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Generalized Ince Gaussian Beams

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ABSTRACT

In this work we present a detailed analysis of the tree families of generalized Gaussian beams, which are the generalized Hermite, Laguerre, and Ince Gaussian beams. The generalized Gaussian beams are not the solution of a Hermitian operator at an arbitrary z plane. We derived the adjoint operator and the adjoint eigenfunctions. Each family of generalized Gaussian beams forms a complete biorthonormal set with their adjoint eigenfunctions, therefore, any paraxial field can be described as a superposition of a generalized family with the appropriate weighting and phase factors. Each family of generalized Gaussian beams includes the standard and elegant corresponding families as particular cases when the parameters of the generalized families are chosen properly. The generalized Hermite Gaussian and Laguerre Gaussian beams correspond to limiting cases of the generalized Ince Gaussian beams when the ellipticity parameter of the latter tends to infinity or to zero, respectively. The expansion formulas among the three generalized families and their Fourier transforms are also presented.

Keywords: Gaussian beams, Ince–Gaussian, Laguerre–Gaussian, Hermite–Gaussian, Paraxial wave equation

1. INTRODUCTION

The standard and elegant Hermite–Gaussian, Laguerre–Gaussian, and Ince–Gaussian beams constitute the three orthogonal and biorthogonal, respectively, complete families of paraxial solutions for the scalar Helmholtz equation. The elegant solutions differ from the standard solutions in that the former contain polynomials with a complex argument but coinciding with that of the Gaussian function, whereas in the latter the argument is real.[1–5]

As other solutions that satisfy the paraxial wave equation, Pratesi and Ronchi in Ref. 6 and, Wunsche in Ref. 7 presented independently the generalized Hermite-Gaussian beams (gHGBs) and generalized Laguerre-Gaussian beams (gLGBs) in which the argument of the polynomials is complex in general. The standard Gaussian beams and elegant Gaussian beams are paticular cases of these generalized solutions.

In this work we present a detailed analysis of the tree complete families of exact and biorthogonal generalized solutions to the paraxial wave equation including the recently derived generalized Ince Gaussian beams (gIGBs). Previous works about the gHGBs and gLGBs do not stress that these families of solutions of the paraxial wave equation while are complete are not orthogonal but biorthogonal.

We show that, at an arbitrary z plane, the generalized beams are not solutions of a Hermitian operator and therefore are not orthogonal functions. We derived the adjoint operator and the adjoint eigenfunctions for the tree generalized families. Each family of generalized Gaussian beams (gGBs) form a complete biorthogonal set with their adjoint eigenfunctions, therefore, any paraxial field can be described as a superposition of gGBs with the appropriate weighting and phase factors. We derive the normalization coefficients to make each family not only biorthogonal but also biorthonormal. Each family of gGBs includes the standard and elegant families as particular cases when the parameters of the generalized families are chosen adequately.

The generalized Hermite and Laguerre Gaussian beams correspond to limiting cases of the generalized Ince Gaussian beams when the ellipticity parameter of the last ones tends to infinity or to zero, respectively. The expansion formulas among the three generalized families and their Fourier transforms are also derived.

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2. GENERALIZED GAUSSIAN BEAMS

In this section we introduce the gGBs that are exact solutions to the paraxial wave equation and their adjoint gGBs. Previous works[6,7] had not studied the adjoint generalized solutions, however, as we will show, by analyzing the relations among the gGBs and their adjoint beams we will be able to find a physical interpretation of the parameters which describe the gGBs.

To derive the gGBs and their adjoint beams we proceed as follows: For a paraxial field traveling in the z direction we write, $U = \Psi(x, y, z) \exp(ikz)$, where (x, y) are the transverse coordinates and Ψ is a slowly varying complex envelope that satisfies the paraxial wave equation

$$\left(\nabla_T^2 + 2ik\frac{\partial}{\partial z}\right)\Psi(x, y, z) = 0, \tag{1}$$

where ∇_T^2 is the transverse Laplacian, and k is the wave number. The lowest-order solution of the paraxial wave equation is the fundamental Gaussian beam,

$$\mathrm{GB}(r, z, \gamma) = \frac{1}{q_\gamma(z)} \exp\left(\frac{-r^2}{\gamma q_\gamma(z)}\right), \tag{2}$$

where r is the transverse radius, $q_\gamma = q_\gamma(z) \equiv 1 + iz/z_\gamma$ is a complex parameter, $z_\gamma = k\gamma/2$ is the Rayleigh range and γ is a complex parameter which real part $\mathrm{Re}(\gamma) = \omega_0^2$ where ω_0 is the beam width at the waist plane.

To derive the gGBs we propose, based on the known generalized solution,[6,7] the following ansatz

$$\Psi(x, y, z) = A(z)\psi(U, V)GB(r, z, \gamma), \tag{3}$$

where

$$A(z) = \left(\frac{q_\alpha^*(z)}{q_\gamma(z)}\right)^{p/2}, \tag{4}$$

p is a non-negative integer separation constant, $\psi(U, V)$ is a function to be determined, and (U, V) is a complex scaled transverse coordinate system related to the transverse Cartesian system by

$$(U, V) = c(z)(x, y), \tag{5}$$

$$c(z) \equiv \left[\frac{\alpha^* + \gamma}{\alpha^*\gamma}\frac{1}{q_\alpha^* q_\gamma}\right]^{1/2}, \tag{6}$$

where α is a complex parameter, $q_\alpha = q_\alpha(z) \equiv 1 + iz/z_\alpha$ and $z_\alpha = k\alpha/2$. The physical meaning of these quantities will be clear later. It is important to notice that the transverse coordinate system (U, V) is just a complex scaled version of each transverse z plane, but the complex scale constant change as a function of z.

The structure of Eq. (3) is particularly convenient because it has factorized out the whole functional z-dependence of the gGBs. As a consequence, we can apply the $\partial/\partial z$ part of the paraxial wave equation to Eq. (3) and get an eigenvalue equation for Ψ that only depends on the transverse complex scale coordinates (U, V) while the z dependence is now implicit in the parameters of the equation and the scaling. Note that the transverse complex scale coordinates depends implicitly on z therefore is it necessary to take in account its variation respect to the transverse coordinate while appliying the $\partial/\partial z$ operator to Eq.(3). In this way, we get the following eigenvalue equation

$$\mathcal{L}\Psi = -2p\Psi, \tag{7}$$

$$\mathcal{L} = \nabla_{TS}^2 + \frac{4q_\alpha^*\alpha^*}{\alpha^* + \gamma} - \frac{4q_\alpha^* q_\gamma \gamma\alpha^*}{(\alpha^* + \gamma)^2} r_s^2 + 2\left(\frac{q_\alpha^*\alpha^* - q_\gamma\gamma}{\alpha^* + \gamma}\right)\mathbf{r}_s \cdot \nabla_{TS}, \tag{8}$$

where $\nabla_{TS} = (\partial/\partial U, \partial/\partial V)$ is the Del operator, and $\mathbf{r}_s = (U, V)$ is the position vector both respect to the complex scaled transverse coordinates. If we consider this equation for a fixed arbitrary transverse z plane then α, γ, q_α, q_γ, are just complex parameters of the eigenvalue equation.

Our next task is to find the adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$, the eigenfunctions of the operator are the adjoint gGBs. To do this, we first define an inner product between two functions of the complex scaled transverse coordinates as follows

$$\langle F, G \rangle = \int F^* G dx dy, \tag{9}$$

The definition of the adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$, is given by

$$\langle F, \mathcal{L} G \rangle = \left\langle \mathcal{L}^\dagger F, G \right\rangle. \tag{10}$$

Using this definition, we integrate by parts the left hand side of Eq. (10) to get it in the form of the right hand side and extract the adjoint operator $\mathcal{L}^\dagger$. In the integration by parts it is necessary that the product FG vanish at infinity in order to drop the surface terms. We will use this boundary condition on FG to constraint the values of the parameter α and γ below. Then we get

$$\mathcal{L}^\dagger = \nabla^2_{TS^*} + \frac{4q_\gamma^*\gamma^*}{\gamma^* + \alpha} - \frac{4q_\gamma^* q_\alpha \alpha \gamma^*}{(\gamma^* + \alpha)^2} r_{s^*}^2 + 2\left(\frac{q_\gamma^*\gamma^* - q_\alpha \alpha}{\gamma^* + \alpha}\right) \mathbf{r}_{s^*} \cdot \nabla_{TS^*}, \tag{11}$$

where $\nabla_{TS^*} = (\partial/\partial U^*, \partial/\partial V^*)$ and $\mathbf{r}_{s^*} = (U^*, V^*)$, this is

$$(U^*, V^*) = c^*(z)(x, y), \tag{12}$$

$$c^*(z) \equiv \left[\left(\frac{\gamma^* + \alpha}{\gamma^* \alpha}\right) \frac{1}{q_\gamma^* q_\alpha}\right]^{1/2}, \tag{13}$$

Finally the adjoint solutions $\widehat{\Psi}$ are eigenfuntions of the adjoint operator $\mathcal{L}^\dagger$ of $\mathcal{L}$,

$$\mathcal{L}^\dagger \widehat{\Psi} = -2p\widehat{\Psi}, \tag{14}$$

Now we can remark the most important result of this analysis, namely, that the gGBs solutions Ψ and its adjoint counterpart $\widehat{\Psi}$ are simply related by

$$\Psi \underset{\alpha \longleftrightarrow \gamma}{\longleftrightarrow} \widehat{\Psi}. \tag{15}$$

To see this, note that $\mathcal{L}$ and $\mathcal{L}^\dagger$ are related by $\mathcal{L} \longleftrightarrow \mathcal{L}^\dagger$ when $\alpha \longleftrightarrow \gamma$, while the complex scale transverse coordinates (U, V) and their complex conjugate (U^*, V^*) are related in the same way. Due to this relation, the physical interpretation of the parameter α is that $\mathrm{Re}(\alpha) = \widehat{\omega}_0^2$, where $\widehat{\omega}_0$ is the beam width of the adjoint solutions at the waist plane. Summarizing, the generalized Gaussian beam are characterized by two complex parameter α and γ, while $\mathrm{Re}(\gamma)$ is related to the beam width of the generalized Gaussian beam at the waist plane, $\mathrm{Re}(\alpha)$ is related to the beam width of the adjoint generalized Gaussian beam at the waist plane. Since we want the gGBs to vanish at infinity we restrict $\mathrm{Re}(\gamma) > 0$. And from the condition that $\left\langle \widehat{\Psi}, \Psi \right\rangle$ must be finite we obtain the constraint $\mathrm{Re}(1/\alpha) + \mathrm{Re}(1/\gamma) > 0$. With these constraints the gGBs as eigenfuntions of $\mathcal{L}$ form a complete basis of function in which we can expand any square integrable function in an arbitrary transverse z plane, we chose the plane $z = 0$ without lose of generality. While the eigenfunctions of $\mathcal{L}$ are complete they are not orthogonal, however they are biorthogonal respect to the adjoint eigenfunctions of $\mathcal{L}^\dagger$. This is

$$\left\langle \widehat{\Psi}_{\mathbf{a}}, \Psi_{\mathbf{b}} \right\rangle = 0, \text{ if } \mathbf{a} \neq \mathbf{b}, \tag{16}$$

where $\mathbf{a}$ and $\mathbf{b}$ represents all possible mode indices of a given complete family of gGBs. Furthermore, using the proper normalization constants for the gGBs and their adjoint beams, as we will use in the following sections, we can make the relation biorthonormal, this is $\left\langle \widehat{\Psi}_{\mathbf{a}}, \Psi_{\mathbf{b}} \right\rangle = \delta_{\mathbf{ab}}$. Then, given an arbitrary transverse profile $F(x, y)$ at $z = 0$ we can expand its paraxial propagation in terms of gGBs in the following form

$$F(x, y, z) = \sum_{\mathbf{a}} A_{\mathbf{a}} \Psi_{\mathbf{a}}(x, y, z), \tag{17}$$

where the sum is over all possible mode indices $\mathbf{a}$, and $A_{\mathbf{a}}$ is given by

$$A_{\mathbf{a}} = \left\langle \widehat{\Psi}_{\mathbf{a}}, F(x, y) \right\rangle. \tag{18}$$

3. FAMILIES OF GENERALIZED GAUSSIAN BEAMS

In the last section we analyzed the fundamental properties of the gGBs, in this section we present the three unique families of gGBs, namely, the generalized Hermite, Laguerre and Ince Gaussian beams. Evaluating the eigenfunction Eq. (7) with the ansatz Eq. (3) we get the following differential equation for the function $\psi(U,V)$

$$\left(\nabla_{TS}^2 - 2\mathbf{r}_s \cdot \nabla_{TS}\right)\psi = -2p\psi. \tag{19}$$

This eigenvalue equation has separable solutions in three coordinate systems[11]: Cartesian, polar and elliptical. Introducing these solutions in Eq. (3) we get for each coordinate system a complete family of biorthogonal generalized Gaussian beam solutions to the paraxial wave equations.

3.1. Generalized Ince Gaussian Beams

To derive the gIGBs we start by solving Eq. (19) in a complex elliptical coordinate system $\mathbf{r}_s = (\xi, \eta)$. These coordinates are defined as

$$U = \epsilon^{1/2}\cosh\xi\cos\eta, \tag{20a}$$

$$V = \epsilon^{1/2}\sinh\xi\sin\eta, \tag{20b}$$

where the radial ξ and the angular η complex elliptic coordinates, and $\epsilon = (0,\infty)$ is the ellipticity parameter. Solving Eq. (19) by separation of variables $\psi = N(\eta)\,E(\xi)$ leads to the separated equations

$$\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2} - \epsilon\sinh 2\xi\frac{\mathrm{d}E}{\mathrm{d}\xi} = (\mu - p\epsilon\cosh 2\xi)\,E, \tag{21}$$

$$\frac{\mathrm{d}^2N}{\mathrm{d}\eta^2} + \epsilon\sin 2\eta\frac{\mathrm{d}N}{\mathrm{d}\eta} = -(\mu - p\epsilon\cos 2\eta)\,N, \tag{22}$$

where μ is a separation constants, and ϵ will be referred to as the ellipticity parameter.

Equation (22) is known in the theory of periodic differential equations under the name of *Ince equation*, it was studied originally by the mathematician E. G. Ince in 1923.[8] Ince equation is a special case of the most general Hill equation and it has been investigated in detail by F. M. Arscott,[9,10] and it is his notation for the solutions that we use. Notice that Eq. (21) may be derived from Eq. (22) by writing $i\xi$ for η, and vice versa. This reciprocal relation is important because radial solutions $E(\xi)$ may be obtained from angular solutions $N(\eta)$ by making the argument imaginary. There are three parameters in Eq. (22), it is convenient to regard ε as fundamental and μ, p as disposable parameters. The technique to solve analytically Eq. (22) is very similar to that for the better known Mathieu equation.[13] Qualitatively, Eq. (22) differs from Mathieu equation in that for certain values of a and p there exist finite solutions, i.e. solutions expressible as finite trigonometric series or as polynomials in $\sin\eta$ or $\cos\eta$.

Solutions of Eq. (22) are known as the even and odd Ince polynomials of order p and degree m, they are denoted usually as $\mathrm{C}_p^m(\eta,\varepsilon)$ and $\mathrm{S}_p^m(\eta,\varepsilon)$ respectively, where $0 \le m \le p$ for even functions, $1 \le m \le p$ for odd functions, the indices (p,m) have always the same parity, i.e. $(-1)^{p-m} = 1$, and ε is the ellipticity parameter defined earlier.[9] In a search for two-dimensional solutions, only products of functions of the same parity in ξ and η satisfy continuity in the whole plane.

Therefore, substituting this solutions of Eq. (19) into Eq. (3), the generalized Ince Gaussian beams and their adjoint beams are given by

$$\mathrm{gIG}_{p,m}^{e,o}(\mathbf{r},z) = c(0)\left[\frac{q_\alpha^*(z)}{q_\gamma(z)}\right]^{p/2}\left\{\begin{array}{c}\mathcal{C}_g C_p^m(i\xi,\epsilon)C_p^m(\eta,\epsilon)\\ \mathcal{S}_g S_p^m(i\xi,\epsilon)S_p^m(\eta,\epsilon)\end{array}\right\}\mathrm{GB}(\mathbf{r},z,\gamma), \tag{23}$$

$$\widehat{\mathrm{gIG}}_{p,m}^{e,o}(\mathbf{r},z) = c^*(0)\left[\frac{q_\gamma^*(z)}{q_\alpha(z)}\right]^{p/2}\left\{\begin{array}{c}\mathcal{C}_g^* C_p^m(i\xi,\epsilon)C_p^m(\eta,\epsilon)\\ \mathcal{S}_g^* S_p^m(i\xi,\epsilon)S_p^m(\eta,\epsilon)\end{array}\right\}\mathrm{GB}(\mathbf{r},z,\alpha), \tag{24}$$

where the subscript indices e and o refers to even and odd modes respectively and $\mathcal{C}_g$ and $\mathcal{S}_g$ are normalization constants that are given below. And the transverse Cartesian coordinates and the complex elliptic coordinates are related by

$$x = f_g\left(z\right)\cosh\xi\cos\eta, \qquad y = f_g\left(z\right)\sinh\xi\sin\eta, \tag{25}$$

$$f_g\left(z\right) = \epsilon^{1/2}/c(z), \tag{26}$$

it is important to note that since f_g is generally complex, it is only real just in the case where $\gamma = \alpha = \mathrm{Re}(\gamma)$, then ξ and η are complex numbers. The biorthogonality relation at any z plane is given by $\left\langle \mathrm{g}\widehat{\mathrm{IG}}^{\sigma}_{p,m}, \mathrm{gIG}^{\sigma'}_{p',m'}\right\rangle = \delta_{pp'}\delta_{mm'}\delta_{\sigma\sigma'}$ where $\sigma = \{e, o\}$. Figure (1) shows the transverse amplitudes and phases of several odd gLGBs, odd gIGBs, gHGBs at plane $z = z_R$. gIGB tend to gLGBs or gHGBs when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively. The indexes of the patterns are included within the figure.

The normalization constans are given by

$$\mathcal{C}_g\left(p \in 2\mathbb{Z}^*, m, \epsilon\right) = i^m\sqrt{\frac{2}{\pi}}\frac{(p/2)!\ A_0^+}{\mathrm{C}_p^m\left(0,\varepsilon\right)\mathrm{C}_p^m\left(\pi/2,\epsilon\right)}, \tag{27}$$

$$\mathcal{C}_g\left(p \in 2\mathbb{Z}^* + 1, m, \epsilon\right) = i^{m+1}\sqrt{\frac{2\epsilon}{\pi}}\frac{\left[\left(p+1\right)/2\right]!\ A_0^-}{\mathrm{C}_p^m\left(0,\varepsilon\right)\mathrm{C}_p'^m\left(\pi/2,\epsilon\right)}, \tag{28}$$

$$\mathcal{S}_g\left(p \in 2\mathbb{Z}^+, m, \epsilon\right) = i^m\sqrt{\frac{2}{\pi}}\epsilon\frac{\left[\left(p+2\right)/2\right]!\ B_1^+}{\mathrm{S}_p'^m\left(0,\varepsilon\right)\mathrm{S}_p'^m\left(\pi/2,\epsilon\right)}, \tag{29}$$

$$\mathcal{S}_g\left(p \in 2\mathbb{Z}^* + 1, m, \epsilon\right) = i^{m-1}\sqrt{\frac{2\epsilon}{\pi}}\frac{\left[\left(p+1\right)/2\right]!\ B_0^-}{\mathrm{S}_p^m\left(\pi/2,\varepsilon\right)\mathrm{S}_p'^m\left(0,\epsilon\right)}, \tag{30}$$

where $\mathbb{Z}^*$ are the nonegative integers, $\mathbb{Z}^+$ the positive integer, $\mathrm{C}_p'^m(x,\epsilon) = \partial\mathrm{C}_p^m(x,\epsilon)/\partial x$, $\mathrm{S}_p'^m(x,\epsilon) = \partial\mathrm{S}_p^m(x,\epsilon)/\partial x$, $A_0^{+,-}$ and $B_0^{+,-}$ are the first Fourier coefficient of the $\mathrm{C}_p^m(\cdot)$ or the $\mathrm{S}_p^m(\cdot)$ Ince polynomials,[3, 11] the superscripts $+$ and $-$ refer to the parity of p. We have used the following normalization for the Ince polynomials

$$\sum_{r=0}^{p/2}\left(1+\delta_{r,0}\right)\left(\frac{p+2r}{2}\right)!\left(\frac{p-2r}{2}\right)!\left(A_r^+\right)^2 = 1, \tag{31a}$$

$$\sum_{r=0}^{(p-1)/2}\left(\frac{p+2r+1}{2}\right)!\left(\frac{p-(2r+1)}{2}\right)!\left(A_r^-\right)^2 = 1, \tag{31b}$$

$$\sum_{r=1}^{p/2}\left(\frac{p+2r}{2}\right)!\left(\frac{p-2r}{2}\right)!\left(B_r^+\right)^2 = 1, \tag{31c}$$

$$\sum_{r=0}^{(p-1)/2}\left(\frac{p+2r+1}{2}\right)!\left(\frac{p-(2r+1)}{2}\right)!\left(B_r^-\right)^2 = 1. \tag{31d}$$

The Fourier transform defined as

$$\mathcal{F}\left\{F\left(x,y\right)\right\} = \widetilde{F}\left(k_x,k_y\right) = \frac{1}{2\pi}\int U\left(x,y\right)\exp\left(-ik_xx - ik_yy\right)dxdy, \tag{32}$$

of the gIGBs in any transverse z plane is given by

$$\mathrm{g}\widetilde{\mathrm{IG}}^{e,o}_{p,m}\left(k_x,k_y,z\right) = c(0)\frac{\gamma}{2}\left[-i\left(\frac{\gamma}{\alpha^*}\right)^{1/2}\right]^p\left\{\begin{array}{c}\mathcal{C}_gC_p^m(i\widetilde{\xi},\epsilon)C_p^m(\widetilde{\eta},\epsilon)\\ \mathcal{S}_gS_p^m(i\widetilde{\xi},\epsilon)S_p^m(\widetilde{\eta},\epsilon)\end{array}\right\}\exp\left(-\frac{\gamma q_\gamma(z)}{4}k_t^2\right), \tag{33}$$

where $k_x = 2\left[\epsilon/\left(\alpha^*+\gamma\right)\right]^{1/2}\cosh\xi\cos\widetilde{\eta}$ and $k_y = 2\left[\epsilon/\left(\alpha^*+\gamma\right)\right]^{1/2}\cosh\xi\cos\widetilde{\eta}$ and $k_t = \left(k_x^2+k_y^2\right)^{1/2}$. The Fourier transform of the adjoint beams are obtained by interchanging α and γ following the relation between the generalized beams and their adjoints given by Eq. (15).

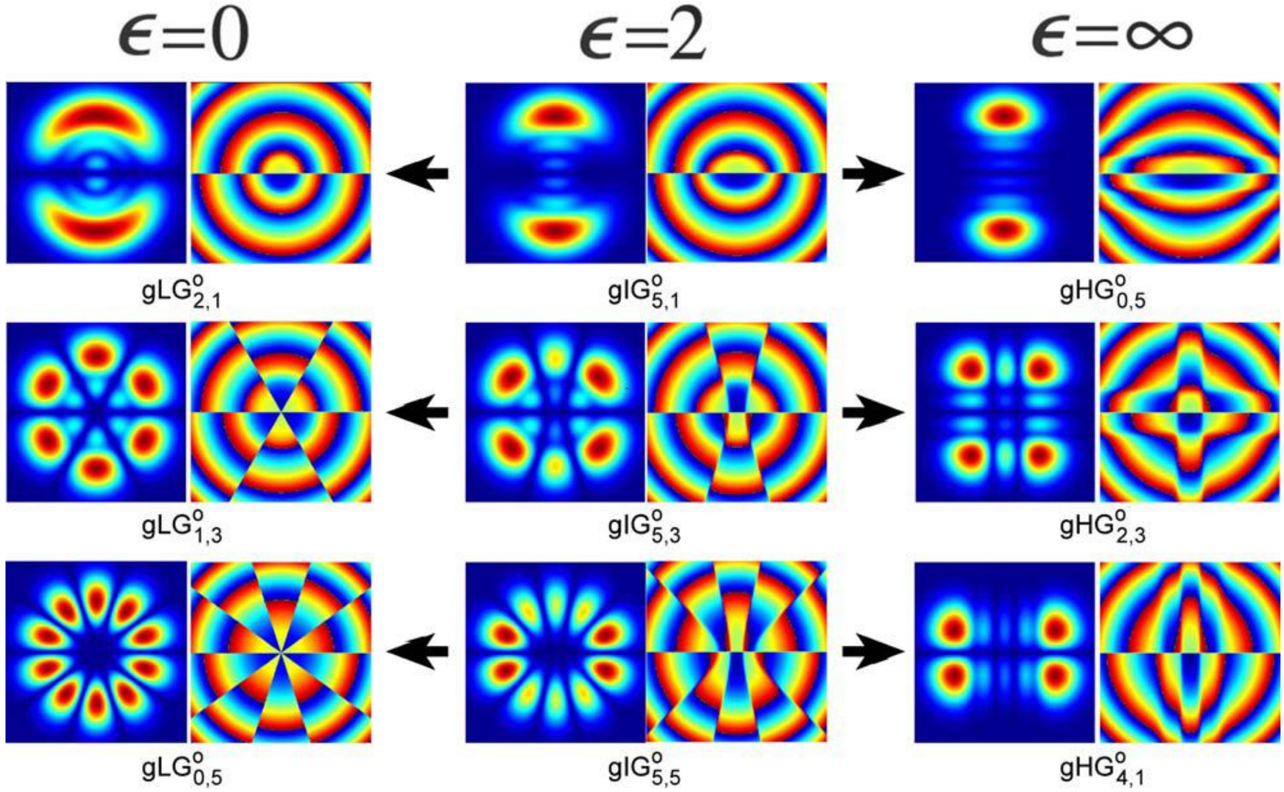

Figure 1. Transverse amplitudes and phases at $z = z_R$ of odd gLGBs, odd gIGBs, gHGBs with $\gamma = (0.0025)^2$ m^2 and $\alpha = \gamma(1+i)$ m^2. The gIGBs correspond to $\epsilon = 2$. Odd gIGB tend to odd gLGBs or gHGBs when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively. The transverse extent of each window is 1 cm.

3.2. Generalized Hermite Gaussian Beams

In the following two subsections we include the explicit expressions for the gHGBs and the gLGBs.[6,7] We also introduce for the first time to our knowledge their adjoint solutions and the proper normalization constants to make each family biorthonormal. The gHGBs and their adjoint functions are

$$\mathrm{gHG}_{n_x,n_y}(\mathbf{r},z) = D_{n_x,n_y} c(0) \left(\frac{q_\alpha^*}{q_\gamma}\right)^{(n_x+n_y)/2} H_{n_x}[c(z)x]\, H_{n_y}[c(z)y]\, \mathrm{GB}(\mathbf{r},z,\gamma), \tag{34}$$

$$\mathrm{g\widehat{HG}}_{n_x,n_y}(\mathbf{r},z) = D_{n_x,n_y} c^*(0) \left(\frac{q_\gamma^*}{q_\alpha}\right)^{(n_x+n_y)/2} H_{n_x}[c^*(z)\,x]\, H_{n_y}[c^*(z)y]\, \mathrm{GB}(\mathbf{r},z,\alpha), \tag{35}$$

where $H_n(\cdot)$ are the n-th order Hermite polynomials and $D_{n_x,n_y} = (2^{n_x+n_y}\pi n_x! n_y! \gamma)^{-1/2}$ is the normalization constant. The biorthogonality relation at any z plane is given by $\left\langle \mathrm{g\widehat{HG}}_{n_x,n_y}, \mathrm{gHG}_{n_x',n_y'} \right\rangle = \delta_{n_x n_x'}\delta_{n_y n_y'}$. In the eigenvalue Eq. (7) $p = n_x + n_y$ for the gHGB. The Fourier transform Eq. (32) of the gHGBs at any transverse z plane is given by

$$\mathrm{g\widetilde{HG}}_{n_x,n_y}(k_x,k_y,z) = D_{n_x,n_y} c(0) \frac{\gamma}{2} \left[-i\left(\frac{\gamma}{\alpha^*}\right)^{1/2}\right]^{n_x+n_y} \times H_{n_x}\left[\frac{(\alpha^*+\gamma)^{1/2}}{2}k_x\right] H_{n_y}\left[\frac{(\alpha^*+\gamma)^{1/2}}{2}k_y\right] \exp\left(-\frac{\gamma q_\gamma(z)}{4}k_t^2\right). \tag{36}$$

3.3. Generalized Laguerre Gaussian Beams

The gLGBs and their adjoint functions are

$$\mathrm{gLG}_{n,l}^{e,o}(\mathbf{r},z)=K_{l,n}c(0)\left(\frac{q_\alpha^*}{q_\gamma}\right)^{(2n+l)/2}\left[c^2(z)r^2\right]^{l/2}L_n^l\left(c^2(z)r^2\right)\left\{\begin{array}{c}\cos l\varphi\\ \sin l\varphi\end{array}\right\}\mathrm{GB}(\mathbf{r},z,\gamma), \tag{37}$$

$$\mathrm{g}\widehat{\mathrm{LG}}_{n,l}^{e,o}(\mathbf{r},z)=K_{l,n}c^*(0)\left(\frac{q_\gamma^*}{q_\alpha}\right)^{(2n+l)/2}\left[c^{*2}(z)r^2\right]^{l/2}L_n^l\left[c^{*2}(z)r^2\right]\left\{\begin{array}{c}\cos l\varphi\\ \sin l\varphi\end{array}\right\}\mathrm{GB}(\mathbf{r},z,\alpha), \tag{38}$$

where $L_n^l(\cdot)$ are the generalized Laguerre polynomials and $K_{l,n}=\left[2n!/\left(1+\delta_{0,l}\right)\pi\left(n+l\right)!\right]^{1/2}$ is the normalization constant. The biorthogonality relation at any z plane is given by $\left\langle \mathrm{g}\widehat{\mathrm{HG}}_{n_x,n_y},\mathrm{gHG}_{n_x',n_y'}\right\rangle=\delta_{n_xn_x'}\delta_{n_yn_y'}$. In the eigenvalue Eq. (7) $p=2l+n$ for the gLGB. The Fourier transform Eq. (32) of the gLGBs at any transverse z plane is given by

$$\mathrm{g}\widetilde{\mathrm{LG}}_{n,l}^{e,o}(k_x,k_y,z)=K_{l,n}c(0)\frac{\gamma}{2}\left[-i\left(\frac{\gamma}{\alpha^*}\right)^{1/2}\right]^{2n+l}\left[\frac{(\alpha^*+\gamma)^{1/2}}{2}k_t\right]^l \times L_n^l\left(\frac{\alpha^*+\gamma}{4}k_t^2\right)\left\{\begin{array}{c}\cos l\varphi\\ \sin l\varphi\end{array}\right\}\exp\left(-\frac{\gamma q_\gamma(z)}{4}k_t^2\right). \tag{39}$$

3.4. Properties of the Generalized Families

In comparison with the standard Gaussian beams the gGBs have the same envelope function. However, the differences are found in the argument of the Hermite, Laguerre or Ince polynomials. Whereas in the standard Gaussian beams the arguments are real for all coordinates values, for the gGBs they are in general complex, with the exceptions of the waist plane when α and γ are real. Therefore, because the Hermite, Laguerre and Ince polynomials have zeros only for real arguments the gGBs lack of nodal lines in comparison with the standard Gaussian beams, this property can be used to distinguish both modes experimentally.

As we already derive the generalized Gaussian beam are characterized by two complex parameter α and γ, that satisfy $\mathrm{Re}(\gamma)>0$ and $\mathrm{Re}(1/\alpha)+\mathrm{Re}(1/\gamma)>0$. In the special case in which $\alpha\to\infty$ the gGBs converge to the elegant Gaussian beams.[4] The elegant Gaussian beams obtained from the gGBs of Fig. (1) in this transition are shown in Fig (2). In the special case in which $\alpha=\gamma$ the gGBs converge to the standard Gaussian beams, which is the only case where the argument of the polynomials is always real. The standard Gaussian beams obtained from the gGBs of Fig. (1) in this transition are shown in Fig (3), note the lack of nodal lines of the gGBs in comparison to the standards ones.

For comparison purposes, in Figs. 2 and 3 we show the transverse amplitudes and phases of several elegant and standard Gaussian beams at plane $z=z_R$. For both families of elegant and standard beams, odd IGB tend to odd LGBs or HGBs when $\epsilon\to0$ or $\epsilon\to\infty$, respectively. The indexes of the patterns are included within the figure.

4. RELATIONS BETWEEN THE GENERALIZED FAMILIES

Let us now examine the relation between gIGBs and gLGBs and gHGBs. The transition from an $\mathrm{gIG}_{p,m}^{e,o}$ eigenmode into a $\mathrm{gLG}_{n,l}^{e,o}$ eigenmode occurs when the elliptic coordinates tend to the circular cylindrical coordinates, *i.e.* when $\epsilon\to0$. In this limit the indices of both modes are related as follows: $l=m$ and $n=(p-m)/2$. On the other hand, the transition from an $\mathrm{gIG}_{p,m}^{e,o}$ eigenmode into a gHG_{n_x,n_y} eigenmode occurs when $\epsilon\to\infty$, in this case the indices are related as follows: for even IG modes $n_x=m$ and $n_y=p-m$, whereas for odd IG modes $n_x=m-1$ and $n_y=p-m+1$.

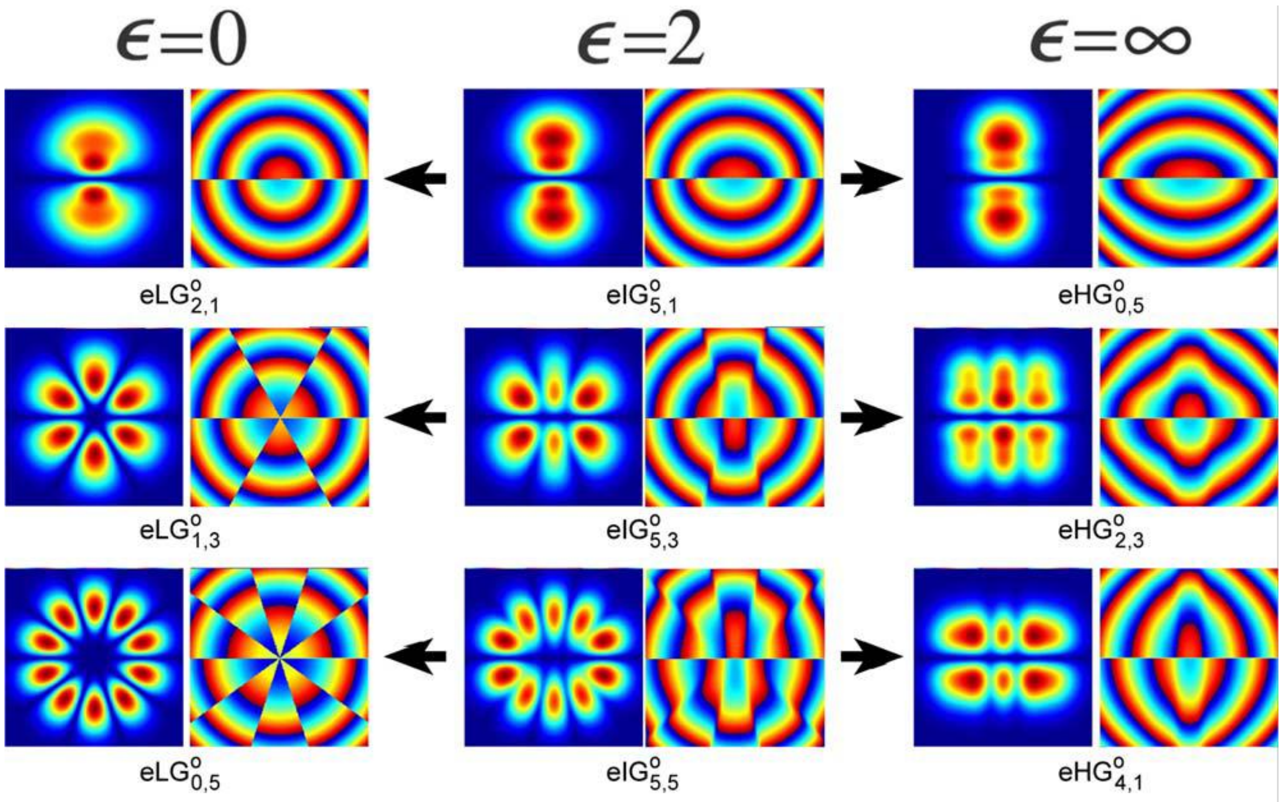

Figure 2. Transverse amplitudes and phases of odd eLGBs, odd eIGBs, eHGBs, at $z = z_R$. The eIGBs correspond to $\epsilon = 2$. eIGB tend to eLGBs or eHGBs when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively. The transverse extent of each window is 1 cm.

The gIG $\Leftrightarrow$ gLG expansions are written as

$$\mathrm{gLG}^{e,o}_{n,l} = \sum_{m=0}^{p=2n+l} B_m \mathrm{gIG}^{e,o}_{p=2n+l,m}, \tag{40}$$

$$\mathrm{gIG}^{e,o}_{p,m} = \sum_{l,n} B_{l,n} \mathrm{gLG}^{e,o}_{n,l}. \tag{41}$$

The coefficients B correspond to the overlap integral between an gIG and a gLG eigenmode, and due to the proposed normalization these coefficients are the same that relate the standar LG and IG beams,[2,3] then we have

$$\left\langle \mathrm{g}\widehat{\mathrm{IG}}^{\sigma}_{p,m}, \mathrm{gLG}^{\sigma'}_{n,l} \right\rangle = \delta_{\sigma'\sigma}\delta_{p,2n+l}(i)^{\delta_{o,e}}\left(-1\right)^{n+l+(p+m)/2}\left[\left(1+\delta_{0,l}\right)(n+l)!n!\right]^{1/2} A^{\sigma}_{(l+\delta_{o,\sigma})/2}\left(\epsilon^{m}_{p}\right), \tag{42}$$

where $A^{\sigma}_{(l+\delta_{o,\sigma})/2}\left(\epsilon^{m}_{p}\right)$ is the $\left(l+\delta_{o,\sigma}\right)/2$-th Fourier coefficient of the $C^{m}_{p}(\cdot)$ or $S^{m}_{p}(\cdot)$ Ince polynomial.[3] Once known the gIG $\Leftrightarrow$ gLG relations, the gIG $\Leftrightarrow$ gHG formulae can be readily obtained by applying the already known LG $\Leftrightarrow$ HG expansions[12] in cascade with the gIG $\Leftrightarrow$ gLG expansions, again due to the proposed normalization of the gGBs and their adjoint beams the expansion formulas between LG $\Leftrightarrow$ HG and gLG $\Leftrightarrow$ gHG have the same coeficients.

Notice that to build up a gLG (or gHG) eigenmode, the constituent gIG eigenmodes must have the same eigenvalue, consequently the expansions between the three families must involve a finite number of degenerate

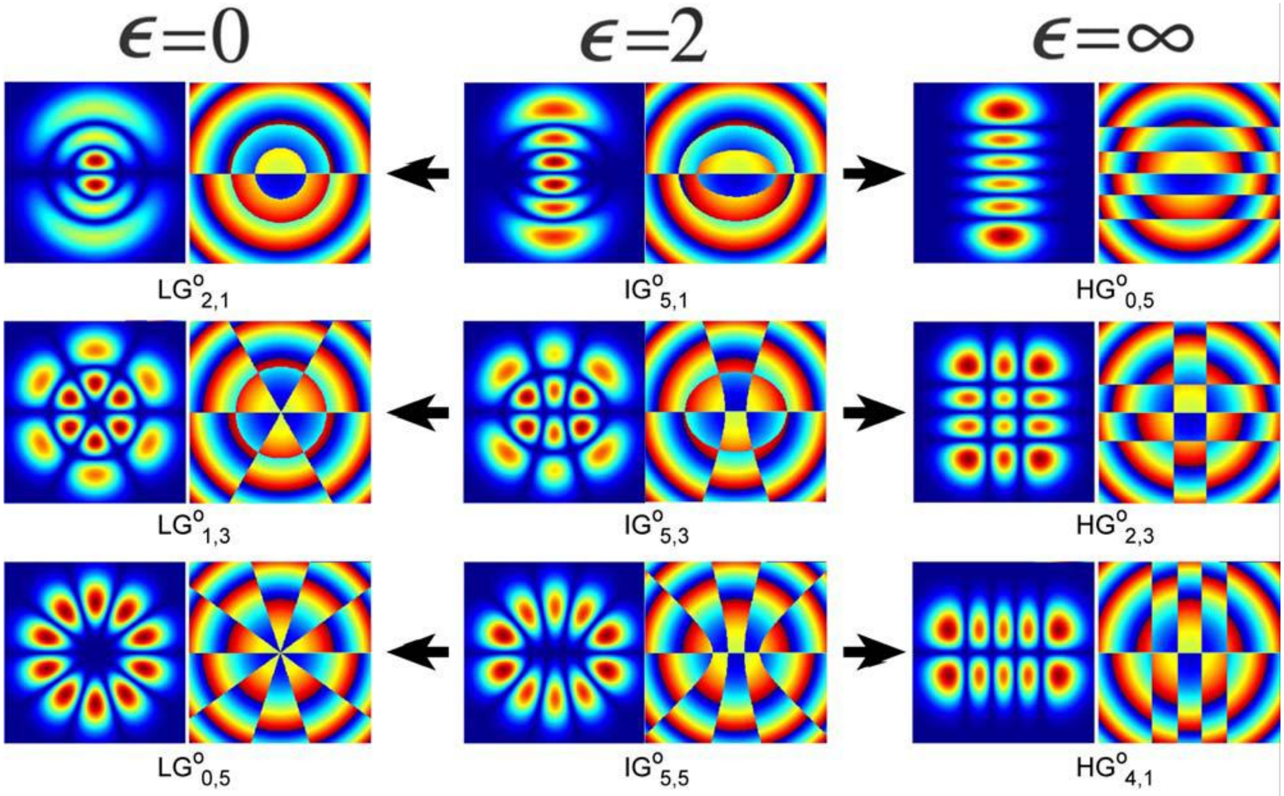

Figure 3. Transverse amplitudes and phases of odd LGBs, od IGBs, HGBs, at $z = z_R$. The IGBs correspond to $\epsilon = 2$. IGB tend to LGBs or HGBs when $\epsilon \to 0$ or $\epsilon \to \infty$, respectively. The transverse extent of each window is 1 cm.

eigenmodes whose indices satisfy the condition $p = 2n + l = n_x + n_y$, for a given p. It is appropriate then to split each family of gLG, gIG and gHG eigenmodes into subsets of degenerate eigenmodes that share the same eigenvalue and parity about the positive x-axis. In group theory language we can say that each subset of modes with the same value of p corresponds to an irreducible representation of $SU(2)$. Let $g\mathbf{L}_p^\sigma$, $g\mathbf{I}_p^\sigma$, and $g\mathbf{H}_p^\sigma$ be the subsets of the (even/odd) gLG, gIG or gHG eigenmodes whose eigenvalue is p, respectively. It is not difficult to see that each subset is composed by

$$N_p = \begin{cases} (p + 2\delta_{\sigma,e})/2, & \text{if } p \text{ is even} \\ (p+1)/2, & \text{if } p \text{ is odd} \end{cases} \tag{43}$$

degenerate eigenmodes that form a complete sub-basis of orthonormal eigenmodes under which any field with eigenvalue p and given parity can be expanded. Therefore any eigenmode of a given subset (e.g. an $\mathrm{gIG}_{p,m}^\sigma$) can be constructed as a linear superposition of the N_p eigenmodes of any of the other two subsets (e.g. $\mathrm{gLG}_{n,l}^\sigma$ or gHG_{n_x,n_y}). The considerations discussed above establish the equivalence between the gIG, gHG and gLG.

The linear relations between subsets $g\mathbf{L}_p^\sigma$, $g\mathbf{I}_p^\sigma$, and $g\mathbf{H}_p^\sigma$ can be written in a matrix notation as follows

$$g\mathbf{I}_p^\sigma = \left[{}_{LI}\mathbf{T}_p^\sigma \right] g\mathbf{L}_p^\sigma \,, \tag{44}$$

$$g\mathbf{H}_p^\sigma = \left[{}_{IH}\mathbf{T}_p^\sigma \right] g\mathbf{I}_p^\sigma \,, \tag{45}$$

$$g\mathbf{L}_p^\sigma = \left[{}_{HL}\mathbf{T}_p^\sigma \right] g\mathbf{H}_p^\sigma \,, \tag{46}$$

where the $N_p \times N_p$ transformation matrices $\left[{}_{AB}\mathbf{T}_p^\sigma \right]$ are real unitary matrices that satisfy $\left[{}_{AB}\mathbf{T}_p^\sigma \right]^{-1} = \left[{}_{AB}\mathbf{T}_p^\sigma \right]^T \equiv \left[{}_{BA}\mathbf{T}_p^\sigma \right]$, and whose columns (and rows) form a basis of N_p orthonormal vectors for the N_p-

th dimensional vector space. Since $\left[{}_{HI}\mathbf{T}_p^\sigma\right] = \left[{}_{LI}\mathbf{T}_p^\sigma\right]\left[{}_{HL}\mathbf{T}_p^\sigma\right]$ only two matrices in Eqs. (44) to (46) are independent.

5. CONCLUSION

In conclusion, we have presented a detailed analysis of the tree families of generalized Gaussian beams, which are the generalized Hermite, Laguerre, and Ince Gaussian beams. A simple relation between the gGBs and their adjoint beams was derived. The normalization constant that make the gGBs and their adjoint beams biorthonormal were obtained. Each family of generalized Gaussian beams includes the standard and elegant corresponding families as particular cases when the parameters of the generalized families are chosen properly. The generalized Hermite Gaussian and Laguerre Gaussian beams correspond to limiting cases of the generalized Ince Gaussian beams when the ellipticity parameter of the latter tends to infinity or to zero, respectively. The expansion formulas among the three generalized families and their Fourier transforms were also presented.

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