On curves with Poritsky property

Alexey Glutsyuk

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Abstract

For a given closed convex planar curve γ with smooth boundary and a given $p > 0$, the string construction is obtained by putting a string surrounding γ of length $p + |\gamma|$ to the plane. Then we pull some point of the string "outwards from γ" until its final position A, when the string becomes stretched completely. The set of all the points A thus obtained is a planar convex curve Γ_p. The billiard reflection T_p from the curve Γ_p acts on oriented lines, and γ is a caustic for Γ_p: that is, the family of lines tangent to γ is T_p-invariant. The action of the reflection T_p on the tangent lines to $\gamma \simeq S^1$ induces its action on the tangency points: a circle diffeomorphism $\mathcal{T}_p : \gamma \to \gamma$. We say that γ has *string Poritsky property,* if it admits a parameter t (called *Poritsky–Lazutkin string length*) in which all the transformations $\mathcal{T}_p$ are translations $t \mapsto t + c_p$. These definitions also make sense for germs of curves γ. Poritsky property is closely related to the famous Birkhoff Conjecture. It is classically known that each conic has string Poritsky property. In 1950 H.Poritsky proved the converse: *each germ of planar curve with Poritsky property is a conic.*

In the present paper we extend this Poritsky's result to germs of curves to all the simply connected complete Riemannian surfaces of constant curvature and to outer billiards on all these surfaces. We also consider the general case of curves with Poritsky property on any two-dimensional surface with Riemannian metric and prove a formula for the derivative of the Poritsky–Lazutkin length as a function of the

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natural length parameter. In this general setting we also prove the following uniqueness result: *a germ of curve with Poritsky property is uniquely determined by its 4-th jet.* In the Euclidean case this statement follows from the above-mentioned Poritsky's result.

Contents

1 Introduction and main results

Consider the billiard in a bounded planar domain $\Omega \subset \mathbb{R}^2$ with a strictly convex smooth boundary. The billiard dynamics T acts on the space of oriented lines intersecting Ω. Namely, let L be an oriented line intersecting Ω, and let A be its last point (in the sense of orientation) of its intersection with $\partial\Omega$. By definition $T(L)$ is the image of the line L under the symmetry with respect to the tangent line $T_A\partial\Omega$, being oriented from the point A inside the domain Ω. A curve $\gamma \subset \mathbb{R}^2$ is a *caustic* of the billiard Ω, if each line tangent to γ is reflected from the boundary $\partial\Omega$ again to a line tangent to γ; in other words, if the curve formed by oriented lines tangent to γ is invariant under the billiard transformation T.

It is well-known that each planar billiard with sufficiently smooth strictly convex boundary has a Cantor family of caustics [6]. Elliptic billiard is *Birkhoff caustic integrable,* that is, an inner neighborhood of its boundary is foliated by closed caustics. The famous Birkhoff Conjecture states the converse: the only Birkhoff caustic integrable planar billiards are elllipses. The Birkhoff Conjecture together with its extension to billiards on surfaces of constant curvature and its version (due to Sergei Tabachnikov) for outer billiards on the latter surfaces are big open problems, see, e.g., [2, 4] and references therein for history and related results.

It is well-known that each smooth convex planar curve γ is a caustic for a family of billiards $\Omega = \Omega_p$, $p \in \mathbb{R}_+$, whose boundaries $\Gamma = \Gamma_p = \partial\Omega_p$ are given by the p-th string constructions, see [14, p.73]. Namely, let $|\gamma|$ denote the length of the curve γ. Take an arbitrary number $p > 0$ and a string of length $p + |\gamma|$ enveloping the curve γ. Let us put a pencil between the curve γ and the string, and let us push it out of γ until a position, when the string, which envelopes γ and the pencil, become stretched. Then let us move the pencil around the curve γ so that the string remains stretched. Thus moving pencil draws a convex curve that is called the *p-th string construction.*

Consider the special case, when γ is an ellipse. Then for every $p > 0$ the curve Γ_p given by the p-th string construction is an ellipse confocal to γ, and γ is a caustic for the billiard in Γ_p. The billiard reflection T_p from the curve Γ_p acts on the lines tangent to γ. It induces the mapping $\mathcal{T}_p : \gamma \to \gamma$ sending each point $A \in \gamma$ to the point of tangency of the curve γ with the line $T_p(T_A\gamma)$. Every ellipse γ admits a canonical bijective parametrization by the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ equipped with parameter t such that for every $p > 0$ small one has $\mathcal{T}_p(t) = t + c_p$, $c_p = c_p(\gamma)$. The property of existence of the above parametrization will be called the *string Poritsky property,* and the parameter t will be called Poritsky–Lazutkin *string length.*

In his seminal paper [10] Hillel Poritsky proved the Birkhoff Conjecture under the following additional assumption: for every two nested caustics γ_λ, γ_μ of the billiard under question the smaller caustic γ_λ is also a caustic of the billiard in the bigger caustic γ_μ. His beautiful geometric proof was based on his remarkable theorem stating that only conics have string Poritsky property, see [10, section 7].

In the present paper we extend the above Poritsky theorem to simply connected complete surfaces of constant curvature (Subsections 1.1 and 2.1) and prove its version for outer billiards and area construction on these surfaces (Subsections 1.2 and 2.2). All the results of the present paper will be stated and proved for germs of curves, and thus, in Subsection 1.1 (2.1) we state the definitions of Poritsky string (area) property for germs. We also study Poritsky property on arbitrary surfaces equipped with Riemannian metric and prove the formula for derivative of the Poritsky string length as a function of the natural lenght (Theorem 1.10 in Subsection 1.3). In this general Riemannian context we show that a C^5-smooth germ of curve with string Poritsky property is uniquely determined by its 4-jet (Theorem 1.11 in Subsection 1.4).

1.1 Poritsky property for string construction and Poritsky–Lazutkin string length

Let Σ be a two-dimensional surface equipped with a Riemannian metric. Let $\gamma \subset \Sigma$ be a smooth curve (germ of a smooth curve at a point $O \in \Sigma$). We consider it to be *convex:* its geodesic curvature should be non-zero. For every given points $A, B \in \gamma$ by C_{AB} we will denote the point of intersection (if any) of the geodesics tangent to γ at A and B respectively. We consider the situation, when A and B are close to each other, and then we can and will choose the above C_{AB} close to A and B in a unique way. Set

$$\lambda(A, B) := \text{ the length of the arc } AB \text{ of the curve } \gamma,$$

$$L(A, B) := |AC_{AB}| + |BC_{AB}| - \lambda(A, B). \tag{1.1}$$

Here for $X, Y \in \Sigma$ close enough to a given point O by $|XY|$ we denote the length of the geodesic segment connecting X and Y.

Definition 1.1 (equivalent definition of string construction) Let $\gamma \subset \Sigma$ be a germ of curve with non-zero geodesic curvature. For every $p \in \mathbb{R}_+$ small enough the subset

$$\Gamma_p := \{C_{AB} \mid L(A, B) = p\} \subset \Sigma$$

is called the *p-th string construction,* see [14, p.73].

Remark 1.2 For every $p > 0$ small enough Γ_p is a well-defined smooth curve, $\Gamma_0 = \gamma$. The curve γ is a caustic for the billiard transformation acting by reflection from the curve Γ_p: a line tangent to γ is reflected from the curve Γ_p to a line tangent to γ [14, theorem 5.1]. One can show that the curves Γ_p with small $p \geq 0$ form a smooth foliation of appropriate domain V with boundary γ. Namely, take a small neighborhood U of the base point of the curve γ. The domain V is the connected component of the complement $U \setminus \gamma$ for which the boundary curve γ is concave.

Definition 1.3 We say that a germ of curve $\gamma \subset \Sigma$ with non-zero geodesic curvature has *string Poritsky property,* if it admits a parametrization by a parameter t (called *Poritsky–Lazutkin string length*) such that for every $p > 0$ there exists a $c = c_p > 0$ such that for every $A, B \in \gamma$ with $L(A, B) = p$ one has $|t(B) - t(A)| = c_p$.

Example 1.4 It is classically known that

(i) for every planar conic $\gamma \subset \mathbb{R}^2$ and every $p > 0$ the p-th string construction Γ_p is a conic confocal to γ;

(ii) all the conics confocal to γ and close enough to it are caustics of the billiard inside the conic Γ_p;

(iii) each planar conic γ has string Poritsky property [10, section 7] [14, p.58];

(iv) conversely, *each planar curve with string Poritsky property is a conic*, by a theorem of H.Poritsky [10, section 7].

One of the results of the present paper extends statement (iv) to billiards on simply connected complete surfaces of constant curvature (by adapting Poritsky's arguments from [10, section 7]) and to outer billiards on the latter surfaces. To state it, let us recall the notion of a conic on a surface of constant curvature.

Without loss of generality we consider simply connected complete surfaces Σ of constant curvature 0, ± 1 and realize each of them in its standard model in the space $\mathbb{R}^3_{(x_1,x_2,x_3)}$ equipped with appropriate quadratic form

$$< Ax, x >, \ A \in \{\mathrm{diag}(1,1,0), \mathrm{diag}(1,1,\pm 1)\}, \ < x, x >= x_1^2 + x_2^2 + x_3^2.$$

- Euclidean plane: $\Sigma = \{x_3 = 1\}$, $A = \mathrm{diag}(1,1,0)$.
- The unit sphere: $\Sigma = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, $A = Id$.
- The hyperbolic plane: $\Sigma = \{x_1^2 + x_2^2 - x_3^2 = -1\} \cap \{x_3 > 0\}$, $A = \mathrm{diag}(1,1,-1)$.

The metric of constant curvature on the surface Σ under question is induced by the quadratic form $< Ax, x >$. The *geodesics* on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^3$. The *conics* on Σ are its intersections with quadrics $\{< Cx, x >= 0\} \subset \mathbb{R}^3$, where C is a real symmetric 3×3-matrix.

Proposition 1.5 *On every surface of constant curvature each conic has string Poritsky property.*

Theorem 1.6 *Conversely, on every surface of constant curvature each germ of curve with string Poritsky property is a conic.*

Proposition 1.5 and 1.6 will be proved in Subsection 2.1.

In the case, when the surface under question is Euclidean plane, Proposition 1.5 was proved in [10, formula (7.1)], and Theorem 1.6 was proved in [10, section 7].

1.2 Poritsky property for outer billiards and area construction

Let $\gamma \subset \mathbb{R}^2$ be a smooth strictly convex closed curve. Let $\mathcal{U}$ be the exterior connected component of the complement $\mathbb{R}^2 \setminus \Gamma$. Recall that the *outer billiard map* $T : \mathcal{U} \to \mathcal{U}$ associated to the curve γ acts as follows. Take a point $A \in \mathcal{U}$. There are two tangent lines to γ through A. Let L_A denote the right tangent line (that is, the image of the line L_A under a small clockwise rotation around the point A is disjoint from the curve γ). Let $B \in \gamma$ denote its tangency point. By definition, the image $T(A)$ is the point of the line L_A central-symmetric to A with respect to the point B.

It is well-known that if γ is an ellipse, then the corresponding outer billiard map is *integrable:* that is, an exterior neighborhood of the curve γ is foliated by invariant closed curves for the outer billiard map so that

γ is a leaf of this foliation. The analogue of Birkhoff Conjecture for the outer billiards, which was suggested by S.Tabachnikov [15, p.101], states the converse: if γ generates an integrable outer billiard, then it is an ellipse. Its algebraic version was recently solved in [3]. For a survey on outer billiards and Tabachnikov's Conjecture see [12, 13, 16] and references therein.

For a given strictly convex smooth curve Γ there exists a one-parametric family of curves γ_p such that γ_p lies in the interior component Ω of the complement $\mathbb{R}^2 \setminus \Gamma$, and the curve Γ is invariant under the outer billiard map T_p generated by γ_p. The curves γ_p are given by the following *area construction* analogous to the string construction. Let $\mathcal{A}$ denote the area of the domain Ω bounded by Γ. For every oriented line L intersecting γ let $\Omega_-(L)$ denote the connected component of the complement $\Omega \setminus L$ for which L is a negatively oriented part of boundary. Let L be a class of parallel and co-directed oriented lines. For every $p > 0$, $p < \mathcal{A}$ let L_p denote the oriented line representing L that intersects Γ and such that $Area(\Omega_-(L_p) = p$. For every given $p > 0$, $p < \frac{1}{2}\mathcal{A}$ the lines L_p corresponding to different classes L form a one-parameter family parametrized by the circle: the azimuth of the line is the parameter. Let γ_p denote the enveloping curve of the latter family, and let T_p denote the outer billiard map generated by γ_p. It is well-known that the curve Γ is T_p-invariant for every p as above [13, corollary 9.5].

In the case, when Γ is an ellipse, all the curves γ_p are ellipses homothetic to Γ with respect to its center. In this case there exists a parametrization of the curve Γ by circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with parameter t such that $T_p : \Gamma \to \Gamma$ is a translation $t \mapsto t + c_p$ in the coordinate t for every p. This follows from the area-preserving property and the fact that for every $p < q$ the ellipse γ_p is T_q-invariant, analogously to the arguments in [10, section 7], [14, p.58]. Similar statements hold for all conics, as in loc. cit.

In our paper we prove the converse statement given by the following theorem, which will be stated in local context, for germs of smooth curves. To state it, let us introduce the following definition.

Definition 1.7 Let Σ be a surface with smooth Riemannian metric, $O \in \Sigma$. Let $\Gamma \subset \Sigma$ be a germ of smooth strictly convex curve at a point O (i.e., its geodesic curvature at O should be non-zero). Let $U \subset \Sigma$ be a disk centered at O that is split by Γ into two components. One of the latter components is convex; let us denote it by V. Consider the curves γ_p given by the above area construction with $p > 0$ small enough and lines replaced by geodesics. The curves γ_p form a germ at O of foliation in the domain V, and its boundary curve Γ is a leaf of this foliation. We say that the curve Γ has *area Poritsky property,* if it admits a local parametrization by parameter t called *area*

Poritsky parameter such that $T_p : \Gamma \to \Gamma$ is a translation $t \mapsto t + c_p$ in the coordinate t for every p.

Proposition 1.8 *(classical) On every surface of constant curvature each conic has area Poritsky property.*

Theorem 1.9 *Conversely, on every surface of constant curvature each curve with area Poritsky property is a conic*[1]*.*

1.3 General formula for the derivative of Poritsky–Lazutkin length

Let Σ be a two-dimensional surface equipped with a Riemannian metric: both Σ and the metric being C^3-smooth.

Theorem 1.10 *Let $\gamma \subset \Sigma$ be a germ of C^3-smooth curve with string Poritsky property. Let s be the natural length parameter of the curve γ, and let t be its Poritsky-Lazutkin string length parameter. Then*

$$\frac{dt}{ds} = \kappa^{\frac{2}{3}} \tag{1.2}$$

up to constant factor.

Theorem 1.10 and its generalization will be proved in Section 4.

1.4 Unique determination by 4-jet

Theorem 1.11 *Let Σ be a C^5-smooth surface equipped with a Riemannian metric. A C^5-smooth germ of curve with string Poritsky property is uniquely determined by its 4-jet.*

Theorem 1.11 will be proved in Section 5.

Remark 1.12 In the case, when Σ is the Euclidean plane, the statement of Theorem 1.11 follows from Poritsky's result [10, section 7] (see statement (iv) of Example 1.4) and the fact that each conic is uniquely determined by its 4-jet. Similarly in the case, when Σ is a surface of constant curvature, the statement of Theorem 1.11 follows from Theorem 1.6.

[1] For *planar* curves with area Poritsky property the statement of Theorem 1.9 was earlier proved by Sergei Tabachnikov (unpublished paper, 2018) by analytic arguments showing that the affine curvature of the curve should be constant. In Section 3 we present a different, geometric proof which works on all the surfaces of constant curvature simultaneously.

2 Birkhoff billiards on surfaces of constant curvature. Proofs of Proposition 1.5 and Theorem 1.6

In Subsection 2.1 we prove Proposition 1.5. The proof of Theorem 1.6, which follows its known proof in the Euclidean case given in [10, section 7], takes the rest of the section. In Subsection 2.2 we present a differential-geometric background material on normal coordinates and an equivalent definition of geodesic curvature used in the proof of Theorem 1.6 and in what follows. In Subsection 2.3 we prove the following coboundary property: we show that for every $A, B \in \gamma$, set $C = C_{AB}$, the ratio of lengths of the geodesic segments AC and BC equals the ratio of values at A and B of one and the same function on γ. In Subsection 2.4 we deduce Theorem 1.6 from the coboundary property by arguments of elementary planimetry by using Ceva's Theorem.

2.1 Proof of Proposition 1.5

We re-state and prove Proposition 1.5 in a more general Riemannian context. To do this, let us recall the following definition.

Definition 2.1 (classical) Let Σ be a surface equipped with a Riemannian metric, $\gamma \subset \Sigma$ be a (germ of) curve with positive geodesic curvature, and let Γ_p denote the family of curves obtained from the curve γ by string construction. We say that γ has *evolution property,* if for every $p_1 < p_2$ the curve Γ_{p_1} is a caustic for the curve Γ_{p_2}.

Example 2.2 It is well-known that each conic on a surface Σ of constant curvature has evolution property, and the corresponding curves Γ_p given by string construction are confocal conics. In the Euclidean case this follows from the classical fact saying that the caustics of a billiard in a conic are confocal conics (Proclus–Poncelet Theorem in the Euclidean case). Analogous statements hold in non-zero constant curvature and in higher dimensions, see [17, theorem 3].

Proposition 2.3 *Let a curve γ on a surface equipped with a Riemannian metric have evolution property. Then it has string Poritsky property. The reflections from the corresponding curves Γ_p commute as transformations acting on the space of oriented geodesics.*

Remark 2.4 In the Euclidean case Proposition 2.3 with a proof is implicitly contained in [10]. Its proof presented below follows arguments analogous to those from [10].

Proof of Proposition 2.3. Without loss of generality we deal with γ as a germ of curve at O and with Γ_p as a germ of family of curves (foliation) at a point O. We orient the leaves Γ_p of this foliation so that the orientation depends continuously on the transverse parameter.

Consider the billiard reflections from the curves Γ_p acting on the manifold of oriented geodesics. They preserve a canonical area form ω on the latter space, one and the same for all the reflections.

For every p let Γ_p^* denote the family or geodesics tangent to Γ_p and oriented as Γ_p. For every $p_1 < p_2$ the curve $\Gamma_{p_1}^*$ is invariant under the reflection from the curve Γ_{p_2} (evolution property). The curves Γ_p^* form a germ of foliation F in the space of oriented geodesics, a foliation by level curves of a regular function ψ. The base point of the germ is the geodesic tangent to γ at O. The area form ω is the product of a transverse invariant measure given by $d\psi$ of the foliation F and a family of length elements λ_p on the leaves Γ_p^*. Each length element λ_p on the invariant curve Γ_p^* should be invariant under the reflections from the curves Γ_q with $q > p$, as are the function ψ and the above area form. Therefore, the reflections from the curves Γ_q^* act by translations along the curves Γ_p^*, $q > p$, in the corresponding length parameter, and hence, commute. In particular, they act on Γ_0^* by translations in the length element λ_0. For every point $A \in \gamma$ let G_A denote the geodesic tangent to γ at A and oriented in the same way, as γ. The correspondence $A \mapsto G_A$ is injective (positivity of geodesic curvature). The length parameter λ_0 of the curve $\gamma^* = \Gamma_0^*$ induces a local parameter denoted by t on the curve γ via the above correspondence $A \mapsto G_A$. The curve γ has string Poritsky property with respect to the parameter t, by Definition 1.3 and the above translation property. Proposition 2.3 is proved. □

Proposition 1.5 follows from Proposition 2.3 and Example 2.2.

2.2 Background material from Riemannian geometry

Let Σ be a two-dimensional surface equipped with a C^3-smooth Riemannian metric g. Let $O \in \Sigma$. Let γ be a C^2-smooth germ of curve at O parametrized by its natural length parameter. Recall that its geodesic curvature $\kappa = \kappa(O)$ equals the orthogonal projection to $(T_O\gamma)^\perp$ of the covariant derivative $\nabla_{\dot{\gamma}}\dot{\gamma}$. In the Euclidean case it coincides with the Euclidean curvature: the inverse of the osculating circle radius.

Consider the exponential chart $\exp : v \mapsto \exp(v)$ parametrizing a neighborhood of the point O by a neighborhood of zero in the tangent plane $T_O\Sigma$. We introduce orthogonal linear coordinates (x, y) on $T_O\Sigma$, which together with the exponential chart, induce **normal coordinates** centered at O, also denoted by (x, y) on a neighborhood of the point O. It is well-known that in normal coordinates the metric has vanishing 1-jet and its Christoffel symbols vanish at O.

Proposition 2.5 *The geodesic curvature $\kappa(O)$ of the germ γ equals its Euclidean geodesic curvature in normal coordinates. If the normal coordinates (x, y) are chosen so that the x-axis is tangent to γ, then γ is the graph of a germ of function:*

$$\gamma = \{y = f(x)\}, \quad f(x) = \frac{\kappa(O)}{2}x^2 + o(x^2), \text{ as } x \to 0. \tag{2.1}$$

The proposition follows from definition and vanishing of the Christoffel symbols at O in normal coordinates.

For every point $A \in \Sigma$ lying in the normal chart (x, y) centered at O and every tangent vector $v \in T_A\Sigma$ set

$$\mathrm{az}(v) := \text{ the azimuth of the vector } v: \text{ its angle with the } x-\text{axis.}$$

Here by angle we mean the angle in the Euclidean metric in the coordinates (x, y). The azimuth of an oriented one-dimensional subspace in $T_A\Sigma$ is defined analogously.

Proposition 2.6 *Consider a point $A \in \Sigma$ close to O and a geodesic α through A parametrized by the natural length parameter s; $s \mapsto \alpha(s)$, $\alpha(0) = A$.*

1) Consider the geodesic α as a planar curve in the coordinates (x, y), and let $\kappa = \kappa(s)$ denote its curvature with respect to the standard Euclidean metric in the coordinates (x, y). One has

$$\kappa(s) = O(\mathrm{dist}(A, O) + |s|), \text{ as } A \to O, \ s \to 0. \tag{2.2}$$

2) Set $v(s) = \dot{\alpha}(s)$. One has

$$\frac{d\,\mathrm{az}(v(s))}{ds} = O(\mathrm{dist}(\alpha(s), O)), \text{ as } \alpha(s) \to O. \tag{2.3}$$

Proof In the coordinates (x, y) the geodesics are solutions of the second order ordinary differential equation saying that $\ddot{\alpha}$ equals a quadratic form from the vector $\dot\alpha$ with coefficients equal to appropriate Christoffel symbols of the metric g, and $|\dot\alpha| = 1$ in the metric g. In the normal coordinates (x, y) the Christoffel symbols vanish at O, and hence, their values at the point $\alpha(s)$ are $O(\mathrm{dist}(\alpha(s), O))$. This together with the above statement implies (2.3).

Let $\widetilde{s}$ denote the Euclidean natural parameter of the curve α, with respect to the standard Euclidean metric in the chart (x, y). Recall that $\kappa(s) = \frac{d\,\mathrm{az}(v(s))}{d\widetilde{s}}$. One has $\frac{d\widetilde{s}}{ds} = 1 + O(\mathrm{dist}(\alpha(s), O)^2)$, since the metric g has vanishing first jet at O. The two latter formulas together with (2.3) imply (2.2). The proposition is proved. □

Proposition 2.7 *Let $\alpha_t, \beta_t \subset \Sigma$ be two families of geodesics parametrized by the natural length parameter s and issued from the same point $A_t = \alpha_t(0) = \beta_t(0)$. Let ϕ_t denote the angle*[2] *between the geodesics at A_t. Let $A_t \to O$ and $\phi_t \to 0$. Then*

$$\mathrm{az}(\dot\alpha_t(s)) - \mathrm{az}(\dot\beta_t(s)) \simeq \phi_t(1 + O(s\,\mathrm{dist}(A_t, O))). \text{ as } A_t \to O \text{ and } s \to 0. \tag{2.4}$$

Proof A geodesic is a solution of a second order differential equation with a given initial condition: a point $A \in \Sigma$ and a unit vector $v = v(0) \in T_A\Sigma$. It depends smoothly on the initial condition. The derivative of the solution in the initial conditions is a linear operator (2x2-matrix) function in s that is a solution of the corresponding linear equation in variations; the corresponding initial condition is the identity matrix. This implies that the left-hand side in (2.4) equals $\phi_t(1+O(s))$. In the case, when the geodesics are issued from the origin O, the azimuths under question remain constant: equal to the azimuths of the initial conditions $v_1 := \dot\alpha(0)$ and $v_2 := \dot\beta(0)$. This implies that the left-hand side in (2.4) equals $\phi_t(1 + u(s, A_t, v_1, v_2))$, where u is a smooth function vanishing whenever either $s = 0$, or $A_t = O$. This implies that $u = O(s\,\mathrm{dist}(A_t, O))$ and proves (2.4). □

Proposition 2.8 *Let Σ be a two-dimensional surface equipped with a C^3-smooth Riemannian metric. Let $O \in \Sigma$, and let γ be a C^2-smooth germ of oriented curve at O with non-zero geodesic curvature at O. For every $A \in \gamma$ let G_A denote the geodesic tangent to γ at A. For every $A, B \in \gamma$ close*

[2]Everywhere below, whenever the contrary is not specified, for every point $A \in \Sigma$ by *angle* between two vectors $v_1, v_2 \in T_A\Sigma$ we mean angle with respect to the Riemannian metric on the surface Σ. The angle between two oriented curves intersecting at A is the angle between their orienting tangent vectors at A.

enough to O let $C = C_{AB}$ denote the point of intersection $G_A \cap G_B$ such that the geodesic arcs AC and BC are also close to O. Let $\alpha(A,B)$ denote the angle between the geodesics G_A and G_B at C, and let $\lambda(A,B)$ denote the length of the arc AB of the curve γ. The geodesic curvature $\kappa(O)$ of the curve γ at O can be found (up to sign) from any of the two following limits:

$$\kappa(O) = \lim_{A,B\to O} \frac{\alpha(A,B)}{\lambda(A,B)}. \tag{2.5}$$

$$\kappa(O) = \lim_{A,B\to O} 2\frac{\text{dist}(B,G_A)}{\lambda(A,B)^2} \tag{2.6}$$

Proof In the Euclidean case formulas (2.5) and (2.6) are classical. Let us check them in the general Riemannian case in normal coordinates (x,y) centered at O chosen so that the x-axis be tangent to γ at O.

Formula (2.6) follows immediately from (2.1).

Let us prove formula (2.5). The azimuths of the tangent vectors to the geodesics G_A and G_B at the points A and B respectively differ by a quantity asymptotic to $\kappa(O)\lambda(A,B)$, as $A,B \to O$, by (2.1). Their intersection point C_{AB} is $O(\lambda(A,B))$-close to A and B. Therefore, the azimuth of the geodesic G_A at C_{AB} differs from its azimuth at A by a smaller quantity $o(\lambda(A,B))$, as $A,B \to O$, by (2.3), and the same statement holds for the geodesic G_B. Finally, the difference of azimuths of the geodesics G_A and G_B at C_{AB} is asymptotic to $\kappa(O)\lambda(A,B)$, as is the above difference of their azimuths at A and B. This implies (2.5). □

2.3 Preparatory coboundary property of length ratio

Let Σ be the surface of constant curvature $K \in \{0,\pm1\}$ under question: either Euclidean plane, or unit sphere in $\mathbb{R}^3$, or hyperbolic plane. Let $O \in \Sigma$, and let $\gamma \subset \Sigma$ be a regular germ of curve through O with positive geodesic curvature with respect to its orientation. For every point $X \in \gamma$ by G_X we denote the geodesic tangent to γ at X. Let $A, B \in \gamma$ be two distinct points close to O such that the curve γ be oriented from B to A. Let $C = C_{AB}$ denote the unique intersection point of the geodesics G_A and G_B and A that is close to O. Set

$$L_+(A,B) := |CA|;\ L_-(A,B) := |CB|;$$

here $|CX|$ is the length of the geodesic arc CX, $X = A, B$. Set

$$\psi(x) = \begin{cases} x, \text{ if } \Sigma \text{ is Euclidean plane,} \\ \sin x \text{ if } \Sigma \text{ is unit sphere,} \\ \sinh x \text{ if } \Sigma \text{ is hyperbolic plane.} \end{cases} \tag{2.7}$$

Proposition 2.9 *Let Σ be as above, $\gamma \subset \Sigma$ be a germ of curve at a point $O \in \Sigma$ with string Poritsky property. There exists a positive smooth function $u(X)$, $X \in \gamma$, such that for every $A, B \in \gamma$ close enough to O one has*

$$\frac{\psi(L_+(A,B))}{\psi(L_-(A,B))} = \frac{u(A)}{u(B)}. \tag{2.8}$$

In the proof of Proposition 2.9 we use the following well-known proposition.

Proposition 2.10 *(classical) Let Σ be a surface of constant curvature: either Euclidean plane, or the unit sphere in $\mathbb{R}^3$, or the hyperbolic plane. For every $r > 0$ the length of a circle of radius r in the metric under question equals $2\pi\psi(r)$, see (2.7)*

Proof The statement of the proposition in the planar case is obvious.

a) Spherical case. Without loss of generality let us place the center O of the circle under question to the north pole $(0, 0, 1)$ in the Euclidean coordinates (x_1, x_2, x_3) on the ambient space. Since each geodesic is a big circle of length 2π and due to symmetry, without loss of generality we consider that $0 < r \leq \frac{\pi}{2}$. Then the disk in Σ centered at O of radius r is 1-to-1 projected to the disk of radius $\sin r$ in the coordinate (x_1, x_2)-plane, and the length of its boundary obviously equals the Euclidean length of the boundary of its projection, that is, $2\pi \sin r$. This proves statement a).

b) Case of hyperbolic plane. We consider the hyperbolic plane in the model of unit disk equipped with the metric $\frac{2|dz|}{1-|z|^2}$ in the complex coordinate z. For every $R > 0$, $R < 1$ the Euclidean circle $\{|z| = R\}$ of radius R is a hyperbolic circle of radius

$$r = \int_0^R \frac{2ds}{1-s^2} = \log\left|\frac{1+R}{1-R}\right|.$$

The hyperbolic length of the same circle equals $L = \frac{4\pi R}{1-R^2}$. Substituting the former formula to the latter one yields

$$R = \frac{e^r - 1}{e^r + 1}, \ L = 2\pi \sinh r$$

and finishes the proof of the proposition. $\square$

Proof of Proposition 2.9. For every $p > 0$ small enough and every $C \in \Gamma_p$ close enough to O there are two geodesics issued from the point C that are tangent to γ. The corresponding tangency points $A = A(C)$ and $B = B(C)$ in γ depend smoothly on the point C, and $C = C_{AB}$. Both latter statements follow from positivity of the geodesic curvature of the curve γ and the Implicit Function Theorem. Let s_p denote the natural length parameter of the curve Γ_p. We set $s = s_0$: the natural length parameter of the curve γ. We write $C = C(s_p)$, and consider $A(C(s_p))$ and $B(C(s_p))$ as functions of s_p: $A = A(s_p)$, $B = B(s_p)$. We will obtain formulas for their derivatives in s_p and we will see that together with the string Poritsky property, the latter formulas will imply the statement of the proposition.

Note that the curves Γ_p have canonical orientation induced by the orientation of the curve γ so that the orienting unit vector field on Γ_p including $\Gamma_0 = \gamma$ depends continuously on the parameter p.

The proof of Proposition 2.9 repeats the arguments from [10, section 7] given there in the Euclidean case. Fix an initial point $C \in \Gamma_p$. Let us normalize the length parameter s_p so that the point $C = C(0)$ correspond to $s_p = 0$. For every $\varepsilon > 0$ let $X(\varepsilon)$ ($Y(\varepsilon)$) denote the intersection point of the geodesics $C(\varepsilon)A(\varepsilon)$ and $C(0)A(0)$ (respectively, $C(\varepsilon)B(\varepsilon)$ and $C(0)B(0)$). One has $X(\varepsilon) \to A(0)$, $Y(\varepsilon) \to B(0)$ as $\varepsilon \to 0$, since all the geodesics $C(\varepsilon)A(\varepsilon)$, $C(\varepsilon)B(\varepsilon)$ are tangent to the same curve γ. Set

$$\alpha := \text{ the angle between } T_{C(0)}\Gamma_p \text{ and the geodesic } A(0)C(0) \text{ at } C(0),$$

$$L_+ := L_+(A(0), B(0)),$$

$$\phi_A(\varepsilon) := \text{ the angle between the geodesics } X(\varepsilon)C(\varepsilon) \text{ and } X(\varepsilon)C(0) \text{ at } X(\varepsilon),$$

$$\phi_B(\varepsilon) := \text{ the angle between the geodesics } Y(\varepsilon)C(\varepsilon) \text{ and } Y(\varepsilon)C(0) \text{ at } Y(\varepsilon);$$

in these formulas the geodesics under question are oriented from $X(\varepsilon)$ to $C(\varepsilon)$ ($C(0)$) etc. so that

$$\phi_A(\varepsilon), \phi_B(\varepsilon) \to 0, \text{ as } \varepsilon \to 0.$$

Claim 1. *One has*

$$\phi_A(\varepsilon) \simeq \varepsilon \sin\alpha(\psi(L_+))^{-1}, \ \phi_B(\varepsilon) \simeq \varepsilon \sin\alpha(\psi(L_-))^{-1}, \text{ as } \varepsilon \to 0. \quad (2.9)$$

Proof We prove only the first equality in (2.9); the proof of the second one is analogous. Let $D(\varepsilon)$ denote the point of the geodesic $X(\varepsilon)C(0)$ lying on the same distance from the point $X(\varepsilon)$, as $C(\varepsilon)$, and on the same side from the point $X(\varepsilon)$, as $C(0)$. Let S_ε denote the circular arc centered at $X(\varepsilon)$ of radius $|X(\varepsilon)C(\varepsilon)|$ that connects the points $D(\varepsilon)$ and $C(\varepsilon)$. The angle ϕ_A equals the length of the arc S_ε divided by $\psi(|X(\varepsilon)C(\varepsilon)|)$, by Proposition 2.10 and homogeneity of the surface Σ: the action of rotations on the space $T_{X(\varepsilon)}\Sigma$ extends to their action on Σ by isometries. The latter value of the function ψ tends to $\psi(|A(0)C(0)|) = \psi(L_+)$, as $\varepsilon \to 0$, by construction. The length of the circular arc S_ε is asymptotic to the distance of the point $C(\varepsilon)$ to the geodesic $X(\varepsilon)C(0) = A(0)C(0)$. The latter distance is asymptotic to $\varepsilon \sin\alpha$, by construction: the length of the arc $C(\varepsilon)C(0) \subset \Gamma_p$ equals ε, while the angle at $C(\varepsilon)$ between the latter arc and the geodesic $X(\varepsilon)C(\varepsilon)$ tends to the angle α, as $\varepsilon \to 0$. Finally the above discussion implies that

$$\phi_A = \frac{|S_\varepsilon|}{\psi(L_+)} \simeq \varepsilon \sin\alpha(\psi(L_+))^{-1}$$

and proves the first formula in (2.9). □

One has

$$s(A(\varepsilon)) - s(A(0)) \simeq \kappa^{-1}(A(0))\phi_A \simeq \varepsilon\kappa^{-1}(A(0)) \sin\alpha(\psi(L_+))^{-1}, \quad (2.10)$$

$$s(B(\varepsilon)) - s(B(0)) \simeq \kappa^{-1}(B(0))\phi_B \simeq \varepsilon\kappa^{-1}(B(0)) \sin\alpha(\psi(L_+))^{-1}, \quad (2.11)$$

by (2.5) and (2.9).

Let now t be the Poritsky length parameter of the curve γ, and let $s = s_0$ be its natural length parameter. Set

$$\nu := \frac{dt}{ds}.$$

This is a function on the curve γ. Recall that

$$t(A(\varepsilon)) - t(A(0)) = t(B(\varepsilon)) - t(B(0)),$$

by the string Poritsky property. Taking asymptotics of the latter equality, as $\varepsilon \to 0$, we get

$$\nu(A(0))(s(A(\varepsilon)) - s(A(0))) \simeq \nu(B(0))(s(B(\varepsilon)) - s(B(0))).$$

Substituting (2.10) and (2.11) to the latter asymptotic equality yields

$$\frac{\nu}{\kappa}(A(0))(\psi(L_+))^{-1}(A(0), B(0)) = \frac{\nu}{\kappa}(B(0))(\psi(L_-))^{-1}(A(0), B(0)).$$

This implies equality (2.8) with

$$u = \frac{\nu}{\kappa}.$$

Proposition 2.9 is proved. □

2.4 Conics and Ceva's Theorem on surfaces of constant curvature. Proof of Theorem 1.6

Definition 2.11 Let Σ be a surface with Riemannian metric. We say that a germ of curve $\gamma \subset \Sigma$ with non-zero geodesic curvature has *tangent incidence property,* if the following statement holds. Let $A', B', C' \in \gamma$ be arbitrary three distinct points close enough to the base point of the germ γ. Let a, b, c denote the geodesics tangent to γ at A', B', C' respectively. Let A, B, C denote the points of intersection $b \cap c$, $c \cap a$, $a \cap b$ respectively. Then the geodesics AA', BB', CC' intersect at one point. See [10, p.462, fig.5] and Fig.1 below.

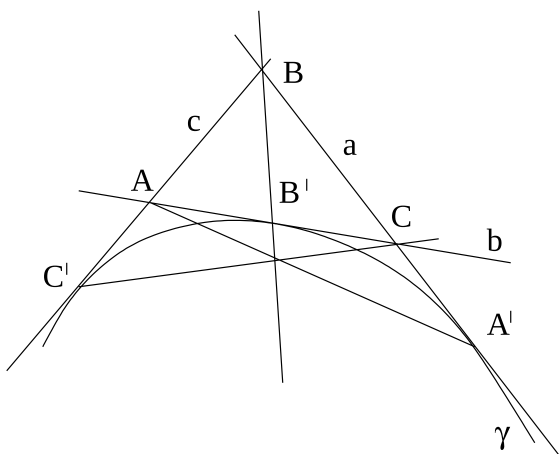

Figure 1: A curve γ with tangent incidence property

Proposition 2.12 *Every curve with string Poritsky property on a surface of constant curvature has tangent incidence property.*

As it is shown below, Proposition 2.12 follows from Proposition 2.9 and the next theorem.

Theorem 2.13 *[8, pp. 3101–3103] (Ceva's Theorem on surfaces of constant curvature.) Let Σ be a simply connected complete surface of constant curvature. Let $\psi(x)$ be the corresponding function in (2.7): the length of* *circle of radius x divided by 2π. Let $A, B, C \in \Sigma$ be three distinct points. Let A', B', C' be respectively some points on the* **sides** *BC, CA, AB of the geodesic triangle ABC. For every $X, Y \in \Sigma$ let $|XY|$ denote the length of the geodesic arc connecting X and Y. Then the geodesics AA', BB', CC' intersect at one point, if and only if*

$$\frac{\psi(|AB'|)}{\psi(|B'C|)}\frac{\psi(|CA'|)}{\psi(|A'B|)}\frac{\psi(|BC'|)}{\psi(|C'A|)} = 1. \tag{2.12}$$

Addendum to Theorem 2.13. *Let now in the conditions of Theorem 2.13 A', B', C' be points on the* **geodesics** *BC, CA, AB respectively so that some two of them, say A', C' do not lie on the corresponding* **sides** *and the remaining third point B' lies on the corresponding side AC.*

1) In the Euclidean and spherical case the geodesics AA', BB', CC' intersect at the same point, if and only if (2.12) holds.

2) In the hyperbolic case (when Σ is of negative curvature) the geodesics AA', BB', CC' intersect at the same point, if and only if some two of them intersect and (2.12) holds.

3) Consider the standard model of the hyperbolic plane Σ in the Minkovski space $\mathbb{R}^3$. Consider the 2-subspaces defining the geodesics AA', BB', CC', and let us denote the corresponding projective lines (i.e., their tautological projections to $\mathbb{RP}^2$) by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ respectively. The projective lines $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ intersect at one point (which may be not the projection of a point in Σ), if and only if (2.12) holds.

Proof Statements 1) and 2) of the addendum follows from Theorem 2.13 by analytic extension, when some two points A' and B' go out of the corressponding sides BC, CA while remaining on the same geodesics BC, CA. Statement 3) is proved analogously. □

Proof of Proposition 2.12. Let O be the base point of the germ γ, and let A', B', C' be its three subsequent points close enough to O. Let a, b, c be respectively the geodesics tangent to γ at them. Then each pair of the latter geodesics intersect at one point close to O. Let A, B, C be the points of intersections $b \cap c$, $c \cap a$, $a \cap b$ respectively. The point B' lies on the geodesic arc $AC \subset b$. This follows from the assumption that the point B' lies between A' and C' on the curve γ and the inequality $\kappa \neq 0$. In a similar way we get that the points A' and C' lie on the corresponding geodesics a and c but outside the sides BC and AB of the geodesic triangle ABC so that A lies between C' and B, and C lies between A' and B. The geodesics BB' and AA' intersect, by the two latter arrangement statements. One has $\frac{\psi(|BA'|)}{\psi(|BC'|)} = \frac{u(A')}{u(C')}$, by (2.8), and similar equalities hold with B replaced by A and C. Multiplying all of the three latter equalities we get that the right-hand side cancels out, and we obtain (2.12). Hence the geodesics AA', BB' and CC' intersect at one point, by statement 2) of the addendum to Theorem 2.13. Proposition 2.12 is proved. $\square$

Theorem 2.14 *Each conic on a surface of constant curvature has tangent incidence property. Vice versa, each curve on a surface of constant curvature that has tangent incidence property is a conic.*

Proof The first, easy statement of the theorem follows from Propositions 1.5 and 2.12. The proof of its second statement repeats the arguments from [10, p.462], which are given in the Euclidean case but remain valid in the other cases of constant curvature without change. Let us repeat them briefly in full generality for completeness of presentation. Let γ be a germ of curve with tangent incidence property on a surface Σ of constant curvature. Let A', B', C' denote three distinct subsequent points of the curve γ, and let a, b, c be respectively the geodesics tangent to γ at these points. Let A, B, C denote respectively the points of intersections $b \cap c$, $c \cap a$, $a \cap b$. Fix the points A' and C'. Consider the pencil $\mathcal{C}$ of conics through A' and C' that are tangent to $T_{A'}\gamma$ and $T_{C'}\gamma$. Then each point of the surface Σ lies in a unique conic in $\mathcal{C}$. Let $\phi \in \mathcal{C}$ denote the conic passing through the point B'.

Claim. *The tangent line $l = T_{B'}\phi$ coincides with $T_{B'}\gamma$.*

Proof Let L denote the geodesic through B' tangent to l. Let C_1 and A_1 denote respectively the points of intersections $L \cap a$ and $L \cap c$. Both curves γ and ϕ have tangent incidence property. Therefore, the three geodesics AA', BB', CC' intersect at the same point denoted X, and the three geodesics AA_1, BB', CC_1 intersect at the same point Y; both X and Y lie on the geodesic BB'. We claim that this is impossible, if $l \neq T_{B'}\gamma$ (or equivalently, if $L \neq b$). Indeed, let to the contrary, $L \neq b$. Let us turn the geodesic b continuously towards L in the family of geodesics b_t through B', $t \in [0, 1]$: $b_0 = b$, $b_1 = L$, the azimuth of the line $T_{B'}b_t$ turns monotonously (clockwise or counterclockwise), as t increases. Let C_t, A_t denote respectively the points of the intersections $b_t \cap a$ and $b_t \cap c$: $C_0 = C'$, $A_0 = A'$. Let X_t denote the point of the intersection of the geodesics AA_t and CC_t: $X_0 = X$, $X_1 = Y$. At the initial position, when $t = 0$, the point X_t lies on the fixed geodesic BB'. As t increases from 0 to 1, the points A and C remain fixes, while the points C_t and A_t move monotonously, so that as C_t moves towards (out from) B along the geodesic a, the point A_t moves out from (towards) B along the geodesic c, see Fig.2. In the first case, when C_t moves towards B and A_t moves out from B, the point X_t moves out of the geodesic BB', to the half-plane bounded by BB' that contains A, and its distance to BB' increases. Hence, $Y = X_1$ does not lie on BB'. The second case is treated analogously. The contradiction thus obtained proves the claim. □

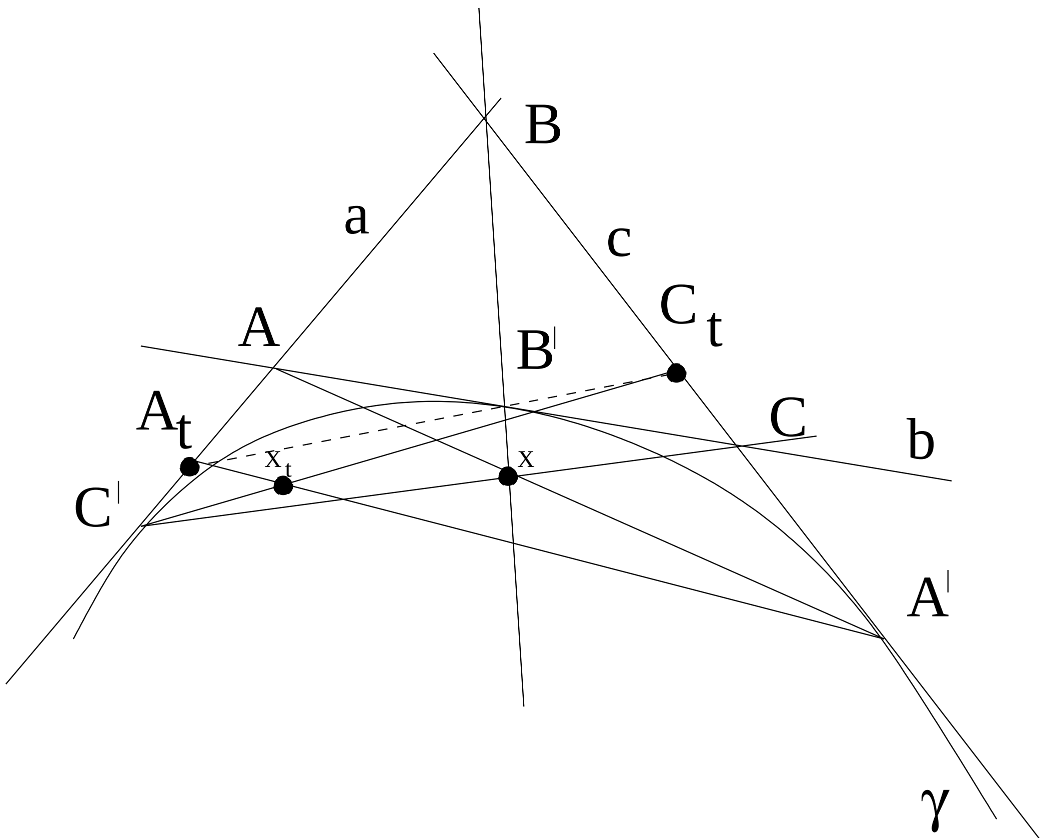

Figure 2: Movement of the intersection point X_t away from the geodesic BB'.

For every point $Q \in \Sigma$ such that the conic $\phi_Q \in \mathcal{C}$ passing through Q is regular, set $l_Q := T_Q\phi_Q$. The lines l_Q form an analytic line field, and its phase curves are the conics from the pencil $\mathcal{C}$. The curve γ is also tangent to the latter line field, by the above claim. Hence, γ is a conic. This proves Theorem 2.14. □

Proof of Theorem 1.6. Let γ be a germ of curve with string Poritsky property on a surface of constant curvature. Then it has tangent incidence property, by Proposition 2.12. Therefore, it is a conic, by Theorem 2.14. Theorem 1.6 is proved. □

3 Case of outer billiards: proof of Theorem 1.9

Everywhere below in the present section Σ is a surface of constant curvature, and $\gamma \subset \Sigma$ is a germ of C^2-smooth curve at a point $O \in \Sigma$ with non-zero geodesic curvature (say, positive with respect to the given orientation of the curve γ).

Proposition 3.1 *Let Σ, O, γ be as above, and let γ have area Poritsky property. Then there exists a function $u : \gamma \to \mathbb{R}_+$ such that for every $A, B \in \gamma$ close enough to O the following statement holds. Let α, β denote the angles between the chord AB with the curve γ at the points A and B respectively. Then*

$$\frac{\sin \beta}{\sin \alpha} = \frac{u(B)}{u(A)}. \tag{3.1}$$

Proof Recall that for every $C, D \in \gamma$ by $\lambda(C, D)$ we denote the length of the arc CD of the curve γ. Fix A and B as above. Set $A(0) = A$. For every small $s > 0$ let $A(s)$ denote the point of the curve γ such that $\lambda(A(s), A(0)) = s$ and the curve γ is oriented from $A(0)$ to $A(s)$. Let $B(s)$ denote the family of points of the curve γ such that the area of the domain bounded by the chord $A(s)B(s)$ and the arc $A(s)B(s)$ of the curve γ remains constant, independent on s. For every s small enough the chord $A(s)B(s)$ intersects the chord $A(0)B(0)$ at a point $X(s)$ tending to the middle of the chord $A(0)B(0)$. This follows from constance of area and homogeneity (constance of curvature) of the surface Σ. Let t denote the area Poritsky parameter of the curve γ. Set

$$u := t'_s = \frac{dt}{ds}.$$

One has

$$t(A(s)) - t(A(0)) = t(B(s)) - t(B(0)) \text{ for every } s,$$

by definition. The left-hand side is asymptotic to $u(A)\lambda(A(0), A(s))$, as $s \to 0$, and analogous statement holds with A replaced by B. Therefore,

$$\frac{\lambda(B(0), B(s))}{\lambda(A(0), A(s))} \simeq \frac{u(A)}{u(B)}, \text{ as } s \to 0. \tag{3.2}$$

The length of the arc $A(0)A(s)$ is asymptotic to the distance of the point $A(s)$ to the geodesic $X(s)A(0) = B(0)A(0)$ divided by $\sin\alpha$, and the same statement holds with A, α replaced by B, β. The distances of the points $A(s)$ and $B(s)$ to the geodesic $A(0)B(0)$ are asymptotic to each other, since the intersection point $X(s)$ of the chords $A(s)B(s)$ and $A(0)B(0)$ tends to the middle of the chord $A(0)B(0)$ and by homogeneity. This implies that the left-hand side in (3.2) tends to the ratio $\frac{\sin\alpha}{\sin\beta}$, as $s \to 0$. This together with (3.2) proves (3.1). □

Proposition 3.2 *Let Σ, O and γ be as at the beginning of the section. Let there exists a function u on γ that satisfies (3.1) for every $A, B \in \gamma$ close to O. Then γ has tangent incidence property, see Definition 2.11.*

Proof Let A', B', C' be three subsequent points of the curve γ. Let a, b, c denote respectively the geodesics tangent to γ at these points. Let A, B, C denote respectively the points of intersections $b \cap c$, $c \cap a$, $a \cap b$ (all the points A', B', C', and hence A, B, C are close enough to the base point O). Let ψ be the same, as in (2.7). One has

$$\frac{\sin(\angle CA'B')}{\sin(\angle CB'A')} = \frac{\psi(|CB'|)}{\psi(|CA'|)} = \frac{u(A')}{u(B')}, \tag{3.3}$$

by (3.1) and Sine Theorem on the Euclidean plane and its analogues for unit sphere and hyperbolic plane applied to the geodesic triangle $CA'B'$, see [5, p.215], [11, theorem 10.4.1]. Similar equalities hold for other pairs of points (B', C'), (C', A'). Multiplying all of them yields relation (2.12): the ratios of values of the function u at A', B', C' cancel out. This together with Theorem 2.13 and its addendum implies that γ has tangent indicence property and proves Proposition 3.2. □

Proof of Theorem 1.9. Let γ be a curve with area Poritsky property on a surface of constant curvature. Then it has tangent incidence property, by Propositions 3.1 and 3.2. Hence, it is a conic, by Theorem 2.14. Theorem 1.9 is proved. □

4 The derivative of the Poritsky length parameter

In this and next sections we deal with an arbitrary Riemannian surface Σ and a germ $\gamma \subset \Sigma$ of regular oriented curve with positive geodesic curvature and string Poritsky property. In the present section we prove Theorem 1.10.

One can prove Theorem 1.10 by direct asymptotic calculations in normal coordinates. We present another proof based on area-preserving property of the billiard ball map and a theorem on interpolating Hamiltonian extending Melrose-Marvizi theorem [7, theorem 3.2]. Their theorem states that the billiard ball map of a $C^\infty-$ smooth planar curve is a shift along appropriate Hamiltonian vector field up to flat terms. In the case of a curve with string Poritsky property one can recover the Poritsky parameter from the above Hamiltonian vector field and deduce Theorem 1.10 in the Euclidean case from the Melrose-Marvizi theorem and its proof. We state and prove a theorem generalizing Theorem 1.10 to more general symplectic mappings that are not necessarily billiard ball maps: the so-called (weakly) billiard-like twist maps (Theorem 4.8 stated in Subsection 4.2 and proved in Subsection 4.4). To do this, we extend the Melrose-Marvizi theorem and its proof to this more general context in Subsection 4.3. The proof of Theorem 1.10 will be given in Subsection 4.5.

4.1 Symplectic properties of billiard ball map

Here we recall a background material on symplecticity of billiard ball map.

Let Σ be a surface with Riemannian metric. Let $\Pi : T\Sigma \to \Sigma$ denote the tautological projection. Let us recall that the *tautological 1-form* α on $T\Sigma$ is defined as follows: for every $(Q, P) \in T\Sigma$ with $Q \in \Sigma$ and $P \in T_Q\Sigma$ for every $v \in T_{(Q,P)}(T\Sigma)$ set

$$\alpha(v) :=< P, \Pi_* v > . \tag{4.1}$$

The differential

$$\omega = d\alpha$$

of the 1-form α is the *canonical symplectic form* on $T\Sigma$.

Let $O \in \Sigma$, and let $\gamma \subset \Sigma$ be a germ of regular oriented curve at O. Let us parametrize it by its natural length parameter s. The corresponding function $s \circ \Pi$ on $T\gamma$ will be also denoted by s. For every $Q \in \gamma$ and $P \in T_Q\gamma$ set

$$\dot\gamma(Q) = \frac{d\gamma}{ds}(Q) := \text{ the directing unit tangent vector to } \gamma \text{ at } Q,$$

$$\sigma(Q, P) :=< P, \dot\gamma(Q) >, \quad y(Q, P) := 1 - \sigma(Q, P).$$

The restriction to $T\gamma$ of the form ω is a symplectic form, which will be denoted by the same symbol ω.

Proposition 4.1 *(see [7, formula (3.1)] in the Euclidean case). The coordinates* (s, y) *on* $T\gamma$ *are symplectic:* $\omega = ds \wedge dy$ *on* $T\gamma$.

Proof The proposition follows from the definition of the symplectic structure $\omega = d\alpha$, α is the same, as in (4.1): in local coordinates (s, σ) one has $\alpha = \sigma ds$, thus, $\omega = d\sigma \wedge ds = ds \wedge dy$. □

Let V denote the Hamiltonian vector field on $T\Sigma$ with the Hamiltonian $||P||^2$: the field V generates the geodesic flow. Consider the unit circle bundle over Σ:

$$S = \mathcal{T}_1\Sigma := \{||P||^2 = 1\} \subset T\Sigma.$$

It is known that for every point $x \in S$ the kernel of the restriction $\omega|_{T_xS}$ is the one-dimensional linear subspace spanned by the vector $V(x)$ of the field V. Each cross-section $W \subset S$ to the field V is identified with the (local) space of geodesics. The symplectic structure ω induces a well-defined symplectic structure on W called the *symplectic reduction.*

Remark 4.2 The symplectic reduction is holonomy invariant: for every arc AB of trajectory of the geodesic flow with endpoints A and B for every two germs of cross-sections W_1 and W_2 through A and B respectively the holonomy mapping $W_1 \to W_2$, $A \mapsto B$ along the arc AB is a symplectomorphism.

Consider the local hypersurface

$$\Gamma = \Pi^{-1}(\gamma) \cap S = (\mathcal{T}_1\Sigma)|_\gamma \subset S.$$

At the points $(Q, P) \in \Gamma$ such that the vector P is transverse to γ the hypersurface Γ is locally a cross-section for the restriction to S of the geodesic flow. Thus, near the latter points the hypersurface Γ carries a canonical symplectic structure given by the symplectic reduction. Set

$$\mathcal{O}_\pm := (O, \pm\dot\gamma(O)) \in \Gamma.$$

For every $(Q, P) \in \Gamma$ close enough to $\mathcal{O}_\pm$ the geodesic issued from the point Q in the direction P (and oriented by P) intersects the curve γ at two points Q and Q'. Let P' denote the orienting unit tangent vector of the latter geodesic at Q'. This defines the germ at $\mathcal{O}_\pm$ of involution

$$\beta : (\Gamma, \mathcal{O}_\pm) \to (\Gamma, \mathcal{O}_\pm), \quad \beta(Q, P) = (Q', P'), \quad \beta^2 = Id, \tag{4.2}$$

that will be called the *billiard ball geodesic correspondence.*

Consider the following open subset in $T\gamma$: the unit ball bundle

$$\mathcal{T}_{\leq 1}\gamma := \{(Q,P) \in T\gamma \mid ||P||^2 \leq 1\}.$$

Let $\pi : (T\Sigma)|_\gamma \to T\gamma$ denote the mapping acting by orthogonal projections

$$\pi : T_Q\Sigma \to T_Q\gamma, \ Q \in \gamma.$$

It induces the following projection also denoted by π:

$$\pi : \Gamma \to \mathcal{T}_{\leq 1}\gamma. \tag{4.3}$$

Let U denote a convex domain with boundary containing γ. Every point $(Q,P) \in \mathcal{T}_{\leq 1}\gamma$ has two π-preimages $(Q, w_\pm)$ in Γ: the vector w_+ (w_-) is directed inside (respectively, outside) the domain U, whenever $||P|| < 1$. The vectors $w_\pm$ coincide, if and only if $||P|| = 1$, and in this case they lie in $T_Q\gamma$. Thus, the mapping $\pi : \Gamma \to \mathcal{T}_{\leq 1}\gamma$ has two continuous inverse branches. Let $\mu_+ := \pi^{-1} : \mathcal{T}_{\leq 1}\gamma \to \Gamma$ denote the inverse branch sending P to w_+, cf. [7, section 2]. The above mappings define the germ of mapping

$$\delta_+ := \pi \circ \beta \circ \mu_+ : (\mathcal{T}_{\leq 1}\gamma, \mathcal{O}_\pm) \to (\mathcal{T}_{\leq 1}\gamma, \mathcal{O}_\pm). \tag{4.4}$$

Recall that Γ carries a canonical symplectic structure given by the above-mentioned symplectic reduction (as a cross-section), and $T\gamma$ carries the standard symplectic structure: the restriction to $T\gamma$ of the form $\omega = ds \wedge dy$.

Theorem 4.3 *(classical) The mappings β, π, and hence, δ_+ given by (4.2)–(4.4) respectively preserve the symplectic structure.*

Proof Symplecticity of the mapping β follows from the definition of symplectic reduction and its holonomy invariance (Remark 4.2). Symplecticity of the projection π follows from the definition of the canonical symplectic structure. In more detail, fix a vector field ν on the bundle $T\Sigma|_\gamma$ that is tangent to its fibers $T_Q\Sigma$ and given by vectors parallel to the orthogonal complement to $T_Q\gamma$. The restriction to $T\Sigma|_\gamma$ of the canonical 1-form α vanishes on ν and is invariant under the flow of the field ν. Therefore, its Lie derivative along the field ν, which is equal to $i_\nu(d\alpha) + d(i_\nu\alpha) = i_\nu\omega$ (the Homotopy Formula), vanishes. This means that the vectors of the field ν lie in the kernel of the restriction to $T\Sigma|_\gamma$ of the symplectic form ω. This implies symplecticity of the mapping π and hence, δ_+. □

Let $I : \Gamma \to \Gamma$ denote the reflection involution

$$I : (Q,P) \mapsto (Q,P^*),$$

$$Q \in \gamma, \ P^* := \text{ the vector symmetric to } P \text{ with respect to the line } T_Q\gamma.$$

Proposition 4.4 *The involutions I and β are C^r-smooth germs of mappings $(\Gamma, \mathcal{O}_\pm) \to (\Gamma, \mathcal{O}_\pm)$, if the surface Σ, its Riemannian metric and the curve γ are C^{r+1}-smooth. The mapping δ_+ is conjugated to their product*

$$\widetilde{\delta}_+ := I \circ \beta = \mu^+ \circ \delta_+ \circ \mu_+^{-1}. \tag{4.5}$$

The proposition follows immediately from definitions.

The billiard transformation T of reflection from the curve γ acts on the space of oriented geodesics that intersect γ and are close enough to the geodesic tangent to γ at O. Each of them intersects γ at two points. To each oriented geodesic G we put into correspondence a point $(Q, P) \in \Gamma = (\mathcal{T}_1\Sigma)|_\gamma$, where Q is its first intersection point with γ (in the sense of the orientation of the geodesic G) and P is the orienting unit vector tangent to G at Q. This is a locally bijective correspondence.

Proposition 4.5 *The billiard mapping T written as a mapping $\Gamma \to \Gamma$ via the above correspondence coincides with $\widetilde{\delta}_+$. Consider the coordinates (s, ϕ) on Γ: $s = s(Q)$ is the natural length parameter of a point $Q \in \gamma$; $\phi = \phi(Q, P)$ is the angle of the vector P with the vector $\dot{\gamma}(Q)$. In the coordinates (s, ϕ) the mappings I, β and $T = \widetilde{\delta}_+$ take the form*

$$I(s, \phi) = (s, -\phi), \ \beta(s, \phi) = (s + \kappa^{-1}(s)\phi + O(\phi^2), -\phi + O(\phi^2)). \tag{4.6}$$

$$\widetilde{\delta}_+(s, \phi) = (s + \kappa^{-1}(s)\phi + O(\phi^2), \phi + O(\phi^2)), \tag{4.7}$$

In the coordinates (s, y) the billiard mapping T coincides with δ_+ and takes the form

$$\delta_+(s, y) = (s + \kappa^{-1}(s)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})). \tag{4.8}$$

Proof All the statements of the proposition except for the formulas follow from definition. Formula (4.6) follows from the definitions of the mappings I and β: a geodesic issued from a point $Q \in \gamma$ at a small angle ϕ with the tangent vector $\dot{\gamma}(Q)$ intersects γ at a point Q' such that $\lambda(Q, Q') = \kappa^{-1}(Q)\phi + O(\phi^2)$, by formula (2.5) and smoothness. Formulas (4.6) and (4.5) imply (4.7), which in its turn implies (4.8). □

4.2 Billiard-like twist maps and Poritsky property of their families

In this and the next subsections we study the following class of area-preserving mappings generalizing the billiard mappings represented in the coordinates (s, y), see (4.8).

Definition 4.6 Consider a germ of mapping

$$F : (\mathbb{R} \times \mathbb{R}_{\geq 0}, (0,0)) \to (\mathbb{R} \times \mathbb{R}_{\geq 0}, (0,0)),$$

$$F : (x, y) \mapsto (x + w(x)\sqrt{y} + O(y), y + O(y^{\frac{3}{2}})), \ w(x) > 0, \tag{4.9}$$

for which the x-axis is a line of fixed points. Let the variable change

$$(x, y) \mapsto (x, \phi), \ y = \phi^2$$

conjugate the germ F with a C^2-smooth germ $\widetilde{F}(x, \phi)$. Let F preserves the standard area form $dx \wedge dy$. Then F will be called *a weakly billiard-like map.* If, in addition to the above assumptions, the mapping $\widetilde{F}$ is a product of two involutions:

$$\widetilde{F} = I \circ \beta, \ I(x, \phi) = (x, -\phi),$$

$$\beta(x, \phi) = (x + w(x)\phi + O(\phi^2), -\phi + O(\phi^2)), \ \beta^2 = Id, \tag{4.10}$$

then F will be called *a (strongly) billiard-like map.*

Definition 4.7 Let F_ε be a family of weakly billiard-like maps depending on a parameter $\varepsilon \in [0, \delta_1)$, $\delta_1 > 0$ and defined on one and the same neighborhood of the origin in $\mathbb{R} \times \mathbb{R}_{\geq 0}$, such that $\widetilde{F}_\varepsilon$ and its derivatives in (x, ϕ) up to the second order are continuous as functions in $(x, y; \varepsilon)$. We say that F_ε has *Poritsky property,* if for every ε small enough the mapping F_ε has an invariant curve γ_ε equipped with a parameter t_ε (which we call the *Poritsky parameter*) that satisfy the following statements.

1) The curves γ_ε are graphs of C^1-smooth functions $\{y = f_\varepsilon(x)\}$ that tend to 0 uniformly with derivative, as $\varepsilon \to 0$; moreover, there exists a positive C^1-smooth function $h(x)$ such that

$$f_\varepsilon(x) = \varepsilon h(x) + o(\varepsilon), \text{ as } \varepsilon \to 0 \tag{4.11}$$

uniformly in $x \in [-\delta, \delta]$ with derivatives for $\delta > 0$ small enough.

2) The parameter functions t_ε on γ_ε considered as functions in x converge with derivatives to a limit C^1-smooth monotonous parameter $t_0 = t(x)$ with $t'(x) > 0$ (called the *limit Poritsky parameter*) uniformly on $[-\delta, \delta]$.

In Subsection 4.4 we prove the following generalization of Theorem 1.10.

Theorem 4.8 *Let a family F_ε of weakly billiard-like maps depending on small parameter $\varepsilon \in [0, \delta_1)$ have Poritsky property, and let $t = t_0(x)$ denote the corresponding limit Poritsky parameter. Let $w(x)$ be the corresponding function from the expression (4.9) for the mapping F_0. Then*

$$\frac{dt}{dx} = cw^{-\frac{2}{3}}(x), \ \ c \equiv const. \tag{4.12}$$

4.3 Interpolating Hamiltonian: a generalization of Melrose-Marvizi theorem

For a function $\zeta(x, y)$ by H_ζ we denote the Hamiltonian vector field with the Hamiltonian ζ. For every vector field v by $\exp(v)$ we denote its unit time flow map acting on the phase space.

Here we prove the following theorem, which extends [7, theorem 3.2] to arbitrary (weakly) billiard-like twist maps and their families.

Theorem 4.9 *Let $F(x, y)$ be a germ of weakly billiard-like map. Then there exists a germ of C^2-smooth function $\zeta(x, y)$ with the following properties.*

$$\zeta(x, y) \simeq a(x)y, \ \ \text{as } y \to 0; \quad a(x) > 0. \tag{4.13}$$

$$F \circ \exp(-\zeta^{\frac{1}{2}} H_\zeta)(x, y) \simeq (x + O(y), y + O(y^2)), \ \ \text{as } y \to 0. \tag{4.14}$$

There exists a $\delta_1 > 0$ such that the asymptotics (4.13) and (4.14) are uniform in $x \in (-\delta_1, \delta_1)$ with derivatives[3].

In addition, the function $a(x)$ from (4.13) is uniquely determined by the function $w(x)$ from (4.9) via the formula

$$a(x) = w^{\frac{2}{3}}(x). \tag{4.15}$$

Addendum 1 to Theorem 4.9. *Let $F(x, y)$ be a strongly billiard-like twist map such that the corresponding conjugated transformation $\widetilde{F}(x, \phi)$, $y = \phi^2$,*

[3] Everywhere where we say that a function $\psi(x, y)$ is $O(y^m)$ (uniformly in $x \in (-\delta, \delta)$) *with derivatives,* as $y \to 0$, we mean that $\psi(x, y), \frac{\partial \psi}{\partial x}(x, y) = O(y^m)$ and $\frac{\partial \psi}{\partial y}(x, y) = O(y^{m-1})$, as $y \to 0$ (uniformly in $x \in (-\delta, \delta)$).

is C^∞-smooth. Then a function ζ as in (4.13), (4.14) can be chosen C^∞-smooth and so that the composition in (4.14) equals $Id + \eta(x,y)$, where $\eta(x,0) = 0$ and the function $\eta(x,y)$ is flat in y:

$$\eta(x,y) = o(y^m) \text{ for every } m \in \mathbb{N} \text{ uniformly with derivatives, as } y \to 0. \tag{4.16}$$

Addendum 2 to Theorem 4.9. *Let in the conditions of Theorem 4.9 $F(x,y;\varepsilon)$ be a family of weakly billiard-like germs depending on a parameter $\varepsilon \in [0,\delta]$. Let the corresponding family of transformations $\widetilde{F}(x,\phi;\varepsilon)$ depend continuously on ε, together with its derivatives in (x,ϕ) up to order 2. Then the corresponding C^2-smooth functions $\zeta = \zeta(x,y;\varepsilon)$ can be chosen to depend continuously on ε (with derivatives up to order 2), and so that asymptotics (4.13), (4.14) are uniform in the parameter ε. If in addition, F is strongly billiard-like and the transformations $\widetilde{F}(x,\phi;\varepsilon)$ are C^∞ in (x,ϕ) and depend continuously on ε with all the derivatives in (x,ϕ), then for every $m \in \mathbb{N}$ the corresponding asymptotics (4.16) is uniform in ε with derivatives in (x,y).*

Proof The proof of Theorem 4.9 and its addendums presented below repeats the arguments from [7, proof of theorem (3.2)] with minor changes.

Step 1. Constructing a C^2-smooth function $t(x,y)$ such that

$$t \circ F - t = O(t^2), \tag{4.17}$$

$$t(x,y) \simeq g_1(x)y, \text{ as } y \to 0, \ g_1(x) > 0. \tag{4.18}$$

The mapping $\widetilde{F}$ can be written in the following form:

$$(x',\phi') = \widetilde{F}(x,\phi), \ x' = x + w(x)\phi + O(\phi^2), \ \phi' = \phi + q(x)\phi^2 + O(\phi^3),$$

and then the corresponding mapping F is written as follows:

$$F(x,y) = (x + w(x)\sqrt{y} + O(y), y + 2q(x)y^{\frac{3}{2}} + o(y^{\frac{3}{2}}).$$

In the coordinates (x,ϕ) every function t as in (4.18) is asymptotic to $g_1(x)\phi^2$, and the corresponding difference in (4.17) has always vanishing ϕ^2-term. In order to make it of order $O(y^2) = O(\phi^4)$, we have to achieve that its ϕ^3-term vanishes as well. This is equivalent to the differential equation

$$w(x)\frac{dg_1(x)}{dx} + 2q(s)g_1(x) = 0, \tag{4.19}$$

which has a unique positive solution up to constant factor. The function $t = g_1(x)y$ is a one we are looking for.

Step 1, case of Addendum 1: $\widetilde{F} = I \circ \beta$ is C^∞-smooth. Then we are looking for a C^∞-smooth function $t(x,y)$ such that

$$t \circ \widetilde{F} - t = o(t^m) \text{ for every } m \in \mathbb{N},\ t \simeq g_1(x)y, \text{ as } y \to 0;\ g_1 > 0.$$

To do this, we first construct a formal Taylor series $g(x,\phi) = \sum_{j=1}^{\infty} g_j(x)\phi^{2j}$ even in ϕ such that the difference $g \circ \widetilde{F} - g = g \circ \beta - g$ is also a series even in ϕ. To achieve this, we have to make sure that its coefficients at odd powers ϕ^{2j+1} vanish. This is true, if and only if the function g_j satisfies a differential equation saying that $\frac{dg_j(x)}{dx}$ equals an expression involving the function g_j and the derivatives of the functions g_k with $k < j$. Such an equation can be always solved. The first equation on the function $g_1(x)$ is the same, as before, and we choose $g_1(x)$ to be any its positive solution. Continuing this procedure yields a series $g(x,\phi)$ we are looking for. Set

$$t := g \circ \beta + g.$$

This is a Taylor series even in ϕ and β-invariant, since β is an involution and the coefficient at ϕ^2 of the series under questioon is $2g_1(x) > 0$, by construction. Take its representative t: a smooth function even in ϕ with the latter asymptotic Taylor series in ϕ. Writing it in the variables (x,y) yields a function we are looking for.

Step 2. Construction of the function ζ. Let $t(x,y)$ be the function constructed on Step 1. The Hamiltonian vector field with the Hamiltonian t has the form

$$H_t = g_1(x)\frac{\partial}{\partial x} - y\frac{dg_1(x)}{dx}\frac{\partial}{\partial y} + o(y),$$

by definition. Fix an arbitrary germ of function τ such that

$$d\tau(H_t) \equiv 1,\ \tau|_{x=0} = 0. \tag{4.20}$$

Then (τ, t) are symplectic coordinates for the form ω: $\omega = d\tau \wedge dt$. The transformation F acts in the coordinates (τ, t) as

$$(\tau, t) \mapsto (\tau + c\sqrt{t} + O(t), t + O(t^2)),\ c = const > 0. \tag{4.21}$$

Indeed, it follows from definition that F sends (τ, t) to $(\tau + c(\tau)\sqrt{t} + O(t), t + O(t^2))$, $c(x) > 0$. The function $c(\tau)$ is constant, since the transformation F is symplectic, thus preserves the form $d\tau \wedge dt$: otherwise, $F^*\omega - \omega = c'(\tau)\sqrt{t}\omega + O(t) \not\equiv 0$, – a contradiction.

Now let us construct the function $\zeta(x,y)=R(t)$, $R(0)=0$, such that

$$\zeta^{\frac{1}{2}}H_{\zeta}=c\sqrt{t}\frac{\partial}{\partial\tau};\ c \text{ is the same, as in (4.21).}$$

In the coordinates (τ,t) one has $H_t=\frac{\partial}{\partial\tau}$. Then in the coordinates (τ,t) one should have $R^{\frac{1}{2}}(t)R'(t)=c\sqrt{t}$, i.e., $R(t)=tc^{\frac{2}{3}})$. By construction,

$$F\circ\exp(-\zeta^{\frac{1}{2}}H_{\zeta}):(\tau,t)\mapsto(\tau+O(t),t+O(t^2)).$$

This implies a similar statement (4.14) in the coordinates (x,y).

Step 2: case of Addendum 1. Let t be the function constructed on Step 1, and let τ be as above. Then in the coordinates (t,τ) the transformation F acts as

$$(\tau,t)\mapsto(\tau+c(t)\sqrt{t},t)+o(t^m),\ \text{for every } m\in\mathbb{N};\ c(0)>0.$$

Independence of the coefficient c on the variable τ follows from symplecticity, as in the above discussion. We construct $\zeta=R(t)$ so that

$$\zeta^{\frac{1}{2}}H_{\zeta}=c(t)\sqrt{t}\frac{\partial}{\partial\tau}.$$

This is equivalent to the equality $R^{\frac{1}{2}}R'(t)=c(t)\sqrt{t}$. Set

$$R(t)=(\frac{3}{2}\int_0^t c(s)s^{\frac{1}{2}}ds)^{\frac{2}{3}}.$$

The function $R(t)$ satisfies the latter differential equation. Set $\zeta=R(t(x,y))$. Then the composition in (4.14) differs from the identity by $o(y^m)$ for every m, as in the above discussion.

Step 3. Calculation of the linear term in y of the function ζ. The function $\zeta(x,y)$ and its Hamiltonian H_{ζ} have the form

$$\zeta(x,y)=a(x)y+O(y^2),\ H_{\zeta}=a(x)\frac{\partial}{\partial x}+O(y),\ a(x)>0, \tag{4.22}$$

by construction. Let η be the function such that $d\eta(H_{\zeta})=1$ and $\eta|_{x=0}=0$. Set

$$\nu(x)=\int_0^x a^{-1}(s)ds.$$

One has

$$\eta(x,y)=\nu(x)+O(y), \tag{4.23}$$

by definition. Then (η, ζ) are symplectic coordinates, as in the above discussion, and in the new coordinates one has $H_\zeta = \frac{\partial}{\partial \eta}$,

$$F : (\eta, \zeta) \mapsto (\eta + \zeta^{\frac{1}{2}} + O(\zeta), \zeta + O(\zeta^2)).$$

Writing the latter transformation in the old coordinates $(x, y) = G(\eta, \zeta)$ yields

$$F : (x, y) \mapsto G(\nu(x) + a^{\frac{1}{2}}(x)\sqrt{y} + O(y), a(x)y + O(y^2)),$$

by (4.22) and (4.23). The x-component of the latter right-hand side equals

$$x + (\nu'(x))^{-1} a^{\frac{1}{2}}(x)\sqrt{y} + O(y) = x + a^{\frac{3}{2}}(x)\sqrt{y} + O(y),$$

by (4.22) and (4.23). On the other hand, it should be equal to $x + w(x)\sqrt{y} + O(y)$. Therefore, $a(x) = w^{\frac{2}{3}}(x)$. This proves (4.15).

Case of Addendum 2: dependence on parameter. The corresponding asymptotics from Theorem 4.9 are uniform in ε, by the above construction of the function ζ. Theorem 4.9 is proved. □

4.4 Weakly billiard-like maps with Poritsky property. Proof of Theorem 4.8

Let F_ε be a family of weakly billiard-like maps with Poritsky property. We know that the mapping $F_\varepsilon|_{\gamma_\varepsilon}$ is a translation in the coordinate t_ε by a constant $c(\varepsilon)$, by the definition of the Poritsky parameter.

Proposition 4.10 *Let F_ε and $c(\varepsilon)$ be as above. Then $c(\varepsilon) \to 0$, as $\varepsilon \to 0$, and there exists a $\nu > 0$ such that for every ε small enough one has*

$$c(\varepsilon) \geq \nu\sqrt{\varepsilon}. \tag{4.24}$$

Proof The curves γ_ε tend to the x-axis as C^1-smooth curves, and the restriction $F_\varepsilon|_{\gamma_\varepsilon}$ tends to the identity mapping of the x-axis, by definition. This together with the convergence of the coordinate t_ε to the parameter t_0 implies that $c(\varepsilon) \to 0$. Inequality (4.24) with appropriate $\nu > 0$ follows from (4.9) and (4.11). □

Let ζ_ε denote the family of functions given by Theorem 4.9 applied to the family F_ε. For every ε set

$$v_\varepsilon := \text{ the projection to } T\gamma_\varepsilon \text{ of the restriction } H_\zeta|_{\gamma_\varepsilon}. \tag{4.25}$$

One has

$$(F_\varepsilon)_* v_\varepsilon = v_\varepsilon + O(\varepsilon), \tag{4.26}$$

since the image of the Hamiltonian field H_{ζ_ε} under the mapping F_ε differs from H_{ζ_ε} by a quantity of order $O(y)$, and $y|_{\gamma_\varepsilon} = O(\varepsilon)$, see (4.11), and the latter asymptotics are uniform in small ε.

Proposition 4.11 *Let v_ε be a continuous family of continuous non-zero vector fields on an interval $(-\delta, \delta)$ that depends on a small parameter $\varepsilon \in [0, \delta_1)$. Let $c(\varepsilon)$ be a continuous function on $[0, \delta_1)$, $c(0) = 0$, satisfying (4.24). Let the translation family $F_\varepsilon(t) := t + c(\varepsilon)$ and the vector field family v_ε satisfy (4.26). Then $v_0 = const$.*

Proof For every ε small enough each F_ε-orbit on $(-\delta, \delta)$ forms a $c(\varepsilon)$-net, $c(\varepsilon) \to 0$, as $\varepsilon \to 0$. The distance between its any two neighbor points is no less than $\nu\sqrt{\varepsilon}$, by (4.24). Hence, each orbit consists of at most $4\delta\varepsilon^{-\frac{1}{2}}$ points. The vectors of the field v_ε at neighbor points of an orbit differ one from the other by a quantity $O(\varepsilon)$, by (4.26). Therefore, for every two points A and B of one and the same orbit one has $v_\varepsilon(A) - v_\varepsilon(B) = O(\varepsilon^{\frac{1}{2}})$. Thus, as ε is small, the restriction of the field v_ε to each orbit is constant up to $O(\sqrt{\varepsilon})$. This together with continuity implies that $v_0 = const$ and proves the proposition. □

Let us consider arcs of the curves γ_ε parametrized by an interval $(-\delta, \delta)$ via the parameter t_ε. The family of vector fields v_ε on γ_ε given by (4.25), thus considered as a family of vector fields on the interval $(-\delta, \delta)$, satisfies the conditions of Proposition 4.11. Therefore, $v_0 = const$. Hence, in the coordinate t on γ_0 the Hamiltonian vector field $H_\zeta|_{\gamma_0}$, which is tangent to γ_0, is constant. On the other hand,

$$\zeta_\varepsilon(x, y) \simeq w^{\frac{2}{3}}(x) y, \text{ as } y \to 0,$$

uniformly in small ε, by (4.15). Therefore, one has

$$H_\zeta = w^{\frac{2}{3}}(x) \frac{\partial}{\partial x} + O(y). \tag{4.27}$$

For $\varepsilon = 0$, formula (4.27) together with constance of the field $H_\zeta|_{\gamma_0}$ in the coordinate t imply that $\frac{dt}{dx} = \widetilde{c} w^{-\frac{2}{3}}(x)$, $\widetilde{c} \equiv const$. This proves (4.12) and Theorem 4.8.

4.5 Derivative of the Poritsky–Lazutkin parameter. Proof of Theorem 1.10

In what follows Σ is a two-dimensional surface with a Riemannian metric, $O \in \Sigma$, and $\gamma \subset \Sigma$ is a germ of curve at O: the surface Σ, its metric and the curve γ being C^3-smooth. The following three propositions will be used in the proof of Theorem 1.10.

Proposition 4.12 *Let Σ, O, γ be as above. The corresponding billiard ball map written in symplectic coordinates (s, y), see Proposition 4.5, takes the form (4.8). Its germ at every point $(s_0, 0)$ is a strongly billiard-like map with $w(s) = \kappa^{-1}(s)$.*

Proof The mapping $\widetilde{\delta}_+$ written in the coordinates (s, ϕ), $\phi = \arccos(1-y)$, is a product of two C^2-smooth involutions, see (4.5). This statement remains valid in the coordinates $(s, \widetilde{\phi})$ with $\widetilde{\phi} = \sqrt{y}$, since the coordinate change $(s, \phi) \mapsto (s, \widetilde{\phi})$ is analytic. The area-preserving property of the mapping δ_+ in the coordinates (s, y) follows from Theorem 4.3. This together with formula (4.8) implies the last statement of Proposition 4.12. □

Let Γ_p denote the family of curves obtained from the curve γ by string construction. Let $Conc$ denote the concave part of a neighborhood of the point O, for which the curve γ is a concave part of boundary; by definition, the curve γ is included in $Conc$.

Proposition 4.13 *The curves Γ_p are phase curves of a C^2-smooth line field on $Conc$.*

Proof Consider the line field on $Conc$ defined as follows. For every $Q \in Conc$ close to γ there are two geodesics tangent to γ. Let $v_1, v_2 \in T_Q\Sigma$ denote the directing unit vectors of the latter geodesics that are directed from Q to the tangency points in γ. Let $L(Q)$ denote the one-dimensional subspace in $T_Q\Sigma$ that is the exterior bisector of the angle between the vectors v_1 and v_2: the vectors v_1 and $-v_2$ are symmetric with respect to the line $L(Q)$. The lines $L(Q)$ are tangent to the curves Γ_p and form a C^2-smooth line field on $Conc$. This proves the proposition. □

The phase curves of the above line field form a C^2-smooth foliation by C^3-smooth curves, and they are exactly the curves Γ_p. Now we replace the parameter p by a new parameter in order to make Γ_p a smooth family of curves including at $p = 0$. Namely consider a smooth cross-section T through O to the above foliation. Let ε denote a smooth coordinate on T

such that $\varepsilon(O) = 0$. From now on for every ε we denote by Γ_ε the leaf of the above foliation through the point of the section T with the coordinate ε. Let F_ε denote the billiard ball maps acting on the space of local oriented geodesics by reflection from the curves Γ_ε. For every ε set

$$\gamma_\varepsilon := \text{ the family of positively oriented geodesics tangent to } \gamma.$$

The curve γ_ε is an invariant curve for each F_ε, since γ is a caustic for the billiard on each curve Γ_ε. For every ε let us write the map $F_\varepsilon = \delta_{+,\varepsilon}$ in the coordinates $(s_\varepsilon, y_\varepsilon)$ associated to the curve Γ_ε, see the end of Subsection 4.1. Namely, recall that to each oriented geodesic G close to the geodesic tangent to γ at O we put into correspondence the first point Q_ε of its intersection with the curve Γ_ε (in the sense of orientation of the geodesic G) and its unit orienting tangent vector $\dot{G}(Q_\varepsilon)$. We set $s_\varepsilon = s(Q)$ to be the coordinate of the point Q_ε in the natural parameter of the curve Γ_ε;

$$\phi_\varepsilon := \text{ the angle between the vectors } \dot{G}(G_\varepsilon) \text{ and } \dot{\Gamma}_\varepsilon(Q_\varepsilon);\ y_\varepsilon := 1 - \sigma.$$

Proposition 4.14 *Let a germ γ have string Poritsky property. Let Γ_ε denote the corresponding re-parametrized family of curves given by the string construction, see the above discussion. Let $F_\varepsilon = \delta_+$ denote the above family of billiard ball maps associated to the curves Γ_ε and written in the coordinates $(s_\varepsilon, y_\varepsilon)$. Let t denote the Poritsky parameter* **of the curve** *γ. Let t_ε denote the parameter on the curves γ_ε induced by t via the correspondence "a geodesic tangent to $\gamma \mapsto$ the tangency point". Then* **the family** *F_ε has Poritsky property with respect to the parameters t_ε.*

Proof The family of mappings F_ε, their invariant curves γ_ε and the parameters t_ε on γ_ε satisfy the conditions of Definition 4.7. Hence, the family F_ε has Poritsky property. □

Proof of Theorem 1.10. The family of billiard ball maps F_ε associated to the curves Γ_ε written in the symplectic coordinates $(s_\varepsilon, y_\varepsilon)$ has Poritsky property. The corresponding limit Poritsky parameter t_0 of the family F_ε, which is a parameter on the curve $\gamma_0 = \gamma$, coincides with its Poritsky length parameter t up to constant factor, by construction and Proposition 4.14. On the other hand, the derivative of the limit Poritsky parameter t_0 with respect to the natural parameter $s = s_0$ of the curve γ equals $w^{-\frac{2}{3}}(s)$ up to constant factor, see (4.12); here $w(s) = \kappa^{-1}(s)$, see (4.8). This implies (1.2) and proves Theorem 1.10. □

5 Osculating curves with string Poritsky property. Proof Theorem 1.11

Here we prove Theorem 1.11, which states that a germ of curve with string Poritsky property is uniquely determined by its 4-jet.

Everywhere below for a curve (function) γ by $j^r_p\gamma$ we denote its r-jet at the point p. Set

$$\mathcal{F}^r := \text{ the space of } r\text{-jets of functions of one variable } x \in \mathbb{R}.$$

Let Σ be a C^m-smooth two-dimensional manifold. For every $r \in \mathbb{Z}_{\geq 0}$, $r \leq m$, set

$$\mathcal{J}^r = \mathcal{J}^r(\Sigma) := \text{ the space of } r\text{-jets of regular curves in } \Sigma.$$

In more detail, a *germ of regular curve* is the graph of a germ of function $\{y = h(x)\}$ in appropriate local chart (x, y). Locally a neighborhood in $\mathcal{J}^r$ of the jet of a given C^r-germ of regular curve is thus identified with a neighborhood of a jet in $\mathcal{F}^r$. One has $\dim \mathcal{F}^r = \dim \mathcal{J}^r = r + 2$. There are local coordinates $(x, b_0, \dots, b_r)$ on $\mathcal{F}^r$ defined by the condition that for every jet $j^r_p h \in \mathcal{F}^r$ one has

$$x(j^r_p h) = p, \ b_0(j^r_p h) = h(p), \ b_1(j^r_p h) = h'(p), \dots, b_r(j^r_p h) = h^{(r)}(p). \quad (5.1)$$

Definition 5.1 (see an equivalent definition in [9, pp.122–123]). Consider the space $\mathcal{F}^r$ equipped with the above coordinates $(x, b_0, \dots, b_r)$. The *Cartan (or contact) distribution* $\mathcal{D}_r$ on $\mathcal{F}^r$ is the field of two-dimensional subspaces in its tangent spaces defined by the system of Pfaffian equations

$$db_0 = b_1 dx, \ db_1 = b_2 dx, \ \dots, \ db_{r-1} = b_r dx. \quad (5.2)$$

For every C^m-smooth surface Σ and every $r \leq m$ the *Cartan (or contact) distribution (plane field) on* $\mathcal{J}_r$, which is also denoted by $\mathcal{D}_r$, is defined by (5.2) locally on its domains identified with open subsets in $\mathcal{F}^r$; the distributions (5.2) defined on intersecting domains V_i, V_j with respect to different charts (x_i, y_i) and (x_j, y_j) coincide and yield a global plane field on $\mathcal{J}_r$.

The main result of the present section is the following theorem, which immediately implies Theorem 1.11. The proof of this implication will be given at the end of the section.

Theorem 5.2 *Let Σ be a C^5-smooth two-dimensional surface with a C^5-smooth Riemannian metric. There exists a C^1-smooth line field $\mathcal{P}$ on $\mathcal{J}^4 = \mathcal{J}^4(\Sigma)$ lying in the Cartan plane field $\mathcal{D}$ such that the 4-jet extension of every C^5-smooth curve on Σ with string Poritsky property (if any) is a phase curve of the field $\mathcal{P}$.*

Let γ be a germ of curve with string Poritsky property at a point $A \in \Sigma$. The Poritsky–Lazutkin parameter t on γ is given by already known formula (1.2). We normalize it so that $t(A) = 0$ and identify points of the curve γ with the corresponding values of the parameter t. Poritsky property implies that the function $L(a, a + t) = L(0, t)$ is independent on a, that is, the function

$$\Lambda(t) := L(0,t) - L(-t,0) \tag{5.3}$$

vanishes. For the proof of Theorem 5.2 we show (in the Main Lemma stated in Subsection 5.2) that for every odd $n > 3$ the differential equation $\Lambda^{(n+1)}(0) = 0$ is equivalent to an equation saying that the coordinate $b_n = b'_{n-1}$ of the n-jet of the curve γ is equal to a function of the other coordinates $(x, b_0, \dots, b_{n-1})$. For $n = 5$ this yields an ordinary differential equation on $\mathcal{J}^4$ satisfied by the 4-jet extension of the curve γ. It will be represented by a line field contained in $\mathcal{D}_4$.

The proof of the Main Lemma takes the most of the section. For its proof we study two curves equipped with appropriately normalized parameter t given by (1.2) that have contact of order n. We express the difference of the $(n + 1)$-jets of the corresponding functions $\Lambda(t)$ at 0 in terms of the difference of the coordinates b_n of the n-jets of the curves. To this end, we consider a local chart (x, y) centered at A with x-axis being tangent to γ at A. We compare different quantities related to both curves, all of them being considered as functions of x: the natural parameters, the curvature etc. (Subsections 4.3 and 4.5). In Subsection 4.4 we present asymptotic properties geodesic of triangles, which will be used in the proofs.

5.1 Differential equations in jet spaces defined by the string Poritsky property and the Main Lemma

Let γ be a germ of C^5-smooth curve at a point O on a C^5-smooth surface Σ equipped with a C^5-smooth Riemannian metric. Let s denote the natural oriented length parameter of the curve γ, $s(O) = 0$. Let κ be its geodesic curvature considered as a function $\kappa(s)$, and let $\kappa > 0$. We already know that if the curve γ has string Poritsky property, then its Poritsky–Lazutkin parameter t is expressed as a function of a point $Q \in \gamma$ in terms of the natural parameter s via the formula

$$t(Q) := \kappa^{\frac{1}{3}}(0) \int_0^{s(Q)} \kappa^{\frac{2}{3}}(s) ds \tag{5.4}$$

up to constant factor and addition of constant, see (1.2). On the other hand, we can define the parameter t given by (5.4) on any curve γ, not necessarily having Poritsky property. We identify the points of the curve γ with the corresponding values of the parameter t defined by (5.4); thus, $t(O) = 0$.

Let $G = G_0$ denote the geodesic tangent to γ at its base point O. We will work in normal coordinates (x, y) centered at O, in which G coincides with the x-axis. For every t let G_t denote the geodesic tangent to γ at the point t, and let C_t denote the point of the intersection $G \cap G_t$.

Let $L(A, B)$ the function of $A, B \in \gamma$ defined in (1.1). We consider $L(A, B)$ as a function of the corresponding parameters $t(A)$ and $t(B)$. Recall that

$$L(0, t) = L(O, \gamma(t)) = |OC_t| + |C_t\gamma(t)| - \lambda(0, t), \tag{5.5}$$

where $|OC_t|$, $|C_t\gamma(t)|$ are the lengths of the geodesic segments $OC_t \subset G$ and $C_t\gamma(t) \subset G_t$ respectively, and $\lambda(0, t) = \lambda(O, \gamma(t))$ is the length of the arc $O\gamma(t)$ of the curve γ. Let $\Lambda(t)$ be the same, as in (5.3). If γ has string Poritsky property, then t is its Poritsky–Lazutkin length parameter (Theorem 1.10), and hence,

$$\Lambda(t) = 0 \text{ for every } t > 0 \text{ small enough.} \tag{5.6}$$

The main part of the proof of Theorem 5.2 is the following lemma.

Lemma 5.3 *Let $n \in \mathbb{N}$, $n \geq 3$. Let Σ be a C^{n+1}-smooth surface equipped with a C^{n+1}-smooth Riemannian metric. Let $E \in \Sigma$, and let (x, y) be coordinates on Σ centered at E and parametrizing some its neighborhood $V = V(E) \subset \Sigma$. Let $\mathcal{J}^n_y(V)$ denote the space of n-jets of curves in V that are graphs of C^n-smooth functions $y = y(x)$; thus, it is naturally identified with an open subset $\mathcal{F}^n_y(V) \subset \mathcal{F}^n$. Let $(x, b_0, \dots, b_n)$ denote the corresponding coordinates on $\mathcal{F}^n_y(V) \simeq \mathcal{J}^n_y(V)$ given by (5.1). Set*

$$J_2 := (x, b_0, b_1, b_2).$$

There exist a C^{n-2}-smooth function $\sigma_n(J_2)$ in $J_2 \in \mathcal{J}^2_y(V)$,

$$\sigma_n \neq 0 \text{ for odd } n > 3; \ \sigma_n \equiv 0 \text{ for } n = 3 \text{ and for every even } n > 3 \tag{5.7}$$

and a polynomial $P_{J_2,n}(b_3,\dots,b_{n-1})$ *with coefficients being* C^{n-2}*-smooth functions in* J_2 *such that every jet* $J^n = (x, b_0, \dots, b_n) \in \mathcal{J}^n_y(V)$ *satisfies the following statement. Let* γ *be a* C^n*-smooth germ of curve representing the jet* J^n*, and let* t *be the parameter on* γ *defined by (5.4). The corresponding function* $\Lambda(t)$ *from (5.3) has asymptotic Taylor polynomial of degree* $n+1$ *at* 0 *of the following type:*

$$\Lambda(t) = \sum_{k\leq n+1} \hat{\Lambda}_k t^k + o(t^{n+1}),$$

$$\hat{\Lambda}_{n+1} = \sigma_n(J_2)b_n - P_{J_2,n}(b_3,\dots,b_{n-1}) \text{ with } J_2 = J_2^n. \tag{5.8}$$

Lemma 5.3 will be proved in Subsection 5.4.

5.2 A preparatory comparison of parameters of osculating curves

As it will be shown below, it suffices to prove the statement of Lemma 5.3 for a germ of curve at a given base point O in normal coordinates (x, y) centered at O and chosen so that the x-axis is tangent to γ at O. Let t be the parameter on the curve γ given by (5.4). Without loss of generality we consider that the geodesic curvature of the curve γ at O equals 1. One can achieve this by multiplying the metric by a constant $C > 0$ (which means that the normal coordinates should be rescaled by homothety). This does not change the parameter t of the curves: the normalization factor $\kappa^{\frac{1}{3}}(0)$ in the definition (5.4) of the parameter t is chosen to guarantee invariance of the parameter t under the above rescalings. Its invariance follows from the fact that in the rescaled metric the geodesic curvature of the curve γ considered as a function of a point in γ is divided by C simultaneously at all points. The geodesic curvature 1 of the curve γ at O is equal to the standard curvature with respect to the Euclidean metric in the normal coordinates (x, y), by Proposition 2.5. This means that

$$\gamma = \{y = f(x)\}, \ f(x) = \frac{x^2}{2} + O(x^3).$$

Consider the coordinates of the n-jet $j^n_O\gamma$: they have the form

$$(0, 0, 0, 1, b_3, \dots, b_n).$$

The function f defining γ has asymptotic Taylor expansion of the type

$$f(x) = \frac{x^2}{2} + \sum_{j=3}^{n} \beta_j x^j + o(x^n), \ \beta_j := \frac{b_j}{j!}. \tag{5.9}$$

We parametrize the curve γ by the coordinate x. In this subsection we consider functions on γ as functions in x. For every x_0 let $s = s(x_0)$ denote the oriented length of the arc of the curve γ parametrized by the segment $[0, x_0]$ of the x-axis: $s(x) > 0$ for $x > 0$; $s(x) < 0$ for $x < 0$. For every function $h(x)$ having asymptotic Taylor expansion up to $o(x^r)$ for a certain r and every $k \leq r$ let $\hat{h}_k$ denote its Taylor coefficient at 0 of degree k:

$$h(x) = \sum_{k \leq r} \hat{h}_k x^k + o(x^r).$$

Proposition 5.4 *The functions $s(x)$, $\kappa(x)$ and $t(x)$ have asymptotic Taylor expansions at 0 up to $o(x^{n+1})$, $o(x^{n-2})$ and $o(x^{n-1})$ respectively. Their Taylor coefficients of the corresponding degrees have the following forms:*

$$\hat{s}_{n+1} = \frac{n}{n+1}\beta_n + P_s(\beta_3, \dots, \beta_{n-1}), \tag{5.10}$$

$$\hat{\kappa}_{n-2} = n(n-1)\beta_n + P_\kappa(\beta_3, \dots, \beta_{n-1}), \tag{5.11}$$

$$\hat{t}_{n-1} = \frac{2n}{3}\beta_n + P_t(\beta_3, \dots, \beta_{n-1}). \tag{5.12}$$

In these three formulas P_s, P_κ and P_t are polynomials.

Proof Let us prove (5.10). Set

$$v(x) := (1, f'(x)) \in T_{(x, f(x))}\gamma.$$

By definition,

$$s(x) = \int_0^x |v(u)| du, \ |v(u)| := \text{ the Riemannian norm of the vector } v(u).$$

The Riemannian scalar square $|v(x)|^2$ is equal to the Euclidean scalar square $1 + (f'(x))^2$ of the same vector $v(x)$ plus a linear combination of products and squares of its components (which are equal to 1, $f'(x)$, $(f'(x))^2$) with coefficients that are C^{n+1}-smooth functions vanishing at 0 of order at least two, and hence, are $O(x^2)$. This follows from the assumption that (x, y) are normal coordinates centered at O, hence our C^{n+1}-smooth metric has the same second jet at O, as the standard Euclidean metric. One has

$$f'(x) = x + 3\beta_3 x^2 + \dots + n\beta_n x^{n-1} + o(x^{n-1}).$$

This together with the prefious statement implies that

$$|v(x)|^2 \text{ has Taylor expansion up to order } o(x^n) \text{ with free term } 1 \tag{5.13}$$

and Taylor coefficients of degrees from 1 to n being expressed as polynomials in $(\beta_3, \dots, \beta_n)$. Therefore, its square root $|v(x)|$ has Taylor expansion of a similar type, with polynomial coefficients, and hence, so its primitive $s(x)$, since the Taylor coefficients of the square root of a function $1 + O(x)$ are polynomials in the coefficients of the function. The lowest Taylor term in $|v(x)|^2$ where the contribution of the coefficient β_n is non-trivial is the Taylor term $2n\beta_n x^n$ of the function $(f'(x))^2$. The corresponding Taylor term in $|v(x)|$ equals $n\beta_n x^n$, and its primitive equals $\frac{n}{n+1}\beta_n x^{n+1}$. This implies (5.10).

Let us prove (5.11). Recall that the geodesic curvature equals

$$\kappa(x) = \frac{D^2\gamma(x)}{|\dot{\gamma}(x)|^3}, \ D^2\gamma(x) = [\nabla_{\dot{\gamma}(x)}\dot{\gamma}(x), \dot{\gamma}(x)], \tag{5.14}$$

$$\gamma(x) = (x, f(x)), \ \dot{\gamma}(x) = v(x),$$

where the vector product is taken with respect to the Riemannian metric on $T_{\gamma(x)}\Sigma$. Formula (5.14) implies that the function $\kappa(x)$ has Taylor expansion up to order $o(x^{n-2})$ with coefficients being polynomials in $(\beta_3, \dots, \beta_n)$, as in the proof of (5.10). Let us now calculate the contribution of the coefficient β_n to the lowest possible Taylor term of the function $\kappa(x)$. One has $D^2\gamma(x) = f''(x)$ up to an expression involving products of $f'(x)$ and Christoffel symbols at $\gamma(x)$. The Christoffel symbols are $O(x)$ (normality of the coordinates (x, y)). Therefore, the lowest term contribution of the coefficient β_n to $D_2\gamma(x)$ is equal to its contribution to $f''(x)$, that is, to $n(n-1)\beta_n x^{n-2}$. Taking into account that the denominator $|v(x)|^3$ in (5.14) is a Taylor series with unit free term, see (5.13), and β_n contributes an $O(x^{n-1})$ there, this yields that the corresponding term in the Taylor expansion of the function $\kappa(x)$ is the same term $n(n-1)\beta_n x^{n-2}$. This implies (5.11).

Now let us prove formula (5.12). By definition,

$$t(x) = \int_0^x \kappa^{\frac{2}{3}}(u)|v(u)|du. \tag{5.15}$$

Existence of Taylor expansion of the function $t(x)$ up to order $o(x^{n-1})$ with coefficients polynomial in $(\beta_3, \dots, \beta_n)$ follows from statement (5.11) and similar statement on $|v(x)|$. The lowest degree contribution of the coefficient β_n to the subintegral expression in (5.15) is equal to $\frac{2}{3}n(n-1)\beta_n x^{n-2}$, by (5.11), and since its contribution to $|v(x)|$ is $O(x^{n-1})$. The primitive of the former contribution is equal to $\frac{2n}{3}\beta_n x^{n-1}$. This proves (5.12) and finishes the proof of Proposition 5.4. □

In what follows we will be also using the inverse function to $t(x)$:

$$x(t), \quad t(x(t)) = t.$$

Corollary 5.5 *The function $x(t)$ admits an asymptotic Taylor expansion at 0 up to order $o(t^{n-1})$, and its degree $n-1$ coeffcient is equal to*

$$\hat{x}_{n-1} = -\frac{2n}{3}\beta_n + P_x(\beta_3, \dots, \beta_n). \tag{5.16}$$

Here $P_x(\beta_3, \dots, \beta_n)$ is a polynomial.

The corollary follows immediately from formula (5.12), since $t \simeq x$.

5.3 Geodesic triangles in normal coordinates

Everywhere below in the present subsection Σ is a two-dimensional surface equipped with a C^3-smooth Riemannian metric g, and $O \in \Sigma$.

Proposition 5.6 *Let $A_u B_u C_u$ be a family of geodesic right triangles in Σ with right angle B_u. Set*

$$c = c_u = |A_u B_u|, \; b = b_u = |A_u C_u|, \; a = a_u = |B_u C_u|, \; \alpha = \alpha_u = \angle B_u A_u C_u.$$

Let the vertices A_u, B_u, C_u tend to a point $O \in \Sigma$ so that $\alpha_u \to 0$, as $u \to u_0$. Then

$$b \simeq c, \; b - c \simeq \frac{a^2}{2c} \simeq \frac{1}{2}c\alpha^2. \tag{5.17}$$

Proof Normal coordinates depend smoothly on the choice of base point. Consider the normal coordinates (x_u, y_u) centered at A_u. The coordinates

$$(X_u, Y_u) := \left(\frac{x_u}{c_u}, \frac{y_u}{c_u}\right)$$

are normal coordinates centered at A_u for the Riemannian metric rescaled by division by c_u. For the rescaled metric one has $|A_u B_u| = 1$. In the rescaled normal coordinates (X_u, Y_u) the metric tends to the Euclidean one, as $u \to u_0$. Then the rescaled metric is Euclidean on $T_{A_u}\Sigma$ with vanishing 1-jet, and its higher term part tends to zero with derivatives, as $u \to u_0$, uniformly on the Euclidean disk of radius 2 in the coordinates (X_u, Y_u).

One has obviously $|A_uB_u| \simeq |A_uC_u|$, and rescaling back, we get the first asymptotic formula in (5.17).

Let S_u denote the circle of radius $|A_uB_u|$ centered at A_u, and let D_u denote its point lying on the geodesic A_uC_u: the arc B_uD_u of the circle S_u is its intersection with the geodesic angle $B_uA_uC_u$. The circle S_u tends to the Euclidean unit circle centered at 0. Thus, its geodesic curvature in the rescaled metric tends to 1. The geodesic segment B_uC_u is tandent to S_u at the point B_u, and $\angle B_uC_uA_u \to \frac{\pi}{2}$. The two latter statement together imply that in the rescaled metric one has

$$|D_uC_u| = |A_uC_u| - |A_uD_u| = |A_uC_u| - |A_uB_u| \simeq \frac{|B_uC_u|^2}{2} \simeq \frac{1}{2}\alpha^2.$$

Rescaling back to the initial metric, we get (5.17). □

5.4 Asymptotics of the function $\Lambda(t)$

Let γ be a germ of C^n-smooth curve at a point $O \in \Sigma$. We use the normal coordinates (x, y) centered at O and the notations from the previous subsection. Recall that we suppose that the germ γ has unit curvature at O (rescaling the metric), and the x-axis is tangent to γ at O. Thus

$$\gamma = \{y = h(x)\}, \ h(x) = \frac{1}{2}x^2 + \text{ higher terms.}$$

In what follows we fix a $b \in \mathbb{R}$ and consider a germ

$$\gamma_{n,b} = \{y = h_{n,b}(x)\}, \ h_{n,b}(x) = h(x) + bx^n + o(x^n).$$

We parametrize the curves γ and $\gamma_{n,b}$ by the corresponding parameters t defined by (5.4). Let $L(0, t)$ and $L_{n,b}(0, t)$ be respectively the corresponding functions (5.5).

In the present subsection we will prove the following lemma. As it will be shown in the next subsection, it implies Lemma 5.3

Lemma 5.7 *One has*

$$L_{n,b}(0,t) - L(0,t) = \frac{(n-2)(n-3)}{12(n+1)}bt^{n+1} + o(t^{n+1}), \ \textit{as } t \to 0. \tag{5.18}$$

Lemma 5.7 is proved below. To do this, we consider the parametrizations

$$\gamma(t) = (x(t), y(t)), \quad \gamma_{n,b}(t) = (x_{n,b}(t), y_{n,b}(t))$$

in the normal coordinates (x, y). We introduce the following notations:

$$P = P(t) := \gamma(t),\ Q = Q(t) := (x_{n,b}(t), h(x_{n,b}(t))) \in \gamma,\ A = A(t) := \gamma_{n,b}(t),$$

$$G(t) := \text{ the geodesic tangent to } \gamma \text{ at } P,\ G(0) = \text{ the } x-\text{axis},$$

$$C = C(t) := G(t) \cap G(0),\ V = V(t) := \{x = x_{n,b}(t)\},$$

$$B = B(t) := G(t) \cap V,\ Q = Q(t) := \gamma \cap V,$$

$$G_{n,b}(t) := \text{ the geodesic tangent to } \gamma_{n,b} \text{ at } A,\ D = D(t) := G_{n,b}(t) \cap G(0),$$

see Fig. 3. In what follows for any two points $E, F \in \Sigma$ close to O the

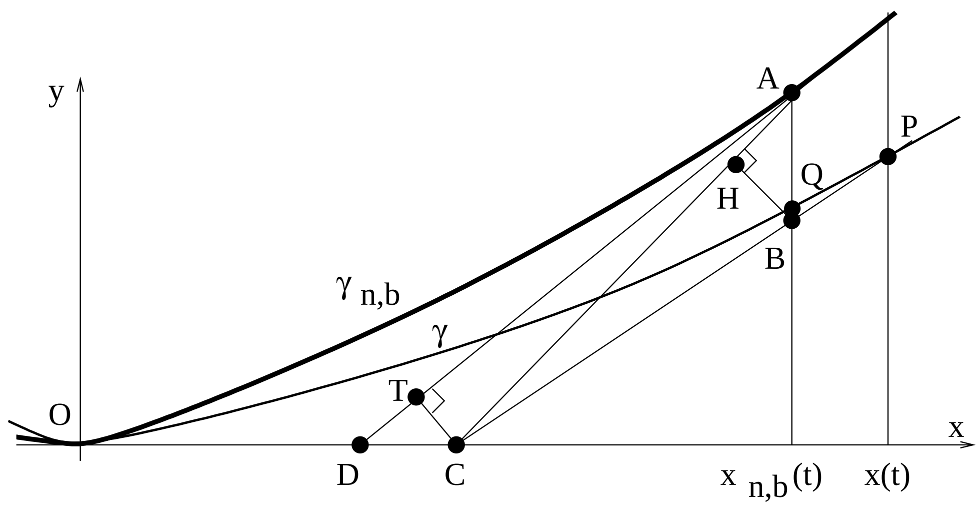

Figure 3: Auxiliary geodesics for calculation of the asymptotic of the difference $L_{n,b}(0, t) - L(0, t)$.

length of the geodesic segment connecting F to E will be denoted by EF. By definition,

$$L(0, t) = OC + CP - \lambda(O, P),\ L_{n,b}(0, t) = OD + DA - \lambda(O, A). \quad (5.19)$$

Recall that $\lambda(O, A)$, $\lambda(O, P)$ are lengths of arcs OA and OP of the curves $\gamma_{n,b}$ and γ respectively. Set

$$L_1 = L_1(t) := OC + CB - \lambda(O, Q),\ L_2 = L_2(t) := OC + CA - \lambda(O, A),$$

$$\Delta_1 = \Delta_1(t) := L_1(t) - L(0, t),\ \Delta_2 = \Delta_2(t) := L_2(t) - L_1(t),$$

$$\Delta_3 = \Delta_3(t) := L_{n,b}(0,t) - L_2(t) :$$

$$L_{n,b}(0,t) - L(0,t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t). \tag{5.20}$$

In what follows we find the asymptotics of each Δ_j. To do thus, we will use the obvious asymptotic formula

$$x(t) \simeq t \simeq h'(x(t)), \text{ as } t \to 0. \tag{5.21}$$

which follows from definition and by the normalization assumption that $\kappa(O) = 1$.

Proposition 5.8 *One has*

$$\Delta_1(t) = O(t^{3(n-1)}) = O(t^{n+2}) \textit{ whenever } n \geq 3. \tag{5.22}$$

Proof The proof of formula (5.22) repeats the proof of formula (5.10) with obvious changes. □

Proposition 5.9 *One has*

$$\Delta_2(t) = \frac{b}{n+1} t^{n+1} + o(t^{n+1}). \tag{5.23}$$

Proof By definition,

$$\Delta_2(t) = OC + CA - \lambda(O,A) - (OC + CB - \lambda(O,Q))$$

$$= (CA - CB) - (\lambda(O,A) - \lambda(O,Q)), \tag{5.24}$$

$$\lambda(O,A) - \lambda(O,Q) = s_{n,b}(x_{n,b}(t)) - s(x_{n,b}(t)) = \frac{n}{n+1} bt^{n+1} + O(t^{n+2}), \tag{5.25}$$

by (5.10) and (5.21). To find the asymptotics of the difference $CA - CB$, let us consider the height denoted by BH of the geodesic triangle ABC, which splits it into two triangles ABH and CBH, see Fig. 3. We use the following asymptotic formula for lengths of their sides:

$$AB \simeq bt^n \simeq BH, \tag{5.26}$$

$$CB \simeq CP \simeq CA \simeq \frac{t}{2} \tag{5.27}$$

$$AH \simeq bt^{n+1} \simeq AC - BC. \tag{5.28}$$

Proof of (5.26). The Euclidean distance in the coordinates (x, y) between the points A and Q is asymptotic to $bx^n_{n,b}(t) \simeq bt^n$, by construction. Therefore, the distance between them in the metric g is asymptotic to the same quantity, since g is Euclidean on $T_O\Sigma$. The Euclidean distance between the points Q and B is of order $O((x(P) - x(B))^2) \simeq O(t^{2(n-1)}) = O(t^{n+1})$, by construction and since $n \geq 3$: $2(n-1) \geq n+1$ for $n \geq 3$. The two latter statements together imply that $AB \simeq bt^n$; this is the first asymptotics in (5.26).

In the proof of the second asymptotics in (5.26) and in what follows we use the two next claims.

Claim 1. *The azimuths of the tangent vectors of the geodesic arcs CA, CP, DA are uniformly asymptotically equivalent to $x(t)+O(t^2) = t+O(t^2)$, as $t \to 0$.*

Proof Let us prove the statement of the claim for the geodesic arc CP; the proof of its statement for the arcs CA and DA is analogous. The slope of the tangent vector to the curve γ at the point P is asymptotic to $x(P) = x(t) \simeq t$, and it is equal to the slope of the tangent vector of the geodesic CP at P. On the other hand, let us apply formula (2.3) to the geodesic arc $\alpha = CP$: its right-hand side is a quantity of order $O(t)$. The length of the arc CP is $O(t)$. Hence, the difference between the azimuths of tangent vectors at any two points of the geodesic arc CP is of order $O(t^2)$. This proves the claim. □

Claim 2. *The angle A of the geodesic triangle ABH is asymptotic to $\frac{\pi}{2} - t + O(t^2)$. Its angle B is asymptotic to $t + O(t^2)$.*

Proof The first statement of the claim follows from Claim 1 applied to CA and the fact that the slopes of the tangent vectors to the geodesic arc BA are all $O(t^2)$-close to $\frac{\pi}{2}$. This follows from formula (2.3) applied to the geodesic arc BA, formula (5.26) and Roll Theorem, which implies that the tangent vector to the geodesic arc BA in at least its one point is vertical. The second statement of the claim follows from the first one and the fact that the sum of angles of the triangle AHB is asymptotic to $\pi + O(Area(ABH)) = \pi + O((BH)^2) = \pi + O(t^2)$ (Gaus-Bonnet Formula). □

The first statement of Claim 2 implies that $AB \simeq HB$, which yields the second asymptotics in (5.26). Formula (5.26) is proved. □

Proof of (5.27). The asymptotics $CP \simeq \frac{x(P)}{2} \simeq \frac{t}{2}$ follows from Claim 1 and the fact that the height of the point P over the x-axis is asymptotic to $\frac{x^2(P)}{2} \simeq \frac{t^2}{2}$. The other asymptotics in (5.27) follow from the above one, formula (5.26) and the fact that $BP = O(t^{n-1})$ (follows from (5.16) and Claim 1). □

Proof of (5.28). The geodesic triangle ABH has right angle at H. This together with Claim 2 and (5.26) implies the first asymptotic formula in (5.28): $AH \simeq AB \sin \angle HAB \simeq bt^{n+1}$. In the proof of the second formula in (5.28) we use the following claim.

Claim 3. *The angle $\phi := \angle BCH = \angle BCA$ of the triangle BCH is asymptotically equivalent to $2bt^{n-1}$.*

Proof The triangle BCH has right angle at H,

$$BH \simeq bt^n, \ BC \simeq \frac{t}{2},$$

by (5.26) and (5.27). This implies that $\phi \simeq bt^n/\frac{t}{2} = 2bt^{n-1}$. Claim 3 is proved. □

Now let us prove the second asymptotic formula in (5.28): $AC - BC \simeq bt^{n+1}$. One has

$$HC - BC \simeq \frac{1}{2} BC\phi^2,$$

by formula (5.17) applied to the family of triangles BCH. The right-hand side in the latter formula is asymptotically equivalent to $b^2t^{2n-1} = O(t^{n+2})$, by (5.27) and Claim 3 and since $2n - 1 \geq n + 2$ for $n \geq 3$. Thus,

$$HC - BC = O(t^{n+2}), \tag{5.29}$$

$$AC - BC = (HC - BC) + AH = AH + O(t^{n+2}) \simeq bt^{n+1},$$

by the first formula in (5.28) proved above. Formula (5.28) is proved. □

Now let us return to the proof of Proposition 5.9. One has

$$\lambda(O, A) - \lambda(O, Q) \simeq \frac{n}{n+1} b(x(Q))^{n+1} \simeq \frac{n}{n+1} bt^{n+1},$$

by (5.10). Substituting this formula and (5.28) to (5.24) yields to

$$\Delta_2(t) = bt^{n+1} - \frac{n}{n+1} bt^{n+1} + o(t^{n+1}) = \frac{b}{n+1} t^{n+1} + o(t^{n+1}).$$

Proposition 5.9 is proved. □

Proposition 5.10 *One has*

$$\Delta_3(t) \simeq \frac{n-6}{12} bt^{n+1}. \tag{5.30}$$

Proof Recall that

$$\Delta_3(t) = L_{n,b}(t) - L_2(t) = OD + DA - \lambda(O,A) - (OC + CA - \lambda(O,A))$$
$$= DA - (DC + CA). \tag{5.31}$$

Here $DC := OC - OD$ is the "oriented length": we'll show that it is indeed positive.

Let CT denote the height of the geodesic triangle DCA. To find an asymptotic formula for the right-hand side in (5.31), we first find asymptotics of the length of the height CT and the angle $\angle DAC$.

Claim 4. *The angle $\alpha := \angle DAC$ is asymptotically equivalent to $\frac{6-n}{3}bt^{n-1}$.*
Proof Consider the following tangent lines of the geodesic arcs AD, AC, BC, CP and the curve γ:

$$\ell_1 := T_A AD = T_A \gamma_{n,b},\ \ell_2 := T_A AC,\ \ell_3 := T_B BC,$$

$$\ell_4 := T_Q \gamma,\ \ell_5 := T_P CP = T_P \gamma.$$

We orient all these lines "to the right". One has

$$\alpha \simeq \mathrm{az}(\ell_2) - \mathrm{az}(\ell_1), \tag{5.32}$$

by definition and since the Riemannian metric at the point A written in the normal coordinates (x,y) tends to the Euclidean one, as $t \to 0$. Let us find asymptotic formula for the above difference of azimuths by comparing azimuths of appropriate pairs of lines $\ell_1,\dots,\ell_5$. One has

$$\mathrm{az}(\ell_4) - \mathrm{az}(\ell_1) \simeq -nbt^{n-1},$$

since the above difference of azimuths is asymptotically equivalent to the difference of the derivatives of the functions $h(x)$ and $h_{n,b}(x) = h(x) + bx^n + o(x^n)$ at the same point $x = x(B) \simeq t$: hence, to $-nbx^{n-1}$. One has

$$\mathrm{az}(\ell_5) - \mathrm{az}(\ell_4) \simeq h'(x(t)) - h'(x_{n,b}(t)) \simeq x(t) - x_{n,b}(t) \simeq \frac{2n}{3}bt^{n-1},$$

by (5.16) and since the function $h'(x) \simeq x$ has unit derivative at 0,

$$\mathrm{az}(\ell_3) - \mathrm{az}(\ell_5) = O(t(x(B) - x(P))) = O(t(x_{n,b}(t) - x(t))) = O(t^n),$$

by (2.3) and (5.16),

$$\mathrm{az}(\ell_2) - \mathrm{az}(\ell_3) \simeq \angle BCA \simeq 2bt^{n-1},$$

by (2.4) and Claim 3. The right-hand sides of the above asymptotic formulas for azimuth differences are all of order t^{n-1}, except one them, which is of smaller order t^n. Summing up all of them yields

$$\alpha \simeq \mathrm{az}(\ell_2) - \mathrm{az}(\ell_1) \simeq \frac{6-n}{3} b t^{n-1}$$

and proves the claim. □

Claim 5. *In the right triangle CDT one has $\angle TDC \simeq t$, $CT \simeq \frac{6-n}{6} b t^n$,*

$$CD - DT \simeq \frac{6-n}{12} b t^{n+1}. \tag{5.33}$$

Proof The first, angle asymptotic follows from Claim 1. The asymptotic for the length of the side CT is found via the adjacent right triangle ACT, from the asymptotic formula $CT \simeq AC\angle CAT$ after substituting the asymptotics $\angle CAT \simeq \frac{6-n}{3} b t^{n-1}$ (Claim 4) and $AC \simeq \frac{t}{2}$. Formula (5.33) follows from formula (5.17) applied to the right triangle CDT and the asymptotic formulas from Claim 5 for the side CT and the angle $\angle TDC$. □

Now let us prove formula (5.30). Recall that

$$\Delta_3(t) = DA - (DC + CA) = (DT - DC) + (AT - AC), \tag{5.34}$$

see (5.31). One has $DT - DC \simeq -\frac{6-n}{12} b t^{n+1}$, by (5.33), and $AT - AC = O(t^{n+2})$, as in the analogous formula (5.29). Substituting the two latter asymptotic formulas to (5.34) yields to (5.30). Proposition 5.10 is proved. □

Proof of Lemma 5.7. Summing up formulas (5.22), (5.23) and (5.30) for $n \geq 3$ and substituting their sum to (5.20) yields to

$$L_{b,n}(t) - L(t) = \Delta_1(t) + \Delta_2(t) + \Delta_3(t)$$

$$= O(t^{n+2}) + \frac{b}{n+1} t^{n+1} + \frac{n-6}{12} b t^{n+1} + o(t^{n+1})$$

$$= (\frac{1}{n+1} + \frac{n-6}{12}) b t^{n+1} + o(t^{n+1}) = \frac{(n-2)(n-3)}{12(n+1)} b t^{n+1} + o(t^{n+1}).$$

This proves Lemma 5.7. □

5.5 Proof of Lemma 5.3

Let Σ be a surface equipped with a Riemannian metric, and let $O \in \Sigma$. Let (x, y) be normal coordinates centered at O.

Proposition 5.11 *Let $n \geq 3$. Let γ and $\gamma_{n,b}$ be two germs of curves centered at O with non-zero geodesic curvature at O such that in the normal coordinates (x, y) they are graphs of functions:*

$$\gamma = \{y = h(x)\}, \quad \gamma_{n,b} = \{y = h_{n,b}(x)\}, \; h_{n,b}(x) - h(x) = bx^n + o(x^n).$$

Consider the differences $\Lambda(t)$ from (5.3) defined for both curves:

$$\Lambda(t) = L(0, t) - L(-t, 0), \quad \Lambda_{n,b}(t) = L_{n,b}(0, t) - L_{n,b}(-t, 0).$$

For every $n \geq 3$ one has

$$\Lambda_{n,b}(t) - \Lambda(t) = \begin{cases} \frac{(n-2)(n-3)}{6(n+1)} bt^{n+1} + o(t^{n+1}) \text{ if } n \text{ is odd,} \\ o(t^{n+1}), \text{ if } n \text{ is even,} \end{cases} \qquad \text{as } t \to 0. \tag{5.35}$$

Proof Applying Lemma 5.7 to the opposite parameter $-t$ of the (oppositely oriented) curves instead of the parameter t yields to

$$L_{n,b}(-t, 0) - L(-t, 0) = (-1)^n \frac{(n-2)(n-3)}{12(n+1)} bt^{n+1} + o(t^{n+1}). \tag{5.36}$$

Therefore, for odd (even) n the main asymptotic terms in (5.36) and (5.18) are opposite (respectively, coincide). Hence, in the expression

$$\Lambda_{n,b}(t) - \Lambda(t) = (L_{n,b}(0, t) - L(0, t)) - (L_{n,b}(-t, 0) - L(-t, 0))$$

they are added (respectively, cancel out), and we get (5.35). □

Proof of Lemma 5.3. Let $E \in \Sigma$, and let $V = V(E) \subset \Sigma$ be its small neighborhood. Let (x, y) be local coordinates centered at E, and let V be contained in the (x, y)-chart. Consider a C^n-smooth germ of curve γ at a point $O = (x_0, y_0) \in V$ with non-zero geodesic curvature and whose tangent line $T_O\gamma$ is not parallel to the y-axis in the coordinates (x, y). Let us rescale the metric by constant factor and introduce normal coordinates $(\widetilde{x}, \widetilde{y})$ centered at O for the rescaled metric so that the geodesic curvature of the curve γ at A becomes equal to 1, and the $\widetilde{x}$-axis be tangent to γ at A. Then γ is the graph of a function

$$\gamma = \{\widetilde{y} = h(\widetilde{x})\}, \quad h(\widetilde{x}) = \frac{1}{2}\widetilde{x}^2 + \frac{1}{3!}\widetilde{b}_3\widetilde{x}^3 + \cdots + \frac{1}{n!}\widetilde{b}_n\widetilde{x}^n + o(\widetilde{x}^n).$$

By definition, the coordinates of the jet $j^n_O\gamma$ associated to the chart $(\widetilde{x}, \widetilde{y})$ are $(0, 0, 0, 1, \widetilde{b}_3, \dots, \widetilde{b}_n)$.

Proposition 5.12 *The function $\Lambda(t)$ associated to the curve γ admits an asymptotic Taylor expansion*

$$\Lambda(t) = \sum_{k=3}^{n+1} \hat{\Lambda}_k t^k + o(t^{n+1}),$$

and its Taylor coefficients are polynomials in $(\widetilde{b}_3, \dots, \widetilde{b}_n)$.

The proof of Proposition 5.12 is analogous to the proof of Proposition 5.4.

One has

$$\hat{\Lambda}_{n+1} = \widetilde{\sigma}_n \widetilde{b}_n - \widetilde{h}_n, \ \sigma_n = \begin{cases} (n+1)! \frac{(n-2)(n-3)}{6(n+1)} \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}; \tag{5.37}$$

here $\widetilde{h}_n = \widetilde{h}_n(\widetilde{b}_3, \dots, \widetilde{b}_{n-1})$ is a polynomial. This follows from Proposition 5.12 and the fact that the Taylor coefficient Λ_{n+1} depends on $\widetilde{b}_n$ in an affine way, as a linear non-homogeneous function with multiplier $\widetilde{\sigma}_n$, by (5.35). Note that $\widetilde{\sigma}_n$ is uniquely defined by the choice of normal coordinates and the choice of rescaling factor of the metric (which is chosen to make the geodesic curvature at the base point being equal to 1). The latter normal coordinate and the curvature, which determine $\widetilde{\sigma}_n$, depend only on the 2-jet of the curve γ. Formula (5.37) implies (5.8) with $\mathcal{P} = \frac{6(n+1)}{(n+1)!}\widetilde{h}_n$. Let $(x_0, y_0, b_1, \dots, b_n)$ be the coordinates of the germ γ associated to the chart (x, y). The coordinate change $(x, y) \mapsto (\widetilde{x}, \widetilde{y})$ is C^n-smooth, which follows from definition and C^{n+1}-smoothness of the metric. It depends on the 2-jet of the curve γ as a parameter and acts on germs of curves so that the corresponding coordinates in jet spaces (i.e., Taylor coefficients) are transformed polynomially with coefficients being C^{n-2}-smooth functions of the parameters. Moreover, the latter transformation is triangular: the higher degree coefficients of the curve-source do not contribute to the lower degree coefficients of the curve-image. This implies that $\hat{\Lambda}_{n+1}$ is expressed as an affine function of the coordinate b_n of the germ γ associated to the chart (x, y). Together with the above discussion, this implies (5.8) and proves Lemma 5.3. □

5.6 Proof of Theorems 5.2 and 1.11

Proof of Theorem 5.2. Let $O \in \Sigma$. Let (x, y) be local coordinates on a neighborhood $V = V(O) \subset \Sigma$. Let $\mathcal{J}^4_y(V)$ denote the space of 4-jets of curves as in Lemma 5.3. Let σ_5 and $h_5 := \mathcal{P}_{J_2,5}(b_3, b_4)$ be the same, as in (5.8). Consider the field of kernels K_4 of the following 1-form ν_4 on $\mathcal{J}^4_y(V)$:

$$\nu_4 := db_4 - \sigma_5^{-1} h_5(x, b_0, b_1, b_2, b_3, b_4)dx; \;\; K_4 := \mathrm{Ker}(\nu_4).$$

Let $\mathcal{D}_4$ denote the canonical distribution on $\mathcal{J}^4_y(V) \simeq \mathcal{F}^4_y(V)$, see (5.2):

$$\mathcal{D}_4 = \mathrm{Ker}(db_0 - b_1dx, db_1 - b_2dx, db_2 - b_3dx, db_3 - b_4dx).$$

Set

$$\mathcal{P} := K_4 \cap \mathcal{D}_{4,V}. \tag{5.38}$$

This is a line field, since the above intersections are obviously transverse and $\dim(\mathcal{D}_4) = 2$. Let γ be an arbitrary C^5-smooth germ of curve γ based at a point $A \in V$ such that the line $T_A\gamma$ is not parallel to the y-axis. Let γ have string Poritsky property. Then it satisfies equation (5.6): $\Lambda(t) \equiv 0$, hence, $\hat{\Lambda}_5 = 0$, thus,

$$\sigma_5(J_2)b_5 - h_5(b_3, b_4) = 0, \tag{5.39}$$

by (5.8). On the other hand, for every n the n-jet extension of the curve γ is tangent to the canonical distribution $\mathcal{D}_n$. Hence, the Pfaffian equation $db_4 = b_5dx$ holds on its 5-jet extension. This together with (5.39) implies that the 4-jet extension of the curve γ is tangent to the hyperplane field defined by the Pfaffian equation $db_4 = \frac{h_5}{\sigma_5}dx$. Thus, γ is tangent to the kernel field K_4, and hence, to $\mathcal{P} = K_4 \cap \mathcal{D}_4$. This proves Theorem 5.2. □

Proof of Theorem 1.11. Two germs of curves with string Poritsky property having the same 4-jet correspond to one and the same point in $\mathcal{J}^4$. Therefore, their 4-jet extensions (families of their 4-jets) coincide with one and the same phase curve of the line field $\mathcal{P}$, by Theorem 5.2 and the Uniqueness Theorem for ordinary differential equations. Therefore, the germs under question coincide. This proves Theorem 1.11. □

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