.9	0.016841	0.032941
1.0	0.016841	0.035958

β_1 = CPT : Classical Plate Theory

β_2 = RTP : Refined Theory (Present)

$M_x = \beta P_o a^2$, $\mu = 0.3$

Table 5.16 Total Distributed Reaction R Along Edges Of A Uniformly Loaded Square Plate

h/a	α_1	α_2	α_3
0.005	-1.02	-1.02	-1.02
0.01	-1.02	-1.02	-1.02
0.05	-1.02	-1.02	-1.02
0.1	-1.02	-1.03	-1.03
0.2	-1.03	-1.03	-1.04
0.3	-1.07	-1.04	-1.05
0.4	-1.07	-1.05	-1.07
0.5	-1.08	-1.05	-1.08
0.6	-1.09	-1.05	-1.08
0.7	-1.09	-1.05	-1.08
0.8	-1.09	-1.05	-1.09
0.9	-1.11	-1.05	-1.09
1.0	-1.09	-1.06	-1.09

α_1 = SIMPLY SUPPORTED SQUARE PLATE.
α_2 = SIMPLY SUPPORTED / CLAMPED SQUARE PLATE.
α_3 = SIMPLY SUPPORTED / FREE SQUARE PLATE.
$R = \alpha (P_o)$, $\mu = 0.3$

APPENDIX

A-1 DERIVATION OF EQUATION (4-28) :

Equations (4-25) to (4-27), can be expressed in the form :

$$a_{11}\ \overline{w} + a_{12}\ \varphi_x + a_{13}\ \varphi_y = c_1\ p \tag{A-1}$$

$$a_{21}\ \overline{w} + a_{22}\ \varphi_x + a_{23}\ \varphi_y = c_2\ p \tag{A-2}$$

$$a_{31}\ \overline{w} + a_{32}\ \varphi_x + a_{33}\ \varphi_y = c_3\ p \tag{A-3}$$

Where :

$$a_{11} = a\frac{\partial}{\partial x}\Delta - S\frac{\partial}{\partial x} \tag{A-4.1}$$

$$a_{12} = b\left(2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - S \tag{A-4.2}$$

$$a_{13} = b\ \frac{\partial^2}{\partial x \partial y} \tag{A-4.3}$$

$$a_{21} = a\frac{\partial}{\partial y}\Delta - S\frac{\partial}{\partial y} \tag{A-4.4}$$

$$a_{22} = b\ \frac{\partial^2}{\partial x \partial y} \tag{A-4.5}$$

$$a_{23} = b\left(2\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right) - S \tag{A-4.6}$$

$$a_{31} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \text{(A-4.7)}$$

$$a_{32} = \frac{\partial}{\partial x} \qquad \text{(A-4.8)}$$

$$a_{33} = \frac{\partial}{\partial y} \qquad \text{(A-4.9)}$$

$$c_1 = c\frac{\partial}{\partial x} \qquad \text{(A-4.10)}$$

$$c_2 = c\frac{\partial}{\partial y} \qquad \text{(A-4.11)}$$

$$c_3 = \frac{-1}{S} \qquad \text{(A-4.12)}$$

$$a = -D + \frac{h^3 F_1 S}{6} \qquad \text{(A-4.13)}$$

$$c = \mu\frac{h^3 F_1}{12(1 - \mu)} \qquad \text{(A-4.14)}$$

To obtain the governing differential equation for $\overline{w}$, we write :

$$\overline{w} = \frac{\begin{vmatrix} c_1 p & a_{11} & a_{13} \\ c_2 p & a_{22} & a_{23} \\ c_3 p & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

or :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \{\overline{w}\} = \begin{vmatrix} c_1 p & a_{11} & a_{13} \\ c_2 p & a_{22} & a_{23} \\ c_3 p & a_{32} & a_{33} \end{vmatrix} \{p\} \qquad \text{(A-5)}$$

By expanding the operators determinants in equation (A-5) , we get for this equation:

$$\left\{(2 b^2 - a b)\Delta^3 + (a S - 2 b S)\Delta^2\right\}\{\overline{w}\} =$$

$$\left\{A \Delta^2 + B \Delta + C\right\}\{p\}$$

or :

$$M' \Delta^3\overline{w} + N' \Delta^2\overline{w} = A \Delta^2 p + B \Delta p + Cp \qquad \text{(A-6)}$$

Thus equation (4-28) is proved .

A-2 DERIVATION OF THE FUNCTION $Y_m(y)$ IN EQN. 4-37 :

Substituting equation (4-36) in equation (4-32), we get :

$$M'\left[\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right]\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\left\{\overline{w}_2\right\} + N'\left[\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right]\left\{\overline{w}_2\right\} = 0 \qquad \text{(A-7)}$$

OR :

$$M'\left[\frac{\partial^6}{\partial x^6} + 3\frac{\partial^6}{\partial x^4 \partial y^2} + 3\frac{\partial^6}{\partial x^2 \partial y^4} + \frac{\partial^6}{\partial y^6}\right]\left\{\overline{w}_2\right\} + N'\left[\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right]\left\{\overline{w}_2\right\} = 0$$

OR :

$$M'\left(- \alpha_m^6 Y_m + 3\alpha_m^4 Y_m(y) - 3\alpha_m^2 Y_m^{(iv)}(y) + Y_m^{(vi)}(y) \right) + N'\left(\alpha_m^4 Y_m - 2\alpha_m^2 Y_m(y) + Y_m^{(iv)}(y) \right) = 0$$

Rearranging the above equation , we get :

$$Y_m^{(vi)} - \left(3\alpha_m^2 - \frac{N'}{M'}\right)Y_m^{(iv)} + \alpha_m^2\left(3\alpha_m^2 - 2\frac{N'}{M'} \right)Y_m - \alpha_m^4\left(\alpha_m^2 - \frac{N'}{M'} \right)Y_m = 0 \qquad \text{(A-8)}$$

The characteristic equation for the above differential equation is :

$$r^6 - \left(3\alpha_m^2 - \frac{N'}{M'} \right) r^4 + \alpha_m^2\left(3\alpha_m^2 - 2\frac{N'}{M'} \right) r^2$$

$$- \alpha_m^4\left(\alpha_m^2 - \frac{N'}{M'}\right) = 0 \qquad \text{(A-9)}$$

A root for the above equation is: $\pm\, \alpha_m$

Thus equation (A-9) can be rewritten as:

$$\left(r^2 - \alpha_m^2\right)\left\{ \left(r^2 - \alpha_m^2\right)\left(r^2 - \left(\alpha^2 - \frac{N'}{M'}\right) \right) \right\} = 0$$ From the above equation, the roots for equation (A-9) are :

$$\pm\, \alpha_m \,,\ \pm\, \alpha_m \,,\ \pm \sqrt{\alpha_m^2 - \frac{N'}{M'}}$$

OR :

$$\pm\, \alpha_m \,,\ \pm\, \alpha_m \,,\ \pm\, \gamma_m$$

where :

$$\gamma_m^2 = \alpha_m^2 - \frac{N'}{M'} \qquad \text{(A-10)}$$

Therefore we get for $Y_m(y)$:

$$Y_m(y) = c_1\, e^{-\alpha_m y} + c_2\, y\, e^{-\alpha_m y} + c_3\, e^{\alpha_m y}$$

$$+ c_4\, y\, e^{\alpha_m y} + c_5\, e^{-\gamma_m y} + c_6\, e^{\gamma_m y} \qquad \text{(A-11)}$$

Since :

$$\sinh(y) = \frac{e^{y} - e^{-y}}{2}$$

$$\cosh(y) = \frac{e^{y} + e^{-y}}{2}$$ And :

$$e^{y} = \sinh(y) + \cosh(y)$$

$$e^{-y} = \cosh(y) - \sinh(y)$$

Then equation (A-11) can be rewritten as :

$$Y_m(y) = A_m \cosh \alpha_m y + B_m \alpha_m y \sinh \alpha_m y + C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y + E_m \cosh \gamma_m y + F_m \sinh \gamma_m y \qquad \text{(A-13)}$$

Thus equation 4-37 is proved .

A-3 DERIVATION OF THE PARTICULAR SOLUTIONS FOR THE BENDING PROBLEM :

To get the particular solutions for this case , the dependent variables may be assumed to be of the form :

$$w_o = \sum w_{oo} \sin \alpha_m x$$

$$u_o = \sum u_{oo} \cos \alpha_m x$$

$$v_o = \sum v_{oo} \sin \alpha_m x$$

$$Q_x = \sum Q_{ox} \cos \alpha_m x$$

$$Q_y = \sum Q_{oy} \sin \alpha_m x$$

$$\varphi_x = \sum \varphi_{ox} \cos \alpha_m x$$

$$\varphi_y = \sum \varphi_{oy} \sin \alpha_m x$$

$$M_x = \sum M_{ox} \cos \alpha_m x$$

$$M_y = \sum M_{oy} \sin \alpha_m x$$

$$M_{xy} = \sum M_{oxy} \cos \alpha_m x$$

$$p = \sum M_{oxy} \cos \alpha_m x \qquad \text{(A-14)}$$

Substituting equations (A-14) into equation (3-7), we get :

$$\frac{dQ_x}{dx} = -p$$

or :

$$Q_{ox} = \frac{p_m}{\alpha_m} \tag{A-15}$$

Let :

$M_m = M_{ox} + M_{oy}$

Then , from equations (3-27.1), (3-27.2), and (A-14), we get :

$$M_m = p_m \left(\frac{1 + \mu}{\alpha_m^2} + \frac{\mu h^3 F_1}{12} \right) \tag{A-16}$$

From the governing equation for w_o (equation (3-35)) we get :

$$w_{oo} = \frac{p_m}{\alpha_m^4 D} \left\{ 1 + \frac{(2 - \mu) h^3 \alpha_m^2 F_1}{12(1 - \mu)} - \frac{\alpha_m^4 D}{N} \right.$$

$$\left. + \frac{\mu h^2 \alpha_m^2}{40(1 - \mu)} + \frac{\mu^2 h^5 \alpha_m^4 F_1}{480(1 - \mu^2)} \right\} \tag{A-17}$$

From equation (3-27.4), we get for φ_{ox}

$$\varphi_{ox} = \left\{ - w_m \alpha_m + \frac{1}{S} \frac{p_m}{\alpha_m} - \frac{1}{N} \alpha_m p_m \right.$$

$$\left. - \frac{1}{N} \alpha_m p_m + \frac{1}{R} \alpha_m M_m \right\} \tag{A-18}$$

From equation (3-27.5), we get for φ_{oy}

$$\varphi_{oy} = 0 \tag{A-19}$$

From the equilibrium equation:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = Q_x$$

we get :

$$\frac{dM_x}{dx} = Q_x$$

From which and with equation (A-15) for Q_{ox} , we get for M_{ox} :

$$M_{ox} = \frac{p_m}{\alpha_m^2} \tag{A-20}$$

From equations (3-27.1), (A-18), and (A-20) , we get for φ_{ox}:

$$\varphi_{ox} = p_m\left(\frac{\mu(1 + \mu)}{E\alpha_m} - \frac{1}{\alpha_m^3 D}\right) \tag{A-21}$$

Similarly by using equation (3-34), we get for Q_{oy} :

$$Q_{oy} = 0 \tag{A-22}$$

And from the equilibrium equation:

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = Q_y$$

we get :

$$M_{oxy} = 0 \tag{A-23}$$

A-4 PHYSICAL INTERPRETATION FOR THE AVERAGE DISPLACEMENTS $\bar{w}$, $\bar{u}$, $\bar{v}$, AND AVERAGE ROTATIONS φ_x and φ_y :

For convenience in formulation and analysis,average displacements $\bar{w}$, $\bar{u}$, $\bar{v}$, and average rotations φ_x and φ_y are introduced . This is similar to introducing moment stress resultants which are actually average stresses :

{Exact Stresses : σ_x , σ_y,...

{Average Stresses : M_x , M_y,...

Similarly :

{Exact Displacements : u,v,w

{Average Displacements : $\bar{u}$, $\bar{v}$, and $\bar{w}$

The average displacement $\bar{u}$ is defined as follows :

$$\bar{u} = \frac{1}{h} \int_{\frac{-h}{2}}^{\frac{+h}{2}} u \, dz \qquad \text{(A-24)}$$

And similarly :

$$\bar{v} = \frac{1}{h} \int_{\frac{-h}{2}}^{\frac{+h}{2}} v \, dz \qquad \text{(A-25)}$$

Equating work of the transverse shear stress τ_{xz} due to displacement w to the work of the transverse shear resultant Q_x due to average displacement $\bar{w}$, one has :

$$\int_{\frac{-h}{2}}^{\frac{+h}{2}} \tau_{xz} \, w \, dz = Q_x \, \bar{w} \qquad \text{(A-26)}$$

On substituting for τ_{xz} and w from equations (3.3) and (3.15), respectively yields for the $\bar{w}$ the expression :

$$\bar{w} = w_o + \frac{p}{N} - \frac{M}{R} \qquad \text{(A-27)}$$

The same result would be obtained if one were to use the work of τ_{yz} stresses.

Defining the average rotations of sections x = constant , y = constant by ψ_x and ψ_y , respectively , one may equate the work of the resultant couple on the average rotation to the work of the corresponding stresses σ_x , σ_y , on the displacements u and v and expressed as :

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_x \; u \; dz = M_x \; \psi_x \tag{A-28}$$

$$\int_{-\frac{h}{2}}^{+\frac{h}{2}} \sigma_y \; v \; dz = M_y \; \psi_y \tag{A-29}$$

The stress expressions to be used for σ_x , σ_y are the initial linear variations ($\sigma_x = \frac{12M_x}{h^3}z$, $\sigma_y = \frac{12M_y}{h^3}z$)

On substituting the linear form of σ_x, and u into equation (A-28) and integrating the results , an expression for ψ_x is obtained as :

$$\psi_x = - \frac{\partial w_o}{\partial x} + \frac{Q_x}{S} - \frac{1}{N}\frac{\partial p}{\partial x} + \frac{1}{R}\frac{\partial M}{\partial x} \tag{A-30}$$

Similarly an expression for ψ_y is obtained as :

$$\psi_y = - \frac{\partial w_o}{\partial y} + \frac{Q_y}{S} - \frac{1}{N}\frac{\partial p}{\partial y} + \frac{1}{R}\frac{\partial M}{\partial y} \tag{A-31}$$

On comparison of equations (A-30) and (A-3.27.4), one notes that

$\psi_x = \varphi_x$,

i.e. :

φ_x is the rotation of a vertical element x = constant of the plate .

Also on comparison of equations (A-31) and (3.27.5), one notes that

$\psi_y = \varphi_y$,

i.e. :

φ_y is the rotation of a vertical element y = constant of the plate .

A-5 PROGRAM LISTING

A-5.1 PROGRAM DISS2 LISTING :

```
C
C
C
C
C
C
C**************************************************
C
C  PROGRAM FOR THE ANALYSIS OF THICK PLATE BENDING PROBLEMS
C  USING LEVY METHOD
C
C  PROGRAM WRITTEN BY : AMMAR KHALEEL HAFEZ MOHAMMED
C                 IN DHAHRAN , SAUDI ARABIA.
C
C**************************************************
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DOUBLE PRECISION  NU,KPD,K4,K5
      DATA NU/0.30/,BAR/1.00/,
     .   MTERM/25/,IBOUND/1/,IPLANE/1/,ISTRES/4/,IPLOT/2/,IDEF/2/,
     .   IPRINT/2/,NPLATE/ 1/,MPLATE/13/,IZMAX/11/,
     .   X/0.50/,Y/0.00/,ZI/0.50/,UU/0.200/,ILOAD/1/
C**************************************************
 18  FORMAT('****************************************
     .****************************************')
      GO TO (170,171) IPLANE
 170 WRITE(6,175) IPLANE
 175 FORMAT('IPLANE =',I2,2X,': EDGE AT Y = + -B/2 IS NOT ALLOWED TO STRET
     .CH IN THE Y-DIRECTION ')
      GO TO 177
 171 WRITE(6,176) IPLANE
 176 FORMAT('IPLANE=',I2,2X,' : EDGE AT Y = + -B/2 IS  ALLOWED TO STRETCH
     . IN THE Y-DIRECTION ')
 177 CONTINUE
      GO TO (70,71,72) IBOUND
 70  WRITE(6,73) IBOUND
      GO TO 76
 71  WRITE(6,74) IBOUND
      GO TO 76
 72  WRITE(6,75) IBOUND
 73  FORMAT('IBOUND = ',I2,2X,': PLATE SIMPLY  SUPPORTED AT Y  =  + ,- B/2')
 74  FORMAT('IBOUND = ',I2,2X,': PLATE CLAMPED          AT Y  =  + ,- B/2')
 75  FORMAT('IBOUND = ',I2,2X,': PLATE FREE             AT Y  =  + ,- B/2')
```

```
76  CONTINUE
     GO TO (400,401,501) ILOAD

400 WRITE(6,402)
     GO TO 404
402 FORMAT(' LOAD :  UNIFORM LOAD ')
401 WRITE(6,403) ZI
     GO TO 404
403 FORMAT(' LOAD :  LINE LOAD APPLIED AT ZI  =',F8.2)
501 WRITE(6,503) UU,ZI
503 FORMAT('LOAD : STRIP LOAD ,WIDTH  =',F8.3,',CENTERED AT ZI  =',F8.3)
404 WRITE(6,188) NU
188 FORMAT('NU   =  ',F6.3)
     WRITE(6,101) BAR
     WRITE(6,122) MTERM
122 FORMAT('M  =  1,3,5,....',I2)
101 FORMAT('B/A  =  ',F10.2)
C    PI = 22.0/7.0
     PI = -1.00
     PI = DARCOS(PI)
     GO TO (490,491,491) IDEF
490 CONTINUE
     WRITE(6,141)
     WRITE(6,492) X,Y
     WRITE(6,141)
492 FORMAT('DEFLECTIONS,X-M,Y-MOM :  ARE EVALUATED AT X  =',F8.2,2X,
    .'Y  =',F8.2)
C
     GO TO 435
491 CONTINUE
     GO TO (370,371,373,373,435) ISTRES
370 CONTINUE
     WRITE(6,141)
     WRITE(6,183) X,Y
183 FORMAT('NOTE : SIGMAX,SIGMAY,& SIGMAZ ARE EVALUATED AT (',F4.1,'A,
    .',F4.1,'B,Z)'  )
     WRITE(6,141)
C    WRITE(6,331)
331 FORMAT('SIGMAX   ',6X,'SIGMAY   ',4X,'SIGMAZ   ')
     GO TO 435
371 WRITE(6,141)
     WRITE(6,180)
     WRITE(6,181)
     WRITE(6,182)
180 FORMAT('NOTE : SIGMAXY IS EVLUATED AT (0   ,B/2,Z)')
181 FORMAT('NOTE : SIGMAXZ IS EVLUATED AT (0   , 0 ,Z)')
182 FORMAT('NOTE : SIGMAYZ IS EVLUATED AT (A/2,B/2,Z)')
     WRITE(6,141)
     WRITE(6,372)
372 FORMAT('SIGMAXY ',6X,'SIGMAXZ ',4X,'SIGMAYZ ')
```

```
      GO TO 435
  373 CONTINUE
      WRITE(6,141)
      WRITE(6,437) X,Y
C     WRITE(6,497)
C     WRITE(6,466) X,Y
C     WRITE(6,479)
      WRITE(6,141)
  497 FORMAT(7X,'Z',8X,'SIGZ-B',5X,'SIGZ-P')
C497  FORMAT(7X,'Z',8X,'SIGXZR',5X,'SIGXZB',7X,'SIGXZP')
C497  FORMAT(7X,'H/A',8X,'XSHERR',5X,'XSHERB',7X,'XSHERP')
  479 FORMAT(7X,'HAR',5X,'XYMOMR',7X,'XYMOMP')
C479  FORMAT(5X,'H/A',6X,'TOTAL INPLANE FORCE NY  ')
C     WRITE(6,440)
C437  FORMAT('NOTE :SIGMA-X IS EVALUATED BY DIFFERENT REFINED THEORIES')
C437  FORMAT('NOTE : SIGY , SIGXY , SIGYZ : ARE EVLUATED AT FREE END:
  437 FORMAT('NOTE : STRESSES ARE EVALUATED AT X  = ',F8.2,2X,'Y  = ',F8.2)
  466 FORMAT('NOTE : A CHECK FOR TOTAL LOAD ON PLATE' )
C437  FORMAT('NOTE : NU IS EVLUATED AT X=0.5*A,Y=0.0' )
C440  FORMAT('Z ',10X,'NU REISSNER',6X,'NU PRESENT')
C437  FORMAT('NOTE : W(X,Y,Z) IS EVLUATED AT X=0.5*A,Y=0.0' )
C440  FORMAT(5X,'X ',10X,'W REISSNER',6X,'W PRESENT',' AT Z/H  =  0.0')
C440  FORMAT('Z ',10X,'W REISSNER',6X,'W PRESENT')
  435 CONTINUE
C**************************************************
      DO 300 IPLATE=NPLATE,MPLATE
C**************************************************
      IF(IPLATE.GT.4) GO TO 134
      GO TO (130,131,132,133) IPLATE
  130 HAR=0.005
      GO TO 136
  131 HAR=0.010
      GO TO 136
  132 HAR=0.050
      GO TO 136
  133 HAR=0.100
      GO TO 136
C134  HAR=0.200*(IPLATE-4)
  134 HAR=0.100*(IPLATE-3)
      IF(IPLATE.GT.13) GO TO 184
      GO TO 136
  184 HAR=IPLATE-12.0
  136 CONTINUE
      WRITE(6,141)
      WRITE(6,367) HAR
  367 FORMAT('H/A  = ',F6.3)
      WRITE(6,477)
C477  FORMAT(7X,'H/A',8X,'YSHERR',5X,'YSHERB',7X,'YSHERP')
C477  FORMAT(7X,'Z',8X,'SIGXYR',5X,'SIGXYB',7X,'SIGXYP')
  477 FORMAT(7X,'Z',9X,'SIGXR',6X,'SIGXB',7X,'SIGXP')
```

```
C477 FORMAT(7X,'Z',5X,'SIGMA-X(B)',5X,'SIGMA-X(P)')
 141 FORMAT('*****************************************')
     WRITE(6,141)
     Z = - 0.600000
C*****************************************************
     DO 250 IZ=1,IZMAX
C*****************************************************
     Z = Z + 0.100000
C*****************************************************
     DO 200 IBALCH=1,2
C*****************************************************

     WBAR  = 0.0
     WBARE = 0.0
     WBARR = 0.0
     WBARRE= 0.0
     XMOM  = 0.0
     YMOM  = 0.0
     XYMOM = 0.0
     XSHER=0.0
     YSHER=0.0
     WR=0.0
     WP=0.0
     EPSXP=0.0
     EPSYP=0.0
     EPSZP=0.0
     EPSXR=0.0
     EPSYR=0.0
     EPSZR=0.0
     APLOAD=0.0
C
     XMOMR  = 0.0
     YMOMR  = 0.0
     XYMOMR = 0.0
     VXR=0.0
     VYR=0.0
     W0   = 0.0
     XMPYM = 0.0
C-----------------------------------
     GO TO (340,341) IBALCH
 340 XSTB=0.0
     YSTB=0.0
     ZSTB=0.0
     XYSTB=0.0
     XZSTB=0.0
     YZSTB=0.0
     GO TO 342
C
 341 XSTP=0.0
     YSTP=0.0
```

```
      ZSTP=0.0
      XYSTP=0.0
      XZSTP=0.0
      YZSTP=0.0
      YNYP=0.0
C
 342  XSTR=0.0
      YSTR=0.0
      ZSTR=0.0
      XYSTR=0.0
      XZSTR=0.0
      YZSTR=0.0
      XLOADR=0.0
      XLOADP=0.0
C*****************************************************
      DO 100 M=1,MTERM,2
C*****************************************************
      IF(HAR.LT.0.10)GO TO 222
      ITHICK=2
      GO TO 223
 222  ITHICK=1
 223  CONTINUE
      GO TO (112,113,113) IPRINT
 112  WRITE(6,18)
      WRITE(6,17) M
 17   FORMAT('          M   = ',I2)
 113  CONTINUE
      GO TO (150,151) IBALCH
 150  F1=6./5.
      F2=-1./2.
      F4=-1./48.
      F3=39./1120.
      GO TO 152
 151  CONTINUE
      CALL    DISS(M,NU,HAR,ALPHA,A1,A2,A3,A4,A5,F1,F2,F3,F4,
     .        Z,F1Z,F1ZP,F2Z,F3Z)
 152  CONTINUE
      CALL     POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .          HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .          X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
      CALL    BENDNG(IBOUND,ITHICK,M,NU,HAR,AP,APB,GAMB,KPD,UU,
     .         BAR,BETA,BETAP,A,B,EE,IPRINT,F1,X,Y,ZI,ILOAD)
      CALL    FORCES(IBOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
     .         BETA,BETAP,A,B,EE,WPAR,WPARE,XM,YM,IPRINT,X,Y,
     .         ZI,UU,ILOAD,XYM,QX,QY)
      CALL    REISS(M,IBOUND,ITHICK,NU,HAR,BAR,AP,APB,GAMB,F1,
     .         WPARR,XMR,YMR,XYMR,WPARRE,VX,VY,SIGXR,SIGYR,UU,
     .         SIGZR,SIGXYR,SIGXZR,SIGYZR,X,Y,Z,ZI,ILOAD,F1ZR)
      CALL    XPLANE(M,IPLANE,IBOUND,NU,HAR,BAR,AP,APB,C1,C2,UP,
     .          XK4,X,Y,ZI,UU,ILOAD,F1,F2)
```

```
      CALL    STRESS(IBOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
     .          F2,F3,F4,F1Z,F2Z,F3Z,BETA,BETAP,A,B,EE,C1,C2,
     .          UP,XK4,SIGX,SIGY,SIGZ,SIGXY,SIGXZ,SIGYZ,IBALCH,
     .   .      X,Y,Z,ZI,UU,ILOAD,F1ZP,QX,QY,YNY)
      WBAR  =WBAR+WPAR
      WBARE=WBARE+WPARE
      WBARR=WBARR+WPARR
      WBARRE=WBARRE+WPARRE
      XMOM  =XMOM+XM
      YMOM  =YMOM+YM
      XYMOM=XYMOM+XYM
      XSHER=XSHER+QX
      YSHER=YSHER+QY
C
C
      APLOAD= APLOAD + PM*DSIN(APX)
C     WRITE(6,330) HAR,PM,APLOAD
C
C
      YNYP=YNYP + YNY
C
      XMOMR=XMOMR+XMR
      YMOMR=YMOMR+YMR
      XYMOMR=XYMOMR+XYMR
      VXR=VXR+VX
      VYR=VYR+VY
      XNU2  =12.*(1.-NU**2.)
C
      GO TO (35,36,40)IBOUND
 35   ALFA1=WBAR
      ALFA1R=WBARR
      GO TO 37
 36   ALFA1  =WBAR/XNU2
      ALFA1R=WBARR/XNU2
      GO TO 37
 40   ALFA1  =WBAR/XNU2
      ALFA1E=WBARE/XNU2
C40   ALFA1  =WBAR
C     ALFA1E=WBARE
      ALFA1R=WBARR/XNU2
      ALFARE=WBARRE/XNU2
 37   BETA1  =XMOM
      GAMA1  =YMOM
      BETA1R=XMOMR
      GAMA1R=YMOMR
C
      GO TO (114,115,115) IPRINT
 114  WRITE(6,125) ALFA1,BETA1,GAMA1
 125  FORMAT('ALFA1 =',E12.5,3X,'BETA1 = ',E12.5,3X,'GAMA1 = ',E12.5)
C
```

```
C NOTE :
C      PNR = P/N
C      RMR = M/R
C WHERE :
C        M = M + M
C            X   Y
C
  115 XMPYM=XM+YM
      K4=4.*F3*HAR**4./AP
      K5=3.*NU/10.*XMPYM*HAR**2.
      GO TO (30,31,31)IBOUND
   31 K4=K4/XNU2
      K5=K5/XNU2
C     WPAR=WPAR/XNU2
   30 PNR=K4
      RMR=K5
      W0=W0+WPAR/XNU2-PNR+RMR
      GO TO (116,117,117) IPRINT
  116 CONTINUE
      WRITE(6,102) W0
  102 FORMAT('W0=',E12.5)
  117 CONTINUE
C****************************************************
C= = = > IPLANE : IS AN INDICATOR WHETHER THE EDGE AT (X,+-B/2)
C              IS OR NOT ALLOWED TO STRETCH IN THE Y-DIRECTION .
C              IF :
C     IPLANE = 1 = = = >EDGE IS NOT ALLOWED TO STRETCH IN THE Y-DIRECTION .
C     IPLANE = 2 = = = >EDGE IS     ALLOWED TO STRETCH IN THE Y-DIRECTION .
C
C= = > NOTE : IPRINT : INDICATOR WHETHER TO PRINT INTERMEDIATE
C              RESULTS FOR FORCES & DEFLECTION OR NOT
C     IPRINT= 1   PRINT INTERMEDIATE RESULTS .
C     IPRINT= 2   DO NOT PRINT INTERMEDIATE RESULTS .
C     IPRINT= 3   DO NOT PRINT FINAL      RESULTS .
C
C= = > NOTE : IDEF   : INDICATOR WHETHER TO PRINT INTERMEDIATE
C              RESULTS FOR DEFLECTION OR NOT
C     IDEF  = 1   PRINT INTERMEDIATE RESULTS .
C     IDEF  = 2   DO NOT PRINT INTERMEDIATE RESULTS .
C
C
C= = > NOTE : ISTRES : INDICATOR WHETHER TO PRINT INTERMEDIATE
C              RESULTS FOR STRESSES OR NOT
C     ISTRES= 1   PRINT INTERMEDIATE RESULTS .
C     ISTRES= 2   DO NOT PRINT INTERMEDIATE RESULTS .
C
C= = > NOTE : IPLOT  : INDICATOR WHETHER TO PRINT RESULTS
C              FOR PLOTTING PURPOSES OR NOT
C     IPLOT = 1   PRINT  RESULTS .
C     IPLOT = 2   DO NOT PRINT  RESULTS .
```

```
C*****************************************************
      XSTR=XSTR + SIGXR
      YSTR=YSTR + SIGYR
      ZSTR=ZSTR + SIGZR
      XYSTR=XYSTR+ SIGXYR
      XZSTR=XZSTR+ SIGXZR
      YZSTR=YZSTR+ SIGYZR
C
      F2R=-1./4.*( 2.*Z - 3.*Z**2 + 2.*Z**4 )
      F3R=39./1120.
      XMPYMR=XMR + YMR
      WR=WR + PM*HAR4*(F2R-F3R)*DSIN(APX)
     .    + 3.*NU*HAR2*XMPYMR*(1./10.-2.*Z**2) + WPARR
C
      EPSX= SIGXR -NU*( SIGYR + SIGZR )
      EPSY= SIGYR -NU*( SIGXR + SIGZR )
      EPSZ= PM*F1ZR*DSIN(APX) - 12.*NU*XMPYMR/HAR2*Z
      EPSXR=EPSXR+EPSX
      EPSYR=EPSYR+EPSY
      EPSZR=EPSZR+EPSZ
C
      GO TO (332,333) IBALCH
  332 XSTB =XSTB + SIGX
      YSTB =YSTB + SIGY
      ZSTB =ZSTB + SIGZ
      XYSTB=XYSTB+ SIGXY
      XZSTB=XZSTB+ SIGXZ
      YZSTB=YZSTB+ SIGYZ
      GO TO 190
  333 XSTP =XSTP + SIGX
      YSTP =YSTP + SIGY
      ZSTP =ZSTP + SIGZ
      XYSTP=XYSTP+ SIGXY
      XZSTP=XZSTP+ SIGXZ
      YZSTP=YZSTP+ SIGYZ
C
      XMPYMP=XM + YM
      WP=WP + PM*HAR4*(F2Z-F3)*DSIN(APX)
     .    + 3.*NU*HAR2*XMPYMP*(1./10.-2.*Z**2) + WPAR
C
      EPSX= SIGX -NU*( SIGY + SIGZ )
      EPSY= SIGY -NU*( SIGX + SIGZ )
      EPSZ= PM*F1Z*DSIN(APX) - 12.*NU*XMPYMP/HAR2*Z
      EPSXP=EPSXP+EPSX
      EPSYP=EPSYP+EPSY
      EPSZP=EPSZP+EPSZ
      XK22=(1.-NU)/(1.+NU)
      GO TO (441,442) ITHICK
  441 EFSIN=0.0
      GO TO 443
```

```
442 EESIN = EE*DSINH(GAMB)
443 XLOADP = XLOADP + 1./XNU2*( 24.*XK22*(DCOS(AP)-1.)/F1/HAR2
   . *( AP/2./GAMB - DSQRT(GAMA2)/HAR/2./GAMB )*EESIN
   . + 6.*(1.-NU)*AP*(DCOS(AP)-1.)/F1/HAR2*(BETA + BETAP) )
C
190 CONTINUE
    GO TO (110,111) IBALCH
110 ALFA1B = ALFA1
    BETA1B = BETA1
    GAMA1B = GAMA1
    W0B = W0
    XYMOMB = XYMOM
    XSHERB = XSHER
    YSHERB = YSHER
    GO TO 100
111 ALFA1P = ALFA1
    BETA1P = BETA1
    GAMA1P = GAMA1
    W0P = W0
    XYMOMP = XYMOM
    XSHERP = XSHER
    YSHERP = YSHER
C*************************************
100 CONTINUE
C*************************************
C*************************************
200 CONTINUE
C*************************************
    GO TO (185,205,205) IDEF
185 GO TO (312,312,205) IPRINT
312 CONTINUE
C   WRITE(6,530) X,ALFA1R,WP
    GO TO (504,505,505) ILOAD
504 WRITE(6,140) ALFA1R,BETA1R,GAMA1R
505 WRITE(6,118) ALFA1B,BETA1B,GAMA1B
    WRITE(6,120) ALFA1P,BETA1P,GAMA1P
118 FORMAT('ALFA1B = ',E12.5,3X,'BETA1B = ',E12.5,3X,'GAMA1B = ',E12.5)
120 FORMAT('ALFA1P = ',E12.5,3X,'BETA1P = ',E12.5,3X,'GAMA1P = ',E12.5)
140 FORMAT('ALFA1R = ',E12.5,3X,'BETA1R = ',E12.5,3X,'GAMA1R = ',E12.5)
    GO TO (187,187,205,207)IBOUND
207 WRITE(6,165) ALFA1E
    WRITE(6,195) ALFARE
165 FORMAT('ALFA1E = ',E12.5)
195 FORMAT('ALFARE = ',E12.5)
187 GO TO (123,205,205) IPRINT
123 WRITE(6,119) W0B
    WRITE(6,121) W0P
119 FORMAT('W0B = ',E12.5)
121 FORMAT('W0P = ',E12.5)
205 CONTINUE
```

```
C************************************
      XSTR = XSTR / HAR2
      YSTR = YSTR / HAR2
      ZSTR = ZSTR / HAR2
      XYSTR = XYSTR / HAR2
      XZSTR = XZSTR / HAR
      YZSTR = YZSTR / HAR
C
      XSTB = XSTB / HAR2
      YSTB = YSTB / HAR2
      ZSTB = ZSTB / HAR2
      XYSTB = XYSTB / HAR2
      XZSTB = XZSTB / HAR
      YZSTB = YZSTB / HAR
C
      XSTP = XSTP / HAR2
      YSTP = YSTP / HAR2
      ZSTP = ZSTP / HAR2
      XYSTP = XYSTP / HAR2
      XZSTP = XZSTP / HAR
      YZSTP = YZSTP / HAR
C
      GO TO (514,515) IPLOT
 514  CONTINUE
C     WRITE(6,330) XYSTR,XYSTB,XYSTP
C     WRITE(6,530) Z,XSTR,XSTB,XSTP
      WRITE(6,530) Z,YSTR,YSTB,YSTP
C     WRITE(6,530) Z,ZSTR,ZSTB,ZSTP
      GO TO 439
 530  FORMAT(8(F10.2,2X))
 515  CONTINUE
      GO TO (360,361,439,438) ISTRES
 360  CONTINUE
      WRITE(6,335)
      WRITE(6,325) Z
      WRITE(6,335)
      WRITE(6,330) XSTR,YSTR,ZSTR
      WRITE(6,330) XSTB,YSTB,ZSTB
      WRITE(6,330) XSTP,YSTP,ZSTP
      GO TO 439
 330  FORMAT(6(F12.5,2X))
 361  CONTINUE
      WRITE(6,335)
      WRITE(6,325) Z
      WRITE(6,335)
      WRITE(6,330) XYSTR,XZSTR,YZSTR
      WRITE(6,330) XYSTB,XZSTB,YZSTB
      WRITE(6,330) XYSTP,XZSTP,YZSTP
 335  FORMAT('**********************')
 325  FORMAT('Z/H = ',F8.5)
```

```
      GO TO 439
  438 CONTINUE
C     WRITE(6,335)
C     WRITE(6,325) Z
C     WRITE(6,335)
C     WRITE(6,330) HAR,YNYP
C     WRITE(6,330) HAR,XYMOMR,XYMOMP
C     WRITE(6,530) Z,ZSTB,ZSTP
C     WRITE(6,530) Z,XYSTR,XYSTB,XYSTP
C     WRITE(6,530) Z,YSTR,YSTB,YSTP
      WRITE(6,330) Z,XSTR,XSTB,XSTP
C     WRITE(6,330) YSTR,XYSTR,YZSTR
C     WRITE(6,330) YSTB,XYSTB,YZSTB
C     WRITE(6,330) YSTP,XYSTP,YZSTP
C     WRITE(6,330) HAR,VXR,XSHERB,XSHERP
C     WRITE(6,330) HAR,APLOAD
C     WRITE(6,478) HAR,XLOADP
  478 FORMAT('H/A = ',F8.4,2X,'TOTAL REACTION ALONG EDGES OF PLATE = ',
     .F8.2)
C     WRITE(6,330) Z,WBARR,WP
C     XNUR = DABS(EPSXR/EPSZR)
C     XNUP = DABS(EPSXP/EPSZP)
C     WRITE(6,530) Z,XNUR,XNUP
  439 CONTINUE
C*************************************
  250 CONTINUE
C*************************************
  300 CONTINUE
C*************************************
      WRITE(6,18)
      STOP
      END
C
C **************************
C *** END OF MAIN PROGRAM ***
C **************************
C************************************************
C************************************************
C************************************************
C
C*** SUBROUTINE ' XPLANE ' TO FIND SOLUTION OF THE IN-PLANE PROBLEM
C          ********
C    I.E : TO DTERMINE THE CONSTANTS C1 AND C2 IN THE EXPRESSION
C
C       FOR THE INPLANE DISPLACEMENTS UBAR & VBAR .
C
      SUBROUTINE XPLANE(M,IPLANE,IBOUND,NU,HAR,BAR,AP,APB,C1,C2,UP,
     .             XK4,X,Y,ZI,UU,ILOAD,F1,F2)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU
```

```
C***************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C***************************************************
      UP = PM*NU*(1.+NU)*F2/AP
      XK1 = (1.-NU)/2.
      XK2 = (1.+NU)/2.
      XK4 = (1.+XK1)/XK2
      XK7 = -1./DSINH(APB)*( APB-XK4*DSINH(APB) )
      XK8 = XK7*DCOSH(APB)+APB*DSINH(APB)
      GO TO (1,2) IPLANE
 1    C2 = -UP/XK8
      C1 = XK7*C2
      GO TO 3
 2    A11 = DCOSH(APB)
      A12 = APB*DSINH(APB)
      A21 = (1.-NU)*DCOSH(APB)
      A22 = (1.-XK4)*DCOSH(APB) + (1.-NU)*APB*DSINH(APB)
      R1 = -UP
      R2 = -(1.-NU)*UP
      C1 = (A22*R1-A12*R2)/(A11*A22-A12*A21)
      C2 = (A11*R2-A21*R1)/(A11*A22-A12*A21)
 3    RETURN
      END
C***************************************************
C***************************************************
C***************************************************
C
C*** SUBROUTINE ' STRESS ' TO EVALUATE:
C         ********
C                THE STRESSES SIGX,SIGY,SIGZ,SIGXY,SIGXZ,&
C                SIGYZ
C   AT A SPECIFIED POINT(X,Y) IN THE PLATE AND
C   ACCORDING TO THE SPECIFIED BOUNDARY CONDITIONS
C
      SUBROUTINE STRESS(IBOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
     .           F2,F3,F4,F1Z,F2Z,F3Z,BETA,BETAP,A,B,EE,C1,C2,
     .           UP,XK4,SIGX,SIGY,SIGZ,SIGXY,SIGXZ,SIGYZ,IBALCH,
     .           X,Y,Z,ZI,UU,ILOAD,F1ZP,QX,QY,YNY)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU,KPD
C***************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C***************************************************
      XNU2 = (1.-NU**2.)
      XNU1 = NU/12./(1.-NU)
      XNUP1 = NU + 1.
```

```
      XNUM1 = NU-1.
      XK1 = 6.*(1.-NU)/F1/HAR2
      XK22 = (1.-NU)/(1.+NU)
C
      APY1 = APY
      APX1 = APX
      GAMY1 = GAMY
      XK7 = -NU*NU*F1/6./XNUM1
      XK8 = -NU/12./XNUM1
      GO TO (333,334) ITHICK
  333 EEBAR1 = 0.0
      EESIN = 0.0
      EECOS = 0.0
      GO TO 335
  334 EEBAR1 = EE*DSINH(GAMY)
      EESIN = EE*DSINH(GAMY)
      EECOS = EE*DCOSH(GAMY)

C
  335 CONTINUE
      GO TO (20,21) IBALCH
   20 G1 = Z/4. - 5.*Z**3/3.
      G2 = -3./10.*Z + 2.*Z**3
      G3 = 5./4.*Z - 5./3.*Z**3
      G4 = - 1./48. - Z*(-336.*NU**2-195.*NU + 195.)/5600./XNUM1
     .  + Z**2/4. - Z**3*(8.*NU**2 + 5.*NU-5.)/20./XNUM1
     .  + Z**5/10.
      F1Z = -1./4.*(2. - 6.*Z + 8.*Z**3)
      F1ZP = 3./2.*( 1. - (2.*Z)**2 )
C     WRITE(6,50) G1,G2,G3,G4
      GO TO 22
   21 G1 =( F1Z - F2 )/F1 - Z
      G2 = 2.*Z**3 - .30*Z
      G3 =( F1Z - F2 )/F1
      G4 = -F3Z + F3*Z + F4
C
   22 CONTINUE
      DWDX2 = - A*AP2*DCOSH(APY) - B*AP2*APY*DSINH(APY)
     .   - EECOS*AP2 - AP2*BETA
      DWDY2 = A*AP2*DCOSH(APY) + B*(AP2*APY*DSINH(APY) + 2.*AP2*DCOSH(APY))
     .   + EECOS*GAMA2/HAR2
      DWDXY = A*AP2*DSINH(APY) + B*(AP2*APY*DCOSH(APY) + AP2*DSINH(APY))
     .   + EEBAR1*AP*DSQRT(GAMA2)/HAR
      XI1 = DWDX2 + NU*DWDY2
      XJ1 = NU*DWDX2 + DWDY2
      XL1 = DWDXY
C
      DFIX3 = -A*AP4*HAR2*DCOSH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY) )
     .   + EECOS*XK22*AP4*HAR2 + AP4*HAR2*BETAP
```

```
      DFIXY2=A*AP4*HAR2*DCOSH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)
     .   + 2.*AP4*HAR2*DCOSH(APY) )
     .   - EECOS*XK22*AP2*GAMA2
      DFIY3=-A*AP4*HAR2*DCOSH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)
     .   + 4.*AP4*HAR2*DCOSH(APY) )
     .   + EECOS*XK22*GAMA2*GAMA2/HAR2
      DFIYX2=A*AP4*HAR2*DCOSH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DCOSH(APY) + APY*AP4*HAR2*DSINH(APY)
     .   + 2.*AP4*HAR2*DCOSH(APY) )
     .   - EECOS*XK22*AP2*GAMA2
      XI2  = DFIX3 + DFIYX2 + NU*DFIXY2 + NU*DFIY3
      XJ2  = NU*DFIX3 + NU*DFIYX2 - DFIXY2 + DFIY3
C
      DFIX2Y=A*AP4*HAR2*DSINH(APY) + B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DSINH(APY) + APY*AP4*HAR2*DCOSH(APY)
     .   + AP4*HAR2*DSINH(APY) )
     .   - EEBAR1*XK22*AP3*HAR*DSQRT(GAMA2)
      DFIY2X=-A*AP4*HAR2*DSINH(APY) - B*(-2.*F1*AP2*HAR2/6./XNUM1
     .   *AP4*HAR2*DSINH(APY) + APY*AP4*HAR2*DCOSH(APY)
     .   + 3.*AP4*HAR2*DSINH(APY) )
     .   + EEBAR1*XK22*AP/HAR*GAMA2*DSQRT(GAMA2)
      XL2  = DFIX2Y + DFIY2X
C
      DFIXDX= A*AP2*DCOSH(APY) + B*(-2.*F1*AP4*HAR2/6./XNUM1*DCOSH(APY)
     .    + APY*AP2*DSINH(APY) )
     .    - EECOS*XK22*AP2 - AP2*BETAP
      DFIYDY=-A*AP2*DCOSH(APY) - B*(-2.*F1*AP4*HAR2/6./XNUM1*DCOSH(APY)
     .    + APY*AP2*DSINH(APY) + 2.*AP2*DCOSH(APY) )
     .    + EECOS*XK22*GAMA2/HAR2
      XI3  = DFIXDX + NU*DFIYDY
      XJ3  = NU*DFIXDX + DFIYDY
C
      DFIXDY=-A*AP2*DSINH(APY)-B*(-2.*F1*AP4*HAR2/6./XNUM1*DSINH(APY)
     .    + APY*AP2*DCOSH(APY) + AP2*DSINH(APY) )
     .    + EEBAR1*XK22*AP*DSQRT(GAMA2)/HAR
      DFIYDX=-A*AP2*DSINH(APY)-B*(-2.*F1*AP4*HAR2/6./XNUM1*DSINH(APY)
     .    + APY*AP2*DCOSH(APY) + AP2*DSINH(APY) )
     .    + EEBAR1*XK22*AP*DSQRT(GAMA2)/HAR
      XL3  = DFIXDY + DFIYDX
C
      XI4  =-AP2*HAR4*PM
      XJ4  =NU*XI4
      XL4  =0.0
C
      DUDX = -C1*AP*DCOSH(APY) - C2*APY*AP*DSINH(APY) - AP*UP
      DVDY = C1*AP*DCOSH(APY)
     .    + C2*( APY*AP*DSINH(APY) + (1.-XK4)*AP*DCOSH(APY) )
      XI5  = DUDX + NU*DVDY
```

```
      XJ5  = NU*DUDX + DVDY
      XJ5BAR = XJ5
      XI5 = XI5*HAR2
      XJ5 = XJ5*HAR2
C
      DUDY = C1*AP*DSINH(APY)
     .    + C2*( APY*AP*DCOSH(APY) + AP*DSINH(APY) )
      DVDX = C1*AP*DSINH(APY)
     .    + C2*( APY*AP*DCOSH(APY) - XK4*AP*DSINH(APY) )
      XL5  = DUDY + DVDX
      XL5 = XL5*HAR2/2.
C
      XI6 = -NU*PM*F1Z*HAR2/XNUM1
      XJ6 = XI6
      XL6 = XL4
C
      XI7 = PM*NU**2*AP2*F1*HAR4/6./XNUM1
      XJ7 = NU*XI7
C-------------------
 83   CONTINUE
      GO TO (24,25) IBALCH
 24   XI7 = 0.0
      XJ7 = 0.0
 25   CONTINUE
      SIGX = 1./XNU2*( XI1*G1 + XK8*XI2*G2 + XI3*G3 + XI4*G4
     .          + XI7*G2 + XI5 ) + XI6
C
      SIGY = 1./XNU2*( XJ1*G1 + XK8*XJ2*G2 + XJ3*G3 + XJ4*G4
     .          + XJ7*G2 + XJ5 ) + XJ6
C
      SIGXY = 1./XNUP1*( XL1*G1 + XK8*XL2*G2 + XL3*G3/2.
     .           + XL4*G4 + XL6*G2 + XL5 )
C
C     YNY = C1*(1.-NU)*AP*DCOSH(APY) + C2*( (1.-XK4)*AP*DCOSH(APY)
C    .   + (1.-NU)*APY*AP*DSINH(APY) )
C    .   + (1.-NU)*AP*UP
      YNY = 1./XNU2*XJ5BAR - NU/XNUM1*PM*F2
C
      SIGX = SIGX*DSIN(APX)
      SIGY = SIGY*DSIN(APX)
      SIGZ = HAR2*PM*F1Z*DSIN(APX)
      SIGXY = SIGXY*DCOS(APX)
      SIGXZ = QX*F1ZP
      SIGYZ = QY*F1ZP
      YNY = YNY*HAR*DSIN(APX)
      RETURN
      END
C----------------------------------------
C----------------------------------------
C----------------------------------------
```

```
C
C*** SUBROUTINE REISS  TO FIND SOLUTION OF THE PROBLEM
C                USING REISSNER'S SHEAR DEFORMATION
C                THEORY.
C
      SUBROUTINE  REISS(M,IBOUND,ITHICK,NU,HAR,BAR,AP,APB,GAMB,F1,
     .            WPARR,XMR,YMR,XYMR,WPARRE,VX,VY,SIGXR,SIGYR,UU,
     .            SIGZR,SIGXYR,SIGXZR,SIGYZR,X,Y,Z,ZI,ILOAD,F1ZR)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU
      XK = (2.-NU)/(1.-NU)
C*****************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C*****************************************************
C
C= = = >  EVALUATE THE CONSTANTS : C4,C5,C6
C   FOR THE VARIOUS B.C.'S  .
C
      GAM2 = AP2*HAR2 + 10.
      GAMY = Y*BAR/HAR*DSQRT(GAM2)
      XK = (2.-NU)/(1.-NU)
C
      APY1 = APY
      APX1 = APX
      GAMY1 = GAMY
      APY2 = 0.0
      APX2 = 0.0
      GAMY2 = 0.0
      GO TO (2,3,4) IBOUND
C
C *--------------------------*
C*** SIMPLY SUPPORTED PLATE AT Y = + ,- B/2 ***
C *--------------------------*
C
 2    CONTINUE
       C4 = 0.0
       C4SH1 = 0.0
       C4SH2 = 0.0
       C4SH3 = 0.0
       C6 = 1./2./DCOSH(APB)
       C5 = -1./DCOSH(APB)*( 1. + XK*HAR2*AP2/10. + APB*DTANH(APB)/2. )
       GO TO 5
C *--------------------------*
C*** CLAMPED PLATE AT Y = + ,- B/2 ***
C *--------------------------*
 3     R1 = -2.*HAR3*AP3/5./DSQRT(GAM2)/(1.-NU)*DCOSH(APB)
     .   *DTANH(GAMB) - APB*DSINH(APB)*DTANH(APB) + APB*DCOSH(APB)
     .   + ( 1.+ 2.*HAR2*AP2/5./(1.-NU) )*DSINH(APB)
```

```
      R2 = -HAR3*AP3/5./DSQRT(GAM2)/(1.-NU)*DTANH(GAMB)
     . + DTANH(APB)*(1.+XK*HAR2*AP2/10.)
      C6 = R2/R1
      C5 = -1./DCOSH(APB)*( 1. + XK*HAR2*AP2/10. + APB*DSINH(APB)*C6 )
      C4SH2 = 4./AP2*( 2.*C6*DCOSH(APB) - 1. )
      GO TO (6,7) ITHICK
    6 C4 = 0.0
      C4SH1 = 0.0
      C4SH3 = 0.0
      GO TO 5
    7 C4 = 4./5.*HAR2/AP/DCOSH(GAMB)*( 2.*C6*DCOSH(APB) - 1. )
      C4SH1 = 4./AP2*HAR/DSQRT(GAM2)/DCOSH(GAMB)*DSINH(GAMY)
     .     *( 2.*C6*DCOSH(APB) - 1. )
      C4SH3 = 4./AP*HAR*DSQRT(GAM2)*(2.*C6*DCOSH(APB) - 1. )*DCOSH(APY)
     .    /DCOSH(APB)
      GO TO 5
C *-----------------------*
C*** FREE PLATE AT Y = +,- B/2 ***
C *-----------------------*
    4 R5 = DSQRT(HAR2*AP2 + 10.)
      R4 = 1./HAR/AP*R5
      R3 = 2.*HAR2*AP2/5.*( 1. - R4*DTANH(APB)/DTANH(GAMB) ) + 3. + NU
     . -2.*APB*(1.-NU)/DSINH(2.*APB)
      C6 = NU*(GAM2)/10./R3/DCOSH(APB)
      C5 = C6/(1.-NU)*( 1. + NU -(1.-NU)*APB/DTANH(APB) )
C
C
      GO TO (8,9) ITHICK
    8 C4 = 0.0
C     C4SH1 = 8./AP3*DSINH(APB)*C6
      C4SH1 = 0.0
      C4SH2 = 0.0
C     C4SH3 = 0.0
      C4SH3 = 8.*( DSINH(APY)*C6/AP2 )
      GO TO 5
    9 C4 = 8./5.*HAR/AP2*R5*DSINH(APB)/DSINH(GAMB)*C6
      C4SH1 = 8./AP3/DSINH(GAMB)*DSINH(APB)*C6*DSINH(GAMY)
      C4SH2 = 8./HAR/AP*DSQRT(GAM2)/AP2/DSINH(GAMB)*DSINH(APB)*C6
     .    *DCOSH(GAMY)
C     C4SH3 = 8./AP2 DSINH(GAMB)*DSINH(APB)*DSINH(GAMY)*C6
      C4SH3 = 8.*( DSINH(APY)*C6/AP2 )
C
C
    5 APD2 = AP/2.
      GO TO (60,61) ITHICK
   60 C4COS = 0.0
      GO TO (100,100,101) IBOUND
  101 C4COS = 8./5.*HAR/AP2*R5*DSINH(APB)/DTANH(GAMB)*C6
  100 C4SIN = 0.0
      GO TO 62
```

```
61   C4COS = C4*DCOSH(GAMY)
     C4SIN = C4*DSINH(GAMY)
62   CONTINUE
C
     WPARR = 48.*(1.-NU**2.)*(1./AP5*(C5*DCOSH(APY) + C6*APY*DSINH(APY) + 1.)
    .   + XK*HAR2/AP3/10.)
C
     WPARRE = 48.*(1.-NU**2.)*( 1./AP5*(C5*DCOSH(APB) + C6*APB*DSINH(APB)
    .    + 1.) + XK*HAR2/AP3/10.)
C
     XMR = C6*8./AP3*(HAR2*AP2/5.-NU)*DCOSH(APY)
    .  + 4./AP3*(1.-NU)*C6*APY*DSINH(APY)
    .  + 4./AP3*(1.-NU)*C5*DCOSH(APY)
    .  + C4COS + 4./AP3*(HAR2*AP2*NU/10./(1.-NU) + 1. )
    .  - PM*NU*HAR2/10./(1.-NU)
C
     YMR = -C6*8./AP3*(HAR2*AP2/5. + 1.)*DCOSH(APY)
    .  - 4./AP3*(1.-NU)*C6*APY*DSINH(APY)
    .  - 4./AP3*(1.-NU)*C5*DCOSH(APY)
    .  + C4COS + 4.*NU/AP3*(HAR2*AP2*XK/10. + 1. )
    .  - PM*NU*HAR2/10./(1.-NU)
C
     XYMR = -C6*4./AP3*(1.-NU)*APB*DCOSH(APY) - 4./AP3*(1.-NU)*(C5
    .   + C6)*DSINH(APY) + C4SH1
C
     VX = -4.*(2.*DCOSH(APY)*C6-1.)/AP2 + C4SH2
C
     VY = -8.*( DSINH(APY1)*C6/AP2 ) + C4SH3
C
C
     WPARR  = WPARR*DSIN(APX)
     WPARRE = WPARRE*DSIN(APX)
     XMR    = XMR*DSIN(APX)
     YMR    = YMR*DSIN(APX)
     XYMR   = XYMR*DCOS(APX)
     VX     = VX*DCOS(APX)
     VY     = VY*DSIN(APX)
C
     FIZR   = -1./4.*( 2. - 3.*(2.*Z) + 8.*Z**3 )
C
     SIGXR  = 12.*XMR*Z
     SIGYR  = 12.*YMR*Z
     SIGZR  = HAR2*FIZR*PM
     SIGZR  = SIGZR*DSIN(APX)
     SIGXYR = 12.*XYMR*Z
     SIGXZR = 3./2.*VX*(1. - 4.*Z**2)
     SIGYZR = 3./2.*VY*(1. - 4.*Z**2)
C
     RETURN
     END
```

```
C-----------------------------------------------
C-----------------------------------------------
C-----------------------------------------------
C
C
C*** SUBROUTINE BOUND  TO EVALUATE THE COEFFICIENT MATRIX
C                ACCORDING TO THE SPECIFIED BOUNDARY
C                CONDITIONS
C   IBOUND : IS AN INDICATOR TO TELL WHAT BOUNDARY CONDITION
C         ,FOR THE EDGE AT Y = +,- B/2. , IS
C         BEING CONSIDERED AS FOLLOWS :
C   IBOUND = 1  = = = = > INDICATES SIMPLY SUPPORTED EDGE
C   IBOUND = 2  = = = = > INDICATES    CLAMPED     EDGE
C   IBOUND = 3  = = = = > INDICATES    FREE       EDGE
C
      SUBROUTINE  BOUND(M,IBOUND,ITHICK,NU,HAR,BAR,AP,APB,GAMB,F1,
     .            PK,BETA,BETAP,AMAT,RH,IFPR,FIX,X,Y)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU
      DIMENSION AMAT(3,3),RH(3),IFPR(3),FIX(3)
C*****************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C*****************************************************
      AHR = 1./HAR
      XK1 = 6.*(1.-NU)/F1/HAR2
      XNU2 = 12.*(1.-NU**2.)
      XK22 = (1.-NU)/(1.+NU)
      GO TO (2,3,4) IBOUND
C *---------------------------*
C*** SIMPLY SUPPORTED PLATE AT Y = +,- B/2 ***
C *---------------------------*
C
C= = = > WBAR(X,+-B/2) = 0.0
C
  2   AMAT(1,1) = DCOSH(APB)
      AMAT(1,2) = APB*DSINH(APB)
      AMAT(1,3) = 1./DTANH(GAMB)
C
C= = = > MY(X,+-B/2) = 0.0
C
      AMAT(2,1) = AP2*DCOSH(APB)
      AMAT(2,2) = 2.*AP2*DCOSH(APB)
     .         + AP2*APB*DSINH(APB)
      AMAT(2,3) = +(AP2*HAR2 + 12./F1)/DTANH(GAMB)
C
C= = = > QX(X,+-B/2) = 0.0
C
C= = = = NOTE : THE ABOVE B.C. COMES FROM THE B.C.:
```

```
C           PHIX(X,+-B/2)=DW/DX + QX/S
C        AND SINCE W(X,+-B/2) = 0.0  THEN
C        DW/DX = 0.0 = = = > QX(X,+-B/2)/S = 0.0
C        OR SIMPLY : QX(X,+-B/2) = 0.0
C
      AMAT(3,1)=0.0
      AMAT(3,2)=2.*AP3*DCOSH(APB)
      AMAT(3,3)=-2.*XK1*AP/(1.+NU)/DTANH(GAMB)
C
      RH(1)=-BETA
      RH(2)= PK*(1.-2./NU)
      RH(3)=+XK1*AP*(BETA + BETAP)
      GO TO 11
C *-----------------*
C*** CLAMPED PLATE AT Y=+,- B/2 ***
C *-----------------*
C
C= = = > WBAR(X,+-B/2)=0.0
C
    3 AMAT(1,1)=DCOSH(APB)
      AMAT(1,2)=APB*DSINH(APB)
      AMAT(1,3)=1./DTANH(GAMB)
C
C= = = > PHIY(X,+-B/2)=0.0
C
      AMAT(2,1)=AP*DSINH(APB)
      AMAT(2,2)=(F1*AP3*HAR2/3./(1.-NU)+AP)*DSINH(APB)
     .      +APB*AP*DCOSH(APB)
      AMAT(2,3)=-(1.-NU)/(1.+NU)/HAR*DSQRT(AP2*HAR2+12./F1)
C
C= = = > DQY/DY + P = 0.0  ; AT Y = +- B/2.
C
C
C= = = = NOTE : THE ABOVE B.C. COMES FROM THE EQUILIBRIUM EQN.
C        DQX/DX + DQY/DY + P = 0.0
C        SINCE FROM THE B.C. :
C           PHIX(X,+-B/2)=DW/DX + QX/S
C        AND SINCE W(X,+-B/2) = 0.0  THEN
C        DW/DX = 0.0 = = = > QX(X,+-B/2)/S = 0.0
C        AND ALSO : DQX/DX = 0.0
C
      AMAT(3,1)=0.0
      AMAT(3,2)=2.*AP4*DCOSH(APB)
      AMAT(3,3)=-2.*XK1/(1.+NU)/HAR2*(AP2*HAR2+12./F1)/DTANH(GAMB)
C
      RH(1)=-BETA
      RH(2)=0.0
      RH(3)=+XNU2*4./AP
      GO TO 11
C *-----------------*
```

```
C*** FREE PLATE AT Y=+,- B/2 ***
C *------------------*
C
C===> MY(X,+-B/2)=0.0
C
  4 CONTINUE
      AMAT(1,1)=+(1.-NU)*AP2*DCOSH(APB)
      AMAT(1,2)=+(1.-NU)*APB*AP2*DSINH(APB)
     .        +F1*AP4*HAR2/3.*DCOSH(APB)
     .        +2.*AP2*DCOSH(APB)
      AMAT(1,3)=-XK22*(GAMA2/HAR2-NU*AP2)/DTANH(GAMB)
C
C===> VY(X,+-B/2)=0.0
C   THE ABOVE EQN. IS OBTAINED FROM THE EQN. :
C     VY=QY-DMXY/DX
C    SINCE AT Y = +-B/2. :
C     DMXY/DX=0.0   & QY=0.0
C*************************************************
      AMAT(2,1)=(1.-NU)*AP3*DSINH(APB)
      AMAT(2,2)=-(1.+NU)*AP3*DSINH(APB)
     .        +(1.-NU)*APB*AP3*DCOSH(APB)
      AMAT(2,3)=+((1.-NU)*AP2 + 12.*XK22/F1/HAR2)
     .        /HAR*DSQRT(GAMA2)
C*************************************************
C
C===> QY(X,+-B/2)=0.0
C   THE ABOVE EQN. IS OBTAINED FROM THE EQN. :
C      DMY/DY - DMXY/DX = QY
C   NOTING THAT :
C           1) DMXY/DX = 0.0 ( SINCE MXY(X,+-B/2.) = 0.0 )
C           2) D2W/DXY = 0.0 ( SINCE DMXY = D2W/DXY = 0.0 )
C**** SEE CHAPTER 4 FOR MORE DETAILS ****
C*************************************************
      AMAT(3,1)=(1.-NU)*AP3*DSINH(APB)
      AMAT(3,2)=-2.*AP3*DSINH(APB)
     .        +(1.-NU)*AP3*DSINH(APB)
     .        +(1.-NU)*APB*AP3*DCOSH(APB)
      AMAT(3,3)=-(1.-2.*XK22)*12./F1/HAR2/HAR*DSQRT(GAMA2)
     .        +((1.-NU)*AP2 )
     .        /HAR*DSQRT(GAMA2)
C*************************************************
C
C     AMAT(3,1)=0.0
C     AMAT(3,2)=+2.*AP3*DSINH(APB)
C     AMAT(3,3)=+(1.-2.*XK22)*12./F1/HAR2/HAR*DSQRT(GAMA2)
C
C     AMAT(3,1)=0.0
C     AMAT(3,2)=-2.*AP3*DSINH(APB)
C     AMAT(3,3)=+12.*XK22/F1/HAR2/HAR*DSQRT(GAMA2)
C
```

```
      RH(1)= -NU*AP2*BETAP + PK
      RH(2)=0.0
      RH(3)=0.0
      GO TO (11,11) ITHICK
 17   CONTINUE
      IFPR(3)=1
      FIX(3) =0.0
 11   CONTINUE
      RETURN
      END
C
C---------------------------------------------------
C---------------------------------------------------
C---------------------------------------------------
C
C*** SUBROUTINE POWERS TO EVALUATE THE POWERS OF : ALPHA , B/A , H/A
C
      SUBROUTINE  POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .            HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .            X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
      IMPLICIT REAL*8(A-H,O-Z)
      PI=-1.00
      PI=DARCOS(PI)
      ALPHA=M*PI
      AP=ALPHA
      AP2=AP**2.
      AP3=AP**3.
      AP4=AP**4.
      AP5=AP**5.
      AP6=AP**6.
      HAR2=HAR**2.
      HAR3=HAR**3.
      HAR4=HAR**4.
      HAR5=HAR**5.
      BAR2=BAR**2.
      BAR3=BAR**3.
      BAR4=BAR**4.
      BAR5=BAR**5.
      GAMA2=AP2*HAR2+12./F1
      APX=AP*X
      APY=AP*BAR*Y
      GAMY=Y*BAR*DSQRT(GAMA2)/HAR
      GO TO (50,51,52) ILOAD
 50   PM =4./AP
      GO TO 53
 51   APZI=AP*ZI
      PM =2.*DSIN(APZI)
      GO TO 53
 52   APZI=AP*ZI
      APU=AP*UU/2.
```

```
      PM=4./AP*DSIN(APZI)*DSIN(APU)
 53   CONTINUE
      RETURN
      END
C
C---------------------------------------
C---------------------------------------
C---------------------------------------
C
C*** SUBROUTINE ' BENDNG ' TO EVALUATE THE CONSTANTS A(M),B(M),& E(M)
C           ********
C                  ACCORDING TO THE SPECIFIED BOUNDARY CONDITIONS
C
      SUBROUTINE BENDNG(IBOUND,ITHICK,M,NU,HAR,AP,APB,GAMB,KPD,UU,
     .           BAR,BETA,BETAP,A,B,EE,IPRINT,F1,X,Y,ZI,ILOAD)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU,K1,K2,K11,K12,K13,K14,
     .           KPD
      DIMENSION AMAT(3,3),SOLT(3),RH(3),IFPR(3),FIX(3),
     .      BMAT(3,3)
C
C***  P0 : IS THE VALUE OF THE UNIFORMLY DISTRIBUTED LOAD ON THE PLATE
C
C     P=4.0*P0/(M*PI)
C****************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C****************************************************
      XK1=6.*(1.-NU)/F1/HAR2
      XK22=(1.-NU)/(1.+NU)
      XNU2=12.*(1.-NU**2.)
      APB=(AP/2.)*BAR
      AHR=1./HAR
C
      GAMB=.5*BAR*AHR*DSQRT(GAMA2)
C
      K11=(2.-NU)*(1+NU)*AP4*HAR4*F1/144.
      K12=(3.-2.*NU)*(1.+NU)*AP2*HAR2*F1/12.
      K13=(1.-NU**2.)
      K14=AP4*(AP2*HAR2*F1/12.+1.0)
      K1=12.0*PM*(K11+K12+K13)/K14
      BETA=K1
      K2=2.*NU*(1.+NU)*F1**2.*HAR4/3./AP/(1.-NU)*AP/4.*PM
      BETAP=-(K1-K2)/(AP2*HAR2*F1/6./(1.-NU) + 1.0)
      PK=PM*NU*(1.+NU)*F1*HAR2
C
C---( NOTE: PK = K*P*H**2. )-----
C
C     BETA=(AA*AP**4.-BB*AP**2.+CC)/(-MP*AP**6.+NP*AP**4.)*P
```

```
C   BETAP=-AP*(BETA-KAP*P*D.S)/U2
C   GAMA=DSQRT(AP**2.-NP/MP)
C*******************************
      GO TO (676,677,677) IPRINT
 676  WRITE(6,101) ALPHA
      WRITE(6,110) GAMB
C     WRITE(6,309) K11,K12
C     WRITE(6,311) K13,K14
      WRITE(6,111) BETA
      WRITE(6,310) K2
      WRITE(6,112) BETAP
 677  CONTINUE
 101  FORMAT('ALPHA=',E15.5)
C309  FORMAT('K11=',E10.5,2X,'K12=',E10.5)
C311  FORMAT('K13=',E10.5,2X,'K14=',E10.5)
 310  FORMAT('K2=',E10.5)
 110  FORMAT('GAMB =',E15.5)
 111  FORMAT('BETA =',E15.5)
 112  FORMAT('BETAP=',E15.5)
C*******************************
      N=3
      NEQNS=N
      DO 64 I=1,N
      RH(I)  =0.0
      IFPR(I)=0
      FIX(I) =0.0
      DO 64 J=1,N
 64   AMAT(I,J)=0.0
C
      CALL      BOUND(M,IBOUND,ITHICK,NU,HAR,BAR,AP,APB,GAMB,F1,
     .          PK,BETA,BETAP,AMAT,RH,IFPR,FIX,X,Y)
C*******************************
      DO 315 I=1,N
      DO 315 J=1,N
 315  BMAT(I,J)=AMAT(I,J)
C*******************************
C     WRITE(6,228) M
C228  FORMAT('M=',I2,3X,'COEFFICIENT MATRIX BEFORE MODIFICATION')
C     DO 121 I=1,N
C     WRITE(6,122) (AMAT(I,J1),J1=1,N)
C121  CONTINUE
 122  FORMAT(3(E12.5,2X))
C     DO 123 I=1,N
C     WRITE(6,124) RH(I)
C123  CONTINUE
C*******************************
C
      GO TO (672,673,673) IPRINT
 672  WRITE(6,227) M
 227  FORMAT('M=',I2,3X,'COEFFICIENT MATRIX AFTER MODIFICATION')
```

```
      DO 723 I=1,N
      WRITE(6,122) (AMAT(I,J1),J1=1,N)
 723  CONTINUE
      DO 226 I=1,N
      WRITE(6,124) RH(I)
 226  CONTINUE
 673  CONTINUE
 124  FORMAT(E12.5)
C********************************
      CALL      GREDU  (NEQNS,AMAT,FIX,RH,IFPR)
      CALL      BAKSU  (NEQNS,AMAT,FIX,RH,IFPR,SOLT)
C     CALL      GREDUC (NEQNS,AMAT,FIX,RH,IFPR)
C     CALL      BAKSUB (NEQNS,AMAT,FIX,RH,IFPR,SOLT)
C********************************
C     CALL      JORDAN(NEQNS,AMAT,RH,SOLT)
C     CALL  DLSARG(N,AMAT,N,RH,1,SOLT)
C     CALL  DLSLRG(N,AMAT,N,RH,1,SOLT)
C********************************
      GO TO (674,675,675) IPRINT
674   WRITE(6,229) M
 229  FORMAT('M=',I2,3X,'COEFFICIENT MATRIX AFTER SOLUTION')
      DO 224 I=1,N
      WRITE(6,122) (AMAT(I,J1),J1=1,N)
 224  CONTINUE
      DO 525 I=1,N
      WRITE(6,124) RH(I)
 525  CONTINUE
 675  CONTINUE
C********************************
      A=SOLT(1)
      B=SOLT(2)
      EE=SOLT(3)
C********************************
      GO TO (205,206,206) IPRINT
 205  RH1 =AMAT(1,1)*A+AMAT(1,2)*B+AMAT(1,3)*EE
      RH2=AMAT(2,1)*A+AMAT(2,2)*B+AMAT(2,3)*EE
      RH3=AMAT(3,1)*A+AMAT(3,2)*B+AMAT(3,3)*EE
      WRITE(6,316) RH1,RH2,RH3
 316  FORMAT('RH1 =',E12.5,2X,'RH2 =',E12.5,2X,'RH3 =',E12.5)
 206  CONTINUE
C********************************
      GO TO (665,203) ITHICK
 665  CONTINUE
      EE=0.0
      GO TO 204
 203  CONTINUE
      EE=EE/DSINH(GAMB)
 204  CONTINUE
C********************************
C     BETAP=BETAP*AP2
```

```
      KPD = PK
      GO TO (678,679,679) IPRINT
 678  WRITE(6,27) A,B,EE
      WRITE(6,312) KPD
      WRITE(6,112) BETAP
 27   FORMAT('A    =',E12.4,2X,'B    =',E12.4,2X,'EE=',E12.4)
 312  FORMAT('KPD =',E10.5)
C
 679  CONTINUE
      RETURN
      END
C
C-----------------------------------------------
C-----------------------------------------------
C-----------------------------------------------
C
C*** SUBROUTINE ' FORCES ' TO EVALUATE:
C           ********
C                 (1) THE DISPLACEMENTS WBAR(M),PHIX(M),&
C                     PHIY(M)
C                 (2) THE FORCES XMOM,YMOM,XYMOM,XSHEAR,&
C                     YSHEAR
C     AT A SPECIFIED POINT(X,Y) IN THE PLATE AND
C     ACCORDING TO THE SPECIFIED BOUNDARY CONDITIONS
C
      SUBROUTINE FORCES(IBOUND,ITHICK,M,HAR,BAR,NU,AP,APB,GAMB,KPD,F1,
     .           BETA,BETAP,A,B,EE,WPAR,WPARE,XM,YM,IPRINT,X,Y,
     .           ZI,UU,ILOAD,XYM,QX,QY)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU,KPD
C*****************************************************
      CALL      POWERS(M,HAR,BAR,PI,ALPHA,AP,AP2,AP3,AP4,AP5,AP6,HAR2,
     .           HAR3,HAR4,HAR5,HAR6,BAR2,BAR3,BAR4,BAR5,GAMA2,
     .           X,Y,APX,APY,GAMY,F1,PM,ILOAD,ZI,UU)
C*****************************************************
      APD2 = AP/2.0
      XK1 = 6.*(1.-NU)/F1/HAR2
      XNU2 = 12.*(1.-NU**2.)
      XK22 = (1.-NU)/(1.+NU)
      APY1 = Y*AP
      APX1 = X*AP
      GAMY1 = GAMY
C
      APY2 = 0.0
      APX2 = 0.0
      GAMY2 = 0.0
C
      GO TO (180,181) ITHICK
 180  EESIN = 0.0
      EECOS = 0.0
```

```
      EEBAR1 = 0.0
      EEBAR2 = 0.0
      EEBAR3 = 0.0
      GO TO 183
  181 EESIN = EE*DSINH(GAMY)
      EECOS = EE*DCOSH(GAMY)
      EEBAR1 = EE*DCOSH(GAMY)
      EEBAR2 = EE*DSINH(GAMY)
  183 CONTINUE
C
      WPAR = A*DCOSH(APY) + B*APY*DSINH(APY) + EECOS + BETA
      WPARE = 0.0
C
      GO TO (162,162,163)IBOUND
  163 CONTINUE
      GO TO (160,161) ITHICK
  160 WPARE = A*DCOSH(APB) + B*APB*DSINH(APB) + BETA
      GO TO 162
  161 WPARE = A*DCOSH(APB) + B*APB*DSINH(APB) + EEBAR2 + BETA
      WPARE = WPARE*DSIN(APX)
C
C
  162 CONTINUE
C
      XM = (1.-NU)*AP2*DCOSH(APY)*A + B*(2.*F1*HAR2*AP4/6.*DCOSH(APY)
     .  - 2.*NU*AP2*DCOSH(APY) + (1.-NU)*APY*AP2*DSINH(APY) )
     .  - XK22*(AP2 - NU/HAR2*GAMA2)*EECOS
     .  - AP2*BETAP + KPD
C
      YM = -(1.-NU)*AP2*DCOSH(APY)*A + B*(-2.*F1*HAR2*AP4/6.*DCOSH(APY)
     .  - 2.*AP2*DCOSH(APY) - (1.-NU)*APY*AP2*DSINH(APY) )
     .  + XK22*(-NU*AP2 + 1./HAR2*GAMA2)*EECOS
     .  - NU*AP2*BETAP + KPD
C
      XYM = 2.*AP2*DSINH(APY)*A + B*(4.*F1*HAR2*AP4/6./(1.-NU)*DSINH(APY)
     .  + 2.*AP2*DSINH(APY) + 2*APY*AP2*DCOSH(APY) )
     .  - 2.*XK22*AP/HAR*DSQRT(GAMA2)*EEBAR2
      XYM = XYM/24./(1.+NU)
C
      QX = 1./XNU2*( -2.*AP3*DCOSH(APY)*B + 12.*XK22*AP/HAR2/F1
     .            *EEBAR1 + XK1*AP*(BETA + BETAP) )
C
      GO TO (100,100,101) IBOUND
  100 QY = 1./XNU2*( -2.*AP3*DSINH(APY)*B + 12.*XK22/HAR2/F1
     .            /HAR*DSQRT(GAMA2)*EEBAR2 )
      GO TO 102
  101 QY = 1./XNU2*( (1.-NU)*AP3*DSINH(APB)*A
     .         + B*( -(1.+NU)*AP3*DSINH(APB)
     .           + (1.-NU)*APB*AP3*DCOSH(APB) )
     .         + EESIN*(-12./F1/HAR2*(1.-2.*XK22) + (1.-NU)*AP2)
```

```
     .          *DSQRT(GAMA2)/HAR  )
C    XYM = A*AP2*DSINH(APY) + B*(AP2*APY*DCOSH(APY) + AP2*DSINH(APY))
C  .     + EEBAR1*AP*DSQRT(GAMA2)/HAR
C    XYM = XYM/24./(1.+ NU)
C
 102 CONTINUE
      WPAR = WPAR*DSIN(APX)
      XM = XM/XNU2*DSIN(APX)
      YM = YM/XNU2*DSIN(APX)
      XYM = XYM*DCOS(APX)
      QX = QX*DCOS(APX)
      QY = QY*DSIN(APX)
      RETURN
      END
C
C-----------------------------------------
C-----------------------------------------
C-----------------------------------------
C
C
C
C
C*** SUBROUTINE DISS TO EVALUATE THE FUNCTION F1(Z) AND ALL
C              RELATED FUNCTIONS AND CONSTANTS
C
      SUBROUTINE DISS(M,NU,HAR,ALPHA,A1,A2,A3,A4,A5,F1,F2,F3,F4,
     .           Z,F1Z,F1ZP,F2Z,F3Z)
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU
      DIMENSION AMAT(3,3),RH(3),IFPR(3),FIX(3),SOLT(3)
C
      NEQNS = 3
      PI = 22.0/7.0
      ALPHA = M*PI
      AP = ALPHA
      HAR2 = HAR*HAR
      AP2 = AP**2.
      AP4 = AP**4.
C
      AA = AP2*(2.-NU)/(1.-NU)
      BB = AP4/(1.-NU**2.)
      DD = DSQRT(AA**2.-4.*BB)
      A = DSQRT( .5*(AA + DD) )
      B = DSQRT( .5*(AA-DD) )
      AH = A*HAR
      AH2 = A*HAR/2.
      BH = B*HAR
      BH2 = B*HAR/2.
      AZ = AH*Z
      BZ = BH*Z
```

```
C
      AMAT(1,1)=1.0
      AMAT(1,2)=2*DSINH(AH2)
      AMAT(1,3)=2*DSINH(BH2)
      AMAT(2,1)=1.0
      AMAT(2,2)=AH*DCOSH(AH2)
      AMAT(2,3)=BH*DCOSH(BH2)
      AMAT(3,1)=1.-NU**2.
      AMAT(3,2)=12.*NU**2.*(2.*DSINH(AH2)/AH**2.-DCOSH(AH2)/AH)
      AMAT(3,3)=12.*NU**2.*(2.*DSINH(BH2)/BH**2.-DCOSH(BH2)/BH)
      DO 10 I=1,NEQNS
      RH(I)=0.0
      FIX(I)=0.0
 10   IFPR(I)=0.0
      RH(1)=1.0
      CALL      GREDUC  (NEQNS,AMAT,FIX,RH,IFPR)
      CALL      BAKSUB  (NEQNS,AMAT,FIX,RH,IFPR,SOLT)
      A1 =SOLT(1)
      A3 =SOLT(2)
      A5 =SOLT(3)
      A2 =0.5/( AH/BH*DSINH(AH2)/DTANH(BH2)  - DCOSH(AH2) )
      A4 =-A2*AH/BH*DSINH(AH2)/DSINH(BH2)
C***********************************************
      C1 =-(A3/AH + A5/BH)
      C2 =2.*(1.+NU)/AP2/HAR**2.*( A2 + A4 ) - ( A2/AH**2. + A4/BH**2. )
C     WRITE(6,52) C1,C2
C52   FORMAT('C1 =',E15.6,2X,'C2 =',E15.6)
C
C
      F1 = A1 - 12.*A3*( 2./AH**2.*DSINH(AH2) - DCOSH(AH2)/AH )
     .    - 12.*A5*( 2./BH**2.*DSINH(BH2) - DCOSH(BH2)/BH )
C
      F31 =A1/40. + C1
      F32 =12./AH**3.*A3
      F33 =DCOSH(AH2)-2.*DSINH(AH2)/AH
      F34 =12./BH**3.*A5
      F35 =DCOSH(BH2)-2.*DSINH(BH2)/BH
      F3 =F31 + F32*F33 + F34*F35
C
      F2 =2./AH*DSINH(AH2)*A2 + 2./BH*DSINH(BH2)*A4
C
      F4 =  A2*2./(AH**3)*DSINH(AH2)
     .  + A4*2./(BH**3)*DSINH(BH2) + C2
C
C     WRITE(6,60) AH,BH
C60   FORMAT('AH =',E12.4,2X,'BH =',E12.4)
C     WRITE(6,12) A1
C     WRITE(6,13) A2
C     WRITE(6,14) A3
C     WRITE(6,15) A4
```

```
C    WRITE(6,16) A5
C12  FORMAT('A1',2X,'=',F20.6)
C13  FORMAT('A2',2X,'=',F20.6)
C14  FORMAT('A3',2X,'=',F20.6)
C15  FORMAT('A4',2X,'=',F20.6)
C16  FORMAT('A5',2X,'=',F20.6)
C    WRITE(6,42) F1
C    WRITE(6,43) F2
C    WRITE(6,44) F3
C    WRITE(6,45) F4
C42  FORMAT('PRESENT  WORK  F1',3X,'=',E15.5)
C43  FORMAT('PRESENT  WORK  F2',3X,'=',E15.5)
C44  FORMAT('PRESENT  WORK  F3',3X,'=',E15.5)
C45  FORMAT('PRESENT  WORK  F4',3X,'=',E15.5)
C
      F1Z=A1*Z + A2*DCOSH(AZ) + A3*DSINH(AZ)
     .  + A4*DCOSH(BZ) + A5*DSINH(BZ)
C
      F1ZP=A1 + A2*AH*DSINH(AZ) + A3*AH*DCOSH(AZ)
     .  + A4*BH*DSINH(BZ) + A5*BH*DCOSH(BZ)
C
      F2Z=Z**2/2.*A1 + A2*DSINH(AZ)/AH + A3*DCOSH(AZ)/AH
     .  + A4*DSINH(BZ)/BH + A5*DCOSH(BZ)/BH + C1
C
      F3Z=Z**3/6.*A1 + A2*DCOSH(AZ)/AH**2
     .  + A3*DSINH(AZ)/AH**2
     .  + A4*DCOSH(BZ)/BH**2
     .  + A5*DSINH(BZ)/BH**2
     .  + C1*Z + C2
      RETURN
      END
C
C*****************************************************
C
C
C
C
      SUBROUTINE GREDUC  (NEQNS,ASTIF,FIXED,ASLOD,IFPRE)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASLOD(3),ASTIF(3,3),
     .      FIXED(3),IFPRE(3)
C
C===> NOTE : NEQNS     : NUMBER OF EQUATIONS TO BE SOLVED = N
C        ASTIF(N,N) : COEFFICIENT MATRIX
C        FIXED(N)   : VECTOR OF PRESCRIBED ( OR KNOWN ) VARIABLES;
C                 FIXED(N)
C        ASLOD(N)   : VECTOR OF R.H.S. OF THE EQUATIONS; ASLOD(N).
C        IFPRE(N)   : VECTOR INDICATING WHETHR A VARIABLE IS
C                 PRESCRIBED OR NOT ; IF :
C                 IFPRE(I)=0 ===> VARIABLE #I IS NOT PRESCRIBED
```

```
C                   IFPRE(I)=1 = = = > VARIABLE #I IS    PRESCRIBED
C
C
      NEQNS=3
      DO 50 IEQNS=1,NEQNS
      IF(IFPRE(IEQNS).GT.0)GO TO 30
C--- CHOOSING THE BIGGEST
      JCOLS=IEQNS
      IF(JCOLS.EQ.NEQNS) GO TO 50
      RMAX=0.0
      DO 101 IROWS=IEQNS,NEQNS
      R=ASTIF(IROWS,JCOLS)
      IF(DABS(R).LE.DABS(RMAX)) GO TO 101
      RMAX=R
      IBIG=IROWS
 101  CONTINUE
      IF (IBIG.EQ.IEQNS) GO TO 500
C--- INTERCHANGING ROWS
      SHIFT2=ASLOD(IEQNS)
      ASLOD(IEQNS)=ASLOD(IBIG)
      ASLOD(IBIG)=SHIFT2
      DO 103 J=1,NEQNS
      SHIFT1=ASTIF(IEQNS,J)
      ASTIF(IEQNS,J)=ASTIF(IBIG,J)
      ASTIF(IBIG,J)=SHIFT1
 103  CONTINUE
C----REDUCE EQUATIONS
 500  PIVOT=ASTIF(IEQNS,IEQNS)
      IF(DABS(PIVOT).LT.1.0E-50)GO TO 60
      IF(IEQNS.EQ.NEQNS)GO TO 50
      IEQN1=IEQNS+1
      DO 20 IROWS=IEQN1,NEQNS
      FACTR=ASTIF(IROWS,IEQNS)/PIVOT
      IF(FACTR.EQ.0.0)GO TO 20
      DO 10 ICOLS=IEQNS,NEQNS
      ASTIF(IROWS,ICOLS)=ASTIF(IROWS,ICOLS)-FACTR*ASTIF(IEQNS,ICOLS)
 10   CONTINUE
      ASLOD(IROWS)=ASLOD(IROWS)-FACTR*ASLOD(IEQNS)
C     WRITE(6,124) ASLOD(IROWS)
C
 20   CONTINUE
C
C
C     WRITE(6,229) IEQNS
C229  FORMAT('IEQNS=',I2,'COEFFICIENT MATRIX AFTER SOLUTION')
C     DO 224 I=1,NEQNS
C     WRITE(6,122) (ASTIF(I,J1),J1=1,NEQNS)
C224  CONTINUE
C     DO 225 I=1,NEQNS
C     WRITE(6,124) ASLOD(I)
```

```
C225 CONTINUE
C122 FORMAT(3(E12.5,2X))
C124 FORMAT(E12.5)
C
C
      GO TO 50
C
C    ADJUST RHS(LOADS) FOR
C    PRESCRIBED DISPLACEMENTS
C
30   DO 40 IROWS=IEQNS,NEQNS
      ASLOD(IROWS)=ASLOD(IROWS)-ASTIF(IROWS,IEQNS)*FIXED(IEQNS)
      ASTIF(IROWS,IEQNS)=0.0
40   CONTINUE
C
C
C    WRITE(6,229) IEQNS
C    DO 324 I=1,NEQNS
C    WRITE(6,122) (ASTIF(I,J1),J1=1,NEQNS)
C324 CONTINUE
C    DO 325 I=1,NEQNS
C    WRITE(6,124) ASLOD(I)
C325 CONTINUE
C
      GO TO 50
60   PRINT 100
100  FORMAT(5X,15HINCORRECT PIVOT)
      STOP
50   CONTINUE
      RETURN
      END
CC
C-------------------------------------------
C    BACK-SUBSTITUTION ROUTINE
C-------------------------------------------
C
      SUBROUTINE BAKSUB (NEQNS,ASTIF,FIXED,ASLOD,IFPRE,DISPL)
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASTIF(NEQNS,NEQNS),IFPRE(NEQNS),
     .        FIXED(NEQNS),DISPL(NEQNS),ASLOD(NEQNS)
C
      NEQN1=NEQNS+1
      DO 30 IEQNS=1,NEQNS
      NBACK=NEQN1-IEQNS
      PIVOT=ASTIF(NBACK,NBACK)
      RESID=ASLOD(NBACK)
      IF(NBACK.EQ.NEQNS)GO TO 20
      NBAC1=NBACK+1
      DO 10 ICOLS=NBAC1,NEQNS
```

```
      RESID = RESID-ASTIF(NBACK,ICOLS)*DISPL(ICOLS)
10    CONTINUE
20    IF(IFPRE(NBACK).LE.0)
     *DISPL(NBACK) = RESID/ASTIF(NBACK,NBACK)
C    *DISPL(NBACK) = RESID/PIVOT
      IF(IFPRE(NBACK).GT.0)DISPL(NBACK) = FIXED(NBACK)
C     IF(IFPRE(NBACK).GT.0)REACT(NBACK) = -RESID
30    CONTINUE
      RETURN
      END
C
C--------------------------------------------
C*** GAUSS-JORDAN REDUCTION ROUTINE
C--------------------------------------------
C
      SUBROUTINE JORDAN(NEQNS,ASTIF,ASLOD,SOL)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASLOD(NEQNS),ASTIF(NEQNS,NEQNS),SOL(NEQNS)
      DO 30 IEQNS = 1,NEQNS
      PIVOT = ASTIF(IEQNS,IEQNS)
      DO 20 IROWS = 1,NEQNS
      FACTR = ASTIF(IROWS,IEQNS)/PIVOT
      IF(IROWS.EQ.IEQNS.OR.FACTR.EQ.0.0) GO TO 20
      DO 10 ICOLS = 1,NEQNS
      ASTIF(IROWS,ICOLS) = ASTIF(IROWS,ICOLS)-FACTR*ASTIF(IEQNS,ICOLS)
10    CONTINUE
      ASLOD(IROWS) = ASLOD(IROWS)-FACTR*ASLOD(IEQNS)
20    CONTINUE
30    CONTINUE
      DO 40 IEQNS = 1,NEQNS
      SOL(IEQNS) = ASLOD(IEQNS)/ASTIF(IEQNS,IEQNS)
40    CONTINUE
      RETURN
      END
C
C--------------------------------------------
C
C--------------------------------------------
C
      SUBROUTINE GREDU   (NEQNS, ASTIF, FIXED, ASLOD, IFPRE)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASLOD(NEQNS),ASTIF(NEQNS,NEQNS),
     .         FIXED(NEQNS),IFPRE(NEQNS)
C
C     GAUSSIAN REDUCTION ROUTINE
C
      DO 50 IEQNS = 1,NEQNS
      IF(IFPRE(IEQNS).EQ.1) GO TO 30
C
C     REDUCE EQUATIONS
```

```
C
      PIVOT=ASTIF(IEQNS,IEQNS)
      IF(DABS(PIVOT).LT.1.0E-50) GO TO 60
      IF(IEQNS.EQ.NEQNS) GO TO 50
      IEQN1=IEQNS+1
      DO 20 IROWS=IEQN1,NEQNS
      FACTR=ASTIF(IROWS,IEQNS)/PIVOT
      IF(FACTR.EQ.0.0) GO TO 20
      DO 10 ICOLS=IEQNS,NEQNS
      ASTIF(IROWS,ICOLS)=ASTIF(IROWS,ICOLS)-FACTR*ASTIF(IEQNS,ICOLS)
   10 CONTINUE
      ASLOD(IROWS)=ASLOD(IROWS)-FACTR*ASLOD(IEQNS)
   20 CONTINUE
C
C
C     WRITE(6,229) IEQNS
C229  FORMAT('IEQNS=',I2,'COEFFICIENT MATRIX AFTER SOLUTION')
C     DO 224 I=1,NEQNS
C     WRITE(6,122) (ASTIF(I,J1),J1=1,NEQNS)
C224  CONTINUE
C     DO 225 I=1,NEQNS
C     WRITE(6,124) ASLOD(I)
C225  CONTINUE
C122  FORMAT(3(E12.5,2X))
C124  FORMAT(E12.5)
C
C
      GO TO 50
C
C     ADJUST RHS(LOADS) FOR PRESCRIBED DISPLACEMENTS
C
   30 DO 40 IROWS=IEQNS,NEQNS
      ASLOD(IROWS)=ASLOD(IROWS)-ASTIF(IROWS,IEQNS)*FIXED(IEQNS)
      ASTIF(IROWS,IEQNS)=0.0
   40 CONTINUE
      GO TO 50
   60 WRITE(6,900) PIVOT,IEQNS
  900 FORMAT(5X,18HINCORRECT PIVOT = ,E20.6,5X,13HEQUATION NO. ,I5)
      STOP
   50 CONTINUE
      RETURN
      END
C
C-----------------------------------------------
C
C-----------------------------------------------
C
      SUBROUTINE BAKSU (NEQNS, ASTIF, FIXED, ASLOD, IFPRE,XDISP)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASTIF(NEQNS,NEQNS),IFPRE(NEQNS),
```

```
     .   FIXED(NEQNS),XDISP(NEQNS),ASLOD(NEQNS)
C
C    BACK-SUBSTITUTION ROUTINE
C
      NEQN1=NEQNS+1
      DO 30 IEQNS=1,NEQNS
      NBACK=NEQN1-IEQNS
      PIVOT=ASTIF(NBACK,NBACK)
      RESID=ASLOD(NBACK)
      IF(NBACK.EQ.NEQNS) GO TO 20
      NBAC1=NBACK+1
      DO 10 ICOLS=NBAC1,NEQNS
      RESID=RESID-ASTIF(NBACK,ICOLS)*XDISP(ICOLS)
   10 CONTINUE
   20 IF(IFPRE(NBACK).EQ.0) XDISP(NBACK)=RESID/PIVOT
      IF(IFPRE(NBACK).EQ.1) XDISP(NBACK)=FIXED(NBACK)
   30 CONTINUE
      RETURN
      END
```

A-5.2 PROGRAM DISS4 LISTING :

```
C*****************************************
C****************************************************
C
C PROGRAM  DISS4 : TO FIND SOLUTION ( DEFLECTION & STRESSES )
C
C             IN THE CASE OF CYLINDRICAL BENDING
C DONE BY AMMAR KHALEEL HAFEDH MOHAMMED  ( IN PH.D DISSERTATION )
C
C****************************************************
      IMPLICIT REAL*8(A-H,O-Z)
      DOUBLE PRECISION  NU,NUP1,NUSM1,NUM1,LH2,LA,K,N,LAMDA,
     .            INCREM
      DATA NU/0.30D0/,E/1.0D0/,HAR/0.00/,INCREM/0.500/,
     .  NPLATE/6 /,NTERM/15/,MP/15/,
     .  IPRINT/2/,IDEF/2/,ISTRES/2/,IBAL/2/,ISIGZ/1/,IFOUR/2/
C
C
      NUP1 = NU + 1.D0
      NUSM1 = 1.0-NU**2
      NUM1  = 1.0-NU
      G = E/(2.D0*(1.D0 + NU))
      PI = 22.D0/7.D0
C
C NOTE : MP  =  IS AN INDICATER TO TELL AT WHAT "M" VALUE WE WANT RESULTS
C     TO BE PRINTED
C     NPLATE  =  IS AN INDICATER TO TELL US FOR HOW MANY PLATE RATIOS W
C     WANT THE RESULTS
C
C     IDEF  =  IS AN INDICATER FOR PRINTING DEFLECTION RESULTS
C   IF  IDEF  =  1 : PRINT DEFLECTIONS
C   IF  IDEF  =  2 : DO NOT PRINT DEFLECTIONS
C
C     ISTRES  =  IS AN INDICATER FOR PRINTING STRESS SIGMAX
C   IF  ISTRES  =  1 : PRINT STRESSES
C   IF  ISTRES  =  2 : DO NOT PRINT STRESSES
C
C
C     ISIGZ  =  IS AN INDICATER FOR PRINTING STRESS SIGMAZ
C   IF  ISIGZ  =  1 : PRINT STRESSES
C   IF  ISIGZ  =  2 : DO NOT PRINT STRESSES
C
C     IPRINT  =  IS AN INDICATER FOR PRINTING INTERMEDIATE RESULTS
C   IF  IPRINT  =  1 : PRINT INTERMEDIATE RESULTS
C   IF  IPRINT  =  2 : DO NOT PRINT INTERMEDIATE RESULTS
C
       WRITE(6,210)
  210 FORMAT('CYLINDRICAL BENDING ')
```

```
      GO TO (212,213) IFOUR
 212  WRITE(6,211)
 211  FORMAT('LOAD P0 = SIN(PI*X/L)')
      GO TO 215
 213  WRITE(6,214)
 214  FORMAT('LOAD P0 =  UNIFORM LOAD')
 215  ABAR=1.D0
      WRITE(6,188) NU
      WRITE(6,18)
 188  FORMAT('NU = ',F6.2)
      GO TO (561,562) IDEF
 561  WRITE(6,102)
 101  FORMAT('******************************')
 102  FORMAT('                         DEFLECTIONS ')
      WRITE(6,101)
      WRITE(6,556)
      GO TO 564
 562  GO TO (565,564) ISTRES
 565  WRITE(6,103)
      WRITE(6,101)
      WRITE(6,555)
 555  FORMAT(7X,'Z/H',8X,'RTP',6X,'EXACT',8X,'PANC',8X,'RTB',8X,'OTHERS'
     .)
 556  FORMAT(5X,' H',6X,'RTP',7X,'EXACT',6X,'RTB',6X,'PANC',6X,'REISS'
     .,6X,'NAGHDI')
 564  GO TO (406,407) ISIGZ
 406  WRITE(6,408)
      WRITE(6,409)
 408  FORMAT('*    H    *  PRESENT *  PRESENT *   EXACT  * EXACT *')
 409  FORMAT('*         *  SIGMAX  *  SIGMAZ  *   SIGMAX * SIGMAZ*')
 407  CONTINUE
C******************************************
      DO 200 I=1,NPLATE
C******************************************
      IF(I.LE.30) GO TO 31
      GO TO 32
 31   HAR=HAR+INCREM
C
C
C NOTE : INCREM  IS THE INCREMENT IN THE A/H RATIO
C
C
      AHR=1.D0/HAR
      GO TO 33
 32   IF(I.EQ.31) AHR=0.D0
      IF(I.GE.31) GO TO 34
      AHR=AHR+2.D0
      GO TO 33
 34   AHR=AHR+100.D0
 33   H=ABAR/AHR
```

```
      GO TO (800,801) ISTRES
800   WRITE(6,101)
      WRITE(6,25) H
      WRITE(6,101)
801   GO TO (404,405) ISIGZ
404   WRITE(6,101)
      WRITE(6,25) H
      WRITE(6,101)
405   D=E*H**3/(12.D0*(1.D0-NU**2))
      WCT=0.D0
      WMT=0.D0
      W0PANC=0.D0
      WREIS=0.D0
      WNAGD=0.D0
      WPT=0.D0
      WBT=0.D0
      W0CHEK=0.D0
      W0EXAK=0.D0
C
C
      NPOINT=11
C*****************************************
      DO 100 J=1,NPOINT
C*****************************************
C
      SIGMAP=0.D0
      SIGMPA=0.D0
      SIGMAE=0.D0
      SIGMAB=0.D0
      SIGMAM=0.D0
      SIGMAO=0.D0
      SIGZP=0.D0
      SIGZE=0.D0
C
      IF(J.EQ.1)GO TO 222
      GO TO 223
222   Z=-0.50*H
223   ZH=Z/H
C
C
C*****************************************
      DO 10 M=1,NTERM,2
C*****************************************
C
C
C PRESENT WORK : DEFLECTION
C
C
      ALPHA=M*PI/ABAR
      AP=ALPHA
```

```
      APH2=ALPHA*H/2.
      APB=ALPHA**2
      APBS=ALPHA**4
      AA=APB*(2.D0-NU)/(1.D0-NU)
      BB=APBS/(1.D0-NU**2)
      DD=DSQRT(AA**2-4.D0*BB)
      A=DSQRT(.5D0*(AA+DD))
      B=DSQRT(.5D0*(AA-DD))
      AH=A*H
      AH2=A*H/2.D0
      BH=B*H
      BH2=B*H/2.D0
C
C
C
      A11=H
      A12=2*DSINH(AH2)
      A13=2*DSINH(BH2)
      A21=1.D0
      A22=A*DCOSH(AH2)
      A23=B*DCOSH(BH2)
      A31=1.0D0-NU**2
      A32=(12.*NU**2/H**3)*(2.*DSINH(AH2)/A**2-H*DCOSH(AH2),A)
      A33=(12.*NU**2/H**3)*(2.*DSINH(BH2)/B**2-H*DCOSH(BH2),B)
      B11=1.D0
      B22=0.D0
      B33=0.D0
      D11=A22*A33-A23*A32
      D12=A21*A33-A23*A31
      D13=A21*A32-A22*A31
      D22=B22*A33-A23*B33
      D23=B22*A32-A22*B33
      D33=A21*B33-B22*A31
      DET1=A11*D11-A12*D12+A13*D13
      DET2=B11*D11-A12*D22+A13*D23
      DET3=A11*D22-B11*D12+A13*D33
      DET4=-A11*D23-A12*D33+B11*D13
      A1=DET2/DET1
      A3=DET3/DET1
      A5=DET4/DET1
      DCOT=1.D0/DTANH(BH2)
C
C
      A2=0.5D0/( A*DSINH(AH2)/( DTANH(BH2)*B ) - DCOSH(AH2) )
      A4=-1.D0*DSINH(AH2)*A*A2/( B*DSINH(BH2) )
C
      GO TO (500,501) IPRINT
500   WRITE(6,18)
      WRITE(6,24) AHR
      WRITE(6,25) H
```

```
      WRITE(6,111) ZH
      WRITE(6,17) M
      WRITE(6,18)
      WRITE(6,102)
24    FORMAT(' A/H  =',F8.2)
25    FORMAT('                    H   =',F10.4)
      WRITE(6,101)
17    FORMAT('          M   =  ',I2)
111   FORMAT('          Z/H  =',F6.2)
C     WRITE(6,12) A1
C     WRITE(6,13) A2
C     WRITE(6,14) A3
C     WRITE(6,15) A4
C     WRITE(6,16) A5
12    FORMAT('A1 = ',E15.6)
13    FORMAT('A2 = ',E15.6)
14    FORMAT('A3 = ',E15.6)
15    FORMAT('A4 = ',E15.6)
16    FORMAT('A5 = ',E15.6)
18    FORMAT('****************************************')
501   F1 = A1-(12./H**3)*A3*(2./A**2*DSINH(AH2)-H*DCOSH(AH2)/
     .A)-(12./H**3)*A5*(2./B**2*DSINH(BH2)-H*DCOSH(BH2)/B)
      C1 = -(A3/A + A5/B)
      C2 = 2.D0*(1.D0 + NU)/APB*(A2 + A4)-(A2/A**2 + A4/
     .    B**2)
C
C     WRITE(6,552) C1
C     WRITE(6,553) C2
552   FORMAT('C1 = ',F20.6)
553   FORMAT('C2 = ',F20.6)
C
      F31 = (H**2/40.)*A1 + C1
      F32 = (12./H**3)*(A3/A**2)
      F33 = (H/A)*DCOSH(AH2)-2.*DSINH(AH2)/A**2
      F34 = (12./H**3)*(A5/B**2)
      F35 = (H/B)*DCOSH(BH2)-2.*DSINH(BH2)/B**2
      F3 = F31 + F32*F33 + F34*F35
C
C
      F2 = 2./A*DSINH(AH2)*A2 + 2./B*DSINH(BH2)*A4
      F4 = 2./A**3*DSINH(AH2)*A2 + 2./B**3*DSINH(BH2)*A4
     .    + C2*H
C
C
      GO TO (600,601) IPRINT
600   WRITE(6,42) F1
      WRITE(6,51) F1B
      WRITE(6,43) F3
      WRITE(6,52) F3B
      WRITE(6,53) F2
```

```
      WRITE(6,54) F4
 42   FORMAT('PRESENT WORK  F1 = ',E15.6)
 43   FORMAT('PRESENT WORK  F3 = ',E15.6)
 51   FORMAT('BALUCH  WORK  F1B = ',E15.6)
 52   FORMAT('BALUCH  WORK  F3B = ',E15.6)
 53   FORMAT('PRESENT WORK  F2 = ',E15.6)
 54   FORMAT('PRESENT WORK  F4 = ',E15.6)
601   S = G/F1
      N = E/F3
      R = 10.*E*H/(3.*NU)
      GO TO (230,231) IFOUR
230   P = 1.0
      GO TO 232
231   P = 4.D0/(M*PI)
232   AM = M*PI/2.D0
C     AM = ALPHA*ABAR/2.
C     W00 = P/(APBS*D)
C     W01 = 1.0 + (2.-NU)*H**3*APB*F1/(12.*(1.-NU))-APBS*D/N
C     W02 = NU*H**2*APB/(40.*(1.-NU))
C     W03 = NU**2*H**5*APBS*F1/(480.*(1.-NU)**2)
C     W04 = NU**2*H**5*APBS*F1/(240.*(1.-NU)**2*(1.+NU))
C     W0 = W00*(W01 + W02-W03 + W04)*DSIN(AM)
      W0CHEC = P/(BB*E)*( AA*A1-BB*C1-A3*(A**3
     .    - A*AA + BB/A ) - A5*( B**3-B*AA
     .     + BB/B ) )*DSIN(AM)
C    .     + BB/B ) )-P*C1/E )*DSIN(AM)
      BMCHEK = H**3*P*A1/(12.*NU)
      BM = D*P*(NUP1/(D*APB)-NU*NUP1**2*F1/E + 2.*NU*NUP1*F1/E)
      BM = BMCHEK
      W0 = ( P*( 1./(APBS*D) + (2.-NU)*NUP1*F1/(APB*E)-1./N )
     ;     + BM/R )*DSIN(AM)
C    ;     + BM/R -P*C1/E )*DSIN(AM)
      W0CHEK = W0CHEC + W0CHEK
      WMT = WMT + W0
      WC = P*DSIN(AM)/(APBS*D)
      WCT = WCT + WC
      WRP = WMT/WCT
      WRCHEK = W0CHEK/WCT
C
C
C  EXACT SOLUTION
C
C
      R3 = -P*DSINH(APH2)/(2.*APB*( DSINH(APH2)*DCOSH(APH2) + APH2 ) )
      R4 = -P*DCOSH(APH2)/(2.*APB*( DSINH(APH2)*DCOSH(APH2)-APH2 ) )
      R1 = -R4*(APH2*DTANH(APH2) + 1.)
      R2 = -R3*(APH2/DTANH(APH2) + 1.)
C
C
C NOTE : IN EXACT SOLUTION ( E )WILL BE REPLACED BY  ( E/(1.-NU**2) )
```

```
C
C
      EEXAC=E/NUSM1
      W0EXAC=(R4*AP*DSIN(AM)/EEXAC)*(2.+NUP1*APH2*DTANH(APH2))
      W0EXAK=W0EXAC + W0EXAK
C     WREXAC=W0EXAK/WCT
      WREXAC=DABS(W0EXAK/WCT)
C
C
C PANC'S WORK
C
C
      LAMDA=ALPHA*DSQRT(2./(1.-NU))
      LA=LAMDA
      LH2=LAMDA*H/2.
      K=2.*E*(LH2-DTANH(LH2))/(LAMDA**3*(1.-NU**2))
      WPANC=P*DSIN(AM)/(APBS*K)
      W0PANC=WPANC+W0PANC
      WRPANC=W0PANC/WCT
C
C END OF PANC'S WORK
C
C*********************
C
C
C BALUCH'S  WORK
C
C*********************
C
      C1B=0.D0
      C2B=-NUP1/APB
      F1B=6.D0/(5.D0*H)
      F3B=39.D0*H/1120.D0
C     F3B=39.D0*H/1120.D0 + C1B
      SB=G/F1B
      NB=E/F3B
C     W00=P/(APBS*D)
C     W01=1.0+(2.-NU)*H**3*APB*F1/(12.*(1.-NU))-APBS*D/N
C     W02=NU*H**2*APB/(40.*(1.-NU))
C     W03=NU**2*H**5*APBS*F1/(480.*(1.-NU)**2)
C     W04=NU**2*H**5*APBS*F1/(240.*(1.-NU)**2*(1.+NU))
C     W0B=W00*(W01+W02-W03+W04)*DSIN(AM)
      BMB=D*P*((1.+NU)/(D*APB)-NU*(1.+NU)**2*F1B/E+2.*NU*
     ;    (1.+NU)*F1B/E)
      W0B=(P*(1./(APBS*D)+(2.-NU)*(1.+NU)*F1B/(APB*E)-1./NB)
     ;    +BMB/R )*DSIN(AM)
      WBT=WBT+W0B
      WRB=WBT/WCT
C*********************
C
```

```
C
C REISSNER SHEAR DEFORMATION THEORY
C
C
C**********************
      WREISS=(1.+APB*H**2*(2.-NU)/(10.*(1.-NU)))*P/(APBS*D)*
     ;      DSIN(AM)
      WREIS=WREISS+WREIS
      WRREIS=WREIS/WCT
C**********************
C
C
C NAGHDI-ESSENBURG TRANSVERSE NORMAL STRAIN THEORY
C
C
C**********************
      WNAGDI=(1.D0 +(8. - 3.*NU*(1.-NU) )*H**2*APB/(40.*(1.-NU))
     ;    -3.*APBS*H**4/1120.)*P/(APBS*D)*DSIN(AM)
      WNAGD=WNAGDI+WNAGD
      WRNAGD=WNAGD/WCT
C
C    WRITE(6,19) WCT
C    WRITE(6,21) WMT
      GO TO (672,503) IPRINT
672  IF(M.GE.MP) GO TO 544
      GO TO 503
544  WRITE(6,22) WRP
C    WRITE(6,64) WRCHEK
      WRITE(6,72) WREXAC
      WRITE(6,56) WRB
      WRITE(6,27) WRPANC
      WRITE(6,29) WRREIS
      WRITE(6,41) WRNAGD
C    WRITE(6,67) BM
C    WRITE(6,68) BMCHEK
C19   FORMAT(' ','W ,CLASSICAL THEORY , WCT =',F15.6)
C21   FORMAT(' ','W ,MODIFIED THEORY  , WMT =',F15.6)
333   FORMAT(' ','BBBBBBBBBBBBBBBBBBB    C2B =',F15.6)
22   FORMAT(' ','PRESENT WORK RATIO   ; WRP  =',F15.6)
72   FORMAT(' ','EXACT SOLUTION RATIO;WREXAC=',F15.6)
64   FORMAT(' ','PRESENT WORK RATIO ;WRCHEK  =',F15.6)
C67   FORMAT(' ','BM  =',F20.6)
C68   FORMAT(' ','BMCHEK  =',F20.6)
56   FORMAT(' ','BALUCH RATIO         ; WRB  =',F15.6)
27   FORMAT(' ',' WRPANC  =',F15.6)
29   FORMAT(' ',' WRREIS  =',F15.6)
41   FORMAT(' ',' WRNAGD  =',F15.6)
C
C
C PRESENT WORK - STRESSES : SIGMAX
```

```
C
C
      WRITE(6,101)
      WRITE(6,103)
      WRITE(6,101)
 103  FORMAT('                             STRESSES     ')
 503  AZ=A*Z
      BZ=B*Z
      F1Z=A1*Z+A2*DCOSH(AZ)+A3*DSINH(AZ)+A4*DCOSH(BZ)
     .    +A5*DSINH(BZ)
      F3Z=(A1/6.)*Z**3+A2/A**2*DCOSH(AZ)+A3/A**2*
     .    DSINH(AZ)+A4/B**2*DCOSH(BZ)+A5/B**2*DSINH(BZ)
     .    +C1*Z+C2
C     W2P=W0/DSIN(AM)+P*C1/E
      W2P=W0/DSIN(AM)
      SIGXP=( (E*APB*W2P/NUSM1)*Z - P*(2.D0-NU)/NUM1*F1Z
     .      + (P*APB/NUSM1)*F3Z -(2.*NU*APB*BM/(H**3
     .      *NUSM1))*Z**3 - P/(NUSM1*H)*( (NU**2-NU-2.)*F2+
     .      APB*F4 ) )*DSIN(AM)
      SIGMAP=SIGMAP + SIGXP
C     SIGXP=SIGMAP
C
C PRESENT WORK - STRESSES : SIGMAZ
C
      SIGZP = SIGZP + P*F1Z*DSIN(AM)
C
C
C EXACT SOLUTION - STRESSES : SIGMAX
C
C
      APZ=-ALPHA*Z
      SIGXE=APB*( R1*DSINH(APZ) + R2*DCOSH(APZ) + R3*(2.*
     .      DCOSH(APZ)+APZ*DSINH(APZ)) + R4*(2.*DSINH(APZ)
     .      +APZ*DCOSH(APZ)) )*DSIN(AM)
      SIGMAE=SIGMAE + SIGXE
C     SIGXE=SIGMAE
C
C EXACT SOLUTION - STRESSES : SIGMAX
      SIGZE = SIGZE - APB*DSIN(AM)*( R1*DSINH(APZ) + R2*DCOSH(APZ)
     .            + R3*APZ*DSINH(APZ) + R4*APZ*DCOSH(APZ) )
C
C
C
C PANC'S SOLUTION : STRESSES
C
C
      F1ZP=0.5 *( LA*Z*DCOSH(LH2)-DSINH(LA*Z) )/( LH2*
     ;     DCOSH(LH2)-DSINH(LH2) )
      W2PA=WPANC/DSIN(AM)
      SIGXPA=( (E*APB*W2PA/NUSM1 )*Z - ( 2.*P/NUM1 )*F1ZP)
```

```
     ;       *DSIN(AM)
      SIGMPA=SIGMPA + SIGXPA
C    SIGXPA=SIGMPA
C
C
C  BALUCH'S  SOLUTION : STRESSES
C
C
      SIGB0 = APB*H**2/(48.*NUSM1)
      SIGB1 = 12./(APB*H**2) - 3.0/5.0
     ;     + APB*H**2*(168.*NU**2-195.)/(5600.*NUSM1)
      SIGB2 = -APB*H**2/(4.*NUSM1)
      SIGB3 = 4. + (APB*H**2/NUSM1)*(5.-4.*NU**2)/20.
      SIGB4 = -APB*H**2/(10.*NUSM1)
      SIGXB = ( SIGB0 + SIGB1*ZH + SIGB2*ZH**2 + SIGB3*ZH**3
     ;       +SIGB4*ZH**5 )*P*DSIN(AM)
      SIGMAB = SIGMAB + SIGXB
      GO TO(592,593)IBAL
592   WRITE(6,18)
      WRITE(6,25) H
      WRITE(6,111) ZH
      WRITE(6,17) M
      WRITE(6,590) APB,AM,P
      WRITE(6,591) SIGB0,SIGB1,SIGB2,SIGB3,SIGB4,SIGXB,SIGMAB
590   FORMAT(3F10.5)
591   FORMAT(7F10.5)
C    SIGXB=SIGMAB
C
C
C  BALUCH'S MODIFIED  SOLUTION : STRESSES
C
C
593   F1ZB = -0.5D0 + (3./2.)*ZH - 2.*ZH**3
C     F3ZB = -0.25*Z**2 + 0.25/H*Z**3 - 0.1/H**3*Z**5
      F3ZB = -0.25*Z**2 + 0.25/H*Z**3 - 0.1/H**3*Z**5 + C1B*Z + C2B
      F2B = -0.5*H
C     F4B = -H**3/48.
      F4B = -H**3/48. + C2B*H
      W2B = W0B/DSIN(AM)
      SIGXBM = ( (E*APB*W2B/NUSM1)*Z - P*(2.D0-NU)/NUM1*F1ZB
     ;        + (P*APB/NUSM1)*F3ZB -(2.*NU*APB*BMB/(H**3
     ;        *NUSM1))*Z**3 - P/(NUSM1*H)*( (NU**2-NU-2.)*F2B +
     ;        APB*F4B ) )*DSIN(AM)
      SIGMAM = SIGMAM + SIGXBM
C    SIGXBM = SIGMAM
C
C
C  STRESSE BY OTHER PLATE THEORIES :
C                  KIRCHOFF THIN PLATE , REISSNER SHEAR
C                  DEFORMATION PLATE THEORY, AND NAGHDI-ESSENBURG
```

```
C                  TRANSVERSE NORMAL STRAIN THEORY                    DIS04970
C                                                                     DIS04980
C                                                                     DIS04990
      SIGXO=( 12.*P/(APB*H**2) )*ZH*DSIN(AM)                          DIS05000
      SIGMAO=SIGMAO + SIGXO                                           DIS05010
C     SIGXO=SIGMAO                                                    DIS05020
C                                                                     DIS05030
      GO TO (670,10) IPRINT                                           DIS05040
670   IF(M.GE.MP) GO TO 644                                           DIS05050
      GO TO 10                                                        DIS05060
644   WRITE(6,104) SIGMAP                                             DIS05070
      WRITE(6,105) SIGMAE                                             DIS05080
      WRITE(6,106) SIGMPA                                             DIS05090
      WRITE(6,107) SIGMAB                                             DIS05100
C     WRITE(6,109) SIGMBM                                             DIS05110
      WRITE(6,108) SIGMAO                                             DIS05120
104   FORMAT(' ','PRESENT WORK    : SIGMAP  =',F15.6)                 DIS05130
105   FORMAT(' ','EXACT SOLUITION : SIGMAE  =',F15.6)                 DIS05140
106   FORMAT(' ','PANC     WORK   : SIGMPA  =',F15.6)                 DIS05150
107   FORMAT(' ','BALUCH   WORK   : SIGMAB  =',F15.6)                 DIS05160
109   FORMAT(' ','BALUCH MOD. WORK: SIGMBM  =',F15.6)                 DIS05170
108   FORMAT(' ','OTHER THEORIES  : SIGMAO  =',F15.6)                 DIS05180
C                                                                     DIS05190
10    CONTINUE                                                        DIS05200
C                                                                     DIS05210
      GO TO (667,668) ISTRES                                          DIS05220
667   CONTINUE                                                        DIS05230
C     WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA,SIGMAB                     DIS05240
      WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA,SIGMAB,SIGMAO              DIS05250
C     WRITE(6,558) ZH,SIGMAP,SIGMAE,SIGMPA                            DIS05260
C     WRITE(6,558) ZH,SIGMAP,SIGMAE                                   DIS05270
558   FORMAT(7(F10.2,2X))                                             DIS05280
C668  IF(I.EQ.31) AHR=0.D0                                             DIS05290
668   GO TO (400,401) ISIGZ                                           DIS05300
400   WRITE(6,402) ZH,SIGZP,SIGZE                                     DIS05310
402   FORMAT(3(F10.3,2X))                                             DIS05320
C                                                                     DIS05330
401   Z=Z+0.1*H                                                       DIS05340
100   CONTINUE                                                        DIS05350
C                                                                     DIS05360
      GO TO (502,200) IDEF                                            DIS05370
502   CONTINUE                                                        DIS05380
C     WRITE(6,225) H,WRP,WREXAC,WRB,WRPANC                            DIS05390
      WRITE(6,225) H,WRP,WREXAC,WRB,WRPANC,WRREIS,WRNAGD              DIS05400
C     WRITE(6,225) H,WRP,WREXAC,WRPANC                                DIS05410
225   FORMAT(7(F8.2,2X))                                              DIS05420
200   CONTINUE                                                        DIS05430
      STOP                                                            DIS05460
      END                                                             DIS05470
```

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