On Siegel's problem for E-functions

S Fischler, T. Rivoal

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Abstract

Siegel defined in 1929 two classes of power series, the E-functions and G-functions, which generalize the Diophantine properties of the exponential and logarithmic functions respectively. In 1949, he asked whether any E-function can be represented as a polynomial with algebraic coefficients in a finite number of confluent hypergeometric series with rational parameters. The case of E-functions of differential order less than 2 was settled in the affirmative by Gorelov in 2004, but Siegel's question is open for higher order. We prove here that if Siegel's question has a positive answer, then the ring $\mathbf{G}$ of values taken by analytic continuations of G-functions at algebraic points must be a subring of the relatively "small" ring $\mathbf{H}$ generated by algebraic numbers, $1/\pi$ and the values of the derivatives of the Gamma function at rational points. Because that inclusion seems unlikely (and contradicts standard conjectures), this points towards a negative answer to Siegel's question in general. As intermediate steps, we first prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function, which completes previous results of ours. We then prove that the coefficients of the asymptotic expansion of a confluent hypergeometric series with rational parameters are in $\mathbf{H}$. Finally, we prove a similar result for G-functions.

1 Introduction

Siegel [24, p. 223] introduced in 1929 the notion of E-function as a generalization of the exponential and Bessel functions. We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1. *A power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ is an E-function if*

(i) $F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

(ii) There exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.

(iii) There exists $D > 0$ and a sequence of integers d_n, with $1 \leq d_n \leq D^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.

Siegel's original definition was in fact slightly more general than above and we shall make some remarks about this in §2.1. Note that (i) implies that the a_n's all lie in a certain number field $\mathbb{K}$, so that in (ii) there are only finitely many Galois conjugates $\sigma(a_n)$ of a_n to consider, with $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ (assuming for simplicity that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$). E-functions are entire, and they form a ring stable under $\frac{d}{dz}$ and $\int_0^z$. A power series $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is said to be a G-function if $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. Algebraic functions over $\overline{\mathbb{Q}}(z)$ regular at 0 and polylogarithms (defined in §2.2) are examples of G-functions.

The generalized hypergeometric series is defined as

$$
{}_pF_q\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n \tag{1.1}
$$

where $p, q \geq 0$ and $(a)_0 := 1$, $(a)_n := a(a+1)\cdots(a+n-1)$ if $n \geq 1$. The parameters a_j and b_j are in $\mathbb{C}$, with the restriction that $b_j \notin \mathbb{Z}_{\leq 0}$ so that $(b_j)_n \neq 0$ for all $n \geq 0$. We shall also denote it by ${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z]$. Siegel proved that, for any integer $p \geq 1$, the *confluent* hypergeometric series

$$
{}_pF_p\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; z\right] \tag{1.2}
$$

is an E-function (in the sense of this paper) when $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j. The simplest example is ${}_1F_1[1; 1; z] = \exp(z)$. If $a_j \in \mathbb{Z}_{\leq 0}$ for some j, then the series reduces to a polynomial. Any polynomial with coefficients in $\overline{\mathbb{Q}}$ of hypergeometric functions of the form ${}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; \lambda z]$, with parameters $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$, is an E-function.

The E-functions

$$
L(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!}, \; H(z) := \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!}, \; J_0(z) := \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2}
$$

are not of the hypergeometric type (1.2), even with z changed to λz for some $\lambda \in \overline{\mathbb{Q}}$, but we have

$$
\begin{aligned}
L(z) &= e^{(3-2\sqrt{2})z} \cdot {}_1F_1\big[1/2; 1; 4\sqrt{2}z\big], \\
H(z) &= ze^z \cdot {}_2F_2\big[1, 1; 2, 2; -z\big], \\
J_0(z) &= e^{-iz} \cdot {}_1F_1[1/2; 1; 2iz].
\end{aligned}
$$

(See [1, p. 509, 13.6.1] and [23].) These puzzling identities, amongst others, naturally suggest to study further the role played by hypergeometric series in the theory of E-functions. In fact, Siegel had already stated in [25] a problem that we reformulate as the following question.

Question 1 (Siegel). *Is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of hypergeometric functions of the form ${}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; \lambda z]$, with parameters $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$?*

It must be understood that λ and p can take various values in the polynomial. Siegel's original statement is given in §2.1 along with some comments. Gorelov [13, p. 514, Theorem 1] proved that the answer to Siegel's question is positive if the E-function (in the above sense, not Siegel's original one) satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ of order ≤ 2. He used the pioneering results of André [3] on E-operators. Gorelov's theorem was reproved in [23] with a method also based on André's results, but somewhat different in the details. It seems difficult to generalize any one of these two approaches when the order is ≥ 3, though Gorelov [14] also obtained partial results in the case of E-functions solution of a linear inhomogeneous differential equation of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$, like $H(z)$ above.

In this paper, we adopt another point of view on Siegel's question. Let us first define two subrings of $\mathbb{C}$; the former was introduced and studied in [9].

Definition 2. $\mathbf{G}$ *denotes the ring of G-values, i.e. the values taken at algebraic points by the analytic continuations of all G-functions.*

$\mathbf{H}$ *denotes the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and the values $\Gamma^{(n)}(r)$, $n \geq 0$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.*

Here, $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ is the usual Gamma function that can be analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We can now state our main result.

Theorem 1. *At least one of the following statements is true:*

(*i*) $\mathbf{G} \subset \mathbf{H}$;

(*ii*) *Siegel's question has a negative answer.*

We provide in §2.2 another description of the ring $\mathbf{H}$, and explain there why the inclusion $\mathbf{G} \subset \mathbf{H}$ (and therefore a positive answer to Siegel's question) seems very unlikely; as Y. André, F. Brown and J. Fresán pointed out to us, this inclusion contradicts standard conjectures.

The paper is organized as follows. In §2, we comment on Siegel's original formulation of his problem and make some remarks on the ring $\mathbf{H}$. In §3, we prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function (Theorem 3). In §4, we prove that the coefficients of the asymptotic expansion of hypergeometric series ${}_pF_p$ with rational parameters are in $\mathbf{H}$ (Theorem 4). We complete the proof of Theorem 1 in §5 by comparing the results of the previous sections. Finally, we consider in §6 an analogous problem for G-functions and prove a similar result to Theorem 1.

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2 Comments on Theorem 1

2.1 Siegel's formulation of his problem

In [25, Chapter II, §9], Siegel proved that the hypergeometric series of the type (1.2) with rational parameters are E-functions, and named them "hypergeometric E-functions". He then wrote on page 58: *Performing the substitution $x \mapsto \lambda x$ for arbitrary algebraic λ and taking any polynomial in x and finitely many hypergeometric E-functions, with algebraic coefficients, we get again an E-function satisfying a homogeneous linear differential equation whose coefficients are rational function of x. It would be interesting to find out whether all such E-functions can be constructed in the preceding manner.*

Siegel obviously considered E-functions in his sense, which we recall here: in Definition 1, (i) is unchanged but (ii) and (iii) have to be replaced by

(ii') For any $\varepsilon > 0$ and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there exists $N(\varepsilon, \sigma) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon, \sigma)$, $|\sigma(a_n)| \leq n!^{\varepsilon}$.

(iii') There exists a sequence of integers $d_n \neq 0$ such that $d_n a_m$ are algebraic integers for all $m \leq n$ and such that for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon)$, $|d_n| \leq n!^{\varepsilon}$.

Again, by (i), there are only finitely many σ to consider for a given E-function. We have chosen to formulate his problem for E-functions in the restricted sense of Definition 1 because the proof of Theorem 1 is based on results which are currently proven only in this sense. However, *a fortiori*, Theorem 1 obviously holds *verbatim* if one considers E-functions in Siegel's sense. Note also that the function $1 - z$ is equal to the hypergeometric series ${}_1F_1[-1; 1; z]$ so that Siegel could have formulated his problem in terms of hypergeometric series only, as we did. Despite the apparences, the E-function $\sinh(z) = \frac{1}{2z}(e^z - e^{-z})$ is not a counter-example to Siegel's problem because $\frac{1}{2z}(e^z - 1) = {}_1F_1[1; 2; z]$; there is no unicity of the representation of E-functions by polynomials in hypergeometric ones.

Moreover, the series in (1.2) may be an E-function even if some of its parameters are not rational numbers. For instance, for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$,

$${}_1F_1\left[\begin{matrix}\alpha+1\\ \alpha\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(1)_n(\alpha)_n} z^n = \sum_{n=0}^{\infty} \frac{\alpha+n}{\alpha} \cdot \frac{z^n}{n!} = \big(1 + \frac{z}{\alpha}\big)e^z$$

is an E-function. Thus, even though Siegel did not consider such examples, the notion of "hypergeometric E-functions" could be interpreted in a broader way than he did in his problem. Galochkin [12] proved the following non-trivial characterization, where E-functions are understood in Siegel's sense. (See [22] for a different proof for E-functions in the sense of the present paper).

Theorem (Galochkin)**.** *Let $p \geq 1$, $a_1, \ldots, a_p, b_1, \ldots, b_p \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{2p}$ be such that $a_i \neq b_j$ for all i, j. Then, the hypergeometric series ${}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; z]$ is an E-function if and only if the following two conditions hold:*

(*i*) *The* a_j*'s and* b_j*'s are all in* $\overline{\mathbb{Q}}$*;*

(*ii*) *The* a_j*'s and* b_j*'s which are not rational (if any) can be grouped in* $k \leq p$ *pairs* $(a_{j_1}, b_{j_1}), \ldots, (a_{j_k}, b_{j_k})$ *such that* $a_{j_\ell} - b_{j_\ell} \in \mathbb{N}$*.*

It follows that hypergeometric E-functions with arbitrary parameters are in fact $\overline{\mathbb{Q}}$-linear combinations of hypergeometric E-functions with rational parameters. Hence, there is no loss of generality in considering the latter instead of the former in Siegel's problem.

Another generalization of Siegel's problem is the following. When $q \geq p \geq 1$, $r := q - p + 1 \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, the function

$$ {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z^{q-p+1} \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n (rn)!}{(1)_n (b_1)_n \cdots (b_q)_n} \frac{z^{rn}}{(rn)!} \tag{2.1} $$

is an E-function. The special case $q = p$ is that of confluent hypergeometric series, but this family includes also Bessel's function

$$ J_0(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!^2} = {}_1F_2 \left[\begin{matrix} 1, 1 \\ 1 \end{matrix} ; (iz/2)^2 \right]. $$

We recall that $J_0(z) = e^{-iz} \cdot {}_1F_1[1/2; 1; 2iz]$ so that $J_0(z)$ is an example for Siegel's problem but this is not known for other parameters in the function (2.1) in general. It is natural to ask the following question: is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form (2.1) with z replaced with λz, $\lambda \in \overline{\mathbb{Q}}$? It must be understood that λ, p, q and $q - p$ can take various values in the polynomial.

2.2 The ring H

For $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we define the Digamma function

$$ \Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+x} \Big), $$

where γ is Euler's constant $\lim_{n \to +\infty} (\sum_{k=1}^{n} 1/k - \log(n))$, and the Hurwitz zeta function

$$ \zeta(s, x) := \frac{(-1)^s}{(s-1)!} \Psi^{(s-1)}(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad s \in \mathbb{N},\ s \geq 2. $$

The polylogarithms are defined by

$$ \mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad s \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}, $$

where the series converges for $|z| \leq 1$ (except at $z = 1$ if $s = 1$). The Beta function is defined as

$$\mathrm{B}(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y \in \mathbb{C}$ which are not singularities of Beta coming from the poles of Γ at non-positive integers.

In this section, we shall prove the following result.

Proposition 1. *The ring* $\mathbf{H}$ *is generated by* $\overline{\mathbb{Q}}$, γ, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$ ($s \in \mathbb{N}^*$, $r \in \mathbb{Q}$, $(s, e^{2i\pi r}) \neq (1, 1)$), $\log(q)$ ($q \in \mathbb{N}^*$) *and* $\Gamma(r)$ ($r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$).

For any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\Gamma(r)$ *is a unit of* $\mathbf{H}$.

Proof. We first prove that for any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\Gamma(r)$ is a unit of $\mathbf{H}$. Indeed, if $r \in \mathbb{N}^*$, then $\Gamma(r) \in \mathbb{N}^*$ and $1/\Gamma(r) \in \mathbb{Q} \subset \mathbf{H}$. If $r \in \mathbb{Q} \setminus \mathbb{Z}$, then by the reflection formula [5, p. 9, Theorem 1.2.1], we have

$$\frac{1}{\Gamma(r)} = \frac{1}{\pi} \sin(\pi r)\Gamma(1 - r) \in \mathbf{H}$$

because $1/\pi \in \mathbf{H}$, $\sin(\pi r) \in \overline{\mathbb{Q}} \subset \mathbf{H}$ and $\Gamma(1 - r) \in \mathbf{H}$.

From the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ we obtain that, for any integer $s \geq 1$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$,

$$\Psi^{(s)}(r) = \frac{\Gamma^{(s+1)}(r)}{\Gamma(r)} - \sum_{k=0}^{s-1} \binom{s}{k} \frac{\Gamma^{(s-k)}(r)}{\Gamma(r)} \Psi^{(k)}(r).$$

Since $\Gamma(r)$ is a unit of $\mathbf{H}$, we have $\psi(r) \in \mathbf{H}$ and it follows immediately by induction on s that $\zeta(s, r) = \frac{(-1)^s}{(s-1)!} \Psi^{(s-1)}(r) \in \mathbf{H}$ for any $s \geq 2$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. In particular $\gamma = -\Psi(1)$ and the values of the Riemann zeta function $\zeta(s) = \zeta(s, 1)$ ($s \geq 2$) are all in $\mathbf{H}$. Note that γ is not expected to be in $\mathbf{G}$ but that $\zeta(s) \in \mathbf{G}$ for all $s \geq 2$.

We have for any $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and any $n \in \mathbb{N}$,

$$\Psi(x + n) = \Psi(x) + \sum_{k=0}^{n-1} \frac{1}{k + x}, \tag{2.2}$$

and the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ also implies by induction that, for any $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$\Gamma^{(s)}(x) = \Gamma(x) P_s\big(\Psi(x), \zeta(2, x), \ldots, \zeta(s, x)\big) \tag{2.3}$$

for some $P_s \in \mathbb{Q}[X_1, \ldots, X_s]$. Furthermore, set $p, q \in \mathbb{N}$, $0 < p \leq q$, and $\mu := \exp(2i\pi/q)$.

Then,

$$\Psi\Big(\frac{p}{q}\Big) = -\gamma - \log(q) - \sum_{n=1}^{q-1} \mu^{-np}\,\mathrm{Li}_1(\mu^n), \tag{2.4}$$

$$\mathrm{Li}_1(\mu^p) = -\frac{1}{q}\sum_{n=1}^{q} \mu^{np}\,\Psi\Big(\frac{n}{q}\Big), \quad p \neq q \tag{2.5}$$

$$\zeta\Big(s, \frac{p}{q}\Big) = q^{s-1}\sum_{n=1}^{q} \mu^{-np}\,\mathrm{Li}_s(\mu^n), \quad s \geq 2 \tag{2.6}$$

$$\mathrm{Li}_s(\mu^p) = \frac{1}{q^s}\sum_{n=1}^{q} \mu^{np}\,\zeta\Big(s, \frac{n}{q}\Big), \quad s \geq 2. \tag{2.7}$$

We refer to [5, p. 14] for details. From (2.5) and (2.7), we deduce that $\mathrm{Li}_s(\mu^p) \in \mathbf{H}$ for any $s \geq 1$ (with $(s, \mu^p) \neq (1,1)$); then (2.4) implies in turn that $\log(q) \in \mathbf{H}$. The numbers $\log(q)$ and $\mathrm{Li}_s(\mu^p)$ are also in $\mathbf{G}$.

The set of Identities (2.2)–(2.7) shows that $\mathbf{H}$ coincides with the ring generated by $\overline{\mathbb{Q}}$, $\gamma = -\Psi(1)$, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$ ($s \in \mathbb{N}^*$, $r \in \mathbb{Q}$, $(s, e^{2i\pi r}) \neq (1,1)$), $\log(q)$ ($q \in \mathbb{N}^*$) and $\Gamma(r)$ ($r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$). □

Other units of $\mathbf{H}$ can be easily identified, which are also units of $\mathbf{G}$ (see [9, §2.2]): non-zero algebraic numbers and the values of the Beta function $\mathrm{B}(x,y)$ at rational numbers x, y at which it is defined and non-zero. It follows that $\pi = \Gamma(1/2)^2 = \mathrm{B}(1/2, 1/2)$ and more generally $\Gamma(a/b)^b = (a-1)!\prod_{j=1}^{b-1} \mathrm{B}(a/b, ja/b)$, $a, b \in \mathbb{N}^*$, are units of $\mathbf{H}$. By the Chowla-Selberg formula [20, p. 230, Corollary 2], periods of CM elliptic curves defined over $\overline{\mathbb{Q}}$ are also units of $\mathbf{H}$.

If Siegel's problem has a positive answer, Theorem 1 yields $\mathbf{G} \subset \mathbf{H}$: any element of $\mathbf{G}$ can be written as a polynomial, with algebraic coefficients, in the numbers γ, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\Gamma(r)$ of Proposition 1. This seems extremely doubtful: we recall that $\mathbf{G}$ contains all the multiple zeta values

$$\zeta(s_1, s_2, \ldots, s_n) := \sum_{k_1 > k_2 > \cdots > k_n \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_n^{s_n}},$$

where the integers s_j are such that $s_1 \geq 2, s_2 \geq 1, \ldots, s_n \geq 1$, all values at algebraic points of (multiple) polylogarithms, all elliptic and abelian integrals, etc. For now, we have proved that $\mathbf{G} \cap \mathbf{H}$ contains the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and all the values $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\mathrm{B}(x,y)$, and it is in fact possible that both rings are equal.

It is interesting to know what can be deduced from the standard conjectures in the domain, such as the Bombieri-Dwork conjecture "G-functions come from geometry", Grothendieck's periods conjecture, its extension to exponential periods by Fresán-Jossen, and the Rohrlich-Lang conjecture on the values of the Gamma function; see [4, Partie III] and [11, p. 201, Conjecture 8.2.5]. In a private communication to the authors, Y. André wrote the following argument, which he has autorised us to reproduce here. It shows that $\mathbf{G} \subset \mathbf{H}$ cannot hold under these standard conjectures:

Because of the presence of γ, the inclusion $\mathbf{G} \subset \mathbf{H}$ does not contradict Grothendieck's periods conjecture but it certainly contradicts its extension to exponential motives. More precisely, in the description of $\mathbf{H}$ given in Proposition 1, we find γ (a period of an exponential motive E_γ, which is a non-classical extension of the Tate motive [11, §12.8]), $1/\pi$ (a period of the Tate motive), $\mathrm{Li}_s(e^{2i\pi r})$ (periods of a mixed Tate motive over $\mathbb{Z}[1/r]$), $\log(q)$ (a period of a 1-motive over $\mathbb{Q}$), and $\Gamma(r)$ whose suitable powers are periods of Abelian varieties with complex multiplication by $\mathbb{Q}(e^{2i\pi r})$. On the one hand, let M be the Tannakian category of mixed motives over $\overline{\mathbb{Q}}$ generated by all these motives. On the other hand, consider a non CM elliptic curve over $\overline{\mathbb{Q}}$ and E its motive. The periods of E are in $\mathbf{G}$: indeed, it is enough to consider the Gauss hypergeometric solutions centered at $1/2$, and to observe that the periods of the fiber at $1/2$ of the Legendre family can be expressed using values of the Beta function at rational points by the Chowla-Selberg formula, and in particular are algebraic in π and $\Gamma(1/4)$. If $\mathbf{G} \subset \mathbf{H}$, the periods of E are in $\mathbf{H}$. By the exponential periods conjecture, E would be in M, which is impossible since the motivic Galois group of M is pro-resoluble, while that of E is GL_2.

We conclude this section with a question of J. Fresán: at which differential order can we expect to find a counter-example to Siegel's problem? Based on the above remarks, it seems unlikely that all the values $\mathrm{Li}_s(\alpha)$ are in $\mathbf{H}$, where the integer $s \geq 1$ and $\alpha \in \overline{\mathbb{Q}}$, $|\alpha| < 1$. From the proof of Theorem 3 below, we deduce that if $\mathrm{Li}_s(\alpha) \notin \mathbf{H}$, then the E-function

$$\sum_{n=2}^{\infty} \Big(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \Big) \frac{z^n}{n!}$$

is such a counter-example. It is of differential order at most $s+2$ because it is in the kernel of the differential operator $P(\theta - 2) + zQ(\theta - 1) + z^2 R(\theta) \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$, where $\theta := z\frac{d}{dz}$ and

$$P(x) := (x+2)(x+1)^{s+1}, \quad Q(x) := (x+1)(\alpha x^s - (x+1)^s), \quad R(x) := \alpha x^s.$$

It is thus possible that a counter-example to Siegel's problem already exists at the order 3. However, the function $H(z) := \sum_{n=0}^{\infty} (\sum_{k=1}^{n} \frac{1}{k}) \frac{z^n}{n!}$ is an example of order 3 to the problem (see the Introduction) and this shows that one must be careful and not draw hasty conclusions here.

3 Elements of G as coefficients of asymptotic expansions of E-functions

3.1 Definition of asymptotic expansions

As in [10], the asymptotic expansions used throughout this paper are defined as follows.

Definition 3. *Let $\theta \in \mathbb{R}$, and $\Sigma \subset \mathbb{C}$, $S \subset \mathbb{C}$, $T \subset \mathbb{N}$ be finite subsets. Given complex numbers $c_{\rho,\alpha,i,n}$, we write*

$$f(x) \approx \sum_{\rho\in\Sigma} e^{\rho x} \sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} \log(1/x)^i \tag{3.1}$$

and say that the right-hand side is the asymptotic expansion of $f(x)$ in a large sector bisected by the direction θ, if there exist $\varepsilon, R, B, C > 0$ and, for any $\rho \in \Sigma$, a function $f_\rho(x)$ holomorphic in

$$U = \Big\{ x \in \mathbb{C},\ |x| \geq R,\ \theta - \frac{\pi}{2} - \varepsilon \leq \arg(x) \leq \theta + \frac{\pi}{2} + \varepsilon \Big\},$$

such that

$$f(x) = \sum_{\rho\in\Sigma} e^{\rho x} f_\rho(x)$$

and

$$\Big| f_\rho(x) - \sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} \log(1/x)^i \Big| \leq C^N N! |x|^{B-N}$$

for any $x \in U$ and any $N \geq 1$.

This means (see [21, §§2.1 and 2.3]) that for any $\rho \in \Sigma$,

$$\sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} \log(1/x)^i \tag{3.2}$$

is 1-summable in the direction θ and its sum is $f_\rho(x)$. Using a result of Watson (see [21, §2.3]), the sum $f_\rho(x)$ is then determined by its asymptotic expansion (3.2). Therefore the expansion on the right-hand side of (3.1) determines $f(x)$, up to analytic continuation. The converse holds too: [10, Lemma 1] asserts that a given function $f(x)$ can have at most one asymptotic expansion in the sense of Definition 3. Of course we assume implicitly (throughout this paper) that Σ, S and T in (3.1) cannot trivially be made smaller, and that for any α there exist ρ and i with $c_{\rho,\alpha,i,0} \neq 0$.

3.2 Computing asymptotic expansions of E-functions

In this section, we state [10, Theorem 5] which enables one to determine the asymptotic expansion of an E-function. We refer to [10] for more details.

Let $E(x) = \sum_{n=1}^{\infty} a_n x^n$ be an E-function such that $E(0) = 0$; consider $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$. Denoting by $\overline{\mathcal{F}} : \mathbb{C}[z, \frac{\mathrm{d}}{\mathrm{d}z}] \to \mathbb{C}[x, \frac{\mathrm{d}}{\mathrm{d}x}]$ the Fourier transform of differential operators, i.e. the morphism of $\mathbb{C}$-algebras defined by $\overline{\mathcal{F}}(z) = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\mathcal{F}}(\frac{\mathrm{d}}{\mathrm{d}z}) = -x$, there exists a G-operator $\mathcal{D}$ such that $\overline{\mathcal{F}}\mathcal{D}E = 0$, and we have $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D} g = 0$ where δ is the degree of $\mathcal{D}$. We denote by Σ the set of all finite singularities of $\mathcal{D}$ and let

$$\mathcal{S} = \mathbb{R} \setminus \{\arg(\rho - \rho'), \rho, \rho' \in \Sigma, \rho \neq \rho'\} \tag{3.3}$$

where all the values modulo 2π of the argument of $\rho-\rho'$ are considered, so that $\mathcal{S}+\pi=\mathcal{S}$.

We fix $\theta\in\mathbb{R}$ with $-\theta\in\mathcal{S}$ (so that the direction θ is not anti-Stokes, i.e. not singular, see for instance [18, p. 79]). For any $\rho\in\Sigma$ we denote by $\Delta_\rho=\rho-e^{-i\theta}\mathbb{R}_+$ the closed half-line of angle $-\theta+\pi$ mod 2π starting at ρ. Since $-\theta\in\mathcal{S}$, no singularity $\rho'\neq\rho$ of $\mathcal{D}$ lies on Δ_ρ: these half-lines are pairwise disjoint. We shall work in the simply connected cut plane obtained from $\mathbb{C}$ by removing the union of these half-lines. We agree that for $\rho\in\Sigma$ and z in the cut plane, $\arg(z-\rho)$ will be chosen in the open interval $(-\theta-\pi,-\theta+\pi)$. This enables one to define $\log(z-\rho)$ and $(z-\rho)^\alpha$ for any $\alpha\in\mathbb{Q}$.

Now let us fix $\rho\in\Sigma$. Combining theorems of André, Chudnovski and Katz (see [3, p. 719]), there exist (non necessarily distinct) rational numbers $t_1^\rho,\ldots,t^\rho_{J(\rho)}$, with $J(\rho)\geq 1$, and G-functions $g^\rho_{j,k}$, for $1\leq j\leq J(\rho)$ and $0\leq k\leq K(\rho,j)$, such that a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta\mathcal{D}$ around ρ (in the above-mentioned cut plane) is given by the functions

$$f^\rho_{j,k}(z-\rho)=(z-\rho)^{t_j^\rho}\sum_{k'=0}^{k} g^\rho_{j,k-k'}(z-\rho)\frac{\log(z-\rho)^{k'}}{k'!} \tag{3.4}$$

for $1\leq j\leq J(\rho)$ and $0\leq k\leq K(\rho,j)$. Since $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta\mathcal{D}g=0$ we can expand g in this basis:

$$g(z)=\sum_{j=1}^{J(\rho)}\sum_{k=0}^{K(\rho,j)}\varpi^\rho_{j,k}f^\rho_{j,k}(z-\rho)$$

with connection constants $\varpi^\rho_{j,k}$; Theorem 2 of [9] yields $\varpi^\rho_{j,k}\in\mathbf{G}$.

We denote by $\{u\}\in[0,1)$ the fractional part of a real number u, and agree that all derivatives of this or related functions taken at integers will be right-derivatives. We let

$$y_{\alpha,i}(z)=\sum_{n=0}^{\infty}\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}y^i}\Big(\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)}\Big)_{|y=\alpha}z^n\in\mathbb{Q}[[z]] \tag{3.5}$$

for $\alpha\in\mathbb{Q}$ and $i\in\mathbb{N}$. We also denote by $\star$ the Hadamard (coefficientwise) product of formal power series in z, and we consider

$$\eta^\rho_{j,k}(1/x)=\sum_{m=0}^{k}(y_{t_j^\rho,m}\star g^\rho_{j,k-m})(1/x)\in\overline{\mathbb{Q}}[[1/x]] \tag{3.6}$$

for any $1\leq j\leq J(\rho)$ and $0\leq k\leq K(j,\rho)$. Then [10, Theorem 5] is the following result, where $\widehat{\Gamma}:=1/\Gamma$.

Theorem 2. *In a large sector bisected by the direction θ we have the following asymptotic expansion:*

$$E(x)\approx$$
$$\sum_{\rho\in\Sigma}e^{\rho x}\sum_{j=1}^{J(\rho)}\sum_{k=0}^{K(j,\rho)}\varpi^\rho_{j,k}x^{-t_j^\rho-1}\sum_{i=0}^{k}\Big(\sum_{\ell=0}^{k-i}\frac{(-1)^\ell}{\ell!}\widehat{\Gamma}^{(\ell)}(1-\{t_j^\rho\})\eta^\rho_{j,k-\ell-i}(1/x)\Big)\frac{\log(1/x)^i}{i!}. \tag{3.7}$$

3.3 G-values as coefficients of asymptotic expansions

We can now state and prove the main result of this section.

Theorem 3. *For any $\xi \in \mathbf{G}$, there exists an E-function $E(z)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $E(x)$ in a large sector bisected by θ.*

Proof. Let $\xi \in \mathbf{G}$; we may assume $\xi \neq 0$. Using [9, Theorem 1], there exists a G-function $h(z)$ holomorphic at $z = 1$ such that $h(1) = \xi$. Let $g(z) = \frac{h(1/z)}{z(z-1)}$. This function has a Taylor expansion at ∞ of the form $\sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$, and $E(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ is an E-function. Using the results of [10] recalled in §3.2 we shall compute (partially) its asymptotic expansion at infinity in a large sector bisected by the direction θ, for any $\theta \in [-\pi, \pi)$ outside a finite set; we shall prove that the coefficient of e^x in this expansion is equal to ξ. With this aim in mind, we keep the notation of §3.2, including $\mathcal{D}$ and θ.

We let $\rho = 1$ (eventhough we still write ρ for better readability), and consider a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}$ around ρ with functions $f^{\rho}_{j,k}$ and $g^{\rho}_{j,k}$ as in §3.2. By Frobenius' method, upon shifting t^{ρ}_j by an integer we may assume that $g^{\rho}_{j,0}(0) \neq 0$. Moreover, upon performing $\overline{\mathbb{Q}}$-linear combinations of the basis elements and a permutation of the indices, we may assume that $t^{\rho}_1 < \ldots < t^{\rho}_{J(\rho)}$ so that the solutions $f^{\rho}_{j,k}$ have pairwise distinct asymptotic behaviours at 0, namely $f^{\rho}_{j,k}(s) \sim \frac{g^{\rho}_{j,0}(0)}{k!} s^{t^{\rho}_j} \log(s)^k$. At last, dividing each $f^{\rho}_{j,k}$ with $g^{\rho}_{j,0}(0)$ we may assume that $g^{\rho}_{j,0}(0) = 1$ for any j.

Now consider the expansion

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi^{\rho}_{j,k} f^{\rho}_{j,k}(z-\rho). \tag{3.8}$$

Let $T = \{(j,k),\, \varpi^{\rho}_{j,k} \neq 0\}$. Since g is not identically zero, T is not empty. Let $j_0 \in \{1, \ldots, J(\rho)\}$ be the minimal value such that $(j_0, k) \in T$ for some k, and let k_0 be the maximal value such that $(j_0, k_0) \in T$. Then on the right-hand side of Eq. (3.8), the leading term as $z \to \rho$ is given by $(j,k) = (j_0, k_0)$, so that

$$g(z) \sim \frac{\varpi^{\rho}_{j_0,k_0}}{k_0!} (z-\rho)^{t^{\rho}_{j_0}} \log(z-\rho)^{k_0} \tag{3.9}$$

since $g^{\rho}_{j_0,0}(0) = 1$. Now recall that $g(z) = \frac{h(1/z)}{z(z-1)}$ with $h(1) = \xi \neq 0$ and $\rho = 1$; therefore $g(z) \sim \frac{\xi}{z-1}$. Comparing this with Eq. (3.9) yields $t^{\rho}_{j_0} = -1$, $k_0 = 0$, and $\varpi^{\rho}_{j_0,0} = \xi$.

Let us consider the asymptotic expansion given by Theorem 2, and especially the coefficient of e^x that we denote by α. This coefficient comes from the multiple sum in Eq. (3.7). In this sum, we have $\varpi^{\rho}_{j,k} = 0$ for any $j < j_0$ and any k (by definition of j_0), so that these terms do not contribute to the value of α. For any $j > j_0$ we have $t^{\rho}_j > t^{\rho}_{j_0} = -1$ so that $-t^{\rho}_j - 1 < 0$ and the corresponding terms do not contribute either. Therefore the value of

α is given only by the terms corresponding to $j = j_0$ (with $t^{\rho}_{j_0} = -1$):

$$\alpha = \sum_{k=0}^{K(\rho,j_0)} \varpi^{\rho}_{j_0,k} \sum_{\ell=0}^{k} \frac{(-1)^{\ell}}{\ell!} \widehat{\Gamma}^{(\ell)}(1) \eta^{\rho}_{j_0,k-\ell}(0).$$

Now recall that by definition, $k_0 = 0$ is the maximal value of k such that $\varpi^{\rho}_{j_0,k} \neq 0$. Therefore the previous sum has (at most) one non-zero term: the one corresponding to $k = 0$. Since $\widehat{\Gamma}(1) = 1$ and $\varpi^{\rho}_{j_0,0} = \xi$ we have $\alpha = \xi \eta^{\rho}_{j_0,0}(0) = \xi y_{-1,0}(0) g^{\rho}_{j_0,0}(0) = \xi$ using Eqs. (3.5) and (3.6). This concludes the proof that the coefficient of e^x in the asymptotic expansion of $E(x)$ is equal to ξ.

4 Asymptotic expansion of the generalized hypergeometric series

In this section, we prove the following result (recall that asymptotic expansions have been defined in §3.1).

Theorem 4. *Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z) = {}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then $f(z)$ has an asymptotic expansion*

$$f(x) \approx \sum_{\rho \in \Sigma} e^{\rho x} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} \log(1/x)^i$$

in a large sector bisected by θ, with $\Sigma \subset \{0, 1\}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ both finite, and coefficients $c_{\rho,\alpha,i,n}$ in $\mathbf{H}$.

Proof. If one of the a_j's is in $\mathbb{Z}_{\leq 0}$, the hypergeometric series is in $\mathbb{C}[z]$ and the conclusion clearly holds with $c_{\rho,\alpha,i,n}$ in $\overline{\mathbb{Q}}$. From now on, as for the b_j's, we assume that none of the a_j's is in $\mathbb{Z}_{\leq 0}$.

Let

$$R(s) = R(\underline{a}, \underline{b}; s) := \frac{\prod_{j=1}^{p} \Gamma(a_j + s)}{\prod_{j=1}^{p} \Gamma(b_j + s)} \Gamma(-s).$$

The poles of $R(s)$ are located at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, p$, and at $\mathbb{Z}_{\geq 0}$. We define the series

$$L_p(\underline{a}, \underline{b}; z) := \sum_{j=1}^{p} \sum_{k=0}^{\infty} \mathrm{Residue}\big(R(s) z^s, s = -a_j - k\big).$$

Set $\nu := \sum_{j=1}^{p} a_j - \sum_{j=1}^{p} b_j$, $b_{p+1} := 1$ and

$$e_{k,m} := e_{k,m}(\underline{a}, \underline{b}) := \sum_{j=1}^{p+1} (1 - \nu + b_j + m)_{k-m} \frac{\prod_{i=1}^{p} (a_i - b_j)}{\prod_{i=1, i \neq j}^{p+1} (b_i - b_j)}.$$

We define a sequence $C_k := C_k(\underline{a}, \underline{b})$ by induction:

$$C_0 := 1, \quad C_k := \frac{1}{k} \sum_{m=0}^{k-1} e_{k,m} C_m,$$

and the formal series

$$K_p(\underline{a}, \underline{b}; z) := e^z \sum_{k=0}^{\infty} C_k(\underline{a}, \underline{b}) z^{\nu - k}.$$

By [15, p. 283, Theorem], reproved in [26, p. 113, Theorem 4.1, Eq. (4.4)], we have in fact

$$C_k(\underline{a}, \underline{b}) = \sum_{k_1 \geq 0, \dots, k_p \geq 0, K_p = k} \frac{(1 - a_p)_{k_p} \prod_{j=1}^{p-1} (a_{j+1} + b_{j+1} - a_j)_{k_j} \prod_{j=1}^{p} (B_j + K_{j-1})_{k_j}}{\prod_{j=1}^{p} k_j!},$$

where for every j, $B_j = \sum_{m=1}^{j} b_m$ and $K_j = \sum_{m=1}^{j} k_m$. It follows in particular that $K_p(\underline{a}, \underline{b}; z) \in e^z z^\nu \mathbb{Q}[\underline{a}, \underline{b}][[1/z]]$.

In [19, p. 212], it is shown that as $z \to \infty$ in the sector $-\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}$, we have the asymptotic expansion

$${}_pF_p \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix}; z \right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big),$$

while if $z \to \infty$ in the sector $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$, we have

$${}_pF_p \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix}; z \right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{-i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big).$$

These two expansions satisfy Definition 3 above: they hold in a *large sector* bisected by any $\theta \in (-\pi, 0)$, respectively any $\theta \in (0, \pi)$, and $L_p(\underline{a}, \underline{b}; e^{\pm i\pi} z)$ and $e^{-z} K_p(\underline{a}, \underline{b}; z)$ are 1-summable in the direction θ. Indeed, it is well-known that any hypergeometric series ${}_pF_p[\underline{a}, \underline{b}; z]$ admits an asymptotic expansion ([1]) in the sense of Definition 3, while Lemma 1 of [10] ensures that a function admits at most one expansion of this type in any given large sector bisected by a given direction.

These asymptotic expansions are refined versions of Barnes and Wright's fundamental works [6, 27] and are consequences of the general expansion of Meijer G-function [19, Chapter V]. Note that Meijer G-function is not related to Siegel's G-functions, though by specialization of its parameter the former provides examples of the latter. In the next two subsections, we provide more explicit expressions for the function $L_p(\underline{a}, \underline{b}; z)$ under the assumption that the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, in order to prove that all coefficients of the asymptotic expansion belong to $\mathbf{H}$.

[1]When the a_j's and b_j's are in $\mathbb{Q}$ as in our application, ${}_pF_p[\underline{a}, \underline{b}; z]$ is an E-function and Theorem 2 above proves the existence of this expansion. But the proof of this theorem does not need this assumption for the hypergeometric series, so that the general case follows as well.

4.1 R has simple poles

If the a_j's are pairwise distinct modulo $\mathbb{Z}$, then the poles of $R(s)$ are simple, and we have

$$L_p(\underline{a}, \underline{b}; z) = \sum_{j=1}^{p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(a_j + k) \prod_{i=1, i \neq j}^{p} \Gamma(a_i - a_j - k)}{k! \prod_{i=1}^{p} \Gamma(b_i - a_j - k)} z^{-a_j - k}.$$

When the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} L_p(\underline{a}, \underline{b}; z)$ is thus equal to a finite sum

$$\sum_j z^{-a_j} f_j(z)$$

with $f_j(z) \in \mathbf{H}[[1/z]]$. Note that the element $1/\pi \in \mathbf{H}$ appears through the use of the reflection formula $\frac{1}{\Gamma(s)} = \frac{1}{\pi} \sin(\pi s) \Gamma(1-s)$ for rational values of s.

4.2 R has multiple poles

We assume that the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Up to reordering the a_j's, we can group them in ℓ groups as follows: for $m = 0, \ldots, \ell - 1$, we have

$$a_{j_m+1}, a_{j_m+2}, \ldots, a_{j_{m+1}} \text{ equal} \bmod \mathbb{Z}, \quad a_{j_m+1} \text{ the smallest one in the group,}$$

where the a_{j_m} are pairwise distinct mod $\mathbb{Z}$ for $m = 1, \ldots, \ell$, and $0 = j_0 < j_1 < j_2 < \cdots < j_\ell = p$.

Then, for every $j \in \{j_m + 1, \ldots, j_{m+1}\}$, we have

$$\Gamma(a_j + s) = (a_{j_m+1} + s)_{a_j - a_{j_m+1}} \Gamma(a_{j_m+1} + s).$$

Set $d_m := j_m - j_{m-1} \geq 1$, $c_m := a_{j_{m-1}+1}$ and

$$P(s) := \prod_{m=0}^{\ell-1} \left(\prod_{j=j_m+1}^{j_{m+1}} (a_{j_m+1} + s)_{a_j - a_{j_m+1}} \right) \in \mathbb{Q}[s].$$

Hence,

$$R(s) = P(s) \Gamma(-s) \frac{\prod_{m=1}^{\ell} \Gamma(c_m + s)^{d_m}}{\prod_{m=1}^{p} \Gamma(b_m + s)}.$$

To compute the residue of $R(s) z^{-s}$ at $s = -c_n - k$ for given $n \in \{1, \ldots, \ell\}$ and $k \in \mathbb{Z}_{\geq 0}$, we write

$$\Gamma(c_n + s) = \frac{\Gamma(c_n + s + k + 1)}{(c_n + s)_k (c_n + s + k)}$$

and define

$$\Phi_{c_n,k}(s) := z^{-s} P(s) \Gamma(-s) \frac{\prod_{m=1, m \neq n}^{\ell} \Gamma(c_m + s)^{d_m}}{\prod_{m=1}^{p} \Gamma(b_m + s)} \cdot \frac{\Gamma(c_n + s + k + 1)^{d_n}}{(c_n + s)_k^{d_n}}$$

which is holomorphic at $s = -c_n - k$. We thus deduce from

$$R(s)z^{-s} = \frac{\Phi_{c_n,k}(s)}{(c_n + s + k)^{d_n}}$$

that

$$\mathrm{Residue}\big(R(s)z^{-s}, s = -c_n - k\big) = \frac{1}{(d_n - 1)!}\Phi_{c_n,k}^{(d_n-1)}(-c_n - k).$$

It follows that $\frac{\prod_{j=1}^{p}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} L_p(\underline{a}, \underline{b}; z)$ is equal to a finite sum

$$\sum_{j,\ell} z^{-a_j} \log(1/z)^{\ell} f_{j,\ell}(z)$$

with $f_{j,\ell}(z) \in \mathbf{H}[[1/z]]$. This concludes the proof of Theorem 4.

5 Application to Siegel's problem

We now complete the proof of Theorem 1. Assume that Siegel's question has an affirmative answer, and let $\xi \in \mathbf{G}$. Theorem 3 provides an E-function $E(z)$ and a finite set $S \subset (-\pi, \pi)$ such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is a coefficient of the asymptotic expansion of $E(z)$ in a large sector bisected by θ. Now an affirmative answer to Siegel's question yields n ${}_pF_p$ hypergeometric series $f_1, \ldots, f_n$ with rational parameters, n algebraic numbers $\lambda_1, \ldots, \lambda_n$, and a polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$, such that $E(z) = P(f_1(\lambda_1 z), \ldots, f_n(\lambda_n z))$. Choose $\theta \in (-\pi, \pi) \setminus S$ such that $\theta + \arg(\lambda_i) \notin \pi\mathbb{Z}$ for any $i \in \{1, \ldots, n\}$. Then Theorem 4 implies that for any i, the asymptotic expansion of $f_i(\lambda_i z)$ in a large sector bisected by θ has coefficients in $\mathbf{H}$. The same holds for $E(z) = P(f_1(\lambda_1 z), \ldots, f_n(\lambda_n z))$ because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra. Since such an asymptotic expansion is unique (see §3.1), the coefficient ξ belongs to $\mathbf{H}$. This concludes the proof.

6 A Siegel type problem for G-functions

We recall that $\sum_{n=0}^{\infty} a_n z^n$ is a G-function if $\sum_{n=0}^{\infty} a_n z^n/n!$ is an E-function (in the sense of this paper). G-functions form a ring stable under $\frac{d}{dz}$ and $\int_0^z$; they are not entire in general, they have a finite number of singularities and they can be analytically continued in a cut plane with cuts at these singularities. Moreover, given any algebraic function $\alpha(z)$ over $\overline{\mathbb{Q}}(z)$ holomorphic at 0 and any G-function $f(z)$, the functions $\alpha(z)$ and $f(z\alpha(z))$ are G-functions.

For any integer $p \geq 0$, the hypergeometric series

$${}_{p+1}F_p \left[\begin{matrix} a_1, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix}; z \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{(1)_n (b_1)_n \cdots (b_p)_n} z^n \tag{6.1}$$

is a G-function when $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j; Galochkin's classification can obviously be adapted to describe all the hypergeometric G-functions of type ${}_{p+1}F_p$. The simplest examples are ${}_1F_0[a;\cdot;z] = (1-z)^a$ ($a \in \mathbb{Q}$) and ${}_2F_1[1,1;2;z] = -\log(1-z)/z$. If $a_j \in \mathbb{Z}_{\leq 0}$ for some j, then the series reduces to a polynomial. Any polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form $\mu(z)\,{}_{p+1}F_p[a_1,\ldots,a_{p+1};b_1,\ldots,b_p;\lambda(z)]$ is a G-function, where the parameters $a_j, b_j \in \mathbb{Q}$, and $\mu(z), \lambda(z)$ are algebraic over $\overline{\mathbb{Q}}(z)$ and holomorphic at $z=0$, with $\lambda(0)=0$.

In the spirit of Siegel's problem for E-functions, it is natural to ask the following question.

Question 2. *Is it possible to write any G-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form $\mu(z)\cdot{}_{p+1}F_p[a_1,\ldots,a_{p+1};b_1,\ldots,b_p;\lambda(z)]$, with $p \geq 0$, $a_j, b_j \in \mathbb{Q}$, $\mu(z), \lambda(z)$ algebraic over $\overline{\mathbb{Q}}(z)$ and holomorphic at $z=0$, and $\lambda(0)=0$?*

We prove in this section a result similar to that for E-functions (recall that the inclusion $\mathbf{G} \subset \mathbf{H}$ is very unlikely: see §2.2).

Theorem 5. *At least one of the following statements is true:*

(*i*) $\mathbf{G} \subset \mathbf{H}$;

(*ii*) *Question 2 has a negative answer under the further assumption that the algebraic functions λ have a common singularity in $\overline{\mathbb{Q}}^* \cup \{\infty\}$ at which they all tend to ∞.*

Our method seems inoperant if this further assumption on the λ's is dropped. This problem is related to a conjecture of Dwork [8, p. 784] concerning the classification of certain operators in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ of order 2, which was disproved by Krammer [17]; later on, Bouw-Möller [7] gave other counter-examples, of a different nature. Dwork's conjecture said that a globally nilpotent operator of order 2 either has a basis of algebraic solutions over $\overline{\mathbb{Q}}(z)$ or is an algebraic pullback of the hypergeometric equation for the ${}_2F_1$ with rational parameters. We will not define here globally nilpotent operators (see the references), but they are conjectured to coincide with G-operators, i.e. operators in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ which are minimal for some non-zero G-function. It is known that operators "coming from geometry" are G-operators and globally nilpotent, by results of André [2, p. 111] and Katz [16, Theorem 10.0] respectively. The Krammer and Bouw-Möller operators "come from geometry" and in [7, §9], the authors even produced explicit G-functions solutions of their operators which are neither algebraic functions nor algebraic pullbacks of a ${}_2F_1$ with rational parameters. However, this does not rule out the possibility that these G-functions could be polynomials in more variables in ${}_{p+1}F_p$ hypergeometric functions with various values of $p \geq 1$.

Finally, if there exist $\alpha \in \overline{\mathbb{Q}}$, $|\alpha| < 1$, and $s \in \mathbb{N}$ such that $\mathrm{Li}_s(\alpha) \notin \mathbf{H}$, then the proof given below shows that $\mathrm{Li}_s(\frac{\alpha z}{z-\alpha})$ provides a counter-example, of differential order $s+1$, to Question 2 with the restriction in Theorem 5.

6.1 G-values as connection constants of G-functions

Given a non-zero G-function $f(z)$, let L denote a non-zero operator in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ such that $Lf(z) = 0$ and of minimal order for f. By standard results of André, Chudnovsky and Katz recalled in [3, §3] or [9, §4.1], L is fuchsian with rational exponents and, at any $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, L admits a $\mathbb{C}$-basis of solutions of the form

$$F(z-\alpha) = \sum_{e\in E}\sum_{k\in K}(z-\alpha)^e \log(z-\alpha)^k f_{e,k}(z-\alpha)$$

where $E \subset \mathbb{Q}$, $K \subset \mathbb{N}$ are finite sets, and the $f_{e,k}(z)$ are G-functions; if $\alpha = \infty$, $z-\alpha$ has to be replaced by $1/z$. We call such a basis an ACK basis of L at α. The determination of $\log(z-\alpha)$ is fixed but somewhat irrelevant to our purpose; the monodromy matrices of L around its singularities and ∞ all have coefficients in $\overline{\mathbb{Q}}[\pi]$.

Given an element $F(z-\beta)$ of an ACK basis of L at some point $\beta \in \overline{\mathbb{Q}} \cup \{\infty\}$ and an ACK basis $F_1(z-\alpha), \ldots, F_\mu(z-\alpha)$ of L at $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, we can connect locally around α an analytic continuation of $F(z-\beta)$ (in a suitable cut plane) to this ACK basis:

$$F(z-\beta) = \sum_{j=1}^{\mu} \omega_j F_j(z-\alpha), \tag{6.2}$$

where, following a general terminology, the complex numbers $\omega_1, \ldots, \omega_\mu$ are *connection constants*. In [9, Theorem 2], we proved that $\omega_1, \ldots, \omega_\mu$ are in fact in $\mathbf{G}$. We prove here a converse result.

Theorem 6. *Let $\xi \in \mathbf{G} \setminus \{0\}$ and $\alpha \in \overline{\mathbb{Q}}^* \cup \{\infty\}$. There exists a non-zero G-function $F(z)$ solution of $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$, of minimal order for F, and an ACK basis $F_1(z-\alpha), \ldots, F_\mu(z-\alpha)$ of L at α such that the analytic continuation of $F(z)$ in a suitable cut plane is given by*

$$\sum_{j=1}^{\mu} \omega_j F_j(z-\alpha),$$

where $\omega_1 = F(\alpha) = \xi$.

Proof. Let $\xi \in \mathbf{G} \setminus \{0\}$. We first assume that $\alpha \neq \infty$. By [9, Theorem 1], there exists a non-zero G-function $G(z)$ of radius of convergence $> |\alpha|$ such that $G(\alpha) = \xi$. Let $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ be of minimal order for G. Let $F_1(z-\alpha), \ldots, F_\mu(z-\alpha)$ be an ACK basis of L at α. Up to relabeling the basis, we can assume without loss of generality that there exists $\lambda \le \mu$ such that

$$G(z) = \sum_{j=1}^{\lambda} \omega_j F_j(z-\alpha) \tag{6.3}$$

in a cut plane locally around $z = \alpha$, where the $\omega_j \in \mathbf{G}$ are all non-zero. Up to performing $\overline{\mathbb{Q}}$-linear combinations of the F_j's, we can assume without loss of generality that for all $j \ge 2$, $F_j(z-\alpha) = o(F_1(z-\alpha))$ locally around $z = \alpha$. (Doing so, $(F_1(z-\alpha), \ldots, F_\mu(z-\alpha))$

remains an ACK basis of L at α.) Hence, $G(z) \sim \omega_1 F_1(z-\alpha)$ as $z \to \alpha$ in the cut plane. Because $G(\alpha) = \xi \neq 0$ and $\omega_1 \neq 0$, this implies that $\kappa := F_1(0) \in \overline{\mathbb{Q}}^*$ by [9, Lemma 5]. Upon replacing F_1 with $\kappa^{-1}F_1$, we may assume that $\kappa = 1$; then we have $\xi = G(\alpha) = \omega_1$.

To work around ∞, we fix $\alpha \in \overline{\mathbb{Q}}^*$ and keep the same notations; we consider now $H(z) := G(\frac{\alpha z}{z-\alpha})$ and $H_j(\frac{1}{z}) := F_j(\frac{\alpha z}{z-\alpha} - \alpha)$. Then H is a non-zero G-function solution of a differential operator $M \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ minimal for H (trivially deduced from L). Now, $H_1(\frac{1}{z}), \dots, H_\mu(\frac{1}{z})$ is an ACK basis of M at ∞ and, by (6.3), we have

$$H(z) = \sum_{j=1}^{\lambda} \omega_j H_j\Big(\frac{1}{z}\Big)$$

still with $\omega_1 = \xi$ and $H(z) \to \xi$ as $z \to \infty$ in a suitable cut plane. $\square$

6.2 Analytic continuation of the hypergeometric series ${}_{p+1}F_p(z)$

In this section, we prove the following result.

Theorem 7. *Let $f(z) = {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then, the analytic continuation of $f(z)$ to the domain defined by $|\arg(-z)| < \pi$ and $|z| > 1$ is given by*

$$\sum_{j=1}^{p+1} \sum_{\ell} z^{-a_j} \log(1/z)^\ell f_{j,\ell}(1/z)$$

where the sum over the integer ℓ is finite and each $f_{j,\ell}(z) \in \mathbf{H}[[z]]$ converges for $|z| < 1$.

Proof. Let

$$R(s) = R(\underline{a}, \underline{b}; s) := \frac{\prod_{j=1}^{p+1} \Gamma(a_j + s)}{\prod_{j=1}^{p} \Gamma(b_j + s)} \Gamma(-s).$$

The poles of $R(s)$ are located at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, p+1$, and at $\mathbb{Z}_{\geq 0}$. We define the series

$$M_p(\underline{a}, \underline{b}; z) := \sum_{j=1}^{p+1} \sum_{k=0}^{\infty} \mathrm{Residue}\big(R(s)(-z)^s, s = -a_j - k\big),$$

which converges for any z such that $|\arg(-z)| < \pi$ and $|z| > 1$.

Then the analytic continuation of f to the domain defined by $|\arg(-z)| < \pi$ and $|z| > 1$ is given by

$$f(z) = \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p+1} \Gamma(a_j)} M_p(\underline{a}, \underline{b}; z).$$

See the discussion in [19, §5.3.1], and Eqs. (5) and (17) there in particular. The same method as in §4.1 and §4.2 shows that

$$M_p(\underline{a}, \underline{b}; z) = \sum_{j=1}^{p+1} \sum_{\ell} z^{-a_j} \log(1/z)^\ell f_{j,\ell}(1/z)$$

where the sum over the integer $\ell \geq 0$ is finite and each $f_{j,\ell}(z) \in \mathbf{H}[[z]]$ converges for $|z| < 1$. This completes the proof. $\square$

Note that when the a_j's are pairwise distinct mod $\mathbb{Z}$, the poles of $R(s)$ at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$ are all distinct and we simply have

$$f(z) = \sum_{j=1}^{p+1} (-z)^{-a_j} \frac{\prod_{k=1,k\neq j}^{p+1} \Gamma(a_k - a_j)}{\prod_{k=1,k\neq j}^{p+1} \Gamma(a_k)} \\ \times \frac{\prod_{k=1}^{p} \Gamma(b_k)}{\prod_{k=1}^{p} \Gamma(b_k - a_j)} {}_{p+1}F_p \left[\begin{matrix} a_j, 1 - b_1 + a_j, \dots, 1 - b_p + a_j \\ 1 - a_1 + a_j, \dots * \dots, 1 - a_{p+1} + a_j \end{matrix} ; -\frac{1}{z} \right]$$

where $*$ means that the term $1 - a_j + a_j$ is omitted.

6.3 Proof of Theorem 5

We start with some general considerations. Let $F(z)$ be a G-function and $L \in \overline{\mathbb{Q}}(z)[\frac{d}{dz}] \setminus \{0\}$ be its minimal operator. Given a cut plane and $\alpha \in \overline{\mathbb{Q}}^* \cup \{\infty\}$, the local behaviour around α of the analytic continuation of F is described by an ACK basis of L at α. In particular, if $|z|$ is large enough, the analytic continuation of F is of the form

$$\sum_{e\in E} \sum_{k\in K} \sum_{n\geq 0} c_{e,k,n} z^{-e-n} \log(1/z)^k \tag{6.4}$$

where $c_{e,\ell,n} \in \mathbf{G}$, $E \subset \mathbb{Q}$, $K \subset \mathbb{N}$ are finite sets (recall that the connection constants ω_j in Eq. (6.2) belong to $\mathbf{G}$, by [9, Theorem 2]). Since the monodromy matrices of L around its singularities and around ∞ all have coefficients in $\overline{\mathbb{Q}}[\pi] \subset \mathbf{G}$, any analytic continuation of F is in fact of the form (6.4) at ∞.

Let now $f(z) = {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Let $\mu(z), \lambda(z) \in \overline{\overline{\mathbb{Q}}(z)}$ be holomorphic at $z = 0$, with $\lambda(0) = 0$ and $\lambda(z) \to \infty$ as $z \to \infty$. A more precise result than (6.4) can be obtained for the G-function $g(z) := \mu(z) f(\lambda(z))$. We first recall that the analytic continuations of $\mu(z)$ and $\lambda(z)$ in suitable cut planes admit convergent Puiseux expansions at ∞ of the form

$$\sum_{n\geq -m} a_n z^{-n/d} \in \overline{\mathbb{Q}}[[1/z^{1/d}]]$$

for some integers $m \geq 0$ and $d \geq 1$. Moreover, there exists $n \leq -1$ such that $a_n \neq 0$ for λ because $\lambda(z) \to \infty$ as $z \to \infty$. Using Theorem 7, we then deduce that $g(z)$ admits an analytic continuation at ∞ of the form (6.4) with $c_{e,k,n} \in \mathbf{H}$. Again, because $\overline{\mathbb{Q}}[\pi] \subset \mathbf{H}$, any analytic continuation of g is of the form (6.4) at ∞ with $c_{e,k,n} \in \mathbf{H}$.

For $j = 1, \dots, N$, consider $g_j(z) := \mu_j(z) f_j(\lambda_j(z))$ where $f_j(z)$ is a ${}_{p+1}F_p$ hypergeometric series with rational parameters and $\mu_j(z), \lambda_j(z) \in \overline{\overline{\mathbb{Q}}(z)}$ are holomorphic at $z = 0$, with $\lambda_j(0) = 0$ and $\lambda_j(z) \to \infty$ as $z \to \infty$. For any polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_N]$, it follows from the above discussion that any analytic continuation of the G-function $P(g_1(z),\dots,g_N(z))$ is also of the form (6.4) at ∞ with $c_{e,k,n}\in\mathbf{H}$.

Let us now assume that Question 2 has a positive answer when all the λ's tend to ∞ as $z\to\infty$. Given $\xi\in\mathbf{G}\setminus\{0\}$, consider the non-zero G-function $F(z)$ given by Theorem 6 for $\alpha=\infty$: in a suitable cut plane, its analytic continuation is of the form (6.4) with $c_{0,0,0}=\xi$. On the other hand, we have

$$F(z)=P\big(g_1(z),\dots,g_N(z)\big)$$

in a neighborhood of $z=0$, where the polynomial P and the g_j's are as above. The properties of this specific analytic continuation of $F(z)$ and of those of the right-hand side imply that $\xi\in\mathbf{H}$. Hence $\mathbf{G}\subset\mathbf{H}$ in this case.

If the λ's all tend to ∞ as $z\to\beta\in\overline{\mathbb{Q}}^*$, then the above argument can be adapted using Puiseux expansions of the μ's and λ's of the form $\sum_{n\ge -m}a_n(z-\beta)^{n/d}\in\overline{\mathbb{Q}}[[(z-\beta)^{1/d}]]$, a G-function F given by Theorem 6 with $\alpha=\beta$ and an ACK basis at β.

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