Some formulas for determinants of tridiagonal matrices in terms of finite generalized continued fractions

Feng Qi, Wang Wen, Lim Dongkyu, Bai-Ni Guo

SOME FORMULAS FOR DETERMINANTS OF TRIDIAGONAL MATRICES IN TERMS OF FINITE GENERALIZED CONTINUED FRACTIONS

FENG QI, WEN WANG, DONGKYU LIM, AND BAI-NI GUO

Abstract. In the paper, by virtue of induction and properties of determinants, the authors discover explicit and recurrent formulas of evaluations for determinants of general tridiagonal matrices in terms of finite generalized continued fractions and apply these formulas to evaluations for determinants of the Sylvester matrix and two Sylvester type matrices.

Contents

1. Introduction

A finite generalized continued fraction is an expression of the form

$$q_0+\cfrac{p_1}{q_1+\cfrac{p_2}{q_2+\cfrac{p_3}{\ddots\cfrac{\ddots}{q_{m-2}+\cfrac{p_{m-1}}{q_{m-1}+\cfrac{p_m}{q_m}}}}}},$$

where $q_0,q_1,\dots,q_m$ and $p_1,p_2,\dots,p_m$ can generally be integers, real numbers, complex numbers, or functions. It can also be written in the forms

$$q_0+\mathop{K}_{\ell=1}^{m}\frac{p_\ell}{q_\ell}=q_0+\frac{p_1}{q_1+}\frac{p_2}{q_2+}\cdots\frac{p_{m-1}}{q_{m-1}+}\frac{p_m}{q_m}=q_0+\sum_{\ell=1}^{m}\frac{p_\ell|}{|q_\ell}.$$

In this paper, we adopt the second form above. For more information on continued fractions, please refer to [7, 14] and closely related references therein.

2010 *Mathematics Subject Classification.* Primary 15A15; Secondary 11C20, 15B05, 15B99, 65F40.

Key words and phrases. determinant; tridiagonal matrix; induction; explicit formula; recurrent formula; finite generalized continued fraction; Sylvester matrix; Sylvester type matrix.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Generally, a tridiagonal matrix of order n for $n \in \mathbb{N}$ is defined by

$$D_n = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_2 & a_3 & b_3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & a_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-3} & b_{n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c_{n-3} & a_{n-2} & b_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{n-1} & a_n \end{pmatrix} = \left(d_{i,j}\right)_{1 \le i,j \le n}, \tag{1.1}$$

where

$$d_{i,j} = \begin{cases} a_i, & 1 \le i = j \le n; \\ b_i, & 1 \le i = j - 1 \le n - 1; \\ c_j, & 1 \le j = i - 1 \le n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

In [15, 16, 17], the determinant $|D_n|$ and some special cases were discussed, computed, and applied to several problems in combinatorial number theory. In [2, 5, 6, 9, 15, 16, 17], there have been some computation of the inverse and determinant of the general tridiagonal matrix D_n. For more information about this topic, please refer to [4, 8, 12, 13] and closely related references therein.

Let $n \ge 2$ and

$$P_n = \begin{pmatrix} \alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\ \alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\ \alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\ \alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n \end{pmatrix} = \left(p_{i,j}\right)_{1 \le i,j \le n}, \tag{1.2}$$

where

$$p_{i,j} = \begin{cases} \alpha_i, & 1 \le i \le n, j = 1; \\ \beta_i, & 2 \le i = j \le n; \\ \gamma_i, & 1 \le i = j - 1 \le n - 1; \\ 0, & \text{otherwise.} \end{cases}$$

In this paper, by virtue of induction and properties of determinants, we will discover explicit and recurrent formulas for evaluations of determinants $|P_n|$ and $|D_n|$ and will apply these formulas to evaluations for determinants of the Sylvester matrix and two Sylvester type matrices.

2. Recurrent and explicit formulas for $|P_n|$

Now we start out to establish recurrent and explicit formulas for the determinant $|P_n|$ of the matrix P_n.

Theorem 2.1. *Let $n \ge 2$ and $\beta_k \ne 0$ for $2 \le k \le n$. Then the determinant $|P_n|$ of the matrix P_n can be computed recursively by*

$$|P_n| = \lambda_{1,n} \prod_{k=2}^{n} \beta_k, \tag{2.1}$$

where

$$\lambda_{k,n} = \alpha_k - \frac{\gamma_k}{\beta_{k+1}} \lambda_{k+1,n}, \quad 1 \le k \le n-1 \tag{2.2}$$

and $\lambda_{n,n} = \alpha_n$.

Proof. When $n = 2$, we have

$$|P_2| = \begin{vmatrix} \alpha_1 & \gamma_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\gamma_1$$

and

$$\lambda_{1,2} \prod_{k=2}^{2} \beta_k = \lambda_{1,2}\beta_2 = \left(\alpha_1 - \frac{\gamma_1\lambda_{2,2}}{\beta_2}\right)\beta_2 = \left(\alpha_1 - \frac{\gamma_1\alpha_2}{\beta_2}\right)\beta_2 = \alpha_1\beta_2 - \alpha_2\gamma_1 = |P_2|.$$

The formula (2.1) is thus valid for $n = 2$.

We now assume that the formula (2.1) is valid for $n = m - 1$, that is,

$$|P_{m-1}| = \lambda_{1,m-1} \prod_{k=2}^{m-1} \beta_k.$$

When $n = m$, expanding $|P_m|$ according to the first rank and employing the above assumption for $n = m - 1$ yield

$$|P_m| = \alpha_1 \prod_{k=2}^{m} \beta_k - \gamma_1 \begin{vmatrix} \alpha_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_3 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\ \alpha_4 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-3} & 0 & 0 & \cdots & \beta_{m-3} & \gamma_{m-3} & 0 & 0 \\ \alpha_{m-2} & 0 & 0 & \cdots & 0 & \beta_{m-2} & \gamma_{m-2} & 0 \\ \alpha_{m-1} & 0 & 0 & \cdots & 0 & 0 & \beta_{m-1} & \gamma_{m-1} \\ \alpha_m & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_m \end{vmatrix}$$

$$= \alpha_1 \prod_{k=2}^{m} \beta_k - \gamma_1\lambda_{2,m} \prod_{k=3}^{m} \beta_k = \left(\alpha_1 - \frac{\gamma_1}{\beta_2}\lambda_{2,m}\right) \prod_{k=2}^{m} \beta_k = \lambda_{1,m} \prod_{k=2}^{m} \beta_k.$$

By induction, the formula (2.1) follows. The proof of Theorem 2.1 is complete. □

Theorem 2.2. *For $n \ge 2$, the determinant $|P_n|$ of the matrix P_n can be computed explicitly by*

$$|P_n| = \alpha_1 \prod_{k=2}^{n} \beta_k - \sum_{k=2}^{n} (-1)^k \left(\prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n} \beta_m \right) \alpha_k. \tag{2.3}$$

Proof. From the recurrence relation (2.2), it follows that

$$\begin{aligned} \lambda_{1,n} &= \alpha_1 - \frac{\gamma_1}{\beta_2}\lambda_{2,n} \\ &= \alpha_1 - \frac{\gamma_1}{\beta_2}\left(\alpha_2 - \frac{\gamma_2}{\beta_3}\lambda_{3,n}\right) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl[\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl(\alpha_3 - \frac{\gamma_3}{\beta_4}\lambda_{4,n}\biggr)\biggr]\\
&= \cdots\\
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl[\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl(\alpha_3 - \frac{\gamma_3}{\beta_4}\biggl[\alpha_4 - \cdots - \frac{\gamma_{\ell-1}}{\beta_\ell}\biggl(\alpha_\ell - \frac{\gamma_\ell}{\beta_{\ell+1}}\lambda_{\ell+1,n}\biggr)\biggr]\biggr)\biggr]\\
&= \cdots\\
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl[\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl(\alpha_3 - \frac{\gamma_3}{\beta_4}\biggl[\alpha_4 - \cdots - \frac{\gamma_{\ell-1}}{\beta_\ell}\biggl(\alpha_\ell - \cdots\\
&\quad - \frac{\gamma_{n-3}}{\beta_{n-2}}\biggl[\alpha_{n-2} - \frac{\gamma_{n-2}}{\beta_{n-1}}\biggl(\alpha_{n-1} - \frac{\gamma_{n-1}}{\beta_n}\lambda_{n,n}\biggr)\biggr]\biggr)\biggr]\biggr)\biggr]\\
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl[\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl(\alpha_3 - \frac{\gamma_3}{\beta_4}\biggl[\alpha_4 - \cdots - \frac{\gamma_{\ell-1}}{\beta_\ell}\biggl(\alpha_\ell - \cdots\\
&\quad - \frac{\gamma_{n-3}}{\beta_{n-2}}\biggl[\alpha_{n-2} - \frac{\gamma_{n-2}}{\beta_{n-1}}\biggl(\alpha_{n-1} - \frac{\gamma_{n-1}}{\beta_n}\alpha_n\biggr)\biggr]\biggr)\biggr]\biggr)\biggr]\\
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl(\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl[\alpha_3 - \frac{\gamma_3}{\beta_4}\biggl(\alpha_4 - \cdots - \frac{\gamma_{\ell-1}}{\beta_\ell}\biggl[\alpha_\ell - \cdots\\
&\quad - \frac{\gamma_{n-3}}{\beta_{n-2}}\biggl(\alpha_{n-2} - \frac{\gamma_{n-2}}{\beta_{n-1}}\alpha_{n-1} + \frac{\gamma_{n-2}\gamma_{n-1}}{\beta_{n-1}\beta_n}\alpha_n\biggr)\biggr]\biggr)\biggr]\biggr)\\
&= \alpha_1 - \frac{\gamma_1}{\beta_2}\biggl(\alpha_2 - \frac{\gamma_2}{\beta_3}\biggl[\alpha_3 - \frac{\gamma_3}{\beta_4}\biggl(\alpha_4 - \cdots - \frac{\gamma_{\ell-1}}{\beta_\ell}\biggl[\alpha_\ell - \cdots\\
&\quad - \biggl(\frac{\gamma_{n-3}}{\beta_{n-2}}\alpha_{n-2} - \frac{\gamma_{n-3}\gamma_{n-2}}{\beta_{n-2}\beta_{n-1}}\alpha_{n-1} + \frac{\gamma_{n-3}\gamma_{n-2}\gamma_{n-1}}{\beta_{n-2}\beta_{n-1}\beta_n}\alpha_n\biggr)\biggr]\biggr)\biggr]\biggr)\\
&= \cdots\\
&= \alpha_1 - \sum_{k=2}^{n}(-1)^k\Biggl(\prod_{\ell=2}^{k}\frac{\gamma_{\ell-1}}{\beta_\ell}\Biggr)\alpha_k
\end{aligned}
$$

for $n \ge 2$. Substituting this into (2.1) and simplifying arrive at (2.3). The proof of Theorem 2.2 is complete. □

Remark 2.1. Applying $\alpha_k = k$, $\beta_k = k$, and $\gamma_k = k$ to the explicit formula (2.3) in Theorem 2.2 reveals

$$
\begin{vmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
2 & 2 & 2 & 0 & \cdots & 0 & 0 & 0 & 0\\
3 & 0 & 3 & 3 & \cdots & 0 & 0 & 0 & 0\\
4 & 0 & 0 & 4 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
n-3 & 0 & 0 & 0 & \cdots & n-3 & n-3 & 0 & 0\\
n-2 & 0 & 0 & 0 & \cdots & 0 & n-2 & n-2 & 0\\
n-1 & 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & n-1\\
n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n
\end{vmatrix}
= \frac{1-(-1)^n}{2}n!.
$$

3. Explicit and recurrent formulas for $|D_n|$

We are now in a position to derive explicit and recurrent formulas for the determinant $|D_n|$.

Theorem 3.1. *For $n \in \mathbb{N}$, the determinant $|D_n|$ of the tridiagonal matrix D_n can be explicitly and recurrently computed by*

$$
\begin{aligned}
|D_n| = a_1a_2 + (a_1 - b_1c_1)\prod_{m=3}^{n}\left[a_m + \underset{\ell=1}{\overset{m-2}{K}} \frac{(-b_{m-\ell}c_{m-\ell})}{a_{m-\ell}}\right] \\
- \sum_{k=3}^{n}\left[\prod_{\ell=1}^{k-1}(b_\ell c_\ell)\right] \frac{\prod_{m=k+1}^{n}\left[a_m + K_{\ell=1}^{m-2} \frac{(-b_{m-\ell}c_{m-\ell})}{a_{m-\ell}}\right]}{\prod_{m=2}^{k-1}\left[a_m + K_{\ell=1}^{m-2} \frac{(-b_{m-\ell}c_{m-\ell})}{a_{m-\ell}}\right]}
\end{aligned}
\tag{3.1}
$$

and

$$
|D_n| = \eta_{1,n}\left(a_2 + \prod_{k=3}^{n}\left[a_k + \underset{\ell=1}{\overset{k-2}{K}} \frac{(-b_{k-\ell}c_{k-\ell})}{a_{k-\ell}}\right]\right), \tag{3.2}
$$

where $K_{\ell=q}^{p}$ for $p<q$ is understood to be zero,

$$
\eta_{1,n} = -1 - \frac{b_1}{a_2}\eta_{2,n}, \quad \eta_{2,n} = c_1 - \frac{b_2}{a_3 - \frac{b_2c_2}{a_2}}\eta_{3,n}, \quad \eta_{3,n} = -\frac{c_1c_2}{a_2} - \frac{b_3}{a_4 - \frac{b_3c_3}{a_3 - \frac{b_2c_2}{a_2}}}\eta_{4,n},
$$

$$
\eta_{k,n} = (-1)^k \frac{\prod_{\ell=1}^{k-1} c_\ell}{a_2 + \prod_{\ell=3}^{k-1}\left[a_\ell + K_{m=1}^{\ell-2} \frac{(-b_{\ell-m}c_{\ell-\ell})}{a_{\ell-m}}\right]} - \frac{b_k}{a_{k+1} + K_{\ell=1}^{k-1} \frac{(-b_{k-\ell+1}c_{k-\ell+1})}{a_{k-\ell+1}}}\eta_{k+1,n}
$$

for $4 \le k \le n-1$, and

$$
\eta_{n,n} = (-1)^n \frac{\prod_{\ell=1}^{n-1} c_\ell}{a_2 + \prod_{k=3}^{n-1}\left[a_k + K_{\ell=1}^{k-2} \frac{(-b_{k-\ell}c_{k-\ell})}{a_{k-\ell}}\right]}.
$$

Proof. The determinant $|D_n|$ of the tridiagonal matrix D_n in (1.1) can be reformulated as

$$
|D_n| = \begin{vmatrix}
a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\frac{c_1c_2}{a_2} & 0 & a_3 - \frac{b_2c_2}{a_2} & b_3 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c_3 & a_4 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n-3} & b_{n-3} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & c_{n-3} & a_{n-2} & b_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{n-1} & a_n
\end{vmatrix}
$$

$$
=\begin{vmatrix}
a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
c_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0\\
-\frac{c_1c_2}{a_2} & 0 & a_3-\frac{b_2c_2}{a_2} & b_3 & \cdots & 0 & 0 & 0 & 0\\
\frac{c_1c_2c_3}{a_2a_3-b_2c_2} & 0 & 0 & a_4-\frac{a_2b_3c_3}{a_2a_3-b_2c_2} & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & a_{n-3} & b_{n-3} & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & c_{n-3} & a_{n-2} & b_{n-2} & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{n-1} & a_n
\end{vmatrix}
$$

$$
=\cdots
$$

$$
=\begin{vmatrix}
\alpha_1 & b_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\alpha_2 & \beta_2 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0\\
\alpha_3 & 0 & \beta_3 & b_3 & \cdots & 0 & 0 & 0 & 0\\
\alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
\alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & b_{n-3} & 0 & 0\\
\alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & b_{n-2} & 0\\
\alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & b_{n-1}\\
\alpha_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_n
\end{vmatrix},
$$

where

$$
\beta_2=a_2,\quad \beta_3=a_3-\frac{b_2c_2}{\beta_2},\quad \beta_4=a_4-\frac{b_3c_3}{\beta_3},\quad \dots,\quad \beta_{n-3}=a_{n-3}-\frac{b_{n-4}c_{n-4}}{\beta_{n-4}},
$$

$$
\beta_{n-2}=a_{n-2}-\frac{b_{n-3}c_{n-3}}{\beta_{n-3}},\quad \beta_{n-1}=a_{n-1}-\frac{b_{n-2}c_{n-2}}{\beta_{n-2}},\quad \beta_n=a_n-\frac{b_{n-1}c_{n-1}}{\beta_{n-1}}
$$

and

$$
\alpha_1=a_1,\quad \alpha_2=c_1,\quad \alpha_3=-\frac{c_2}{\beta_2}\alpha_2,\quad \alpha_4=-\frac{c_3}{\beta_3}\alpha_3,\quad \dots,\quad \alpha_{n-3}=-\frac{c_{n-4}}{\beta_{n-4}}\alpha_{n-4},
$$

$$
\alpha_{n-2}=-\frac{c_{n-3}}{\beta_{n-3}}\alpha_{n-3},\quad \alpha_{n-1}=-\frac{c_{n-2}}{\beta_{n-2}}\alpha_{n-2},\quad \alpha_n=-\frac{c_{n-1}}{\beta_{n-1}}\alpha_{n-1}.
$$

The sequences β_k and α_k for $k\ge3$ can be represented by generalized continued fractions

$$
\beta_k=a_k-\cfrac{b_{k-1}c_{k-1}}{a_{k-1}-\cfrac{b_{k-2}c_{k-2}}{a_{k-2}-\cfrac{b_{k-3}c_{k-3}}{a_{k-3}-\cfrac{b_{k-4}c_{k-4}}{\ddots\,\cfrac{\ddots}{a_4-\cfrac{b_3c_3}{a_3-\frac{b_2c_2}{a_2}}}}}}}=a_k+\underset{\ell=1}{\overset{k-2}{K}}\frac{(-b_{k-\ell}c_{k-\ell})}{a_{k-\ell}}
$$

and

$$
\alpha_k=(-1)^k\frac{\prod_{\ell=1}^{k-1}c_\ell}{\prod_{\ell=2}^{k-1}\beta_\ell}.
$$

Making use of (2.3) arrives at

$$
|D_n|=\Biggl(\beta_2+\prod_{k=3}^n\beta_k\Biggr)\alpha_1-\Biggl(b_1\prod_{m=3}^n\beta_m\Biggr)\alpha_2-\sum_{k=3}^n(-1)^k\Biggl(\prod_{\ell=1}^{k-1}b_\ell\prod_{m=k+1}^n\beta_m\Biggr)\alpha_k
$$

$$\begin{aligned}
&= a_1\left(a_2 + \prod_{k=3}^{n}\left[a_k + \mathop{K}_{\ell=1}^{k-2}\frac{(-b_{k-\ell}c_{k-\ell})}{a_{k-\ell}}\right]\right) - b_1c_1\prod_{m=3}^{n}\left[a_m + \mathop{K}_{\ell=1}^{m-2}\frac{(-b_{m-\ell}c_{m-\ell})}{a_{m-\ell}}\right]\\
&\quad - \sum_{k=3}^{n}\prod_{\ell=1}^{k-1}b_\ell\prod_{m=k+1}^{n}\left[a_m + \mathop{K}_{\ell=1}^{m-2}\frac{(-b_{m-\ell}c_{m-\ell})}{a_{m-\ell}}\right]\frac{\prod_{\ell=1}^{k-1}c_\ell}{\prod_{\ell=2}^{k-1}\left[a_\ell + K_{i=1}^{\ell-2}\frac{(-b_{\ell-i}c_{\ell-i})}{a_{\ell-i}}\right]}
\end{aligned}$$

which can be reformulated as (3.1).

Utilizing (2.1) and (2.2) yields (3.2). The proof of Theorem 3.1 is complete. □

4. Remarks

Remark 4.1. In 1854, J. J. Sylvester found that

$$|M_n(x)| = \begin{vmatrix}
x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
n & x & 2 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & n-1 & x & 3 & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & n-2 & x & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & x & n-2 & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & 3 & x & n-1 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 2 & x & n\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & x
\end{vmatrix} = \prod_{k=0}^{n}(x+n-2k).$$

See [3, p. 1018] and closely related references therein. Applying (3.1) to $|M_n(x)|$ yields

$$\begin{aligned}
|M_n(x)| &= x^2 + (x-n)\prod_{m=3}^{n}\left[x + \mathop{K}_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]\\
&\quad - \sum_{k=3}^{n}\left[\prod_{\ell=1}^{k-1}\ell(n-\ell+1)\right]\frac{\prod_{m=k+1}^{n}\left[x + K_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]}{\prod_{m=2}^{k-1}\left[x + K_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]}\\
&= x^2 + (x-n)\prod_{m=3}^{n}\left[x + \mathop{K}_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]\\
&\quad - n!\sum_{k=3}^{n}\frac{(k-1)!}{(n-k+1)!}\frac{\prod_{m=k+1}^{n}\left[x + K_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]}{\prod_{m=2}^{k-1}\left[x + K_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}\right]}\\
&\triangleq x^2 + (x-n)\prod_{m=3}^{n}S(x;m,n) - n!\sum_{k=3}^{n}\frac{(k-1)!}{(n-k+1)!}\frac{\prod_{m=k+1}^{n}S(x;m,n)}{\prod_{m=2}^{k-1}S(x;m,n)}.
\end{aligned}$$

Now we try to explicitly compute

$$S(x;m,n) = x + \mathop{K}_{\ell=1}^{m-2}\frac{-(m-\ell)(n-m+\ell+1)}{x}.$$

When $m=3$, it is easy to obtain that

$$S(x;3,n) = \frac{x^2-2(n-1)}{x} \triangleq \frac{b_1}{a_1}.$$

When $m=4$, using the above result for $S(x;3,n)$, we can obtain

$$S(x;4,n)=\frac{b_1x-3(n-2)a_1}{b_1}=\frac{x(x^2-5n+8)}{x^2-2n+2}\triangleq\frac{b_2}{a_2}.$$

If we assume $S(x;k+1,n)=\frac{b_{k-1}}{a_{k-1}}$, then, by induction, we obtain

$$S(x;k+2,m)=\frac{b_k}{a_k}=\frac{b_{k-1}x-a_{k-1}(k+1)(n-k)}{b_{k-1}}.$$

We note that $a_{k-1}=b_{k-2}$. Then

$$b_k-b_{k-1}x+b_{k-2}(k+1)(n-k)=0.$$

Further replacing k by $k+2$ leads to

$$b_{k+2}-b_{k+1}x+(k+3)(n-k-2)b_k=0.$$

By the method used in [15, Theorem 3.1], the characteristic equation is

$$y^2-xy+(k+3)(n-k-2)=0$$

which has solutions

$$y=\frac{x\pm\sqrt{x^2-4(k+3)(n-k-2)}}{2}.$$

Consequently, it follows that

$$b_k=A\left(\frac{x+\sqrt{x^2-4(k+3)(n-k-2)}}{2}\right)^{k-1}+B\left(\frac{x-\sqrt{x^2-4(k+3)(n-k-2)}}{2}\right)^{k-1},$$

where

$$A=-\frac{2x^3-2(5n-8)-(x^2-2n+2)(x+\sqrt{x^2-20n+80}\,)}{2\sqrt{x^2-20n+80}}$$

and

$$B=\frac{2x^3-2(5n-8)-(x^2-2n+2)(x-\sqrt{x^2-20n+80}\,)}{2\sqrt{x^2-20n+80}}.$$

In a word, by a new method, we supply an alternative form for the Sylvester determinant $|M_n(x)|$.

In [3], by means of left eigenvector method, the following determinants of tridiagonal matrices similar to the Sylvester matrix were collected and calculated:

$$|M_n(x,y)|=\begin{vmatrix}
x & 1 & 0 & 0 & \cdots & 0 & 0 & 0\\
n & x+y & 2 & 0 & \cdots & 0 & 0 & 0\\
0 & n-1 & x+2y & 3 & \cdots & 0 & 0 & 0\\
0 & 0 & n-2 & x+3y & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & x+(n-2)y & n-1 & 0\\
0 & 0 & 0 & 0 & \cdots & 2 & x+(n-1)y & n\\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & x+ny
\end{vmatrix}$$
$$=\prod_{k=0}^{n}\left(x+\frac{ny}{2}+\frac{n-2k}{2}\sqrt{y^4+4}\right),$$

$$
\begin{aligned}
|M_n(x,y;u,v)| &= \begin{vmatrix}
x & u & 0 & 0 & \cdots & 0 & 0 & 0 \\
nv & x+y & 2u & 0 & \cdots & 0 & 0 & 0 \\
0 & (n-1)v & x+2y & 3u & \cdots & 0 & 0 & 0 \\
0 & 0 & (n-2)v & x+3y & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & x+(n-2)y & (n-1)u & 0 \\
0 & 0 & 0 & 0 & \cdots & 2v & x+(n-1)y & nu \\
0 & 0 & 0 & 0 & \cdots & 0 & v & x+ny
\end{vmatrix} \\
&= \prod_{k=0}^{n}\left(x+\frac{ny}{2}+\frac{n-2k}{2}\sqrt{y^4+4uv}\right).
\end{aligned}
$$

These evaluations can be computed alternatively by Theorem 3.1.

Remark 4.2. The condition $\beta_k \neq 0$ for $2 \le k \le n$ in Theorem 2.1 is removed off in Theorem 2.2. Therefore, the explicit formula (2.3) is better than the recursive formulas (2.1) and (2.2).

Remark 4.3. The explicit formula (2.3) can be simply rearranged as

$$
|P_n| = \sum_{k=1}^{n}(-1)^{k+1}\left(\prod_{\ell=1}^{k-1}\gamma_\ell \prod_{m=k+1}^{n}\beta_m\right)\alpha_k,
$$

where the empty product is understood to be 1 as usual.

Remark 4.4. Let

$$
U_n = \begin{pmatrix}
a_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & d_1 \\
a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & d_2 \\
a_3 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & d_3 \\
a_4 & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & d_4 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
a_{n-3} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 & d_{n-3} \\
a_{n-2} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} & d_{n-2} \\
a_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & c_{n-1} \\
a_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_n
\end{pmatrix} = (u_{i,j})_{1\le i,j\le n},
$$

where

$$
u_{i,j} = \begin{cases}
a_i, & 1 \le i \le n, j = 1; \\
b_i, & 2 \le i = j \le n; \\
c_i, & 1 \le i = j-1 \le n-1; \\
d_i, & 1 \le i \le n-2, j = n; \\
0, & \text{otherwise}.
\end{cases}
$$

The determinant $|U_n|$ can be reformulated as

$$|U_n| = \begin{vmatrix} a_1 & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & d_1 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & d_2 \\ a_3 & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & d_3 \\ a_4 & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & d_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 & d_{n-3} \\ a_{n-2} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} & d_{n-2} \\ a_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & c_{n-1} \\ a_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 - \frac{a_n d_1}{b_n} & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_2 - \frac{a_n d_2}{b_n} & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 - \frac{a_n d_3}{b_n} & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 & 0 \\ a_4 - \frac{a_n d_4}{b_n} & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} - \frac{a_n d_{n-3}}{b_n} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 & 0 \\ a_{n-2} - \frac{a_n d_{n-2}}{b_n} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} & 0 \\ a_{n-1} - \frac{a_n c_{n-1}}{b_n} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} & 0 \\ a_n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & b_n \end{vmatrix}$$

$$= b_n \begin{vmatrix} a_1 - \frac{a_n d_1}{b_n} & c_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 - \frac{a_n d_2}{b_n} & b_2 & c_2 & 0 & \cdots & 0 & 0 & 0 \\ a_3 - \frac{a_n d_3}{b_n} & 0 & b_3 & c_3 & \cdots & 0 & 0 & 0 \\ a_4 - \frac{a_n d_4}{b_n} & 0 & 0 & b_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-3} - \frac{a_n d_{n-3}}{b_n} & 0 & 0 & 0 & \cdots & b_{n-3} & c_{n-3} & 0 \\ a_{n-2} - \frac{a_n d_{n-2}}{b_n} & 0 & 0 & 0 & \cdots & 0 & b_{n-2} & c_{n-2} \\ a_{n-1} - \frac{a_n c_{n-1}}{b_n} & 0 & 0 & 0 & \cdots & 0 & 0 & b_{n-1} \end{vmatrix}.$$

Applying Theorem 2.1 and 2.2 and Remark 4.3 directly results in

$$|U_n| = \sum_{k=1}^{n-2} (-1)^{k+1} (a_k b_n - d_k a_n) \prod_{\ell=1}^{k-1} c_\ell \prod_{m=k+1}^{n-1} b_m + (-1)^n (a_{n-1} b_n - d_{n-1} a_n) \prod_{\ell=1}^{n-2} c_\ell$$

and

$$|U_n| = \Lambda_{1,n-1} \prod_{k=2}^{n} b_k,$$

where

$$\Lambda_{k,n-1} = a_k - \frac{a_n}{b_n} d_k - \frac{c_k}{b_{k+1}} \Lambda_{k+1,n-1}, \quad 1 \le k \le n-2$$

and $\Lambda_{n-1,n-1} = a_{n-1} - \dfrac{a_n}{b_n} c_{n-1}$.

Remark 4.5. The determinant $|P_n|$ of the matrix P_n in (1.2) can be reformulated as

$$
\begin{aligned}
|P_n| &= \begin{vmatrix}
\alpha_1 & \gamma_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_3 & 0 & \beta_3 & \gamma_3 & \cdots & 0 & 0 & 0 & 0 \\
\alpha_4 & 0 & 0 & \beta_4 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\alpha_{n-3} & 0 & 0 & 0 & \cdots & \beta_{n-3} & \gamma_{n-3} & 0 & 0 \\
\alpha_{n-2} & 0 & 0 & 0 & \cdots & 0 & \beta_{n-2} & \gamma_{n-2} & 0 \\
\alpha_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-1} & \gamma_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}} & \beta_n - \frac{\alpha_n\gamma_{n-1}}{\alpha_{n-1}}
\end{vmatrix} \\
&= \cdots \\
&= \begin{vmatrix}
\alpha_1 & \gamma_1 & 0 & \cdots & 0 & 0 & 0 \\
0 & \beta_2 - \frac{\alpha_2\gamma_1}{\alpha_1} & \gamma_2 & \cdots & 0 & 0 & 0 \\
0 & -\frac{\alpha_3\beta_2}{\alpha_2} & \beta_3 - \frac{\alpha_3\gamma_2}{\alpha_2} & \cdots & 0 & 0 & 0 \\
0 & 0 & -\frac{\alpha_4\beta_3}{\alpha_3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \gamma_{n-3} & 0 & 0 \\
0 & 0 & 0 & \cdots & \beta_{n-2} - \frac{\alpha_{n-2}\gamma_{n-3}}{\alpha_{n-3}} & \gamma_{n-2} & 0 \\
0 & 0 & 0 & \cdots & -\frac{\alpha_{n-1}\beta_{n-2}}{\alpha_{n-2}} & \beta_{n-1} - \frac{\alpha_{n-1}\gamma_{n-2}}{\alpha_{n-2}} & \gamma_{n-1} \\
0 & 0 & 0 & \cdots & 0 & -\frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}} & \beta_n - \frac{\alpha_n\gamma_{n-1}}{\alpha_{n-1}}
\end{vmatrix} \\
&= \alpha_1 \begin{vmatrix}
\beta_2 - \frac{\alpha_2\gamma_1}{\alpha_1} & \gamma_2 & \cdots & 0 & 0 & 0 \\
-\frac{\alpha_3\beta_2}{\alpha_2} & \beta_3 - \frac{\alpha_3\gamma_2}{\alpha_2} & \cdots & 0 & 0 & 0 \\
0 & -\frac{\alpha_4\beta_3}{\alpha_3} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \gamma_{n-3} & 0 & 0 \\
0 & 0 & \cdots & \beta_{n-2} - \frac{\alpha_{n-2}\gamma_{n-3}}{\alpha_{n-3}} & \gamma_{n-2} & 0 \\
0 & 0 & \cdots & -\frac{\alpha_{n-1}\beta_{n-2}}{\alpha_{n-2}} & \beta_{n-1} - \frac{\alpha_{n-1}\gamma_{n-2}}{\alpha_{n-2}} & \gamma_{n-1} \\
0 & 0 & \cdots & 0 & -\frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}} & \beta_n - \frac{\alpha_n\gamma_{n-1}}{\alpha_{n-1}}
\end{vmatrix} \\
&\triangleq \alpha_1 |Q_{n-1}|.
\end{aligned}
$$

Accordingly, by Theorems 2.1 and 2.2, we can derive that the determinant $|Q_{n-1}|$ of the tridiagonal matrix Q_{n-1} satisfies

$$|Q_{n-1}| = \frac{|P_n|}{\alpha_1} = \frac{\lambda_{1,n}}{\alpha_1} \prod_{k=2}^{n} \beta_k \tag{4.1}$$

and

$$|Q_{n-1}| = \frac{|P_n|}{\alpha_1} = \prod_{k=2}^{n} \beta_k - \frac{1}{\alpha_1} \sum_{k=2}^{n} (-1)^k \left(\prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n} \beta_m \right) \alpha_k. \tag{4.2}$$

Further taking

$$\begin{cases} a_k = \beta_{k+1} - \dfrac{\alpha_{k+1}\gamma_k}{\alpha_k}, & 1 \le k \le n-1; \\ b_k = \gamma_{k+1}, & 1 \le k \le n-2; \\ c_k = -\dfrac{\alpha_{k+2}\beta_{k+1}}{\alpha_{k+1}}, & 1 \le k \le n-2 \end{cases} \tag{4.3}$$

in equalities (4.1) and (4.2) procures

$$|D_{n-1}| = \frac{\lambda_{1,n}}{\alpha_1} \prod_{k=2}^{n} \beta_k \tag{4.4}$$

and

$$|D_{n-1}| = \prod_{k=2}^{n} \beta_k - \frac{1}{\alpha_1} \sum_{k=2}^{n} (-1)^k \left(\prod_{\ell=1}^{k-1} \gamma_\ell \prod_{m=k+1}^{n} \beta_m \right) \alpha_k. \tag{4.5}$$

From the second equation in (4.3), it is easy to see that $\gamma_k = b_{k-1}$ for $2 \le k \le n-1$. If one can derive another two relations from (4.3) to express α_k for $1 \le k \le n$ and β_k for $2 \le k \le n$ in terms of a_k for $1 \le k \le n-1$, b_k for $1 \le k \le n-2$, and c_k for $1 \le k \le n-2$, then, by substituting these three relations into (4.4) and (4.5) an alternative and explicit form for evaluation of the determinant $|D_n|$ of the general tridiagonal matrix D_n would be obtained. This is an open problem we leave to the interested readers.

Remark 4.6. In [1, Lemma 1.1] and [11, Lemma 2.1], it was obtained that

$$\begin{vmatrix} d_1 & d_2 & d_3 & d_4 & \cdots & d_{n-2} & d_{n-1} & d_n \\ a & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & a & b & 0 & \cdots & 0 & 0 & 0 \\ 0 & c & a & b & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & a & b \end{vmatrix} = \sum_{k=1}^{n} (-1)^{k-1} d_k b^{n-k} (bc)^{(k-1)/2} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right), \tag{4.6}$$

where $U_k(x)$, generated [18, 19] by

$$\frac{1}{1-2xt+t^2} = \sum_{k=0}^{\infty} U_k(x) t^k, \quad |x| < 1, \quad |t| < 1,$$

is the kth Chebyshev polynomials of the second kind. Letting $d_1 = d_2 = \cdots = d_{n-1} = 0$ and $d_n = 1$ and rearranging, the formula (4.6) becomes

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n} = (bc)^{n/2} U_n\left(\frac{a}{2\sqrt{bc}}\right). \tag{4.7}$$

This is different from

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n} = \begin{cases} \dfrac{\bigl(a+\sqrt{a^2-4bc}\,\bigr)^{n+1}-\bigl(a-\sqrt{a^2-4bc}\,\bigr)^{n+1}}{2^{n+1}\sqrt{a^2-4bc}}, & a^2\ne 4bc \\ (n+1)\left(\dfrac{a}{2}\right)^n, & a^2=4bc \end{cases} \tag{4.8}$$

and

$$\begin{vmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & \cdots & 0 & 0 \\ 0 & c & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{vmatrix}_{n\times n} = \prod_{j=1}^{n}\left(b+2a\sqrt{\frac{c}{a}}\cos\frac{j\pi}{n+1}\right) \tag{4.9}$$

established and collected in [15, pp. 130] and [17, Theoem 4].

Comparing (4.7) with (4.8) and (4.9), letting $b=c=1$ and $a=2x$, and simplifying yield

$$\begin{aligned} U_n(x) &= \prod_{j=1}^{n}\left(1+2\sqrt{2x}\cos\frac{j\pi}{n+1}\right) \\ &= \begin{cases} \dfrac{\bigl(x+\sqrt{x^2-1}\,\bigr)^{n+1}-\bigl(x-\sqrt{x^2-1}\,\bigr)^{n+1}}{2^{n+1}\sqrt{x^2-1}}, & x^2\ne 1 \\ (n+1)x^n, & x^2=1 \end{cases} \end{aligned}$$

which are new explicit formulas for the Chebyshev polynomials of the second kind $U_n(x)$.

Remark 4.7. On 21 September 2019, we are reminded of the paper [10] in which an explicit formula for the elements of the inverse of a tridiagonal matrix and an efficient and fast computing method to obtain the elements of the inverse of a tridiagonal matrix by backward continued fractions were investigated.

Acknowledgements. The third author was supported by the National Research Foundation of Korea (NRF-2018R1D1A1B07041846), South Korea.

References

[1] D. Bozkurt, C. M. Da Fonseca, and F. Yılmaz, *The determinants of circulant and skew-circulant matrices with tribonacci numbers*, Math. Sci. Appl. E-Notes **2** (2014), no. 2, 67–75.

[2] Z. R. Chen, *Inversion of general tridiagonal matrices*, J. Numer. Methods Comput. Appl. **8** (1987), no. 3, 158–164. (Chinese)

[3] W. Chu, *Fibonacci polynomials and Sylvester determinant of tridiagonal matrix*, Appl. Math. Comput. **216** (2010) 1018–1023; available online at `https://doi.org/10.1016/j.amc.2010.01.089`.

[4] D. F. G. Coelho, V. S. Dimitrov, and L. Rakai, *Efficient computation of tridiagonal matrices largest eigenvalue*, J. Comput. Appl. Math. **330** (2018), 268–275; available online at `https://doi.org/10.1016/j.cam.2017.08.008`.

[5] M. El-Mikkawy and A. Karawia, *Inversion of general tridiagonal matrices*, Appl. Math. Lett. **19** (2006), no. 8, 712–720; available online at `https://doi.org/10.1016/j.aml.2005.11.012`.

[6] M. A. El-Shehawey, Gh. A. El-Shreef, and A. Sh. Al-Henawy, *Analytical inversion of general periodic tridiagonal matrices*, J. Math. Anal. Appl. **345** (2008), no. 1, 123–134; available online at `https://doi.org/10.1016/j.jmaa.2008.04.002`.

[7] L. Euler, *An essay on continued fractions*, translated from the Latin by B. F. Wyman and M. F. Wyman, Math. Systems Theory **18** (1985), no. 4, 295–328; available online at https://doi.org/10.1007/BF01699475.

[8] C. F. Fischer and R. A. Usmani, *Properties of some tridiagonal matrices and their application to boundary value problems*, SIAM J. Numer. Anal. **6** (1969), no. 1, 127–142; available online at https://doi.org/10.1137/0706014.

[9] Y. Huang and W. F. McColl, *Analytical inversion of general tridiagonal matrices*, J. Phys. A **30** (1997), no. 22, 7919–7933; available online at https://doi.org/10.1088/0305-4470/30/22/026.

[10] E. Kılıç, *Explicit formula for the inverse of a tridiagonal matrix by backward continued fractions*, Appl. Math. Comput. **197** (2008), no. 1, 345–357; available online at https://doi.org/10.1016/j.amc.2007.07.046.

[11] E. Kırklar and F. Yılmaz, *A general formula for determinants and inverses of r-circulant matrices with third order recurrences*, Math. Sci. Appl. E-Notes **7** (2019), no. 1, 1–8.

[12] S. Kouachi, *Eigenvalues and eigenvectors of some tridiagonal matrices with non-constant diagonal entries*, Appl. Math. (Warsaw) **35** (2008), no. 1, 107–120; available online at https://doi.org/10.4064/am35-1-7.

[13] S. Kouachi, *Eigenvalues and eigenvectors of tridiagonal matrices*, Electron. J. Linear Algebra **15** (2006), Article 8, 115–133; available online at https://doi.org/10.13001/1081-3810.1223.

[14] G. Panti, *Slow continued fractions, transducers, and the Serret theorem*, J. Number Theory **185** (2018), 121–143; available online at https://doi.org/10.1016/j.jnt.2017.08.034.

[15] F. Qi, V. Čerňanová, and Y. S. Semenov, *Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **81** (2019), no. 1, 123–136.

[16] F. Qi, V. Čerňanová, X.-T. Shi, and B.-N. Guo, *Some properties of central Delannoy numbers*, J. Comput. Appl. Math. **328** (2018), 101–115; available online at https://doi.org/10.1016/j.cam.2017.07.013.

[17] F. Qi and A.-Q. Liu, *Alternative proofs of some formulas for two tridiagonal determinants*, Acta Univ. Sapientiae Math. **10** (2018), no. 2, 287–297; available online at https://doi.org/10.2478/ausm-2018-0022.

[18] F. Qi, D.-W. Niu, and D. Lim, *Notes on explicit and inversion formulas for the Chebyshev polynomials of the first two kinds*, Miskolc Math. Notes **20** (2019), no. 2, in press; available online at https://doi.org/10.18514/MMN.2019.2976.

[19] F. Qi, Q. Zou, and B.-N. Guo, *The inverse of a triangular matrix and several identities of the Catalan numbers*, Appl. Anal. Discrete Math. **13** (2019), no. 2, 518–541; available online at https://doi.org/10.2298/AADM190118018Q.

(Qi) College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China; School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China

Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: https://qifeng618.wordpress.com

(Wang) School of Mathematics and Statistics, Hefei Normal University, Hefei 230601, Anhui, China
Email address: wwen2014@mail.ustc.edu.cn

(Lim) Department of Mathematics Education, Andong National University, Andong 36729, Republic of Korea

Email address: dgrim84@gmail.com, dklim@andong.ac.kr
URL: http://orcid.org/0000-0002-0928-8480

(Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, Henan, China

Email address: bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com