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ASYMPTOTIC STABILIZATION OF CONTINUOUS-TIME PERIODIC STOCHASTIC SYSTEMS BY FEEDBACK CONTROL BASED ON PERIODIC DISCRETE-TIME OBSERVATIONS

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Abstract. In 2013, Mao initiated the study of stabilization of continuous-time hybrid stochastic differential equations (SDEs) by feedback control based on discrete-time state observations. In recent years, this study has been further developed while using a constant observation interval. However, time-varying observation frequencies have not been discussed for this study. Particularly for non-autonomous periodic systems, it's more sensible to consider the time-varying property and observe the system at periodic time-varying frequencies, in terms of control efficiency. This paper introduces a periodic observation interval sequence, and investigates how to stabilize a periodic SDE by feedback control based on periodic observations, in the sense that, the controlled system achieve L^p-stability for $p > 1$, almost sure asymptotic stability and pth moment asymptotic stability for $p \geq 2$. This paper uses the Lyapunov method and inequalities to derive the theory. We also verify the existence of the observation interval sequence and explains how to calculate it. Finally, an illustrative example is given after a useful corollary. By considering the time-varying property of the system, we reduce the observation frequency dramatically and hence reduce the observational cost for control.

1. Introduction.

In the past decades, stochastic differential equations have been playing an important role in many areas such as engineering, finance and population ecology. Hybrid SDEs (SDEs with Markovian switching) have been widely used for modelling systems that may undergo abrupt changes in structures and parameters, which can be caused by environmental disturbances or accidents. Automatic control and stability analysis of SDEs have been studied by many authors (e.g. [2, 5, 7, 8, 9, 13, 14, 17, 18, 19, 21, 26, 30, 32]).

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Consider a continuous-time hybrid SDE in the Itô sense

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{1}$$

on $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the system state, $B(t) = (B_1(t), \cdots, B_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain (please see Section 2 for formal definitions) which represents the system mode. If system (1) is not stable and need to be stabilized by a feedback control, a traditional (or regular) choice $u(x(t), r(t), t)$ requires continuous-time observations of state $x(t)$ and mode $r(t)$, which is unrealistic as the observations are often of discrete-time and can be expensive for implementation.

As a result, Mao [16] initiated the study of stabilization of (1) by feedback control based on discrete-time state observations. Later, the observation interval was increased, more stabilities and uncertain systems were also investigated ([6, 20, 33, 34]). Recently, observation of system mode has also been discretized ([10, 12, 25]) and the controlled system regarding to (1) becomes

$$dx(t) =[f(x(t), r(t), t) + u(x([t/\tau]\tau), r([t/\tau]\tau), t)]dt + g(x(t), r(t), t)dB(t), \tag{2}$$

where τ is a positive number representing observation interval, $[t/\tau]$ denotes the integer part of t/τ. So the controller u needs observations of x and r only at time points $0, \tau, 2\tau, 3\tau, \cdots$.

Although such a type of feedback control based on discrete-time observations is more realistic and costs less than the traditional one, it fails to consider the time-varying property of the system. If the controlled system is non-autonomous (i.e., f or g or u depends on time explicitly), then a time-varying observation frequency obviously makes more sense than the constant one. Intuitively, when the system state or mode change rapidly, we should observe them very frequently and vice versa.

A particular interest for a time-varying system is its periodicity. Periodic phenomena are all around us, such as satellite orbit, seasons, wave vibration, etc. Stochastic models involving periodicity have been studied by many authors (see e.g. [1, 3, 22, 23, 27, 28, 29, 31]) due to their wide applications in many areas. In addition, a simple generalization of the existing analysis techniques cannot be applied to the time-varying observations, for stabilization of periodic SDEs.

Motivated by above discussion, this paper investigates how to stabilize a non-autonomous periodic[1] SDE with or without Markovian switching, by a periodic feedback control based on periodic discrete-time observations, to achieve $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$-stability, pth moment asymptotic stability for $p \geq 2$, and almost sure asymptotic stability.

If the system coefficients and controller are all periodic, then it makes sense to use periodic observations. As we know, the observation interval is the time length between two observations. Define our periodic observation interval sequence to be $\{\tau_j\}_{j\geq 1}$ such that

$$\tau_{kM+j} = \tau_j$$

for a positive integer M, $\forall k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, M$. This means the system is observed at time points $0, \tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3, \cdots$.

[1]We say an SDE is non-autonomous periodic if its coefficients change with time explicitly periodically.

Note that for any $t \geq 0$, there is a positive integer k such that

$$\sum_{j=1}^{k} \tau_j \leq t < \sum_{j=1}^{k+1} \tau_j,$$

then we can define a step function

$$\delta_t := \sum_{j=1}^{k} \tau_j. \tag{3}$$

Consequently, our controlled system regarding to (1) has the form

$$dx(t) = [f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]dt + g(x(t), r(t), t)dB(t). \tag{4}$$

Similarly, given an unstable periodic SDE in the Itô sense

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \tag{5}$$

we can design a feedback control u and make the controlled system

$$dx(t) = [f(x(t), t) + u(x(\delta_t), t)]dt + g(x(t), t)dB(t) \tag{6}$$

stable with appropriate observation frequencies.

By considering the time-varying property into the analysis, we increase the observation intervals. In other words, we reduce the observation frequency and the cost of control.

The remainder of this paper is organised as follows. Section 2 explained the notations. Sections 3 and 4 establish the theory for hybrid SDEs and single-mode SDEs respectively. Before the conclusion in Section 6, Section 5 presents a numerical example for illustration.

2. **Notation.**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is increasing and right continuous with $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\emptyset$ denote the empty set. Let $\mathbb{R}_+$ denote the set of all non-negative real numbers $[0, \infty)$. We write the transpose of a matrix or vector A as A^T. Denote the m-dimensional Brownian motion defined on the probability space by $B(t) = (B_1(t), \cdots, B_m(t))^T$. For a vector x, $|x|$ means its Euclidean norm. For a matrix Q, its trace norm $|Q| = \sqrt{\text{trace}(Q^T Q)}$ and its operator norm $\|Q\| = \max\{|Qx| : |x| = 1\}$. For a real symmetric matrix Q, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ mean its smallest and largest eigenvalues respectively. For a subset A of Ω, denote by I_A its indicator function; namely $I_A(\omega) = 1$ when $\omega \in A$ and 0 otherwise. Denote by $L^p(\Omega; \mathbb{R}^n)$ the family of $\mathbb{R}^n$-valued random variables x such that $\mathbb{E}|x|^p < \infty$. For $a, b \in \mathbb{R}$, $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Let $r(t)$ for $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator matrix $\Gamma = (\gamma_{ij})_{N\times N}$, whose elements γ_{ij} are the transition rates from state i to j for $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. Define a positive number $\overline{\gamma} := -\min_{i\in\mathbb{S}} \gamma_{ii}$.

For any $t \in [\sum_{j=1}^{k} \tau_j, \sum_{j=1}^{k+1} \tau_j)$, define a step function $\kappa_t := \tau_{k+1}$. This means $\delta_t \leq t < \delta_t + \kappa_t$. For example, when $t \in [0, \tau_1)$, we have $\delta_t = 0$ and $\kappa_t = \tau_1$; when $t \in [\tau_1, \tau_1 + \tau_2)$, we have $\delta_t = \tau_1$ and $\kappa_t = \tau_2$; when $t \in [\tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3)$, we have $\delta_t = \tau_1 + \tau_2$ and $\kappa_t = \tau_3$; $\cdots$. Obviously κ_t is a periodic step function of time.

Define two positive parameters depending on the moment order p:

$$\rho = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1,2), \\ [\frac{p(p-1)}{2}]^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases}$$

and

$$\xi = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1,2), \\ \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases}$$

3. **SDEs with Markovian switching.** Consider an n-dimensional periodic hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{7}$$

on $t \geq 0$, with initial values $x(0) = x_0 \in L^p(\Omega; \mathbb{R}^n)$ for $p > 1$ and $r(0) = r_0 \in \mathbb{S}$. Here

$$f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n \times m}.$$

The given system may not be stable and our aim is to design a feedback control $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ for stabilization.

The controlled system corresponding to (7) has the form

$$dx(t) = [f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]dt + g(x(t), r(t), t)dB(t). \tag{8}$$

Assumption 3.1. Assume that $f(x, i, t)$, $g(x, i, t)$ and $u(x, i, t)$ are all periodic with respect to time t. Assume f, g, u and κ_t have a common period T.

The assumption that T is a period of κ_t means $\kappa_t = \kappa_{t+kT}$ for $k = 0, 1, 2, \cdots$ and $\sum_{j=1}^{M} \tau_j = T$.

Assumption 3.2. Assume that the coefficients $f(x, i, t)$ and $g(x, i, t)$ are both locally Lipschitz continuous on x (see e.g. [19]), and they both satisfy the following linear growth condition

$$|f(x, i, t)| \leq K_1(t)|x| \quad \text{and} \quad |g(x, i, t)| \leq K_2(t)|x| \tag{9}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $K_1(t)$ and $K_2(t)$ are periodic non-negative continuous functions with period T.

Note (9) implies that

$$f(0, i, t) = 0 \quad \text{and} \quad g(0, i, t) = 0 \tag{10}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.3. Assume

$$|u(x, i, t) - u(y, i, t)| \leq K_3(t)|x - y| \tag{11}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $K_3(t)$ is a periodic non-negative continuous function with period T. Moreover, we also assume

$$u(0, i, t) = 0 \tag{12}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.3 implies that $u(x,i,t)$ is globally Lipschitz continuous on x and satisfies the following linear growth condition

$$|u(x,i,t)| \le K_3(t)|x| \tag{13}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Let

$$\overline{K_1} = \max_{0\le t\le T} K_1(t), \quad \overline{K_2} = \max_{0\le t\le T} K_2(t) \quad \text{and} \quad \overline{K_3} = \max_{0\le t\le T} K_3(t).$$

Denote the largest observation interval $\max_{j\ge 1}\tau_j$ by $\tau_{\max}$. For stabilization purpose, we define the following initial values

$$\begin{aligned} &x(s) = x_0, \;\; r(s) = r_0, \;\; f(x,i,s) = f(x,i,0), \\ &u(x,i,s) = u(x,i,0) \;\text{ and }\; g(x,i,s) = g(x,i,0) \end{aligned}$$

for all $(x,i,s) \in \mathbb{R}^n \times \mathbb{S} \times [-\tau_{\max}, 0)$.

Notice that the controlled system (8) can be written as a stochastic differential delay equation

$$dx(t) = \big[f(x(t),r(t),t) + u\big(x(t-\eta(t)), r(t-\eta(t)), t\big)\big]dt + g(x(t),r(t),t)dB(t),$$

where $\eta(t) = t - \delta_t \in [0,\tau_{\max})$ is a Borel measurable function. Then the Lipschitz condition and linear growth condition required by Assumptions 3.2 and 3.3 guarantees the existence and uniqueness of the solution and $\mathbb{E}|x(t)|^p < \infty$ for all $t \ge 0$ and $p > 1$ (see e.g. Theorem 7.3 and page 304 in [19]).

Let $V(x,i,t)$ be a Lyapunov function periodic with respect to t, and we require $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$. Then define $\mathcal{L}V : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x,i,t) = &V_t(x,i,t) + V_x(x,i,t)[f(x,i,t) + u(x,i,t)] \\ &+ \frac{1}{2}\mathrm{trace}[g^T(x,i,t)V_{xx}(x,i,t)g(x,i,t)] + \sum_{k=1}^N \gamma_{ik}V(x,k,t). \end{aligned} \tag{14}$$

We impose an assumption on the Lyapunov function.

Assumption 3.4. Assume that there is a Lyapunov function $V(x,i,t)$ and a positive continuous function $\lambda(t)$ which have common period T, constants $l > 0$ and $p > 1$ such that

$$\mathcal{L}V(x,i,t) + l|V_x(x,i,t)|^{\frac{p}{p-1}} \le -\lambda(t)|x|^p \tag{15}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times [0,T]$.

Let $\underline{\lambda} = \min_{0\le t\le T}\lambda(t)$.

Let us divide $[0,T]$ into $Z-1$ subintervals, where $Z \ge 2$ is an arbitrary integer, by choosing a partition $\{T_j\}_{1\le j\le Z}$ with $T_1 = 0$ and $T_Z = T$. Then we define the following three step functions on $t \ge 0$ with periodic T:

$$\begin{aligned} \hat{K}_{1t} &= \sup_{T_j\le s\le T_{j+1}} K_1(s) \quad \text{for } T_j \le t < T_{j+1}, \\ \hat{K}_{2t} &= \sup_{T_j\le s\le T_{j+1}} K_2(s) \quad \text{for } T_j \le t < T_{j+1}, \\ \hat{K}_{3t} &= \sup_{T_j\le s\le T_{j+1}} K_3(s) \quad \text{for } T_j \le t < T_{j+1}, \end{aligned} \tag{16}$$

where $j = 1, \cdots, Z-1$.

Define a periodic function [2]

$$\beta(t) := \beta(\kappa_t, t) = \lambda(t) - \frac{1}{p}(\frac{p-1}{pl})^{p-1}$$
$$\times \Big(K_3^p(t)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_t}) + \frac{2^{p-1}\kappa_t^{\frac{p}{2}}\hat{K}_{3t}^p}{1-8^{p-1}\kappa_t^p\hat{K}_{3t}^p}[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_t}) + 2^{p-1}]$$
$$\times [2^{p-1}\kappa_t^{\frac{p}{2}}K_1^p(t) + \rho K_2^p(t) + 4^{p-1}\kappa_t^{\frac{p}{2}}K_3^p(t)]\Big). \tag{17}$$

3.1. **Main results.**

Theorem 3.5. *Fix the moment order $p > 1$. Let Assumptions 3.1, 3.2, 3.3 and 3.4 hold. Divide $[0, T]$ into $Z - 1$ subintervals with $T_1 = 0$ and $T_Z = T$. Choose the observation interval sequence $\{\tau_j\}_{1\le j\le M}$ sufficiently small such that $\kappa_t \le T_{j+1} - T_j$ for $t \in [T_j, T_{j+1})$ [3] where $j = 1, 2, \cdots, Z-1$ and*

$$\beta(t) > 0 \quad and \quad \kappa_t\hat{K}_{3t} < 8^{-\frac{p-1}{p}} \quad for \quad \forall t \in [0, T), \tag{18}$$

where $\beta(t)$ has been defined in (17). Then the controlled system (8) is $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$-stable in the sense

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty \tag{19}$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

We will explain why such an observation interval sequence exists and how to calculate it step by step after the proof.

Proof. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. Applying the generalized Itô formula to $V(x(t), r(t), t)$ gives

$$dV(x(t), r(t), t) = LV(x(t), r(t), t)dt + dM(t)$$

for $t \ge 0$, where $M(s)$ is a continuous local martingale with $M(0) = 0$ and

$$LV(x(t), r(t), t)$$
$$= V_t(x(t), r(t), t) + V_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]$$
$$+ \frac{1}{2}\text{trace}[g^T(x(t), r(t), t)V_{xx}(x(t), r(t), t)g(x(t), r(t), t)] + \sum_{k=1}^N \gamma_{ik}V(x, k, t). \tag{20}$$

Since $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, we can use the generalized Itô formula and get

$$\mathbb{E}V(x(t), r(t), t) = V_0 + \int_0^t \mathbb{E}LV(x(s), r(s), s)ds, \tag{21}$$

where $V_0 = V(x(0), r(0), 0)$.

We can rewrite $LV(x(s), r(s), s)$ as

$$LV(x(s), r(s), s)$$
$$= \mathcal{L}V(x(s), r(s), s) - V_x(x(s), r(s), s)[u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)]. \tag{22}$$

[2]It can be seen that T is a period of $\beta(t)$.

[3]In other words, the length of the subinterval cannot be shorter than the observation interval.

By Young's inequality, we can derive that

$$
\begin{aligned}
&-V_x(x(s),r(s),s)[u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)] \\
\leq&\Big[\varepsilon|V_x(x(s),r(s),s)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \\
&\times\Big[\varepsilon^{1-p}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p\Big]^{\frac{1}{p}} \\
\leq& l|V_x(x(s),r(s),s)|^{\frac{p}{p-1}} \\
&+\frac{1}{p}(\frac{p-1}{pl})^{p-1}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p,
\end{aligned}
\tag{23}
$$

where $l=\frac{p-1}{p}\varepsilon$ for $\forall\varepsilon>0$.

Using the elementary inequality $|a+b|^p\leq 2^{p-1}(|a|^p+|b|^p)$ for $a,b\in\mathbb{R}$ and $p>1$, we have

$$
\begin{aligned}
&\mathbb{E}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p \\
\leq&2^{p-1}\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p \\
&+2^{p-1}\mathbb{E}|u(x(\delta_s),r(s),s)-u(x(s),r(s),s)|^p.
\end{aligned}
\tag{24}
$$

According to Lemma 1 in [25], for any $t\geq t_0$, $v>0$ and $i\in\mathbb{S}$,

$$
\mathbb{P}(r(s)\neq i \text{ for some } s\in[t,t+v]\Big|r(t)=i)\leq 1-e^{-\overline{\gamma}v}. \tag{25}
$$

By Assumption 3.3, we have

$$
\begin{aligned}
&\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p \\
=&\mathbb{E}\Big[\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p\Big|\mathcal{F}_{\delta_s})\Big] \\
\leq&\mathbb{E}\Big[2^pK_3^p(s)|x(\delta_s)|^p\mathbb{E}\Big(I_{\{r(s)\neq r(\delta_s)\}}|\mathcal{F}_{\delta_s}\Big)\Big] \\
\leq&2^{2p-1}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})[\mathbb{E}|x(s)|^p+\mathbb{E}|x(\delta_s)-x(s)|^p].
\end{aligned}
\tag{26}
$$

Substituting (26) into (24) gives

$$
\begin{aligned}
&\mathbb{E}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p \\
\leq&2^{3p-2}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})\mathbb{E}|x(s)|^p \\
&+[2^{3p-2}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}K_3^p(s)]\mathbb{E}|x(\delta_s)-x(s)|^p.
\end{aligned}
\tag{27}
$$

Substitute (27) into (23). Then substitute the result into (22). By Assumption 3.4, we obtain that

$$
\begin{aligned}
&\mathbb{E}LV(x(s),r(s),s) \\
\leq&-[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})]\mathbb{E}|x(s)|^p \\
&+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(\delta_s)-x(s)|^p.
\end{aligned}
\tag{28}
$$

Note that $t-\delta_t\leq\kappa_t$ for all $t\geq 0$. By the Itô formula, Hölder's inequality, the Burkholder-Davis-Gundy inequality (see e.g. [15, p.40]) and [15, Theorem 7.1 on page 39], we obtain that (see e.g. [6])

$$\begin{aligned}&\mathbb{E}|x(t)-x(\delta_t)|^p \le 2^{p-1}\kappa_t^{\frac{p-2}{2}}\\ &\times \mathbb{E}\int_{\delta_t}^t \Big[\kappa_t^{\frac{p}{2}}|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^p+\rho|g(x(s),r(s),s)|^p\Big]ds. \qquad (29)\end{aligned}$$

By Assumptions 3.2 and 3.3, we have that for any $s\in[\delta_s,\delta_s+\kappa_s)$,

$$\begin{aligned}&\mathbb{E}|x(s)-x(\delta_s)|^p\\ \le&2^{p-1}\kappa_s^{\frac{p-2}{2}}\int_{\delta_s}^s[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)]\mathbb{E}|x(z)|^pdz\\ &+8^{p-1}\kappa_s^{p-1}\int_{\delta_s}^s K_3^p(z)dz[\mathbb{E}|x(s)-x(\delta_s)|^p+\mathbb{E}|x(s)|^p].\end{aligned}$$

Since condition (18) guarantees $8^{p-1}\kappa_s^p\hat{K}_{3s}^p<1$, we can rearrange it and get

$$\begin{aligned}&\mathbb{E}|x(s)-x(\delta_s)|^p\\ \le&\frac{8^{p-1}\kappa_s^p\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\mathbb{E}|x(s)|^p\\ &+\frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\int_{\delta_s}^s\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^pdz. \qquad (30)\end{aligned}$$

Recall $\tau_{\max}=\max_{j\ge1}\tau_j$. Let $x(s)=x_0$, $r(s)=r_0$, $K_1(s)=K_1(0)$, $K_2(s)=K_2(0)$ and $K_3(s)=K_3(0)$ for all $(x,i,s)\in\mathbb{R}^n\times\mathbb{S}\times[-\tau_{\max},0)$. In addition, note that for $\forall z\in[\delta_s,s]$, we have $\kappa_z=\kappa_s$ and $K_3(s)\le\hat{K}_{3s}=\hat{K}_{3z}$. Since $s-\kappa_s<\delta_s$, it's easy to show that for a non-negative bounded function $F(t)$,

$$\begin{aligned}&\int_0^t\int_{\delta_s}^s F(z)dzds\le\int_0^t\int_{s-\kappa_s}^s F(z)dzds\\ \le&\int_{-\kappa_z}^t F(z)\int_z^{z+\kappa_z}dsdz\le\int_{-\kappa_s}^t\kappa_zF(z)dz\le C+\int_0^t\kappa_zF(z)dz. \qquad (31)\end{aligned}$$

Here we explain the second inequality: $\int_0^t\int_{s-\kappa_s}^s F(z)dzds$ is an integral on domain $\{z\times s: s-\kappa_s\le z\le s, 0\le s\le t\}$. In other words, $z\le s\le z+\kappa_s$. Then we can derive that $-\kappa_s\le s-\kappa_s\le z\le s\le t$. Notice the integral domain

$$\{z\times s: s-\kappa_s\le z\le s, 0\le s\le t\}\subset\{z\times s: z\le s\le z+\kappa_s, -\kappa_s\le z\le t\}.$$

Recall that $\kappa_z = \kappa_s$, we can obtain the second inequality in (31).
Then

$$
\begin{aligned}
&\int_0^t K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3z}^p}\\
&\quad\times\int_{\delta_s}^s[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)]\mathbb{E}|x(z)|^p dzds\\
\le&\int_0^t\int_{\delta_s}^s\hat{K}_{3z}^p[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_z})+2^{p-1}]\frac{2^{p-1}\kappa_z^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_z^p\hat{K}_{3z}^p}\\
&\quad\times\Big[2^{p-1}\kappa_z^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dzds\\
\le& C+\int_0^t\frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}\Big]\\
&\quad\times\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)\Big]\mathbb{E}|x(s)|^p ds.
\end{aligned}
$$

where, and in the remaining part of this paper, C denotes a positive constant that may change from line to line but its special form is not used.

So

$$
\begin{aligned}
&\int_0^t K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(s)-x(\delta_s)|^p ds\\
\le& C+\int_0^t\frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}\Big]\\
&\quad\times\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)+4^{p-1}\kappa_s^{\frac{p}{2}}K_3^p(s)\Big]\mathbb{E}|x(s)|^p ds. \qquad (32)
\end{aligned}
$$

Substitute (32) into (28), then substitute the result into (22). By (17), we have

$$
\begin{aligned}
&\mathbb{E}V(x(t),r(t),t)\\
=&V_0+\int_0^t\mathbb{E}LV(x(s),r(s),s)ds\\
\le& C-\int_0^t\Big[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})\Big]\mathbb{E}|x(s)|^p ds\\
&+\int_0^t\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(s)-x(\delta_s)|^p ds\\
\le& C-\int_0^t\beta(s)\mathbb{E}|x(s)|^p ds.
\end{aligned}
$$

By definition of the Lyapunov function V, we have that for $\forall t\ge 0$,

$$0\le\mathbb{E}V(x(t),r(t),t)\le C-\int_0^t\beta(s)\mathbb{E}|x(s)|^p ds.$$

Then $\int_0^t\beta(t)\mathbb{E}|x(t)|^p dt\le C$. Let $\underline{\beta}=\inf_{0\le t<T}\beta(t)(>0)$. Then we have

$$\underline{\beta}\int_0^\infty\mathbb{E}|x(s)|^p ds\le\int_0^\infty\beta(t)\mathbb{E}|x(t)|^p dt\le C.$$

Hence we obtain assertion (19). $\square$

We use the same observation frequency in one subinterval of $[0,T]$. Observation interval sequence can be calculated[4] in three steps:

Step 1. The first step is to divide $[0,T]$ into $Z-1$ subintervals and we propose two ways to do it.
One is simple even division: all $Z-1$ subintervals have the same length and $T_j = \frac{j-1}{Z-1}T$ for $1 \le j \le Z$.
The other way is by an auxiliary function $\tilde{\tau_a}(t)$, which satisfies

$$\begin{aligned} 0 \le &\lambda(t) - \frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big(K_3^p(t)2^{3p-2}(1-e^{-\overline{\gamma}\tilde{\tau_a}(t)}) \\ &+ \frac{2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)}{1-8^{p-1}\tilde{\tau_a}^p(t)K_3^p(t)}[2^{3p-2}(1-e^{-\overline{\gamma}\tilde{\tau_a}(t)})+2^{p-1}] \\ &\times [2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_1^p(t)+\rho K_2^p(t)+4^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)]\Big). \end{aligned} \tag{33}$$

Notice that, when $\tilde{\tau_a}=0$, the right-hand-side of (33) equals to $\lambda(t)$ and it decreases as $\tilde{\tau_a}(t)$ increases. To get a low observation frequency (wide observation interval), we want to set $\tilde{\tau_a}(t)$ as large as possible with (33) holds. Then divide $[0,T]$ into $Z-1$ subintervals according to the shape of $\tilde{\tau_a}(t)$. We want the supremum and the infimum of $\tilde{\tau_a}(t)$ in each subinterval are relatively close. This means, if $\tilde{\tau_a}(t)$ changes slowly over a time interval, then we can set a wide subinterval in this time interval; otherwise if $\tilde{\tau_a}(t)$ changes rapidly over a time interval, then we need to set several narrow subintervals in this time interval.

Step 2. For the jth subinterval (i.e., for $t \in [T_j, T_{j+1})$), find a function $\tilde{\tau_j}(t) \in (0, 8^{-\frac{p-1}{p}\hat{K}_{3t}})$ with $\inf_{t\in[T_j,T_{j+1})}\tilde{\tau_j}(t) > 0$ such that

$$\inf_{t\in[T_j,T_{j+1})} \beta(\tilde{\tau_j}(t),t) > 0, \tag{34}$$

where β has been defined in (17).
Find $\tilde{\tau_j}(t)$ for all $1 \le j \le Z-1$.

Step 3. For the jth subinterval $(1 \le j \le Z-1)$, choose a positive integer N_j such that

$$\frac{T_{j+1}-T_j}{N_j} < \inf_{t\in[T_j,T_{j+1})} \tilde{\tau_j}(t).$$

Then let

$$\underline{\kappa}_j = \frac{T_{j+1}-T_j}{N_j}.$$

So the observation interval is $\underline{\kappa}_j$ and we observe N_j times on the jth subinterval. In other words, the system is observed at $t = T_j, T_j+\underline{\kappa}_j, T_j+2\underline{\kappa}_j, \cdots, T_j+N_j\underline{\kappa}_j$, where $T_j + N_j\underline{\kappa}_j = T_{j+1}$.
Find N_j and $\underline{\kappa}_j$ for all $1 \le j \le Z-1$.

Consequently, our observation interval sequence for one period $[0,T)$ is:
$\tau_1 = \underline{\kappa}_1, \cdots, \tau_{N_1} = \underline{\kappa}_1,$
$\tau_{N_1+1} = \underline{\kappa}_2, \cdots, \tau_{N_1+N_2} = \underline{\kappa}_2,$
$\vdots$
$\tau_{N_1+\cdots+N_{Z-2}+1} = \underline{\kappa}_{Z-1}, \cdots, \tau_{N_1+\cdots+N_{Z-1}} = \underline{\kappa}_{Z-1}.$
Besides, we always observe once at $t = kT$ where $k = 0,1,2,\cdots$.

[4] Obviously there are other ways to guarantee that the observation interval sequence satisfies the conditions in Theorem 3.5. Here we only show one method.

Now let us explain why we can find a positive sequence satisfying condition (18). When observation interval $\kappa_t = 0$, $\beta(t) = \lambda(t)$. When $\kappa_t < 8^{-\frac{p-1}{p}\hat{K}_{3t}}$, κ_t, $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\overline{\gamma}$ are all negative related to $\beta(t)$. Increase of κ_t leads to decrease of $\beta(t)$. Large values of $K_1(t)$, $K_2(t)$ and large $K_3(t)$ would lead to small κ_t, to guarantee $\beta(t) > 0$ for any $0 \le t < T$. Notice that increase of $\hat{K}_{3t}$ leads to decrease of $\beta(t)$. Then $\tilde{\tau_a}(t)$ defined in (33) is larger than the observation interval. Specifically,

$$0 < \min_{1 \le j \le Z-1} \kappa_j \le \min_{1 \le j \le Z-1} \inf_{t \in [T_j, T_{j+1})} \tilde{\tau_j}(t) \le \inf_{0 \le t < T} \tilde{\tau_a}(t).$$

Under the condition (18), large $K_1(t), K_2(t)$ and $K_3(t)$ would lead to small κ_t. Notice that large values of $K_1(t), K_2(t)$ and $K_3(t)$ indicate large values of coefficients, which imply rapid change of the system state $x(t)$. In other words, (18) requires that when system changes fast, observations need to be more frequently. Similarly, a large $\overline{\gamma}$ is corresponding to a small κ_t, under the condition (18). This means if the system mode switches rapidly, then observations need to be very frequently.

Theorem 3.6. *Fix the moment order $p \ge 2$. Under the same assumptions and conditions of Theorem 3.5, the solution of the controlled system (8) satisfies*

$$\lim_{t \to \infty} \mathbb{E}|x(t)|^p = 0 \tag{35}$$

and

$$\lim_{t \to \infty} x(t) = 0 \quad a.s. \tag{36}$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. In other words, the controlled system (8) is asymptotically stable in pth moment and almost surely.

Proof. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. By the Itô formula, Assumptions 3.2 and 3.3, we have that for any $t \ge 0$,

$$\mathbb{E}(|x(t)|^p) \le |x_0|^p + \int_0^t \Big(p\overline{K_1}\mathbb{E}|x(s)|^p + p\overline{K_3}\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \theta\overline{K_2}^2\mathbb{E}|x(s)|^p\Big)ds, \tag{37}$$

where $\theta = \frac{p(p-1)}{2}$.

By the Young inequality, we get

$$|x(s)|^{p-1}|x(\delta_s)| \le \frac{2^{p-1}}{p^p}\Big[[(p-1)^p + 2^{p-1}]|x(s)|^p + 2^{p-1}|x(s) - x(\delta_s)|^p\Big]. \tag{38}$$

Substituting this into (37) gives

$$\mathbb{E}(|x(t)|^p) \le |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t \mathbb{E}|x(s) - x(\delta_s)|^p ds. \tag{39}$$

Recall that C's denote positive constants that may change from line to line.

Denote $\sup_{0\le t<T}(\kappa_t \hat{K}_{3t})$ by H, then (18) guarantees $8^{\frac{p-1}{p}} H < 1$. By (30) and (31), we have

$$\begin{aligned}
&\int_0^t \mathbb{E}|x(s)-x(\delta_s)|^p ds\\
\le&\frac{8^{p-1}\tau_{\max}^p\overline{K_3}^p}{1-8^{p-1}H^p}\int_0^t \mathbb{E}|x(s)|^p ds\\
&+\frac{2^{p-1}\tau_{\max}^{\frac{p-2}{2}}(2^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_1}^p+\rho\overline{K_2}^p)}{1-8^{p-1}H^p}\int_0^t\int_{\delta_s}^s \mathbb{E}|x(z)|^p dz ds\\
\le& C+\frac{2^{p-1}\tau_{\max}^{\frac{p}{2}}}{1-8^{p-1}H^p}[2^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_1}^p+\rho\overline{K_2}^p+4^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_3}^p]\int_0^t \mathbb{E}|x(s)|^p ds. \qquad (40)
\end{aligned}$$

Substituting this into (39) yields

$$\mathbb{E}|x(t)|^p \le C+|x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds. \qquad (41)$$

So by Theorem 3.5, we have

$$\mathbb{E}|x(t)|^p \le C \quad \forall t \ge 0. \qquad (42)$$

In addition, it follows from the Itô formula that

$$\begin{aligned}
&\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\\
\le&\int_{t_1}^{t_2}\Big(p\overline{K_1}\mathbb{E}|x(t)|^p+p\overline{K_3}\mathbb{E}\Big[|x(t)|^{p-1}|x(\delta_t)|\Big]+\theta\overline{K_2}^2\mathbb{E}|x(t)|^p\Big)dt.
\end{aligned}$$

Then by (38), (40) and (42), we get that for any $0\le t_1<t_2<\infty$,

$$\Big|\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\Big| \le C\int_{t_1}^{t_2}\mathbb{E}|x(t)|^p dt \le C(t_2-t_1). \qquad (43)$$

Apply Barbalat's lemma (see e.g. [24, page 123]) to $\int_0^t \mathbb{E}|x(s)|^p ds$, by combining the uniform continuity of $\mathbb{E}|x(t)|^p$, which is shown in (43), with Theorem 3.5. Thus we can obtain assertion (35). The proof of almost sure asymptotic stability (36) is similar to the proof of Theorem 2 in [11] and Theorem 3.4 in [34]. So we only present the sketch of the proof in the Appendix. □

In practice, a frequent choice of Lyapunov functions is quadratic functions, for example, $V(x(t),r(t),t)=(x^T(t)Q_{r(t)}x(t))^{\frac{p}{2}}$ where $Q_{r(t)}$ are positive-definite $n\times n$ matrices for $p\ge 2$. By calculating its partial derivatives, we propose the following alternative assumption and corollary.

Assumption 3.7. Assume that there exist positive-definite symmetric matrices $Q_i\in\mathbb{R}^{n\times n}$ $(i\in\mathbb{S})$ and a periodic positive continuous function $b(t)$ such that

$$\begin{aligned}
&p(x^TQ_ix)^{\frac{p}{2}-1}\Big(x^TQ_i[f(x,i,t)+u(x,i,t)]+\frac{1}{2}\text{trace}[g^T(x,i,t)Q_ig(x,i,t)]\Big)\\
&+p(\frac{p}{2}-1)[x^TQ_ix]^{\frac{p}{2}-2}|g^TQ_ix|^2+\sum_{j=1}^N\gamma_{ij}[x^TQ_jx]^{\frac{p}{2}}\le -b(t)|x|^p, \qquad (44)
\end{aligned}$$

for all $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times[0,T]$.

We can see that T is a period of $b(t)$. Let $\underline{b}=\min_{0\le t\le T} b(t)$.

Corollary 1. *If Assumption 3.4 are replaced by Assumption 3.7, then Theorems 3.5 and 3.6 still hold for* $p \geq 2$ *with* $\lambda(t) = b(t) - ld$ *where* $d = [p \max_{i\in\mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i)]^{\frac{p}{p-1}}$ *and* $l < \underline{b}/d$.

4. **Single-mode SDEs.** We also establish similar theory for single-mode SDEs. Consider an n-dimensional periodic SDE

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{45}$$

on $t \geq 0$, with initial value $x(0) = x_0 \in \mathbb{R}^n$. Here

$$f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

If the given system is not stable, then we can design a feedback control $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ to make the controlled system

$$dx(t) = \big(f(x(t), t) + u(x(\delta_t), t)\big)dt + g(x(t), t)dB(t) \tag{46}$$

stable.

Assumption 4.1. Assume that $f(x, t)$, $g(x, t)$ and $u(x, t)$ are all periodic with respect to time t. Assume f, g, u and κ_t have a common period T.

Assumption 4.2. Assume that the coefficients $f(x, t)$ and $g(x, t)$ are both locally Lipschitz continuous on x. We also assume that $f(x, t)$ and $g(x, t)$ both satisfy the following linear growth conditions on x

$$|f(x, t)| \leq K_1(t)|x| \quad \text{and} \quad |g(x, t)| \leq K_2(t)|x| \tag{47}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, where $K_1(t)$ and $K_2(t)$ are periodic non-negative continuous functions with period T.

Assumption 4.3. Assume

$$|u(x, t) - u(y, t)| \leq K_3(t)|x - y| \tag{48}$$

for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, where $K_3(t)$ is a periodic non-negative continuous function with period T. We also assume that

$$u(0, t) = 0 \tag{49}$$

for all $t \in \mathbb{R}_+$.

Let $V(x(t), t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ be a Lyapunov function periodic with respect to t. Then define $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathcal{L}V(x, t) = V_t(x, t) + V_x(x, t)[f(x, t) + u(x, t)] + \frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)]. \tag{50}$$

Assumption 4.4. Assume that there is a Lyapunov function $V(x, t)$ and a positive continuous function $\lambda(t)$ which have common period T, constants $l > 0$ and $p > 1$ such that

$$\mathcal{L}V(x, t) + l|V_x(x, t)|^{\frac{p}{p-1}} \leq -\lambda(t)|x|^p \tag{51}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Theorem 4.5. *Fix the moment order $p > 1$. Let Assumptions 4.1, 4.2, 4.3 and 4.4 hold. Divide $[0,T]$ into $Z-1$ subintervals with $T_1 = 0$ and $T_Z = T$. Choose the observation interval sequence $\{\tau_j\}_{1\le j\le M}$ sufficiently small such that $\kappa_t \le T_{j+1} - T_j$ for $t \in [T_j, T_{j+1})$ where $j = 1, 2, \cdots, Z-1$ and*

$$\beta(t) > 0 \quad \text{and} \quad \kappa_t \hat{K}_{3t} < 8^{-\frac{p-1}{p}} \quad \text{for} \quad \forall t \in [0,T), \tag{52}$$

where

$$\begin{aligned}\beta(t) := \beta(\kappa_t, t) = \lambda(t) - &\frac{[2(p-1)]^{p-1}\hat{K}_{3s}^p}{p^p(1-8^{p-1}\kappa_t^p\hat{K}_{3t}^p)} l^{1-p}\kappa_t^{\frac{p}{2}} \\ &\times\Big[2^{p-1}\kappa_t^{\frac{p}{2}}K_1^p(t) + \rho K_2^p(t) + 4^{p-1}\kappa_t^{\frac{p}{2}}K_3^p(t)\Big].\end{aligned} \tag{53}$$

Then the controlled system (46) satisfies (19) for all initial data $x_0 \in \mathbb{R}^n$.

Theorem 4.5 can be proved in the same way as Theorem 3.5.

Theorem 4.6. *Fix the moment order $p \ge 2$. Under the same assumptions of Theorem 4.5, the solution of the controlled system (46) satisfies (35) and (36) for any initial data $x_0 \in \mathbb{R}^n$. In other words, the controlled system (46) is asymptotically stable in pth moment and almost surely.*

Assumption 4.7. Assume that there exist positive-definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a periodic positive continuous function $b(t)$ such that

$$\begin{aligned}&p(x^TQx)^{\frac{p}{2}-1}\Big(x^TQ[f(x,t)+u(x,t)] + \frac{1}{2}\text{trace}[g^T(x,t)Qg(x,t)]\Big) \\ &\quad + p(\frac{p}{2}-1)[x^TQx]^{\frac{p}{2}-2}|g^TQx|^2 \le -b(t)|x|^p,\end{aligned} \tag{54}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T)$.

Corollary 2. *If Assumption 4.4 are replaced by Assumption 4.7, then Theorems 4.5 and 4.6 still hold for $p \ge 2$, with $\lambda(t) = b(t) - ld$ where $d = [p\lambda_{\max}^{\frac{p}{2}}(Q)]^{\frac{p}{p-1}}$, $l < \underline{b}/d$ and $\underline{b} = \inf_{0\le t<T} b(t)$.*

5. **Example.** Now we illustrate our theory with an example.

Consider a 2-dimensional SDE

$$dx(t) = F(x(t),t)x(t)dt + G(t)x(t)dB(t) \tag{55}$$

on $t \ge 0$, where $x(t) = (x_1(t), x_2(t))^T$ and $B(t)$ is a scalar Brownian motion. Here the coefficients are

$$F(x,t) = [1.5 + \cos(\frac{\pi}{6}t)] \begin{bmatrix} 0 & \sin(x_1) \\ \cos(x_2) & 0 \end{bmatrix}$$

and

$$G(t) = [1 + \sin(\frac{\pi}{6}t - 2.8)] \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

The upper plot in Figure 1 shows that the original system (55) is not stable.

Coefficients $f(x,t) = F(x(t),t)x(t)$ and $g(x,t) = G(t)x(t)$ have common period $T = 12$. Assumption 4.2 holds with $K_1(t) = 1.5 + \cos(\frac{\pi}{6}t)$ and $K_2(t) = 1 + \sin(\frac{\pi}{6}t - 2.8)$. Then we can design a feedback control $u(x,t)$ and find an observation intervals to make the controlled system

$$dx(t) = [F(x(t),t)x(t) + u(x(\delta_t),t)]dt + G(t)x(t)dB(t) \tag{56}$$

mean square asymptotically stable.

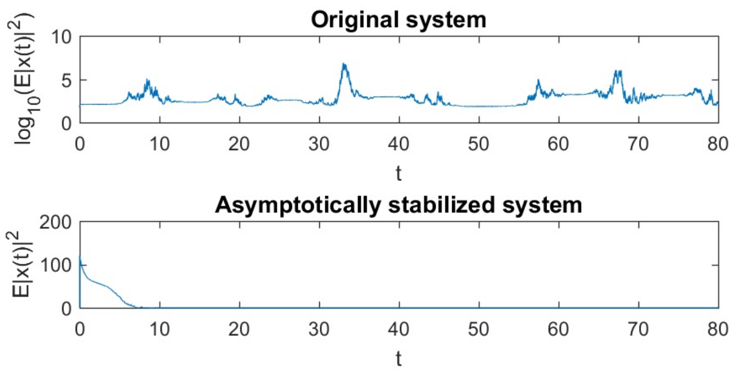

FIGURE 1. Sample averages of $|x|^2$ from 500 simulated paths by the Euler-Maruyama method with step size $1e-5$ and random initial values. Upper plot shows original system (55); lower plot shows controlled system (56) for mean square asymptotically stabilization with corresponding observation frequencies.

Let $u(x,t) = A(x,t)x$ where $A : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ with bounded norm. Then assumption 4.3 holds with $K_3(t) \geq \max_{x\in\mathbb{R}^2} \|A(x,t)\|$. Let $V(x,t) = x^TQx$ where $Q = I$, the 2×2 identity matrix, then Corollary 2 can be applied. The left-hand-side of (54) is

$$\begin{aligned}&2[x^T(f(x,t)+u(x,t)) + \frac{1}{2}g^T(x,t)g(x,t)]\\ =&2x^T(F(x,t)+A(x,t))x + x^TG^T(t)G(t)x\\ \leq& x^T\tilde{Q}x \leq \lambda_{\max}(\tilde{Q})|x|^2,\end{aligned}\tag{57}$$

where

$$\tilde{Q} = F(x,t) + F^T(x,t) + A(x,t) + A^T(x,t) + G^T(t)G(t).$$

To satisfy Assumption 4.7, we design $A(x,t)$ to make $\tilde{Q}$ negative definite.

Let

$$A(x,t) = \begin{bmatrix} B_1(x,t) & B_2(x,t) \\ B_2(x,t) & B_1(x,t) + 0.1\sin(\frac{\pi}{6}t) \end{bmatrix},$$

where $B_1(x,t) = -0.25K_2^2(t) - 0.5$ and $B_2(x,t) = -0.25K_1(t)K_2^2(t)[\sin(x_1) + \cos(x_2)]$. Then

$$\tilde{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 + 0.2\sin(\frac{\pi}{6}t) \end{bmatrix}.$$

So $b(t) = -\lambda_{\max}(\tilde{Q}) = \min\{1 - 0.2\sin(\frac{\pi}{6}t), 1\}$, $\underline{b} = 0.8$ and $d = 4$.

Choose $l = 0.1$, then $\lambda(t) = b(t) - 0.4$. Obviously, $T = 12$ is a common period of f, g, u, b and λ. Figure 2 shows parameters $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\lambda(t)$.

Now we calculate $\{\tau_j\}_{j\geq 1}$ for mean square asymptotic stabilization. The auxiliary function $\tilde{\tau_a}(t)$ is calculated through

$$\begin{aligned}\lambda(t) \geq& \frac{[2(p-1)]^{p-1}}{p^p(1 - 8^{p-1}\tilde{\tau_a}^p(t)K_3^p(t))} l^{1-p}K_3^p(t)\tilde{\tau_a}^{\frac{p}{2}}(t)\\ &\times\Big[2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_1^p(t) + \rho K_2^p(t) + 4^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)\Big].\end{aligned}$$

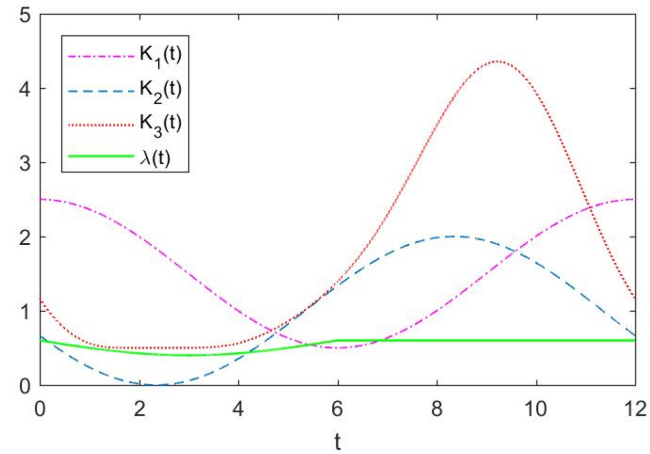

FIGURE 2. Plot of parameters $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\lambda(t)$.

It's plotted as a dashed blue line in Figure 3. Then according to its shape, we divide

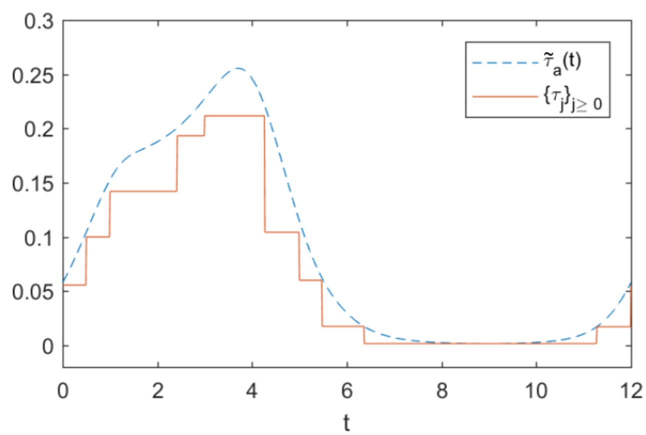

FIGURE 3. Plot of observation intervals. The dashed blue line shows the auxiliary function and the solid orange line is observation interval sequence.

$[0, 12]$ into 10 subintervals. The calculated observation interval for each subinterval is shown as the solid orange line in Figure 3. Table 1 presents the partition, the observation interval and observation times for each subinterval.

Table 1 shows that on the first subinterval $[0, 0.5)$, the system needs to be observed once every 0.05556 time units for 9 times; the 6th row means that when $t \in [3 + 12k, 4.27 + 12k)$ for $k = 0, 1, 2, \cdots$, the system needs to be observed once every 0.21167 time units for 6 times.

This observation frequency setting gives $\min_{0\leq t\leq T} \beta(t) = 1.98e - 06 > 0$, so all the conditions on observation intervals have been satisfied. By Corollary 2, the

Table 1. Period partition, observation interval and observation times in each subinterval.

Subinterval	Observation interval	Observation times
[0, 0.5)	0.05556	9
[0.5, 1)	0.1	5
[1, 2.42)	0.142	10
[2.42, 3)	0.19333	3
[3, 4.27)	0.21167	6
[4.27, 5)	0.10429	7
[5, 5.48)	0.06	8
[5.48, 6.37)	0.01745	51
[6.37, 11.28)	0.00164	2988
[11.28, 12)	0.01714	42

controlled system (56) is asymptotically stable in mean square and almost surely. In addition, the lower plot of Figure 1 agrees with it.

The largest and the smallest observation intervals are 0.00164 and 0.21167 respectively. Existing theory gives a constant observation interval 0.00089 [4, Chapter 3]. Our shortest observation intervals are still wider than the constant one. This is because parameters $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\lambda(t)$ do not reach their minimum values at the same time point, which can be seen from Figure 2.

6. **Conclusion.** This paper has discussed the asymptotic stabilization of periodic SDEs by feedback control based on periodic discrete-time observations. The stabilities analyzed include L^p-stability for $p > 1$, asymptotic stability in pth moment for $p \geq 2$, and almost sure asymptotic stability.

The main contributions of this paper are: (1) considering the time-varying property of the system into stability analysis and using time-varying observation frequencies for stabilization of periodic SDEs; (2) reducing the observational cost for control by reducing the observation frequencies dramatically.

Appendix. Following is the sketch of the proof of assertion (36).

Proof. It follows from Theorem 3.5 that

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{58}$$

We claim that

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{59}$$

Otherwise, we must have

$$\mathbb{P}\Big(\limsup_{t\to\infty} |x(t)| > 0\Big) > 0.$$

Consequently, for event $\Omega_1 := \Big\{\limsup_{t\to\infty} |x(t)| > 2\varepsilon\Big\}$, we can find a number $\varepsilon > 0$ such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon. \tag{60}$$

Let $h > |x_0|$ and $\tau_h = \inf\{t \geq 0 : |x(t)| \geq h\}$ with $\inf \emptyset = \infty$.

Notice that

$$\mathbb{E}|x(t \wedge \tau_h)|^p = h^p \mathbb{P}(\tau_h \leq t) + \mathbb{E}\Big(|x(t)|^p I\{t < \tau_h\}\Big). \tag{61}$$

By (37), we have that for any $t \geq 0$,

$$\begin{aligned}&\mathbb{E}(|x(t\wedge\tau_h)|^p)\\ \leq&|x_0|^p+\int_0^t\Big(p\overline{K_1}\mathbb{E}|x(s)|^p+p\overline{K_3}\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big]+\theta\overline{K_2}^2\mathbb{E}|x(s)|^p\Big)ds.\end{aligned}$$

If follows from (38), (39), (40), (41) and Theorem 3.5 that as $t, h \to \infty$, we still have

$$\mathbb{E}|x(t\wedge\tau_h)|^p \leq C.$$

Then by (61), we have

$$h^p\mathbb{P}(\tau_h<\infty)\leq C.$$

Choose h sufficiently large so that $\mathbb{P}(\tau_h < \infty) \leq \frac{C}{h^p} \leq \varepsilon$.
Let $\Omega_2 = \{|x(t)| < h \text{ for all } 0 \leq t < \infty\}$. Then $\mathbb{P}(\Omega_2) \geq 1-\varepsilon$. By (60), we have

$$\mathbb{P}(\Omega_1\cap\Omega_2)\geq 2\varepsilon. \tag{62}$$

Define a sequence of stopping times:

$$\begin{aligned}\alpha_1&=\inf\{t\geq 0:|x(t)|^p\geq 2\varepsilon\},\\ \alpha_{2k}&=\inf\{t\geq\alpha_{2k-1}:|x(t)|^p\leq\varepsilon\},\quad k=1,2,\cdots,\\ \alpha_{2k+1}&=\inf\{t\geq\alpha_{2k}:|x(t)|^p\geq 2\varepsilon\},\quad k=1,2,\cdots.\end{aligned}$$

By definitions of α_{2k-1} and α_{2k}, we have $|x(t)|^p \geq \varepsilon$ for $\alpha_{2k-1} \leq t \leq \alpha_{2k}$. Hence by Theorem 3.5, we can derive that

$$\infty>\mathbb{E}\int_0^\infty|x(t)|^pdt\geq\varepsilon\sum_{i=1}^\infty\mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty,\tau_h=\infty\}}[\alpha_{2k}-\alpha_{2k-1}]\Big). \tag{63}$$

Let $F(t) = f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)$ and $G(t) = g(x(t), r(t), t)$ for $t \geq 0$. By Assumptions 3.2 and 3.3, there is a $K_h > 0$ for any $h > 0$ such that $|F(t)|^p \vee |G(t)|^p \leq K_h$ for all $t \geq 0$ and $|x(t)| \vee |x(\delta_t)| \leq h$.

We can obtain from Hölder's inequality and the Burkholder-Davis-Gundy inequality that for any $s > 0$,

$$\mathbb{E}\Big(I_A\sup_{0\leq t\leq s}\Big|x(\tau_h\wedge(\alpha_{2k-1}+t))-x(\tau_h\wedge\alpha_{2k-1})\Big|^p\Big)\leq 2^{p-1}K_hs^{\frac{p}{2}}(s^{\frac{p}{2}}+\xi), \tag{64}$$

where $A = \{\tau_h \wedge \alpha_{2k-1} < \infty\}$.

Use the elementary inequality $|a^p - b^p| \leq p|a-b|(a^{p-1} + b^{p-1})$ for $\forall a, b \geq 0$ and $p \geq 1$ (see e.g. [19, page 53]). Note that $\Big||x| - |y|\Big| \leq |x-y|$ for any $x, y \in \mathbb{R}^n$. Let $\theta = \frac{\varepsilon}{2ph^{p-1}}$, then we have

$$\Big||x|^p-|y|^p\Big|<\varepsilon \text{ whenever } |x-y|<\theta,\ |x|\vee|y|\leq h. \tag{65}$$

Choose s sufficiently small for $2^{p-1}K_hs^{\frac{p}{2}}(s^{\frac{p}{2}}+\xi) < \theta^{-p}\varepsilon$. By Chebyshev's inequality and (64), we have

$$\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\leq t\leq s}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|\geq\theta\Big\}\Big)\leq\varepsilon. \tag{66}$$

It can be seen from definitions of Ω_1 and Ω_2, (62) and (66) that

$$\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\leq t\leq s}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|<\theta\Big\}\Big)\geq\varepsilon.$$

Let

$$\tilde{\Omega}_k = \Big\{ \sup_{0 \le t \le s} \big||x(\alpha_{2k-1}+t)|^p - |x(\alpha_{2k-1})|^p\big| < \varepsilon \Big\}.$$

Then (65) implies that

$$\mathbb{P}\Big(\{\alpha_{2k-1} < \infty, \tau_h = \infty\} \cap \tilde{\Omega}_k\Big) \ge \varepsilon. \tag{67}$$

It follows from the definition of α_k, (63), (65) and (67) that

$$\begin{aligned} \infty &> \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty, \tau_h=\infty\}}[\alpha_{2k} - \alpha_{2k-1}]\Big) \\ &\ge \varepsilon s \sum_{i=1}^{\infty} \mathbb{P}\Big(\{\alpha_{2k-1} < \infty, \tau_h = \infty\} \cap \tilde{\Omega}_k\Big) = \infty, \end{aligned}$$

which is a contradiction. Hence (59) must hold. The proof is complete. □

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REFERENCES

[1] L. Arnold and C. Tudor, Stationary and almost periodic solutions of almost periodic affine stochastic differential equations, *Stochastics and Stochastic Reports*, **64** (1998), 177–193.

[2] G. K. Basak, A. Bisi and M. K. Ghosh, Stability of a random diffusion with linear drift, *J. Math. Anal. Appl.*, **202** (1996), 604–622.

[3] P. H. Bezandry and T. Diagana, *Almost periodic stochastic processes*, Springer Science and Business Media, 2011.

[4] R. Dong, *Stabilization of Stochastic Differential Equations by Feedback Controls Based on Discrete-time Observations*, PhD thesis, University of Strathclyde, UK, 2019.

[5] R. Dong, Almost sure exponential stabilization by stochastic feedback control based on discrete-time observations, *Stochastic Analysis and Applications*, **36** (2018), 561–583.

[6] R. Dong and X. Mao, On Pth Moment Stabilization of Hybrid Systems by Discrete-time Feedback Control, *Stochastic Analysis and Applications*, **35** (2017), 803–822.

[7] L. Hu, Y. Ren and T. Xu, p-Moment stability of solutions to stochastic differential equations driven by G-Brownian motion, *Applied Mathematics and Computation*, **230** (2014), 231–237.

[8] C. Huang, Y. He, L. Huang and W. Zhu, pth moment stability analysis of stochastic recurrent neural networks with time-varying delays, *Information Sciences*, **178** (2008), 2194–2203.

[9] Y. Ji and H. J. Chizeck, Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control*, **35** (1990), 777–788.

[10] Y. Li, J. Lu, C. Kou, X. Mao and J. Pan, Robust stabilization of hybrid uncertain stochastic systems by discretetime feedback control. *Optimal Control Applications and Methods*, **38** (2017), 847–859.

[11] X. Li, and X. Mao, A note on almost sure asymptotic stability of neutral stochastic delay differential equations with Markovian switching, *Automatica*, **48** (2012), 2329–2334.

[12] J. Lu, Y. Li, X. Mao and Q. Qiu, Stabilization of Hybrid Systems by Feedback Control Based on Discrete-Time State and Mode Observations, *Asian Journal of Control*, **19** (2017), 1943–1953.

[13] X. Mao, Stability of stochastic differential equations with Markovian switching, *Sto. Proc. Their Appl.*, **79** (1999), 45–67.

[14] X. Mao, Exponential stability of stochastic delay interval systems with Markovian switching, *IEEE Trans. Auto. Control*, **47** (2002), 1604–1612.

[15] X. Mao, *Stochastic differential equations and applications*, 2^{nd} edition, Horwood Publishing Limited, Chichester, 2007.

[16] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica*, **49** (2013), 3677–3681.

[17] X. Mao, Almost sure exponential stabilization by discrete-time stochastic feedback control, *IEEE Transactions on Automatic Control*, **61** (2016), 1619–1624.

[18] X. Mao, G. Yin and C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations, *Automatica*, **43** (2007), 264–273.

[19] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[20] X. Mao, W. Liu, L. Hu, Q. Luo and J. Lu, Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time state observations, *Systems & Control Letters*, **73** (2014), 88–95.

[21] Y. Niu, D. W. C. Ho and J. Lam, Robust integral sliding mode control for uncertain stochastic systems with time-varying delay, *Automatica*, **41** (2005), 873–880.

[22] R. Rifhat, L. Wang and Z. Teng, Dynamics for a class of stochastic SIS epidemic models with nonlinear incidence and periodic coefficients, *Physica A: Statistical Mechanics and its Applications*, **481** (2017), 176–190.

[23] J. L. Sabo and D. M. Post, Quantifying periodic, stochastic, and catastrophic environmental variation, *Ecological Monographs*, **78** (2008), 19–40.

[24] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*, Prentice Hall, New Jersey, 1991.

[25] G. Song, B. C. Zheng, Q. Luo and X. Mao, Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time observations of state and mode, *IET Control Theory Appl.*, **11** (2016), 301–307.

[26] M. Sun, J. Lam, S. Xu and Y. Zou, Robust exponential stabilization for Markovian jump systems with mode-dependent input delay, *Automatica*, **43** (2007), 1799–1807.

[27] I. Tsiakas, Periodic stochastic volatility and fat tails, *Journal of Financial Econometrics*, **4** (2005), 90–135.

[28] C. Wang and R. P. Agarwal, Almost periodic solution for a new type of neutral impulsive stochastic LasotaWazewska timescale model, *Applied Mathematics Letters*, **70** (2017), 58–65.

[29] C. Wang, R. P. Agarwal and S. Rathinasamy, Almost periodic oscillations for delay impulsive stochastic Nicholsons blowflies timescale model, *Computational and Applied Mathematics*, **37** (2018), 3005–3026.

[30] G. Wang, Z. Wu and J. Xiong, A linear-quadratic optimal control problem of forward-backward stochastic differential equations with partial information, *IEEE Transactions on Automatic Control*, **60** (2015), 2904–2916.

[31] T. Wang and Z. Liu, Almost periodic solutions for stochastic differential equations with Lévy noise, *Nonlinearity*, **25** (2012), 2803.

[32] L. Wu, P. Shi and H. Gao, State estimation and sliding mode control of Markovian jump singular systems, *IEEE Trans on Automatic Control*, **55** (2010), 1213–1219.

[33] S. You, L. Hu, W. Mao and X. Mao, Robustly exponential stabilization of hybrid uncertain systems by feedback controls based on discrete-time observations, *Statistics & Probability Letters*, **102** (2015), 8–16.

[34] S. You, W. Liu, J. Lu, X. Mao and Q. Qiu, Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Control Optim.*, **53** (2015), 905–925.

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