REINTERPRETING §56 OF FREGE'S
THE FOUNDATIONS OF ARITHMETIC

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In the introductory sections of *The Foundations of Arithmetic* Frege claims that his aim in this book is to determine whether the concept "number" is definable. He believes that an answer to this question will assist him in determining the nature of the laws of arithmetic, specifically, whether they are analytic or *a posteriori*.[1] Michael Dummett expresses concern with some of the conclusions Frege reaches in the *Grundlagen* in *Frege: Philosophy of Mathematics*, his recent study of Frege's contribution to the foundations of arithmetic.

Noteworthy is Dummett's claim that "§56 may be stigmatized as the weakest in the whole *Grundlagen*."[2] In this paper I will argue that Dummett has misread Frege's intentions in this and the surrounding sections of the *Grundlagen*. Specifically, Dummett treats these sections as an argument—a poor one at that—against the so-called adjectival strategy for explaining number statements. These sections, I will argue, are concerned with proving that numbers must be objects. Thus, Frege appeals to the definitions which Dummett identifies with the adjectival strategy merely for rhetorical purposes. He is not trying to show that the adjectival strategy is untenable, but rather, he wants to show what adequate definitions of numbers must be like.

I will begin with an explanation of what Dummett takes Frege's intentions to be in §56. Then, I will explain how, according to Dummett, Frege fails to accomplish what he intended to accomplish. I will then propose an alternative reading of this section of the *Grundlagen*. Finally, I will discuss the relevance of Dummett's criticisms to the alternative reading.

SECTION I   *Frege's Intentions*

In this section I will explain what Dummett thinks Frege's intentions are in §56. This will involve an explanation of what Dummett calls the adjectival strategy for explaining number statements. I will also explain what Dummett believes Frege must do in order to accomplish all he intends to accomplish.

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[1] Gottlob Frege, *The Foundations of Arithmetic*, Evanston: Northwestern University Press, 1980, page 5.
[2] Michael Dummett, *Frege: Philosophy of Mathematics*, Cambridge: Harvard University Press, 1991, page 105. All references to Dummett are to this book.

Dummett first tells us that Frege's intention in §56 is to reject the definitions that he has provided in the previous section. (101) Dummett treats §56 as part of a series of sections addressing a single problem.

> In §56-61 Frege is attempting to establish that arithmetical statements containing numerals must be taken at face-value and hence that the radical adjectival strategy is not feasible. Ontologically expressed, he is trying to establish that numbers must be regarded as objects. (101)

Thus, as Dummett construes things, the definitions in §55 are definitions that are compatible with the adjectival strategy.

What then is the adjectival strategy for explaining number statements? According to Dummett, the adjectival strategy is one of two possible strategies for explaining how "to make explicit the content both of ascriptions of numbers and of the propositions encountered in number theory". (99) And, when using the adjectival strategy we "first explain the adjectival use of number-words, and then explain the corresponding numerical terms by reference to it". (99) As Dummett explains,

> a radical version of the adjectival strategy would be to refuse to take numerical terms at face-value. According to this strategy, equations and other arithmetical statements in which numerals apparently figure as singular terms are to be explained ... by *transforming* them into sentences in which number-words occur only adjectivally. (99)

This strategy, Dummett claims, is to be distinguished from what he calls the substantival strategy. According to this latter strategy we "explain the use of numerals as singular terms, and then explain the corresponding number-adjectives by reference to *it*". (99) The substantival view commits us to the thesis that numbers are objects.

Having set out Frege's project in §56 and surrounding sections, Dummett then proceeds to specify what Frege must do in order to accomplish all he aims to accomplish. According to Dummett, in order to prove that numbers are objects, Frege must "convict the definitions he proposed in §55 of being logically amiss". (102)

Thus, as Dummett construes things, Frege's intention in §56 is to show that the definitions provided in the previous section are unacceptable. This, in turn, will show that the adjectival view of numbers is unacceptable, and thus that numbers are in fact objects.

SECTION II   *Dummett's Problems With §56*

In this section I will explain what Dummett takes to be the central problems with Frege's presentation in §56. Dummett identifies four problems.

First, Dummett accuses Frege of begging the question. That is, given that Frege intends to show that number statements cannot be adequately accounted for with the resources available in the adjectival strategy, and consequently, that numbers are objects, it would beg the question if Frege rejected the definitions provided in §55 on grounds that presuppose that numbers are objects. But, according to Dummett, this is exactly what Frege does.[3]

To understand Dummett's second criticism some background is required. In §55 Frege presents three definitions, a definition of 0, a definition of 1, and a definition of the successor function. Dummett's concern is that the criticisms Frege raises against these definitions in §56 really only address the third definition, the definition of the successor function. Consequently, as far as Dummett is concerned, Frege has not adequately shown that the adjectivalist's definitions for 0 and 1 are unacceptable. (102) Dummett does grant that the definition for "there are just (n + 1) ..." is unacceptable, but he claims that Frege "had no case against those of "there are 0 ..." and "there is just 1 ...''". (105)

The third problem that Dummett has with §56 is that Frege treats the adjectival view as a disguised version of the substantival view. (109) Hence, rather than showing that the adjectival view is inadequate, Dummett suggests that Frege has misrepresented the adjectival view.[4]

Dummett's final criticism of § 56 is that Frege fails to prove that numbers are in fact objects. (105) Consequently, according to Dummett, Frege "has in no way shown the adjectival strategy impossible, as he is purporting to have done". (109) In fact, Dummett argues that

> contrary to the impression [Frege] contrives to convey in §§56-
> 57, the radical strategy can be pursued for a considerable
> distance: the definitions Frege actually gives in §§71-83 can
> readily be mimicked in the adjectival mode. (131)

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[3] See pages 102 and 106. Dummett states this most clearly in the passage on page 106. According to Dummett, the second criticism that Frege raises in §56 against the definitions proposed by the adjectival strategy is with respect to their inability to account for the identity of a number with itself over time. But, according to Dummett, "identity, as Frege understood it, is a relation between **objects**: the complaint [that Frege raises] therefore assumes what has yet to be proved, that numbers are objects". (emphasis is mine)

[4] I felt it was important to distinguish this criticism from the first. After all, one could beg the question without misrepresenting the view one is criticizing, or similarly one could misrepresent the view one is criticizing without raising criticisms that beg the question.

These are the criticisms underlying Dummett's claim regarding the weakness of §56.

SECTION III   *An Alternative Reading*

In this section, I would like to present and defend an alternative reading of §56 of the *Grundlagen*. On the interpretation that I will be defending, Frege is not trying to show that the adjectival strategy cannot make sense of our use of number statements, as Dummett suggests, but rather, his intention is to show that only the substantival strategy can provide us with adequate definitions of numbers. And it is because of this that Frege concludes that numbers must be objects.

Let us begin by considering Frege's description of sections 55-61 in his "Analysis of Contents". These sections are grouped under the heading: "Every individual number is a self-subsistent object". These sections address three distinct projects. First, in §55 Frege presents three definitions—Dummett refers to these as the definitions of the adjectival strategy. Second, sections 56 and 57 provide a discussion of the three definitions presented in §55. In the first of these sections Frege argues that "the attempted definitions ... define a predicate in which the number is only an element."[5] And in the latter section, Frege claims to be showing that "a statement of number should be regarded as an identity between numbers."[6] Finally, in sections 58 to 61 Frege addresses two objections that he anticipates to his thesis that numbers are self-subsistent objects.

Clearly, if Frege is addressing objections to his thesis from section 58 onward, he must be assuming that he has already advanced the thesis that numbers are objects in the preceding sections. I believe that he has, to his satisfaction. In fact, as I will argue in the remainder of this section, I believe that §56 is intended to show the necessity of this thesis.

In §56 Frege raises two difficulties with the definitions proposed in the previous section. They are as follows: (1) the definitions do not enable us to distinguish numbers from non-numbers, for example, they do not allow us to determine whether Julius Caesar is a number or not; and, (2) the definitions do not enable us to identify a number with itself over time. These difficulties, Frege believes, are the sorts of difficulties that adequate definitions of numbers must overcome. That is, any set of definitions that are plagued by these difficulties are inadequate. And, according to Frege, only the substantival strategy could possibly provide us with definitions that avoid these difficulties.

Evidence for my reading of this section is provided in the last sentence of §56. According to Frege, with the definitions provided in the previous

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5 Gottlob Frege, *op.cit.*, page x. This would be an odd criticism to raise against definitions that, on Dummett's reading, purport to do no more.

6 Ibid, page x.

section "we have no authority to pick out 0 and 1 here as self-subsistent objects that can be recognized as the same again."[7] Clearly, if Frege's intention is to show that the definitions provided by the adjectival strategy are inadequate, he would not have raised this criticism. Hence, what I am suggesting is that by §56 Frege is already determined to defend the substantival account of number statements. He is not trying to prove that the definitions of the adjectival strategy are problematic, as Dummett suggests.

Why then did he propose the definitions in the previous section? These, I believe, were proposed strictly for rhetorical purposes. These definitions bring to our attention the sorts of qualities adequate definitions of the numbers must have. Hence, by proposing these definitions, Frege can present the substantival strategy as necessary. That is, Frege is not suggesting that we choose between two competing accounts of number statements, as Dummett suggests. Rather, at this point in the *Grundlagen*, Frege is convinced that numbers are objects and presents the alternative definitions in order for his reader to see that only if numbers are objects can we possibly make sense of number statements.

Frege's discussion in §57 offers additional support for my reading. He begins the section arguing that the definitions of §55 treat numbers as predicates.[8] Clearly, if Frege were trying to argue that the adjectival view is untenable he would not have raised this as an objection; this really would beg the question. Rather, this complaint is only sensible if he is already presupposing that numbers are objects.

Further, in this same section, Frege contrasts "a concept of number usable for purposes of science" with "the language of everyday life [in which] numbers appear also in attributive constructions."[9] Our concern, Frege argues, is with the former, not the latter.[10] At this point Frege cannot possibly be trying to refute the adjectival view, a view which maintains that all number statements can be recast in attributive constructions, and that such constructions better reflect the ontology of numbers. The content of §57 clearly suggests that Frege has already settled on the substantival strategy.

Consequently, contrary to what Dummett suggests, Frege is not trying to argue that the substantival view is better than the adjectival view. Rather, his claim is that a particular set of definitions—those presented in §55—are inadequate because they fail to meet our expectations of definitions of numbers. And these expectations can only be satisfied if we assume that numbers are objects.

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7 Ibid, page 68.

8 Ibid, page 68.

9 Ibid, page 69.

10 Ibid, page 69.

SECTION IV *Reassessing Dummett's Criticisms*

In this section, I will assess the relevance of Dummett's criticisms raised in section II of this paper to this alternative reading of §56.

First, consider Dummett's criticism that Frege begs the question. Given that Frege is arguing that the substantival view is the only view that can provide us with acceptable definitions of numbers and that he proposes the adjectival definitions merely for rhetorical purposes, Frege cannot be accused of begging the question. Dummett is correct to say that Frege presupposes in §56 that numbers are objects, but this does not beg the question, for Dummett is mistaken about what Frege is trying to do in this section. The question Dummett takes to be Frege's question is not, in fact, Frege's question. As I suggested in the previous section, Frege's intention is not to show that the adjectival strategy won't do. Rather, his intention is to show that the substantival view is correct **because** it avoids the sorts of short-comings that are characteristic of the definitions provided in §55. The definitions are proposed in order to show us what acceptable definitions of numbers must do.

Second, consider Dummett's complaint that Frege's criticisms in §56 only address the third definition. Given my proposed reading of §56, the criticisms raised in this section are not directed toward any particular definition. Rather, the criticisms are intended to show that numbers must be self-subsistent objects. Further, the last sentence of §56 which I discussed in some detail earlier is explicitly about the definitions of 0 and 1. Consequently, given my reading of §56, this criticism is not relevant.

Consider Dummett's third criticism, that Frege regards the adjectival view as a disguised version of the substantival view. Clearly, on the reading I have proposed, Frege is not claiming that the definitions proposed in §55 implicitly assume that numbers are objects. Rather, Frege's claim is that these definitions make it apparent to us that numbers must be objects. After all, unless they are objects we cannot make sense of number statements. Consequently, this criticism is also irrelevant.

Dummett's final criticism is the one that I believe to be the most detrimental to my proposed reading of §56. Recall that this criticism states that Frege has failed to prove that numbers are in fact objects. I am inclined to agree with Dummett. One is left with the impression that Frege has not actually proved that numbers are objects. Insofar as Frege has provided us with a proof, it is of the transcendental sort. Frege's argument in §56 seems to be as follows: unless numbers are objects, we would neither (1) be able to distinguish numbers from non-numbers, nor (2) be able to identify a number with itself over time; but, we can both distinguish numbers from non-numbers and identify a number with itself over time; therefore, numbers must be objects. I believe it is questionable whether numbers must be objects in order to distinguish numbers from non-numbers, and to identify a

number with itself over time. Consequently, Dummett's final criticism is still a concern on my reading of §56.

At the end of §61 Frege gives a clearer indication of what he thinks his thesis entails. He claims that "it is a fact that the number 4 is exactly the same for everyone who deals with it."[11] If this is what Frege means when he claims that numbers are objects, one can still wonder, with Dummett, whether the adjectival strategy cannot also accommodate this fact—that each number is exactly the same for everyone who deals with it.[12] You will recall that Dummett believes that in order to show that numbers are objects Frege must show that the definitions of the adjectival strategy are logically amiss. It is questionable whether Frege has accomplished this.

CONCLUDING REMARKS

In this paper I have tried to defend an alternative reading of §56 of Frege's *Grundlagen*, one that rescues Frege from Dummett's charge that this section is the weakest in the whole book. On my reading, Frege is not presenting arguments against the adjectival strategy. Rather, Frege presents the definitions in §55 in order to convince his reader that numbers must be objects. In §56 Frege suggests that these definitions contain two short-comings that adequate definitions of numbers must overcome. And these short-comings, he argues, can only be avoided if numbers are objects. Further, I have argued that my alternative reading defuses three of Dummett's four criticisms of §56, consequently challenging Dummett's claim that this section should be stigmatized as the weakest in the whole book. Nevertheless, I am inclined to agree with Dummett that the adjectival strategy is more robust than Frege suggests. Frege's arguments seem far from conclusive. Thus, it may be that the adjectival strategy can provide us with definitions that can do everything we can expect from adequate definitions of numbers.[13]

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[11] Ibid, page 72.

[12] Obviously Frege must have more in mind when he claims that numbers are objects, for, unless numbers are objects in some stronger sense, Frege's proof for the infinity of numbers cannot work.