Robust Controller Design for Stochastic Nonlinear Systems via Convex Optimization

Hiroyasu Tsukamoto, *Member, IEEE*, and Soon-Jo Chung, *Senior Member, IEEE*

***Abstract*—This paper presents ConVex optimization-based Stochastic steady-state Tracking Error Minimization (CV-STEM), a new state feedback control framework for a class of Itô stochastic nonlinear systems and Lagrangian systems. Its strength lies in computing the control input by an optimal contraction metric, which greedily minimizes an upper bound of the steady-state mean squared tracking error of the system trajectories. Although the problem of minimizing the bound is nonlinear, its equivalent convex formulation is proposed utilizing state-dependent coefficient parameterizations of the nonlinear system equation. It is shown using stochastic incremental contraction analysis that the CV-STEM provides a sufficient guarantee for exponential boundedness of the error for all time with L2-robustness properties. For the sake of its sampling-based implementation, we present discrete-time stochastic contraction analysis with respect to a state- and time-dependent metric along with its explicit connection to continuous-time cases. We validate the superiority of the CV-STEM to PID, H-infinity, and given nonlinear control for spacecraft attitude control and synchronization problems.**

***Index Terms*—Stochastic optimal control, Optimization algorithms, Robust control, Nonlinear systems, LMIs.**

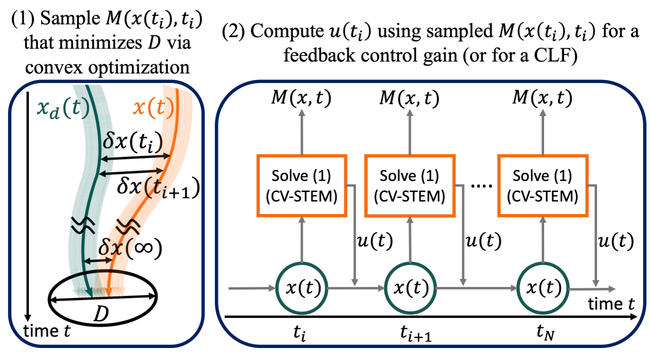

Fig. 1: Illustration of the CV-STEM control: $M(x, t)$ denotes the optimal contraction metric; $x(t)$ and $x_d(t)$ are controlled and desired system trajectories; $u(t)$ is the control input computed by $M(x, t)$ (see Sec. III for details).

I. Introduction

STABLE and optimal feedback control of Itô stochastic nonlinear systems [1] is an important, yet challenging problem in designing autonomous robotic explorers operating with sensor noise and external disturbances. Since the probability density function of stochastic processes governed by Itô stochastic differential equations exhibits non-Gaussian behavior characterized by the Fokker-Plank equation [1], [2], feedback control schemes developed for deterministic nonlinear systems could fail to meet control performance specifications in the presence of stochastic disturbances.

A. Contributions

The main purpose of this paper is to propose ConVex optimization-based Stochastic steady-state Tracking Error Minimization (CV-STEM), a new framework to design an optimal contraction metric for feedback control of Itô stochastic nonlinear systems and stochastic Lagrangian systems depicted in Fig. 1. Contrary to Lyapunov theory, which gives a sufficient condition for exponential convergence, the existence of a contraction metric leads to a necessary and sufficient characterization of exponential incremental stability of nonlinear system trajectories [3], [4]. We explore this approach further to obtain an optimal contraction metric for controlling Itô stochastic nonlinear systems. This paper builds upon our prior work [5] but provides more rigorous proofs and explanations on how we convexify the problem of minimizing D on Fig. 1 in a mean squared sense. We also investigate its stochastic incremental stability properties and the impact of sampling-based implementation on its control performance both in detail, introducing some additional theorems and simulation results. The construction and contributions of our proposed method are summarized as follows.

1) The CV-STEM design is based on a convex combination of multiple State-Dependent Coefficient (SDC) forms of a nonlinear system equation (i.e. $f(x,t)$ written as $A(x,t)x$ [6]–[8], where $A(x,t)$ is not necessarily unique). The main advantage of our control synthesis algorithm lies in solving an optimization problem, the objective of which is to find an optimal contraction metric that greedily minimizes an upper bound of the steady-state mean squared tracking error of Itô stochastic nonlinear system trajectories, and thereby construct an optimal feedback control gain and Control Lyapunov Function (CLF) [9]–[11] (see Fig. 1). Although the problem of minimizing the bound is originally nonlinear, we reformulate it as an equivalent convex optimization problem with the State-Dependent Riccati Inequality (SDRI) constraint expressed as an LMI [12], so we can use various computationally-efficient numerical methods [12]–[15]. We also propose one way to utilize non-unique choices of SDC forms for verifying the controllability of the system. This result is a significant improvement over the observer design [16], whose optimization-cost function uses a linear combination of observer parameters

The authors are with the Graduate Aerospace Laboratories (GALCIT), California Institute of Technology, 1200 E California Blvd, Pasadena, CA, USA. E-mail: {htsukamoto, sjchung}@caltech.edu, Code: https://github.com/astrohiro/cvstem.

without accounting for the contraction constraint, which we express as an LMI [12] in this paper. This approach is further extended to the control of stochastic Lagrangian systems with a nominal exponentially stabilizing controller, and its superiority to the prior work [17], [18], PID, and $\mathcal{H}_\infty$ control [19]–[21] is shown using results of numerical simulations on spacecraft attitude control and synchronization.

2) It is proven using stochastic incremental contraction analysis that the trajectory under the CV-STEM feedback control exponentially converges to the desired trajectory in a mean squared sense with a non-vanishing error term (which will be minimized as explained above). It is also shown that the controller is robust against external deterministic disturbances which often appear in parametric uncertain systems, and that the tracking error has a finite $\mathcal{L}_2$ gain with respect to the noise and disturbances acting on the system. We note that the mean-square bound does not imply the asymptotic almost-sure bounds although finite time bounds could be obtained [1], [22], as the CV-STEM-based Lyapunov function is not a supermartingale due to the non-vanishing steady-state error term.

3) Discrete-time stochastic incremental contraction analysis with respect to a state- and time-dependent metric is derived for studying the effect of sampling-based implementation of the CV-STEM on its control performance. It is proven that stochastic incremental stability of discrete-time systems reduces to that of continuous-time systems if the time interval is sufficiently small. It is shown in the numerical simulations that the CV-STEM sampling period Δt can be relaxed to $\Delta t \leq 25$ (s) for spacecraft attitude control and $\Delta t \leq 350$ (s) for spacecraft tracking and synchronization control without impairing its performance.

4) Some extensions of the CV-STEM are proposed to explicitly incorporate input constraints and to avoid solving the convex optimization problem at every time instant.

B. Related Work

CLFs [9]–[11] as well as feedback linearization [11], [23], [24] are among the most widely used tools for controlling nonlinear systems perturbed by deterministic disturbances. As there is no general analytical scheme for finding a CLF, several techniques are proposed to find them utilizing some special structure of the systems in question [25]–[29]. The state-dependent Riccati equation method [6]–[8] can also be viewed as one of these techniques and is applicable to systems that are written in SDC linear structure. Building on these ideas for deterministic systems, a stochastic counterpart of the Lyapunov methods is proposed in [30] to design CLF-based state and output feedback control of stochastic nonlinear systems [31], [32]. For a class of strict-feedback and output-feedback stochastic nonlinear systems, there exists a more systematic way of asymptotic stabilization in probability using a backstepping-based controller [33], [34]. However, one drawback of these approaches is that they are primarily directed toward stability with some implicit inverse optimality guarantees.

Some theoretical methodologies have been developed to explicitly incorporate optimality into their feedback control formulation. These include $\mathcal{H}_\infty$ control [20], [21], [35], which attempts to minimize the $\mathcal{H}_\infty$ norm for the sake of optimal disturbance attenuation. Although it is originally devised for linear systems [36]–[41], its nonlinear analogues are obtained in [20], [21] and then expanded to stochastic nonlinear systems [19] unifying the results on the $\mathcal{L}_2$ gain analysis based on the Hamilton-Jacobi equations and inequalities [11]. Although we could design feedback control schemes optimally for specific types of systems such as Hamiltonian systems with stochastic disturbances [42] or linearized and discretized stochastic nonlinear systems [43], finding the solution to the stochastic nonlinear state feedback $\mathcal{H}_\infty$ optimal control problem is not trivial in general.

The CV-STEM addresses this issue by numerically sampling an optimal contraction metric and CLF that greedily minimize an upper bound of the steady-state mean squared tracking error of Itô stochastic nonlinear system trajectories. We select this as an objective function, instead of integral objective functions which often appear in optimal control problems, as it gives us an exact convex optimization-based control synthesis algorithm. Also, since the problem has the SDRI as its constraint, the CV-STEM control is robust against both deterministic and stochastic disturbances and ensures that the tracking error is exponentially bounded for all time. We remark that this approach is not intended to supersede but to be utilized on top of existing methodologies on constructing desired control inputs using stochastic nonlinear optimal control techniques [1], [44]–[47] as this is a type of feedback control scheme. In particular, stochastic model predictive control [48], [49] with guaranteed stability [50], [51] *assumes* the existence of a stochastic CLF, whilst our approach explicitly *constructs* an optimal CLF which could be used for the stochastic CLF with some modifications on the non-vanishing error term in our formulation.

The tool we use for analyzing incremental stability [4] in this paper is contraction analysis [3], [52], [53], where its stochastic version is derived in [16], [22]. Contraction analysis for discrete-time and hybrid systems is provided in [3], [54], [55] and its stochastic counterpart is investigated in [56] with respect to a state-independent metric. In this paper, we describe discrete-time incremental contraction analysis with respect to a state- and time-dependent metric. Since the differential (virtual) dynamics of δx used in contraction analysis is a Linear Time-Varying (LTV) system, global exponential stability can be studied using a quadratic Lyapunov function of δx, $V = \delta x^T M(x,t)\delta x$ [3], as opposed to the Lyapunov technique where V could be any function of x. Therefore, designing V reduces to finding a positive definite metric $M(x,t)$ [28], [57], which enables the aforementioned convex optimization-based control of Itô stochastic nonlinear systems.

C. Paper Organization

The rest of this paper is organized as follows. Section II introduces stochastic incremental contraction analysis and presents its discrete-time version with a state- and time-dependent metric. In Sec. III, the CV-STEM control for Itô stochastic nonlinear systems is presented and its stability is analyzed using contraction analysis. In Sec. IV, this approach is extended to the control of stochastic Lagrangian systems. Section V proposes several extensions of the CV-STEM control synthesis. The aforementioned two simulation examples are reported in Sec. VI. Section VII concludes the paper.

D. Notation

For a vector $x \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times m}$, we let $\|x\|$, δx, $\partial_\mu x$, $\|A\|$, $\|A\|_F$, $\mathrm{Im}(A)$, $\mathrm{Ker}(A)$, A^+, and $\kappa(A)$ denote the Euclidean norm, infinitesimal variation of x, partial derivative of x with respect to μ, induced 2-norm, Frobenius norm, image of A, kernel of A, Moore–Penrose inverse, and condition number, respectively. For a square matrix A, we use the notation $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ for the minimum and maximum eigenvalues of A, $\mathrm{Tr}(A)$ for the trace of A, $A \succ 0$, $A \succeq 0$, $A \prec 0$, and $A \preceq 0$ for the positive definite, positive semi-definite, negative definite, negative semi-definite matrices, respectively, and $\mathrm{sym}(A) = A + A^T$. For a vector $x \in \mathbb{R}^n$ and a positive definite matrix $A \in \mathbb{R}^{n\times n}$, we denote a norm $\sqrt{x^TAx}$ as $\|x\|_A$. Also, $I \in \mathbb{R}^{n\times n}$ represents the identity matrix, $E[\cdot]$ denotes the expected value operator, and $E_x[\cdot]$ denotes the conditional expected value operator with x fixed. The $\mathcal{L}_p$ norm in the extended space $\mathcal{L}_{pe}$, $p \in [1,\infty]$, is defined as $\|(y)_\tau\|_{\mathcal{L}_p} = \left(\int_0^\tau \|y(t)\|^p\right)^{1/p} < \infty$ for $p \in [1,\infty)$ and $\|(y)_\tau\|_{\mathcal{L}_\infty} = \sup_{t\geq 0}\|(y(t))_\tau\| < \infty$ for $p = \infty$, where $(y(t))_\tau$ is a truncation of $y(t)$, i.e., $(y(t))_\tau = 0$ for $t > \tau$ and $(y(t))_\tau = y(t)$ for $0 \leq t \leq \tau$ with $\tau \in [0,\infty)$.

II. Stochastic Incremental Stability via Contraction Analysis

We introduce contraction analysis that will be used for stability analysis in Sec. III and IV. We also present new theorems for analyzing stochastic incremental stability of discrete-time nonlinear systems with respect to a state- and time-dependent Riemannian metric, along with its explicit connection to contraction analysis of continuous-time systems.

Contraction analysis studies incremental stability [4], i.e., stability of system trajectories with respect to each other by means of differential (virtual) dynamics unlike Lyapunov theory. This allows us to utilize approaches for LTV systems theory, yielding a convex optimization-based framework for optimal Lyapunov function construction in Sec. III and IV.

A. Continuous-time Dynamical Systems

Consider the following continuous-time nonlinear system and its virtual dynamics:

$$\dot{x} = f(x,t),\quad \delta\dot{x} = \frac{\partial f(x,t)}{\partial x}\delta x \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$, $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, and $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$.

Lemma 1: The system (1) is contracting (i.e. all the solution trajectories exponentially converge to a single trajectory globally from any initial condition), if there exists a uniformly positive definite metric $M(x,t) = \Theta(x,t)^T\Theta(x,t)$, $M(x,t) \succ 0,\ \forall x,t$, with a smooth coordinate transformation of the virtual displacement $\delta z = \Theta(x,t)\delta x$ s.t.

$$\dot{M}(x,t) + \mathrm{sym}\left(M(x,t)\frac{\partial f}{\partial x}\right) \preceq -2\gamma_c M(x,t),\ \forall x,t \tag{2}$$

where $\gamma_c > 0$. If the system (1) is contracting, then we have $\|\delta z(t)\| = \|\Theta(x,t)\delta x(t)\| \leq \|\delta z(0)\|e^{-\gamma_c t}$.

Proof: See [3], [58]. ■

Next, consider the nonlinear system (1) with stochastic perturbation given by the Itô stochastic differential equation

$$dx = f(x,t)dt + G(x,t)dW,\ x(0) = x_0 \tag{3}$$

where $G : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times d}$ is a matrix-valued function, $W(t)$ is a d-dimensional Wiener process, and x_0 is a random variable independent of $W(t)$ [59]. In this paper, we assume that $\exists L_1 > 0,\ \forall t,\ \forall x_1,\ x_2 \in \mathbb{R}^n$ s.t. $\|f(x_1,t) - f(x_2,t)\| + \|G(x_1,t) - G(x_2,t)\|_F \leq L_1\|x_1 - x_2\|$ and $\exists L_2 > 0,\ \forall t,\ \forall x_1 \in \mathbb{R}^n$ s.t. $\|f(x_1,t)\|^2 + \|G(x_1,t)\|_F^2 \leq L_2(1 + \|x_1\|^2)$ for the sake of existence and uniqueness of the solution to (3). Now, consider the following two systems with trajectories $\xi_1(t)$ and $\xi_2(t)$ driven by two independent Wiener processes $W_1(t)$ and $W_2(t)$:

$$d\xi = \begin{bmatrix} f(\xi_1,t) \\ f(\xi_2,t)\end{bmatrix}dt + \begin{bmatrix} G_1(\xi_1,t) & 0 \\ 0 & G_2(\xi_2,t)\end{bmatrix}\begin{bmatrix} dW_1 \\ dW_2\end{bmatrix} \tag{4}$$

where $\xi(t) = [\xi_1(t)^T, \xi_2(t)^T]^T \in \mathbb{R}^{2n}$. The following theorem analyzes stochastic incremental stability of the two trajectories $\xi_1(t)$ and $\xi_2(t)$ with respect to each other in the presence of stochastic noise. The trajectories of (3) are parameterized as $x(0,t) = \xi_1$ and $x(1,t) = \xi_2$. Also, we define $G(x,t)$ as $G(x(0,t),t) = G_1(\xi_1,t)$ and $G(x(1,t),t) = G_2(\xi_2,t)$.

Theorem 1: Suppose that there exist bounded positive constants $\underline{m}$, $\overline{m}$, g_1, g_2, $\overline{m}_x$, and $\overline{m}_{x^2}$ s.t. $\underline{m} \leq \|M(x,t)\| \leq \overline{m}$, $\|G_1(x,t)\|_F \leq g_1$, $\|G_2(x,t)\|_F \leq g_2$, $\|\partial(M_{ij})/\partial x\| \leq \overline{m}_x$, and $\left\|\partial^2(M_{ij})/\partial x^2\right\| \leq \overline{m}_{x^2},\ \forall x,t$. Suppose also that (2) holds (i.e. the deterministic system (1) is contracting). Consider the generalized squared length with respect to a Riemannian metric $M(x(\mu,t),t)$ defined by

$$V(x,\partial_\mu x,t) = \int_0^1 \frac{\partial x}{\partial \mu}^T M(x(\mu,t),t)\frac{\partial x}{\partial \mu}d\mu \tag{5}$$

s.t. $V(x,\partial_\mu x,t) \geq \underline{m}\|\xi_1 - \xi_2\|^2$. Then we have

$$\mathscr{L}V \leq -2\gamma_1 V + \underline{m}C_c \tag{6}$$

for $\gamma_1 = \gamma_c - ((g_1^2 + g_2^2)/2\underline{m})(\varepsilon_c\overline{m}_x + \overline{m}_{x^2}/2)$ and $C_c = (\overline{m}/\underline{m} + \overline{m}_x/(\varepsilon_c\underline{m}))(g_1^2 + g_2^2)$, where $\mathscr{L}$ is an infinitesimal differential generator defined in [16], γ_c is the contraction rate for the deterministic system (1), and $\varepsilon_c > 0$ is an arbitrary constant. Further, if we have $\gamma_1 > 0$, (6) implies that the mean squared distance between the two trajectories of (4), whose initial conditions given by a probability distribution $p(a_0,b_0)$ are independent of $W_1(t)$ and $W_2(t)$, is exponentially bounded as follows:

$$E\left[\|\xi_1(t) - \xi_2(t)\|^2\right] \leq \frac{C_c}{2\gamma_1} + \frac{E[V(x(0),\partial_\mu x(0),0)]e^{-2\gamma_1 t}}{\underline{m}}. \tag{7}$$

Proof: Using the property $\mathrm{Tr}(AB) \leq \|A\| \mathrm{Tr}(B)$ for $A, B \succeq 0$, we have $\mathrm{Tr}(G_i(\xi_i,t)^T M(\xi_i,t) G_i(\xi_i,t)) \leq \overline{m} g_i^2$. This relation along with the proof given in Lemma 2 of [16] completes the derivation of (7). ■

Remark 1: The contraction rate γ_1 and uncertainty bound C_c depend on the choice of an arbitrary constant ε_c. One way to select ε_c is to solve $dF/d\varepsilon_c = 0$ with $F(\varepsilon_c) = C_c/(2\gamma_1)$, whose solution minimizes the steady-state bound $F(\varepsilon_c)$ with the constraint $\gamma_1 > 0$ [16]. Also, C_c is a function of $\overline{m}/\underline{m}$ and this fact facilitates the convex optimization-based control synthesis in Sec. III and IV.

B. Main Result 1: Connection between Continuous and Discrete Stochastic Incremental Contraction Analysis

We have a similar result to Lemma 1 for the following discrete-time nonlinear system and its virtual dynamics:

$$x_{k+1} = f_k(x_k, k), \quad \delta x_{k+1} = \frac{\partial f_k(x_k, k)}{\partial x_k} \delta x_k \tag{8}$$

where $x_k \in \mathbb{R}^n$ and $f_k : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^n$.

Lemma 2: The system (8) is contracting if there exists a uniformly positive definite metric $M_k(x_k, k) = \Theta_k(x_k, k)^T \Theta_k(x_k, k)$, $M_k(x_k, k) \succ 0,\ \forall x_k, k$, with a smooth coordinate transformation of the virtual displacement $\delta z_k = \Theta_k(x_k, k) \delta x_k$ s.t.

$$\frac{\partial f_k}{\partial x_k}^T M_{k+1}(x_{k+1}, k+1) \frac{\partial f_k}{\partial x_k} \preceq (1 - \gamma_d) M_k(x_k, k),\ \forall x_k, k \tag{9}$$

where $\gamma_d \in (0, 1)$. If the system (8) is contracting, then we have $\|\delta z_k\| = \|\Theta_k(x_k, k) \delta x_k\| \leq \|\delta z_0\| (1 - \gamma_d)^{\frac{k}{2}}$.

Proof: See [3], [55], [58]. ■

We now present a discrete-time version of Theorem 1, which can be extensively used for proving stability of discrete-time and hybrid stochastic nonlinear systems, along with known results for deterministic systems [54], [55]. Consider the discrete-time nonlinear system (8) with stochastic perturbation modeled by the stochastic difference equation

$$x_{k+1} = f_k(x_k, k) + G_k(x_k, k) w_k \tag{10}$$

where $G_k : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^{n \times d}$ is a matrix-valued function and w_k is a d-dimensional sequence of zero mean uncorrelated normalized Gaussian random variables. Consider the following two systems with trajectories $\xi_{1,k}$ and $\xi_{2,k}$ driven by two independent stochastic perturbation $w_{1,k}$ and $w_{2,k}$:

$$\xi_{k+1} = \begin{bmatrix} f_k(\xi_{1,k}, k) \\ f_k(\xi_{2,k}, k) \end{bmatrix} + \begin{bmatrix} G_{1,k}(\xi_{1,k}, k) & 0 \\ 0 & G_{2,k}(\xi_{2,k}, k) \end{bmatrix} \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix} \tag{11}$$

where $\xi_k = [\xi_{1,k}^T, \xi_{2,k}^T]^T \in \mathbb{R}^{2n}$. The following theorem analyzes stochastic incremental stability for discrete-time nonlinear systems, but we remark that this is different from [56], [60] in that the stability is studied in a differential sense and its Riemannian metric is state- and time-dependent. We parameterize x_k and G_k in (10) as $x_k(\mu = 0) = \xi_{1,k}$, $x_k(\mu = 1) = \xi_{2,k}$, $G_k(x_k(\mu = 0), k) = G_{1,k}(\xi_{1,k}, k)$, and $G_k(x_k(\mu = 1), k) = G_{2,k}(\xi_{2,k}, k)$.

Theorem 2: Suppose that the system (11) has the following bounds, $\underline{m} I \preceq M_k(x_k, k) \preceq \overline{m} I,\ \forall x_k, k,\ \|G_{1,k}(\xi_{1,k}, k)\|_F \leq g_{1d}$, and $\|G_{2,k}(\xi_{2,k}, k)\|_F \leq g_{2d},\ \forall \xi_{1,k}, \xi_{2,k}, k$, where $\overline{m}$, g_{1d}, and g_{2d} are bounded positive constants. Suppose also that (9) holds for the discrete-time deterministic system (8) and there exists $\gamma_2 \in (0, 1)$ s.t. $\gamma_2 \leq 1 - (\overline{m}/\underline{m})(1 - \gamma_d)$, where γ_d is the contraction rate of (8). Consider the generalized squared length with respect to a Riemannian metric $M_k(x_k(\mu), k)$ defined as

$$v_k(x_k, \partial_\mu x_k, k) = \int_0^1 \frac{\partial x_k}{\partial \mu}^T M_k(x_k(\mu), k) \frac{\partial x_k}{\partial \mu} d\mu \tag{12}$$

s.t. $v_k(x_k, \partial_\mu x_k, k) \geq \underline{m} \|\xi_{1,k} - \xi_{2,k}\|_2^2$. Then the mean squared distance between the two trajectories of the system (11) is bounded as follows:

$$E_{\zeta_0}\left[\|\xi_{1,k} - \xi_{2,k}\|^2\right] \leq \frac{1 - \tilde{\gamma}_d^k}{1 - \tilde{\gamma}_d} C_d + \frac{\tilde{\gamma}_d^k}{\underline{m}} v_0. \tag{13}$$

where $C_d = (\overline{m}/\underline{m})(g_{1d}^2 + g_{2d}^2)$ and $\tilde{\gamma}_d = 1 - \gamma_2 \in (0, 1)$. The subscript ζ_0 means that x_0, $\partial_\mu x_0$, and t_0 are fixed.

Proof: Consider a Lyapunov-like function v_{k+1} in (12), where we use $v_k = v_k(x_k, \partial_\mu x_k, k)$ and $M_k = M_k(x_k, k)$ for notational simplicity. Using the bounds along with (10), we have

$$\begin{aligned} v_{k+1} &\leq \overline{m} \int_0^1 \left\| \frac{\partial f_k}{\partial x_k} \frac{\partial x_k}{\partial \mu} + \frac{\partial G_k}{\partial \mu} w_k \right\|^2 d\mu \qquad (14) \\ &\leq \frac{\overline{m}}{\underline{m}} (1 - \gamma_d) \int_0^1 \frac{\partial x_k}{\partial \mu}^T M_k \frac{\partial x_k}{\partial \mu} d\mu \\ &+ \overline{m} \int_0^1 \left(2 \frac{\partial x_k}{\partial \mu}^T \frac{\partial f_k}{\partial x_k}^T \frac{\partial G_k}{\partial \mu} w_k + w_k^T \frac{\partial G_k}{\partial \mu}^T \frac{\partial G_k}{\partial \mu} w_k \right) d\mu \end{aligned}$$

where f_k and G_k denote $f_k(x_k, k)$ and $G_k(x_k, k)$, respectively. Taking the conditional expected value of (14) with x_k, $\partial_\mu x_k$, and k fixed, we have that

$$\begin{aligned} &E_{\zeta_k}[v_{k+1}] \leq \gamma_m v_k + \overline{m} E_{\zeta_k}\left[\int_0^1 w_k^T \frac{\partial G_k}{\partial \mu}^T \frac{\partial G_k}{\partial \mu} w_k d\mu \right] \\ &\leq \gamma_m v_k + \sum_{i=1,2} \overline{m} E_{\zeta_k}\left[\mathrm{Tr}\left(w_{i,k} w_{i,k}^T G_{i,k}^T G_{i,k} \right) \right] \\ &\leq \gamma_m v_k + \overline{m} \sum_{i=1,2} \mathrm{Tr}\left(G_{i,k}^T G_{i,k} \right) \leq \tilde{\gamma}_d v_k + \underline{m} C_d. \qquad (15) \end{aligned}$$

where $\gamma_m = \overline{m}/\underline{m}(1 - \gamma_d)$ and x_k, $\partial_\mu x_k$, and k are denoted as ζ_k. Since there exists $\gamma_2 \in (0, 1)$ s.t. $\gamma_m \leq 1 - \gamma_2$, the property $E_{\zeta_{k-2}}[v_k] = E_{\zeta_{k-2}}[E_{\zeta_{k-1}}[v_k]]$ gives us that

$$E_{\zeta_{k-2}}[v_k] \leq \tilde{\gamma}_d^2 v_{k-2} + \underline{m} C_d + \underline{m} C_d \tilde{\gamma}_d \tag{16}$$

where $\tilde{\gamma}_d = 1 - \gamma_2$. Continuing this operation with the relation $\underline{m} E_{\zeta_0}\left[\|\xi_{1,k} - \xi_{2,k}\|^2\right] \leq E_{\zeta_0}[v_k]$ yields

$$E_{\zeta_0}\left[\|\xi_{1,k} - \xi_{2,k}\|^2\right] - \frac{\tilde{\gamma}_d^k}{\underline{m}} v_0 \leq C_d \sum_{i=0}^{k-1} \tilde{\gamma}_d^i = \frac{1 - \tilde{\gamma}_d^k}{1 - \tilde{\gamma}_d} C_d.$$

Rearranging terms gives (13). ■

Let us now consider the case where the time interval $\Delta t = t_{k+1} - t_k$ is sufficiently small, i.e., $\Delta t \gg (\Delta t)^2$. Then the continuous-time stochastic system (3) can be discretized as

$$\begin{aligned} x_{k+1} =& x_k + \int_{t_k}^{t_{k+1}} f(x(t),t)dt + G(x(t),t)dW(t) \\ \simeq& x_k + f(x_k,t_k)\Delta t + G(x_k,t_k)\Delta W_k \end{aligned} \tag{17}$$

where $x_k = x(t_k)$, $\Delta W_k = \sqrt{\Delta t} w_k$, and w_k is a d-dimensional sequence of zero mean uncorrelated normalized Gaussian random variables. When $\Delta t \gg (\Delta t)^2$, $f_k(x_k,k)$ and $G_k(x_k,k)$ in (10) can be approximated as $f_k(x_k,k) = x_k + f(x_k,t_k)\Delta t$ and $G_k(x_k,k) = \sqrt{\Delta t} G(x_k,t_k)$. In this situation, we have the following theorem that connects stochastic incremental stability of discrete-time systems with that of continuous-time systems.

Theorem 3: Suppose that (15) in Theorem 2 holds with $\tilde{\gamma}_d = 1 - \gamma_2 \in (0,1)$. Then the expected value of v_{k+1} up to first order in Δt is given as $E_{\zeta_k}[v_{k+1}] = v_k + \Delta t \mathscr{L} v_k$, where $\mathscr{L}$ is an infinitesimal differential generator defined in [16]. Furthermore, the following inequality holds:

$$\mathscr{L} v_k(x_k,\partial_\mu x_k,t_k) \leq -\frac{\gamma_2}{\Delta t} v_k(x_k,\partial_\mu x_k,t_k) + \underline{m}\tilde{C}_c \tag{18}$$

where $\tilde{C}_c$ is a positive constant given as

$$\tilde{C}_c = \frac{C_d}{\Delta t} = \frac{\overline{m}}{\underline{m}\Delta t}(g_{1d}^2 + g_{2d}^2) = \frac{\overline{m}}{\underline{m}}(g_1^2 + g_2^2) \tag{19}$$

with g_1 and g_2 defined in Theorem 1.

Proof: M_{k+1} up to first order in Δt is written as

$$\begin{aligned} M_{k+1} =& \frac{\partial M_k}{\partial t_k}\Delta t + \sum_{i=1}^{n} \frac{\partial M_k}{\partial (x_k)_i}(f_{c,k}\Delta t + G_{c,k}\Delta W_k)_i \\ &+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 M_k}{\partial (x_k)_i \partial (x_k)_j}(G_{c,k}\Delta W_k)_i (G_{c,k}\Delta W_k)_j + M_k \end{aligned} \tag{20}$$

where $f_{c,k}$ and $G_{c,k}$ are defined as $f_{c,k} = f(x_k,t_k)$ and $G_{c,k} = G(x_k,t_k)$ for notational simplicity. The subscripts i and j denote the ith and jth element of the corresponding vectors. Similarly, $\partial x_{k+1}/\partial \mu$ up to first order in Δt can be computed as

$$\frac{\partial x_{k+1}}{\partial \mu} = \frac{\partial x_k}{\partial \mu} + \frac{\partial f_{c,k}}{\partial x_k}\frac{\partial x_k}{\partial \mu}\Delta t + \frac{\partial G_{c,k}}{\partial \mu}\Delta W_k. \tag{21}$$

Substituting (20) and (21) into $E_{\zeta_k}[v_{k+1}]$ yields

$$\begin{aligned} E_{\zeta_k}[v_{k+1}] =& E_{\zeta_k}\left[\int_0^1 \frac{\partial x_{k+1}}{\partial \mu}^T M_{k+1} \frac{\partial x_{k+1}}{\partial \mu} d\mu\right] \\ =& v_k + (V_d + V_s)\Delta t + \mathcal{O}(\Delta t^{3/2}) \end{aligned} \tag{22}$$

where V_d and V_s are given by

$$V_d = \int_0^1 \frac{\partial x_k}{\partial \mu}^T \left(\frac{\partial f_{c,k}}{\partial x_k}^T M_k + \dot{M}_k + M_k \frac{\partial f_{c,k}}{\partial x_k}\right)\frac{\partial x_k}{\partial \mu} d\mu \tag{23}$$

with $\dot{M}_k = \partial M_k/\partial t_k + \sum_{i=1}^n (\partial M_k/\partial (x_k)_i) f_{c,k}$ and

$$\begin{aligned} V_s = \int_0^1 \Bigg[& \sum_{i=1}^{n}\sum_{j=1}^{n} (M_k)_{ij} \left(\frac{\partial G_{c,k}}{\partial \mu}\frac{\partial G_{c,k}}{\partial \mu}^T\right)_{ij} \\ &+ 2\frac{\partial (M_k)_i}{\partial (x_k)_j}\frac{\partial x_k}{\partial \mu}\left(G_{c,k}\frac{\partial G_{c,k}}{\partial \mu}^T\right)_{ij} \\ &+ \frac{1}{2}\frac{\partial x_k}{\partial \mu}^T \frac{\partial^2 M_k}{\partial (x_k)_i \partial (x_k)_j}\frac{\partial x_k}{\partial \mu}(G_{c,k}G_{c,k}^T)_{ij}\Bigg] d\mu. \end{aligned} \tag{24}$$

We note that the properties of w_k as a d-dimensional sequence of zero mean uncorrelated normalized Gaussian random variables are used to derive the above equality. Since $V_d + V_s = \mathscr{L} v_k$ where $\mathscr{L}$ is the infinitesimal differential generator, we have $E_{\zeta_k}[v_{k+1}] = v_k + \Delta t \mathscr{L} v_k$. Thus, the condition $E_{\zeta_k}[v_{k+1}] \leq (1-\gamma_2)v_k + \underline{m}C_d$ given by (15) in Theorem 2 reduces to the following inequality:

$$\mathscr{L} v_k(x_k,\partial_\mu x_k,t_k) \leq -\frac{\gamma_2}{\Delta t} v_k(x_k,\partial_\mu x_k,t_k) + \underline{m}\frac{C_d}{\Delta t}. \tag{25}$$

Finally, (25) with the relations $\tilde{C}_c = C_d/\Delta t$ and $G_k(x_k,k) = \sqrt{\Delta t} G(x_k,t_k)$ results in (18) and (19). ■

Remark 2: The positive constant $\tilde{C}_c$ is equal to the positive constant C_c in Theorem 1 when $\overline{m}_x = 0$. This is due to the fact that we used an upper bound of $\|M_k\|$ when obtaining the first line of (14) in Theorem 2.

In practical control applications, we use the same control input at $t = t_k$ for a finite time interval $t \in [t_k, t_{t+1})$. Theorems 1 and 3 indicate that if Δt is sufficiently small, a discrete-time stochastic controller can be viewed as a continuous-time counterpart with contraction rate $2\gamma_1 = \gamma_2/\Delta t$. We will illustrate how to select the sampling period Δt large enough without deteriorating the CV-STEM control performance in Sec. VI. Also, the steady-state mean squared tracking error for both discrete and continuous cases can be expressed as a function of the condition number of the metric $M(x,t)$, which is useful in designing convex optimization-based control synthesis as shall be seen in Sec. III and IV.

III. Main Result 2: CV-STEM Control with Stability and Optimization

This section presents the CV-STEM control for general input-affine nonlinear stochastic systems. We note that this is not for finding an optimal control trajectory and input, which can be used as desired values in the present control design. Incremental stability of this feedback control scheme is analyzed using contraction analysis given in Theorems 1 and 3. Since the differential dynamics of δx used in contraction analysis can be viewed as an LTV system, it enables assuming an optimal differential Lyapunov function of the form $\delta x^T M(x,t)\delta x$ without loss of generality [3], and thereby finding $M(x,t)$ via convex optimization.

In Sec. III-E, we present a convex optimization problem for finding the optimal contraction metric for the CV-STEM control, which greedily minimizes an upper bound of the steady-state mean squared tracking error of Itô stochastic nonlinear system trajectories. It is shown that this problem is equivalent to the original nonlinear optimization problem of minimizing the upper bound.

A. Problem Formulation

Consider the following Itô stochastic nonlinear systems with a control input u, perturbed by a d-dimensional Wiener process $W_u(t)$:

$$\begin{aligned} dx =& f(x,t)dt + B(x,t)udt + G_u(x,t)dW_u \\ dx_d =& f(x_d,t)dt + B(x_d,t)u_d dt. \end{aligned} \tag{26}$$

where $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $B : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times m}$, $G_u : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times d}$, and $x_d : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $u_d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are the desired trajectory and input, respectively. The dynamical system of the desired states is deterministic as x_d and u_d are assumed to be given.

Remark 3: Since $\dot{x}_d - f(x_d,t) \in \operatorname{Im} B(x_d,t)$ holds for a feasible desired trajectory, u_d can be obtained as $u_d = B(x_d,t)^+(\dot{x}_d - f(x_d,t))$ where $(\cdot)^+$ denotes the Moore-Penrose inverse. This is the unique least squares solution (LSS) to $B(x_d,t)u_d = \dot{x}_d - f(x_d,t)$ when $\operatorname{Ker} B(x_d,t) = \{0\}$ and an LSS with the smallest Euclidean norm when $\operatorname{Ker} B(x_d,t) \neq \{0\}$. The desired input u_d can also be found by solving an optimal control problem [1], [44]–[51], [61] and a general system with $\dot{x} = f(x,u)$ can be transformed into an input-affine form by treating $\dot{u}$ as another input.

In the proceeding discussion, we assume that $f(x,t) = 0$ at $x = 0$ and that f is a continuously differentiable function. This allows us to use the following lemma.

Lemma 3: Let Ω be the state set that is a bounded open subset of some Euclidean space s.t. $0 \in \Omega \subseteq \mathbb{R}^n$. Under the assumptions $f(0) = 0$ and $f(x)$ is a continuously differentiable function of x on Ω, there always exists at least one continuous nonlinear matrix-valued function $A(x)$ on Ω s.t. $f(x) = A(x)x$, where $A : \Omega \to \mathbb{R}^{n\times n}$ is found by mathematical factorization and is non-unique when $n > 1$.

Proof: See [8]. ■

Using Lemma 3, (26) is expressed as

$$\begin{aligned} dx =& A(\varrho,x,t)xdt + B(x,t)udt + G_u(x,t)dW_u \\ dx_d =& A(\varrho,x_d,t)x_d dt + B(x_d,t)u_d dt \end{aligned} \tag{27}$$

where $\varrho = (\varrho_1,\cdots,\varrho_{s_1})$ are the coefficients of the convex combination of SDC parameterizations $A_i(x,t)$, i.e.,

$$A(\varrho,x,t) = \sum_{i=1}^{s_1} \varrho_i A_i(x,t). \tag{28}$$

Writing the system dynamics (26) in SDC form provides a design flexibility to mitigate effects of stochastic noise while verifying that the system is controllable as shall be seen later.

B. Feedback Control Design

We consider the following feedback control scheme (to be optimized in Sec. III-E):

$$\begin{aligned} u =& -K(x,t)(x-x_d) + u_d \\ =& -R(x,t)^{-1}B(x,t)^T M(x,t)(x-x_d) + u_d \end{aligned} \tag{29}$$

where $R(x,t) \succ 0$ is a weighting matrix on the input u and $M(x,t)$ is a positive definite matrix which satisfies the following matrix inequality for $\gamma > 0$:

$$\begin{aligned} &\dot{M}(x,t) + \operatorname{sym}(M(x,t)A(\varrho,x,t)) + \gamma M^2(x,t) \\ &- M(x,t)B(x,t)R(x,t)^{-1}B(x,t)^T M(x,t) \preceq 0. \end{aligned} \tag{30}$$

Define $A_{cl}(\varrho,y,t)$, $\Delta A(\varrho,y,t)$, and $\Delta B(y,t)$ [7] as

$$\begin{aligned} A_{cl}(\varrho,y,t) =& A(\varrho,y+x_d,t) - B(y+x_d,t)K(y+x_d,t) \\ \Delta A(\varrho,y,t) =& A(\varrho,y+x_d,t) - A(\varrho,x_d,t) \\ \Delta B(y,t) =& B(y+x_d,t) - B(x_d,t). \end{aligned} \tag{31}$$

Substituting (29) into (27) yields

$$de = f_e(e,t)dt + G_u(e+x_d,t)dW_u \tag{32}$$

where $e = x - x_d$ and

$$f_e(e,t) = A_{cl}(\varrho,e,t)e + \Delta A(\varrho,e,t)x_d + \Delta B(e,t)u_d.$$

Lemma 4: Suppose that the deterministic system is perturbed as follows:

$$\dot{x} = f(x,t) + B(x,t)(u+d). \tag{33}$$

If there exists a positive definite solution $M(x,t)$ to the inequality (30) with $R(x,t) = S(x,t)^2 \succ 0$ and $S(x,t) \succ 0$, then the system with inputs $\mu_1 = S(x,t)d$, $\mu_2 = (\sqrt{2/\gamma})\Delta_d$ and an output $y = (\sqrt{\gamma/2})M(x,t)(x - x_d)$, where $\Delta_d = \Delta A x_d + \Delta B u_d$, is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to 1 for each input μ_1 and μ_2.

Proof: See Appendix A. ■

C. Incremental Stability Analysis

As we discussed earlier in Sec. II, even when a control input at $t = t_k$ is applied during a finite time interval $t \in [t_k, t_{k+1})$, Theorem 3 along with Theorem 2 guarantees that the discrete-time controller leads to an analogous result to the continuous-time case (29) if Δt_k is sufficiently small. Thus, we perform stability analysis for continuous-time dynamical systems. Let us define a deterministic virtual system of (27) as follows:

$$\dot{y} = f_v(y,t) = A_{cl}(\varrho,e,t)y + \Delta A(\varrho,y,t)x_d + \Delta B(y,t)u_d. \tag{34}$$

where (34) has $y = e$ and $y = 0$ as its particular solutions. The virtual dynamics of (34) is expressed as

$$\delta\dot{y} = A_{cl}(\varrho,e,t)\delta y + \phi(\varrho,y,t)\delta y \tag{35}$$

where $\phi(\varrho,y,t) = \partial(\Delta A x_d + \Delta B u_d)/\partial y$. Using $f_v(y,t)$, the virtual system of (32) with respect y is defined as

$$dy = f_v(y(\mu,t),t)dt + G(y(\mu,t),t)dW \tag{36}$$

where $\mu \in [0,1]$ is introduced to parameterize the trajectories $y = e$ and $y = 0$, i.e., $y(\mu = 0,t) = e$, $y(\mu = 1,t) = 0$, $G(y(0,t),t) = G_u(e+x_d,t)$, and $G(y(1,t),t) = 0_{n\times d}$. It can be seen that (36) has $y = e$ and $y = 0$ as its particular solutions because we have

- $f_v = f_e(e,t)$ and $G = G_u(e+x_d,t)$ when $y = e$.

- $f_v = \Delta A(\varrho, 0, t)x_d + \Delta B(0, t)u_d = 0$ and $G = 0_{n\times d}$ when $y = 0$.

Now we introduce the following theorem for exponential boundedness of the mean squared tracking error of system trajectories (27).

Theorem 4: Suppose there exist bounded positive constants $\underline{m}$, $\overline{m}$, $\overline{m}_x$, $\overline{m}_{x^2}$, and g_u s.t. $\underline{m} \leq \|M(x,t)\| \leq \overline{m}$, $\|\partial(m_{ij})/\partial x\| \leq \overline{m}_x$, $\left\|\partial^2(m_{ij})/\partial x^2\right\| \leq \overline{m}_{x^2}$, and $\|G_u(x,t)\|_F \leq g_u$, $\forall x, t$ where $\underline{m} = \inf_{x,t}\lambda_{\min}(M(x,t))$, $\overline{m} = \sup_{x,t}\lambda_{\max}(M(x,t))$, and m_{ij} is the (i,j) component of $M(x,t)$. Suppose also that there exists $\alpha > 0$ s.t.

$$\gamma M^2 + MBR^{-1}B^TM - \phi^TM - M\phi - 2\alpha_g I \succeq 2\alpha M \tag{37}$$

where $2\alpha_g = g_u^2(\overline{m}_x\varepsilon + \overline{m}_{x^2}/2)$ with an arbitrary positive constant ε, and the arguments ϱ, x, and t are dropped for notational simplicity. If there exists a positive definite solution $M(x,t)$ to the inequality (30), then the mean squared distance between the trajectories of (27) under the feedback control (29) is exponentially bounded as follows:

$$E\left[\|x_d - x\|^2\right] \leq \frac{C}{2\alpha} + \frac{E[V(x(0), \partial_\mu y(0), 0)]e^{-2\alpha t}}{\underline{m}} \tag{38}$$

where $V(x, \partial_\mu y, t) = \int_0^1 I_V(x, \partial_\mu y, t)d\mu$ with

$$I_V(x, \partial_\mu y, t) = \frac{\partial y}{\partial \mu}^T M(x,t)\frac{\partial y}{\partial \mu} \tag{39}$$

and $C = (\overline{m}/\underline{m})g_u^2 + (\overline{m}_x g_u^2)/(\varepsilon\underline{m})$.

Proof: For notational simplicity, let $I_V = I_V(x, \partial_\mu y, t)$, $A = A(\varrho, x, t)$, $B = B(x,t)$, $R = R(x,t)$, $G = G(y,t)$, $M = M(x,t)$, and $\phi = \phi(\varrho, y, t) = \partial(\Delta A x_d)/\partial y + \partial(\Delta B u_d)/\partial y$. Define an infinitesimal differential generator as

$$\begin{aligned}\mathscr{L}V = \int_0^1 &\frac{\partial I_V}{\partial t} + \sum_{i=1}^n\left(\frac{\partial I_V}{\partial x_i}f_i + \frac{\partial I_V}{\partial(\partial_\mu y_i)}\frac{\partial f_v}{\partial y}\frac{\partial y}{\partial \mu}\right)\\ &+ \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n\left[\frac{\partial^2 I_V}{\partial x_i\partial x_j}(G_u(x,t)G_u(x,t)^T)_{ij}\right.\\ &+ 2\frac{\partial^2 I_V}{\partial x_i\partial(\partial_\mu y_j)}\left(G_u(x,t)\frac{\partial G(y,t)}{\partial \mu}^T\right)_{ij}\\ &\left.+ \frac{\partial^2 I_V}{\partial(\partial_\mu y_i)(\partial_\mu y_j)}\left(\frac{\partial G(y,t)}{\partial \mu}\frac{\partial G(y,t)}{\partial \mu}^T\right)_{ij}\right]d\mu\end{aligned} \tag{40}$$

where f_i is the ith component of $f(x,t)$. Since we have

$$\begin{aligned}\frac{\partial I_V}{\partial t} + \sum_{i=1}^n\frac{\partial I_V}{\partial x_i}f_i =& \frac{\partial y}{\partial \mu}^T\dot{M}\frac{\partial y}{\partial \mu}\\ \sum_{i=1}^n\frac{\partial I_V}{\partial(\partial_\mu y_i)}\frac{\partial f_v}{\partial y}\frac{\partial y}{\partial \mu} =& \frac{\partial y}{\partial \mu}^T\mathrm{sym}(M(A_{cl}(\rho, e, t) + \phi))\frac{\partial y}{\partial \mu}\end{aligned}$$

where I_V is given in (39), the equation (40) reduces to

$$\begin{aligned}\mathscr{L}V = \int_0^1 &\frac{\partial y}{\partial \mu}^T(\dot{M} + A^TM + MA - 2MBR^{-1}B^TM\\ &+ \phi^TM + M\phi)\frac{\partial y}{\partial \mu}d\mu + V_2.\end{aligned} \tag{41}$$

The computation of V_2 and its upper bound $\overline{V}_2 = 2\alpha_g\int_0^1\|\partial y/\partial \mu\|^2 d\mu + \underline{m}C$ is given in Appendix B. Substituting (30) into (41) yields

$$\begin{aligned}\mathscr{L}V \leq \int_0^1 &\frac{\partial y}{\partial \mu}^T(-\gamma M^2 - MBR^{-1}B^TM\\ &+ \phi^TM + M\phi)\frac{\partial y}{\partial \mu}d\mu + V_2.\end{aligned} \tag{42}$$

Thus, using (37) and $V_2 \leq \overline{V}_2$, we have that

$$\begin{aligned}\mathscr{L}V \leq& -2\int_0^1\frac{\partial y}{\partial \mu}^T(\alpha M + \alpha_g I)\frac{\partial y}{\partial \mu}d\mu\\ &+ 2\alpha_g\int_0^1\left\|\frac{\partial y}{\partial \mu}\right\|^2 d\mu + \underline{m}C\\ =& -2\alpha V + \underline{m}C.\end{aligned} \tag{43}$$

Theorem 1 along with (43) completes the derivation of (38). ∎

Remark 4: The Euclidean norm of the state vector has to be upper bounded by a constant [7], [62] in order for (37) to have a positive definite solution and for $\|\phi\|$ to be bounded [16], [62]. This assumption is satisfied by many engineering applications [16] and does not imply any assumption on the incremental stability of the proposed controller. Also, the result of Theorem 4 does not imply the asymptotic almost-sure bounds as $V(x, \partial_\mu y, t)$ is not a supermartingale due to the non-vanishing term $\underline{m}C$ in (43). Finite time bounds can be obtainable using the supermartingale inequality (see [1, pp. 86], [22]).

D. Robustness against Stochastic and Deterministic Disturbances

We also show that the tracking error has a finite $\mathcal{L}_2$ gain with respect to the noise and disturbances acting on the system, i.e., the proposed controller is robust against external deterministic and stochastic disturbances analogously to Lemma 4. Consider the following nonlinear system under these disturbances:

$$dx = f(x,t)dt + B(x,t)udt + d(x,t)dt + G_u(x,t)dW_u. \tag{44}$$

The virtual system is defined as

$$dy = f_v(y,t)dt + d_y(y,t)dt + G(y,t)dW \tag{45}$$

where $d_y(e,t) = d(x,t)$ and $d_y(0,t) = 0$. One important example of these systems is a parametric uncertain system, where $d(x,t)$ is given as $d(x,t) = f_{true}(x,t) - f(x,t)$ with f_{true} being the system with true parameter values. Thus, the following corollary allows us to apply adaptive control techniques including [63], [64] on top of our method. In particular, it is shown in [63] that we can use contraction metrics to estimate unknown parameters θ when $G_u(x,t) = 0$ and $d(x,t) = h(x,t)\theta$ for a given h.

Corollary 1: The controller (29) with the constraints (30) and (37) is robust against external disturbances in (44) and satisfies the following $\mathcal{L}_2$ norm bound on the tracking error e:

$$E_{y_0}[\|(e)_\tau\|_{\mathcal{L}_2}^2] \leq \frac{\|e(0)\|_{M(0)}^2 + \frac{\overline{m}}{\varepsilon_1}E_{y_0}[\|(d)_\tau\|_{\mathcal{L}_2}^2] + C_m\tau}{2\alpha_1} \tag{46}$$

where $C_m = \underline{m}C$ and $\alpha_1 = \alpha\underline{m} - \varepsilon_1\overline{m}/2$ with some positive constant ε_1 that guarantees $\alpha_1 > 0$.

Proof: Using the controller (29) with (30) and (37),

$$\begin{aligned}\mathscr{L}V &\leq -2\alpha V + \underline{m}C + 2\overline{m}\int_0^1 \left\|\frac{\partial y}{\partial \mu}\right\|\left\|\frac{\partial d_y}{\partial \mu}\right\| d\mu \qquad (47)\\ &\leq -(2\alpha\underline{m} - \varepsilon_1\overline{m})\int_0^1 \left\|\frac{\partial y}{\partial \mu}\right\|^2 d\mu + C_m + \frac{\overline{m}}{\varepsilon_1}\|d(x,t)\|^2\end{aligned}$$

where the inequality $2a'b' \leq \varepsilon_1^{-1}a'^2 + \varepsilon_1 b'^2$ for scalars a', b' and $\varepsilon_1 > 0$ is used with $a' = \|\partial d_y/\partial\mu\|$ and $b' = \|\partial y/\partial\mu\|$. Since ε_1 is arbitrary, let us select ε_1 s.t. $\alpha_1 = \alpha\underline{m} - \varepsilon_1\overline{m}/2 > 0$. Applying the Dynkin's formula [1, pp. 10] to (47), we have

$$\begin{aligned}&E_{y_0}[V(x,\partial_\mu y,t)] - V(x(0),\partial_\mu y(0),0)\\ &\leq E_{y_0}\left[\int_0^t \left(-2\alpha_1\|x(\tau)-x_d(\tau)\|^2 + \underline{m}C\right.\right.\\ &\left.\left.+\frac{\overline{m}}{\varepsilon_1}\|d(x(\tau),\tau)\|^2\right) d\tau\right]. \qquad (48)\end{aligned}$$

Using $E_{y_0}[V(x,\partial_\mu y,t)] > 0$ and $V(x(0),\partial_\mu y(0),0) = \|x(0)-x_d(0)\|^2_{M(0)}$ yields the desired inequality (46). ■

Remark 5: Corollary 1 implies that the CV-STEM control low is finite-gain $\mathcal{L}_2$ stable and input-to-state (ISS) in a mean squared sense (see Lemma 4 in [58]). However, unlike the deterministic case, where $dV^p/dt = pV^{p-1}dV/dt$ can be used to prove the finite-gain $\mathcal{L}_p$ stability for $p \in [1,\infty)$, we have $\mathscr{L}V \neq pV^{p-1}\mathscr{L}V$. Directly computing $\mathscr{L}V^p$ using (40) gives us the stability property of the proposed controller for general p but it is left as future work due to space limitations.

E. ConVex optimization-based Stochastic steady-state Tracking Error Minimization (CV-STEM) Control

We formulate a convex optimization problem to find the optimal contraction metric $M(x,t)$, which greedily minimizes an upper bound of the steady-state mean squared distance in (38) of Theorem 4. This choice of $M(x,t)$ makes the stabilizing feedback control scheme (29) optimal in some sense.

Assumption 1: From now on, we assume the following.

1) α and ε are selected by a user. In particular, ε can be chosen in a way that it minimizes the steady-state bound as explained in Remark 1.
2) α_g is fixed, i.e., $\overline{m}_x$, $\overline{m}_{x^2}$, and g_u are given.
3) An upper bound of (38) as $t\to\infty$ is minimized instead of (38) itself.
4) The objective value is minimized greedily at each step.

1) Objective Function: As a result of Theorem 4, we have

$$\lim_{t\to\infty} E\left[\|x_d - x\|^2\right] \leq \frac{C}{2\alpha} = \frac{g_u^2}{2\alpha}\left(\frac{\overline{m}}{\underline{m}} + c_1\frac{1}{\underline{m}}\right) \qquad (49)$$

where $c_1 = \overline{m}_x/\varepsilon$. Since $\underline{m} = \inf_{x,t}\lambda_{\min}(M(x,t))$ and $\overline{m} = \sup_{x,t}\lambda_{\max}(M(x,t))$ depend on the future values of $M(x,t)$, the problem of directly minimizing (49) becomes an infinite horizon problem. Instead of solving it, we greedily minimize the current steady-state upper bound (49) to find an optimal $M(x,t)$ at the current time step as stated in Assumption 1. Namely, we drop inf and sup in the objective function (49). The following lemma is critical in deriving the CV-STEM control framework.

Lemma 5: The greedy objective function, i.e., the value inside the bracket of (49) without inf and sup, is upper bounded as follows:

$$\frac{\lambda_{\max}(M)}{\lambda_{\min}(M)} + \frac{c_1}{\lambda_{\min}(M)} \leq \kappa(W) + c_1\kappa(W)^2\lambda_{\min}(W) \qquad (50)$$

where $W(x,t) = M(x,t)^{-1}$ and $\kappa(\cdot)$ is the condition number.

Proof: Rewriting the left-hand side of (50) using κ gives

$$\frac{\lambda_{\max}(M)}{\lambda_{\min}(M)} + \frac{c_1}{\lambda_{\min}(M)} \leq \kappa(M) + c_1\frac{\kappa(M)^2}{\lambda_{\max}(M)} \qquad (51)$$

where $1 \leq \kappa(M) \leq \kappa(M)^2, \forall M$ by definition of κ is used to upper-bound the term $c_1\kappa(M)/\lambda_{\max}(M)$. Substituting $\kappa(M) = \kappa(W)$ and $\lambda_{\max}(M) = 1/\lambda_{\min}(W)$ into (51) completes the proof. ■

Remark 6: We saw that the steady-state tracking error as a result of discrete-time stochastic contraction analysis in Theorem 2 is also a function of the condition number of the metric $M_k(x_k,t_k)$. This fact with the result of Theorem 3 justifies the continuous-time control design to minimize the objective function written by the condition number of the metric $M(x,t)$, although the optimization-based controller has to be implemented in a discrete way in practical applications.

2) Convex Constraints: Let us introduce additional variables χ and ν defined as

$$I \preceq \tilde{W} \preceq \chi I \qquad (52)$$

where $\tilde{W} = \nu W$ and $\nu > 0$.

Lemma 6: Suppose that the coefficients of the SDC parameterizations ϱ are fixed. Given a positive constant ν, the SDRI constraint (30) is equivalent to the following constraint:

$$-\dot{\tilde{W}} + A\tilde{W} + \tilde{W}A^T + \tilde{\gamma}I - \nu BR^{-1}B^T \preceq 0 \qquad (53)$$

where $\tilde{\gamma} = \nu\gamma$. Similarly, the constraint (37) is equivalent to the following LMI constraint:

$$\begin{bmatrix}\tilde{\gamma}I + \nu BR^{-1}B^T - \tilde{W}\phi^T - \phi\tilde{W} - 2\alpha\tilde{W} & \tilde{W}\\ \tilde{W} & \frac{\nu}{2\alpha_g}I\end{bmatrix} \succeq 0. \qquad (54)$$

Proof: Since $\nu > 0$ and $W(x,t) \succ 0$, multiplying (30) and (37) by ν and then by $W(x,t)$ from both sides preserves matrix definiteness. Also, the resultant inequalities are equivalent to the original ones [12, pp. 114]. For the SDRI constraint (30), these operations yield the desired inequality (53). For the constraint (37), these operations give us that

$$\tilde{\gamma}I + \nu BR^{-1}B^T - \tilde{W}\phi^T - \phi\tilde{W} - \frac{2\alpha_g}{\nu}\tilde{W}^2 \succeq 2\alpha\tilde{W}. \qquad (55)$$

Applying the Schur's complement lemma [12, pp. 7] to (55) results in the desired LMI constraint (54). ■

3) Convex Optimization Formulation: We are now ready to state our main result on the convex optimization-based sampling of optimal contraction metrics.

Theorem 5: Suppose α, g_u, and c_1 in (49) are given. Then the problem of greedily minimizing a steady-state upper bound of $E[\|x-x_d\|^2]$ in Theorem 4 is defined as follows:

$$\mathcal{J}_{nl}^* = \min_{\gamma>0, W\succ 0, M\succ 0} \kappa(W) + c_1\kappa(W)^2\lambda_{\min}(W) \quad \text{s.t. (30), (37), and } M(x,t)=W(x,t)^{-1}. \tag{56}$$

Further, the following convex optimization problem

$$\mathcal{J}_{cv}^* = \min_{\substack{\tilde{\gamma}>0,\nu>0,\tau\in\mathbb{R}\\ \chi\in\mathbb{R},\tilde{W}\succ 0}} \tau \quad \text{s.t. (52), (53), (54), and } \begin{bmatrix} \tau-\chi & \chi \\ \chi & \frac{\nu}{c_1}\end{bmatrix}\succeq 0. \tag{57}$$

is equivalent to (56), i.e., $\mathcal{J}_{nl}^* = \mathcal{J}_{cv}^*$.

Proof: The first part (56) follows from Lemma 5, which derives an upper bound of the steady-state mean squared distance (38) under the conditions (30) and (37). For the second part, consider the following two optimization problems:

$$\mathcal{J}_{n2c}^* = \min_{\substack{\tilde{\gamma}>0,\nu>0\\ \chi\in\mathbb{R},\tilde{W}\succ 0}} \chi + c_1\frac{\chi^2}{\nu} \quad \text{s.t. (52), (53), and (54)} \tag{58}$$

and

$$\hat{\mathcal{J}}_{n2c}^* = \min_{\substack{\tilde{\gamma}>0,\nu>0\\ \chi\in\mathbb{R},\tilde{W}\succ 0}} \chi + c_1\frac{\chi^2}{\nu} \quad \text{s.t. (53), (54), } \lambda_{\min}(\tilde{W})=1, \text{and } \lambda_{\max}(\tilde{W})=\chi. \tag{59}$$

The rest of the proof is outlined as follows: we first prove $\mathcal{J}_{nl}^* = \mathcal{J}_{n2c}^*$ by showing a) $\mathcal{J}_{nl}^* = \hat{\mathcal{J}}_{n2c}^* \geq \mathcal{J}_{n2c}^*$ and b) $\mathcal{J}_{nl}^* \leq \mathcal{J}_{n2c}^*$, and then prove c) $\mathcal{J}_{n2c}^* = \mathcal{J}_{cv}^*$ to obtain the desired relation $\mathcal{J}_{nl}^* = \mathcal{J}_{cv}^*$.

a) $\mathcal{J}_{nl}^* = \hat{\mathcal{J}}_{n2c}^* \geq \mathcal{J}_{n2c}^*$: Let us denote the feasible set of (58) as $\mathcal{S}_{n2c}$ and that of (59) as $\hat{\mathcal{S}}_{n2c}$. Due to the constraint (52), which can be rewritten as $\lambda_{\min}(\tilde{W})\geq 1$ and $\lambda_{\max}(\tilde{W})\leq\chi$, we have $\hat{\mathcal{S}}_{n2c}\subseteq\mathcal{S}_{n2c}$. This indicates that $\hat{\mathcal{J}}_{n2c}^* \geq \mathcal{J}_{n2c}^*$ as (58) and (59) use the same objective function. Also, using $\nu = 1/\lambda_{\min}(W)$ and $\chi=\lambda_{\max}(\tilde{W})=\kappa(W)$, $\forall \nu,\chi\in\hat{\mathcal{S}}_{n2c}$ by definition, $\hat{\mathcal{J}}_{n2c}^*$ can be expressed as

$$\hat{\mathcal{J}}_{n2c}^* = \min_{\tilde{\gamma}>0,\nu>0,\tilde{W}\succ 0} \kappa(W) + c_1\kappa(W)^2\lambda_{\min}(W) \quad \text{s.t. (53) and (54).} \tag{60}$$

Since (53) and (54) are equivalent to (30) and (37), respectively, as proved in Lemma 6, (56) and (60) imply that $\mathcal{J}_{nl}^* = \hat{\mathcal{J}}_{n2c}^*$. Thus, we have $\mathcal{J}_{nl}^* = \hat{\mathcal{J}}_{n2c}^* \geq \mathcal{J}_{n2c}^*$ as desired.

b) $\mathcal{J}_{nl}^* \leq \mathcal{J}_{n2c}^*$: For $\tilde{W}\in\mathcal{S}_{n2c}$, we have

$$\begin{aligned}\kappa(W)+c_1\kappa(W)^2\lambda_{\min}(W) &= \frac{\lambda_{\max}(\tilde{W})}{\lambda_{\min}(\tilde{W})} + c_1\frac{(\lambda_{\max}(\tilde{W}))^2}{\nu\lambda_{\min}(\tilde{W})} \\ &\leq \lambda_{\max}(\tilde{W}) + c_1\frac{(\lambda_{\max}(\tilde{W}))^2}{\nu} \leq \chi + c_1\frac{\chi^2}{\nu}.\end{aligned} \tag{61}$$

where $\kappa(W)=\kappa(\tilde{W})$ and $\lambda_{\min}(W)=\lambda_{\min}(\tilde{W})/\nu$ are used for the first equality, and (52) expressed as $\lambda_{\min}(\tilde{W})\geq 1$ and $\lambda_{\max}(\tilde{W})\leq\chi$ is used for the second and third inequalities, respectively. Since (61) holds for any decision variable in $\mathcal{S}_{n2c}$, we have $\mathcal{J}_{nl}^* \leq \mathcal{J}_{cv}^*$ by (56) and (58).

c) $\mathcal{J}_{n2c}^* = \mathcal{J}_{cv}^*$: The epigraph form [13, pp. 134] of (58) is given as

$$\mathcal{J}_{n2c}^* = \min_{\substack{\tilde{\gamma}>0,\nu>0,\tau\in\mathbb{R}\\ \chi\in\mathbb{R},\tilde{W}\succ 0}} \tau \quad \text{s.t. (52), (53), (54), and } \tau\geq\chi+c_1\frac{\chi^2}{\nu} \tag{62}$$

Applying the Schur's complement lemma [12, pp. 7] to the last constraint of (62) results in $\mathcal{J}_{n2c}^* = \mathcal{J}_{cv}^*$. ■

Remark 7: Although (57) is convex, it is infinite dimensional due to $\dot{\tilde{W}}$. We could address this issue by computing $\dot{\tilde{W}}$ along the trajectory or by approximating the contraction metric as a liner combination of given basis functions [28]. These techniques will be briefly discussed in Sec. V.

The coefficients of the SDC parameterizations ϱ can also be treated as a decision variable as can be seen in the following proposition.

Proposition 1: Introducing new variables $\tilde{W}_{\varrho_i}\succ 0$ and $\tilde{\varrho}_i = \nu\varrho_i$ where $\tilde{W}_{\varrho_i}=\varrho_i\tilde{W}$, the bilinear matrix inequalities (53) and (54) in terms of $\tilde{W}$ and ϱ with $\nu>0$ can be relaxed as follows:

$$-\dot{\tilde{W}} + \sum_{i=1}^{s_1} A_i\tilde{W}_{\varrho_i} + \sum_{i=1}^{s_1}\tilde{W}_{\varrho_i}A_i^T + \tilde{\gamma}I - \nu BR^{-1}B^T \preceq 0 \tag{63}$$

and

$$\begin{bmatrix}\tilde{\gamma}I+\nu BR^{-1}B^T-\Phi-2\alpha\tilde{W} & \tilde{W} \\ \tilde{W} & \frac{\nu}{2\alpha_g}I\end{bmatrix}\succeq 0. \tag{64}$$

where Φ is given by

$$\begin{aligned}\Phi = &\sum_{i=1}^{s_1}\tilde{W}_{\varrho_i}\frac{\partial(\Delta A_i x_d)}{\partial q}^T + \sum_{i=1}^{s_1}\frac{\partial(\Delta A_i x_d)}{\partial q}\tilde{W}_{\varrho_i} \\ &+ \tilde{W}\frac{\partial(\Delta B u_d)}{\partial q}^T + \frac{\partial(\Delta B u_d)}{\partial q}\tilde{W}\end{aligned}$$

with $\Delta A(\varrho,x,t) = \sum_{i=1}^{s_1}\varrho_i\Delta A_i(x,t) = \sum_{i=1}^{s_1}\varrho_i(A_i(x,t)-A_i(x_d,t))$. We also need some additional relaxed constraints to ensure controllability and $\tilde{W}_{\varrho_i}=\varrho_i\tilde{W}$, i.e.,

$$\begin{aligned}&\tilde{W},\tilde{W}_{\varrho_i}\succ 0,\ \sum_{i=1}^{s_1}\tilde{W}_{\varrho_i}=\tilde{W},\ \mathrm{sym}\begin{bmatrix}\nu I & \tilde{W} \\ \tilde{\varrho}_i I & \tilde{W}_{\varrho_i}\end{bmatrix}\succeq 0, \\ &\sum_{i=1}^{s_1}\tilde{\varrho}_i=\nu,\ \tilde{\varrho}_i\in[0,\nu],\ cc_k(\tilde{\varrho},x)\leq 0,\ \forall i,\ \forall k=1,\cdots,n_c\end{aligned} \tag{65}$$

where $cc_j(\varrho,x)\leq 0$, $\forall k=1,\cdots,n_c$ denotes convex constraints to maintain the controllability of the pair (A,B).

Proof: The first two inequalities (63) and (64) follow from the desired equality $\tilde{W}_{\varrho_i}=\varrho_i\tilde{W}$ and $A(\varrho,x,t)=\sum_{i=1}^{s_1}\varrho_i A_i(x,t)$. See [16] for the derivation of (65). ■

4) Summary of CV-STEM Control Design: The CV-STEM control of a class of Itô stochastic nonlinear systems is designed as (29), where the optimal contraction metric $M(x) = \nu\tilde{W}(x)^{-1}$ is selected by the convex optimization problem (57) in Theorem 5. The coefficients of SDC parameterizations ϱ can also be used to preserve controllability by considering the relaxed problem with the constraints (63), (64), and (65) in Proposition 1, where the decision variables are $\tilde{\gamma} > 0$, $\nu \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\chi \in \mathbb{R}$, $\tilde{W} \succ 0$, $\tilde{W}_{\varrho i} \succ 0$, and $\tilde{\varrho}_i$.

This control design provides a convex optimization-based methodology for computing the contraction metric that greedily minimizes an upper bound of the steady-state mean squared tracking error (38) in Theorem 4. As proved in Corollary 1, it is also robust against external disturbances and has the $\mathcal{L}_2$ norm bound on the tracking error. In practice, (57) of Theorem 5 can be implemented using computationally-efficient numerical techniques such as the polynomial-time interior point method for convex programming [12]–[15] and the SDRI solvers [65]–[69]. Although the control parameters are supposed to be updated by (57) at each time instant due to the state-and time-dependent constraints, its sampling period can be relaxed to larger values to allow online implementation of the CV-STEM as will be seen in Sec. VI. Further, the controllability constraint can be incorporated into this framework [16] as in Proposition 1, utilizing the non-unique choice of SDC parametrizations.

IV. Main Result 3: CV-STEM Control Design for Lagrangian Systems

In this section, we consider stochastic Lagrangian systems equipped with an exponentially-stabilizing tracking controller [24]. We propose a robust optimization-based controller that can handle stochastic disturbances and guarantee exponential boundedness of the mean squared tracking error of system trajectories.

A. Problem Formulation and Feedback Control Design

Let us consider the following Lagrangian system with a stochastic disturbance:

$$\mathcal{M}(q)d\dot{q} + (C(q,\dot{q})\dot{q} + G(q))dt = \mathcal{B}(q,\dot{q})udt + \Gamma(x,t)dW \tag{66}$$

where $q : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $G : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\Gamma : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times d}$ with the same assumptions on the existence and uniqueness of the solution stated in Sec. II. We note that the matrix $C(q,\dot{q})$ is selected to make $\dot{\mathcal{M}} - 2C$ skew-symmetric, so we have a useful property s.t. $z^T(\dot{\mathcal{M}} - 2C)z = 0,\ \forall z \in \mathbb{R}^n$. A feedback controller u for this system is designed as a combination of an exponentially stabilizing nominal controller u_n and a stochastic controller u_s s.t.

$$\begin{aligned} u =& u_n + u_s \qquad (67)\\ u_n =& \mathcal{B}(q,\dot{q})^+(\mathcal{M}(q)\ddot{q}_r + C(q,\dot{q})\dot{q}_r + G(q) - K(t)(\dot{q}-\dot{q}_r))\\ u_s =& -K_s(x)s = -R(x)^{-1}B(x)^TM(x)s \end{aligned}$$

where $\dot{q}_r = \dot{q}_d - \Lambda(q - q_d)$, $s = \dot{q} - \dot{q}_r$, $x = [q^T, \dot{q}^T]^T$, and

$$\begin{aligned} &A(x) = -\mathcal{M}(q)^{-1}(C(q,\dot{q}) + K(t))\\ &B(x) = \mathcal{M}(q)^{-1}\mathcal{B}(q,\dot{q})\\ &\dot{M} + MA + A^TM - MBR^{-1}B^TM + \gamma M^2 \preceq 0. \end{aligned} \tag{68}$$

with $M \succ 0$, $\gamma > 0$, and $R(x) \succ 0$ being a weighting matrix on the input u_s. When $\mathcal{B}\mathcal{B}^+ = I$, applying (67) to (66) yields the following closed loop system:

$$\begin{aligned} &\mathcal{M}(q)ds + (C(q,\dot{q}) + K(t))sdt\\ &= -\mathcal{B}(q,\dot{q})K_s(x)sdt + \Gamma(x,t)dW. \end{aligned} \tag{69}$$

Lemma 7: Suppose that the deterministic system is perturbed as follows:

$$\mathcal{M}(q)\dot{s} + (C(q,\dot{q}) + K(t))s = \mathcal{B}(q,\dot{q})(u_s + d). \tag{70}$$

If there exists a positive definite solution $M(x)$ to (68) with $R(x) = S(x)^2 \succ 0$ and $S(x) \succ 0$, then the system with an input $\mu = S(x)d$ and an output $y = \sqrt{\gamma}M(x)s$ is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to 1.

Proof: Following the same proof as in Appendix A with the Lyapunov function $V_M = s^TMs$, we have $\dot{V}_M \leq -\|y\|^2 + \|\mu\|^2$ due to (68). This relation along with the comparison lemma [11, pp. 211] gives us the desired result. ■

Remark 8: Since the system with the output $y = \sqrt{\gamma}M(x)s$ and input $\mu = S(x)d$ is clearly zero-state observable [20], it is exponentially stable when $d = 0$.

B. Incremental Stability Analysis

Let us define a virtual system of (66) as follows:

$$\begin{aligned} &\mathcal{M}(q)dy + (C(q,\dot{q}) + K(t))y(\mu,t)dt\\ &= -\mathcal{B}(q,\dot{q})K_s(x)y(\mu,t)dt + \Gamma_y(y(\mu,t),t)dW \end{aligned} \tag{71}$$

where $\mu \in [0,1]$ is introduced to parameterize the trajectories $y = s$ and $y = 0$, i.e., $y(\mu = 0, t) = s$, $y(\mu = 1, t) = 0$, $\Gamma_y(y(0,t),t) = \Gamma(x,t)$, and $\Gamma_y(y(1,t),t) = 0_{n\times d}$. Note that (71) has $y = s$ and $y = 0$ as particular solutions as a result of this parameterization. The following theorem analyzes a stochastic contraction property of the Lagrangian system (66) under the feedback control (67) similarly to Theorem 4.

Theorem 6: Suppose there exist $\overline{\ell}_x$, $\overline{\ell}_{x^2}$, and g_B s.t. $\|\mathcal{M}(q)^{-1}\Gamma(x,t)\|_F \leq g_B$, $\|\partial((\mathcal{M}(q) + \sigma M(x))_{ij})/\partial x\| \leq \overline{\ell}_x$, and $\left\|\partial^2((\mathcal{M}(q) + \sigma M(x))_{ij})/\partial x^2\right\| \leq \overline{\ell}_{x^2}$, $\forall x$, where $\overline{\ell}_x$, $\overline{\ell}_{x^2}$, and g_B are bounded. Suppose also that there exist $\alpha_\ell > 0$ and $\sigma > 0$ s.t.

$$\begin{aligned} &\mathcal{B}(q,\dot{q})R(x)^{-1}B(x)^TM(x) + M(x)B(x)R(x)^{-1}\mathcal{B}(q,\dot{q})^T\\ &+ \sigma(\gamma M(x)^2 + M(x)B(x)R(x)^{-1}B(x)^TM(x)) - 2\alpha_\gamma I\\ &\succeq 2\alpha_\ell(\mathcal{M}(q) + \sigma M(x)) \end{aligned} \tag{72}$$

where $2\alpha_\gamma = g_B^2\left(\overline{\ell}_x\varepsilon_\ell + \overline{\ell}_{x^2}/2\right)$ with an arbitrary positive constant ε_ℓ. If there exists a positive definite solution $M(x,t)$ to the inequality (68), then the mean-squared distance of the composite state s is bounded as follows:

$$E[\|s\|^2] \leq \frac{E[V(x(0), \partial_\mu y(0), 0)]e^{-2\overline{\alpha}t} + \frac{C_\ell}{2\overline{\alpha}}}{\inf_{t\geq 0}\lambda_{\min}(\mathcal{M}(q) + \sigma M(x))} \tag{73}$$

where $V(x,\partial_\mu y)$ is given by

$$V(x,\partial_\mu y)=\int_0^1 \frac{\partial y}{\partial \mu}^T(\mathcal{M}(q)+\sigma M(x))\frac{\partial y}{\partial \mu}d\mu \tag{74}$$

with $C_\ell = g_B^2\sup_{t\geq 0}(\lambda_{\max}(\mathcal{M}(q)+\sigma M(x)))+\overline{\ell}_x g_B^2/\varepsilon_\ell$, $\overline{\alpha}=\alpha_\ell+\underline{k}/\sup_t\lambda_{\max}(\mathcal{M}(q)+\sigma M(x))$, and $\underline{k}I\prec K(t),\forall t$.

Proof: Following the same proof given in Theorem 4, the condition (68) gives us that

$$\begin{aligned}\mathscr{L}V\leq&-\sigma\int_0^1\frac{\partial y}{\partial \mu}^T(\gamma M^2+MBR^{-1}B^TM)\frac{\partial y}{\partial \mu}d\mu \qquad (75)\\ &-2\int_0^1\frac{\partial y}{\partial \mu}^T(K+2\mathcal{B}K_s)\frac{\partial y}{\partial \mu}d\mu+2\alpha_\gamma\int_0^1\left\|\frac{\partial y}{\partial \mu}\right\|^2 d\mu+C_\ell\end{aligned}$$

where the skew-symmetric property of $\dot{\mathcal{M}}-2C$ is used to obtain the above inequality. Using (72), we have

$$\begin{aligned}\mathscr{L}V\leq&-2\alpha_\ell V-2\int_0^1\frac{\partial y}{\partial \mu}^T K\frac{\partial y}{\partial \mu}d\mu+C_\ell\\ \leq&-2\overline{\alpha}V+C_\ell.\end{aligned} \tag{76}$$

Thus, applying Theorem 1 yields the desired result (73). ■

C. Robustness against Stochastic and Deterministic Disturbances

Analogously to Lemma 7, consider a Lagrangian system with deterministic and stochastic disturbances

$$\begin{aligned}&\mathcal{M}(q)d\dot{q}+(C(q,\dot{q})\dot{q}+G(q))dt\\ &=\mathcal{B}(q,\dot{q})qdt+d(x,t)dt+\Gamma(q,\dot{q})dW.\end{aligned} \tag{77}$$

Again, an important example of these systems is a parametric uncertain system.

Corollary 2: Let $\mathcal{M}_0=\mathcal{M}(0)+\sigma M(0)$. The controller (67) with the constraints (68) and (72) is robust against the external disturbances and satisfies the following $\mathcal{L}_2$ norm bound on the tracking error:

$$E_{y_0}[\|(s)_\tau\|_{\mathcal{L}_2}^2]\leq\frac{\|s(0)\|_{\mathcal{M}_0}^2+\frac{\overline{\ell}}{\underline{\varpi}\varepsilon_2}E_{y_0}[\|(d)_\tau\|_{\mathcal{L}_2}^2]+C_\ell\tau}{2\alpha_2} \tag{78}$$

where $\underline{\ell}I\preceq\mathcal{M}(q)+\sigma M(x)\preceq\overline{\ell}I$, $\underline{\varpi}I\preceq\mathcal{M}(q)$, $\forall x$, and $\alpha_2=\overline{\alpha}\underline{\ell}-\varepsilon_2\overline{\ell}/(2\underline{\varpi})$ with $\varepsilon_2>0$ that guarantees $\alpha_2>0$.

Proof: Following the same proof as in Corollary 1, we have $\mathcal{L}V\leq-2\alpha_2\int_0^1\|\partial y/\partial\mu\|^2d\mu+\overline{\ell}/(\underline{\varpi}\varepsilon_2)\|d(x,t)\|^2+C_\ell$, where y is the virtual state and V is given in (74). The rest follows from the Dynkin's formula [1, pp. 10]. ■

D. Convex Optimization Formulation

As a result of Theorem 6, we have

$$\lim_{t\to\infty}E[\|s\|^2]\leq\frac{g_B^2\sup_{t\geq0}(\lambda_{\max}(\mathcal{M}+\sigma M))+\frac{\overline{\ell}_x g_B^2}{\varepsilon_\ell}}{2\overline{\alpha}\inf_t\lambda_{\min}(\mathcal{M}+\sigma M)}. \tag{79}$$

We propose one way to formulate a convex optimization problem to find the optimal contraction metric which minimizes an upper bound of the right-hand side of (79) under the following conditions.

Assumption 2: In addition to the conditions given in Assumption 1, we assume that $\sigma=1$, which is possible as we can optimally select the value of γ.

1) Objective Function: Under Assumption 2, we have the following lemma on the greedy objective function as in Lemma 5 of Sec. III-E.

Lemma 8: The greedy objective function, i.e., (79) without sup, inf, and constants g_B and $\overline{\alpha}$, is bounded as follows:

$$\frac{\lambda_{\max}(\mathcal{M}+M)+\frac{\overline{\ell}_x}{\varepsilon_\ell}}{\lambda_{\min}(\mathcal{M}+M)}\leq\kappa(W)+c_2\kappa(W)^2\lambda_{\min}(W) \tag{80}$$

where $W(x)=M(x)^{-1}$ and $c_2=\lambda_{\max}(\mathcal{M})+\overline{\ell}_x/\varepsilon_\ell$.

Proof: Using the relations $\lambda_{\max}(\mathcal{M}+M)\leq\lambda_{\max}(\mathcal{M})+\lambda_{\max}(M)$ and $\lambda_{\min}(\mathcal{M}+M)\geq\lambda_{\min}(\mathcal{M})+\lambda_{\min}(M)\geq\lambda_{\min}(M)$ [70, pp. 242], we have

$$\frac{\lambda_{\max}(\mathcal{M}+M)+\frac{\overline{\ell}_x}{\varepsilon_\ell}}{\lambda_{\min}(\mathcal{M}+M)}\leq\frac{\lambda_{\max}(M)}{\lambda_{\min}(M)}+\frac{c_2}{\lambda_{\min}(M)} \tag{81}$$

Applying Lemma 5 to (81) completes the proof. ■

2) Equivalent Convex Optimization Problem: Let us introduce $\nu>0$, $\chi,\tau\in\mathbb{R}$, and $\tilde{W}=\nu W\succ0$ constrained as

$$I\preceq\tilde{W}\preceq\chi I,\quad\begin{bmatrix}\tau-\chi & \chi\\ \chi & \frac{\nu}{c_2}\end{bmatrix}\succeq0. \tag{82}$$

Analogously to Theorem 5, we have the following results.

Theorem 7: Suppose that $\overline{\alpha}$, g_B, and c_2 are given. Then the problem of greedily minimizing an upper bound of the mean squared distance (73) of Theorem 6 is defined as follows:

$$\mathcal{J}_{nl\ell}^*=\min_{\gamma>0,W\succ0,M\succ0}\kappa(W)+c_2\kappa(W)^2\lambda_{\min}(W) \tag{83}$$

s.t. (68), (72), and $M(x,t)=W(x,t)^{-1}$.

Further, the following convex optimization problem

$$\mathcal{J}_{cv\ell}^*=\min_{\substack{\tilde{\gamma}>0,\nu>0,\tau\in\mathbb{R}\\ \chi\in\mathbb{R},\tilde{W}\succ0}}\tau \tag{84}$$

$$\text{s.t. }\dot{\tilde{W}}+A\tilde{W}+\tilde{W}A^T-\nu BR^{-1}B^T+\tilde{\gamma}I\preceq0 \tag{85}$$

$$\begin{bmatrix}\tilde{\mathcal{M}}_\ell & \tilde{W}\\ \tilde{W} & \frac{\nu}{2}(\alpha_\ell\mathcal{M}+\alpha_\gamma I)^{-1}\end{bmatrix}\succeq0\text{ and (82)} \tag{86}$$

where $\tilde{\gamma}=\nu\gamma$ and $\tilde{\mathcal{M}}_\ell=\mathrm{sym}(\tilde{W}\mathcal{B}R^{-1}B^T)+\tilde{\gamma}I+\nu BR^{-1}B^T-2\alpha_\ell\tilde{W}$, is equivalent to (83), i.e., $\mathcal{J}_{nl\ell}^*=\mathcal{J}_{cv\ell}^*$.

Proof: The first part follows from Lemma 8. The constraints (68) and (72) are equivalent to (85) and the first constraint of (86), respectively, as shown in Lemma 6. The rest follows from the same proof as in Theorem 5. ■

In summary, the CV-STEM control of stochastic Lagrangian systems is designed as (67), where the optimal contraction matric $M(x)=\nu\tilde{W}(x)^{-1}$ is selected by the convex optimization problem (84) in Theorem 7.

V. Main Result 4: CV-STEM with Input Constraints and Other Extensions

Several extensions of algorithms to compute the optimal contraction metric for the feedback control of Itô stohastic nonlinear systems are discussed in this section.

A. Input Constraints

We propose two ways to incorporate input constraints into the convex optimization problem (57) of Theorem 5 and (84) of Theorem 7 without losing their convexity.

1) Input Constraints through the Feedback Gain: Let us consider the case when the input constraint can be relaxed to $\|u(t)\| \leq u_{\max}$, where $u(t)$ is defined in (29) and $u_{\max} > 0$ is given.

Proposition 2: A sufficient condition for the input constraint $\|u(t)\| \leq u_{\max}$, $\forall t \geq 0$ with a given $u_{\max} \geq \|u_d(t)\|$ is expressed as follows:

$$\nu\|R^{-1}B^T\|\|e(t)\| \leq (u_{\max} - \|u_d(t)\|)\lambda_{\min}(\tilde{W}), \ \forall t, x \tag{87}$$

where $e(t) = x(t)-x_d(t)$ and the arguments (x, t) are dropped for notational simplicity. Further, this is a convex constraint in terms of the decision variables of (57) in Theorem 5.

Proof: Using the relations $M = \nu\tilde{W}^{-1}$ and $\|\tilde{W}^{-1}\| \leq 1/\lambda_{\min}(\tilde{W})$ due to (52), we have

$$\begin{aligned}\|u\| =& \| - K(x - x_d) + u_d\| = \|\nu R^{-1}B^T\tilde{W}^{-1}e\| + \|u_d\| \\ \leq& \frac{\nu\|R^{-1}B^T\|\|e\|}{\lambda_{\min}(\tilde{W})} + \|u_d\|.\end{aligned} \tag{88}$$

Thus, a sufficient condition for $\|u(t)\| \leq u_{\max}$, $\forall t \geq 0$ reduces to (87). Also, this is convex in terms of ν and $\tilde{W}$ as $u_{\max} - \|u_d\| \geq 0$ by assumption and $\lambda_{\min}(\tilde{W})$ is a concave function [13, pp. 118]. ■

Proposition 2 allows us to implement $\|u(t)\| \leq u_{\max}$, $\forall t \geq 0$ in (57) and (84) without losing their convexity.

2) Input Constraints through CLFs: Let us take (84) as an example. Although u_s is given by $u_s = -K_s s$ in (67), this form of u_s is not optimal in any sense. Instead, we find u_s which minimizes its Euclidean norm, assuming $M(x, t)$ and γ are obtained by solving (84). The following proposition allows us to optimally incorporate input constraints without dramatically changing the CV-STEM stability and optimality properties.

Proposition 3: Consider the following convex optimization problem to minimize $\|u_s\|$ with an input constraint $u_s \in \mathcal{U}_s$, where $\mathcal{U}_s$ is a given convex set:

$$u_s^* = \arg\min_{\substack{u_s\in\mathcal{U}_s\\ \delta\in\mathbb{R}}} u_s^T u_s + \delta^2 \tag{89}$$

$$\begin{aligned}\text{s.t. } & s^T(2\alpha_\ell(\mathcal{M} + M) + \dot{M} + MA + A^TM + 2\alpha_\gamma I)s \\ & + 2s^T(\mathcal{B} + MB)u_s \leq \delta\end{aligned} \tag{90}$$

where M is given by (84) and the dependence on $x = [q^T, \dot{q}^T]^T$ is omitted for notational simplicity. Then we have

$$E[\|s\|^2] \leq \frac{V(x(0), s(0))e^{-2\overline{\alpha}t} + \frac{C_\ell+\delta}{2\overline{\alpha}}}{\inf_{t\geq0}\lambda_{\min}(\mathcal{M}(q) + M(x))}. \tag{91}$$

where $V(x, s) = s^T(\mathcal{M}(q) + M(x))s$. Also, we have $\delta = 0$ when $\mathcal{U}_s = \mathbb{R}^m$.

Proof: As in the proof of Theorem 6 with $\sigma = 1$, we have

$$\begin{aligned}\mathscr{L}V \leq& - 2s^TKs + s^T(\dot{M} + MA + A^TM + 2\alpha_\gamma I)s \\ & + 2s^T(\mathcal{B} + MB)u_s + C_\ell.\end{aligned} \tag{92}$$

This inequality with the condition (90) gives $\mathscr{L}V \leq 2\overline{\alpha}V + C_\ell + \delta$, which yields (79) by Theorem 1. The last part of this proposition follows from the fact that $u_s = -K_s s$ is a feasible solution of (89) when $\mathcal{U}_s = \mathbb{R}^m$ and $\delta = 0$ for M given by solving (84). ■

Remark 9: The decision variable δ is introduced to avoid infeasibility due to the input constraint $u_s \in \mathcal{U}_s$. Also, for $\mathcal{U}_s = \mathbb{R}^m$, (89) reduces to a quadratic program and has a computationally-efficient analytical solution [13].

B. Finite-Dimensional Formulation of (57) and (84)

In order to solve (57) and (84), we need $\dot{\tilde{W}}$, $\overline{m}_x$, $\overline{m}_{x^2}$, and ϕ at each time instant. Assuming that an initial value of $\tilde{W}$ is given, $\dot{\tilde{W}}$ can be computed by backward difference approximation, $\dot{\tilde{W}}(t_k) \simeq (\tilde{W}(x(t_k)) - \tilde{W}(x(t_{k-1})))/dt$, where $\tilde{W}(x(t_k))$ is a decision variable of the current convex optimization problem and $\tilde{W}(x(t_{k-1}))$ is a given constant as a result of the convex optimization at the previous time step t_{k-1}. We can perform similar operations for computing $\overline{m}_x$, $\overline{m}_{x^2}$, and ϕ at each time instant.

For practical applications, it is also possible to simply neglect them or assign approximate values to each variable [16], although the resultant control parameters could be sub-optimal in these cases.

C. Computationally-Efficient CV-STEM Algorithms

Since solving (57) or (84) at every time step can be computational intractable for some systems, we propose several ways to update the contraction metric less frequently.

1) Relaxed CV-STEM Algorithm: This method updates the control parameters only when one of the constraints in (57) or (84) is violated, or when the objective value at the current iteration is larger than that at the previous iteration. Since this will not change the stability proof, the controller still guarantees exponential boundedness of the mean squared tracking error of system trajectories. This approach will be demonstrated in Sec. VI along with the discussion on how to select the sampling period Δt of the CV-STEM control.

2) Approximate CV-STEM Algorithm: We could approximate the sampled CV-STEM solutions offline assuming the form of a contraction metric in a given hypothesis function space. One candidate is the polynomial basis function space, which leads to the SOS programming-based search algorithm [27]–[29]. However, its application is limited by the facts that it is developed for systems with a polynomial vector field and that the problem size grows exponentially with the number of variables and basis functions [71].

D. Coefficients of SDC Parameterizations

There are two variations of (57) with the relaxed constraints (63), (64), and (65) in Proposition 1, when selecting ϱ of SDC parameterizations. We can either set them to some given values *a priori* to preserve the controllability of the parameterization, or pre-compute a constant solution M offline using constant parameterizations of A [16].

VI. Numerical Simulation

The performance of the CV-STEM is evaluated in the following two problems, where convex optimization problems are solved using *cvx* toolbox in Matlab [72], [73]. Since running an optimization algorithm at every time step is unrealistic in practice, the relaxed CV-STEM in Sec. V is used in this section along with the discussion on the sampling period Δt introduced in Theorem 3. The computation of $d\tilde{Q}/dt$ is performed by backward difference approximation. A Python implemetation of the CV-STEM algorithm is available at https://github.com/astrohiro/cvstem.

A. Spacecraft Attitude Control

We first consider the spacecraft attitude dynamical system given in [74] with stochastic disturbances.

1) Simulation Setup: The spacecraft state (modified Rodrigues parameters) is initialized as $q(0) = [0.9, -0.9, 0.7]^T$, $\dot{q}(0) = [0.6, 0.7, -0.5]^T$, and $G_u(x,t)$ in (26) is given as $G_u(x,t) = 0.2 \times [0,0,0,1,1,1]^T$. We initialize $\tilde{W}$ by solving the CV-STEM without the $d\tilde{W}/dt$ term. The desired trajectories are defined as $q_{1d} = 0.3\sin(2\pi(0.1)t)$, $q_{2d} = 0.2\sin(2\pi(0.2)t + \pi/6)$, and $q_{3d} = 0$ and the CV-STEM is applied with $\alpha = 10^{-3}$ and $R = I$. The input constraint in Proposition 2 is used with $u_{\max} = 700$. The same simulation is performed for PID, $\mathcal{H}_\infty$ [20], and a nonlinear controller with an exponential stability guarantee [17], where the PID gains are selected as $K_P = 1300I$, $K_I = 300I$ and $K_D = 1300I$. We use $K_r = 100I$ and $\Lambda = I$ for the controller in [17]. The sampling period $\Delta t = 0.1$ is used for the CV-STEM and $\mathcal{H}_\infty$.

2) Simulation Results: Figures 2 shows tracking errors of each state for the CV-STEM, the controller in [17], PID, and $\mathcal{H}_\infty$ control, smoothed by the 150-point moving average filter. Figure 3 shows the normalized steady-state tracking error $\lim_{t\to 50}\|x(t)-x_d(t)\|^2$ and control effort $\int_0^{50} u(t)dt$ of each controller averaged over 60 simulations, where $x = [q^T, \dot{q}^T]^T$. It also includes those of the CV-STEM control with different sampling periods Δt to see the impact of discrete-time implementation of the proposed algorithm. It should be noted $\lim_{t\to\infty}\|x(t)-x_d(t)\|^2$ is what we attempt to minimize and it is computed by the average over the values of last 150 steps at each simulation to account for the stochasticity in the system. Table I summarizes the steady-state tracking error and control effort for each controller depicted as horizontal lines in Fig. 3.

It is shown that the proposed controller achieves a smaller steady-state tracking error than that of the controller in [17], PID, and $\mathcal{H}_\infty$ with smaller amount of control effort as shown in Fig. 2, 3 and Table I. Also, the error of the CV-STEM with its sampling period $\Delta t \leq 35$ (s) remains smaller than the other three even with the smaller control effort for $\Delta t \leq 25$ (s) as can be seen in Fig. 3. This fact implies that the CV-STEM control framework could be used in real-time with an onboard computer that solves the optimization within the period $\Delta t \leq 25, 35$ (s) whilst maintaining its superior performance. For example, solving the convex optimization takes less than 1.0s with a Macbook Pro laptop (2.2 GHz Intel Core i7, 16 GB 1600 MHz DDR3 RAM).

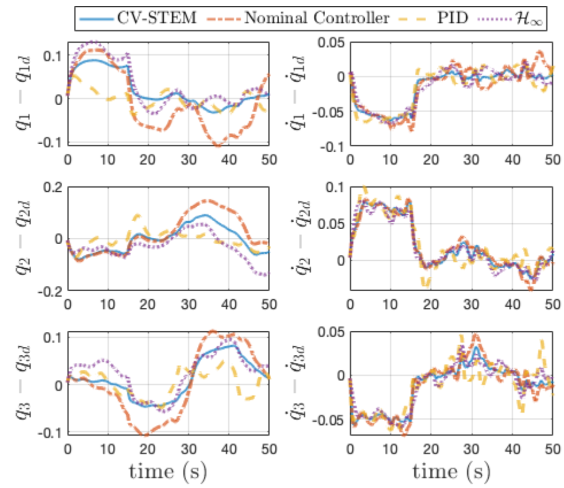

Fig. 2: Tracking errors of Modified Rodrigues parameters

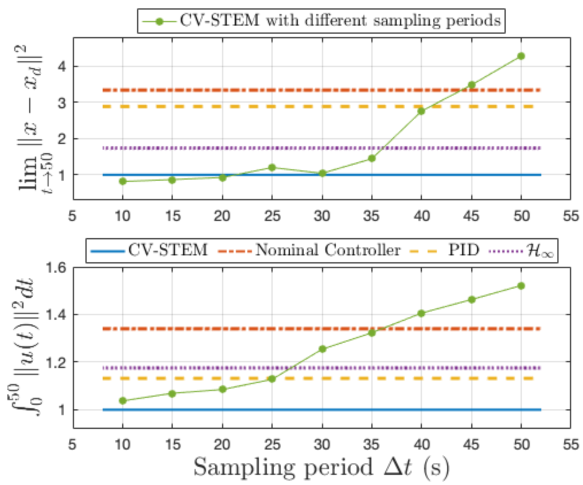

Fig. 3: Steady-state tracking errors and control effort for spacecraft attitude control: Values in the figure are computed by the average over 60 simulations and normalized by one at the CV-STEM performances. The steady-state error is computed by the average over the values of last 150 steps at each simulation to account for the stochasticity in the system.

B. Multi-Agent System

Next, we consider tracking and synchronization control of multiple spacecraft (5 agents) orbiting the earth. The detailed equation of motion and definition of symbols used in this simulation can be found in [58].

1) Simulation Setup: The desired trajectory of the leader agent is given as $x_d(t) = 2.0\sin(\omega t + \phi_{e_0})$, $y_d(t) = 2.0\cos(\omega t + \phi_{e_0})$, and $z_d(t) = 0$. See [58] for how to

TABLE I: Control performances (spacecraft attitude control): Values in this table are computed as explained in Fig. 3

	CV-STEM	Controller [17]	PID	$\mathcal{H}_\infty$
Steady-state error	1	3.3395	2.8849	1.7384
Control effort	1	1.3403	1.1319	1.1755

TABLE II: Control performances (spacecraft tracking and synchronization control): Values in this table are computed as explained in Fig. 3

	CV-STEM	Controller [18]	PID	$\mathcal{H}_\infty$
Steady-state error	1	11.176	34.997	49.903
Control effort	1	0.7946	1.2701	1.0496

construct synchronized desired orbits of the follower agents. We use $\Gamma(x,t) = [1,\cdots,1]^T \in \mathbb{R}^{np\times 1}$ for the diffusion term defined in (66), where $n = 3$ (3 dimensional space) and $p = 5$ (5 agents). The tracking gain K_1 and the synchronization gain K_2 in [18] are selected as $K_1 = 5I$ and $K_2 = 2I$ with $\alpha = 10^{-3}$ and $R = I$ for the CV-STEM control. The spacecraft positions are initialized as uniformly distributed random variables over a cube with side length 0.4 ($-0.2 \leq x_j, y_j, z_j \leq 0.2$), velocities are as $[\dot{x}_j, \dot{y}_j, \dot{z}_j]^T = [0,0,0]^T$, and $\tilde{W}$ is as $\tilde{W}(0) = I$, for all agents j. The gain for the composite states in [18] is selected as $\Lambda_j = I$, $\forall j$. Similarly to the first simulation, the input constraint in Proposition 2 is used with $u_{\max} = 1.0$. For comparison, the nominal nonlinear controller in [18], PID, and $\mathcal{H}_\infty$ control are also applied to this problem with $K_P = 7I$, $K_I = 0I$, and $K_D = 11I$. The sampling period $\Delta t = 0.5$ is used for the CV-STEM and $\mathcal{H}_\infty$.

2) Simulation Results: Figure 4 shows a comparison between the controlled and desired trajectories in the LVLH frame for the CV-STEM, the controller in [18], PID, and $\mathcal{H}_\infty$. Figure 5 shows the normalized steady-state tracking error and control effort of each controller and the CV-STEM with different sampling periods Δt, averaged over 60 simulations. Again, the steady-state errors are computed by the average over the values of last 150 steps at each simulation. Table II summarizes the control performances depicted as horizontal lines in Fig. 5.

Figures 4 and 5 indicate that the CV-STEM control performs better than the controller in [17], PID, and $\mathcal{H}_\infty$ in terms of the steady-state tracking error. Due to the formulation $u = u_n + u_s$, its control effort is 0.8 times larger than that of the nonlinear controller [18] in this case as shown in Table II. Furthermore, the error of the CV-STEM stays smaller than the others for the sampling period $\Delta t \leq 450$ (s) with control effort smaller than that of PID and $\mathcal{H}_\infty$. In particular, it is less than 1.7 times as large as that of the nominal CV-STEM with $\Delta t = 0.5$ (s) for $\Delta t \leq 350$ (s). This is a promising outcome for the real-time implementation of the CV-STEM control, as the aforementioned Macbook Pro laptop (2.2 GHz Intel Core i7, 16 GB 1600 MHz DDR3 RAM) solves the optimization within 1.5s.

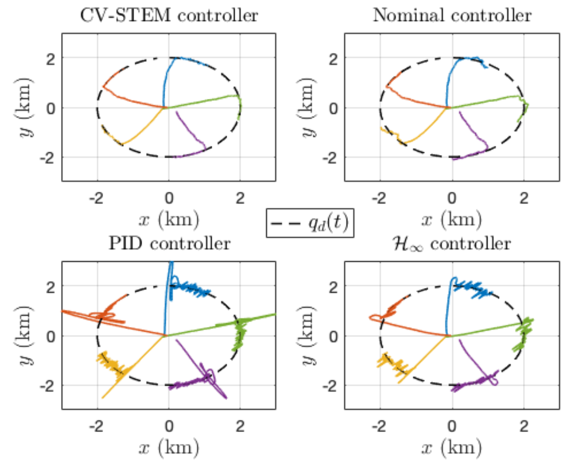

Fig. 4: Controlled and desired trajectories in the LVLH frame

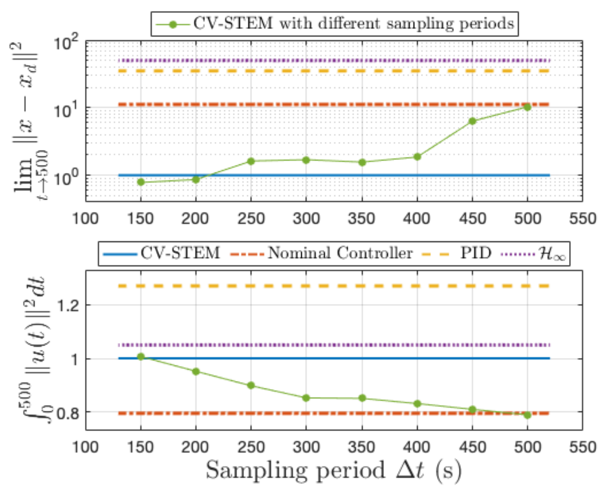

Fig. 5: Steady-state tracking error and control effort for spacecraft tracking and synchronization control: Values in this figure are computed as explained in Fig. 3.

VII. Conclusion

In this paper, we present CV-STEM, a new framework to construct an optimal contraction metric for feedback control of Itô stochastic nonlinear systems and stochastic Lagrangian systems, expressed in SDC extended linear structure. It computes the metric by solving a convex optimization problem, which is proven to be equivalent to its nonlinear counterpart of greedily minimizing an upper bound of the steady-state mean squared tracking error of the system trajectories. It is shown by stochastic incremental contraction analysis that the mean squared error is exponentially bounded for all time, and that the CV-STEM control is robust against stochastic and deterministic disturbances. We also propose discrete-time stochastic contraction analysis with a state- and time-dependent metric to validate the sampling-based implementation of the algorithm. In numerical simulations, the CV-STEM control outperforms PID, $\mathcal{H}_\infty$, and nonlinear controllers developed for spacecraft attitude control and synchronization problems in terms of the steady-state tracking error, with the large enough sampling period which enables its real-time implementation.

Appendix A
Proof of Lemma 4

Proof: Let us omit the arguments x and t for notational simplicity. Differentiating $V_M = e^T M e$ with $e = x - x_d$ under the condition (30) yields

$$\dot{V}_M \leq e^T(-\gamma M^2 - MBR^{-1}B^T M)e + 2e^T M(\Delta_d + Bd)$$

where $\Delta_d = \Delta A x_d + \Delta B u_d$. Adding and subtracting $\|\mu_1\| = \|Sd\|^2$ where $R = S^2$ and completing the square, we have

$$\dot{V}_M \leq -\|y\|^2 + \|\mu_1\|^2 - \|\mu_1 - S^{-1}B^T M e\|^2 + 2e^T M \Delta_d.$$

where $y = (\sqrt{\gamma/2})M(x,t)e$. Using $\mu_2 = (\sqrt{2/\gamma})\Delta_d$,

$$\begin{aligned}\dot{V}_M \leq& -\|y\|^2 + \|\mu_1\|^2 - \frac{1}{2}\gamma\left\|Me - \frac{2\Delta_d}{\gamma}\right\|^2 + \frac{2}{\gamma}\|\Delta_d\|^2 \\ \leq& -\|y\|^2 + \|\mu_1\|^2 + \|\mu_2\|^2. \end{aligned} \tag{93}$$

By the comparison lemma [11, pp. 211], this reduces to

$$\|y_\tau\|_{\mathcal{L}_2} \leq \|(\mu_1)_\tau\|_{\mathcal{L}_2} + \|(\mu_2)_\tau\|_{\mathcal{L}_2} + \sqrt{V_M(x(0))} \tag{94}$$

which completes the proof. ■

Appendix B
Computation of V_2 and $\overline{V}_2$ in Theorem 4

Using (40), V_2 in Theorem 4 can be computed as follows:

$$\begin{aligned} V_2 =& \int_0^1 \left[\sum_{i,j} m_{ij}\left(\frac{\partial G}{\partial \mu}\frac{\partial G}{\partial \mu}^T\right)_{ij} + 2(M_i)_{x_j}\frac{\partial y}{\partial \mu}\left(G_u \frac{\partial G}{\partial \mu}^T\right)_{ij} \right. \\ &\left. + \frac{1}{2}\frac{\partial y}{\partial \mu}^T M_{x_i x_j}\frac{\partial y}{\partial \mu}(G_u G_u^T)_{ij}\right] d\mu \end{aligned} \tag{95}$$

where M_i is the ith row of M. Following the proof of Lemma 2 in [16], we have

$$\begin{aligned} V_2 \leq& \overline{m} g_u^2 + \int_0^1 2\overline{m}_x g_u^2 \left\|\frac{\partial y}{\partial \mu}\right\| + \frac{1}{2}\overline{m}_{x^2} g_u^2 \left\|\frac{\partial y}{\partial \mu}\right\|^2 d\mu \\ \leq& 2\alpha_g \int_0^1 \left\|\frac{\partial y}{\partial \mu}\right\|^2 d\mu + \underline{m}C = \overline{V}_2 \end{aligned} \tag{96}$$

where $2\alpha_g = g_u^2(\overline{m}_x \varepsilon + \overline{m}_{x^2}/2)$ and $C = (\overline{m}/\underline{m})g_u^2 + (\overline{m}_x g_u^2)/(\varepsilon \underline{m})$. The first inequality in (96) is due to $\mathrm{Tr}(AB) \leq \|A\|\,\mathrm{Tr}(B)$ for $A, B \succeq 0$, and the second inequality follows from the relation $2a'b' \leq \varepsilon^{-1}a'^2 + \varepsilon b'^2$ for any scalars a', b', and $\varepsilon > 0$. Thus, V_2 is upper bounded by $\overline{V}_2$ as desired.

Acknowledgment

This work was in part funded by the Jet Propulsion Laboratory, California Institute of Technology and the Raytheon Company.

References

[1] H. J. Kushner, *Stochastic Stability and Control*. Academic Press New York, 1967.
[2] V. Palleschi, F. Sarri, G. Marcozzi, and M. Torquati, "Numerical solution of the Fokker-Planck equation: A fast and accurate algorithm," *Phys. Lett. A*, vol. 146, no. 7, pp. 378 – 386, 1990.
[3] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for nonlinear systems," *Automatica*, vol. 34, no. 6, pp. 683 – 696, 1998.
[4] D. Angeli, "A Lyapunov approach to incremental stability properties," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 410–421, Mar. 2002.
[5] H. Tsukamoto and S.-J. Chung, "Convex optimization-based controller design for stochastic nonlinear systems using contraction analysis," in *58th IEEE Conf. Decis. Control*, Dec. 2019, pp. 8196–8203.
[6] J. R. Cloutier, "State-dependent Riccati equation techniques: An overview," in *Proc. Amer. Control Conf.*, vol. 2, Jun. 1997, pp. 932–936.
[7] H. T. Banks, B. M. Lewis, and H. T. Tran, "Nonlinear feedback controllers and compensators: A state-dependent Riccati equation approach," *Comput. Optim. Appl.*, vol. 37, no. 2, pp. 177–218, Jun. 2007.
[8] T. Çimen, "Survey of state-dependent Riccati equation in nonlinear optimal feedback control synthesis," *J. Guid. Control Dyn.*, vol. 35, no. 4, pp. 1025–1047, Jul. 2012.
[9] E. Sontag, "A Lyapunov-like characterization of asymptotic controllability," *SIAM J. Control Optim.*, vol. 21, no. 3, pp. 462–471, 1983.
[10] E. D. Sontag, "A 'universal' construction of Artstein's theorem on nonlinear stabilization," *Syst. Control Lett.*, vol. 13, no. 2, pp. 117 – 123, 1989.
[11] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice-Hall, 2002.
[12] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA: SIAM, Jun. 1994, vol. 15.
[13] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, Mar. 2004.
[14] A. Ben-Tal and A. S. Nemirovskiaei, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*. Philadelphia, PA, USA: SIAM, 2001.
[15] J. Mattingley and S. Boyd, "Real-time convex optimization in signal processing," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 50–61, May 2010.
[16] A. P. Dani, S.-J. Chung, and S. Hutchinson, "Observer design for stochastic nonlinear systems via contraction-based incremental stability," *IEEE Trans. Autom. Control*, vol. 60, no. 3, pp. 700–714, Mar. 2015.
[17] S. Bandyopadhyay, S.-J. Chung, and F. Y. Hadaegh, "Nonlinear attitude control of spacecraft with a large captured object," *J. Guid. Control Dyn.*, vol. 39, no. 4, pp. 754–769, Jan. 2016.
[18] S.-J. Chung and J.-J. E. Slotine, "Cooperative robot control and concurrent synchronization of Lagrangian systems," *IEEE Trans. Robot.*, vol. 25, no. 3, pp. 686–700, Jun. 2009.
[19] W. Zhang and B. Chen, "State feedback $\mathcal{H}_\infty$ control for a class of nonlinear stochastic systems," *SIAM J. Control Optim.*, vol. 44, no. 6, pp. 1973–1991, 2006.
[20] A. J. Van der Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 770–784, Jun. 1992.
[21] A. Isidori and A. Astolfi, "Disturbance attenuation and $\mathcal{H}_\infty$-control via measurement feedback in nonlinear systems," *IEEE Trans. Autom. Control*, vol. 37, no. 9, pp. 1283–1293, Sep. 1992.
[22] Q. Pham, N. Tabareau, and J.-J. E. Slotine, "A contraction theory approach to stochastic incremental stability," *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 816–820, Apr. 2009.
[23] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin, Heidelberg: Springer-Verlag, 1995.
[24] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Upper Saddle River, NJ: Pearson, 1991.
[25] R. A. Freeman and P. V. Kokotovic, "Optimal nonlinear controllers for feedback linearizable systems," in *Proc. Amer. Control Conf.*, vol. 4, Jun. 1995, pp. 2722–2726.
[26] J. A. Primbs, V. Nevistic, and J. C. Doyle, "Nonlinear optimal control: A control Lyapunov function and receding horizon perspective," *Asian J. Control*, vol. 1, pp. 14–24, 1999.
[27] S. Prajna, A. Papachristodoulou, and F. Wu, "Nonlinear control synthesis by sum of squares optimization: A Lyapunov-based approach," in *Proc. Asian Control Conf.*, vol. 1, Jul. 2004, pp. 157–165.
[28] E. M. Aylward, P. A. Parrilo, and J.-J. E. Slotine, "Stability and robustness analysis of nonlinear systems via contraction metrics and SOS programming," *Automatica*, vol. 44, no. 8, pp. 2163 – 2170, 2008.
[29] G. Chesi, "LMI techniques for optimization over polynomials in control: A Survey," *IEEE Trans. Autom. Control*, vol. 55, no. 11, pp. 2500–2510, Nov. 2010.
[30] P. Florchinger, "Lyapunov-like techniques for stochastic stability," in *Proc. 33rd IEEE Conf. Decis. Control*, vol. 2, Dec. 1994, pp. 1145–1150.
[31] H. Deng, M. Krstic, and R. J. Williams, "Stabilization of stochastic nonlinear systems driven by noise of unknown covariance," *IEEE Trans. Autom. Control*, vol. 46, no. 8, pp. 1237–1253, Aug. 2001.
[32] S.-J. Liu and J.-F. Zhang, "Output-feedback control of a class of stochastic nonlinear systems with linearly bounded unmeasurable states," *Int. J. Robust Nonlinear Control*, vol. 18, no. 6, pp. 665–687, 2008.
[33] H. Deng and M. Krstic, "Stochastic nonlinear stabilization – I: A backstepping design," *Syst. Control Lett.*, vol. 32, no. 3, pp. 143 – 150, 1997.
[34] H. Deng and M. Krstic, "Output-feedback stochastic nonlinear stabilization," *IEEE Trans. Autom. Control*, vol. 44, no. 2, pp. 328–333, Feb. 1999.

[35] T. Basar and P. Bernhard, *$\mathcal{H}_\infty$ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*. Birkhauser, 1995.
[36] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Autom. Control*, vol. 34, no. 8, pp. 831–847, Aug. 1989.
[37] P. P. Khargonekar and M. A. Rotea, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: A convex optimization approach," *IEEE Trans. Autom. Control*, vol. 36, no. 7, pp. 824–837, Jul. 1991.
[38] L. Xie and E. de Souza Carlos, "Robust $\mathcal{H}_\infty$ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Autom. Control*, vol. 37, no. 8, pp. 1188–1191, Aug. 1992.
[39] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 896–911, Jul. 1997.
[40] D. Hinrichsen and A. Pritchard, "Stochastic $\mathcal{H}_\infty$," *SIAM J. Control Optim.*, vol. 36, no. 5, pp. 1504–1538, 1998.
[41] B.-S. Chen and W. Zhang, "Stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control with state-dependent noise," *IEEE Trans. Autom. Control*, vol. 49, no. 1, pp. 45–57, Jan. 2004.
[42] W. Q. Zhu, "Nonlinear stochastic dynamics and control in Hamiltonian formulation," *Appl. Mech. Rev.*, vol. 59, no. 4, pp. 230–248, Jul. 2006.
[43] E. Todorov and W. Li, "A generalized iterative LQG method for locally-optimal feedback control of constrained nonlinear stochastic systems," in *Proc. Amer. Control Conf.*, vol. 1, Jun. 2005, pp. 300–306.
[44] S. Peng, "A general stochastic maximum principle for optimal control problems," *SIAM J. Control Optim.*, vol. 28, no. 4, pp. 966–979, 1990.
[45] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, 2nd ed. Athena Scientific, 2000.
[46] E. Tse, Y. Bar-Shalom, and L. Meier, "Wide-sense adaptive dual control for nonlinear stochastic systems," *IEEE Trans. Autom. Control*, vol. 18, no. 2, pp. 98–108, Apr. 1973.
[47] D. S. Bayard, "A forward method for optimal stochastic nonlinear and adaptive control," *IEEE Trans. Autom. Control*, vol. 36, no. 9, pp. 1046–1053, Sep. 1991.
[48] A. Mesbah, "Stochastic model predictive control: An overview and perspectives for future research," *IEEE Control Syst. Mag.*, vol. 36, no. 6, pp. 30–44, Dec. 2016.
[49] A. Mesbah, S. Streif, R. Findeisen, and R. D. Braatz, "Stochastic nonlinear model predictive control with probabilistic constraints," in *Proc. Amer. Control Conf.*, Jun. 2014, pp. 2413–2419.
[50] M. Mahmood and P. Mhaskar, "Lyapunov-based model predictive control of stochastic nonlinear systems," *Automatica*, vol. 48, no. 9, pp. 2271 – 2276, 2012.
[51] E. A. Buehler, J. A. Paulson, and A. Mesbah, "Lyapunov-based stochastic nonlinear model predictive control: Shaping the state probability distribution functions," in *Amer. Control Conf.*, 2016, pp. 5389–5394.
[52] W. Wang and J.-J. E. Slotine, "On partial contraction analysis for coupled nonlinear oscillators," *Biol. Cybern.*, vol. 92, no. 1, pp. 38–53, Jan. 2005.
[53] J. Jouffroy and J.-J. E. Slotine, "Methodological remarks on contraction theory," in *43rd IEEE Conf. Decis. Control*, vol. 3, Dec. 2004, pp. 2537–2543.
[54] J.-J. E. Slotine, W. Wang, and K. El Rifai, "Contraction analysis of synchronization in networks of nonlinearly coupled oscillators," in *Int. Symp. Math. Theory Netw. Syst.*, Jul. 2004.
[55] W. Lohmiller and J.-J. E. Slotine, "Nonlinear process control using contraction theory," *AIChE Journal*, vol. 46, pp. 588 – 596, Mar. 2000.
[56] Q. Pham, "Analysis of discrete and hybrid stochastic systems by nonlinear contraction theory," in *Int. Conf. Control Automat. Robot. Vision*, Dec. 2008, pp. 1054–1059.
[57] H. Tsukamoto and S.-J. Chung, "Neural contraction metrics for robust estimation and control: A convex optimization approach," *IEEE Control Syst. Lett., under revision*, 2020.
[58] S.-J. Chung, S. Bandyopadhyay, I. Chang, and F. Y. Hadaegh, "Phase synchronization control of complex networks of Lagrangian systems on adaptive digraphs," *Automatica*, vol. 49, no. 5, pp. 1148 – 1161, 2013.
[59] L. Arnold, *Stochastic Differential Equations: Theory and Applications*. Wiley, 1974.
[60] T.-J. Tarn and Y. Rasis, "Observers for nonlinear stochastic systems," *IEEE Trans. Autom. Control*, vol. 21, no. 4, pp. 441–448, Aug. 1976.
[61] D. E. Kirk, *Optimal Control Theory: An Introduction*. Dover Publications, Apr. 2004.
[62] C. Jaganath, A. Ridley, and D. S. Bernstein, "A SDRE-based asymptotic observer for nonlinear discrete-time systems," in *Proc. Amer. Control Conf.*, vol. 5, Jun. 2005, pp. 3630–3635.
[63] B. T. Lopez and J.-J. E. Slotine, "Contraction metrics in adaptive nonlinear control," arXiv:1912.13138, 2019.
[64] Y. K. Nakka, A. Liu, G. Shi, A. Anandkumar, Y. Yue, and S.-J. Chung, "Chance-constrained trajectory optimization for safe exploration and learning of nonlinear systems," arXiv:2005.04374, 2020.
[65] A. Laub, "A Schur method for solving algebraic Riccati equations," *IEEE Trans. Autom. Control*, vol. 24, no. 6, pp. 913–921, Dec. 1979.
[66] D. Kleinman, "On an iterative technique for Riccati equation computations," *IEEE Trans. Autom. Control*, vol. 13, no. 1, pp. 114–115, Feb. 1968.
[67] D. Kleinman, "Stabilizing a discrete, constant, linear system with application to iterative methods for solving the Riccati equation," *IEEE Trans. Autom. Control*, vol. 19, no. 3, pp. 252–254, Jun. 1974.
[68] D. Lainiotis, "Generalized Chandrasekhar algorithms: Time-varying models," *IEEE Trans. Autom. Control*, vol. 21, no. 5, pp. 728–732, Oct. 1976.
[69] B. D. O. Anderson and J. B. Moore, *Optimal control: Linear Quadratic Methods*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1990.
[70] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2012.
[71] W. Tan, "Nonlinear control analysis and synthesis using sum-of-squares programming," Ph.D. dissertation, Univ. California, Berkeley, 2006.
[72] M. Grant and S. Boyd, "CVX: Matlab software for disciplined convex programming, version 2.1," http://cvxr.com/cvx, Mar. 2014.
[73] M. C. Grant and S. P. Boyd, "Graph implementations for nonsmooth convex programs," in *Recent Advances Learn. Control*. Springer London, 2008, pp. 95–110.
[74] S.-J. Chung, U. Ahsun, and J.-J. E. Slotine, "Application of synchronization to formation flying spacecraft: Lagrangian approach," *J. Guid. Control Dyn.*, vol. 32, no. 2, pp. 512–526, Mar. 2009.

Hiroyasu Tsukamoto (M'19) received the B.S. degree in aerospace engineering from Kyoto University, Kyoto, Japan, in 2017 and the M.S. degree in space engineering from California Institute of Technology (Caltech), Pasadena, CA, USA, in 2018. He is currently pursuing the Ph.D. degree in space engineering at Caltech. His research interests include autonomous decision making and control of general nonlinear systems, machine learning-based optimal control, estimation, and motion planning of aerial swarms and autonomous aerospace systems. Mr. Tsukamoto is a recipient of the Caltech Vought Fellowship and the Funai Overseas Scholarship for graduate studies.

Soon-Jo Chung (M'06–SM'12) received the B.S. degree (*summa cum laude*) in aerospace engineering from the Korea Advanced Institute of Science and Technology, Daejeon, South Korea, in 1998, and the S.M. degree in aeronautics and astronautics and the Sc.D. degree in estimation and control from Massachusetts Institute of Technology, Cambridge, MA, USA, in 2002 and 2007, respectively.

He is currently Bren Professor of Aerospace and a Jet Propulsion Laboratory Research Scientist in the California Institute of Technology, Pasadena, CA, USA. He was with the faculty of the University of Illinois at Urbana-Champaign (UIUC) during 20092016. His research interests include spacecraft and aerial swarms and autonomous aerospace systems, and in particular, on the theory and application of complex nonlinear dynamics, control, estimation, guidance, and navigation of autonomous space and air vehicles.

Dr. Chung was the recipient of the UIUC Engineering Deans Award for Excellence in Research, the Beckman Faculty Fellowship of the UIUC Center for Advanced Study, the U.S. Air Force Office of Scientific Research Young Investigator Award, the National Science Foundation Faculty Early Career Development Award, and three Best Conference Paper Awards from the IEEE and the American Institute of Aeronautics and Astronautics. He is an Associate Editor of IEEE TRANSACTIONS ON ROBOTICS, IEEE TRANSACTIONS ON AUTOMATIC CONTROL (since November 2019), and Journal of Guidance, Control, and Dynamics.