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FRACTAL SETS SATISFYING THE STRONG OPEN SET CONDITION IN COMPLETE METRIC SPACES

Abstract. Let K be a Hutchinson fractal in a complete metric space X, invariant under the action S of the union of a finite number of Lipschitz contractions. The Open Set Condition states that X has a non-empty subinvariant bounded open subset V, whose images under the maps are disjoint. It is said to be *strong* if V meets K. We show by a category argument that when $K \not\subset V$ and the restrictions of the contractions to V are open, the strong condition implies that $\check{V} = \bigcap_{n=0}^{\infty} S^n(V)$, termed the *core* of V, is non-empty. In this case, it is an invariant, proper, dense, subset of K, made up of points whose addresses are unique. Conversely, $\check{V} \neq \emptyset$ implies the SOSC, without any openness assumption.

Keywords: address, Baire category, fractal, scaling function, scaling operator, strong open set condition.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Suppose that X is a complete metric space and w_i, $i \in \{1, 2, \dots N\}$, are contracting self-maps of X, each having Lipschitz constant less than one. Call the w_i *scaling maps*, and define the *scaling operator* S on 2^X as

$$S(E) = \bigcup_{i=1}^{N} w_i(E),$$

for $E \subset X$. Say E is *subinvariant* under scaling if $S(E) \subseteq E$, *invariant* if equality holds.

These notions were introduced by Hutchinson, in his fundamental paper [3]. There, he established the existence of a unique non-empty bounded closed subset K of X that is invariant under scaling, and showed it to be compact. The set K is termed the *fractal*, or *invariant set*, determined by the scaling maps. Familiar examples are the Sierpinski Gasket, the von Koch Curve, and Barnsley's Fern.

Along with fractal sets, Hutchinson also introduced fractal measures. Each probability distribution $P = (p_1, p_2, \ldots, p_N)$ on the index set defines a *Markov operator*

$$M_P\mu = \sum_{i=1}^{N} p_i \cdot \mu \circ w_i^{-1},$$

acting on finite measures μ on the Borel σ-algebra $\mathcal{B}(X)$ and transforming them into measures of a similar nature. The operator is well-defined, since the scaling maps, being continuous, are Borel-measurable.

A regular Borel probability measure μ_P for which $M_P\mu_P = \mu_P$ is called an invariant, or *fractal* measure. Hutchinson [3], 733 proved the existence and uniqueness of such measures, assuming that P is non-degenerate, and established that K is their topological support.

The images of under the scaling maps may overlap, be disjoint, or they may "just touch", as they do in the foregoing examples. The last notion means, intuitively, that, although the images may *touch*, they do not significantly *overlap.*

To capture it, standard practice has been to invoke the *Open Set Condition* (OSC), introduced by Moran [6]. It posits the existence of a non-empty subinvariant bounded open set V, whose images under the scaling maps are disjoint. The first two examples above clearly have this property, and the third one is reputed to possess it.

However, Lalley [4] recognized that if it were possible for $K \subset \partial V$, the separation property required of the images of V will have lost its effect on K. Accordingly, he added to the OSC the further requirement that $K \cap V \neq \emptyset$, and called it the *Strong Open Set Condition*(SOSC). In the case in which the scaling maps are similitudes, or, at least, conformal, Peres et al. [7] proved it to be, ultimately, a consequence of the OSC, generalizing an earlier result of Schief [8].

The aim of this paper is to study the SOSC in the general setting of complete metric spaces. In contrast to Schief [9], we allow the scaling maps to be arbitrary strict contractions.

The principal object of our study is what we call the *core* $\check{V}$ of the set V occurring in the OSC. It is defined by the formula

$$\check{V} = \bigcap_{n=0}^{\infty} S^n(V),$$

where S^n denotes the n-th iterate of S.

The main result is that, under the SOSC, $\check{V}$ is non-empty, whenever the restriction of the scaling maps to V are open: this includes the case of homeomorphisms. It is then a dense, proper, invariant subset of K. Conversely, $\check{V} \neq \emptyset$ implies the SOSC.

It is a pleasure to acknowledge the priority of Andy Lasota and his collaborators [5] in considering, for the case of Polish spaces, the notion of the core in an even more general setting than that of Hutchinson fractals (although we have some reservations about what they claim). It, nevertheless, follows from their Zero-One Law, op. cit, 346, that the SOSC implies that $\overline{\mu}_P \circ \check{V} = 1$ holds (even in the non-separable case), where $\overline{\mu}_P$ denotes the completion of the invariant measure μ_P. We shall present these measure-theoretic results in [2].

2. THE BEHAVIOR OF S UNDER ITERATION

For any natural number n, and any string of indices $i_1, i_2, \ldots, i_n$, with values in $\{1, 2, \ldots, N\}$, set

$$E_{i_1 i_2 \ldots i_n} = w_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_n}(E),$$

where $E \subset X$. Then $S^n(E)$ can be expressed as

$$S^n(E) = \bigcup_{i_1, i_2, \ldots, i_n} E_{i_1 i_2 \ldots i_n},$$

where the $i_1, i_2, \ldots, i_n$ vary independently over $\{1, 2, \ldots, N\}$.

Proposition 2.1 ([3, p. 724]). *If E is non-empty and subinvariant, then, for every non-negative integer n there holds $K \subset S^n(\mathrm{cl}E)$, where* cl *denotes the closure.*

Theorem 2.2 ([3, p. 724]). *If E is non-empty and bounded, then $S^n(E) \to K$ in the Hausdorff semi-metric, as $n \to \infty$.*

The following crucial result, already well-known in the case of precompact E, shows that the inclusion in Proposition 2.1 becomes, in the limit, equality, when E is bounded.

Theorem 2.3. *If E is a non-empty, bounded, subinvariant set, then*

$$K = \bigcap_{m=0}^{\infty} S^m(\mathrm{cl}E).$$

Proof. By Theorem 2.2, $S^n(\mathrm{cl}E) \to K$ in the Hausdorff semi-metric, as $n \to \infty$. For each $\varepsilon > 0$, let K_ε denote the union of open balls of radius ε whose centers lie in K. Now, prescribe ε at will. Since the sets $S^n(\mathrm{cl}E)$ form a decreasing sequence as n increases, and K is their Hausdorff limit, $S^n(\mathrm{cl}E) = \bigcap_{m=0}^{n} S^m(\mathrm{cl}E) \subset K_\varepsilon$, from some n on. Thus, $\bigcap_{m=0}^{\infty} S^m(\mathrm{cl}E) \subset K_\varepsilon$. As ε was arbitrary and K is compact, an indirect argument shows that $\bigcap_{m=0}^{\infty} S^n(\mathrm{cl}E) \subset K$. However, by Proposition 2.1, $K \subset \bigcap_{m=0}^{\infty} S^m(\mathrm{cl}E)$. Hence, $K = \bigcap_{m=0}^{\infty} S^m(\mathrm{cl}E)$, as asserted. □

3. THE OPEN SET CONDITIONS AND THE CORE

The scaling maps satisfy the *Open Set Condition* (OSC), if there is a non-empty, bounded, open set V that is subinvariant, and its images under the scaling maps are disjoint. It is called the *Strong OSC* (SOSC) if, further, K meets V.

These conditions are only of interest when the images of K under the scaling maps fail to be disjoint. To ensure that, the constraint $K \not\subset V$ must be imposed, and we do so henceforth, with one sole exception. It is suspended in Remark 4.5, in order to describe what would occur in its absence

Proposition 3.1. *Under the SOSC,* $K = \mathrm{cl}[K \cap S^n(V)]$, *for every non-negative integer* n.

Proof. The inclusion $K \supset \mathrm{cl}[K \cap S^n(V)]$ is immediate, since K is closed, and the reverse inclusion is a consequence of Proposition 2.1, applied to $K \cap V$, which is subinvariant. □

Proposition 3.2. *There holds* $K \backslash S^n(V) \subset \partial S^n(V)$, *for every non-negative integer* n.

Proof. Applying Proposition 2.1 to V yields $K \subset S^n(\mathrm{cl}V)$, for every n. The continuity of S^n implies that $S^n(\mathrm{cl}V) \subset \mathrm{cl}S^n(V)$. Hence,

$$K \setminus S^n(V) \subset \mathrm{cl}S^n(V) \setminus S^n(V) \subset \mathrm{cl}S^n(V) \setminus \mathrm{int}S^n(V) = \partial S^n(V),$$

for every n, where $\mathrm{int}S^n(V)$ denotes the interior of $S^n(V)$. □

Note that the case $n = 0$ gives $K \setminus V \subset \partial V$, so that $K \subset \partial V$, when $K \cap V = \emptyset$.

Definition 3.3. *The* core $\check{V}$ *of* V *is defined as*

$$\check{V} = \bigcap_{n=0}^{\infty} S^n(V).$$

Main Theorem 3.4. *If the SOSC holds and the images of* V *under the scaling maps are open, the core* $\check{V}$ *is non-empty.*

Proof. It follows from the definition of $\check{V}$ that

$$\bigcup_{n=0}^{\infty} [K \setminus S^n(V)] = K \setminus \bigcap_{n=0}^{\infty} S^n(V) = K \setminus \check{V},$$

so the equation

$$\bigcup_{n=0}^{\infty} [K \setminus S^n(V)] = K$$

holds if and only if $\check{V}$ is empty. We show by a category argument that the assumption of equality leads to a contradiction.

In fact, if any one of the sets $K \setminus S^n(V)$, contained an interior point, that point would possess an open neighborhood that does not meet $S^n(V)$. However, by Proposition 3.2, the same point would also belong to $\partial S^n(V)$. Thus, each of its open neighborhoods would meet $S^n(V)$, leading to a contradiction. Accordingly, none of the sets $K \setminus S^n(V)$ can have an interior point.

As K is compact, it is complete in the induced metric. By Baire's Category Theorem, it thus is of second category. Since the scaling maps are assumed to be open, the sets $K \cap S^n(V)$ are open relative to K. Consequently, their relative complements $K \setminus S^n(V)$ are closed. Since they do not have interior points, they are nowhere dense in K. (The same conclusion could have been drawn from Proposition 3.1, for the complements of these sets with respect to K, $K \cap S^n(V)$, are open and dense.) Thus, the hypothesis that $\check{V}$ is empty implies that K is the countable union of nowhere dense closed sets. Hence, it is of first category, in contradiction to Baire's Category Theorem. Consequently, $\check{V}$ must be non-empty, as claimed. □

Remark 3.5. *Under the foregoing hypotheses, together with Theorem 4.4 below, $\check{V}$ is a dense G_δ in a complete metric space, so it has the power of the continuum.*

4. PROPERTIES OF THE CORE

Assume throughout this section that the OSC holds. That allows the core $\check{V}$ to be defined. We now show that it possesses some remarkable features. Although we present them here in full generality, that is merely apparent, for they are only of interest when the set $\check{V}$ is non-empty. As we shall see, that will occur if the SOSC prevails, and only then.

Theorem 4.1. *The core is invariant under scaling, i.e., $S(\check{V}) = \check{V}$.*

Proof. By the OSC, V is subinvariant: $S(V) \subset V$. Applying S repeatedly to both sides gives $S^{n+1}(V) \subset S^n(V)$, for each $n \geq 1$. Hence, the sets $S^n(V)$ form a decreasing sequence, with increasing n, and

$$w_i \circ \bigcap_{n=0}^{m} S^n(V) = w_i \circ S^m(V) = \bigcap_{n=0}^{m} w_i \circ S^n(V),$$

for every m. Sending $m \to \infty$ on the right,

$$w_i \circ \bigcap_{n=0}^{m} S^n(V) \supset \bigcap_{n=0}^{\infty} w_i \circ S^n(V),$$

for each m, so that

$$w_i \circ \bigcap_{n=0}^{\infty} S^n(V) \supset \bigcap_{n=0}^{\infty} w_i \circ S^n(V).$$

Similarly, sending $m \to \infty$ on the left gives

$$w_i \circ \bigcap_{n=0}^{\infty} S^n(V) \subset \bigcap_{n=0}^{\infty} w_i \circ S^n(V),$$

and thus equality holds. Taking the union over i and recalling the definition of $\check{V}$,

$$S(\check{V}) = \bigcup_{i=1}^{N} w_i(\check{V}) = \bigcup_{i=1}^{N} \bigcap_{n=0}^{\infty} w_i \circ S^n(V).$$

By the OSC, the images of V under the scaling maps are disjoint. Since $S^n(V) \subset V$ for each n, it follows that

$$w_i \circ S^n(V) \cap w_j \circ S^{n'}(V) = \emptyset,$$

for any pair of non-negative integers n, n', whenever $i \neq j$. Therefore, it is licit to permute the union with the intersection and get

$$S(\check{V}) = \bigcap_{n=0}^{\infty} \bigcup_{i=1}^{N} w_i \circ S^n(V) = \bigcap_{n=0}^{\infty} S^{n+1}(V).$$

As $S(V) \subset V$, the last expression equals $\bigcap_{n=0}^{\infty} S^n(V) = \check{V}$, and the invariance of $\check{V}$ is established. □

Let n be any natural number. For any string of indices $i_1, i_2, \ldots, i_n$ in $\{1, 2, \ldots, N\}^n$, set

$$E_{i_1 i_2 \ldots i_n} = w_{i_1} \circ w_{i_2} \circ \ldots \circ w_{i_n}(E),$$

where $E \subset X$. Then $S^n(E)$ can be expressed as

$$S^n(E) = \bigcup_{i_1, \ldots, i_n} E_{i_1 i_2 \ldots i_n},$$

where $i_1, i_2, \ldots, i_n$ vary independently over $\{1, 2, \ldots, N\}$.

Corollary 4.2. *For each n and varying $i_1, i_2, \ldots, i_n$, the $\check{V}_{i_1 i_2 \ldots i_n}$ form a partition of $\check{V}$.*

Proof. Since $\check{V}$ is invariant,

$$\check{V} = S^n(\check{V}) = \bigcup_{i_1, \ldots, i_2} \check{V}_{i_1 i_2 \ldots i_n}.$$

The OSC implies that the $V_{i_1 i_2 \ldots i_n}$ are disjoint. Since $\check{V} \subset V$, the same is true of the $\check{V}_{i_1 i_2 \ldots i_n}$. □

Theorem 4.3. $\check{V} \subset K$.

Proof. Clearly, $S^n(V) \subset S^n(\text{cl}V)$, for every non-negative integer n. Since the $S^n(V)$ form a decreasing sequence, $S^m(V) \subset S^n(\text{cl}V)$, for every $m \geq n$. Taking the intersection over m yields $\check{V} \subset S^n(\text{cl}V)$, for every n. Intersecting over n, the right side converges to K, by Theorem 2.3. Hence, $\check{V} \subset K$, as asserted.

To show that the inclusion is proper, observe that $\check{V} \subset V = \text{int}V$, so that no point in ∂V can lie in $\check{V}$. However, applying Proposition 2.1 to V gives $K \subset \text{cl}V$, and, by our convention, $K \not\subset V$. Accordingly, $K \cap \partial V \neq \emptyset$, hence there are points in K that do not lie in $\check{V}$. □

Recall that an *address* of a point x in K is a string of digits $i_1(x), i_2(x), \dots, i_n(x), \dots$ with values in $\{1, 2, \dots, N\}^\infty$, such that

$$x = \bigcap_{n=0}^{\infty} K_{i_1(x) i_2(x) \dots i_n(x)}.$$

Hutchinson [3], 724, proved that each point in K has at least one address. It may, in general, have many.

Theorem 4.4. *If x belongs to $\check{V}$, then its address is unique.*

Proof. Otherwise, there is a natural number n and two distinct strings of indices, $i_1(x), i_2(x), \dots, i_n(x)$ and $j_1(x), j_2(x), \dots, j_n(x)$ in $\{1, 2, \dots, N\}^n$, such that x lies in $K_{i_1(x) i_2(x) \dots i_n(x)}$ and in $K_{j_1(x) j_2(x) \dots j_n(x)}$. Since, by Theorem 4.2, $\check{V} \subset K$,

$$K_{i_1(x) i_2(x) \dots i_n(x)} \cap \check{V} = \check{V}_{i_1(x) i_2(x) \dots i_n(x)}, \quad \text{and} \quad K_{j_1(x) j_2(x) \dots j_n(x)} \cap \check{V} = \check{V}_{j_1(x) j_2(x) \dots j_n(x)},$$

x lies in $\check{V}_{i_1(x) i_2(x) \dots i_n(x)}$ and $\check{V}_{j_1(x) j_2(x) \dots j_n(x)}$. However, by Corollary 4.2, these two sets are disjoint. Contradiction. □

Proposition 4.5. *If $K \cap V = \emptyset$, then $\check{V} = \emptyset$.*

Proof. If $K \cap V = \emptyset$, then $K \subset \partial V$, but $\check{V} \subset \mathrm{int} V$, so $K \cap \check{V} = \emptyset$. Since K is non-empty, that is only possible if $\check{V} = \emptyset$. □

Hence, the SOSC is a *necessary* condition for $\check{V}$ to be non-empty. Theorem 3.4 shows that when the restrictions of the scaling maps to V are open, it is also *sufficient.* We show in [2], Theorem 3.6, that by using measure-theoretic methods, the requirement of openness can be dropped.

Theorem 4.6. *If $\check{V}$ is non-empty, then it is dense in K.*

Proof. Since $\check{V}$ is bounded and subinvariant, Proposition 2.1 applies and, by Theorem 4.2,

$$K \subset S(\mathrm{cl}\check{V}) \subset \mathrm{cl} S(\check{V}) \subset \mathrm{cl}\check{V} \subset \mathrm{cl} K = K.$$

Hence, $\mathrm{cl}\check{V} = K$. □

Remark 4.7. *The assumption that $K \not\subset V$ was used only to show that $\check{V}$ is a proper subset of K. Were it not imposed, the images of K under the scaling maps would not only be disjoint, but K and $\check{V}$ would actually coincide. To see this, write $K \subset V$, and apply S^n to each side. Since K is invariant, that would give $K \subset S^n(V)$, for each n. Taking the intersection over n would make $K \subset \check{V}$. However, Theorem 4.2 states that $\check{V} \subset K$, and, therefore, $\check{V} = K$.*

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