Retrodictive State Generation and Quantum Measurement

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Abstract. Retrodictive quantum states are states that propagate backwards in time from a measurement event. Although retrodictive quantum mechanics appears to be very different from the usual predictive formalism, in that propagation of states into the past appears to violate causality, this is not so. Indeed causality is not manifest in the time direction of propagation of quantum states at all. Instead, causality is ensured by the different normalization conditions applied to the preparation and measurement device operators. It is this difference that introduces the arrow of time into quantum mechanics. Retrodictive states are useful for applications such as measurement, predictive quantum state engineering and quantum communication. Here we show how any optical retrodictive state that can be expressed to a good approximation in a finite-dimensional Hilbert space can be generated from predictive coherent states, a lossless multiport device and photodetectors. The composition of the retrodictive state can be controlled by adjusting the input predictive coherent states. This allows, for example, projection synthesis for an optical state to be achieved with the exotic reciprocal binomial reference state replaced by a simple coherent state.

INTRODUCTION

In the ideal classical case, if we prepare a system in some state and then measure it before significant evolution can occur, the prepared and measured states will be identical. We can thus unambiguously assign either the prepared or measured state to the system between the times of preparation and measurement. In quantum mechanics with its intrinsic uncertainties, however, the situation is different. Even for ideal preparation, perfect measurement and no evolution, the prepared and measured states can differ significantly so the assignment of the state is not so clear cut. The traditional formulation of quantum mechanics is predictive. We assign a state to a system based on the outcome of a preparation event and allow this state to evolve to a later time until it is measured. The usual probability postulate allows us to calculate the probabilities of various measurement outcomes. The retrodictive formalism [1, 2] of quantum mechanics is equally valid, however, and can be used for calculating the probabilities of preparation events given a particular measurement event. In this formalism, the state between preparation and measurement is assigned on the basis of the measurement outcome. This formalism also leads to results that are verifiable by experiment and is perfectly consistent with the predictive formalism. Thus we appear to have a choice in how we assign the state between preparation and measurement. This choice can be quite sharp. For example, for the Schrödinger's Cat experiment the state of the cat between preparation and measurement in the predictive formalism is a superposition of alive and dead. In the retrodictive formalism the cat is simply alive or dead depending on what the subsequent measurement will reveal. Most physicists, however, feel more comfortable in assigning a predictive state on the basis that a predictive state can be viewed as having been determined by the action of the preparation device and then propagates forwards in time in agreement with our notion of causality. A retrodictive state, on the other hand, is shaped by a future event, that is, the measurement event. Thus a retrodictive state that travels backwards in time appears to violate causality. In this paper we examine this problem explicitly and find there is no conflict. Retrodictive states are interesting in their own right and can, indeed, be quite useful. We also explore a general method for their generation and look at the application of retrodictive states to the measurement of the phase of light.

CAUSALITY IN QUANTUM MECHANICS

It is well known that, in the absence of measurement, the quantum dynamics of a closed system is time symmetric. The arrow of time is usually considered to be associated with the measurement process. In this section we investigate how causality is embedded in quantum mechanics. We consider the situation where Alice prepares a system in some state and sends a label *i* representing the preparation event to a computer. After a time short enough to ensure that the system has not evolved significantly, Bob performs a measurement on it and may or may not send the outcome *j* of the measurement to the computer. If the computer receives an input from both Alice and Bob, it records the combined preparation and measurement event (*i*, *j*). The process is repeated many times on identical systems with Alice preparing any states she chooses. The computer composes a list of combined events. The fundamental postulate of the probabilistic interpretation of symmetric quantum mechanics is that the probability of a combined event (*i*, *j*), as measured by the occurrence frequency on the list, is given by [2]

$$P^{\Lambda\Gamma}(i,j) = \frac{\mathrm{Tr}(\hat{\Lambda}_i \hat{\Gamma}_j)}{\mathrm{Tr}(\hat{\Lambda}\hat{\Gamma})} \tag{1}$$

where $\hat{\Lambda} = \sum_i \hat{\Lambda}_i$ and $\hat{\Gamma} = \sum_j \hat{\Gamma}_j$. The operators, which act on the Hilbert space of the system, are positive or negative definite to ensure the probability is positive. The trace over the Hilbert space of the system ensures that we extract a number that is independent of the order of the two non-commuting operators, thereby maintaining the symmetry. The normalization denominator ensures the probability is between zero and unity. The operators, which we refer to as preparation or measurement device operators, include a description of the state prepared or measured and also the likelihood that the corresponding preparation or measurement event is recorded.

The formula (1) is quite general and includes the case in which some of the measurement results are discarded by Bob and thus not included in the statistics. We easily find from (1) that the probability for the measurement event *i* to be recorded is

$$P^{\Lambda\Gamma}(i) = \sum_j P^{\Lambda\Gamma}(i,j) = \frac{\mathrm{Tr}(\hat{\Lambda}_i \hat{\Gamma})}{\mathrm{Tr}(\hat{\Lambda}\hat{\Gamma})} . \tag{2}$$

Bob has some control over $\hat{\Gamma}$ by selecting the type of measuring device used and whether or not he records particular measurement events. He thus has some control over $P^{\Lambda\Gamma}(i)$ and can use this to send a message to Alice by altering $P^{\Lambda\Gamma}(i)$. As Alice has to wait until the list is composed, Alice can only receive the message after Bob has sent it. Let us now assume that both Alice and Bob faithfully record all the preparation and measurement events that occur. Then $P^{\Lambda\Gamma}(i)$ is equal to the probability that Alice actually prepares a particular state and she can ascertain this probability before looking at the list. For example suppose the series of preparation events takes one hour and Bob waits one day before making the corresponding series of measurements. In this case if Bob can influence $P^{\Lambda\Gamma}(i)$ by his choice of measuring device, he has a means of sending a message to Alice backwards in time, in violation of our notion of causality. For example if the system is a spin, Bob could alter the direction of the field in his Stern-Gerlach apparatus or he may decide not to perform a measurement at all. To preserve causality $\hat{\Gamma}$ must commute with all operators, such as unitary operators for example, that describe a change in the measurement device. It must therefore be proportional to the unit operator $\hat{1}$, that is $\hat{\Gamma} = k\hat{1}$. Then (2) becomes completely independent of $\hat{\Gamma}$ for faithfully recorded measurements, which we refer to as unbiased measurements. We could also prevent Alice from sending a message to Bob by ensuring that $\hat{\Lambda} \propto \hat{1}$, but in accord with causality, we do not wish to do this. Thus causality induces an asymmetry. It is useful to define a normalized set of measurement device operators $\hat{\Pi}_j = \hat{\Gamma}_j / k$.

From (1) and (2) we can obtain the probability of measurement event *j* if preparation event *i* occurs as

$$P^{\Lambda\Gamma}(j\,|\,i) = \frac{P^{\Lambda\Gamma}(i,j)}{P^{\Lambda\Gamma}(i)} = \frac{\mathrm{Tr}(\hat{\Lambda}_i\hat{\Gamma}_j)}{\mathrm{Tr}(\hat{\Lambda}_i\hat{\Gamma})}\,. \tag{3}$$

Then inserting the causality condition $\hat{\Gamma} = k\hat{1}$, we obtain for a faithfully recording measurement procedure,

$$P^{\Lambda\Gamma}(j\,|\,i) = \frac{\mathrm{Tr}(\hat{\Lambda}_i\hat{\Pi}_j)}{\mathrm{Tr}(\hat{\Lambda}_i)} = \mathrm{Tr}(\hat{\rho}_i\hat{\Pi}_j) \tag{4}$$

where $\hat{\rho}_i = \hat{\Lambda}_i / \mathrm{Tr}(\hat{\Lambda}_i)$ must be a positive operator with a trace of unity and is thus a density operator. $\hat{\Pi}_j = \hat{\Gamma}_j / k$ must also be positive definite and sums to unity. It is thus an element of a probability operator measure (POM)[3]. This introduced asymmetry in the normalizations of $\hat{\rho}_i$ and $\hat{\Pi}_j$ ensures causality. We see from (2) with $\hat{\Gamma}$ proportional to the unit operator that the probability $P^{\Lambda}(i) = \mathrm{Tr}(\hat{\Lambda}_i)/\mathrm{Tr}(\hat{\Lambda})$ is independent of the future measurement event and we thus call this the *a priori* probability that Alice chooses event *i*. Thus causality in this form ensures that Alice's choice is influenced only by past events. As $\hat{\Lambda}_i$ can be multiplied by any constant without affecting the measured probabilities, we can normalize it so that $\mathrm{Tr}\Lambda = 1$. This allows us to write from the above expressions $\hat{\Lambda}_i = P^{\Lambda}(i)\hat{\rho}_i$, giving a convenient interpretation of the preparation device operator $\hat{\Lambda}_i$.

TIME DIRECTION OF STATE PROPAGATION

Let us assume now that the difference between the measurement time t_m and the preparation time t_p is long enough for some unitary time evolution to take place. What state should we assign to the system during this time? Traditionally we assign the prepared state $\hat{\rho}_i$ at time t_p immediately following preparation and assume this involves unitarily to $\hat{U}(t,t_p)\hat{\rho}_i\hat{U}^{\dagger}(t,t_p)$ at some later time *t* before measurement. This forward-time evolution tradition has its roots in our concept of causality. There is no experimental justification for it however. In the end we measure probabilities. Incorporating the time dependence into (4) gives

$$P^{\Lambda\Gamma}(j\,|\,i) = \mathrm{Tr}[\hat{U}(t_m,t_p)\hat{\rho}_i\hat{U}^{\dagger}(t_m,t_p)\hat{\Pi}_j]\,. \tag{5}$$

While this result can be verified by experiments, from the cyclic property of the trace we can also write it as

$$P^{\Lambda\Gamma}(j\,|\,i) = \mathrm{Tr}[\hat{\rho}_i\hat{U}^{\dagger}(t_m,t_p)\hat{\Pi}_j\hat{U}(t_m,t_p)] \tag{6}$$

which can be interpreted as the unnormalised measured state propagating backwards in time from the measurement time until it is projected onto the prepared state at the preparation time. We refer to such a state as a retrodictive state. This interpretation yields the same experimental results and thus states propagating backwards in time do not violate causality. *Causality in quantum mechanics is not manifest in the time direction of state propagation, it is ensured by the difference in normalization conditions for the preparation and measurement device operators.* It is this difference, and not the direction of time of the state propagation, that determines the time direction of information propagation.

RETRODICTIVE STATES AND MEASUREMENT

Assume that the measurement events *j* that are retained in the statistics form just a subset of all measurement events. That is, if the result of a particular measurement does not belong to the subset *j* then the result is discarded by Bob and not recorded. Consider the case where the measurement events *j* generate retrodictive states that evolve backwards in time to become $\hat{\Gamma}_j(t_p)$ at the time of preparation and let the state $\hat{\rho}_i = \hat{\Lambda}_i / \mathrm{Tr}(\hat{\Lambda}_i)$ be associated with the preparation event *i*. The probability that a measurement event in the set of retained results is *j* if the preparation event is *i* is given, from (3), by

$$P^{\Lambda\Gamma}(j\,|\,i) = \frac{\mathrm{Tr}[\hat{\rho}_i \hat{\Gamma}_j(t_p)]}{\sum_j \mathrm{Tr}[\hat{\rho}_i \hat{\Gamma}_j(t_p)]} \tag{7}$$

Suppose now that the subset $\hat{\Gamma}_j(t_p)$ of retrodictive states sums to the unit operator on a limited Hilbert subspace and that the state $\hat{\rho}_i$ is reasonably well contained in this subspace. Then (7) becomes simply

$$P^{\Lambda\Gamma}(j\,|\,i) = \mathrm{Tr}[\hat{\rho}_i \hat{\Gamma}_j(t_p)] \tag{8}$$

This is just the probability associated with the measurement of an observable on the limited Hilbert space represented by a POM with elements $\hat{\Gamma}_j(t_p)$. Thus, provided the outcome of the measurement is one of the subset *j*, we can say that this measurement is a successful single-shot measurement for a system in predictive state $\hat{\rho}_i$ of the observable represented by this POM with the result of the measurement being the value associated with $\hat{\Gamma}_j(t_p)$. As a specific example, suppose we wish to measure the phase of a very weak state of light. We refer here to the canonical phase, which is the complement of photon number. The corresponding observable is the Hermitian phase operator [4] in an ($s+1$)-dimensional Hilbert space, where s determines the accuracy required. The retrodictive states required are $|\theta_j\rangle\langle\theta_j|$ where $|\theta_j\rangle$ are the truncated phase states with $j = 0,1,\ldots,s$. With the phase window chosen as [0, 2π], the associated phase eigenvalues are $j2\pi/(s+1)$. We would need to associate each value of j with a particular identifiable measurement event.

Another use of retrodictive states is for projection synthesis [5], by means of which the probability distribution of some observable can be measured. Here we need only generate a single phase state, for example, and measure the probability of the measurement event associated with its generation. The probability distribution is then obtained by recording this probability as the phase is shifted by incremental steps with a phase shifter.

RETRODICTIVE STATE GENERATION

We limit ourselves to discussing optical states. The standard quantum measuring device is a photodetector. If we have a detector in each mode of interest, then the retrodictive state at the measurement time will be a product of states of the type $|n\rangle$ where *n* is the number of photons detected by a particular detector. In order to have reasonable detection efficiency, we limit our detectors to being able to distinguish among zero, one and more than one photocounts. Thus $n = 0$ or 1 for the detection events of interest. We now require a way of ensuring that the retrodictive product state evolves backwards in time to the required form $\hat{\Gamma}_j(t_p)$. We consider first the simpler case where $\hat{\Gamma}_j(t_p)$ represents a single state, such as a truncated phase state, arising from one measurement event.

Consider a lossless multiport device comprising mirrors, beam splitters and phase shifters with input modes and output modes labelled 0, 1, …, *N* and a detector in each output mode. This is illustrated in Fig. 1. In input modes 1, …, *N* are adjustable predictive control states which will be used to control the retrodictive state that is generated in input mode 0. We let this combined control state be $|C\rangle = |c_1\rangle_1 |c_2\rangle_2 \ldots |c_N\rangle_N$. To be specific we consider the measurement event to be the detector in mode 0 detecting zero photons with each other detector detecting one photon. The retrodictive state produced is $|\Psi\rangle = |0\rangle_0 |1\rangle_1 |1\rangle_2 \ldots |1\rangle_N$. The evolution of this state back through the multiport can be described by a unitary operator acting on the total space of the $N+1$ optical modes. At the input of the device the state becomes $\hat{S}^\dagger|\Psi\rangle$, which will in general be a multimode entangled state. At this point it is projected onto the prepared control state $|C\rangle$ to leave the unnormalized retrodictive state in input mode 0 given by

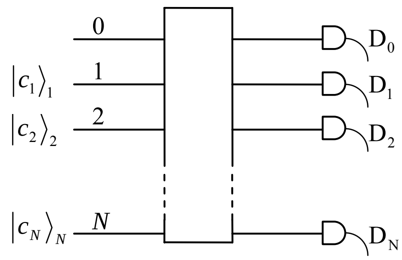

FIGURE 1. Schematic diagram of a multiport device for generating retrodictive states of light. There are $N + 1$ input modes and $N + 1$ output modes. In input modes 1, 2, …, N are N predictive control states. In output modes 0, 1, 2, …, N are photodetectors D_0, D_1, …, D_N. If the control states are appropriately chosen, the (N + 1)-mode retrodictive state associated with a particular photon detection pattern at the detectors is transformed into the required single-mode retrodictive state in input mode 0.

$\langle C|\hat{S}^\dagger|\Psi\rangle$. We require this state to be proportional to a truncated phase state, for example. To achieve this, we can adjust $|C\rangle$ or $\hat{S}^\dagger$ or both. Adjusting the latter means changing the multiport hardware. Fortunately an important theorem of Reck *et al.* [6] shows that it is possible to construct a multiport to give any desired unitary transformation $\hat{S}^\dagger$. It turns out [7] that in adjusting $|C\rangle$ and $\hat{S}^\dagger$ we have more flexibility than we actually need to produce any retrodictive state that we like in a Hilbert space of $N + 1$ dimensions. This allows us to choose a convenient multiport configuration and convenient control states. The natural control states to choose are the easily produced coherent states with adjustable intensities and phases. The latter can be adjusted by simple phase shifters. Even after making this choice there is still sufficient flexibility to alter the multiport to optimize the probability of the detection event required for a successful retrodictive state generation. The details of this optimization procedure can be found in [7]. In addition to optimizing the multiport configuration for a particular state, we can find a usefully versatile configuration which does not differ greatly from the optimum configuration for a broad range of retrodictive states. For this, it is useful to write the action of the multiport unitary operator $\hat{S}^\dagger$ in the form [7]

$$\hat{S}^\dagger \hat{a}_n^\dagger \hat{S} = \sum_{m=0}^{N} U_{nm}^* \hat{a}_m^\dagger \tag{9}$$

where $\hat{a}_n^\dagger$ is the photon annihilation operator for mode n and U_{nm} are the elements of a unitary matrix. The useful versatile configuration is such that the multiport device performs a discrete Fourier transform, that is, U_{nm} is given by

$$U_{nm} = \frac{\exp[inm2\pi/(N+1)]}{\sqrt{N+1}} \tag{10}$$

and the set of transformed operators then form a discrete Fourier transform pair. Such a multiport can be thought of as the natural generalization of the 50:50 symmetric beam splitter. The N coherent control states required can be obtained from a single coherent state by means of a series of beam splitters and phase shifters. This can incorporated into the multiport device rather than being used separately. Thus we can achieve projection synthesis with just a *single coherent* control state and $N - 1$ vacuum inputs. This can be compared with the original projection synthesis device comprising a beam splitter and a reciprocal binomial control state [5].

In the above example in which we applied retrodictive state generation to projection synthesis for the measurement of a probability distribution, there was a substantial amount of flexibility in the choice of control states and the multiport configuration. To obtain a single-shot measurement in the sense of our discussion of expression (8), we must sacrifice this degree of flexibility. Howbeit, we find that it is still possible to make a single-shot measurement of optical phase. This can be achieved with a multiport that performs a discrete Fourier transform as described above with photodetectors that can distinguish among zero, one and many photons. The required control states are a binomial state with appropriate phase in one input mode, say mode 1, and vacuum states in input modes $2,3,\ldots,N$. The retrodictive state generated is projected onto the state to be measured in input mode 0. A successful measurement occurs when all the photodetectors except one record one photocount and the other detector, D_m say, records zero. In this case the retrodictive state generated in input mode 0 is the truncated phase state $|\theta_m\rangle$. The associated phase eigenvalue is $m2\pi/(N+1)$, which is the result of the measurement. The details are given in Ref. [8].

CONCLUSION

We have examined how causality, or the arrow of time, is contained in quantum mechanics. We have found that it lies in the differing normalization conditions on the preparation device and measurement device operators. It is *not* associated with the time direction of propagation of the state. Thus it is as justifiable to assign a state on the basis of a measurement event as on the basis of a preparation event. States assigned on the basis of measurement events are called retrodictive states and propagate backwards in time from measurement to preparation. Even though they are shaped by future events, they cannot be used to send a message into the past. They are useful, however, for other purposes such as measurement, quantum communication and predictive state engineering. The last use arises because exotic retrodictive states are easier to generate than their predictive counterparts [7]. Consider a preparation device that, in the predictive formalism, produces a two-mode entangled state. If we generate a retrodictive state in one of these modes, the predictive state in the other mode will be given by the projection of the retrodictive state onto the two-mode entangled predictive state. By suitably controlling the retrodictive state, the desired predictive state can be engineered in the other mode. In this paper we have concentrated on the use of retrodictive states for the measurement of optical states, both for obtaining the probability distribution and for single-shot measurement, with the particular example being the phase of light. We conclude that retrodictive quantum states are as legitimate as their far more common predictive counterparts. Moreover, for some applications they are far more useful.

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REFERENCES

1. Watanabe, S., *Rev. Mod. Phys*., 27, 179 (1955); Aharonov, Y., Bergman, P. G., and Lebowitz, J. L., *Phys. Rev.,* 134, B1410 (1964); Barnett, S. M., Pegg, D. T., and Jeffers, J., *J. Mod. Optics,* 47, 1779 (2000); Barnett, S. M., Pegg, D. T., Jeffers, J., and Jedrkiewicz, O. , *Phys. Rev. Lett.,* 86, 2455 (2001).
2. Pegg, D. T., Barnett, S. M., and Jeffers, J., *J. Mod. Optics,* 49, 913 (2002).
3. Helstrom, C. W., *Quantum Detection and Estimation Theory*, Academic Press, New York, 1976.
4. Pegg, D. T., and Barnett, S. M., *Europhysics Lett*., 6, 483 (1988); Barnett, S. M., and Pegg, D. T., *J. Mod. Optics*, 36, 7, (1989); Pegg, D. T., and Barnett, S. M., *Phys. Rev. A*, 39, 1665 (1989).
5. Barnett, S. M. and Pegg, D. T., *Phys. Rev. Lett.,* 76, 4148 (1996).
6. Reck, M., Zeilinger A., Bernstein H. J., and Bertani P., *Phys. Rev. Lett.,* 73, 58 (1994).
7. Pregnell, K. L. and Pegg, D. T., *J. Mod. Optics,* 51, 1613, (2004).
8. Pregnell, K. L. and Pegg, D. T., *Phys. Rev. Lett.,* 89, 173601 (2002).