Geometry of color perception. Part 2: perceived colors from real quantum states and Hering's rebit

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RESEARCH

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Abstract

Inspired by the work of Resnikoff, which is described if full details in the first part of this two-part paper, we give a quantum description of the space $\mathcal{P}$ of perceived colors. We show that $\mathcal{P}$ is the effect space of a rebit, a real quantum qubit, whose state space $\mathcal{S}$ is isometric to the hyperbolic Klein disk $\mathcal{K}$. This chromatic state space of perceived colors can be represented as a Bloch disk of real dimension 2 that coincides with the Hering disk given by the color opponency mechanism. Attributes of perceived colors, hue and saturation, are defined in terms of Von Neumann entropy.

Keywords: Color Perception; Quantum states; Jordan Algebras; Quantum Rebit.

1 Introduction

> "The structure of our scientific cognition of the world is decisively determined by the fact that this world does not exist in itself, but is merely encountered by us as an object in *the correlative variance of subject and object*" [1].

The mathematical description of human color perception mechanisms is a longstanding problem addressed by many of the most influential figures of the mathematical physics [1], [2], [3], [4]... The reader will find an overview of the main historical contributions at the beginning of [5] where H. L. Resnikoff points out that the space, that we denote $\mathcal{P}$, of perceived colors is one of the very first examples of abstract manifold mentioned by B. Riemann in his habilitation [6], "a pregnant remark". As suggested by H. Weyl [1], it is actually very tempting to characterize the individual color perception as a specific correlative interaction between an abstract space of perceived colors and an embedding space of physical colors. This raises the question of defining intrinsically, in the sense of Riemannian geometry, the space of perceived colors from basic largely accepted axioms. These axioms, which date back to the works of H. G. Grassmann and H. Von Helmhotz [2], [7], state that $\mathcal{P}$ is a regular convex cone of dimension 3. It is worth noting that convexity reflects the property that one must be able to perform mixtures of perceived colors or, in other words, of color states [8]. What makes the work of H. L. Resnikoff [5] particularly enticing is the remarkable conclusions that he derives by adding the sole axiom that $\mathcal{P}$ is homogeneous under the action of the linear group of background illumination changes [9]. We will discuss at the end of this work, in section 6, the relevance of this statement. To the best of our knowledge, this axiom, which involves an external context, has never been verified by psychophysical experiments.

It endows $\mathcal{P}$ with the rich structure of a symmetric cone [10]. With this additional axiom, and the hypothesis that the distance on $\mathcal{P}$ is given by a Riemannian metric invariant under background illumination changes, H. L. Resnikoff shows that $\mathcal{P}$ can only be isomorphic to one of the two following Riemannian spaces: the product $\mathcal{P}_1 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ equipped with a flat metric, namely the Helmhotz-Stiles metric [11], and $\mathcal{P}_2 = \mathbb{R}^+ \times SL(2,\mathbb{R})/SO(2,\mathbb{R})$ equipped with the Rao-Siegel metric of constant negative curvature [12], [13]. Let us recall that the quotient $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ is isomorphic to the Poincaré hyperbolic disk $\mathcal{D}$. The first space is the usual metric space of the colorimetry, while the second one seems to be relevant to explain psychophysical phenomena such as the ones described by H. Yilmaz in [14] and [15]. In the sequel, we focus on the latter.

The starting point of this work originates from the second part of [5] dedicated to Jordan algebras. Contrary to H. L. Resnikoff we suppose at first that the perceived color space $\mathcal{P}$ can be described from the state space of a quantum system characterized by a Jordan algebra $\mathcal{A}$ of real dimension 3 [16], [17], [18], [19]. This is our only axiom. Jordan algebras are non associative commutative algebras that have been classified by P. Jordan, J. Von Neumann and E. Wigner [20] under the assumptions that they are of finite dimension and formally real. They are considered as a fitting alternative to the usual associative noncommutative algebraic framework for the geometrization of quantum mechanics [21], [22], [23]. Not so surprisingly in view of what precedes, $\mathcal{A}$ is necessarily isomorphic to one of the two following Jordan algebras: the algebra $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ or the algebra $\mathcal{H}(2,\mathbb{R})$ of symmetric real 2 by 2 matrices. It appears that the two geometric models of H. L. Resnikoff can be recovered from this fact by simply taking the positive cone of $\mathcal{A}$. The Jordan algebra $\mathcal{H}(2,\mathbb{R})$ carries a very special structure being isomorphic to the spin factor $\mathbb{R} \oplus \mathbb{R}^2$. It can be seen as the non associative algebra linearly spanned by the unit 1 and a spin system of the Clifford algebra of $\mathbb{R}^2$ [19], [24]. The main topic of this work is to exploit these structures to highlight the quantum nature of the space $\mathcal{P}$ of perceived colors. Actually, the quantum description that we propose gives its full meaning to the relevant remark of [25], p. 5.

Although the geometry of the second model $\mathcal{P}_2$ of H. L. Resnikoff is much richer than the geometry of the first model $\mathcal{P}_1$, very few works are devoted to the possible implications of hyperbolicity in color perception. One of the main objectives of this contribution is to show that the model $\mathcal{P}_2$ is perfectly adapted to explain the coherence between the trichromatic and color opponency theories. We show that the space $\mathcal{P}$ is the effect space of a so-called rebit, a real quantum qubit, whose state space $\mathcal{S}$ is isometric to the hyperbolic Klein disk $\mathcal{K}$. Actually, $\mathcal{K}$ is isometric to the Poincaré disk $\mathcal{D}$, but its geodesics are visually very different, being the straight chords of the unit disk. Klein geometry appears naturally when considering the spin factor $\mathbb{R} \oplus \mathbb{R}^2$ and the 3-dimensional Minkowski future lightcone $\mathcal{L}^+$ whose closure is the state cone of the rebit. We show that the chromatic state space $\mathcal{S}$ can be represented as a Bloch disk of real dimension 2 that coincides with the Hering disk given by the color opponency mechanism. This Bloch disk is an analog, in our real context, of the Bloch ball that describes the state space of a two-level quantum system of a spin-$\frac{1}{2}$ particule. This suggests that the processing performed by the color opponent ganglion cells is similar to spin inversions.

Following this quantum interpretation, we give precise definitions of the two chromatic attributes of a perceived color, hue and saturation, in terms of Von Neumann entropy.

As explained by P. A. M. Dirac in [26], p. 18, physical phenomena justify the need for considering complex Hilbert spaces in quantum mechanics. Alternatively, the structures we deal with in the sequel are real and we may consider the space $\mathcal{P}$ as a non trivial concrete example of an effect space of a real quantum system. The reader will find more information on real-vector-space quantum theory and its consistency regarding optimal information transfer in [27].

Finally, since the spin factors and the corresponding Clifford algebras share the same representations (and the same squares), one may envisage to adapt to the present context the tools developed in [28] for the harmonic analysis of color images.

2 Mathematical preliminaries

We introduce in this section the mathematical notions needed in the sequel. They mainly concern the properties of Jordan algebras. The reader will find more detailed information on this subject in [16], [18], [19], [29] or in the seminal work of P. Jordan [30]. A Jordan algebra $\mathcal{A}$ is a real vector space equipped with a commutative bilinear product $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, $(a, b) \longmapsto a \circ b$, satisfying the following Jordan identity:

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a) \ . \tag{1}$$

This Jordan identity ensures that the power of any element a of $\mathcal{A}$ is well-defined ($\mathcal{A}$ is power associative in the sense that the subalgebra generated by any of its elements is associative). Since a sum of squared observables never vanishes, one logically requires that if a_1, a_2 ,..., a_n are elements of $\mathcal{A}$ such that:

$$a_1^2 + a_2^2 + \cdots + a_n^2 = 0 \ , \tag{2}$$

then $a_1 = a_2 = \cdots = a_n = 0$. The algebra $\mathcal{A}$ is then said to be formally real. This property endows $\mathcal{A}$ with a partial ordering: $a \leq b$ if and only if $b - a$ is a sum of squares, and therefore the squares of $\mathcal{A}$ are positive. Formally real Jordan algebras of finite dimension are classified [20]: every such algebra is the direct sum of so-called simple Jordan algebras. Simple Jordan algebras are of the following types: the algebras $\mathcal{H}(n, \mathbb{K})$ of hermitian matrices with entries in the division algebra $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (the algebra of quaternions), the algebra $\mathcal{H}(3, \mathbb{O})$ of hermitian matrices with entries in the division algebra $\mathbb{O}$ (the algebra of octonions), and the spin factors $\mathbb{R} \oplus \mathbb{R}^n$, with $n \geq 0$. The Jordan product on $\mathcal{H}(n, \mathbb{K})$ and $\mathcal{H}(3, \mathbb{O})$ is defined by:

$$a \circ b = \frac{1}{2}(ab + ba) \ . \tag{3}$$

The spin factors form “the most mysterious of the four infinite series of [simple] Jordan algebras” [31]. They were introduced for the first time under this name by D. M. Topping [32] and are defined as follows. The spin factor $J(V)$ of a given

n-dimensional real inner product space V is the Jordan algebra freely generated by V with the relations:

$$v^2 = \|v\|^2 \ , \tag{4}$$

for all v in V. Due to commutativity, this implies:

$$v \circ w = \langle v, w \rangle \ , \tag{5}$$

for all v and w in V. Consequently, $J(V)$ is isomorphic to the direct sum $\mathbb{R} \oplus V$ with the product:

$$(\alpha + v) \circ (\beta + w) = (\alpha\beta + \langle v, w \rangle + \alpha w + \beta v) \ , \tag{6}$$

where α and β are reals. The following result is well known [31].

Proposition 1 *Let $\mathbb{K}$ be the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$. The spin factor $J(\mathbb{K} \oplus \mathbb{R})$ is isomorphic to the Jordan algebra $\mathcal{H}(2, \mathbb{K})$.*

Proof. An explicit isomorphism is given by the map:

$$\phi : \mathcal{H}(2, \mathbb{K}) \longrightarrow J(\mathbb{K} \oplus \mathbb{R}) \tag{7}$$

$$\begin{pmatrix} \alpha + \beta & x \\ x^* & \alpha - \beta \end{pmatrix} \longmapsto (\alpha + x + \beta) \ , \tag{8}$$

with x in $\mathbb{K}$ and x^* the conjugate of x. □

Now, we focus on the algebra $\mathcal{H}(2, \mathbb{R})$ and the spin factor $J(\mathbb{R} \oplus \mathbb{R})$, both being isomorphic to $\mathbb{R} \oplus \mathbb{R}^2$. The latter is equipped with the Minkowski metric [33]:

$$(\alpha + v) \cdot (\beta + w) = \alpha\beta - \langle v, w \rangle \ , \tag{9}$$

where α and β are reals and v and w are vectors of $\mathbb{R}^2$. It turns out that Proposition 1 has a fascinating reformulation: the algebra of observables of a 2-dimensional real quantum system is isomorphic to the 3-dimensional Minkowski spacetime. Let us recall that the lightcone $\mathcal{L}$ of $\mathbb{R} \oplus \mathbb{R}^2$ is the set of elements $a = (\alpha + v)$ that satisfy:

$$a \cdot a = 0 \ , \tag{10}$$

and that a light ray is a 1-dimensional subspace of $\mathbb{R} \oplus \mathbb{R}^2$ spanned by an element of $\mathcal{L}$. Every such light ray is spanned by a unique element of the form $(1 + v)/2$ with v a unit vector of $\mathbb{R}^2$. Actually, the space of light rays coincides with the projective space $\mathbb{P}_1(\mathbb{R})$. In other words, we have the following result.

Proposition 2 *There is a one to one correspondance between the light rays of the spin factor $\mathbb{R} \oplus \mathbb{R}^2$ and the rank one projections of the Jordan algebra $\mathcal{H}(2, \mathbb{R})$.*

Proof. The correspondance is given by:

$$(1 + v)/2 \longmapsto \frac{1}{2} \begin{pmatrix} 1 + v_1 & v_2 \\ v_2 & 1 - v_1 \end{pmatrix} , \tag{11}$$

where $v = v_1 e_1 + v_2 e_2$ is a unit vector of $\mathbb{R}^2$. □

We will see in the next section that this result has a meaningful interpretation: there is a one to one correspondance between the light rays of the spin factor $\mathbb{R} \oplus \mathbb{R}^2$ and the pure state density matrices of the algebra $\mathcal{H}(2, \mathbb{R})$.

The positive cone $\mathcal{C}$ of the Jordan algebra $\mathcal{A}$ is the set of its positive elements, namely:

$$\mathcal{C} = \{a \in \mathcal{A},\ a > 0\} \ . \tag{12}$$

It can be shown that $\mathcal{C}$ is the interior of the positive domain of $\mathcal{A}$ defined as the set of squares of $\mathcal{A}$. The convex cone $\mathcal{C}$ is symmetric: it is regular, homogeneous and self-dual [10]. The positive cone $\mathcal{H}^+(2, \mathbb{R})$ of the algebra $\mathcal{H}(2, \mathbb{R})$ is the set of positive-definite symmetric matrices.

3 Quantum preliminaries

The positive cone $\mathcal{C}$ is the set of positive observables. A state of $\mathcal{A}$ is a linear functional:

$$\langle \cdot \rangle : \mathcal{A} \longrightarrow \mathbb{R} \ , \tag{13}$$

that is nonnegative: $\langle a \rangle \geq 0,\ \forall a \geq 0$, and normalized: $\langle 1 \rangle = 1$. Since $\mathcal{A}$ is formally real, the pairing:

$$\langle a, b \rangle = \mathrm{Trace}(\mathrm{L}(a)(b)) = \mathrm{Trace}(a \circ b) \ , \tag{14}$$

is a real-valued inner product and one can identify any state with a unique element ρ of $\mathcal{A}$ by setting:

$$\langle a \rangle = \mathrm{Trace}(\rho \circ a) \ , \tag{15}$$

where $\rho \geq 0$ and $\mathrm{Trace}(\rho) = 1$. Such a ρ, for $\mathcal{A} = \mathcal{H}(2, \mathbb{R})$, is a so-called state density matrix [8] and describes a mixed state or mixture. Formula (15) gives the expectation value of the observable a in the state with density matrix ρ.

Regarding Proposition 1, the positive state density matrices of the algebra $\mathcal{H}(2, \mathbb{R})$ are in one to one correspondance with the elements of the future lightcone:

$$\mathcal{L}^+ = \{a = (\alpha + v),\ \alpha > 0,\ a \cdot a > 0\} \ . \tag{16}$$

that are of the form $a = (1 + v)/2$. One way to qualify states is to introduce the Von Neumann entropy [8], [34]. It is given by:

$$S(\rho) = -\mathrm{Trace}(\rho \log \rho) \ . \tag{17}$$

It appears that $S(\rho) = 0$ if and only if ρ satisfies $\rho \circ \rho = \rho$. The zero entropy state density matrices characterize pure states that afford a maximum of information.

Among the other state density matrices, one is of particular interest. It is given by $\rho_0 = Id_2/2$ or $\rho_0 = (1+0)/2$ (Id_2 is the identity matrix) and is characterized by:

$$\rho_0 = \underset{\rho}{\operatorname{argmax}} S(\rho) \ . \tag{18}$$

The mixed state with density matrix ρ_0 is called the state of maximum entropy, $S(\rho_0)$ being equal to $\log 2$. It provides the minimum of information. Using (15), we have:

$$\langle a \rangle_0 = \frac{\text{Trace}(a)}{2} \ . \tag{19}$$

Now, given an observable a, a acts on the state $\langle \cdot \rangle_0$ by the formula:

$$a : \langle \cdot \rangle_0 \longmapsto \langle a \circ \cdot \rangle = \langle \cdot \rangle_{0,a} \ , \tag{20}$$

Since for any state ρ, the element 2ρ is an observable, we get:

$$\langle a \rangle_{0,2\rho} = \langle 2\rho \circ a \rangle_0 = \text{Trace}(\rho \circ a) = \langle a \rangle \ , \tag{21}$$

for all observable a. This means that any state with density matrix ρ can be obtained from the state of maximal entropy with density matrix ρ_0 using the action of the observable 2ρ.

The pure state density matrices of the algebra $\mathcal{A} = \mathcal{H}(2, \mathbb{R})$ are of the form:

$$\frac{1}{2} \begin{pmatrix} 1+v_1 & v_2 \\ v_2 & 1-v_1 \end{pmatrix} \ , \tag{22}$$

where $v = v_1 e_1 + v_2 e_2$ is a unit vector of $\mathbb{R}^2$. They are in one to one correspondance with the light rays of the 3-dimensional Minkowski spacetime, see Proposition 2.

A classical representation of quantum states is the Bloch body [35]. An element ρ of $\mathcal{H}(2, \mathbb{R})$ is a state density matrix if and only if it can be written as:

$$\rho(v_1, v_2) = \frac{1}{2}(Id_2 + v \cdot \sigma) = \frac{1}{2}(Id_2 + v_1\sigma_1 + v_2\sigma_2) \ , \tag{23}$$

where $\sigma = (\sigma_1, \sigma_2)$ with:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \tag{24}$$

and $v = v_1 e_1 + v_2 e_2$ is a vector of $\mathbb{R}^2$ with $\|v\| \leq 1$. The matrices σ_1 and σ_2 are Pauli-like matrices. In the usual framework of quantum mechanics, that is when the observable algebra is the algebra $\mathcal{H}(2, \mathbb{C})$ of 2 by 2 hermitian matrices with complex entries, the Bloch body is the unit Bloch ball in $\mathbb{R}^3$. It represents the states of the two-level quantum system of a spin-$\frac{1}{2}$ particule, also called a qubit. In the present context, the Bloch body is the unit disk of $\mathbb{R}^2$ associated to a rebit. We give now more details on this system using the classical Dirac notations [26], bra, ket, etc... Let us denote $|u_1\rangle$, $|d_1\rangle$, $|u_2\rangle$ and $|d_2\rangle$ the four state vectors defined by:

$$|u_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \; |d_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; , \; |u_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; , \; |d_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \; . \quad (25)$$

We have:

$$\sigma_1 = |u_1\rangle\langle u_1| - |d_1\rangle\langle d_1| \; , \; \sigma_2 = |u_2\rangle\langle u_2| - |d_2\rangle\langle d_2| \; . \quad (26)$$

The state vectors $|u_1\rangle$ and $|d_1\rangle$, resp. $|u_2\rangle$ and $|d_2\rangle$, are eigenstates of σ_1, resp. σ_2, with eigenvalues 1 and -1.

Using polar coordinates $v_1 = r\cos\theta$, $v_2 = r\sin\theta$, we can write $\rho(v_1, v_2)$ as:

$$\rho(r,\theta) = \frac{1}{2} \begin{pmatrix} 1 + r\cos\theta & r\sin\theta \\ r\sin\theta & 1 - r\cos\theta \end{pmatrix} \quad (27)$$

$$= \frac{1}{2}\{(1 + r\cos\theta)|u_1\rangle\langle u_1| + (1 - r\cos\theta)|d_1\rangle\langle d_1| + (r\sin\theta)|u_2\rangle\langle u_2| \quad (28)$$

$$-(r\sin\theta)|d_2\rangle\langle d_2|\} \; . \quad (29)$$

This gives, for instance:

$$\rho(1,0) = |u_1\rangle\langle u_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; , \quad (30)$$

$$\rho(1,\pi) = |d_1\rangle\langle d_1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \; , \quad (31)$$

$$\rho(1,\pi/2) = |u_2\rangle\langle u_2| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \; , \quad (32)$$

$$\rho(1,3\pi/2) = |d_2\rangle\langle d_2| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \; . \quad (33)$$

More generally:

$$\rho(1,\theta) = |(1,\theta)\rangle\langle(1,\theta)| \; , \quad (34)$$

with:

$$|(1,\theta)\rangle = \cos(\theta/2)|u_1\rangle + \sin(\theta/2)|d_1\rangle \; . \quad (35)$$

This means that we can identify the pure state density matrices $\rho(1,\theta)$ with the state vectors $|(1,\theta)\rangle$ and also with the points of the unit disk boundary of coordinate θ. The state of maximal entropy, given by the state density matrix:

$$\rho_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; , \quad (36)$$

is the mixture:

$$\rho_0 = \frac{1}{4}|u_1\rangle\langle u_1| + \frac{1}{4}|d_1\rangle\langle d_1| + \frac{1}{4}|u_2\rangle\langle u_2| + \frac{1}{4}|d_2\rangle\langle d_2| \quad (37)$$

$$= \frac{1}{4}\rho(1,0) + \frac{1}{4}\rho(1,\pi) + \frac{1}{4}\rho(1,\pi/2) + \frac{1}{4}\rho(1,3\pi/2) \ , \quad (38)$$

with equal probabilities. Using (27), we can write every state density matrix as a mixture:

$$\rho(r,\theta) = \rho_0 + \frac{r\cos\theta}{2}\left(\rho(1,0) - \rho(1,\pi)\right) + \frac{r\sin\theta}{2}\left(\rho(1,\pi/2) - \rho(1,3\pi/2)\right) \ . \quad (39)$$

Such a mixture is given by the point of the unit disk of polar coordinates (r,θ). It is important to notice that the four state density matrices $\rho(1,0)$, $\rho(1,\pi)$, $\rho(1,\pi/2)$ and $\rho(1,3\pi/2)$ correspond to two pairs of state vectors $(|u_1\rangle, |d_1\rangle)$, $(|u_2\rangle, |d_2\rangle)$, the state vectors $|u_i\rangle$ and $|d_i\rangle$, for $i = 1, 2$, being linked by the "up and down" Pauli-like matrix σ_i.

In the usual framework, that is when $\mathcal{A}$ is $\mathcal{H}(2,\mathbb{C})$, the three Pauli matrices are associated to the three directions of rotations in $\mathbb{R}^3$. In our case, there are only two Pauli-like matrices. The interpretation in terms of rotations ceases to be relevant since there is no space with a rotation group of dimension 2. This makes rebits somewhat strange. We show in Section 5 that the opponency color mechanism of E. Hering is given by such a rebit when dealing with chromatic opponencies.

The pure and mixed states play a crucial role in the measurements: "... that is, after the interaction with the apparatus, the system-plus-apparatus behaves like a mixture... It is in this sense, and in this sense alone, that a measurement is said to change a pure state into a mixture" [36] (see also the cited reference [37]). It seems actually that the problem of measurements in quantum mechanics was one of the main motivations of P. Jordan for the introduction of his new kind of algebras: "Observations not only disturb what has to be measured, they produce it... We ourselves produce the results of measurements" [36] p. 161, [38], [39].

4 The Riemannian geometry of $\mathcal{C}$ and $\mathcal{L}^+$

Now, we give further information on the underlying geometry of the Jordan algebra $\mathcal{A}$ from both the points of view discussed above, that is $\mathcal{A}$ as the algebra $\mathcal{H}(2,\mathbb{R})$ and $\mathcal{A}$ as the spin factor $\mathbb{R}\oplus\mathbb{R}^2$. We consider the level set:

$$\mathcal{C}_1 = \{X \in \mathcal{H}^+(2,\mathbb{R}), \mathrm{Det}(X) = 1\} \ , \quad (40)$$

Every X in $\mathcal{C}_1$ can be written as:

$$X = \begin{pmatrix} \alpha + v_1 & v_2 \\ v_2 & \alpha - v_1 \end{pmatrix} \ , \quad (41)$$

with $v = v_1e_1 + v_2e_2$ a vector of $\mathbb{R}^2$ satisfying $\alpha^2 - \|v\|^2 = 1$ and $\alpha > 0$. Using the one to one correspondance:

$$X = \begin{pmatrix} \alpha + v_1 & v_2 \\ v_2 & \alpha - v_1 \end{pmatrix} \longmapsto (\alpha + v) \ , \quad (42)$$

the level set $\mathcal{C}_1$ is sent to the level set:

$$\mathcal{L}_1 = \{a = (\alpha + v) \in \mathcal{L}^+,\ a \cdot a = 1\} \ , \tag{43}$$

of the future lightcone $\mathcal{L}^+$. It is well known that the projection:

$$\pi_1 : \mathcal{L}_1 \longrightarrow \{\alpha = 0\} \ , \tag{44}$$

defined by:

$$\pi_1(\alpha + v) = (0 + w) \ , \tag{45}$$

with $w = w_1 e_1 + w_2 e_2$ and:

$$w_i = \frac{v_i}{1+\alpha} \ , \tag{46}$$

for $i = 1, 2$, is an isometry between the level set $\mathcal{L}_1$ and the Poincaré disk $\mathcal{D}$ [40]. Simple computations show that the matrix X can be written:

$$X = \begin{pmatrix} \dfrac{1 + 2w_1 + (w_1^2 + w_2^2)}{1 - (w_1^2 + w_2^2)} & \dfrac{2w_2}{1 - (w_1^2 + w_2^2)} \\ \dfrac{2w_2}{1 - (w_1^2 + w_2^2)} & \dfrac{1 - 2w_1 + (w_1^2 + w_2^2)}{1 - (w_1^2 + w_2^2)} \end{pmatrix} \ , \tag{47}$$

in the w-parametrization.

Proposition 3 *Let X be an element of $\mathcal{C}_1$ written under the form (47), we have:*

$$\frac{\text{Trace}\left[(X^{-1}dX)^2\right]}{2} = 4\left(\frac{(dw_1)^2 + (dw_2)^2}{(1 - (w_1^2 + w_2^2)^2}\right) = ds^2_{\mathcal{D}} \ . \tag{48}$$

Proof. Cayley-Hamilton theorem implies the following equality, where A denotes a 2 by 2 matrix:

$$(\text{Trace}(A))^2 = \text{Trace}(A^2) + 2\text{Det}(A) \ . \tag{49}$$

We apply this equality to the matrix $A = X^{-1}dX$. The matrix X can be written as:

$$X = \left(\frac{1 + |z|^2}{1 - |z|^2}\right) I_2 + \frac{X_1}{1 - |z|^2} \ , \tag{50}$$

where:

$$X_1 = \begin{pmatrix} 2w_1 & 2w_2 \\ 2w_2 & -2w_1 \end{pmatrix} \ , \tag{51}$$

and $z = w_1 + iw_2$. We have:

$$X^{-1} = \left(\frac{1 + |z|^2}{1 - |z|^2}\right) I_2 - \frac{X_1}{1 - |z|^2} \ , \tag{52}$$

and:

$$dX = d\left(\frac{1+|z|^2}{1-|z|^2}\right) I_2 + \frac{dX_1}{1-|z|^2} + \frac{d(|z|^2)}{(1-|z^2|)^2} X_1 \ . \tag{53}$$

Consequently:

$$X^{-1}dX = \frac{2d(|z|^2)}{(1-|z|^2)^2} I_2 - \frac{d(|z|^2)}{(1-|z|^2)^2} X_1 + \frac{(1+|z|^2)dX_1}{(1-|z|^2)^2} - \frac{X_1 dX_1}{(1-|z|^2)^2} \ . \tag{54}$$

Since $\mathrm{Trace}(X_1) = \mathrm{Trace}(dX_1) = 0$ and $\mathrm{Trace}(X_1 dX_1) = 4d(|z|^2)$, then:

$$\left[\mathrm{Trace}(X^{-1}dX)\right]^2 = 0 \ , \tag{55}$$

and:

$$\mathrm{Trace}\left[(X^{-1}dX)^2\right] = -2\mathrm{Det}(X^{-1}dX) = -2\mathrm{Det}(dX) \ . \tag{56}$$

We have also:

$$dX = \frac{d(|z|^2)}{(1-|z|^2)^2} \begin{pmatrix} 1+|z|^2+2w_1 & 2w_2 \\ 2w_2 & 1+|z|^2-2w_1 \end{pmatrix} \tag{57}$$

$$+\frac{1}{(1-|z|^2)} \begin{pmatrix} d(|z|^2)+2dw_1 & 2dw_2 \\ 2dw_2 & d(|z|^2)-2dw_1 \end{pmatrix} \ . \tag{58}$$

Simple computations lead to:

$$\mathrm{Det}(dX) = -4\left(\frac{(dw_1)^2+(dw_2)^2}{(1-|z|^2)^2}\right) \ , \tag{59}$$

and end the proof. □

This proposition means that $\mathcal{C}_1$ equipped with the normalized Rao-Siegel metric, i. e. $\mathrm{Trace}\left[(X^{-1}dX)^2\right]/2$, is isometric to the Poincaré disk $\mathcal{D}$ of constant negative curvature equal to -1. Actually $\mathcal{C}$ is foliated by the level sets of the determinant with leaves that are isometric to $\mathcal{D}$. This description, which is analog to the one considered by H. L. Resnikoff in [5], does not take into account the specific role of the state density matrices of the algebra $\mathcal{H}(2,\mathbb{R})$.

Another classical result of hyperbolic geometry asserts that the projection:

$$\varpi_1 : \mathcal{L}_1 \longrightarrow \{\alpha = 1\} \ , \tag{60}$$

defined by:

$$\varpi_1(\alpha + v) = (1 + x) \ , \tag{61}$$

with $x = x_1 e_1 + x_2 e_2$ and:

$$x_i = \frac{v_i}{\alpha} \ , \tag{62}$$

for $i = 1, 2$, is an isometry between the level set $\mathcal{L}_1$ and the Klein disk $\mathcal{K}$, the Riemannian metric of which is given by:

$$ds^2_{\mathcal{K}} = \frac{(dx_1)^2 + (dx_2)^2}{1 - (x_1^2 + x_2^2)} + \frac{(x_1 dx_1 + x_2 dx_2)^2}{(1 - (x_1^2 + x_2^2))^2} \ , \tag{63}$$

see [40]. An isometry between $\mathcal{K}$ and $\mathcal{D}$ is defined by:

$$x_i = \frac{2w_i}{1 + (w_1^2 + w_2^2)} \ , \tag{64}$$

$$w_i = \frac{x_i}{1 + \sqrt{1 - (x_1^2 + x_2^2)}} \ , \tag{65}$$

for $i = 1, 2$. In other words, we have the following commutative diagram of isometries:

$$\begin{array}{ccccc} \mathcal{L}_1 & \longrightarrow & \mathcal{L}_1 & \longrightarrow & \mathcal{C}_1 \\ {\scriptstyle \varpi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\ \mathcal{K} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D} \end{array} \tag{66}$$

Let us recall that the state density matrices of the quantum system we consider can be identified with the elements:

$$a = (1 + v)/2 \ , \tag{67}$$

of the spin factor $\mathbb{R} \oplus \mathbb{R}^2$ with $\|v\|^2 \leq 1$. Let us denote:

$$\varpi_{1/2} : \mathcal{L}_{1/2} = \{a = (\alpha + v)/2,\ \alpha > 0,\ a \cdot a = 1/4\} \longrightarrow \{\alpha = 1/2\} \ , \tag{68}$$

the projection given by:

$$\varpi_{1/2}((\alpha + v)/2) = (1 + v/\alpha)/2 \ . \tag{69}$$

We have:

$$\varpi_{1/2}((\alpha + v)/2) = \varpi_1(\alpha + v)/2 \ . \tag{70}$$

This means that the map:

$$\varphi : \mathcal{K} \longmapsto \mathcal{K}_{1/2} \ , \tag{71}$$

defined by:

$$\varphi(x_1, x_2) = (x_1, x_2)/2 \ , \tag{72}$$

is an isometry between $\mathcal{K}$ and:

$$\mathcal{K}_{1/2} = \{x/2 \in \mathbb{R}^2,\ \|x\|^2 < 1\}\ , \tag{73}$$

the Riemannian metric on the latter being given by:

$$ds^2_{\mathcal{K}_{1/2}} = (\varphi^{-1})^* ds^2_{\mathcal{K}} = \frac{(dx_1)^2 + (dx_2)^2}{1/4 - (x_1^2 + x_2^2)} + \frac{(x_1 dx_1 + x_2 dx_2)^2}{(1/4 - (x_1^2 + x_2^2))^2}\ . \tag{74}$$

One can verify that $\mathcal{L}^+$ is foliated by the level sets $\alpha = \textit{constant}$ with leaves that are isometric to the Klein disk $\mathcal{K}$. This description is more appropriate than the above one to characterize perceived colors from real quantum states since the state space $\mathcal{S}$ is naturally embedded in $\overline{\mathcal{L}^+}$, see (78) below.

Let us recall some basic facts about the geometry of the Klein disk. Contrary to the Poincaré disk, the geodesics of $\mathcal{K}$ are straight lines and more precisely, the chords of the unit disk. An important feature of the Klein metric is that it coincides with the Hilbert metric defined as follows. Let p and q be two interior points of the disk and let r and s be the two points of the boundary of the disk such that the segment $[r, s]$ contains the segment $[p, q]$. The Hilbert distance between p and q is defined by:

$$d_H(p, q) = \frac{1}{2} \log[r, p, q, s]\ , \tag{75}$$

where:

$$[r, p, q, s] = \frac{\|q - r\|}{\|p - r\|} \times \frac{\|p - s\|}{\|q - s\|}\ , \tag{76}$$

is the cross-ratio of the four points r, p, q and s [41] (in (76), $\|\cdot\|$ is the Euclidean norm).

5 Perceived colors and chromatic states

We describe the space $\mathcal{P}$ of perceived colors under the only hypothesis that $\mathcal{P}$ can be described from the state space of a quantum system characterized by the Jordan algebra $\mathcal{H}(2, \mathbb{R})$. As explained before, we exploit the fact that $\mathcal{H}(2, \mathbb{R})$ is isomorphic to the spin factor $\mathbb{R} \oplus \mathbb{R}^2$. Let us recall that this description does not involve any reference to physical colors or to an observer.

The state space $\mathcal{S}$ is the unit disk embedded in the space of state density matrices by:

$$s = (v_1, v_2) \longmapsto \rho(v_1, v_2) = \frac{1}{2} \begin{pmatrix} 1 + v_1 & v_2 \\ v_2 & 1 - v_1 \end{pmatrix}\ . \tag{77}$$

and in the Klein disk $\mathcal{K}_{1/2}$ of the closure $\overline{\mathcal{L}^+}$ of the future lightcone $\mathcal{L}^+$ by:

$$s = (v_1, v_2) \longmapsto \frac{1}{2}(1 + v) = 1/2 + (v_1/2, v_2/2)\ . \tag{78}$$

In order to describe perceived colors, i.e. measured colors, it is necessary to characterize all the possible measurements that can be performed on the states. We adopt here the viewpoint of the generalized probability theory [42].

We denote $\mathcal{C}(\mathcal{S})$ the state cone defined by:

$$\mathcal{C}(\mathcal{S}) = \left\{ \alpha \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix},\ \alpha \geq 0,\ s = (v_1, v_2) \in \mathcal{S} \right\} . \tag{79}$$

This cone is self-dual, that is:

$$\mathcal{C}(\mathcal{S}) = \mathcal{C}^*(\mathcal{S}) = \{a \in \mathcal{A},\ \forall b \in \mathcal{C}(\mathcal{S}),\ \langle a, b\rangle \geq 0\} , \tag{80}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{A}$ given by (14). By definition, an effect is an element e of $\mathcal{C}^*(\mathcal{S})$ such that $e(s) \leq 1$ for all s in $\mathcal{S}$. Such an effect e can be seen as an affine function $e : \mathcal{S} \longrightarrow [0, 1]$ with $0 \leq e(s) \leq 1$ for all s. It is the most general way of assigning a probability to all states. Effects correspond to positive operator valued measures. In the present settings, every effect is given by a vector $e = (a_1, a_2, a_3)$ such that:

$$0 \leq e \cdot \begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix} \leq 1 , \tag{81}$$

for all $s = (v_1, v_2)$ in $\mathcal{S}$. The measurement effect associated to e is the operator:

$$E = a_3 Id_2 + a_1\sigma_1 + a_2\sigma_2 , \tag{82}$$

that must satisfy $0 \leq E \leq Id_2$. This last condition implies that $0 \leq a_3 \leq 1$, with $a_1^2 + a_2^2 \leq a_3^2$ and $a_1^2 + a_2^2 \leq (1 - a_3)^2$. We denote $\mathcal{E}(\mathcal{S})$ the effect space of $\mathcal{S}$, that is the set of all effects on $\mathcal{S}$. Note that the so-called unit effect, $e_1 = (0, 0, 1)$, satisfies $e_1(s) = 1$ for all s in $\mathcal{S}$.

A perceived color $c = (a_1, a_2, a_3)$ is by definition an effect on $\mathcal{S}$, that is an element of the effect space $\mathcal{E}(\mathcal{S})$. Since $\mathcal{C}(\mathcal{S}) = \mathcal{C}^*(\mathcal{S})$, a perceived color c is an element of the state cone of $\mathcal{S}$, this one being the closure $\overline{\mathcal{L}^+}$ of the future lightcone $\mathcal{L}^+$. The element $c/(2a_3) = (a_1/2a_3, a_2/2a_3, 1/2)$, $a_3 \neq 0$, belongs to the Klein disk $\mathcal{K}_{1/2}$ of $\overline{\mathcal{L}^+}$. This suggests to define the colorimetric attributes of c as follows.

– The real a_3, with $0 \leq a_3 \leq 1$, is the magnitude of the perceived color c.

– The element $s_c = (a_1/a_3, a_2/a_3) \in \mathcal{S}$ is the chromatic state of c.

– A perceived color with a unit chromatic state is a pure perceived color.

– The saturation of a perceived color c is given by the Von Neumann entropy of its chromatic state.

– A perceived color whose chromatic state is the state of maximal entropy is achromatic.

Given a state $(v_1, v_2) \in \mathcal{S}$, the perceived colors which have this state as chromatic state form the intersection:

$$c_s = \mathcal{E}(\mathcal{S}) \cap \left\{ \begin{pmatrix} a_3 v_1 \\ a_3 v_2 \\ a_3 \end{pmatrix}, \ 0 \leq a_3 \leq 1 \right\} . \tag{83}$$

The maximum value of a perceived color $c = (a_1, a_2, a_3)$ is:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} a_1/a_3 \\ a_2/a_3 \\ 1 \end{pmatrix} = \frac{a_1^2 + a_2^2}{a_3} + a_3 = a_3(1 + r^2) , \tag{84}$$

with $r^2 = (a_1^2 + a_2^2)/a_3^2$. We must have:

$$0 \leq r^2 \leq \frac{1 - a_3}{a_3} \leq 1 . \tag{85}$$

If $0 < a_3 < 1/2$, the measure of a perceived color $c = (a_1, a_2, a_3)$ on his chromatic state $(a_1/a_3, a_2/a_3)$ gives the probability $a_3(1+r^2)$, this one being well defined for all $0 < r \leq 1$. In particular, primary colors are measured with the maximum probability $2a_3$. In this case, the magnitude is not high enough to allow measurements with probability 1. In this case, the perceived color is under-estimated.

If $a_3 = 1/2$, the measure of a perceived color $c = (a_1, a_2, 1/2)$ on his chromatic state $(2a_1, 2a_2)$ gives the probability $(1 + r^2)/2$. This probability is well-defined for all $0 < r \leq 1$. It is maximal, equal to 1, if and only if c is a primary color. In this case, the perceived color is ideally-estimated.

If $1/2 < a_3 < 1$, the measure of a perceived color $c = (a_1, a_2, a_3)$ on his chromatic state $(a_1/a_3, a_2/a_3)$ gives the probability $a_3(1+r^2)$. This probability is well defined if and only if equation (85) is satisfied. In particular, primary colors can not be measured on their chromatic states. For instance, if $a_3 = 2/3$, then r should be less than or equal to $\sqrt{2}/2$ and perceived colors with chromatic states of norm equal to $\sqrt{2}/2$ are measured with probability 1. In this case, the perceived color is over-estimated.

An achromatic perceived color $c = (0, 0, a_3)$ measured on a chromatic state gives the probability a_3 and this independently of the considered chromatic state. Such a perceived color does not take into account chromaticity. The unit perceived color $c = e_1$ is the saturated achromatic perceived color.

We explain now how to describe the Hering color opponency mechanism [43], [44], from the rebit system introduced at the end of Section 3. Let us first quote D. H. Krantz [44]: "E. Hering noted that colors can be classified as reddish or greenish or neither, but that redness and greenness are not simultaneously attributes of a color. If we add increasing amounts of a green light to a reddish light, the redness of the mixture decreases, disappears, and gives way to greenness. At the point where redness is gone and greenness is not yet present, the color may be yellowish, bluish, or achromatic. We speak of a partial chromatic equilibrium, with respect to red/green... Similarly, yellow and blue are identified as opponent hues..." We rename

$|g\rangle = |u_1\rangle$, $|r\rangle = |d_1\rangle$, $|b\rangle = |u_2\rangle$ and $|y\rangle = |d_2\rangle$ the four state vectors characterizing the rebit. The opponency mechanism is given by the two matrices σ_1 and σ_2. More precisely, the state vector:

$$|(1,\theta)\rangle = \cos(\theta/2)|g\rangle + \sin(\theta/2)|r\rangle \ , \tag{86}$$

satisfies:

$$\langle(1,\theta)|\sigma_1|(1,\theta)\rangle = \cos\theta, \ \langle(1,\theta)|\sigma_2|(1,\theta)\rangle = \sin\theta \ . \tag{87}$$

This means that if $\cos\theta > 0$, then the pure chromatic state $s(\theta)$ of the Bloch disk with coordinate θ is greenish, and if $\cos\theta < 0$, then $s(\theta)$ is reddish. For $\theta = \pi/2$, or $\theta = 3\pi/2$, $s(\theta)$ is achromatic in the opposition green/red. In the same way, if $\sin\theta$ is positive, then $s(\theta)$ is bluish, and if $\sin\theta$ is negative, then $s(\theta)$ is yellowish. For $\theta = 0$, or $\theta = \pi$, $s(\theta)$ is achromatic in the blue/yellow opposition. The phenomenon "that redness and greenness are not simultaneously attributes of a color", for instance, is a trivial consequence of the fact that $\langle(1,\theta)|\sigma_1|(1,\theta)\rangle$ can not be simultaneously positive and negative.

It appears in consequence that the quantum model that we propose allows to recover axiomatically that a chromatic pure state, that is a hue, is given by a pair of splittings similar to the two spin up and down inversions of a rebit. Following L. E. J. Brouwer, "Newton's theory of color analyzed light rays in their medium, but Goethe and Schopenhauer, more sensitive to the truth, considered color to be the polar splitting by the human eye" [45] (see also [46] and [47]).

6 Group actions and homogeneity

Let us first recall that the special Lorentz group $SO^+(1,2)$ is the identity component of the group $O(1,2)$, this latter being the matrix Lie group that preserves the quadratic form:

$$\|(\alpha + v)\|_{\mathcal{M}} = \alpha^2 - \|v\|^2, \tag{88}$$

where $(\alpha+v)$ belongs to the spin factor $\mathbb{R}\oplus\mathbb{R}^2$. The fact that $SO^+(1,2)$ acts linearly on $\mathcal{L}^+$ means that it acts projectively on the set of lines of $\mathcal{L}^+$ and consequently on the points of the Klein disk $\mathcal{K}_{1/2}$, [48]. Moreover, this projective action gives the isometries of $\mathcal{K}_{1/2}$.

The subgroup of $SO^+(1,2)$ that fixes $(1+0)$ may be identified with the group of rotations $SO(2)$ and in fact every element g of $SO^+(1,2)$ can be decomposed in a unique way as [49]:

$$g = b_\zeta r_\xi \ , \tag{89}$$

where b_ζ is a boost map and r_ξ is a proper rotation. More precisely, if we consider the coordinates (α, v_1, v_2) in $\mathcal{L}^+$, the matrix associated to b_ζ is given by:

$$M(b_\zeta) = \begin{pmatrix} \cosh(\zeta_0) & \zeta_x\sinh(\zeta_0) & \zeta_y\sinh(\zeta_0) \\ \zeta_x\sinh(\zeta_0) & 1+\zeta_x^2(\cosh(\zeta_0)-1) & \zeta_x\zeta_y(\cosh(\zeta_0)-1) \\ \zeta_y\sinh(\zeta_0) & \zeta_x\zeta_y(\cosh(\zeta_0)-1) & 1+\zeta_y^2(\cosh(\zeta_0)-1) \end{pmatrix} \ , \tag{90}$$

where (ζ_x, ζ_y) is a unit vector of $\mathbb{R}^2$ and ζ_0 is the rapidity of the boost. It should be noted that the set of boosts is not a subgroup of the special Lorentz group. The matrix associated to r_ξ is given by:

$$M(r_\xi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & -\sin\xi \\ 0 & \sin\xi & \cos\xi \end{pmatrix} , \tag{91}$$

In order to illustrate the projective action of the boost b_ζ on the Klein disk $\mathcal{K}_{1/2}$, let us consider a simple example (quite more complicated computations give similar results in the general case). We choose $\zeta_x = 1$, $\zeta_y = 0$, and denote $\overline{\zeta} = \tanh(\zeta_0)$. The image (α, w_1, w_2) of a vector $(1/2, \cos\theta/2, \sin\theta/2)$ is given by:

$$\begin{cases} 2\alpha = \cosh(\zeta_0) + \sinh(\zeta_0)\cos\theta \\ 2w_1 = \sinh(\zeta_0) + \cosh(\zeta_0)\cos\theta \\ 2w_2 = \sin\theta \ . \end{cases} \tag{92}$$

This means that the image of the boundary point $(\cos\theta/2, \sin\theta/2)$ is the boundary point (v_1, v_2) with:

$$\begin{cases} 2v_1 = \dfrac{\overline{\zeta} + \cos\theta}{1 + \overline{\zeta}\cos\theta} \\ 2v_2 = \dfrac{(1 - \overline{\zeta}^2)^{1/2}\sin\theta}{1 + \overline{\zeta}\cos\theta} \ . \end{cases} \tag{93}$$

One may notice that the map sending the point $(\cos\theta/2, \sin\theta/2)$ to the point (v_1, v_2) is an element of the group $PSL(2, \mathbb{R})$.

Now, let us explain how these computations are related to the experiments 2 and 3 described by H. Yilmaz in [15]. The image of the point $\overline{R} = (1/2, 0)$ is the point $\overline{R}' = \overline{R} = (1/2, 0)$ (we use the notations of [15]). So, the point remains unchanged. The image of the point $\overline{Y} = (0, 1/2)$ is the point $\overline{Y}' = (\overline{\zeta}/2, (1 - \overline{\zeta}^2)^{1/2}/2)$ and the point $\overline{Y}$ has moved on the boundary, the angle ϕ of [15] (p. 12) being given by: $\sin\phi = \overline{\zeta}$. This means that the experiments 2 and 3 of H. Yilmaz can be interpreted as a relativistic aberration effect on colors [50].

The fact that boost maps act on pure states with $PSL(2, \mathbb{R})$ transformations is not surprising in view on the following result[1].

Proposition 4 *Every pure state generates a one parameter subgroup of boosts.*

Proof. As seen before, the state density matrix of a pure state is given by:

$$\rho(v_1, v_2) = \frac{1}{2}(Id_2 + v \cdot \sigma) \ , \tag{94}$$

where $v = (v_1, v_2)$ is a unit vector and $\sigma = (\sigma_1, \sigma_2)$. The matrices σ_1 and σ_2 are symmetric traceless matrices that are usually chosen to be the two first generators

[1]It is tempting to draw parallels between this results and the more general point of view of [51].

of the Lie algebra $sl(2,\mathbb{R})$ of the group $SL(2,\mathbb{R})$. Note that they do not generate a sub Lie algebra of $sl(2,\mathbb{R})$. The matrix:

$$A(\rho,\zeta_0) = \exp\left(\zeta_0 \frac{v\cdot\sigma}{2}\right) \ , \tag{95}$$

is a symmetric element of $PSL(2,\mathbb{R})$, ζ_0 being a real parameter. Let us recall that the $PSL(2,\mathbb{R})$ action on $\mathcal{H}(2,\mathbb{R})$ is defined by:

$$X \longmapsto AXA^t \ . \tag{96}$$

We have clearly: $\mathrm{Det}(AXA^t) = \mathrm{Det}(X)$. Since σ_1 and σ_2 are elements of $\mathcal{H}(2,\mathbb{R})$, we can consider the matrices given by:

$$\sigma_i \longmapsto A(\rho,\zeta_0)\sigma_i A(\rho,\zeta_0) \ , \tag{97}$$

for $i = 0, 1, 2$, with $\sigma_0 = Id_2$. It can be shown that the 3×3 matrix with coefficients:

$$M(\rho,\zeta_0)_{ij} = \frac{1}{2}\mathrm{Trace}\left(\sigma_i A(\rho,\zeta_0)\sigma_j A(\rho,\zeta_0)\right) \ , \tag{98}$$

is a boost b_ζ with $\zeta = \tanh(\zeta_0)(v_1, v_2)$. Let us verify it on a simple example where $v_1 = 1$ and $v_2 = 0$. In this case:

$$A(\rho,\zeta_0) = \exp\left(\zeta_0 \frac{v_1\sigma_1}{2}\right) = \exp\left(\zeta_0 \frac{\sigma_1}{2}\right) = \begin{pmatrix} e^{\zeta_0/2} & 0 \\ 0 & e^{-\zeta_0/2} \end{pmatrix} \ . \tag{99}$$

We only need to compute the coefficient $B(\sigma,\zeta_0)_{i,j}$ for $i \leq j$. We have:

$$\begin{cases} M(\sigma,\zeta_0)_{00} = \frac{1}{2}\mathrm{Trace}\left(A^2(\rho,\zeta_0)\right) = \cosh(\zeta_0) \\ M(\sigma,\zeta_0)_{01} = \frac{1}{2}\mathrm{Trace}\left(A(\rho,\zeta_0)\sigma_1 A(\rho,\zeta_0)\right) = \sinh(\zeta_0) \\ M(\sigma,\zeta_0)_{02} = \frac{1}{2}\mathrm{Trace}\left(A(\rho,\zeta_0)\sigma_2 A(\rho,\zeta_0)\right) = 0 \\ M(\sigma,\zeta_0)_{11} = \frac{1}{2}\mathrm{Trace}\left(\sigma_1 A(\rho,\zeta_0)\sigma_1 A(\rho,\zeta_0)\right) = \cosh(\zeta_0) \\ M(\sigma,\zeta_0)_{12} = \frac{1}{2}\mathrm{Trace}\left(\sigma_1 A(\rho,\zeta_0)\sigma_2 A(\rho,\zeta_0)\right) = 0 \\ M(\sigma,\zeta_0)_{22} = \frac{1}{2}\mathrm{Trace}\left(\sigma_2 A(\rho,\zeta_0)\sigma_2 A(\rho,\zeta_0)\right) = 1 \ . \end{cases} \tag{100}$$

This means that $M(\rho,\zeta_0) = M(b_\zeta)$ with $\zeta = \tanh(\zeta_0)(1,0)$, see equation (90). □

One can easily verify that the image of the vector $(1/2, 0, 0)$ of $\mathcal{L}^+$ by the boost $b_\zeta = \tanh(\zeta_0)(1,0)$ is the vector $(\cosh(\zeta_0)/2, \sinh(\zeta_0)/2, 0)$. In consequence, the state of maximal entropy $\rho_0 = (0,0)$ is sent to the state $(\tanh(\zeta_0)/2, 0)$. This extends to general boosts.

We can summarize these computations in the following way. As before, we consider the state space $\mathcal{S}$ as the Klein disk $\mathcal{K}_{1/2}$ of the closure $\overline{\mathcal{L}^+}$ of the future lightcone $\mathcal{L}^+$ by using the map:

$$s = (v_1, v_2) \longmapsto \frac{1}{2}(1+v) = 1/2 + (v_1/2, v_2/2) \ . \tag{101}$$

Every pure state ρ generates a one parameter subgroup of boosts, the parameter ζ_0 being the rapidity. Actually, every boost can be obtained in this way. Boost maps act on the Klein disk $\mathcal{K}_{1/2}$ by isometries. If we consider $\mathcal{S}$ as embedded in the space of state density matrices by:

$$s = (v_1, v_2) \longmapsto \rho(v_1, v_2) = \frac{1}{2}\begin{pmatrix} 1+v_1 & v_2 \\ v_2 & 1-v_1 \end{pmatrix} , \tag{102}$$

the one parameter subgroup of boosts is obtained by considering the action of $PSL(2,\mathbb{R})$ on $\mathcal{H}(2,\mathbb{R})$. It is important to notice that we use only the matrices σ_0, σ_1 and σ_2, *i.e.* only information from $\mathcal{S}$. Since every state can be obtained from the state of maximal entropy, boosts, or equivalently pure states, act transitively on S. However, one has to pay attention to the fact that boost maps do not form a subgroup of the special Lorentz group which is reflected by the fact that σ_1 and σ_2 do not form a sub Lie algebra of the Lie algebra $sl(2,\mathbb{R})$.

This point of view is quite different from the approach adopted in [5]. As mentioned in the introduction and explained in [9], one of the key arguments of H. Resnikoff is the existence of a transitive action of the group denoted $GL(\mathcal{P})$ on the space $\mathcal{P}$ of perceived colors. This group is supposed to be composed of all linear changes of background illumination. In what preceds, we make use of the action of $PSL(2,\mathbb{R})$ on $\mathcal{H}(2,\mathbb{R})$, see (96). But the matrices $A(\rho,\zeta_0)$ of $PSL(2,\mathbb{R})$ that are used are also symmetric, due to the fact that σ_1 and σ_2 are symmetric. Actually, the action (96) can be also viewed as the action:

$$X \longmapsto AXA , \tag{103}$$

of the Jordan algebra $\mathcal{H}(2,\mathbb{R})$ on itself. This is precisely the action:

$$Q(A) : X \longmapsto (2L(A)^2 - L(A^2))X , \tag{104}$$

of the quadratic representation of A on X [10]. But once again, the matrices X that we consider are σ_0, σ_1 and σ_2. The matrices σ_1 and σ_2 are not elements of the positive cone $\mathcal{H}^+(2,\mathbb{R})$. It appears in consequence that the homogeneity of $\mathcal{H}^+(2,\mathbb{R})$ is not so important in our approach. Instead of postulating the existence of a group of linear changes of background illumination, we have shown that the quantum description that we propose naturally leads to consider boost maps as illumination changes. These illumination changes are isometries of the Klein disk $\mathcal{K}_{1/2}$.

7 Conclusion

We have shown that the space $\mathcal{P}$ of perceived colors can be interpreted as the effect space of a real quantum system, more precisely a rebit. The state space $\mathcal{S}$ of this system is isometric to the Klein disk $\mathcal{K}$ of constant negative curvature. The chromatic state space of perceived colors is a Bloch disk of real dimension 2 that coincides with the Hering disk given by the color opponency mechanism. This quantum description creates a deep connection between the works of H. Yilmaz [14], [15] and H.L. Resnikoff [5]. In particular, it allows to recover in a natural way the following result of H. L. Resnikoff: the surface of constant magnitude are isometric to the Klein disk of constant negative curvature equal to -1 [5]. Finally, one may envisage to characterize the individual color vision from a convex subset of the effect space $\mathcal{E}(\mathcal{S})$ and/or a convex subset of the state space $\mathcal{S}$ endowed with the Hilbert metric. Further investigations will be devoted to the properties of this metric space in connection with MacAdam ellipses [52].

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