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α_2-LABELING OF GRAPHS

Abstract. We show that if a graph G on n edges allows certain special type of rosy labeling (a.k.a. ρ-labeling), called α_2-labeling, then for any positive integer k the complete graph K_{2nk+1} can be decomposed into copies of G. This notion generalizes the α-labeling introduced in 1967 by A. Rosa.

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Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \dots, G_s$ of K_n, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. Such a decomposition is called *cyclic* if there exists a graph isomorphism φ such that $\varphi(G_i) = G_{i+1}$ for $i = 0, 1, \dots, s-1$ and $\varphi(G_s) = G_0$.

A. Rosa [5] introduced in 1967 certain types of vertex labelings as important tools for decompositions of complete graphs K_{2n+1} into graphs with n edges.

A *labeling* of a graph G with n edges is an injection from $V(G)$, the vertex set of G, into a subset S of the set $\{0, 1, 2, \dots, 2n\}$ of elements of the additive group Z_{2n+1}. Let ρ be the injection. The *length* of an edge xy is defined as $\ell(x, y) = \min\{\rho(x) - \rho(y), \rho(y) - \rho(x))\}$. The subtraction is performed in Z_{2n+1} and hence $0 < \ell(x, y) \leq n$. If the set of all lengths of the n edges is equal to $\{1, 2, \dots, n\}$ and $S \subseteq \{0, 1, \dots, 2n\}$, then ρ is a *rosy labeling* (called originally ρ-valuation by A. Rosa); if $S \subseteq \{0, 1, \dots, n\}$ instead, then ρ is a *graceful labeling* (called β-valuation by A. Rosa). A graceful labeling ρ is said to be an *α-labeling* if there exists a number ρ_0 with the property that for every edge $xy \in G$ with $\rho(x) < \rho(y)$ it holds that $\rho(x) \leq \rho_0 < \rho(y)$. Obviously, G must be bipartite to allow an α-labeling. For an exhaustive survey of graph labelings, see [2] by J. Gallian.

Each graceful labeling is of course also a rosy labeling. The following theorem was proved by A. Rosa in [5].

Theorem 1. *A cyclic G-decomposition of K_{2n+1} for a graph G with n edges exists if and only if G has a rosy labeling.*

The main idea of the proof is the following. K_{2n+1} has exactly $2n+1$ edges of length i for every $i = 1, 2, \dots, n$ and each copy of G contains exactly one edge of each length. The cyclic decomposition is constructed by taking a labeled copy of G, say G_0, and then adding an element $i \in Z_{2n+1}$ to the label of each vertex of G_0 to obtain a copy G_i for $i = 1, 2, \dots, 2n$.

For graphs with an α-labeling, even stronger result was proved by A. Rosa.

Theorem 2. *If a graph G with n edges has an α-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

It is easy to observe that there are graphs that allow rosy or graceful labelings but not α-labelings. The smallest example is the 3-*comet* consisting of three paths P_3 with their endvertices glued together in one vertex of degree 3.

Nice generalization of Theorem B was proved by S. El-Zanati, C. Vanden Eynden, and N. Punnim [1]. They relaxed the properties of the α-labeling to obtain a ρ^+-labeling as follows. A labeling of a bipartite graph G with bipartition X, Y is called a ρ^+-*labeling* if it is a rosy labeling with the additional property that for every edge $xy \in E(G)$ with $x \in X, y \in Y$ it holds that $\rho^+(x) < \rho^+(y)$. The difference between these labelings is that while in an α-labeling we require *all* vertices in X to have the labels smaller than *every* vertex in Y, in a ρ^+-labeling we only require that all *neighbors* of each given vertex $y \in Y$ have their labels smaller than $\rho^+(y)$. Moreover, we can use labels from the set $\{0, 1, \dots, 2n\}$ while in α-labeling only from the set $\{0, 1, \dots, n\}$.

Theorem 3. *If a bipartite graph G with n edges has a ρ^+-labeling, then there exists a G-decomposition of K_{2nk+1} for any positive integer k.*

We will generalize α-labeling in a different way by defining an α_2-labeling.

Definition 4. We say that a bipartite graph G with n edges has an α_2-*labeling* if

- α_2 is a rosy labeling with the label set L;
- $L = L_0 \cup L_1 \cup L_2$ and $L_i \cap L_j = \emptyset$ for $i \neq j$;
- there exist integers λ_1, λ_2 such that $0 \leq l_0 \leq \lambda_1 < l_1 \leq \lambda_2 < l_2$ for all labels $l_i \in L_i, i = 0, 1, 2$;
- if xy is an edge of G and $\alpha_2(x) < \alpha_2(y)$, then $\alpha_2(x) \in L_i$ and $\alpha_2(y) \in L_{i+1}$ for $i \in \{0, 1\}$ and $\alpha_2(y) - \alpha_2(x) \leq n$.

Notice that when restricted to a pair of sets L_i and L_{i+1}, the labeling is "alpha-like" in the sense that the length of an edge is always equal to the difference between the higher and the lower label (in that order).

Now we show that the existence of an α_2-labeling of G guarantees a decomposition in the same way as an α-labeling.

Theorem 5. *Let G be a graph on n edges that allows an α_2-labeling. Then for any positive integer k there exists a G-decomposition of the complete graph K_{2nk+1}.*

Proof. We want to show that for any k there is a graph G' consisting of k edge-disjoint copies of G with a rosy labeling ρ. These copies may share vertices.

Denote the copies by $G_0, G_1, \dots, G_{k-1}$. For a vertex z with $\alpha_2(z) \in L_j$, define the label of its copy z_i belonging to G_i as $\rho(z_i) = ijn + \alpha_2(z)$. Let $x_i y_i$ be the image of an edge xy of G belonging to G_i. It follows from the definition of α_2 that then $\alpha_2(x) \in L_j, \alpha_2(y) \in L_{j+1}$ and therefore $\rho(x_i) = ijn + \alpha_2(x)$ and $\rho(y_i) = i(j+1)n + \alpha_2(y)$.

First we need to prove that ρ is injective. Because G' consists of k *edge disjoint* copies of G, distinct copies G_i, G_j can share vertices. We then only need to show that if x_i and y_i are images of different vertices x and y of G, respectively, then $\rho(x_i) \neq \rho(y_i)$. For the sake of contradiction we suppose that $x_i \neq y_i$ and $\rho(x_i) = \rho(y_i)$. First suppose that both $\alpha_2(x), \alpha_2(y) \in L_j$ for some $j \in \{0,1,2\}$. Then $\rho(x_i) = ijn + \alpha_2(x)$ while $\rho(y_i) = ijn + \alpha_2(y)$. But because $\rho(x_i) = \rho(y_i)$, we get $\rho(x_i) = ijn + \alpha_2(x) = ijn + \alpha_2(y) = \rho(y_i))$ which immediately yields $\alpha_2(x) = \alpha_2(y)$. But this contradicts our assumption that α_2 is a rosy labeling since a rosy labeling must be injective.

Now let $\alpha_2(x) \in L_j$ and $\alpha_2(y) \in L_{j+1}$ for $j \in \{0,1\}$. Then $\rho(x_i) = ijn + \alpha_2(x)$ while $\rho(y_i) = i(j+1)n + \alpha_2(y)$ and we get $\rho(x_i) = ijn + \alpha_2(x) = i(j+1)n + \alpha_2(y) = \rho(y_i)$ which yields $\alpha_2(x) = in + \alpha_2(y)$. But $\alpha_2(y) \leq 2n$ and $i \leq k-1$. Therefore, $\alpha_2(x) = in + \alpha_2(y) \leq (k-1)n + 2n = kn + n \leq 2kn$ whenever $k > 0$. Because this is performed in Z_{2kn+1}, it follows that $\alpha_2(x) \geq \alpha_2(y)$. Since α_2 is a rosy labeling, it must be injective and $\alpha_2(x) \neq \alpha_2(y)$. Thus $\alpha_2(x) > \alpha_2(y)$, which contradicts our assumption that $\alpha_2(x) \in L_j$ and $\alpha_2(y) \in L_{j+1}$ for $j \in \{0,1\}$.

Similarly, if $\alpha_2(x) \in L_0$ and $\alpha_2(y) \in L_2$, then $\rho(x_i) = \alpha_2(x)$ while $\rho(y_i) = 2in + \alpha_2(y)$ and we get $\alpha_2(x) = 2in + \alpha_2(y)$. Now $\alpha_2(y) \leq 2n$ and $i \leq k-1$. Therefore, $\alpha_2(x) = 2in + \alpha_2(y) \leq 2(k-1)n + 2n = 2kn$ and again in Z_{2kn+1} it follows that $\alpha_2(x) \geq \alpha_2(y)$. Because α_2 is injective, $\alpha_2(x) \neq \alpha_2(y)$ and hence $\alpha_2(x) > \alpha_2(y)$. This contradicts our assumption that $\alpha_2(x) \in L_0$ and $\alpha_2(y) \in L_2$. Therefore, ρ is an injection.

Now we want to show that each copy G_i contains n edges of length $in+1, in+2, \dots, in+n$. By our formula, the length of an edge $x_i y_i$ (with the original vertices x, y satisfying $x \in L_j, y \in L_{j+1}$) is equal to $\ell(x_i y_i) = \rho(y_i) - \rho(x_i) = i(j+1)n + \alpha_2(y) - (ijn + \alpha_2(x)) = in + \alpha_2(y) - \alpha_2(x) = in + \ell(xy)$. Since G contains edges of lengths ranging from 1 to n, G_i contains edges of lengths $in+1, in+2, \dots, in+n$. Therefore, G' contains edges of lengths from 1 to kn, which completes the proof. □

As an illustration of the decomposition method based on the α_2-labeling we now prove the following simple result about lobsters. We recall here that a *lobster* is a tree that can be converted into a caterpillar by deleting all vertices of degree one, and a *caterpillar* is a tree that can be converted into a path or a single vertex by deleting all vertices of degree one. It is known that not all lobsters allow α-labelings (see [5]).

Theorem 6. *Every lobster L with n edges allows an α_2-labeling.*

Proof. It is well known (see [5]) that all caterpillars allow α-labelings. So, if L is a caterpillar, we are done. Therefore we may assume that L is a lobster with n edges which is not a caterpillar. Let L' be the caterpillar arising from L by deleting all vertices of degree one. Let $X_0 = \{x_0, x_1, \dots, x_s\}$ and $X_1 = \{x_{s+1}, x_{s+2}, \dots, x_m\}$ be the bipartition of L' with an α-labeling defined as $\alpha(x_i) = i$ for $i = 0, 1, \dots, m$. Let the number of neighbors of x_i of degree one in L be r_i and denote the set of these pendant neighbors of x_i belonging to L but not to L' by $Y_i = \{y_{i1}, y_{i2}, \dots, y_{ir_i}\}$ for every $i = 0, 1, \dots, m$ where obviously Y_i is empty if $r_i = 0$.

Now we want to construct an α_2-labeling in which $V_0 = X_0, V_1 = X_1 \cup Y_0 \cup Y_1 \cup \dots \cup Y_s$, and $V_2 = Y_{s+1}, Y_{s+2}, \dots, Y_m$. First we set $\alpha_2(x_i) = \alpha(x_i) = i$ for $i = 0, 1, \dots, m$. Then we define the labeling for the vertices y_{ij} as follows:

$$\alpha_2(y_{0j}) = m + j$$

(if any vertices y_{0j} exist) and for $i = 1, 2, \dots, m$

$$\alpha_2(y_{ij}) = m + i + \sum_{l=0}^{i-1} r_l + j$$

for all appropriate pairs i, j. One can now check that the labeling is injective and each length $1, 2, \dots, n$ where $n = m + \sum_{l=0}^{m} r_i$ appears exactly once. It should be also obvious that the maximum label used is less than $2n + 1$, because it is at most equal to $m + m + \sum_{l=0}^{m-1} r_l + r_m = 2m + (n - m) = m + n < 2n$. Finally, it should be clear that the sets V_0, V_1, and V_2 satisfy the definition of the α_2-labeling. □

The following corollary follows immediately from Theorems 2 and 3.

Corollary 7. *Every lobster with n edges decomposes K_{2nk+1} for every positive k.*

This result by itself, however, is not new. It was proved by S.I. El-Zanati, C. Vanden Eynden and N. Punnim in [1] (who used earlier results by A. Lladó, G. Ringel, and O. Serra [3], and by A. Lladó and S. C. López [4]) that every lobster allows a ρ^+-labeling, which implies the result in the Corollary. They even conjecture that every bipartite graph allows a ρ^+-labeling. Their conjecture leads us to the following question.

Problem. Does the existence of an α_2-labeling of a bipartite graph G imply the existence of a ρ^+-labeling of G?

Finally, we remark that it may be tempting to try to generalize the definition for label sets $L_0, L_1, \dots, L_t$ with $t > 2$. While such a generalization may work for some special classes of graphs, it needs to be defined carefully. For instance, if we have $n = 10, t = 3$ and $k = 2$, then by using a formula analogous to the formula above for the vertices with $\alpha_2(x) = 20 \in L_3$ and $\alpha_2(y) = 9 \in L_0$ we get in K_{41} in the copy G_1 the labels equal to $\rho(x_1) = 1 \cdot 3 \cdot 10 + 20 = 50 \equiv 9 \mod 41$ and $\rho(y_1) = 1 \cdot 0 \cdot 10 + 9 = 9$ and we have mapped two different vertices of G_1 onto the same vertex of K_{41}.

Therefore, this type of labeling may be used case by case for some classes of graphs even in a more general form, but one has to always make sure that the situation described above will not occur.

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