Unirational algebraic groups

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Abstract

It is well know that any linear smooth connected algebraic group defined over a perfect field is unirational. Over a nonperfect field k, there are non-unirational linear algebraic group. In this article, we study the geometry of the unirational k-groups with a focus on the unipotent case.

We obtain that for a commutative group the property of being unirational is invariant by separable field extension. We describe a general commutative unirational unipotent k-group as the quotient of some particular unipotent k-group obtain by purely inseparable Weil restriction of the multiplicative group. A consequence is that the Picard group of an unirational solvable k-group is finite, and that the restricted Picard functor of an unirational unipotent k-group is representable by an *étale* k-algebraic group. Finally, we give examples of unipotent unirational k-group that have only two unirational k-subgroups: themselves and the zero group.

Keywords
Unirational, Picard group, Imperfect field, Unipotent group, Weil restriction.

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Contents

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Introduction

Let k be a field. In this article, we study smooth connected k-group schemes of finite type, we call them k-*group*.

Background

Let us recall that an integral k-variety is called *rational* if it is birational to some affine n-space $\mathbb{A}_k^n = \operatorname{Spec}(k[t_1,\dots,t_n])$. An integral k-variety X is called *unirational* if there is a dominant rational map $\mathbb{A}_k^n \dashrightarrow X$ for some non-negative integer n.

If k is a perfect field, then any linear k-group is unirational [Bor12, V Th. 18.2]. And any unipotent k-group is k-split (i.e. admits a central composition series with successive quotients isomorphic to the additive group $\mathbb{G}_{a,k}$); hence any unipotent k-group is isomorphic as a k-scheme to an affine n-space $\mathbb{A}_k^n$ (see [DG70, Th. IV.4.4.1, and Cor. IV.2.2.10]), thus the unipotent k-groups are all rational.

Throughout the rest of this article, we assume that k is a imperfect field of characteristic $p > 0$, then the situation is way more complex. The k-split unipotent k-group are still rational, and the reductive k-group are still unirational [Bor12, V Th. 18.2]. Moreover, if G is a perfect (i.e. $G = [G : G]$) k-group, then G is unirational [CGP15, Pro. A.2.11]. But neither the k-pseudo-reductive groups, nor the k-wound unipotent k-groups are in general unirational. We begin with a classical example:

Example. We consider $t \in k \smallsetminus k^p$, and G the k-subgroup of $\mathbb{G}_{a,k}^2$ defined as:

$$G := \left\{(x,y) \in \mathbb{G}_{a,k}^2 \mid y^p = x + tx^p\right\}.$$

Then, G is a unipotent k-group of dimension 1, such that $G_{k(t^{1/p})} \cong \mathbb{G}_{a,k(t^{1/p})}$. We denote C the (canonical) regular completion of G, then

$$C = \left\{[x:y:z] \in \mathbb{P}_k^2 \mid y^p = xz^{p-1} + tx^p\right\}.$$

Then $C \smallsetminus G$ is a unique point of residue field $k\left(t^{1/p}\right) \neq k$, thus G is not isomorphic to $\mathbb{G}_{a,k}$.

Moreover, the arithmetic genus of C is $(p-1)(p-2)/2$. Hence, if $p = 2$, then $C \cong \mathbb{P}_k^1$ thus G is rational. But, if $p > 3$, then G is not rational.

The notion complementary to k-split is k-wound: a unipotent k-group U is called k-*wound* if U does not admit a central k-subgroup isomorphic to $\mathbb{G}_{a,k}$. The group of the example above is a k-wound unipotent k-group. The k-wound unipotent k-groups have strange properties: while some are unirational; others have only one unirational k-subgroup, the trivial group $\{0\}$.

In a previous article [Ach19], we studied of the subtle relationship between the notion of unirationality and the unipotent k-groups. let X, and Y be k-schemes (resp. k-group schemes). We call X *a form* of Y if there is a field extension K/k such that X_K is isomorphic as a K-scheme (resp. K-group scheme) to Y_K. Following an idea of M. Raynaud, we consider the "restricted Picard functor" i.e. for X a form of $\mathbb{A}_k^d$ the contravariant functor:

$$\begin{array}{cccc} \operatorname{Pic}^+_{X/k}: & (\text{Smooth Scheme}/k)^\circ & \to & (\text{group}) \\ & T & \mapsto & \frac{\operatorname{Pic}(X\times_k T)}{\operatorname{Pic}(T)}. \end{array}$$

If X is a unirational form of $\mathbb{A}_k^n$ which admits a regular completion, then Pic^+_X is representable by an *étale* unipotent k-algebraic group, and hence $\operatorname{Pic}(X)$ is a finite group [Ach19, Th. 4.11]. As the underlying scheme of a unipotent k-group is a form of $\mathbb{A}_k^n$, this theorem apply to the unirational unipotent k-group which admits a regular completion (a regular completion exists in dimension $n \leqslant 3$ and conjecturally in any dimension).

Main results and outline of the article

In the first section of this article we gather some results that will be used in the rest of the article. In Subsection 1.1, we make a quick *dévissage* of the structure of the unirational k-group. In Subsection 1.2, we discuss an important example: if k'/k is a finite purely inseparable field extension, then $\mathcal{R}_{k'/k}(\mathbb{G}_{m,k'})/\mathbb{G}_{m,k}$ is a unirational unipotent k-wound k-group (here $\mathcal{R}_{k'/k}$ denote the Weil restriction). This example had already study by J. Oesterlé [Oes84, VI §5]. In Subsection 1.4 we present the method used in Section 2 to study the commutative unirational group. This method is based on the universal mapping property of the rigidified Picard functor [BLR90, 10.3 Th. 2].

In Subsection 2.1, we state and proof the following theorem:

Theorem (2.3)**.** *Let G be a commutative k-group, we consider l/k a separable field extension (not necessary algebraic). Then G is unirational if and only if G_l is unirational.*

Then, in subsection 2.2 we study the unirational commutative unipotent k-groups. The main result of this Subsection is Proposition 2.5:

Proposition (2.5)**.** *We consider a unirational commutative k-wound unipotent k-group U. Then, there are some finite extensions $k_1,\dots,k_n$ of k and a fppf morphism of k-group:*

$$\prod_{i=1}^{n}\mathcal{R}_{l_i/k}\left(\mathcal{R}_{k_i/l_i}(\mathbb{G}_{m,k_i})/\mathbb{G}_{m,l_i}\right)\to U,$$

where l_i is the separable closure of k in k_i.

We use Proposition 2.5 to reduce the study of a general unirational commutative k-wound unipotent k-group to the case of a product of unipotent k-group defined by Weil restriction (which have already been study in Subsection 1.2).

In Subsection 2.3 we study the Picard group of the unirational k-group and the restricted Picard functor of the unirational unipotent k-group.

Theorem (2.8)**.** *If G is a unirational solvable linear k-group, then* $\mathrm{Pic}(G)$ *is a finite group, and* $\mathrm{Ext}^1(G,\mathbb{G}_{m,k})=\mathrm{Pic}(G)$.

If U is a unirational unipotent k-group, we denote $t(U)$ the smallest non-negative integer n such that U is of p^n-torsion. then:

(i) The group $\mathrm{Pic}(U)$ *is a finite group of $p^{t(U)}$-torsion, and* $\mathrm{Ext}^1(U,\mathbb{G}_{m,k})=\mathrm{Pic}(U)$.

(ii) The functor $\mathrm{Pic}^+_{U/k}$ *is representable by an étale unipotent k-algebraic group.*

(iii) For any smooth k-scheme with a k-rational point W, the morphism

$$p_1^*\times p_2^*:\mathrm{Pic}(X)\times\mathrm{Pic}(W)\to\mathrm{Pic}(X\times_k W)$$

is a group isomorphism.

The affirmation on the Picard group and on the extensions by the multiplicative group is a particular case of a theorem due to O. Gabber [Ros18, Th. 1.3]. The second part of this Theorem improve [Ach19, Th. 4.11]: we do not need the hypothesis that U admits a regular completion.

In Subsection 2.4, we study the question [Ach19, Que. 4.9]: Let G and G' be two unirational commutative k-groups, is any commutative extensions of G by G' unirational?

We prove that if [Ach19, Que. 4.9] is true for any G and G' unipotent, then [Ach19, Que. 4.9] is true (Proposition 2.11). A definitive answer to this question will either need a deeper understanding of the structure of the commutative unirational unipotent k-groups, or new geometric arguments that the author is missing.

Finally, in Subsection 2.5, we study *minimal* unipotent unirational k-group, i.e. the unirational k-group U that only admit $\{0\}$ and U as unirational k-subgroup. We obtain the following theorem:

Theorem (2.14)**.** *We consider a minimal unirational unipotent k-group U. Then, there is $t \in k_s$ such that $t^p \notin k(t)$, an integer $n \geqslant 1$, and a fppf k-group morphism*

$$\mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right) \to U$$

where $k'' = k\left(t^{1/p^n}\right)$, and $k' = k\left(t^{1/p^{n-1}}\right)$.

Moreover, $\mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right)$ is a minimal unirational k-group.

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Conventions

We consider a imperfect field k of characteristic $p > 0$. We fix an algebraic closure $\overline{k}$ of k, and we denote by $k_s \subset \overline{k}$ the separable closure of k in $\overline{k}$. The function field of an integral curve defined over a finite field is called a *global function field* i.e. a finite extension of $\mathbb{F}_q(t)$ for some $q = p^n$. By a *local field*, we mean the completion of a global field at one of its place, i.e. a finite extension of $\mathbb{F}_q((t))$ for some $q = p^n$.

All schemes are assumed to be separated and locally noetherian. For every scheme X, we denote the structural sheaf of X by $\mathcal{O}_X$. We denote the ring of regular functions on X by $\mathcal{O}(X)$, and the multiplicative group of invertible regular functions on X by $\mathcal{O}(X)^*$. For every $x \in X$, we denote the stalk of $\mathcal{O}_X$ at x by $\mathcal{O}_{X,x}$, and the residue field of $\mathcal{O}_{X,x}$ by $\kappa(x)$.

A morphism that is is faithfully flat of finite presentation is called *fppf*. The morphisms considered between two k-schemes are morphisms over k. An *algebraic variety* is a scheme of finite type over $\mathrm{Spec}(k)$. In order to lighten our notation, we will denote the product $X \times_{\mathrm{Spec}(k)} Y$ for X and Y two k-schemes by $X \times_k Y$. And for any field extension K/k, we denote the base change $X \times_k \mathrm{Spec}(K)$ by X_K. We denote the function field of an integral variety X by $\kappa(X)$.

A k-scheme is said to be *smooth* if it is formally smooth [EGAIV4, Def. 17.1.1], separated and locally of finite type over $\mathrm{Spec}(k)$. A group scheme locally of finite type over k will be called a *k-locally algebraic group*. A group scheme of finite type over k will be called a *k-algebraic group*. A smooth connected k-algebraic group will be called a *k-group*. A unipotent k-group U over k is said to be *k-wound* if every morphism of k-scheme $\mathbb{A}^1_k \to U$ is constant (with image a point of $U(k)$). An equivalent definition of k-wound is: U does not have a central subgroup isomorphic to $\mathbb{G}_{a,k}$ [CGP15, Pro. B.3.2].

1 Preliminary

1.1 Unirational k-groups

In this subsection, we consider an unirational k-group G and make a quick *dévissage* of its structure.

Let us recall that any k-group is the extension of a pseudo-abelian varieties by a linear k-group (see [Tot13] for the definition of pseudo-abelian varieties). Thus, an unirational k-group is an extension of an unirational pseudo-abelian varieties by a linear k-group. Moreover, the only unirational pseudo-abelian variety is the trivial one [Tot13, Th. 5.1]. Hence, an unirational k-group is linear.

Let G be an unirational k-group. We denote $\mathcal{D}(G)$ the derived k-subgroup of G (see [DG70, II §5 Th. 4.8] for the definition and the existence), then

$$1 \to \mathcal{D}(G) \to G \to G/\mathcal{D}(G) \to 1$$

is an exact sequence of unirational k-group. Indeed, the quotient $G/\mathcal{D}(G)$ is a commutative unirational k-group. And, there is an *fppf* morphism $G \times_k G \to \mathcal{D}(G)$, hence $\mathcal{D}(G)$ is unirational.

Next, we consider a commutative unirational k-group G, then there is an exact sequence:

$$0 \to T \to G \to U \to 0,$$

where T is a k-torus (hence, T is unirational) and U is an unirational unipotent k-group [DG70, IV.3.1 Th. 1 and Cor. 4].

Hence, we have essentially reduced the study to the following cases: the commutative unirational unipotent k-group and the perfect (i.e. $\mathcal{D}(G) = G$) k-group. Let us remark that any perfect k-group is unirational [CGP15, Pro. A.2.11]. More precisely, if G is a perfect k-group, then there is a finite number of k-tori $T_1, \dots, T_n$ and a *fppf* morphism of k-variety $T_1 \times_k \cdots \times_k T_n \to G$. In this article we manly deal with the commutative unirational unipotent case, we hope to study in details the perfect case in a future article.

Remark 1.1. In dimension 1, the unirational unipotent k-group U are already classified: either U isomorphic to the additive group $\mathbb{G}_{a,k}$; or the characteristic of the field is 2 and U is isomorphic the k-subgroup of $\mathbb{G}_{a,k}^2$ defined as $\{(x,y) \in \mathbb{G}_{a,k}^2 \mid y^2 = x + tx^2\}$ for some $t \in k \smallsetminus k^2$ [KMT74, Th. 6.9.2].

The remark above could make us think that for a unipotent k-wound k-group being unirational is an exception that only appear in low characteristics. In the next subsection we study examples of unirational unipotent k-wound k-groups in any (positive) characteristics.

1.2 An example of unirational k-wound unipotent k-group

First, we recall the definition of the Weil restriction (also call scalar restriction). We consider S a base scheme, S' a S-scheme and X' a S'-scheme. We call the covariant functor:

$$\begin{array}{rccl} \mathcal{R}_{S'/S}(X'): & (\text{Scheme}/S)^\circ & \to & (\text{Set}) \\ & T & \mapsto & X'(T \times_S S'), \end{array}$$

the Weil restriction of X' to S. If $X' = X \times_S S'$ for some S-scheme X, then there is a canonical natural transformation $X \to \mathcal{R}_{S'/S}(X')$.

In this article, we consider the case where S and S' are affine scheme. If $S = \text{Spec}(R)$, and $S' = \text{Spec}(R')$, then the Weil restriction is denoted $\mathcal{R}_{R'/R}(X')$ instead of $\mathcal{R}_{S'/S}(X')$. Moreover, if R' is a projective R-module of finite type, and if X' is a affine R'-scheme, then $\mathcal{R}_{R'/R}(X')$ is representable by an affine R-scheme [DG70, I §1 Pro. 6.6].

Example 1.2. We consider a purely inseparable extension k' of k of degree p^n, and a k-torus T of dimension d.

The Weil restriction $\mathcal{R}_{k'/k}(T_{k'})$ is an affine k-group, and the canonical morphism $T \to \mathcal{R}_{k'/k}(T_{k'})$ is a closed immersion. We consider

$$U := \mathcal{R}_{k'/k}(T_{k'})/T.$$

J. Oesterlé studied this k-group: U is a unirational k-wound unipotent k-group [Oes84, VI §5.1 Lem.], and the k'-group $U_{k'}$ is k'-split [Oes84, Cor. A.3.5].

Moreover, we can compute explicitly the Picard group of the group U of the example above. There is an exact sequence:

$$0 \to \mathcal{O}(U)^* \to \mathcal{O}\left(\mathcal{R}_{k'/k}(T_{k'})\right)^* \to \widehat{T} \to \text{Pic}(U) \to \text{Pic}\left(\mathcal{R}_{k'/k}(T_{k'})\right),$$

where $\widehat{T}$ denote the character group of T [Ach19, Pro. 2.18]. Let us consider two special cases:

(i) If T is k-split i.e. $T \cong \mathbb{G}_{m,k}^d$, then $\text{Pic}(U) = (\mathbb{Z}/p^n\mathbb{Z})^d$, and $\text{Pic}^+_{U/k} = (\mathbb{Z}/p^n\mathbb{Z})^d_k$ where $(\mathbb{Z}/p^n\mathbb{Z})^d_k$ denote the constant group associated to $(\mathbb{Z}/p^n\mathbb{Z})^d$ [Ach19, Pro. 4.1].

(ii) If $p > 2$, and T is anisotropic of dimension 1, then T split over a quadratic extension l of k [Spr09, 12.3.8 Exa. (2)]. Hence, the Picard group of $\mathcal{R}_{k'/k}(T_{k'})$ is a 2-torsion group (see e.g. [Bri15, Lem. 2.4]). And as $\mathrm{Pic}(U)$ is a p-torsion group, the map $\mathrm{Pic}(U) \to \mathrm{Pic}\left(\mathcal{R}_{k'/k}(T_{k'})\right)$ is trivial. Finally, as $\widehat{T} = \{0\}$, then $\mathrm{Pic}(U) = \{0\}$.

In general $\mathrm{Pic}(U)$ is a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^d$, and $\mathrm{Pic}^+_{U/k}$ is an *étale* k-algebraic group that is a k-form of the constant group $(\mathbb{Z}/p^n\mathbb{Z})^d_k$.

Example 1.3. Let us consider the field $k = \mathbb{F}_p(a,b)$, and U the k-subgroup of $\mathbb{G}^3_{a,k}$ defined as

$$U := \{(x,y,z) \in \mathbb{G}^3_{a,k} \mid x + ax^p + by^p = x + z^p\}.$$

Then, U is a k-wound unipotent k-group and $\mathrm{Pic}(U) = \{0\}$ [Tot13, Exa. 9.7]. If $p = 2$, then U is rational, else $p > 3$ and U is not unirational (in fact it is k-strongly wound, see next subsection).

Hence, we have examples of unirational k-wound unipotent k-group in any characteristics.

The first indication that the $\mathcal{R}_{k'/k}(T_{k'})/T$ form an important family come from a result of J. Oesterlé: if K is global function field, then the K-wound unipotent K-group of dimension strictly inferior to $p-1$ have a finite number of K-rational point [Oes84, Th. VI.3.1]. Thus, such a group is not unirational, and it does not contain any unirational subgroup of positive dimension.

If $[K' : K] = p$ and T is a K-torus of dimension 1, then $\mathcal{R}_{K'/K}(T_{K'})/T$ is K-wound unirational and of dimension $p-1$. This suggests that these groups play a special role among the K-wound unirational unipotent K-group. Hence, J. Oesterlé asks the following question:

Question. *[Oes84, p. 80]*

We consider an unipotent K-group G. Does $G(K)$ infinite, imply that G admits an unirational K-subgroup of positive dimension? Better, does G admits a subgroup of the type $\mathcal{R}_{K'/K}(T_{K'})/T$?

Inspired by this question, D. T. Nguyễn asks related Questions [Ngu11, Que. 1, 2, 3 and 4]. In Subsection 2.2, we will see that J. Oesterlé intuition is correct: these unipotent groups indeed a special role among commutative unirational unipotent K-group. We will come back to J. Oesterlé question in Subsection 2.5.

1.3 Unirational and k-strongly wound commutative k-groups

In this Subsection, we give a last motivation for the study of unirational unipotent k-group.

Proposition. *[Ach19, Pro. 4.4]*

Let G a commutative k-group, there is a unique maximal unirational k-subgroup of G denoted by G_{ur}.

Moreover, if X is a geometrically reduced unirational k-variety, then any morphism $X \to G$ whose image contains the identity element of G factor via G_{ur}.

Example 1.4. (i) If G is a k-abelian variety, then $G_{ur} = \{0\}$.

(ii) If G is a k-pseudo abelian variety, then $G_{ur} = \{0\}$.

(iii) If T is a k-torus, then $T_{ur} = T$.

(iv) If U is a commutative k-split unipotent k-group, then $U_{ur} = U$.

(v) If U is a nontrivial k-form of $\mathbb{G}_{a,k}$. Then, either $\mathrm{char}(k) = 2$ and U is isomorphic to the subgroup of $\mathbb{G}^2_{a,k}$ defined by the equation $y^2 = x + ax^2$ where $a \notin k^2$; and then U is rational, thus $U_{ur} = U$ [KMT74, Th. 6.9.2]. Or, U is not rational, and $U_{ur} = \{0\}$.

(vi) Let U be a nontrivial k-form of $\mathbb{G}_{a,k}$. We consider G an extension of U by $\mathbb{G}_{m,k}$. Then, either U is rational, and $G_{ur} = G$. Or U is not rational, and $G_{ur} = \mathbb{G}_{m,k}$.

(vii) If T is a k-torus, and k'/k is a finite purely inseparable field extension. Then, the unipotent k-group $U = \mathcal{R}_{k'/k}(T_{k'})/T$ is unirational, hence $U_{ur} = U$.

($viii$) We consider K a global field, and U a commutative unipotent K-group of dimension strictly lower than $p-1$. Then, $U_{ur} = \{0\}$ [Oes84, Th. VI.3.1].

A commutative k-group G such that $G_{ur} = \{0\}$ is unipotent and k-wound, thus we will call such k-group k-*strongly wound*.

Proposition. *[Ach19, Pro. 4.8]*

Let G be a commutative k-group. Then, G admits a unique quotient G_{sw} such that G_{sw} is k-strongly wound and any morphism $G \to H$ to a k-strongly wound k-group H factors in a unique way into $G \to G_{sw}$ followed by a morphism $G_{sw} \to H$. Moreover, the kernel of $G \to G_{sw}$ contains G_{ur}.

There is a natural question: is the extension of an unirational commutative k-group by an other unirational commutative k-group still unirational [Ach19, Que. 4.9]? If the answer to the question above is yes, then the canonical morphism $G/G_{ur} \to G_{sw}$ is an isomorphism. We will go back to this question in Subsection 2.4.

1.4 Rigidified Picard functor, universal mapping property

In this subsection, we gather results from [BLR90, Chap. 8 and Chap. 10], we will use them in Subsection 2.1 and 2.2. First, we define the rigidified Picard functor and state the exact sequence (1.1). Then, we state the universal mapping property of the rigidified Picard functor [BLR90, 10.3 Th. 2]. In [BLR90, §10.3], this universal mapping property has been use to prove a characterisation of the k-strongly wound k-group [BLR90, 10.3 Th. 1]; in this article we use the exact same property to study the commutative unirational k-group.

We consider a k-scheme X, we define the relative Picard functor $\mathrm{Pic}_{X/k}$ as the fppf-sheaf associated to the functor:

$$\begin{array}{ccc} (\mathrm{Scheme}/k)^{\circ} & \to & (\mathrm{Group}) \\ T & \mapsto & \frac{\mathrm{Pic}(X\times_k T)}{\mathrm{Pic}(T)}. \end{array}$$

We assume that X is a proper k-scheme, then $\mathrm{Pic}_{X/k}$ is represented by a k-locally algebraic group [BLR90, 8.2 Th. 3].

Definition 1.5. Rigidified Picard functor [BLR90, §8.1]

First we define a subscheme $Y \subset X$ which is finite to be a *rigidificator* (also called *rigidifier*) of $\mathrm{Pic}_{X/k}$ if for all k-schemes T the map $\mathcal{O}(X_T) \to \mathcal{O}(Y_T)$ induced by the inclusion of schemes $Y_T \to X_T$ is injective.

Let Y be a rigidificator of $\mathrm{Pic}_{X/k}$. We define a *rigidified line bundle* on X along Y, to be a pair $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle on X and α is an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \mathcal{L}_{|Y}$.

Let $(\mathcal{L}, \alpha)$ and $(\mathcal{L}', \alpha')$ be two rigidified line bundles on X along Y. A morphism of rigidified line bundle $f : (\mathcal{L}, \alpha) \to (\mathcal{L}', \alpha')$ is a morphism of line bundle $f : \mathcal{L} \to \mathcal{L}'$ such that $f_{|Y} \circ \alpha = \alpha'$.

The *rigidified Picard functor* is the functor

$$\left(\mathrm{Pic}_{X/k}, Y\right) : (\mathrm{Scheme}/k)^{\circ} \to (\mathrm{Group})$$

which associates to the k-scheme T the group of isomorphisms of rigidified line bundles on X_T along Y_T.

We consider a rigidificator Y of $\mathrm{Pic}_{X/k}$, there is a natural transformation

$$\begin{array}{cccc} \delta : & \mathcal{R}_{Y/k}(\mathbb{G}_{m,Y}) & \to & (\mathrm{Pic}_{X/k}, Y) \\ & a \in \mathcal{O}(Y \times_k T)^* & \mapsto & (\mathcal{O}_{X\times_k T}, \mathrm{mult_a}), \end{array}$$

where the map $\mathrm{mult_a} : \mathcal{O}_{X\times_k T} \xrightarrow{\sim} \mathcal{O}_{X\times_k T}$ is the multiplication by $a \in \mathcal{O}(Y \times_k T)^*$. There is also a map $(\mathrm{Pic}_{X/k}, Y) \to \mathrm{Pic}_{X/k}$ which forgets the rigidification and whose kernel is the image of δ.

Then, we relate the rigidified Picard functor and the Picard functor.

Proposition 1.6. *We assume that X is a geometrically reduced proper k-variety, and Y is a rigidificator. Then, the quotient $\mathcal{R}_{Y/k}(\mathbb{G}_{m,Y})/\mathcal{R}_{X/k}(\mathbb{G}_{m,X})$ is represented by an affine k-locally algebraic group. The functor $(\mathrm{Pic}_{X/k}, Y)$ is represented by a k-locally algebraic group. And, the sequence*

$$0 \to \mathcal{R}_{Y/k}(\mathbb{G}_{m,Y})/\mathcal{R}_{X/k}(\mathbb{G}_{m,X}) \xrightarrow{\delta} (\mathrm{Pic}_{X/k}, Y) \to \mathrm{Pic}_{X/k} \to 0 \tag{1.1}$$

is an exact sequence of k-locally algebraic group.

Remark 1.7. We have previously obtain a relative version of the proposition above [Ach17, Pro. 3.7]. Proposition [Ach17, Pro. 3.7] applied to geometrically integral projective k-variety, but in Subsection 2.1 we need a result that applied to k-variety that are not necessary geometrically irreducible .

The change are minimal compare to the proof of [Ach17, Pro. 3.7]. The only difference is that instead of using Theorem [BLR90, 8.1 Th. 1] to obtain the representability of the Picard functor of X we use Theorem [BLR90, 8.1 Th. 3] (as the base scheme is the spectrum of a field). Hence, for the proof of Proposition 1.6, we refer the reader to the proof of [Ach17, Pro. 3.7].

The rigidified Picard functor is a modern presentation of the generalized Jacobian varieties; we are going to use their universal mapping property (see [Ser12, Chap. V] for an historical reference). A modern reference for the universal mapping property is Theorem [BLR90, 10.3 Th. 2].

We assume that X is a projective, irreducible, geometrically reduced curve over k. We consider G a commutative k-group, and $X \overset{\varphi}{\dashrightarrow} G$ be a rational map. We denote the smooth locus of X by U (U is open and dense in X). Let Y be a rigidificator of X, there is a canonical map $U \smallsetminus Y \to (\mathrm{Pic}_{X/k}, Y)$. Thus, there is a rational map $X \overset{i_Y}{\dashrightarrow} (\mathrm{Pic}_{X/k}, Y)$. The rigidificator Y is called a *conductor* for φ if and only if there is a morphism of k-locally algebraic group $\Phi : (\mathrm{Pic}_{X/k}, Y) \to G$ such that the diagram:

$$\begin{array}{ccc} X & \overset{i_Y}{\dashrightarrow} & \left(\mathrm{Pic}_{X/k}, Y\right)^0 \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\Phi} \\ & & G, \end{array}$$

is commutative; then, the map Φ is uniquely determined. Finally, there is a conductor for φ and there even is a smallest one. The later is called the conductor of φ.

There is no generic smoothness in characteristic $p > 0$. The example below show that, even when $\left(\mathrm{Pic}_{X/k}, Y\right)^0$ and G are smooth, the morphism Φ can be non-smooth.

Example 1.8. Let $k = \mathbb{F}_2(a)$, we consider the projective curve C defined as a close subset of the projective plane $\mathbb{P}^2_k = \mathrm{Proj}(k[x,y,z])$ by the homogeneous equation

$$y^2 - ax^2 - zx = 0.$$

As C is a regular curve of arithmetic genius 0 with a k-rational point, it is isomorphic to the projective lane $\mathbb{P}^1_k$. We denote the open subset of $\mathbb{P}^2_k$ consisting of the homogeneous ideal not containing the principal ideal (z) by $D_+(z)$. We consider the k-morphism:

$$\begin{array}{rccc} f: & C \cap D_+(z) & \to & \mathbb{G}_{a,k} \\ & [x:y:1] & \mapsto & x. \end{array}$$

We see f as a rational morphism from C to $\mathbb{G}_{a,k}$. The only point P_∞ of C not in $C \cap D_+(z)$ is of homogeneous coordinate $[1 : a^{1/2} : 0]$. Then, commutative diagram imply by the mapping property is the following:

$$\begin{array}{ccc} \mathbb{P}^1_k & \overset{i_{P_\infty}}{\dashrightarrow} & \mathcal{R}_{k'/k}(\mathbb{G}_{m,k'})/\mathbb{G}_{m,k} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle\Phi} \\ & & \mathbb{G}_{a,k}, \end{array}$$

where $k' = k\left(a^{1/2}\right)$ is the residue field of P_∞ and $\mathcal{R}_{k'/k}(\mathbb{G}_{m,k'})/\mathbb{G}_{m,k}$ is the unique unipotent k-group that have $C \cap D_+(z)$ as it underlying k-scheme. Hence, the kernel of Φ is the non smooth unipotent k-algebraic group α_2 of underlying scheme $\operatorname{Spec}\left(k[y]/(y^2)\right)$.

Example 1.9. The Kernel of Φ is not necessary connected either. With the same notation as in Example 1.8, we consider the k-morphism

$$\begin{array}{rccc} g: & C \cap D_+(z) & \to & \mathbb{G}_{a,k} \\ & [x:y:1] & \mapsto & y. \end{array}$$

Likewise, we see g as a rational morphism from C to $\mathbb{G}_{a,k}$. Then, the kernel of the morphism Φ induced by g is the constant k-group $(\mathbb{Z}/2\mathbb{Z})_k$.

2 Unirational commutative k-groups

2.1 Unirationality and separable field extension

The property k-strongly wound is invariant by finite separable field extension [BLR90, 10.3 Rem. 4]. Hence, it is natural to ask if being unirational is also invariant by separable field extension.

Lemma 2.1. *Let G be a commutative k-group, we consider l/k a finite separable field extension. If G_l is unirational, then G is unirational.*

Proof. Let us consider a rational map $\mathbb{P}^1_l \overset{f}{\dashrightarrow} G_l$ such that the image of f contains the neutral element of G_l. Then, f generates a l-subgroup $\Gamma(f)$ of G_l [SGAIII1, VI.B Pro. 7.1 and Cor. 7.2.1]. As G_l is unirational, we have $\Gamma\left((f_j)_{j\in J}\right) = G_l$ where the $(f_j)_{j\in J}$ are the rational morphisms such that the image of f_j contains the neutral element of G_l. Moreover, we can consider a finite set $I \subset J$ such that $\Gamma\left((f_i)_{i\in I}\right) = U$.

We now consider $\mathbb{P}^1_l$ as an algebraic variety over k, it is a smooth irreducible projective k-curve that is not geometrically irreducible (except if $l = k$). And f_i induces a rational map of k-scheme $\mathbb{P}^1_l \overset{F_i}{\dashrightarrow} G$. Then, there is a conductor Y_i of F_i such that the diagram

$$\begin{array}{ccc} \mathbb{P}^1_l & \overset{i_{Y_i}}{\dashrightarrow} & \left(\operatorname{Pic}_{\mathbb{P}^1_l/k}, Y_i\right)^0 \\ & \overset{F_i}{\searrow} & \downarrow \Phi_i \\ & & G, \end{array}$$

is commutative. Moreover,

$$\left(\mathbb{P}^1_l\right) \times_k \operatorname{Spec}\left(\overline{k}\right) \cong \coprod_{i=1}^{[l:k]} \mathbb{P}^1_{\overline{k}}.$$

Thus $\operatorname{Pic}^0_{\mathbb{P}^1_l/k} = \{0\}$, and the exact sequence (1.1) imply:

$$\left(\operatorname{Pic}_{\mathbb{P}^1_l/k}, Y_i\right)^0 = \mathcal{R}_{Y_i/k}\left(\mathbb{G}_{m,Y_i}\right)/\mathcal{R}_{\mathbb{P}^1_l/k}\left(\mathbb{G}_{m,\mathbb{P}^1_l}\right).$$

Finally, $\mathcal{R}_{Y_i/k}\left(\mathbb{G}_{m,Y_i}\right)$ is a rational k-group (indeed $\mathbb{G}_{m,k}$ is an open subscheme of $\mathbb{A}^1_k$, so $\mathcal{R}_{Y_i/k}\left(\mathbb{G}_{m,Y_i}\right)$ is an open subscheme of some $\mathbb{A}^n_k$), hence $\left(\operatorname{Pic}_{\mathbb{P}^1_l/k}, Y_i\right)^0$ is an unirational k-group. And, as $G = \Gamma\left((F_i)_{i\in I}\right)$, the morphism

$$\prod_{i\in I}\left(\operatorname{Pic}_{\mathbb{P}^1_l/k}, Y_i\right)^0 \to G$$

is surjective, thus it is a *fppf* morphism of k-group [EGAIV2, Th. 6.9.1]. And the k-group G is unirational. □

Lemma 2.2. *Let l/k be a separable field extension and X a k-algebraic variety such that X_l is unirational. Then, there is a finite separable extension L/k such that X_L is unirational.*

Proof. We follow a classical "principle of finite extensions" method.

First, the field l is the direct limit of the finitely generated k-subalgebras of l. We consider a dominant morphism $f : U \to X_l$, where U is an open of $\mathbb{A}^n_l$. Then, there is a finitely generated k-algebra A such that U is defined on A [EGAIV3, Th. 8.3.11] i.e. there is an open V of $\mathbb{A}^n_A$ that is isomorphic to U after pull-back by $\mathrm{Spec}(l) \to \mathrm{Spec}(A)$. And, up to replacing A by another finitely generated k-subalgebra of l containing A, we can assume that the morphism f is defined over A [EGAIV3, Th. 8.8.2 (i)]. We will just write $F : V \to X_A$.

Let us consider B a k-subalgebra of l containing A. We denote E_B the set of point $s \in \mathrm{Spec}(B)$ such that the morphism $F_s : V_{\kappa(s)} \to X_{\kappa(s)}$ induce on the fibre by F is dominant. Then, E_B is constructible [EGAIV3, Pro. 9.6.1], and $E_B = \mathrm{Spec}(B)$ for some finitely generated k-subalgebra of l [EGAIV3, Cor. 8.3.5].

As l is a separable extension of k, the k-subalgebra B of l is geometrically reduced. And, there is a k_s-rational point in $\mathrm{Spec}(B)$ [Liu06, Pro. 3.2.20]. Thus, we can consider a finite separable extension L and a L-rational point e, i.e a morphism $e : \mathrm{Spec}(L) \to \mathrm{Spec}(B)$. The morphism e induces a rational L-morphism $\mathbb{P}^n_L \dashrightarrow X_L$. As $e \in E_B$ this morphism is dominant, hence X_L is unirational. ☐

Theorem 2.3. *Let G be a commutative k-group, we consider l/k a separable field extension. Then, G is unirational if and only if G_l is unirational.*

Proof. The fact that G unirational imply G_l unirational is obvious. For the converse, we can assume that l/k is finite (Lemma 2.2) and then we conclude via Lemma 2.1. ☐

2.2 Commutative unirational unipotent k-groups

Let us begin with a preliminary lemma:

Lemma 2.4. *We consider l/k a finite field extension, and U a l-wound rational unipotent l-group. Then, $\mathcal{R}_{l/k}(U)$ is a k-wound rational unipotent k-group.*

Proof. We denote $V = \mathcal{R}_{l/k}(U)$, then V is an affine k-group. And,

$$V_{\overline{k}} \cong \mathcal{R}_{l \otimes_k \overline{k}/\overline{k}} \left(U \times_k \mathrm{Spec}(\overline{k}) \right).$$

As U is a unipotent l-group and $\overline{k} = \overline{l}$, then $U \times_k \mathrm{Spec}(\overline{k}) \cong \mathbb{A}^d_{l \times_k \overline{k}}$ as a $l \otimes_k \overline{k}$-scheme (where $d = \dim(U)$). Thus, $V_{\overline{k}} \cong \mathbb{A}^{d[l:k]}_{\overline{k}}$ as $\overline{k}$-scheme. Hence, V is a unipotent k-group [DG70, IV.4.4.1 Lazard Th.]. Moreover, a morphism $\mathbb{A}^1_k \to V$ corresponds to a morphism $\mathbb{A}^1_l \to U$; thus such a morphism is constant and V is k-wound.

Finally, as U is rational, there is an open O of $\mathbb{A}^n_k$ (for some $n \geqslant 0$) and an open immersion of l-scheme $f : O \to U$, then $\mathcal{R}_{l/k}(f) : \mathcal{R}_{l/k}(O) \to \mathcal{R}_{l/k}(U)$ is an open immersion of k-scheme [BLR90, 7.6 Pro. 2 (i)]. As $\mathcal{R}_{l/k}(O)$ is an open subscheme of $\mathbb{A}^{n[l:k]}_k$, the k-group V is rational. ☐

Proposition 2.5. *We consider a commutative unirational k-wound unipotent k-group U. Then, there are some finite extensions $k_1, \ldots, k_n$ of k and a fppf morphism of k-group:*

$$\prod_{i=1}^{n} \mathcal{R}_{l_i/k} \left(\frac{\mathcal{R}_{k_i/l_i}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,l_i}} \right) \to U,$$

where l_i is the separable closure of k in k_i.

Proof. As already remarked in the proof of Lemma 2.1, the group U is generated by the image of the rational map f from $\mathbb{P}^1_k$ to U such that $f\left(\mathbb{P}^1_k\right)$ contains the identity of U; we consider a finite set I such that $\Gamma\left((f_i)_{i\in I}\right) = U$.

We fix $i \in I$, there is a rigidificator Y_i of $\mathrm{Pic}_{\mathbb{P}^1_k/k}$ and a commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^1_k & \overset{i_{Y_i}}{\dashrightarrow} & \left(\mathrm{Pic}_{\mathbb{P}^1_k/k}, Y_i\right)^0 \\ & \underset{f_i}{\searrow} & \downarrow F_i \\ & & U, \end{array}$$

where F_i is a k-group morphism [BLR90, 10.3 Th. 2]. As $\mathrm{Pic}^0_{\mathbb{P}^1_k/k} = \{0\}$, using the exact sequence (1.1), we compute the neutral component of the rigidified Picard functor:

$$\left(\mathrm{Pic}_{\mathbb{P}^1_k/k}, Y_i\right)^0 = \frac{\mathcal{R}_{Y_i/k}\left(\mathbb{G}_{m,Y_i}\right)}{\mathbb{G}_{m,k}}.$$

As U is k-wound, the subscheme Y_i is reduced [BLR90, 10.3 Cor. 3]. Thus, as a set $Y_i = \{P_{i,1}, \dots, P_{i,n_i}\}$ for a finite number of closed point of $\mathbb{P}^1_k$. Then $k_{i,j} = \kappa(P_{i,j})$ is a finite extension of k, we denote $l_{i,j}$ the separable closure of k in $k_{i,j}$. With these notations:

$$\mathcal{R}_{Y_i/k}\left(\mathbb{G}_{m,Y_i}\right) = \prod_{j=1}^{n_i} \mathcal{R}_{k_{i,j}/k}\left(\mathbb{G}_{m,k_{i,j}}\right).$$

Moreover, $\mathcal{R}_{k_{i,j}/k}\left(\mathbb{G}_{m,k_{i,j}}\right) = \mathcal{R}_{l_{i,j}/k}\left(\mathcal{R}_{k_{i,j}/l_{i,j}}\left(\mathbb{G}_{m,k_{i,j}}\right)\right)$. And, there is an exact sequence of $l_{i,j}$-groups:

$$0 \to \mathbb{G}_{m,l_{i,j}} \to \mathcal{R}_{k_{i,j}/l_{i,j}}\left(\mathbb{G}_{m,k_{i,j}}\right) \to U_{i,j} \to 0,$$

where $U_{i,j}$ is a $l_{i,j}$-wound unipotent $l_{i,j}$-group. We apply the Weil restriction $\mathcal{R}_{l_{i,j}/k}$ to the above sequence, and we obtain:

$$0 \to \mathcal{R}_{l_{i,j}/k}\left(\mathbb{G}_{m,l_{i,j}}\right) \to \mathcal{R}_{l_{i,j}/k}\left(\mathcal{R}_{k_{i,j}/l_{i,j}}\left(\mathbb{G}_{m,k_{i,j}}\right)\right) \to \mathcal{R}_{l_{i,j}/k}\left(U_{i,j}\right) \to 0,$$

this is also an exact sequence of k-group [CGP15, Cor. A.5.4]. Moreover,

$$U_{i,j} = \mathcal{R}_{k_{i,j}/l_{i,j}}\left(\mathbb{G}_{m,k_{i,j}}\right)/\mathbb{G}_{m,l_{i,j}}$$

is a $l_{i,j}$-wound rational unipotent $l_{i,j}$-group. Thus, $\mathcal{R}_{l_{i,j}/k}\left(U_{i,j}\right)$ is a k-wound rational unipotent k-group (Lemma 2.4). And, $\mathcal{R}_{l_{i,j}/k}\left(\mathbb{G}_{m,l_{i,j}}\right)$ is a k-torus. As the only k-group morphism from a k-torus to a unipotent k-group is the zero morphism, the morphism

$$\prod_{j=1}^{n_i} \mathcal{R}_{l_{i,j}/k}\left(\mathbb{G}_{m,l_{i,j}}\right) \to \prod_{j=1}^{n_i} \mathcal{R}_{k_{i,j}/k}\left(\mathbb{G}_{m,k_{i,j}}\right) \to \frac{\prod_{j=1}^{n_i} \mathcal{R}_{k_{i,j}/k}\left(\mathbb{G}_{m,k_{i,j}}\right)}{\mathbb{G}_{m,k}} \xrightarrow{F_i} U$$

is trivial. Hence, we obtain a k-group morphism:

$$\prod_{j=1}^{n_i} \mathcal{R}_{l_{i,j}/k}\left(U_{i,j}\right) \to U.$$

Finally, as the image of the f_i generates the group U, the (finite) product morphism

$$\prod_{i\in I}\prod_{j=1}^{n_i} \mathcal{R}_{l_{i,j}/k}\left(U_{i,j}\right) \to U$$

is surjective, thus it is a *fppf* morphism of k-group [EGAIV2, Th. 6.9.1]. □

Remark 2.6. (*i*) A commutative unirational k-wound unipotent k-group is in general not isomorphic to some product

$$\prod_{i=1}^{n} \mathcal{R}_{l_i/k}\left(\frac{\mathcal{R}_{k_i/l_i}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,l_i}}\right).$$

Indeed, the Picard group of such a unipotent group is nontrivial (if at least one of the k_i/l_i is non trivial). But, we have examples of commutative unirational k-wound unipotent k-group with trivial Picard group.

(*ii*) With the arguments of the proof of Proposition 2.5, we have the following results: let G be a commutative unirational k-group, then there is a finite dimension k-algebra A and a *fppf* morphism

$$\mathcal{R}_{A/k}(\mathbb{G}_{m,A}) \to G.$$

Proposition 2.5 asserts that any commutative k-wound unirational unipotent k-group is the quotient of a commutative k-wound rational unipotent k-group of a type that have been explicitly studied in Subsection 1.2. In the Proposition 2.7 below we use this idea to study the Picard group of the commutative unirational unipotent k-group.

Proposition 2.7. *If U is a commutative unirational unipotent k-group, then* $\mathrm{Pic}(U)$ *is finite, and the restricted Picard functor* $\mathrm{Pic}^{+}_{U/k}$ *is representable by an étale unipotent k-algebraic group.*

Proof. First, we only need to prove the proposition when U is k-wound [Ach19, Pro. 2.16]. Next, we assume that $k = k_s$. Then, there is a *fppf* morphism

$$f : \prod_{i=1}^{n} \frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,k}} \to U.$$

And, the pull-back by f

$$f^{*} : \mathrm{Pic}(U) \to \mathrm{Pic}\left(\prod_{i=1}^{n} \frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,k}}\right).$$

is an injective morphism [Ros18, v1 Lem. 3.2]. Moreover, as the $\frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i/k})}{\mathbb{G}_{m,k}}$ are rational, we have:

$$\mathrm{Pic}\left(\prod_{i=1}^{n} \frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,k}}\right) = \prod_{i=1}^{n} \mathrm{Pic}\left(\frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,k}}\right) = \prod_{i=1}^{n} \frac{\mathbb{Z}}{[k_i : k]\mathbb{Z}}.$$

Thus, $\mathrm{Pic}(U)$ is finite.

Next, we show that the restricted Picard functor $\mathrm{Pic}^{+}_{U/k}$ is represented by $\mathrm{Pic}(U)_k$, the constant group associated to $\mathrm{Pic}(U)$ over k. The morphism

$$R := \prod_{i=1}^{n} \frac{\mathcal{R}_{k_i/k}(\mathbb{G}_{m,k_i})}{\mathbb{G}_{m,k}} \xrightarrow{f} U$$

induces a natural transformation $f^{*} : \mathrm{Pic}^{+}_{U/k} \to \mathrm{Pic}^{+}_{R/k}$. We consider T a smooth irreducible k-scheme, then

$$\mathrm{Pic}^{+}_{U/k}(T) = \frac{\mathrm{Pic}(U \times_k T)}{\mathrm{Pic}(T)} = \mathrm{Pic}\left(U_{\kappa(T)}\right).$$

The first equality is the definition of the restricted Picard functor, the second equality come from [Ach19, Lem. 2.17]. Moreover, the morphism $f^{*}_{\kappa(T)} : \mathrm{Pic}\left(U_{\kappa(T)}\right) \to \mathrm{Pic}\left(R_{\kappa(T)}\right)$ is still injective. Let us fix a point $e \in T(k)$, such a point exists as T is smooth and $k = k_s$. Then, e induces a morphism $E_U : U \to U \times_k T$, and the pull-back $E_U^{*} : \frac{\mathrm{Pic}(U \times_k T)}{\mathrm{Pic}(T)} \to \mathrm{Pic}(U)$ is surjective (and same with R). Thus, we have a commutative square:

$$\begin{array}{ccc} \mathrm{Pic}(U) & \overset{f^{*}}{\hookrightarrow} & \mathrm{Pic}(R) \\ {\scriptstyle E_U^{*}}\uparrow & & \| {\scriptstyle E_R^{*}} \\ \mathrm{Pic}^{+}_{U/k}(T) & \underset{f_T^{*}}{\hookrightarrow} & \mathrm{Pic}^{+}_{R/k}(T). \end{array}$$

Hence, $\mathrm{Pic}^+_{U/k}(T) = \mathrm{Pic}(U) = \mathrm{Pic}(U)_k(T)$. More generally, if T is any smooth k-scheme, then $T = \coprod_j T_j$ where the T_j are the open irreducible components of T. As the T_j are smooth irreducible k-scheme, we have

$$\mathrm{Pic}^+_{U/k}(T) = \prod_j \mathrm{Pic}^+_{U/k}(T_j) = \prod_j \mathrm{Pic}(U)_k(T_j) = \mathrm{Pic}(U)_k\left(\coprod_j T_j\right).$$

To finish the proof, we need a Galois descent argument. As this exact argument has already been written in details in a previous article, we refer the reader to [Ach19, §1.3]. □

2.3 Picard group of unirational solvable k-groups

First, we recall some facts on the extension of a k-group by the multiplicative group (see [Ach19, §3.1] for more details). We consider a k-group G, we denote $\mathrm{Ext}^1(G, \mathbb{G}_{m,k})$ the group of equivalence classes of the (necessary central) extension of G by $\mathbb{G}_{m,k}$.

There is a natural map $\varphi : \mathrm{Ext}^1(G, \mathbb{G}_{m,k}) \to \mathrm{Pic}(G)$. Indeed, an extension of G by $\mathbb{G}_{m,k}$ can be seen as a $\mathbb{G}_{m,k}$-torsor over G, and the isomorphism classes of $\mathbb{G}_{m,k}$-torsor are classified by the cohomology group $\mathrm{H}^1(G, \mathbb{G}_{m,k}) = \mathrm{Pic}(G)$.

Moreover, if $a \in G$ then a induces a translation morphism $\mathcal{T}_a : G_{\kappa(a)} \to G$, and there is a canonical morphism $q : G_{\kappa(a)} \to G$. We say that an element $\mathcal{L} \in \mathrm{Pic}(G)$ is invariant by translation by a if $\mathcal{T}_a^*(\mathcal{L}) = q^*(\mathcal{L})$ in $\mathrm{Pic}(G_{\kappa(a)})$.

Then, φ identifies $\mathrm{Ext}^1(G, \mathbb{G}_{m,k})$ with the subgroup of the element of $\mathrm{Pic}(G)$ invariant by translation by G. As G is connected, if $\mathrm{Pic}(G_K)$ is finite for any field extension K/k, every element is invariant by translation; hence, $\mathrm{Ext}^1(G, \mathbb{G}_{m,k}) = \mathrm{Pic}(G)$.

We consider a unipotent k-group U, we denote $t(U)$ the smallest non-negative integer n such that U is of p^n-torsion.

Theorem 2.8. *If G is a unirational solvable linear k-group, then* $\mathrm{Pic}(G)$ *is a finite group, and* $\mathrm{Ext}^1(G, \mathbb{G}_{m,k}) = \mathrm{Pic}(G)$.

Moreover, if U is an unirational unipotent k-group, then:

(i) The group $\mathrm{Pic}(U)$ *is a finite group of $p^{t(U)}$-torsion, and* $\mathrm{Ext}^1(U, \mathbb{G}_{m,k}) = \mathrm{Pic}(U)$.

(ii) The functor $\mathrm{Pic}^+_{U/k}$ *is representable by an étale unipotent k-algebraic group.*

(iii) For any smooth k-scheme with a k-rational point W, the morphism

$$p_1^* \times p_2^* : \mathrm{Pic}(U) \times \mathrm{Pic}(W) \to \mathrm{Pic}(U \times_k W)$$

is a group isomorphism.

Proof. We consider an unirational commutative k-group G, then G is an extension of an unirational commutative unipotent k-group U by a k-torus T. Moreover, the sequence of Picard group:

$$\mathrm{Pic}(U) \to \mathrm{Pic}(G) \to \mathrm{Pic}(T_{\kappa(G)}),$$

is exact [Bri15, Lem. 2.13] (the morphisms are the obvious one). Both $\mathrm{Pic}(U)$ (Proposition 2.7) and $\mathrm{Pic}\left(T_{\kappa(G)}\right)$ are finite, hence $\mathrm{Pic}(G)$ is finite.

Next, we consider a solvable unirational k-group G, as already remarked in Subsection 2.2, G is the extension of the unirational commutative group $G/\mathcal{D}(G)$ and the unirational solvable k-group $\mathcal{D}(G)$. As above, the sequence

$$\mathrm{Pic}\left(G/\mathcal{D}(G)\right) \to \mathrm{Pic}(G) \to \mathrm{Pic}\left(\mathcal{D}(G)_{\kappa(G)}\right),$$

is exact. Hence, by induction on the dimension of G, the Picard group of G is finite, and $\mathrm{Ext}^1(G, \mathbb{G}_{m,k}) = \mathrm{Pic}(G)$.

Now, we consider U a unipotent unirational k-group. The affirmation (i) is straightforward (a unipotent k-group is solvable), the estimation of the torsion is [Ach19, Pro. 3.9].

If U is commutative then (ii) is Proposition 2.7. Else, like above, we use an induction argument. If $\mathrm{Pic}^+_{\mathcal{D}(U)/k}$ is representable by an étale unipotent k-group, then the exact sequence of k-group

$$1 \to \mathcal{D}(U) \to U \to U_{ab} = U/\mathcal{D}(U) \to 1,$$

induces a sequence of group functor

$$0 \to \mathrm{Pic}^+_{U_{ab}/k} \to \mathrm{Pic}^+_{U/k} \to \mathrm{Pic}^+_{\mathcal{D}(U)/k}.$$

This sequence is an exact sequence of group functor [Ach19, Pro. 4.17]. If $k = k_s$, we immediately see that $\mathrm{Pic}^+_{U/k}$ is represented by the constant group $\mathrm{Pic}(U)_k$. For the general case, we use the Galois descent argument of [Ach19, §1.3].

Finally, (iii) is a consequence of (ii) and [Ach19, Cor. 4.13]. □

2.4 Commutative extensions of unirational k-groups

In this subsection we go back to the question asked in Subsection 1.3:

Question. *[Ach19, Que. 4.9]*

Let G and G' be two unirational commutative k-groups. Is any commutative extensions of G by G' unirational?

If the answer to [Ach19, Que. 4.9], in particular the answer to the question below is also true:

Question 2.9. *Let U and U' be two unirational commutative unipotent k-groups. Is any commutative extensions of U by U' unirational?*

Lemma 2.10. *Let E be a commutative linear k-group, there is a canonical exact sequence:*

$$0 \to T_E \to E \to U_E \to 0, \tag{2.1}$$

where T_E is a k-torus and U_E is a commutative unipotent k-group.

Then, the k-group E is unirational if and only if U_E is unirational.

Proof. If E is unirational, then its quotient U_E is also unirational. Let us show the converse.

First, we can assume that k is separably closed, hence that T_E is a split k-torus (Theorem 2.3, and [Spr09, 13.1.1 Pro.]). Next, there is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_E \cong \mathbb{G}^r_{m,k} & \longrightarrow & E & \longrightarrow & U_E & \longrightarrow & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & X & \longrightarrow & \eta, & & \end{array}$$

where η denote the generic point of U_E and $X \to \eta$ is the generic fibre of $E \to U_E$. Then, X is a $\mathbb{G}^r_{m,\eta}$-torsor over $\eta = \mathrm{Spec}\left(\kappa\left(U_E\right)\right)$. As such torsor is trivial, we have $\kappa(E) = \kappa(X) \cong \kappa\left(U_E\right)(t_1, \dots, t_r)$. Finally, if U_E is unirational, then $\kappa\left(U_E\right) \subseteq k(u_1, \dots, u_n)$, thus $\kappa(E) \subseteq k(u_1, \dots, u_n, t_1, \dots, t_r)$ and E is a unirational k-group. □

Proposition 2.11. *If the answer to Question 2.9 is true, then the answer to Question [Ach19, Que. 4.9] is also true.*

Proof. We consider G and G' two commutative unirational k-group, then there are two exact sequences of commutative k-groups:

$$0 \to T \to G \to U \to 0,$$
$$0 \to T' \to G' \to U' \to 0,$$

where T and T' are k-torus, U and U' are commutative unirational unipotent k-groups. Let E be a commutative extension of G by G', then the exact sequence

$$0 \to G' \to E \to G \to 0,$$

induces an exact sequence of unipotent k-group

$$0 \to U' \to U_E \to U \to 0.$$

As E is unirational if and only if U_E is unirational (Lemma 2.10), we have reduced the question to the case of extension of two commutative unirational unipotent k-groups. □

While the generic fibre is a trivial torsor in the proof of Lemma 2.10, for an extension of unipotent group it can be nasty. Below we give an example of an extension of $\mathbb{G}_{a,k}$ by a k-wound unipotent k-group with nontrivial generic fibre.

Example 2.12. Let $k = \mathbb{F}_p(t)$, and $k' = \mathbb{F}_p\left(t^{1/p}\right)$. We denote the unipotent k-group $\mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)/\mathbb{G}_{m,k}$ by U. Then:

$$0 \to U \to \mathbb{G}_{a,k}^p \xrightarrow{P} \mathbb{G}_{a,k} \to 0$$

is an exact sequence of rational unipotent k-group, where $\mathbb{G}_{a,k}^p = \mathrm{Spec}\left(k[x_0, \ldots, x_{p-1}]\right)$ and $P(x_0, \ldots, x_{p-1}) = x_0^p + t x_1^p + \cdots + t^{p-1} x_{p-1}^p - x_{p-1}$ [Oes84, IV Pro. 5.3]. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & \mathbb{G}_{a,k}^p & \longrightarrow & \mathbb{G}_{a,k} & \longrightarrow & 0 \\ & & & & {\scriptstyle f}\uparrow & & \uparrow & & \\ & & & & X & \longrightarrow & \eta, & & \end{array}$$

where η denote the generic point of $\mathbb{G}_{a,k}$ and $X \to \eta$ is the generic fibre of $\mathbb{G}_{a,k}^p \xrightarrow{P} \mathbb{G}_{a,k}$. Then, the pull-back morphism $f^* : \mathrm{Pic}\left(\mathbb{G}_{a,k}^p\right) \to \mathrm{Pic}\,(X)$ is surjective [Ach19, Lem. 2.17], thus $\mathrm{Pic}\,(X) = \{0\}$. And as $\mathrm{Pic}\,(U_\eta) = \mathbb{Z}/p\mathbb{Z}$, the generic fibre is not the trivial U_η-torsor over η. In particular, X is not a unirational η-variety.

2.5 Minimal unirational unipotent k-groups

We call an unirational unipotent k-group U *minimal* if any unirational k-subgroup V of U is trivial (i.e. $V = U$ or $V = \{0\}$). For example, $\mathbb{G}_{a,k}$ is minimal.

Lemma 2.13. *Let U be a minimal unirational unipotent k-group, then U is a k-form of $\mathbb{G}_{a,k}^{\dim(U)}$ (i.e. U is commutative and of p-torsion). Moreover, either $U \cong \mathbb{G}_{a,k}$ or U is k-wound.*

Proof. Minimal unirational unipotent k-group are commutative (else $\mathcal{D}(U)$ is a nontrivial unirational k-subgroup). Let us recall that any commutative unipotent k-group U admits a descending central series:

$$\{0\} = [p^n U] \subset \cdots \subset [pU] \subset U,$$

where $[pU]$ is the image of $p.Id : U \to U$. As U is unirational, the $[p^k U]$ are unirational and the successive quotients are unirational commutative p-torsion unipotent k-groups. Hence, a minimal unirational unipotent k-group is a form of $\mathbb{G}_{a,k}^{\dim(U)}$ [CGP15, Pro. B.1.13].

Finally, if U is not k-wound, then it admits $\mathbb{G}_{a,k}$ as a k-subgroup. If U is not isomorphic to $\mathbb{G}_{a,k}$, this contradicts minimality. □

In the rest of this subsection we are going to prove the following theorem:

Theorem 2.14. *We consider a minimal unirational unipotent k-group U. Then, there are $t \in k_s$ such that $t \notin k_s^p$, an integer $n \geqslant 1$, and a fppf k-group morphism:*

$$\mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right) \to U,$$

where $k'' = k\left(t^{1/p^n}\right)$, and $k' = k\left(t^{1/p^{n-1}}\right)$. Moreover, $\mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right)$ is a minimal unirational k-group.

For convenience, we are going to cut the proof of this theorem in three parts: Proposition 2.15, 2.18 and 2.20.

Proposition 2.15. *We consider a minimal unirational unipotent k-group U. Then, there are $t \in k_s$ such that $t \notin k_s^p$, an integer $n \geqslant 1$, and a fppf k-group morphism:*

$$\mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right) \to U,$$

where $k'' = k\left(t^{1/p^n}\right)$, and $k' = k\left(t^{1/p^{n-1}}\right)$.

Proof. According to Proposition 2.5, there is a finite number of finite algebraic extension k_i/k and a *fppf* morphism:

$$\prod_{i=1}^{n} \mathcal{R}_{l_i/k}\left(\frac{\mathcal{R}_{k_i/l_i}\left(\mathbb{G}_{m,k_i}\right)}{\mathbb{G}_{m,l_i}}\right) \to U.$$

As U is minimal, we can choose an integer i such that

$$\mathcal{R}_{l_i/k}\left(\frac{\mathcal{R}_{k_i/l_i}\left(\mathbb{G}_{m,k_i}\right)}{\mathbb{G}_{m,l_i}}\right) \xrightarrow{F_i} U,$$

is an *fppf* morphism. For convenience, we are going to denote l for l_i. As k_i is the residue field of a closed point of $\mathbb{P}^1_k$ (c.f. proof of Proposition 2.5), there is a $\tau \in \overline{k}$ such that $k_i = k(\tau)$, and then $l = k\left(\tau^{p^r}\right)$ for some integer $r > 0$.

Moreover, for any intermediate extension $k_i/k''/l$, there is a canonical closed immersion

$$\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathbb{G}_{m,l}} \xrightarrow{\varphi_{k''}} \frac{\mathcal{R}_{k_i/l}\left(\mathbb{G}_{m,k_i}\right)}{\mathbb{G}_{m,l}}.$$

So, by composition, we obtain a morphism

$$\psi_{k''} = F_i \circ \mathcal{R}_{l/k}\left(\varphi_{k''}\right) : \mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathbb{G}_{m,l}}\right) \to U.$$

As U is minimal, the morphism $\psi_{k''}$ is either *fppf* or the zero morphism. Hence, we can consider an intermediate extension $k_i/k''/l$ such that $\psi_{k''}$ is *fppf*, and for any intermediate extension $k''/k'/l$ such that $k' \subsetneq k''$, the morphism $\psi_{k'}$ is the zero morphism. Hence we consider $k'' = k\left(\tau^{p^m}\right)$, and $k' = k\left(\tau^{p^{m+1}}\right)$ with $r > m \geqslant 0$ such that $\psi_{k'}$ is the zero morphism, and $\psi_{k''}$ is *fppf*. Then, there is a *fppf* morphism:

$$\mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathbb{G}_{m,l}}\right)\Big/\mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k'/l}\left(\mathbb{G}_{m,k'}\right)}{\mathbb{G}_{m,l}}\right) \to U.$$

And,

$$\mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathbb{G}_{m,l}}\right)\Big/\mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k'/l}\left(\mathbb{G}_{m,k'}\right)}{\mathbb{G}_{m,l}}\right) = \mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathcal{R}_{k'/l}\left(\mathbb{G}_{m,k'}\right)}\right).$$

Finally, $\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right) = \mathcal{R}_{k'/l}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)\right)$, thus

$$\mathcal{R}_{l/k}\left(\frac{\mathcal{R}_{k''/l}\left(\mathbb{G}_{m,k''}\right)}{\mathcal{R}_{k'/l}\left(\mathbb{G}_{m,k'}\right)}\right) = \mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)}{\mathbb{G}_{m,k'}}\right).$$

This finish the proof of the proposition with $t = \tau^{p^r}$, and $n = r - m$. ☐

Let us study the groups that appear in the statement of Proposition 2.15.

Lemma 2.16. *We consider the k-group $U = \mathcal{R}_{k'/k}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right)$ as in the statement of Proposition 2.15 and an algebraic extension K/k. Then:*

(i) If k'' and K are linearly disjoint over k, then U_K is K-wound.

(ii) If U_K is K-split, then there is a field morphism $k'' \to K$ over k.

Proof. Let us show (i). If k'' and K are linearly disjoint over k, then

$$U_K = \mathcal{R}_{k'\otimes_k K/K}\left(\frac{\mathcal{R}_{k''\otimes_k K/k'\otimes_k K}\left(\mathbb{G}_{m,k''\otimes_k K}\right)}{\mathbb{G}_{m,k'\otimes_k K}}\right).$$

And the tensor products $k''\otimes_k K$, and $k'\otimes_k K$ are both fields. As, $k''\otimes_k K/k'\otimes_k K$ is a purely inseparable extension, $\mathcal{R}_{k''\otimes_k K/k'\otimes_k K}\left(\mathbb{G}_{m,k''\otimes_k K}\right)/\mathbb{G}_{m,k'\otimes_k K}$ is a $k'\otimes_k K$-wound unipotent $k'\otimes_k K$-group, hence U_K is K-wound.

We consider K/k be a field extension, we denote $E = k'' \cap K$. Then $E = k\left(t^{1/p^r}\right)$ for some $n \geqslant r \geqslant 0$ [Sta19, Lemma 0EXP]. If E is a subfield of k', then there is an *fppf* morphism

$$U_E \to \mathcal{R}_{k'/E}\left(\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right).$$

Hence, U_E is not E-wound, moreover as k'' and K are linearly independent over E, the K-group U_K is not K-wound either. If $E = k''$, then there is an exact sequence:

$$0 \to \mathcal{U} \to U_{k''} \to \left[\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right]_{k''} \to 0,$$

where $\mathcal{U}$ is a k''-split unipotent k''-group [Oes84, Cor. A.3.5]. Moreover, the unipotent k''-group $\left[\mathcal{R}_{k''/k'}\left(\mathbb{G}_{m,k''}\right)/\mathbb{G}_{m,k'}\right]_{k''}$ is also k''-split, hence $U_{k''}$ is k''-split. □

A minimal splitting field exists for unipotent group of dimension 1 [Rus70, Lem. 1.1]. But, in general there is no minimal splitting field (see the example below).

Example 2.17. Let us fix $k = \mathbb{F}_p(t_1, t_2)$, we consider U the unipotent k-subgroup of $\mathbb{G}_{a,k}^3 = \mathrm{Spec}(k[x,y,z])$, defined by the p-polynomial $t_1 y^{p^2} + t_2 z^p - x^p - x = 0$. Then, there are two exact sequences of k-group:

$$0 \to G \to U \xrightarrow{p_z} \mathbb{G}_{a,k} \to 0,$$

and

$$0 \to G' \to U \xrightarrow{p_y} \mathbb{G}_{a,k} \to 0,$$

where $p_z : (x,y,z) \mapsto z$ and $p_y : (x,y,z) \mapsto y$ are projections. Then, the kernel G of p_z is isomorphic to the form of $\mathbb{G}_{a,k}$ defined by the equation $y^{p^2} = x + t_1^{p-1}x^p$, likewise the kernel G' of p_y is isomorphic to the form of $\mathbb{G}_{a,k}$ defined by the equation $z^p = x + t_2^{p-1}x^p$.

Thus, U_K is K-split for $K = k\left(t_1^{1/p^2}\right)$ and $K = k\left(t_2^{1/p}\right)$. If there is a smallest field K such that U_K is K-split, then U would be k-split. But U is k-wound [CGP15, Lem. B.1.7].

Proposition 2.18. *Let k'/k be a purely inseparable extension of degree p. Then, the unirational unipotent k-group $\mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)/\mathbb{G}_{m,k}$ is minimal.*

Remark 2.19. Let us remark that in two particular case Proposition 2.18 is already well known. Indeed, if k is a local function field, see [Oes84, VI 3.1 Th.]. If k is a global function field, see [Ngu11, Th. 10, and Cor. 11] and [Ros18, Pro. 5.16].

Proof. As minimal over k_s imply minimal over k, we can assume that k is separably closed. We denote $\mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)/\mathbb{G}_{m,k}$ as U, we consider the k-group

$$V = \mathcal{R}_{K'/k}\left(\mathcal{R}_{K''/K'}\left(\mathbb{G}_{m,K''}\right)/\mathbb{G}_{m,K'}\right),$$

where $K'' = k\left(t^{1/p^n}\right)$, and $K' = k\left(t^{1/p^{n-1}}\right)$ for some $t \in k \smallsetminus k^p$. We have to prove that any morphism of k-group $f : V \to U$ is either constant or *fppf* (Proposition 2.15).

First, if k' and K'' are linearly independent, then $f_{k'}$ is the trivial morphism. Hence, f is constant. Let's assume that it's not the case, i.e. that k' is a subfield of K''. Then, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_{m,k} & \longrightarrow & \mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right) & \longrightarrow & U & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle N_{K'/k}} & & \uparrow{\scriptstyle \mathcal{R}(N_{K''/K'})} & & & & \\
0 & \longrightarrow & \mathcal{R}_{K'/k}\left(\mathbb{G}_{m,K'}\right) & \longrightarrow & \mathcal{R}_{K''/k}\left(\mathbb{G}_{m,K''}\right) & \longrightarrow & V & \longrightarrow & 0
\end{array}$$

where the lines are exacts. The vertical morphisms $N_{K'/k}$ and $\mathcal{R}\left(N_{K''/K'}\right)$ are both *fppf* morphism induced by the p^{n-1}-power character of the multiplicative group, according to the universal property of quotient they induce a morphism $N : V \to U$ that is also *fppf*.

Next, the exact sequence of commutative k-group scheme:

$$0 \to \mathbb{G}_{m,k} \to \mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right) \to U \to 0,$$

imply the exact sequence of commutative group [DG70, III §4 Pro. 4.4 and Pro. 4.5]:

$$0 \to \mathbb{G}_{m,k}\left(V\right) \to \mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)\left(V\right) \to U\left(V\right) \to \mathrm{Pic}(V).$$

As V is a unipotent k-group, the identity element of $e \in V(k)$ induce an isomorphism:

$$\mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)\left(V\right)/\mathbb{G}_{m,k}\left(V\right) \cong \mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)\left(k\right)/\mathbb{G}_{m,k}\left(k\right).$$

Thus, $\mathcal{R}_{k'/k}\left(\mathbb{G}_{m,k'}\right)\left(V\right)/\mathbb{G}_{m,k}\left(V\right) \cong U(k)$. Let us compute $\mathrm{Pic}(V)$, we use the same exact sequence as in Subsection 1.2:

$$0 \to \mathcal{O}(V)^* \to \mathcal{O}(\mathcal{R}_{K''/k}\left(\mathbb{G}_{m,K''}\right))^* \to \widehat{\mathcal{R}_{K'/k}\left(\mathbb{G}_{m,K'}\right)} \to \mathrm{Pic}(V) \to \mathrm{Pic}\left(\mathcal{R}_{K''/k}\left(\mathbb{G}_{m,K''}\right)\right).$$

Here, $\mathcal{O}(V)^* = k^*$, moreover $\mathcal{O}(\mathcal{R}_{K''/k}\left(\mathbb{G}_{m,K''}\right))^*/k^*$ is a $\mathbb{Z}$-module generated by the norm $N_{K''/k}$. And, as $\mathcal{R}_{K''/k}\left(\mathbb{G}_{m,K''}\right)$ identify with an open subscheme of $\mathbb{A}_k^{[K'':k]}$, its Picard group is trivial. Hence, $\mathrm{Pic}(V)$ is $\mathbb{Z}/p\mathbb{Z}$ and its elements are induce by power of the norm $N_{K'/k}$. Finally, let us look at the morphism f. If the image of f in $\mathrm{Pic}(V)$ is the identity element of $\mathrm{Pic}(V)$, then it is a constant morphism, else it is *fppf*. □

Proposition 2.20. *We consider $t \in k_s$ such that $t \notin k_s^p$, and an integer $n \geqslant 1$, we denote $k'' = k\left(t^{1/p^n}\right)$, and $k' = k\left(t^{1/p^{n-1}}\right)$.*

If T is a k'-tori of dimension 1, then $\mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}(T_{k''})}{T}\right)$ is a minimal unirational unipotent k-group.

Proof. First, as $\frac{\mathcal{R}_{k''/k'}(T_{k''})}{T}$ is a k'-wound rational k'-group, then $\mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}(T_{k''})}{T}\right)$ is also a k-wound rational unipotent k-group (Lemma 2.4).

Let us show minimality. We consider $\tau \in k_s$ such that $\tau^p \notin k(\tau)$, an integer $m \geqslant 1$, we denote $K'' = k\left(\tau^{1/p^m}\right)$, and $K' = k\left(\tau^{1/p^{m-1}}\right)$. We have to show that any non zero morphism

$$\mathcal{R}_{K'/k}\left(\frac{\mathcal{R}_{K''/K'}\left(\mathbb{G}_{m,K''}\right)}{\mathbb{G}_{m,K'}}\right) \xrightarrow{f} \mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}\left(T_{k''}\right)}{T}\right)$$

is *fppf* (Proposition 2.15).

We are going to reduce the problem to the case where t and τ are in the base field, i.e. k'' and K'' are purely inseparable extensions. Let L be the Galois closure of the compositum $k(t).k(\tau)$, then $k(t) \otimes_k L = \prod_{\sigma \in X(k(t))} L^\sigma$ where $X(k(t))$ denotes the set of morphisms of k-algebra $\sigma : k(t) \to L$, and L^σ denote L with the $k(t)$-algebra structure defined by σ.

Likewise, $k(\tau) \otimes_k L = \prod_{\gamma \in X(k(\tau))} L^\gamma$, with the same notations as above. Then, if we denote $U = \mathcal{R}_{K'/k(\tau)}\left(\frac{\mathcal{R}_{K''/K'}(\mathbb{G}_{m,K''})}{\mathbb{G}_{m,K'}}\right)$, we have $\mathcal{R}_{K'/k}\left(\frac{\mathcal{R}_{K''/K'}(\mathbb{G}_{m,K''})}{\mathbb{G}_{m,K'}}\right) = \mathcal{R}_{k(\tau)/k}(U)$. And,

$$\mathcal{R}_{k(\tau)/k}(U)_L = \mathcal{R}_{k(\tau)\times_k L/L}(U \times_k \operatorname{Spec}(L)) = \mathcal{R}_{\prod_{X(k(\tau))} L^\gamma / L}\left(\coprod_{\gamma \in X(k(\tau))} U \times_{k(\tau)} \operatorname{Spec}(L^\gamma)\right) = \prod_{\gamma \in X(k(\tau))} U \times_{k(\tau)} \operatorname{Spec}(L^\gamma).$$

And likewise for $V = \mathcal{R}_{k'/k(t)}\left(\frac{\mathcal{R}_{k''/k'}(T_{k''})}{T}\right)$. So, after extension to L, the k-morphism f induces a L-morphism:

$$\prod_{\gamma \in X(k(\tau))} U \times_{k(\tau)} \operatorname{Spec}(L^\gamma) \xrightarrow{f_L} \prod_{\sigma \in X(k(t))} V \times_{k(t)} \operatorname{Spec}(L^\sigma).$$

Thus, f_L induces a family of morphisms of L-groups: for $(\gamma, \sigma) \in X(k(\tau)) \times X(k(t))$,

$$U \times_{k(\tau)} \operatorname{Spec}(L^\gamma) \xrightarrow{f_\sigma^\gamma} V \times_{k(t)} \operatorname{Spec}(L^\sigma).$$

Moreover, the group $G = \operatorname{Gal}(L/k)$ acts naturally on both $X(k(\tau))$ and $X(k(t))$, and f_L is G-equivariant. As f is not the zero morphism, f_L is also not the zero morphism. Hence, at least one of the f_σ^γ is non zero. By transitivity of the action of G on $X(k(t))$, we obtain a family of non zero morphisms

$$\mathcal{R}_{L'/L}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right) \to \mathcal{R}_{l'/L}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right),$$

where $L' = L\left(\tau^{1/p^{m-1}}\right)$, and $L'' = L\left(\tau^{1/p^m}\right)$, and $l' = L\left(t^{1/p^{n-1}}\right)$, and finally $l'' = L\left(t^{1/p^n}\right)$.

Let us now consider a non zero L-morphism

$$\mathcal{R}_{L'/L}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right) \xrightarrow{F} \mathcal{R}_{l'/L}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right).$$

We are going to make a recurrence argument on the degree of the extension l'/L. If $l' = L$, then the unirational L-group $\mathcal{R}_{l''/l'}(T_{l''})/T_{l'}$ is minimal (Proposition 2.18), hence F is *fppf*.

Else, $\left[\mathcal{R}_{L'/L}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right)\right]_{L''}$ is a L''-split unipotent L''-group [Oes84, Pro. A.3.5]. As $F_{L''}$ is non zero, the L''-group $\mathcal{R}_{l'/L}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right)_{L''}$ is not L''-wound. Hence L'' and l'' are not linearly disjoint (Lemma 2.16), so $L\left(t^{1/p}\right) = L\left(\tau^{1/p}\right)$, and we denote this field by $\tilde{L}$.

According to the functorial definition of the Weil restriction, the L-morphism F corresponds to some non zero $\tilde{L}$-morphism:

$$\left[\mathcal{R}_{L'/L}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right)\right]_{\tilde{L}} \xrightarrow{\mathcal{F}} \mathcal{R}_{l'/\tilde{L}}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right).$$

As this morphism is non-zero and the right hand side is $\tilde{L}$-wound, the left hand side is not $\tilde{L}$-split. Hence, $\tilde{L} \subseteq L'$, and there is an exact sequence:

$$0 \to \mathcal{U} \to \left[\mathcal{R}_{L'/L}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right)\right]_{\tilde{L}} \to \mathcal{R}_{L'/\tilde{L}}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right) \to 0,$$

where $\mathcal{U}$ is a $\tilde{L}$-split unipotent group [Oes84, Cor. A.3.5].

As $\mathcal{U}$ is $\tilde{L}$-split and $\mathcal{R}_{l'/\tilde{L}}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right)$ is $\tilde{L}$-wound, the morphism $\mathcal{F}$ induces a non zero $\tilde{L}$-morphism:

$$\mathcal{R}_{L'/\tilde{L}}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right) \xrightarrow{\mathfrak{F}} \mathcal{R}_{l'/\tilde{L}}\left(\frac{\mathcal{R}_{l''/l'}(T_{l''})}{T_{l'}}\right).$$

By recurrence hypothesis, the morphism $\mathfrak{F}$ is a *fppf* between smooth $\tilde{L}$-group. Thus, $\mathfrak{F}\left(\mathcal{R}_{L'/\tilde{L}}\left(\frac{\mathcal{R}_{L''/L'}(\mathbb{G}_{m,L''})}{\mathbb{G}_{m,L'}}\right)(\tilde{L}_s)\right)$ is Zariski dense in $\mathcal{R}_{l'/\tilde{L}}\left(\frac{\mathcal{R}_{l''/l'}(\mathbb{G}_{m,l''})}{\mathbb{G}_{m,l'}}\right)$. And, Finally, $F = \mathcal{R}_{\tilde{L}/L}(\mathfrak{F})$ is dominant (the image of the L_s rational point is dense). As F is a group morphism it is surjective, hence it is *fppf* [EGAIV2, Th. 6.9.1]. □

We consider, $U = \mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}(\mathbb{G}_{m,k''})}{\mathbb{G}_{m,k'}}\right)$ where k'/k and k''/k' are nontrivial purely inseparable extensions as in the statement of Proposition 2.15. Then, U is a minimal unirational unipotent k-group (Proposition 2.18, and Proposition 2.20). Let us consider $V(T,K) = \mathcal{R}_{K/k}(T_K)/T$ where T is a k-torus and K/k is a purely inseparable extension. If U and $V(T,K)$ are isomorphic, then for dimension reason $[K:k] = p$. And, $V(T,K)_K$ is K-split, but U_K isn't. Hence, they can't be isomorphic. Finally, as V is unirational and U minimal the only subgroup of U of type $V(T,K)$ is $\{0\}$. Thus, the answer to the second part of J. Oesterlé Question [Oes84, p. 80] is negative. We now have a slightly more general question:

Question 2.21. *Are the minimal unirational unipotent k-group all isomorph to a group of the type $\mathcal{R}_{k'/k}\left(\frac{\mathcal{R}_{k''/k'}(T_{k''})}{T}\right)$ where T is a dimension 1 k'-torus, $k'' = k\left(t^{1/p^n}\right)$, and $k' = k\left(t^{1/p^{n-1}}\right)$ for some $t \in k_s \smallsetminus k_s^p$, and $n \geqslant 1$?*

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