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CURRENT ON AN INFINITELY LONG CYLINDRICAL ANTENNA

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ABSTRACT

The infinitely long circular cylindrical antenna driven at some cross section by a localized electromotive force, V , circumscribing the cylinder in a peripheral band is considered. The asymptotic expression for the current at large distances from the driving e.m.f. is derived using the saddle point method. It is shown that the amplitude of this current is proportional to the reciprocal of the logarithm of axial distance from the driving e.m.f.

The infinitely long cylindrical antenna considered here consists of a perfectly conducting circular cylinder excited by a localized belt of axially directed electric field at the surface of the cylinder. For mathematical simplicity this impressed electric field is taken to be a delta function. As shown in Figure 1, cylindrical coordinates are used and the impressed belt of electric field is localized in the plane $z = 0$.

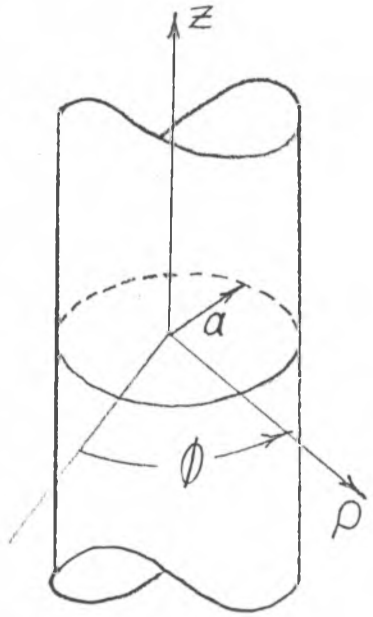


Figure 1.

The problem of determining the current distribution, input impedance and far field of the infinitely long cylindrical antenna has interested several investigators. In a paper by Papas[1], an expression for the asymptotic form of the current distribution for large z is derived. It is shown that the total current flowing in the z-direction is proportional to $1/\ln(z/ka^2)$ for large z . His result is based on a saddle point integration of a certain contour integral representation of the field under the assumption $\rho \gg a$. However, as was pointed out by Schelkunoff[2], this procedure is not valid since, in order to obtain an expression for the current on the cylinder, the field must be evaluated at $\rho = a$ which contradicts the assumed condition $\rho \gg a$. Nevertheless it may be shown that, at least for small radii, the correct z-dependence of the current is identical to that derived by Papas and it is the purpose of this report to carry through a proof of this.

The current on an infinitely long antenna is given by[3]

$$I(z) = i\omega\epsilon_0 \, aV \int_{-\infty}^{\infty} \frac{H_0'(ka\sqrt{1-h^2})e^{ikzh}}{\sqrt{1-h^2}\, H_0(ka\sqrt{1-h^2})} dh \tag{1}$$

where a is the radius of the cylinder, k is the free-space wave number, V is the driving voltage. The value of this integral for large kz and small ka is of interest. For small arguments, the approximation

$$\frac{H_0'(ka\sqrt{1-h^2})}{H_0(ka\sqrt{1-h^2})} \approx \frac{1}{ka\sqrt{1-h^2}\, \ln(\frac{\gamma ka\sqrt{1-h^2}}{2})} \tag{2}$$

holds. Thus the integral in equation 1 can be divided into three regions to yield

$$\frac{I(z)}{i\omega\epsilon_o\, aV} = \int_{-\infty}^{-M} \frac{H_o'(ka\sqrt{1-h^2})\, e^{ikzh}}{\sqrt{1-h^2}\, H_o(ka\sqrt{1-h^2})}\, dh$$

$$+ \frac{1}{ka}\int_{-M}^{M} \frac{e^{ikzh}}{(1-h^2)\, \ell n(\frac{\gamma ka\sqrt{1-h^2}}{2})}\, dh$$

$$+ \int_{M}^{\infty} \frac{H_o'(ka\sqrt{1-h^2})\, e^{ikzh}}{\sqrt{1-h^2}\, H_o(ka\sqrt{1-h^2})}\, dh \quad . \qquad (3)$$

The number M is determined such that the approximation of equation 2 holds for $-M \leq h \leq M$. If this approximation holds for the magnitude of the argument less than a number N , M is given by $|ka\sqrt{1-M^2}| = N$, or since ka is very small, $kaM = N$, and therefore

$$M = N/ka \quad . \qquad (4)$$

The three integrals of equation 3 will now be evaluated for $kz \gg 1$ and $ka \ll 1$.

Let the second integral be denoted as I_2. In order to evaluate I_2 the saddle point method will be used and because M is a large number, the limits will be replaced by $\pm\infty$. This is valid because the contribution to the integral in crossing the saddle point is not affected by this change in limits since the saddle point to be crossed is not near the end points, $\pm M$. Thus I_2 becomes

$$I_2 = \frac{1}{ka}\int_{-\infty}^{\infty} \frac{e^{ikzh}}{(1-h^2)\, \ell n(\frac{\gamma ka\sqrt{1-h^2}}{2})}\, dh \qquad (5)$$

where the contour of integration is shown in Figure 2.

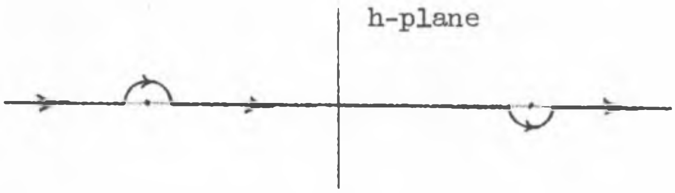


Figure 2.

The transformation $h = \sin \tau$ is now made and equation 5 becomes

$$I_2 = \frac{1}{ka} \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} \frac{e^{ikz \sin \tau}}{\cos \tau \; \ell n(\frac{\gamma ka \cos \tau}{2})} d\tau \tag{6}$$

where the path of integration in the τ-plane is shown in Figure 3.

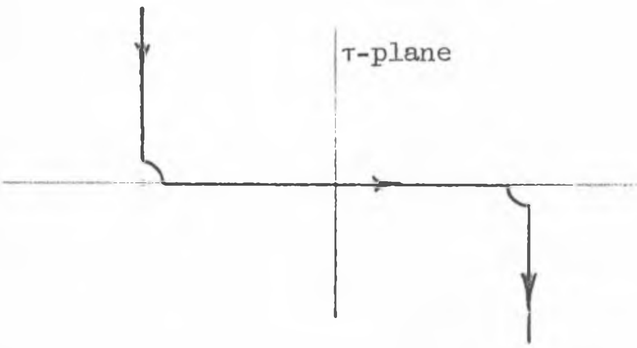


Figure 3.

In order to apply the saddle point method, equation 6 is rewritten in the following form:

$$I_2 = \frac{1}{ka} \int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} \frac{e^{kz[-\frac{1}{kz} \ell n(\cos \tau) + i \sin \tau]}}{\ell n(\frac{\gamma ka \cos \tau}{2})} d\tau \quad . \tag{7}$$

Letting $f(\tau) = -\frac{1}{kz}\ell n(\cos\tau) + i\sin\tau$, the saddle points are clearly determined by the relation

$$f'(\tau_o) = \frac{1}{kz}\frac{\sin\tau_o}{\cos\tau_o} + i\cos\tau_o = 0 \quad . \tag{8}$$

Solving for the saddle points, one obtains

$$\sin\tau_o = \frac{1}{2i\,kz} \pm \sqrt{-\frac{1}{4k^2z^2} + 1} \approx \frac{1}{2ikz} \pm 1 \quad . \tag{9}$$

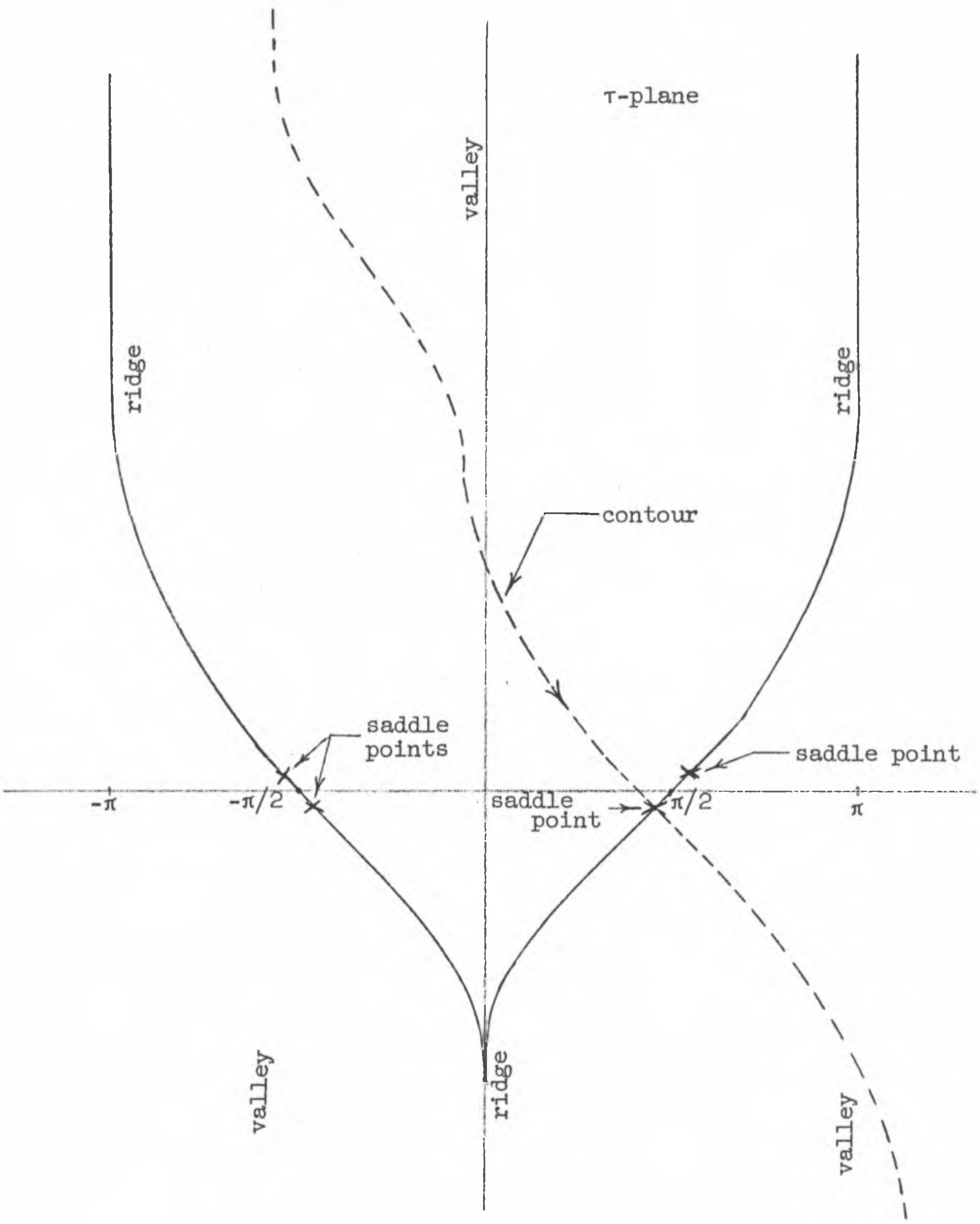
Since kz is assumed to be large, saddle points occur near $\tau = \pm\pi/2$. Near $\pi/2$, $\sin\tau \approx 1 - \frac{1}{2}(\tau - \frac{\pi}{2})^2$ and consequently one obtains

$$\tau_o \approx \frac{\pi}{2} \pm \sqrt{i/kz} \quad . \tag{10}$$

Similarly near $-\pi/2$, $\sin\tau \approx -1 + \frac{1}{2}(\tau + \frac{\pi}{2})^2$, and one obtains

$$\tau_o \approx -\frac{\pi}{2} \pm \sqrt{1/ikz} \quad . \tag{11}$$

Thus there are four saddle points and it is necessary to investigate $f(\tau)$ more closely to choose a correct path of integration. The nature of $Re[f(\tau)]$ is shown in Figure 4. The correct path is the broken line and the saddle point at $\tau_o = \pi/2 - \sqrt{i/kz}$, which is the only one that contributes, is crossed at an angle of $-\pi/4$. From the saddle point method and using $f''(\tau_o) = -2i$ one obtains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ridge
valley

Figure 4.

$$I_2 \approx \frac{1}{ka} \frac{e^{kz[-\frac{1}{kz} \ln\sqrt{i/kz} + i]}}{\ln(\frac{\gamma ka}{2} \sqrt{\frac{i}{kz}})} \sqrt{\frac{2\pi e}{kz\, e^{i\pi}(-2i)}}$$

$$\approx \frac{2\sqrt{\pi}\, e^{ikz}\sqrt{e}}{ika\, \ln(z/ka^2)} \tag{12}$$

Let the first and third integrals of equation 3 be denoted by I_1 and I_3 respectively. It will now be shown that I_1 and I_3 are negligible compared to I_2. From equation 3, I_1 may be written

$$I_1 = -\int_{-\infty}^{-M} \frac{K_o'(ka\sqrt{h^2-1})\, e^{ikzh}}{\sqrt{h^2-1}\, K_o(ka\sqrt{h^2-1})}\, dh \tag{13}$$

where $K_o(x)$ is a modified Bessel function of the second kind. Since M is very large, unity can be neglected with respect to h^2 in the region of integration and equation 13 becomes

$$I_1 = -\int_{\infty}^{M} \frac{K_o'(kah)e^{-ikzh}}{hK_o(kah)}\, dh = \int_{M}^{\infty} \frac{K_o'(kah)e^{-ikzh}}{hK_o(kah)}\, dh \quad . \tag{14}$$

Let $kah = x$ to obtain

$$I_1 = \int_{N}^{\infty} \frac{K_o'(x)\, e^{-i(\frac{kz}{ka})x}}{x\, K_o(x)}\, dx \quad . \tag{15}$$

Since kz/ka is large, this integral may be expanded in a power series in ka/kz by repeatedly integrating by parts, so that I_1 becomes

$$I_1 = \frac{K_o'(N)e^{-i(\frac{kz}{ka})N}}{N\, K_o(N)\, i} (\frac{ka}{kz}) - \frac{d}{dN}\left[\frac{K_o'(N)}{NK_o(N)}\right] e^{-i(\frac{kz}{ka})N} (\frac{ka}{kz})^2 + \cdots \; . \tag{16}$$

Retaining only the largest term I_1 is given to first order by

$$I_1 \approx \frac{K_o'(N)\, e^{-i\left(\frac{kz}{ka}\right)N}}{N\, K_o(N)\, i} \left(\frac{ka}{kz}\right) \tag{17}$$

Thus by comparison with equation 12, it is seen that I_1 becomes negligible with respect to I_2 for large kz and small ka. A similar argument holds for the third integral of equation 3 so that the only contribution is due to I_2 and the current for large kz and small ka is

$$I(z) = \frac{V\, e^{ikz}\, \sqrt{e}}{60\sqrt{\pi}\, \ell n(z/ka^2)} \quad . \tag{18}$$

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2. Private communication with C. H. Papas, April 1959.

3. Schelkunoff, S. A., Proc. IRE 33 (1945), 877.

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