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Modular relations for the Rogers-Ramanujan-Slater type functions of order fifteen and its applications to partitions

Chandrashekar Adiga, A. Vanitha and Nasser Abdo Saeed Bulkhali
Department of Studies in Mathematics
University of Mysore
Manasagangotri
Mysore 570 006
INDIA

e-mail: c_adiga@hotmail.com,
a.vanitha4@gmail.com,
nassbull@hotmail.com.

Abstract

In a manuscript of Ramanujan, published with his Lost Notebook [20] there are forty identities involving the Rogers-Ramanujan functions. In this paper, we establish several modular relations involving the Rogers-Ramanujan functions and the Rogers-Ramanujan-Slater type functions of order fifteen which are analogues to Ramanujan's well known forty identities. Furthermore, we give partition theoretic interpretations of two modular relations.

Keywords and Phrases: Rogers-Ramanujan functions, theta functions, triple product identity, partitions, colored partitions, modular relations.

Mathematical Subject Classification(2010): 33D15, 11P82, 11P84.

1 Introduction

Throughout the paper, we assume $|q| < 1$ and, we use the standard notation

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \quad \text{and} \quad (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

The well-known Rogers-Ramanujan functions are defined for $|q| < 1$ by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}. \tag{1.1}$$

These functions satisfy the famous Rogers-Ramanujan identities

$$G(q) = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \quad \text{and} \quad H(q) = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.2}$$

In [19], Ramanujan remarks, I have now found an algebraic relation between $G(q)$ and $H(q)$, viz.:

$$H(q)\{G(q)\}^{11} - q^2G(q)\{H(q)\}^{11} = 1 + 11q\{G(q)H(q)\}^6.$$

Another interesting formula is

$$H(q)G(q^{11}) - q^2G(q)H(q^{11}) = 1.$$

These two identities are from a list of forty identities involving the Rogers-Ramanujan functions found by Ramanujan. Ramanujan's forty identities for $G(q)$ and $H(q)$ were first brought to the mathematical world by B. J. Birch [11] in 1975. Many of these identities have been established by L. G. Rogers [22], G. N. Watson [26], D. Bressoud [12] and A. J. F. Biagioli [10]. Recently B. C. Berndt et al. [8] offered proofs of 35 of the 40 identities. Most likely these proofs might have given by Ramanujan himself. A number of mathematician tried to find new identities for the Rogers-Ramanujan functions similar to those which have been found by Ramanujan [20], including Berndt and H. Yesilyurt [9] and C. Gugg [14].

Two important analogues of the Rogers-Ramanujan functions are the Ramanujan-Göllnitz-Gordan functions. In addition to that, the Rogers-Ramanujan and Ramanujan-Göllnitz-Gordan functions share some remarkable properties. S. -S. Huang [17] has derived several modular relations analogues to Ramanujan's forty identities for the Rogers-Ramanujan functions. S. -L. Chen and Huang [13] also derived some modular relations for Ramanujan-Göllnitz-Gordan functions. N. D. Baruah, J. Bora and N. Saikia [7], offered new proofs of many of the identities of Chen and Huang [13], their methods yields further new relations as well. In [14], Gugg has established some modular relations for Ramanujan-Göllnitz-Gordan functions. In [15, 16], H. Hahn defined septic analogues of the Rogers-Ramanujan functions as

$$L(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^3;q^7)_\infty(q^4;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.3}$$

$$M(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n}} = \frac{(q^7;q^7)_\infty(q^2;q^7)_\infty(q^5;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{1.4}$$

and

$$N(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{(q^7;q^7)_\infty(q;q^7)_\infty(q^6;q^7)_\infty}{(q^2;q^2)_\infty}. \tag{1.5}$$

In [15, 16], Hahn has established several modular relations for septic analogues of the Rogers-Ramanujan functions and also obtained several relations that are connected with the Rogers-Ramanujan and Göllnitz-Gordan functions. In [6],

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Baruah and Bora have established several modular relations for the nonic analogues of the Rogers-Ramanujan functions which are defined as

$$P(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} q^{3n^2}}{(q^3;q^3)_n (q^3;q^3)_{2n}} = \frac{(q^4;q^9)_\infty (q^5;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.6}$$

$$Q(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n} (1-q^{3n+2}) q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q^2;q^9)_\infty (q^7;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}, \tag{1.7}$$

and

$$R(q) := \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1} q^{3n(n+1)}}{(q^3;q^3)_n (q^3;q^3)_{2n+1}} = \frac{(q;q^9)_\infty (q^8;q^9)_\infty (q^9;q^9)_\infty}{(q^3;q^3)_\infty}. \tag{1.8}$$

They also established several other modular relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordan functions and septic analogues of Rogers-Ramanujan type functions. In [5] Baruah and Bora have established several modular relations involving two functions analogues to the Rogers-Ramanujan functions.

In [3], C. Adiga, K. R. Vasuki and B. R. Srivatsa Kumar have established modular relations involving two functions of Rogers-Ramanujan type. In [25], Vasuki, G. Sharath and K. R. Rajanna have established modular relations for cubic functions and are shown to be connected to the Ramanujan cubic continued fraction. In 2012, Adiga, Vasuki and N. Bhaskar [2] have established modular relations for cubic functions. Vasuki and P. S. Guruprasad [24] have established certain modular relations for the Rogers-Ramanujan type functions of order twelve of which some of them are proved by Baruah and Bora [5] on employing different method. Recently, Adiga and N. A. S. Bulkhali [1] have established several modular relations for the Rogers-Ramanujan type functions of order ten. They also established modular relations that are connected with the Rogers-Ramanujan functions, Göllnitz-Gordan functions and cubic functions, which are analogues to the Ramanujan's forty identities for Rogers-Ramanujan functions. Almost all of these functions which have been studied so far are due to Rogers [21] and L. G. Slater [23].

In [20, p. 33], Ramanujan stated the following identity:

$$\frac{f(aq^3,\ a^{-1}q^3)}{f(-q^2)} = \sum_{n=0}^{\infty} \frac{q^{2n^2}\ (-a^{-1}q;\ q^2)_n\ (-aq;\ q^2)_n}{(q^2;\ q^2)_{2n}}, \tag{1.9}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\ b^{n(n-1)/2}, \quad |ab| < 1, \tag{1.10}$$

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is the general theta function of Ramanujan.

The above result of Ramanujan yields infinitely many identities of Rogers-Ramanujan-Slater type when a is set to $\pm q^r$ for $r \in \mathbb{Q}$. In [18] J. Mc Laughlin, A. V. Sills and P. Zimmer have listed the following Rogers-Ramanujan-Slater type identities:

$$A(q) := \frac{f(-q^7,-q^8)}{f(-q^5)} = \sum_{n=0}^{\infty} \frac{q^{5n^2}\ (q^2;q^5)_n\ (q^3;q^5)_n}{(q^5;q^5)_{2n}}, \tag{1.11}$$

$$B(q) := \frac{f(-q^4,-q^{11})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-1}\ (q^4;q^5)_{n-1}\ (q;q^5)_{n+1}}{(q^5;q^5)_{2n}}, \tag{1.12}$$

$$C(q) := \frac{f(-q^2,-q^{13})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-3}\ (q^2;q^5)_{n-1}\ (q^3;q^5)_{n+1}}{(q^5;q^5)_{2n}}, \tag{1.13}$$

$$D(q) := \frac{f(-q,-q^{14})}{f(-q^5)} = 1 - \sum_{n=1}^{\infty} \frac{q^{5n^2-4}\ (q;q^5)_{n-1}\ (q^4;q^5)_{n+1}}{(q^5;q^5)_{2n}}. \tag{1.14}$$

The main purpose of this paper is to establish several modular relations involving $A(q)$, $B(q)$, $C(q)$ and $D(q)$, which are analogues to Ramanujan's forty identities and further we extract partition theoretic interpretations of two modular relations.

2 Definitions and Preliminary results

In this section, we present some basic definitions and preliminary results on Ramanujan's theta functions.

The function $f(a,b)$ satisfy the following basic properties [4, Entry 18]

$$f(a,b) = f(b,a), \tag{2.1}$$
$$f(1,a) = 2f(a,a^3), \tag{2.2}$$
$$f(-1,a) = 0. \tag{2.3}$$

The well-known Jacobi triple product identity [4, Entry 19] is given by

$$f(a,b) = (-a;ab)_\infty\ (-b;ab)_\infty\ (ab;ab)_\infty. \tag{2.4}$$

Using (1.10), we define

$$f_\delta(a,b) = \begin{cases} f(a,b) & \text{if} \quad \delta \equiv 0 (mod\ 2), \\ f(-a,-b) & \text{if} \quad \delta \equiv 1 (mod\ 2), \end{cases} \tag{2.5}$$

and, if n is an integer,

$$f(a,b) = a^{n(n+1)/2}\ b^{n(n-1)/2}\ f(a(ab)^n, b(ab)^{-n}). \tag{2.6}$$

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The three most interesting special cases of $f(a,b)$ are [4, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{2.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.8}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.9}$$

Also, after Ramanujan, define

$$\chi(q) := (-q;q^2)_\infty. \tag{2.10}$$

The following identity is an easy consequence of Entry 31 [4] when $n = 2$:

$$f(a,b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \tag{2.11}$$

For convenience, we define

$$f_n := f(-q^n) = (q^n;q^n)_\infty,$$

for positive integer n. The following lemma is a consequence of (2.4) and Entry 24 of [4, p. 39].

Lemma 2.1. *We have*

$$\varphi(q) = \frac{f_2^5}{f_1^2\, f_4^2}, \qquad \psi(q) = \frac{f_2^2}{f_1}, \qquad \varphi(-q) = \frac{f_1^2}{f_2}, \qquad \psi(-q) = \frac{f_1\, f_4}{f_2},$$

$$f(q) = \frac{f_2^3}{f_1\, f_4}, \qquad \chi(q) = \frac{f_2^2}{f_1\, f_4} \qquad \textit{and} \qquad \chi(-q) = \frac{f_1}{f_2}.$$

3 Main Result

In this section, we present several modular relations involving $A(q)$, $B(q)$, $C(q)$, and $D(q)$, in the combinations of

$$A_\beta A_\alpha + q^{\frac{2(\alpha+\beta)}{5}} B_\beta B_\alpha + q^{\alpha+\beta} C_\beta C_\alpha + q^{\frac{7(\alpha+\beta)}{5}} D_\beta D_\alpha,$$
$$A_\beta B_\alpha + q^{\frac{(2\beta+3\alpha)}{5}} B_\beta C_\alpha - q^{\beta+\alpha} C_\beta D_\alpha - q^{\frac{(7\beta-2\alpha)}{5}} D_\beta A_\alpha,$$
$$A_\beta C_\alpha - q^{\frac{2(\beta+\alpha)}{5}} B_\beta D_\alpha + q^{\beta-\alpha} C_\beta A_\alpha - q^{\frac{(7\beta-3\alpha)}{5}} D_\beta B_\alpha,$$
$$A_\beta D_\alpha - q^{\frac{(2\beta-7\alpha)}{5}} B_\beta A_\alpha - q^{\beta-\alpha} C_\beta B_\alpha + q^{\frac{(7\beta-2\alpha)}{5}} D_\beta C_\alpha,$$

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where α and β are positive integers and

$$A_k := A(q^k),\ B_k := B(q^k),\ C_k := C(q^k),\ D_k := D(q^k).$$

We prove the following theorem using ideas similar to those of Watson [26]. In Watson's method, one expresses the left sides of the identities in terms of theta functions by using (2.4). After clearing fractions, we see that the right side can be expressed as a product of two theta functions, say with summations indices m and n. One then tries to find a change of indices of the form

$$\alpha m + \beta n = 15M + a \quad \textit{and} \quad \gamma m + \delta n = 15N + b,$$

so that the product on the right side decomposes into the requisite sum of two products of theta functions on the left side.

Theorem 3.1. *We have*

$$A_{14}A_1 + q^6 B_{14}B_1 + q^{15}C_{14}C_1 + q^{21}D_{14}D_1 = \frac{{f_2}^2{f_7}^2}{f_1 f_5 f_{14} f_{70}} - \frac{q f_6 f_{21}}{f_5 f_{70}} - q^3, \tag{3.1}$$

$$A_{13}A_2 + q^6 B_{13}B_2 + q^{15}C_{13}C_2 + q^{21}D_{13}D_2$$
$$= \frac{{f_2}^2{f_{13}}^2}{f_1 f_{10} f_{26} f_{65}} - \frac{q f_6 f_{39}}{f_{10} f_{65}}\sqrt{\frac{f_6 f_{39}}{f_3 f_{78}} - \frac{q^3 f_3 f_{78}}{f_6 f_{39}}} - q^3. \tag{3.2}$$

Proof. Using (1.11) - (1.14), (2.7), (2.8) and Lemma 2.1, we may rewrite (3.1) in the form

$$\begin{aligned} &f(-q^{98},-q^{112})\ f(-q^7,-q^8) + q^6 f(-q^{56},-q^{154})\ f(-q^4,-q^{11}) \\ &+ q^{15} f(-q^{28},-q^{182})\ f(-q^2,-q^{13}) + q^{21} f(-q^{14},-q^{196})\ f(-q,-q^{14}) \\ =&f(q,q^3)\ f(-q^7,-q^7) - q f(-q^6) f(-q^{21}) - q^3 f(-q^5) f(-q^{70}). \end{aligned}$$

Set

$$m + n = 15M + a \ \text{ and } \ m - 14n = 15N + b$$

where a and b will have values from the set $\{0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm7\}$. Then

$$m = 14M + N + (14a + b)/15 \quad \text{and} \quad n = M - N + (a - b)/15.$$

It follows easily that $a = b$, and so $m = 14M + N + a$ and $n = M - N$, where $-7 \le a \le 7$. Thus there is a one-to-one correspondence between the set of all pairs of integers (m, n), $-\infty < m, n < \infty$, and triple of integers (M, N, a), $-\infty < M, N < \infty$, $-7 \le a \le 7$. Using (1.10) and (2.7), we obtain

$$f(1,q)\ f(-q^7,-q^7) = \sum_{m,n=-\infty}^{\infty} (-1)^n\ q^{(m^2+m+14n^2)/2}.$$

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Using (2.2) in the above equation, we obtain

$$\begin{aligned}
&2f(q,q^3)\ f(-q^7,-q^7)\\
&=\sum_{a=-7}^{7} q^{\frac{(a^2+a)}{2}} \sum_{M=-\infty}^{\infty} (-1)^M\ q^{\frac{210M^2+(14+28a)M}{2}} \sum_{N=-\infty}^{\infty} (-1)^N\ q^{\frac{15N^2-(1+2a)N}{2}}\\
&=\sum_{a=-7}^{7} q^{(a^2+a)/2}\ f(-q^{98-14a},-q^{112+14a})\ f(-q^{7-a},-q^{8+a})\\
&=2\{f(-q^{98},-q^{112})\ f(-q^7,-q^8)+qf(-q^{84},-q^{126})\ f(-q^6,-q^9)\\
&\quad +q^3f(-q^{70},-q^{140})\ f(-q^5,-q^{10})+q^6f(-q^{56},-q^{154})\ f(-q^4,-q^{11})\\
&\quad +q^{10}f(-q^{42},-q^{168})\ f(-q^3,-q^{12})+q^{15}f(-q^{28},-q^{182})\ f(-q^2,-q^{13})\\
&\quad +q^{21}f(-q^{14},-q^{196})\ f(-q,-q^{14})\}.
\end{aligned}$$

Dividing both sides by $f(-q^5)f(-q^{70})$ and using the formulas $G(q)=\frac{f(-q^2,-q^3)}{f(-q)}$, $H(q)=\frac{f(-q,-q^4)}{f(-q)}$ (see for example [18, p.11]), we find that

$$\begin{aligned}
&A_{14}A_1+q^6B_{14}B_1+q^{15}C_{14}C_1+q^{21}D_{14}D_1\\
&=\frac{f(q,q^3)\ f(-q^7,-q^7)}{f(-q^5)f(-q^{70})}-\frac{qf(-q^3)f(-q^{42})}{f(-q^5)f(-q^{70})}[G(q^3)G(q^{42})+q^9H(q^3)H(q^{42})]-q^3.
\end{aligned} \tag{3.3}$$

The first published proof of the following identity was given by Rogers [22]:

$$G(q)G(q^{14})+q^3H(q)H(q^{14})=\frac{\chi(-q^7)}{\chi(-q)}. \tag{3.4}$$

Employing (3.4) with q replaced by q^3 in (3.3) and using Lemma 2.1, we get the required result. Proof of (3.2) similar to that of (3.1). □

The proof of the following theorem is strongly depends upon the results of Rogers [22] and Bressoud [12]. We adopt Bressoud's notation, except that we use $q^{\frac{n}{24}}f(-q^n)$ instead of P_n, and the variable q instead of x. Let $g_\alpha^{(p,n)}$ and $\Phi_{\alpha,\beta,m,p}$ be defined as follows:

$$g_\alpha^{(p,n)}:=g_\alpha^{(p,n)}(q)=q^{\alpha\left(\frac{12n^2-12n+3-p}{24p}\right)}\prod_{r=0}^{\infty}\frac{(1-(q^\alpha)^{pr+\frac{p-2n+1}{2}})\ (1-(q^\alpha)^{pr+\frac{p+2n-1}{2}})}{\prod_{k=1}^{p-1}(1-(q^\alpha)^{pr+k})}, \tag{3.5}$$

for any positive odd integer p, integer n, and natural number α, and

$$\Phi_{\alpha,\beta,m,p}:=\Phi_{\alpha,\beta,m,p}(q)=\sum_{n=1}^{p}\sum_{r,s=-\infty}^{\infty}(-1)^{r+s}q^{\frac{1}{2}\{p\alpha(r+m\frac{2n-1}{2p})^2+p\beta(s+\frac{2n-1}{2p})^2\}}, \tag{3.6}$$

where α, β and p are natural numbers, and m is an odd positive integer. Then we can easily obtain the following propositions. We use the standard notation

$$(a_1, a_2, \ldots a_n; q)_\infty := \prod_{j=1}^{n} (a_j; q)_\infty.$$

Proposition 3.2. *[12, eqs. (2.12) and (2.13)]. We have*

$$g_\alpha^{(5,1)} = \quad q^{-\frac{\alpha}{60} G(q^\alpha)} \quad \textit{and} \quad g_\alpha^{(5,2)} = \quad q^{\frac{11\alpha}{60} H(q^\alpha)}.$$

Proposition 3.3. *[15, Proposition 3.6]. We have*

$$g_\alpha^{(7,1)} = q^{-\frac{\alpha}{42}} \frac{f(-q^{2\alpha})}{f(-q^\alpha)} L_\alpha, \tag{3.7}$$

$$g_\alpha^{(7,2)} = q^{\frac{5\alpha}{42}} \frac{f(-q^{2\alpha})}{f(-q^\alpha)} M_\alpha, \tag{3.8}$$

$$g_\alpha^{(7,3)} = q^{\frac{17\alpha}{42}} \frac{f(-q^{2\alpha})}{f(-q^\alpha)} N_\alpha. \tag{3.9}$$

Proposition 3.4. *[6, Proposition 6.3]. We have*

$$g_\alpha^{(9,1)} = q^{-\frac{\alpha}{36}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} P_\alpha, \tag{3.10}$$

$$g_\alpha^{(9,2)} = q^{\frac{\alpha}{12}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)}, \tag{3.11}$$

$$g_\alpha^{(9,3)} = q^{\frac{11\alpha}{36}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} Q_\alpha, \tag{3.12}$$

$$g_\alpha^{(9,4)} = q^{\frac{23\alpha}{36}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} R_\alpha. \tag{3.13}$$

Proposition 3.5. *We have*

$$g_\alpha^{(15,1)} = q^{-\frac{\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} A_\alpha, \tag{3.14}$$

$$g_\alpha^{(15,2)} = q^{\frac{\alpha}{30}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} G(q^{3\alpha}), \tag{3.15}$$

$$g_\alpha^{(15,3)} = q^{\frac{\alpha}{6}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)}, \tag{3.16}$$

$$g_\alpha^{(15,4)} = q^{\frac{11\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} B_\alpha, \tag{3.17}$$

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$$g_\alpha^{(15,5)} = q^{\frac{19\alpha}{30}} \frac{f(-q^{3\alpha})}{f(-q^\alpha)} H(q^{3\alpha}), \tag{3.18}$$

$$g_\alpha^{(15,6)} = q^{\frac{29\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} C_\alpha, \tag{3.19}$$

$$g_\alpha^{(15,7)} = q^{\frac{41\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} D_\alpha. \tag{3.20}$$

Proof. Take $p = 15$, and $n = 1$ in (3.5). Then

$$\begin{aligned} g_\alpha^{(15,1)} &= q^{\alpha(-\frac{1}{30})} \prod_{r=0}^{\infty} \frac{(1-(q^\alpha)^{15r+7})\,(1-(q^\alpha)^{15r+8})}{\prod_{k=1}^{14}(1-(q^\alpha)^{15r+k})} \\ &= \frac{q^{(-\alpha/30)}}{(q^\alpha, q^{2\alpha}, q^{3\alpha}, q^{4\alpha}, q^{5\alpha}, q^{6\alpha}, q^{9\alpha}, q^{10\alpha}, q^{11\alpha}, q^{12\alpha}, q^{13\alpha}, q^{14\alpha}; q^{15\alpha})_\infty} \\ &= q^{-\frac{\alpha}{30}} \frac{f(-q^{5\alpha})}{f(-q^\alpha)} A_\alpha, \end{aligned}$$

which gives (3.14). Similarly we can prove (3.15) - (3.20). □

Lemma 3.6. *[12, Proposition 5.1]. We have*

$$g_\alpha^{(p,n)} = g_\alpha^{(p,-n+1)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,n-2p)}, \quad g_\alpha^{(p,n)} = g_\alpha^{(p,2p-n+1)},$$

$$g_\alpha^{(p,n)} = -g_\alpha^{(p,n-p)}, \quad g_\alpha^{(p,n)} = -g_\alpha^{(p,p-n+1)}, \quad g_\alpha^{(p,(p+1)/2)} = 0.$$

Theorem 3.7. *[12, Proposition 5.4]. For odd $p > 1$,*

$$\Phi_{\alpha,\beta,m,p} = 2q^{\frac{\alpha+\beta}{24}} f(-q^\alpha) f(-q^\beta) \left(\sum_{n=1}^{(p-1)/2} g_\beta^{(p,n)} g_\alpha^{(p,(2mn-m+1)/2)} \right).$$

If we use Lemma 3.6 and Theorem 3.7 with $p = 5, 7, 9$ and 15, respectively, we then deduce the following useful lemmas.

Lemma 3.8. *[12, Corollary 5.7]. We have*

$$\Phi_{\alpha,\beta,1,5} = 2q^{\frac{\alpha+\beta}{40}} f(-q^\alpha) f(-q^\beta) \left(G(q^\beta)G(q^\alpha) + q^{\frac{\alpha+\beta}{5}} H(q^\beta)H(q^\alpha) \right), \tag{3.21}$$

$$\Phi_{\alpha,\beta,2,5} = 2q^{\frac{9\alpha+\beta}{40}} f(-q^\alpha) f(-q^\beta) \left(G(q^\beta)H(q^\alpha) - q^{\frac{-\alpha+\beta}{5}} H(q^\beta)G(q^\alpha) \right). \tag{3.22}$$

Lemma 3.9. *[15, Lemma 3.10]. We have*

$$\Phi_{\alpha,\beta,3,7} = 2q^{\frac{9\alpha+\beta}{56}} f(-q^{2\alpha}) f(-q^{2\beta}) \left(L_\beta M_\alpha - q^{\frac{2\alpha+\beta}{7}} M_\beta N_\alpha - q^{\frac{-\alpha+3\beta}{7}} N_\beta L_\alpha \right). \tag{3.23}$$

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Lemma 3.10. *[6, Lemma 6.9]. We have*

$$\Phi_{\alpha,\beta,1,9} = 2q^{\frac{\alpha+\beta}{72}} f(-q^{3\alpha})f(-q^{3\beta}) \left(P_\beta P_\alpha + q^{\frac{\alpha+\beta}{9}} + q^{\frac{\alpha+\beta}{3}} Q_\beta Q_\alpha + q^{\frac{2\alpha+2\beta}{3}} R_\beta R_\alpha\right), \tag{3.24}$$

$$\Phi_{\alpha,\beta,7,9} = 2q^{\frac{49\alpha+\beta}{72}} f(-q^{3\alpha})f(-q^{3\beta}) \left(P_\beta R_\alpha - q^{\frac{\beta-5\alpha}{9}} + q^{\frac{\beta-2\alpha}{3}} Q_\beta P_\alpha - q^{\frac{2\beta-\alpha}{3}} R_\beta Q_\alpha\right). \tag{3.25}$$

Lemma 3.11. *We have*

$$\begin{aligned}\Phi_{\alpha,\beta,1,15} =& 2q^{\frac{\alpha+\beta}{120}} f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta A_\alpha + q^{\frac{2(\alpha+\beta)}{5}} B_\beta B_\alpha + q^{\alpha+\beta} C_\beta C_\alpha + q^{\frac{7(\alpha+\beta)}{5}} D_\beta D_\alpha + q^{\frac{\alpha+\beta}{5}}\\ &+ q^{\frac{\alpha+\beta}{15}} \frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})G(q^{3\alpha}) + q^{\frac{3(\alpha+\beta)}{5}} H(q^{3\beta})H(q^{3\alpha})]\},\end{aligned} \tag{3.26}$$

$$\begin{aligned}\Phi_{\alpha,\beta,7,15} =& 2q^{\frac{49\alpha+\beta}{120}} f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta B_\alpha + q^{\frac{(2\beta+3\alpha)}{5}} B_\beta C_\alpha - q^{\beta+\alpha} C_\beta D_\alpha - q^{\frac{(7\beta-2\alpha)}{5}} D_\beta A_\alpha - q^{\frac{(\beta-\alpha)}{5}}\\ &- q^{\frac{(2\beta+8\alpha)}{30}} \frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})H(q^{3\alpha}) - q^{\frac{3(\beta-\alpha)}{5}} H(q^{3\beta})G(q^{3\alpha})]\},\end{aligned} \tag{3.27}$$

$$\begin{aligned}\Phi_{\alpha,\beta,11,15} =& 2q^{\frac{121\alpha+\beta}{120}} f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta C_\alpha - q^{\frac{2(\beta+\alpha)}{5}} B_\beta D_\alpha + q^{\beta-\alpha} C_\beta A_\alpha - q^{\frac{(7\beta-3\alpha)}{5}} D_\beta B_\alpha + q^{\frac{(\beta-4\alpha)}{5}}\\ &- q^{\frac{(\beta-14\alpha)}{15}} \frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})G(q^{3\alpha}) + q^{\frac{3(\beta+\alpha)}{5}} H(q^{3\beta})H(q^{3\alpha})]\},\end{aligned} \tag{3.28}$$

$$\begin{aligned}\Phi_{\alpha,\beta,13,15} =& 2q^{\frac{169\alpha+\beta}{120}} f(-q^{5\alpha})f(-q^{5\beta})\\ &\{A_\beta D_\alpha - q^{\frac{(2\beta-7\alpha)}{5}} B_\beta A_\alpha - q^{\beta-\alpha} C_\beta B_\alpha + q^{\frac{(7\beta-2\alpha)}{5}} D_\beta C_\alpha + q^{\frac{(\beta-6\alpha)}{5}}\\ &- q^{\frac{(\beta-11\alpha)}{15}} \frac{f(-q^{3\alpha})f(-q^{3\beta})}{f(-q^{5\alpha})f(-q^{5\beta})}[G(q^{3\beta})H(q^{3\alpha}) - q^{\frac{3(\beta-\alpha)}{5}} H(q^{3\beta})G(q^{3\alpha})]\}.\end{aligned} \tag{3.29}$$

Proof. Applying Theorem 3.7 with $m = 1$ and $p = 15$, we have

$$\begin{aligned}\Phi_{\alpha,\beta,1,15} =& 2q^{\frac{\alpha+\beta}{24}} f(-q^{\alpha})f(-q^{\beta})\{g_\alpha^{(15,1)}g_\beta^{(15,1)} + g_\alpha^{(15,2)}g_\beta^{(15,2)} + g_\alpha^{(15,3)}g_\beta^{(15,3)}\\ &+ g_\alpha^{(15,4)}g_\beta^{(15,4)} + g_\alpha^{(15,5)}g_\beta^{(15,5)} + g_\alpha^{(15,6)}g_\beta^{(15,6)} + g_\alpha^{(15,7)}g_\beta^{(15,7)}\}.\end{aligned} \tag{3.30}$$

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Using (3.14)- (3.20) in (3.30) and then simplifying, we obtain (3.26). The identities (3.27) - (3.29) can be proved in a similar way by setting $m = 7, 11, 13$, respectively, and $p = 15$ in Theorem 3.7. □

Corollary 3.12. *[12, Corollary 5.5 and 5.6]. If $\Phi_{\alpha,\beta,m,p}$ is defined by (3.6), then*

$$\Phi_{\alpha,\beta,m,1} = 0, \tag{3.31}$$

$$\Phi_{\alpha,\beta,1,3} = 2q^{\frac{\alpha+\beta}{24}} f(-q^{\alpha}) f(-q^{\beta}). \tag{3.32}$$

Theorem 3.13. *[12, Corollary 7.3]. Let α_i, β_i, m_i, p_i where $i = 1, 2$, be positive integers with m_1 and m_2 both odd. If $\lambda_1 := (\alpha_1 m_1^2 + \beta_1)/p_1$ and $\lambda_2 := (\alpha_2 m_2^2 + \beta_2)/p_2$, and the conditions*

$$\lambda_1 = \lambda_2, \tag{3.33}$$

$$\alpha_1\beta_1 = \alpha_2\beta_2, \tag{3.34}$$

$$\alpha_1 m_1 \equiv \alpha_2 m_2 (mod\ \lambda_1) \quad or \quad \alpha_1 m_1 \equiv -\alpha_2 m_2 (mod\ \lambda_1) \tag{3.35}$$

hold, then

$$\Phi_{\alpha_1,\beta_1,m_1,p_1} = \Phi_{\alpha_2,\beta_2,m_2,p_2}.$$

Theorem 3.14. *We have*

$$A_7B_2 + q^4B_7C_2 - q^9C_7D_2 - q^9D_7A_2 = \frac{{f_1}^2{f_{14}}^2}{f_2f_7f_{10}f_{35}} + \frac{qf_3f_{42}}{f_{10}f_{35}} + q, \tag{3.36}$$

$$A_{11}B_1 + q^5B_{11}C_1 - q^{12}C_{11}D_1 - q^{15}D_{11}A_1 = \frac{f_1f_{11}}{f_5f_{55}} + \frac{qf_3f_{33}}{f_5f_{55}} + q^2, \tag{3.37}$$

$$\begin{aligned} &A_{26}B_1 + q^{11}B_{26}C_1 - q^{27}C_{26}D_1 - q^{36}D_{26}A_1 \\ &\qquad = \frac{f_2f_{13}}{f_5f_{30}} + \frac{q^2f_3f_{78}}{f_5f_{130}}\sqrt{\frac{f_6f_{39}}{f_3f_{78}} - \frac{q^3f_3f_{78}}{f_6f_{39}}} + q^5, \end{aligned} \tag{3.38}$$

$$A_{14}C_1 - q^6B_{14}D_1 + q^{13}C_{14}A_1 - q^{19}D_{14}B_1 = \frac{f_2f_{21}}{f_5f_{70}} - q^2, \tag{3.39}$$

$$A_7D_2 - B_7A_2 - q^5C_7B_2 + q^9D_7C_2 = \frac{f_3f_{42}}{qf_{10}f_{35}} - \frac{1}{q}. \tag{3.40}$$

Proof. In the following sequel, let N denote the set of positive integers. To prove identity (3.36), set

$$\alpha_1 = 2u,\ \beta_1 = 7u,\ m_1 = 7,\ p_1 = 15u,$$

$$\alpha_2 = u,\ \beta_2 = 14u,\ m_2 = 7,\ p_2 = 9u,$$

in Theorem 3.13, to obtain

$$\Phi_{2u,7u,7,15u} = \Phi_{u,14u,7,9u}, \quad u \in N. \tag{3.41}$$

In particular, by taking $u = 1$ in (3.41) and then using (3.27) and (3.25), we deduce

$$\begin{aligned}
&2q^{7/8}f(-q^{10})f(-q^{35})\{A_7B_2 + q^4B_7C_2 - q^9C_7D_2 - q^9D_7A_2 \\
&- q - \frac{qf(-q^6)f(-q^{21})}{f(-q^{10})f(-q^{35})}[G(q^{21})H(q^6) - q^3H(q^{21})G(q^6)]\} \\
=&2q^{7/8}f_3f_{42}\{P_{14}R_1 + q^4Q_{14}P_1 - q^9R_{14}Q_1 - q\}.
\end{aligned} \tag{3.42}$$

The first published proof of the following identity was given by Rogers [22].

$$G(q^7)H(q^2) - qG(q^2)H(q^7) = \frac{\chi(-q)}{\chi(-q^7)}. \tag{3.43}$$

We also need the following identity proved by Baruah and Bora [6].

$$P_{14}R_1 + q^4Q_{14}P_1 - q^9R_{14}Q_1 = \frac{f_1^2f_{14}^2}{f_2f_3f_7f_{42}} + q. \tag{3.44}$$

Employing (3.43) with q replaced by q^3 and (3.44) in (3.42), and using Lemma 2.1, we get the required result.

To prove identity (3.37), set

$$\alpha_1 = u,\ \beta_1 = 11u,\ m_1 = 7,\ p_1 = 15u,$$

$$\alpha_2 = 11u,\ \beta_2 = u,\ m_2 = 1,\ p_2 = 3u,$$

in Theorem 3.13, to obtain

$$\Phi_{u,11u,7,15u} = \Phi_{11u,u,1,3u}, \quad u \in N \tag{3.45}$$

In particular, by taking $u = 1$ in (3.45) and then using (3.27) and (3.32), we deduce

$$\begin{aligned}
&2q^{1/2}f(-q^5)f(-q^{55})\{A_{11}B_1 + q^5B_{11}C_1 - q^{12}C_{11}D_1 - q^{15}D_{11}A_1 \\
&- q^2 - \frac{qf(-q^3)f(-q^{33})}{f(-q^5)f(-q^{55})}[G(q^{33})H(q^3) - q^6H(q^{33})G(q^3)]\} = 2q^{1/2}f_1f_{11}.
\end{aligned} \tag{3.46}$$

The first published proof of the following identity was given by Rogers [22]. Watson [26] also gave a proof.

$$G(q^{11})H(q) - q^2H(q^{11})G(q) = 1. \tag{3.47}$$

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Employing (3.47) with q replaced by q^3 in (3.46), we get the required result.

To prove identity (3.38), set

$$\alpha_1 = 1,\ \beta_1 = 5u+1,\ m_1 = 7,\ p_1 = u+10,$$

$$\alpha_2 = 1,\ \beta_2 = 5u+1,\ m_2 = 3,\ p_2 = u+2,$$

in Theorem 3.13, to obtain

$$\Phi_{1,5u+1,7,u+10} = \Phi_{1,5u+1,3,u+2}, \quad u \in N. \tag{3.48}$$

In particular, by taking $u = 5$ in (3.48) and then using (3.27) and (3.23), we deduce

$$\begin{aligned} &2q^{5/8} f(-q^5) f(-q^{130})\{A_{26}B_1 + q^{11}B_{26}C_1 - q^{27}C_{26}D_1 - q^{36}D_{26}A_1 \\ &- q^5 - \frac{q^2 f(-q^3) f(-q^{78})}{f(-q^5) f(-q^{130})}[G(q^{78})H(q^3) - q^{15}H(q^{78})G(q^3)]\} \\ =&2q^{5/8} f_2 f_{52}\{L_{26}M_1 - q^4 M_{26}N_1 - q^9 N_{26}L_1\}. \end{aligned} \tag{3.49}$$

The first published proof of the following identity was given by Bressoud [12].

$$G(q^{26})H(q) - q^3 G(q)H(q^{26}) = \sqrt{\frac{\chi(-q^{13})}{\chi(-q)} - q\frac{\chi(-q)}{\chi(-q^{13})}}, \tag{3.50}$$

and Hahn [15] proved the following interesting identity

$$L_{26}M_1 - q^4 M_{26}N_1 - q^9 N_{26}L_1 = \chi(-q^{13})\chi(-q^{26}). \tag{3.51}$$

Employing (3.50) with q replaced by q^3 and (3.51) in (3.49) and using Lemma 2.1, we get the required result.

To prove identity (3.39), put

$$\alpha_1 = u,\ \beta_1 = 14u,\ m_1 = 11,\ p_1 = 15u,$$

$$\alpha_2 = 2u,\ \beta_2 = 7u,\ m_2 = 1,\ p_2 = u,$$

in Theorem 3.13, to obtain

$$\Phi_{u,14u,11,15u} = \Phi_{2u,7u,1,u}, \quad u \in N. \tag{3.52}$$

In particular, by taking $u = 1$ in (3.52) and then using (3.28) and (3.31), we deduce

$$\begin{aligned} &2q^{9/8} f(-q^5) f(-q^{70})\{A_{14}C_1 - q^6 B_{14}D_1 + q^{13}C_{14}A_1 - q^{19}D_{14}B_1 \\ &+ q^2 - \frac{f(-q^3) f(-q^{42})}{f(-q^5) f(-q^{70})}[G(q^{42})G(q^3) + q^9 H(q^{42})H(q^3)]\} = 0. \end{aligned} \tag{3.53}$$

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The first published proof of the following identity was given by Rogers [21].

$$G(q)G(q^{14}) + q^3H(q)H(q^{14}) = \frac{\chi(-q^7)}{\chi(-q)}. \tag{3.54}$$

Employing (3.54) with q replaced by q^3 in (3.53) and using Lemma 2.1, we get the required result.

To prove identity (3.40), set

$$\alpha_1 = 2u,\ \beta_1 = 7u,\ m_1 = 13,\ p_1 = 15u,$$

$$\alpha_2 = u,\ \beta_2 = 14u,\ m_2 = 3,\ p_2 = u,$$

in Theorem 3.13, to obtain

$$\Phi_{2u,7u,13,15u} = \Phi_{u,14u,3,u}, \quad u \in N. \tag{3.55}$$

In particular, by taking $u = 1$ in (3.55) and then using (3.29) and (3.31), we obtain

$$\begin{aligned} &2q^{23/8}f(-q^{10})f(-q^{35})\{A_7D_2 - B_7A_2 - q^5C_7B_2 + q^9D_7C_2 \\ &+ q^{-1} - \frac{q^{-1}f(-q^6)f(-q^{21})}{f(-q^{10})f(-q^{35})}[G(q^{21})H(q^6) - q^3H(q^{21})G(q^6)]\} = 0. \end{aligned} \tag{3.56}$$

The first published proof of the following identity was given by Rogers [21].

$$G(q^7)H(q^2) - qH(q^7)G(q^2) = \frac{\chi(-q)}{\chi(-q^7)}. \tag{3.57}$$

Employing (3.57) with q replaced by q^3 in (3.56) and using Lemma 2.1, we get the required result. □

Remark: Identity (3.1) can also be proved by using Theorem 3.13, (3.26) and (3.24). Identity (3.36) can also be proved by using idea similar to those of Watson [26].

4 Applications to the theory of partitions

In this section, we present partition theoretic interpretations of (3.40) and (3.39). For simplicity, we define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

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Definition 4.1. *A positive integer n has k color if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called "colored partitions".*

For example, if 2 is allowed to have two colors, say r (red), and g (green), then all colored partitions of 3 are 3, $2_r + 1$, $2_g + 1$, $1 + 1 + 1$.
An important fact (see for example [17, p. 211]) is that

$$\frac{1}{(q^u;\ q^v)_\infty^k}$$

is the generating function for the number of partitions of n, where all the parts are congruent to u $(mod\ v)$ and have k colors.

Theorem 4.2. *Let $P_1(n)$ denote the number of partitions of n into parts not congruent to* ±1, ±2, ±5, ±11, ±13, ±17, ±19, ±23, ±25, ±28, ±29, ±30, ±31, ±32, ±37, ±41, ±43, ±47, ±49, ±53, ±55, ±56, ±58, ±59, ±60, ±61, ±62, ±65, ±67, ±71, ±73, ±79, ±83, ±85, ±88, ±89, ±90, ±92, ±95, ±97, ±101, ±103, 105 $(mod\ 210)$*, and parts congruent to* ±42, ±70, ±84 $(mod\ 210)$ *with two colors.*
Let $P_2(n)$ denote the number of partitions of n into parts not congruent to ±1, ±5, ±11, ±13, ±14, ±16, ±17, ±19, ±23, ±25, ±28, ±29, ±30, ±31, ±37, ±41, ±43, ±44, ±46, ±47, ±53, ±55, ±59, ±60, ±61, ±65, ±67, ±71, ±73, ±74, ±76, ±77, ±79, ±83, ±85, ±89, ±90, ±95, ±97, ±101, ±103, ±104, 105 $(mod\ 210)$*, and parts congruent to* ±42, ±70, ±84 $(mod\ 210)$ *with two colors.*
Let $P_3(n)$ denote the number of partitions of n into parts not congruent to ±1, ±5, ±8, ±11, ±13, ±14, ±17, ±19, ±22, ±23, ±25, ±29, ±30, ±31, ±37, ±38, ±41, ±43, ±47, ±52, ±53, ±55, ±59, ±60, ±61, ±65, ±67, ±68, ±71, ±73, ±79, ±82, ±83, ±85, ±89, ±90, ±91, ±95, ±97, ±98, ±101, ±103, 105 $(mod\ 210)$*, and parts congruent to* ±42, ±70, ±84 $(mod\ 210)$ *with two colors.*
Let $P_4(n)$ denote the number of partitions of n into parts not congruent to ±1, ±4, ±5, ±7, ±11, ±13, ±17, ±19, ±23, ±25, ±26, ±29, ±30, ±31, ±34, ±37, ±41, ±43, ±47, ±53, ±55, ±56, ±59, ±60, ±61, ±64, ±65, ±67, ±71, ±73, ±79, ±83, ±85, ±86, ±89, ±90, ±94, ±95, ±97, ±98, ±101, ±103, 105 $(mod\ 210)$*, and parts congruent to* ±42, ±70, ±84 $(mod\ 210)$ *with two colors.*
Let $P_5(n)$ denote the number of partitions of n into parts not congruent to ±1, ±3, ±5, ±6, ±9, ±11, ±12, ±13, ±15, ±17, ±18, ±19, ±21, ±23, ±24, ±25, ±27, ±29, ±30, ±31, ±33, ±36, ±37, ±39, ±41, ±42, ±43, ±45, ±47, ±48, ±51, ±53, ±54, ±55, ±57, ±59, ±60, ±61, ±63, ±65, ±66, ±67, ±69, ±71, ±72, ±73, ±75, ±78, ±79, ±81, ±83, ±84, ±85, ±87, ±89, ±90, ±93, ±95, ±96, ±97, ±99, ±101, ±102, ±103, 105 $(mod\ 210)$*, and parts congruent to* ±70 $(mod\ 210)$ *with two colors.*
Let $P_6(n)$ denote the number of partitions of n into parts not congruent to ±1, ±5, ±10, ±11, ±13, ±17, ±19, ±20, ±23, ±25, ±29, ±30, ±31, ±35, ±37, ±40,

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±41, ±43, ±47, ±50, ±53, ±55, ±59, ±60, ±61, ±65, ±67, ±70, ±71, ±73, ±79, ±80, ±83, ±85, ±89, ±90, ±95, ±97, ±100, ±101, ±103, 105 (mod 210), *and parts congruent to* ±42, ±84 (mod 210) *with two colors.*
Then, for any positive integer $n \geq 9$, *we have*

$$P_1(n) - P_2(n) - P_3(n-5) + P_4(n-9) = P_5(n+1) - P_6(n+1).$$

Proof. Using (1.11) - (1.14) and (2.4) in (3.40) and simplifying we obtain

$$\frac{1}{(q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{14\pm}, q^{15\pm}, q^{16\pm}, q^{18\pm}, q^{20\pm}, q^{21\pm}, q^{22\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{24\pm}, q^{26\pm}, q^{27\pm}, q^{33\pm}, q^{34\pm}, q^{35\pm}, q^{36\pm}, q^{38\pm}, q^{39\pm}, q^{40\pm}, q^{42\pm}, q^{42\pm}, q^{44\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{45\pm}, q^{46\pm}, q^{48\pm}, q^{50\pm}, q^{51\pm}, q^{52\pm}, q^{54\pm}, q^{57\pm}, q^{63\pm}, q^{64\pm}, q^{66\pm}, q^{68\pm}, q^{69\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{70\pm}, q^{70\pm}, q^{72\pm}, q^{74\pm}, q^{75\pm}, q^{76\pm}, q^{77\pm}, q^{78\pm}, q^{80\pm}, q^{81\pm}, q^{82\pm}, q^{84\pm}, q^{84\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{86\pm}, q^{87\pm}, q^{91\pm}, q^{93\pm}, q^{94\pm}, q^{96\pm}, q^{98\pm}, q^{99\pm}, q^{100\pm}, q^{102\pm}, q^{104\pm}; q^{210})_\infty}$$
$$- \frac{1}{(q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{7\pm}, q^{8\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{15\pm}, q^{18\pm}, q^{20\pm}, q^{21\pm}, q^{22\pm}, q^{24\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{26\pm}, q^{27\pm}, q^{32\pm}, q^{33\pm}, q^{34\pm}, q^{35\pm}, q^{36\pm}, q^{38\pm}, q^{39\pm}, q^{40\pm}, q^{42\pm}, q^{42\pm}, q^{45\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{48\pm}, q^{49\pm}, q^{50\pm}, q^{51\pm}, q^{52\pm}, q^{54\pm}, q^{56\pm}, q^{57\pm}, q^{58\pm}, q^{62\pm}, q^{63\pm}, q^{64\pm}, q^{66\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{68\pm}, q^{69\pm}, q^{70\pm}, q^{70\pm}, q^{72\pm}, q^{75\pm}, q^{78\pm}, q^{80\pm}, q^{81\pm}, q^{82\pm}, q^{84\pm}, q^{84\pm}, q^{86\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{87\pm}, q^{88\pm}, q^{91\pm}, q^{92\pm}, q^{93\pm}, q^{94\pm}, q^{96\pm}, q^{98\pm}, q^{99\pm}, q^{100\pm}, q^{102\pm}; q^{210})_\infty}$$
$$- \frac{q^5}{(q^{2\pm}, q^{3\pm}, q^{4\pm}, q^{6\pm}, q^{7\pm}, q^{9\pm}, q^{10\pm}, q^{12\pm}, q^{15\pm}, q^{16\pm}, q^{18\pm}, q^{20\pm}, q^{21\pm}, q^{24\pm}, q^{26\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{27\pm}, q^{28\pm}, q^{32\pm}, q^{33\pm}, q^{34\pm}, q^{35\pm}, q^{36\pm}, q^{39\pm}, q^{40\pm}, q^{42\pm}, q^{42\pm}, q^{44\pm}, q^{45\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{46\pm}, q^{48\pm}, q^{49\pm}, q^{50\pm}, q^{51\pm}, q^{54\pm}, q^{56\pm}, q^{57\pm}, q^{58\pm}, q^{62\pm}, q^{63\pm}, q^{64\pm}, q^{66\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{69\pm}, q^{70\pm}, q^{70\pm}, q^{72\pm}, q^{74\pm}, q^{75\pm}, q^{76\pm}, q^{77\pm}, q^{78\pm}, q^{80\pm}, q^{81\pm}, q^{84\pm}, q^{84\pm}; q^{210})_\infty}$$
$$\times \frac{1}{(q^{86\pm}, q^{87\pm}, q^{88\pm}, q^{92\pm}, q^{93\pm}, q^{94\pm}, q^{96\pm}, q^{99\pm}, q^{100\pm}, q^{102\pm}, q^{104\pm}; q^{210})_\infty}$$

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$$+\frac{q^9}{(q^{2\pm},q^{3\pm},q^{6\pm},q^{8\pm},q^{9\pm},q^{10\pm},q^{12\pm},q^{14\pm},q^{15\pm},q^{16\pm},q^{18\pm},q^{20\pm},q^{21\pm},q^{22\pm},q^{24\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{27\pm},q^{28\pm},q^{32\pm},q^{33\pm},q^{35\pm},q^{36\pm},q^{38\pm},q^{39\pm},q^{40\pm},q^{42\pm},q^{42\pm},q^{44\pm},q^{45\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{46\pm},q^{48\pm},q^{49\pm},q^{50\pm},q^{51\pm},q^{52\pm},q^{54\pm},q^{57\pm},q^{58\pm},q^{62\pm},q^{63\pm},q^{66\pm},q^{68\pm};\ q^{210})_\infty}$$
$$\times\frac{1}{(q^{69\pm},q^{70\pm},q^{70\pm},q^{72\pm},q^{74\pm},q^{75\pm},q^{76\pm},q^{77\pm},q^{78\pm},q^{80\pm},q^{81\pm},q^{82\pm},q^{84\pm};\ q^{210})_\infty}$$
$$\times\frac{1}{(q^{87\pm},q^{88\pm},q^{91\pm},q^{92\pm},q^{93\pm},q^{96\pm},q^{99\pm},q^{100\pm},q^{102\pm},q^{104\pm};\ q^{210})_\infty}$$
$$=\frac{1}{q\ (q^{2\pm},q^{4\pm},q^{7\pm},q^{8\pm},q^{10\pm},q^{14\pm},q^{16\pm},q^{20\pm},q^{22\pm},q^{26\pm},q^{28\pm},q^{32\pm},q^{34\pm},q^{35\pm},q^{38\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{40\pm},q^{44\pm},q^{46\pm},q^{49\pm},q^{50\pm},q^{52\pm},q^{56\pm},q^{58\pm},q^{62\pm},q^{64\pm},q^{68\pm},q^{70\pm},q^{70\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{74\pm},q^{76\pm},q^{77\pm},q^{80\pm},q^{82\pm},q^{86\pm},q^{88\pm},q^{91\pm},q^{92\pm},q^{94\pm},q^{98\pm},q^{100\pm},q^{104\pm};\ q^{210})_\infty}$$
$$-\frac{1}{q\ (q^{2\pm},q^{3\pm},q^{4\pm},q^{6\pm},q^{7\pm},q^{8\pm},q^{9\pm},q^{12\pm},q^{14\pm},q^{15\pm},q^{16\pm},q^{18\pm},q^{21\pm},q^{22\pm},q^{24\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{26\pm},q^{27\pm},q^{28\pm},q^{32\pm},q^{33\pm},q^{34\pm},q^{36\pm},q^{38\pm},q^{39\pm},q^{42\pm},q^{42\pm},q^{44\pm},q^{45\pm};q^{210})_\infty}$$
$$\times\frac{1}{(q^{46\pm},q^{48\pm},q^{49\pm},q^{51\pm},q^{52\pm},q^{54\pm},q^{56\pm},q^{57\pm},q^{58\pm},q^{62\pm},q^{63\pm},q^{64\pm},q^{66\pm};\ q^{210})_\infty}$$
$$\times\frac{1}{(q^{68\pm},q^{69\pm},q^{72\pm},q^{74\pm},q^{75\pm},q^{76\pm},q^{77\pm},q^{78\pm},q^{81\pm},q^{82\pm},q^{84\pm},q^{84\pm},q^{86\pm};\ q^{210})_\infty}$$
$$\times\frac{1}{(q^{87\pm},q^{88\pm},q^{91\pm},q^{92\pm},q^{93\pm},q^{94\pm},q^{96\pm},q^{98\pm},q^{99\pm},q^{102\pm},q^{104\pm};\ q^{210})_\infty}.$$

Note that the six quotients of the above identity represent the generating functions for $P_1(n)$, $P_2(n)$, $P_3(n)$, $P_4(n)$, $P_5(n)$ and $P_6(n)$ respectively. Hence, it is equivalent to

$$\sum_{n=0}^{\infty}P_1(n)q^n-\sum_{n=0}^{\infty}P_2(n)q^n-q^5\sum_{n=0}^{\infty}P_3(n)q^n+q^9\sum_{n=0}^{\infty}P_4(n)q^n$$
$$=\frac{1}{q}\sum_{n=0}^{\infty}P_5(n)q^n-\frac{1}{q}\sum_{n=0}^{\infty}P_6(n)q^n$$

where we set $P_1(0)=P_2(0)=P_3(0)=P_4(0)=P_5(0)=P_6(0)=1$. Equating coefficients of q^n $(n\geq 9)$ on both sides yields the desired result. $\square$

Example 4.3. *The following table illustrates the case $n=10$ in the Theorem 4.2.*

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$P_1(10) = 4$	$10, 7+3, 6+4, 4+3+3$
$P_2(10) = 9$	$10, 8+2, 7+3, 6+4, 6+2+2, 4+4+2,$ $4+3+3, 3+3+2+2, 2+2+2+2+2$
$P_3(5) = 1$	$3+2$
$P_4(1) = 0$	0
$P_5(11) = 2$	$7+4,\ 7+2+2$
$P_6(11) = 8$	$9+2,\ 8+3,\ 7+4,\ 7+2+2, 6+3+2,$ $4+4+3,\ 4+3+2+2,\ 3+3+2+2$

Theorem 4.4. *Let $P_1(n)$ denote the number of partitions of n into parts not congruent to* ± 4, ± 15, ± 19, ± 26, ± 30, ± 34, ± 41, ± 45, ± 49, ± 56, ± 60, ± 64, ± 71, ± 75, ± 79 $(mod\ 165)$*, and parts congruent to* ± 22, ± 33, ± 44, ± 55, ± 66 $(mod\ 165)$ *with two colors.*
Let $P_2(n)$ denote the number of partitions of n into parts not congruent to ± 2, ± 13, ± 15, ± 17, ± 28, ± 30, ± 32, ± 43, ± 45, ± 47, ± 58, ± 60, ± 62, ± 73, ± 75 $(mod\ 165)$*, and parts congruent to* ± 11, ± 22, ± 33, ± 55, ± 66 $(mod\ 165)$ *with two colors.*
Let $P_3(n)$ denote the number of partitions of n into parts not congruent to ± 1, ± 14, ± 15, ± 16, ± 29, ± 30, ± 31, ± 45, ± 46, ± 59, ± 60, ± 61, ± 74, ± 75, ± 76 $(mod\ 165)$*, and parts congruent to* ± 11, ± 33, ± 55, ± 66, ± 77 $(mod\ 165)$ *with two colors.*
Let $P_4(n)$ denote the number of partitions of n into parts not congruent to ± 7, ± 8, ± 15, ± 23, ± 30, ± 37, ± 38, ± 45, ± 52, ± 53, ± 60, ± 67, ± 68, ± 75, ± 82 $(mod\ 165)$*, and parts congruent to* ± 33, ± 44, ± 55, ± 66, ± 77 $(mod\ 165)$ *with two colors.*
Let $P_5(n)$ denote the number of partitions of n into parts not congruent to ± 3, ± 6, ± 9, ± 12, ± 15, ± 18, ± 21, ± 24, ± 27, ± 30, ± 33, ± 36, ± 39, ± 42, ± 45, ± 48, ± 51, ± 54, ± 57, ± 60, ± 63, ± 66, ± 69, ± 72, ± 75, ± 78, ± 81, $(mod\ 165)$*, and parts congruent to* ± 11, ± 22, ± 44, ± 55, ± 77 $(mod\ 165)$ *with two colors.*
Let $P_6(n)$ denote the number of partitions of n into parts not congruent to ± 5, ± 10, ± 15, ± 20, ± 25, ± 30, ± 35, ± 40, ± 45, ± 50, ± 55, ± 60, ± 65, ± 70, ± 75, ± 80 $(mod\ 165)$*, and parts congruent to* ± 11, ± 22, ± 33, ± 44, ± 66 $(mod\ 165)$ *with two colors. Then, for any positive integer $n \geq 15$, we have*

$$P_1(n) + P_2(n-5) - P_3(n-12) - P_4(n-15) = P_5(n-1) + P_6(n-2).$$

Proof. The proof of Theorem 4.4 is similar to that of Theorem 4.2. Expressing (3.39) in q-products and after some simplification we get the desired result.
The following table illustrates the case $n = 15$ in the Theorem 4.4.

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$P_1(15) = 119$
$P_2(10) = 20$
$P_3(3) = 1$
$P_4(0) = 1$
$P_5(14) = 59$
$P_6(13) = 78$

□

Acknowledgements

The authors thank the referee for several helpful comments and suggestions. The first author is thankful to the University Grants Commission, Government of India for the financial support under the grant F.510/2/SAP-DRS/2011. The second author is thankful to DST, New Delhi for awarding INSPIRE Fellowship [No. DST/INSPIRE Fellowship/2012/122], under which this work has been done.

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