FOURIER INTEGRAL OPERATORS ON LIE GROUPOIDS

Jean-Marie Lescure, Stéphane Vassout

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Abstract. As announced in [36], we develop a calculus of Fourier integral G-operators on any Lie groupoid G. For that purpose, we study Lagrangian conic submanifolds of the symplectic groupoid T^*G. This includes their product, transposition and inversion. We also study the relationship between these Lagrangian submanifolds and the equivariant families of Lagrangian submanifolds of $T^*G_x \times T^*G^x$ parametrized by the units $x \in G^{(0)}$ of G. This allows us to select a subclass of Lagrangian distributions on any Lie groupoid G that deserve the name of Fourier integral G-operators (G-FIOs). By construction, the class of G-FIOs contains the class of equivariant families of ordinary Fourier integral operators on the manifolds G_x, $x \in G^{(0)}$. We then develop for G-FIOs the first stages of the calculus in the spirit of Hormander's work. Finally, we illustrate this calculus in the case of manifolds with boundary.

Contents

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1. Introduction

Pseudodifferential operators play a central role in the modern theory of Partial Differential Equations and were initially developed for their efficacy in handling this type of problem, particularly in constructing parametrices for differential operators. It quickly appeared that they would also play an important role in various other mathematical fields, such as index theory, Riemannian and differential geometry, spectral theory, quantization, and so on.

Many generalisations of this calculus have been developed, particularly to handle situations where the space is no longer a smooth compact manifold. For pseudodifferential operators, the theory has been successfully developed by a direct analytic approach for manifolds with boundary [6, 41, 44, 10, 13] or fibred boundaries [40, 33], manifolds with corners [43, 42, 45], manifolds with conical singularities [35, 38, 37, 60, 26, 61], stratified spaces [50], families of operators [5], invariant ΨDOs on Lie groups [62], and the Heisenberg calculus [27, 9], among others.

In the case of longitudinal operators on the leaves of a regular foliation, Connes developped a strategy of desingularization by considering operators acting on a bigger smooth manifold (the holonomy groupoid of the foliation) which are equivariant under some action (the groupoid action). This strategy opened a new systematic approach to the question of defining a pseudodifferential calculus adapted to a geometric situation: following the seminal work of Connes [14, 15], Monthubert-Pierrot [49] and Nistor-Weinstein-Xu [54] developed in full generality the G -invariant pseudodifferential calculus on a Lie groupoid. This general abstract calculus allows to define a ΨDO calculus for a given geometric situation (singular spaces, families, coverings, foliations, deformation groupoids, etc) provided this geometric situation can be geometrically encoded by a Lie groupoid.

This approach has been successful to recover the ΨDO calculus (at least up to smoothing operators) for many geometric situations where the calculus had been previously developed using analysis adapted to each particular situation, for example for manifolds with boundary [47, 54], manifolds with corners [48], invariant ΨDOs on Lie groups [54], and the Heisenberg calculus [63, 56, 57].

The main idea is to change viewpoint and to transfer, if possible, all analytic particularities of a geometric situation to the geometric encoding by the Lie groupoid. This approach also allowed extension of the pseudodifferential calculus to new geometric situations, for example, manifolds with a Lie structure at infinity [2], singular foliations [4, 3]. For manifolds with iterated fibred corners, the calculus was recently defined by a combination of the direct analytic approach, and the groupoid approach [24].

Turning back to the case of ordinary manifolds, in seeking solutions of non elliptic PDEs, a broader calculus containing the pseudodifferential calculus was introduced by Hörmander: the Fourier Integral Operator calculus, based on the notion of Lagrangian distributions. This calculus proved to be a powerful tool for solving, for examples hyperbolic equations, as well as to reduce PDE problems to simpler ones via Egorov's theorem. They appeared also to have an important role to play in spectral theory, index theory, and quantization theory, [32, 68, 69, 71, 28, 29, 34].

Fourier Integral Operators have been extended beyond the case of ordinary manifolds, and using refined analysis, for manifolds with boundary [41], invariant operators on Lie groups [53], manifolds with cylindrical ends [18, 17], manifolds with conical singularities [51, 67, 52], and recently for the Boutet de Monvel calculus on singular manifolds [7, 8, 12, 11].

Having at hand all the previously cited examples of groupoids encoding singular spaces, foliations, among other examples, a natural goal is to develop the calculus of Fourier Integral Operators on arbitrary Lie groupoids, in order to have a conceptual and unified approach to the existing examples, as well as calculi in new situations. This paper contains the realization of this program. The calculus of Fourier Integral Operators that we are going to define is not only a *generalization* of known constructions; it also explains and illuminates the geometrical aspects of the existing calculi. The composition theorem that we give for FIOs on groupoids supports this point. Moreover, another strong motivation for writing down a calculus for FIO on groupoids is to get the appropriate tools to generalize the results about the spectral asymptotics [32, Chapter 29] to situations like stratified spaces (using the groupoids defined in [24]) or measured foliations (using holonomy groupoids). We wish to present our construction of the calculus of FIOs on groupoids in a paper as comprehensive as possible, and this leads us to a long paper. Therefore, the applications to spectral asymptotics are postponed to a forthcoming paper, in which we will begin by studying hyperbolic problems on groupoids.

Our calculus of FIOs on groupoids recovers some of the existing ones. The original calculus on a manifold M is immediately recovered by looking at the pair groupoid $M \times M$. Moreover, the present work recovers and significantly extends the results of [53] for invariant FIOs on a Lie group. Finally, using the b-groupoid of a manifold with boundary, we recover and slightly extend the class of FIOs defined by R. Melrose in [41]: the end of the paper is devoted to the comparison of both constructions.

To define FIOs on general Lie groupoids, pseudodifferential operators [49, 54] (G-ΨDOs) provide a stimulating example: they are exactly the G-operators given by equivariant C^∞ families of pseudodifferential operators acting in the r or s-fibers of G. Importantly, G-ΨDOs are not pseudodifferential operators on the manifold G itself, but they actually coincide with Lagrangian distributions on G subordinate to $A^*G = N^*(G^{(0)}) \subset T^*G$, that is, to the dual of the normal bundle of the embedding $G^{(0)} \hookrightarrow G$.

Another inspiration for the definition of FIOs comes from the paper [36] in which distributions on Lie groupoids are studied (see also [25]). On one hand, distributions on a Lie groupoid G that yield G-operators are characterized and natural sufficient conditions on their wave front sets are given. On the other hand, still in [36], the convolution product of distributions on a Lie groupoid G is analyzed and sufficient conditions for that product to be defined are again given in terms of wave front sets. For the understanding of the present work, it is relevant to note that all the conditions mentioned above, as well as the formula for the wave front set of a convolution product of distributions, have an algebraic nature involving the symplectic groupoid T^*G.

The least we can require is that a Fourier Integral Operator should be an element of $I(G,\Lambda)$ [32, Section 25.1] for some conic Lagrangian submanifold Λ of T^*G, and should be an adjointable G-operator. The article [36] already gives a way to fulfill this constraint: if the conic Lagrangian submanifold $\Lambda \subset T^*G$ does not intersect the kernel of the source and target maps of $T^*G \rightrightarrows A^*G$, then the elements of $I(G,\Lambda)$ provide adjointable G-operators. Note that this is a purely algebraic condition, very simple to check in practice, which boils down to the so-called "no zeros" condition [32, 41] in the case of the pair groupoid $G = X \times X$. We call the conic Lagrangian submanifolds of T^*G fulfilling this condition *G-relations*, in reference to the classical term "canonical relations", and we abbreviate the corresponding Lagrangian distributions as G-FIOs.

A first natural question arises: given a G-relation Λ and a G-FIO $u \in I(G,\Lambda)$, we have at hand a (C^∞, equivariant) family of distributions $u_x \in \mathcal{D}'(G^x \times G_x)$, $x \in G^{(0)}$, and so it is natural to ask whether these distributions are Lagrangian, that is, are ordinary Fourier integral operators on the manifold G_x.

The answer is no in general. Actually, the distributions u_x are still given by oscillatory integrals, but we provide an example where some of them are not Lagrangian distributions. This unstable behavior is fixed by imposing that the underlying G-relation $\Lambda \subset T^*G$ have a projection in G transversal to the canonical (singular) foliation of G. Indeed, this transversality condition implies that Λ gives a (C^∞, equivariant) family of canonical relations $\Lambda_x \in T^*(G^x \times G_x)$, $x \in G^{(0)}$, and each u_x, being expressed as an appropriate pull-back distribution, is then an element of $I(G^x \times G_x, \Lambda_x)$. We call the G-relations enjoying this transversality condition *family G-relations* and we abbreviate the corresponding Lagrangian distributions as G-FFIOs. Note that the transversality property characterizing family G-relations among the general ones is geometric and still very simple to check in practice.

Thus, by construction, G-FFIOs provide C^∞ equivariant families of Lagrangian distributions u_x, $x \in G^{(0)}$ and the next natural goal is to obtain a converse statement. This is achieved after proving that a C^∞ equivariant family of canonical relations $\Lambda_x \in T^*(G^x \times G_x)$, $x \in G^{(0)}$, enjoying some minimal transversality condition, can be "glued" into a single family G-relation $\Lambda \subset T^*G$. This requires some preliminary work on families of Lagrangian submanifolds in the cotangent spaces of the fibers of an arbitrary submersion.

To summarize the previous discussion, G-FIOs provide a class of distributions on G desserving the name of Fourier integral operators on G, and among them we know how to characterize in a simple geometric way those which correspond to C^∞ equivariant families of ordinary FIOs in the s or r fibers of G.

The next natural point is to develop a calculus for G-FIOs. We explore the following issues, as in the classical case:

- existence of an adjoint,
- principal symbol,
- composition,
- module structure over the algebra of pseudodifferential operators,
- Egorov's Theorem,
- C^* continuity (the replacement for ordinary L^2 continuity).

For some of these issues, we could have restricted ourselves to the sub-class of G-FFIOs in order to export all the results available on manifolds to Lie groupoids via the point of view of families. Instead, we have chosen to treat G-FFIOs as single distributions on G to develop the calculus: the statements are simpler, more conceptual, and the central role of the symplectic groupoid T^*G is illuminated. Moreover, most of the results hold for G-FIOs and not just for G-FFIOs.

More precisely, we prove that adjoints of G-FIOs are G-FIOs, and that adjunction replaces the corresponding Lagrangian submanifold by its image under the inverse map of the groupoid $T^*G \rightrightarrows A^*G$. Next, we work out a natural composability assumption on G-relations in order that their product in T^*G is again a G-relation. Then, when Λ_1 and Λ_2 are composable, we prove that the convolution of any distributions $u_j \in I(G, \Lambda_j)$ (that is, the composition of the corresponding G-operators), is a G-FIO subordinate to the product G-relation $\Lambda_1.\Lambda_2$. We observe that the product of family G-relations is not always a family G-relation, and likewise the composition of G-FFIOs need not always be a G-FFIO: we explain how to strengthen the composability assumption on G-relations to fix this problem.

The previous adjunction and composition theorems have direct applications. Firstly, the composition of G-FIOs (resp. G-FFIOs) with pseudodifferential operators are G-FIOs (resp. G-FFIOs), the corresponding Lagrangian submanifold being unchanged. Secondly, for any composable G-relations Λ_1, Λ_2 whose product is contained in the unit space of $T^*G \rightrightarrows A^*G$, that is, $\Lambda_1.\Lambda_2 \subset A^*G$, we get a statement generalizing Egorov's Theorem.

The assumption made in our version of Egorov's Theorem can be viewed as a weak invertibility property for G-relations. Actually, for any G-relation Λ_1, we prove that the existence of a composable G-relation Λ_2 such that $\Lambda_1 * \Lambda_2 = r_\Gamma(\Lambda_1)$ and $\Lambda_2 * \Lambda_1 = s_\Gamma(\Lambda_1)$ (here s_Γ, r_Γ denote the source and target maps of $T^*G \rightrightarrows A^*G$) is equivalent to Λ_1 being a Lagrangian bisection. We call these invertible G-relations.

It then follows that for any invertible G-relation Λ, the G-relation $\Lambda^\star = i_\Gamma(\Lambda)$ (where i_Γ is the inversion of T^*G) is an inverse, and by the composition result it also follows that uu^* is a pseudodifferential operator as soon as $u \in I(G, \Lambda)$. Hence, using known C^*-continuity results for pseudodifferential operators, which rely on the classical Hörmander trick to prove L^2-continuity, we obtain C^*-continuity results for G-FIOs subordinate to invertible G-relations. This also holds for locally invertible G-relations, that is, for G-relations onto which the source and target maps of T^*G are only local diffeomophisms, also known as Lagrangian local bisections [3].

For the sake of clarity in this introduction, we have ignored a technical point about the regularity of G-relations. More precisely, as sets, G-relations are submanifolds, but all the statements above are true for local G-relations, that is, those based on immersed submanifolds (ranges of immersions). As in the classical case, we can not avoid the introduction of immersed submanifolds since the product of two G-relations is the image of some submanifold by a C^∞ map of constant rank. Such images are not in general true submanifolds but are always images of some not necessarily injective immersion: we call them local submanifolds, following the (implicit) suggestion of [32, Prop. C.3.3]. Local submanifolds are countable unions of true submanifolds of the same dimension and it is very convenient for our purposes to handle them in this way.

The paper is organized as follows.

In Section 2, we recall notations and properties for Lie groupoids, provide examples and recall the construction of the cotangent groupoid of Coste-Dazord-Weinstein.

In Section 3, we introduce the notion of local Lagrangian submanifolds, which is needed, as explained before, because the product of two Lagrangian submanifolds is not in general a submanifold. We then recall the local parametrization of Lagrangian manifolds by phase functions. Next we explain a "pushforward" procedure for Lagrangian submanifolds, which is one of the main tools used for the composition of G-FIOs. We end this section by analysing families of Lagrangian submanifolds subordinate to a submersion. In particular we show how such families of Lagrangian submanifolds can be "glued" in a single Lagrangian submanifold, and how a single Lagrangian submanifold can be "sliced" into a family of Lagrangian submanifolds, provided some transversality condition is fulfilled.

Section 4 is devoted to Lagrangian distributions. We begin by recalling the classical definitions and properties, and then we analyse the effect of the aforementioned "pushforward" operation and the family aspect at the level of Lagrangian distributions.

Apart from the family aspect, most of the content of these sections comes from classical results in symplectic geometry and FIO theory, being presented perhaps in a not completely classical way to fit our later use of these results.

The truly new part of this work comes in Sections 5 and 6. In Section 5, we study operations on Lagrangian submanifolds of the symplectic groupoid T^*G. This includes the study of their product, transposition and invertibility property. The notion of G-relation is also introduced and the relationship with equivariant families of Lagrangians is clarified. The G-FIOs are introduced in section 6. Similarly to G-relations, the parallel with equivariant families of FIOs is fully analysed. Furthmermore, we extend the basic calculus of FIOs to G-FIOs, which includes a formula for the product of principal symbols. Section 7 is devoted to the comparison between the calculus we get in the case of the groupoid of the b-calculus and previous constructions by R. Melrose [41].

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2. The cotangent groupoid

2.1. **Main notations.** We will first recall the basic definitions related to Lie groupoids in order to fix our notation. For a complete course on the subject, the unfamiliar reader is invited to consult for instance [39, 46] and references therein.

A *Lie groupoid* is a pair of manifolds $(G, G^{(0)})$ provided with the following C^∞ maps:

- a *source* map $s : G \rightrightarrows G^{(0)}$ and a *target* map $r : G \rightrightarrows G^{(0)}$, both required to be surjective submersions;
- an *inclusion of units* $u : G^{(0)} \longrightarrow G$;
- an *inversion* map $\iota : G \longrightarrow G$;
- a *multiplication* map $m : G^{(2)} = \{(\gamma_1, \gamma_2) \in G^2 \ ; \ s(\gamma_1) = r(\gamma_2)\} \longrightarrow G$.

All these structural maps are required to satisfy the following relations, whenever they make sense:

$$\begin{array}{lcl} r(u(x)) = x \text{ and } s(u(x)) = x & ; & s(\iota(\gamma)) = r(\gamma) \text{ and } r(\iota(\gamma)) = s(\gamma); \\ r(\gamma)\gamma = \gamma \text{ and } \gamma s(\gamma) = \gamma & ; & r(\gamma_1\gamma_2) = r(\gamma_1) \text{ and } s(\gamma_1\gamma_2) = s(\gamma_2); \\ \gamma\gamma^{-1} = r(\gamma) \text{ and } \gamma^{-1}\gamma = s(\gamma) & ; & (\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3). \end{array}$$

Above, we have simplified the presentation by writing $\gamma_1\gamma_2 = m(\gamma_1, \gamma_2)$, $\gamma^{-1} = \iota(\gamma)$ and by identifying $G^{(0)}$ with $u(G^{(0)}) \subset G$.

It follows from this definition that ι is an involution, that m is a surjective submersion and that for any $\gamma \in G$, the element γ^{-1} is the unique solution α of $\gamma\alpha = r(\gamma)$, $\alpha\gamma = s(\gamma)$. These assertions need a proof, and the unfamiliar reader is invited to consult for instance [39, 46] and references therein.

Elements of G are usually called arrows, elements of $G^{(0)}$ units and elements of $G^{(2)}$ composable pairs. We will note $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$ and $G_x^y = G_x \cap G^y$ for any $x, y \in G^{(0)}$. These subsets of G are submanifolds and G_x^x is a Lie group for any $x \in G^{(0)}$. We will also write m_x for the restriction of the multiplication map m to the submanifold $G_x \times G^x \subset G^{(2)}$.

We will also consider, right and left multiplication maps by an element γ in G. Precisely we set $R_\gamma : G_{r(\gamma)} \to G_{s(\gamma)}, \gamma_1 \mapsto \gamma_1\gamma$ and $L_\gamma : G^{s(\gamma)} \to G^{r(\gamma)}, \gamma_2 \mapsto \gamma\gamma_2$.

The Lie algebroid AG of the Lie groupoid G is the normal bundle of $G^{(0)}$ in G, that is

$$AG = T_{G^{(0)}}G/TG^{(0)} = NG^{(0)}. \tag{1}$$

Since $T_{G^{(0)}}G = \ker ds \oplus TG^{(0)} = \ker dr \oplus TG^{(0)}$, the bundle AG is canonically isomorphic to (and if sometimes identified with) $\ker ds|_{G^{(0)}}$ and $\ker dr|_{G^{(0)}}$. The dual Lie algebroid A^*G, that is, the conormal space of $G^{(0)}$ in G, will play an important role in this paper.

Finally, the projection maps $G^2 \to G$, $G^{(2)} \to G$ are denoted respectively by pr_j, $\mathrm{pr}_{(j)}$, $j = 1, 2$ and if E, F are vector bundles over G, we will use the shorthand notation $E \boxtimes F$ to denote ${\mathrm{pr}_{(1)}}^*(E) \otimes {\mathrm{pr}_{(2)}}^*(F) \to G^{(2)}$.

2.2. **Examples.** Many examples of Lie groupoids appear in the litterature [72, 58, 16, 54, 49, 20, 47, 23, 21, 64, 65, 24]. We just give below a few basic examples of these objects.

(1) The pair groupoid of a manifold X is given by $G = X \times X$, $G^{(0)} = X$, the maps r and s being respectively the first and second projections, the multiplication being given by $(x, y)(y, z) = (x, z)$ and the inversion by $(x, y)^{-1} = (y, x)$. This is a simple but fundamental example, for the G-ΨDOs, and the G-FIOs that will be defined in this paper, are respectively in that case the usual ΨDOs and FIOs on the manifold X.
(2) A Lie group G is obviously a Lie groupoid: with $G \rightrightarrows \{e\}$, where e denotes the neutral element.
(3) Let X be a manifold with boundary $\partial X = Y$ and let x be a defining function for Y. The b-groupoid is the following subgroupoid of $X^2 \times \mathbb{R}_+^* \rightrightarrows X$:

$$G_b = \{(p, q, t) \in X^2 \times \mathbb{R}_+^* \ ; \ x(q) = tx(p)\}. \tag{2}$$

We will come back to this example in the last section of this paper.
(4) A fibre bundle, a fibration or a foliation also give rise to natural Lie groupoids. For a regular foliated manifold $(M, \mathcal{F})$, the holonomy groupoid is the quotient of the fundamental group of $\mathcal{F}$ (the paths in the manifold $\mathcal{F}$ up to homotopy) by the relation "having the same holonomy", which means that the germs of diffeomorphisms on the transversal part defined by parallel transport along the paths are the same (see [20] for the case of singular foliations).
(5) Deformation groupoids like adiabatic groupoids or non commutative tangent spaces are groupoids naturally arising in index theory [22].

2.3. **Associated foliations.** It will be useful for our purpose to consider some natural (singular) foliations associated to groupoids.

Let G be a Lie groupoid and consider the equivalence relation on $G^{(0)}$

$$x \sim_{G^{(0)}} y \quad \text{if} \quad G_y^x \neq \emptyset. \tag{3}$$

The equivalence class of $x \in G^{(0)}$, also called the orbit of x is denoted by O_x. We obviously have

$$O_x = r(s^{-1}(x)) = s(r^{-1}(x)) \subset G^{(0)}. \tag{4}$$

It is true that the O_x are all immersed submanifolds [39, Theorem 1.5.11], see also [58, 46]. This defines a singular Stefan foliation $\mathcal{F}_{G^{(0)}}$ (see [39, Section 1.8, p.51]) that will simply be called the canonical foliation of $G^{(0)}$.

The leaves of $\mathcal{F}_{G^{(0)}}$ can be lifted to G using r and this gives rise to another singular Stefan foliation $\mathcal{F}_G$ that we call the canonical foliation of G. Using s instead of r gives the same foliation. The leaves of $\mathcal{F}_G$ are immersed submanifolds and coincide with the equivalence classes of the equivalence relation on G given by

$$\gamma_1 \sim_G \gamma_2 \quad \text{if} \quad G_{r(\gamma_2)}^{s(\gamma_1)} \neq \emptyset. \tag{5}$$

2.4. **The cotangent groupoid.** Given a Lie groupoid G, the cotangent manifold $\Gamma = T^*G$ has a non trivial groupoid structure whose space of units is $\Gamma^{(0)} = A^*G$ [19]. This *cotangent groupoid* is of crucial importance in this paper and we recall its definition [19].

The product in Γ is defined as follows. Given $(\gamma_1, \xi_1) \in T^*G$ and $(\gamma_2, \xi_2) \in T^*G$, then

$$(\gamma_1, \xi_1).(\gamma_2, \xi_2) = (\gamma_1.\gamma_2, \xi_1 \oplus \xi_2) \tag{6}$$

where the linear form $\xi_1 \oplus \xi_2 \in T^*_{\gamma_1.\gamma_2}G$ is defined by

$$\xi_1 \oplus \xi_2(dm(t_1, t_2)) = \xi_1(t_1) + \xi_2(t_2), \quad \forall (t_1, t_2) \in T_{(\gamma_1,\gamma_2)}G^{(2)}. \tag{7}$$

For this to make sense, one just needs to assume first that γ_1 and γ_2 are composable in G and then that the linear form $(\xi_1, \xi_2) \in T^*_{(\gamma_1,\gamma_2)}G^2$ vanishes on $\ker dm$, that is

$$(\xi_1, \xi_2) \in (\ker dm_{(\gamma_1,\gamma_2)})^{\perp}, \tag{8}$$

where the orthogonal is taken in T^*G^2. In other words, the subset of composable pairs in Γ is given by:

$$\Gamma^{(2)} = (\ker dm)^{\perp_{T^*G^2}}$$

and the multiplication map m_Γ, defined by (6) and (7), reads

$$m_\Gamma(\gamma_1, \xi_1, \gamma_2, \xi_2) = (m(\gamma_1, \gamma_2), ({}^t dm_{(\gamma_1,\gamma_2)})^{-1}(\rho(\xi_1, \xi_2)),$$

where ρ denotes the canonical restriction map

$$\rho : T^*_{G^{(2)}}G^2 \longrightarrow T^*(G^{(2)}). \tag{9}$$

One can show that there exists a unique Lie groupoid structure on Γ whose multiplication is indeed given as above (see [19, 39, 59, 36] for details). The remaining structural maps of Γ are the following:

- $s_\Gamma(\gamma, \xi) = (s(\gamma), \overline{s}(\xi))$ with $\overline{s}(\xi) = {}^t d(L_\gamma)_{s(\gamma)}(\xi) \in A^*_{s(\gamma)}G = (T_{s(\gamma)}G/T_{s(\gamma)}G^{(0)})^*$;
- $r_\Gamma(\gamma, \xi) = (r(\gamma), \overline{r}(\xi))$ with $\overline{r}(\xi) = {}^t d(R_\gamma)_{r(\gamma)}(\xi) \in A^*_{r(\gamma)}G = (T_{r(\gamma)}G/T_{r(\gamma)}G^{(0)})^*$;
- $\iota_\Gamma(\gamma, \xi) = (\gamma^{-1}, -({}^t d\iota_\gamma)^{-1}(\xi))$.

Note that all structural maps of Γ are vector bundles homomorphisms. If we denote by

$$p : \Gamma \longrightarrow G \quad ; \quad p^2 : \Gamma^2 \longrightarrow G^2 \quad ; \quad p^{(2)} : \Gamma^{(2)} \longrightarrow G^{(2)} \quad ; \quad p^{(0)} : \Gamma^{(0)} \longrightarrow G^{(0)},$$

the natural vector bundle projection maps, we get the following exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & N^*(G^{(2)}) \arrow[r] \arrow[d, "p^{(2)}"] & \Gamma^{(2)} \arrow[r, "m_\Gamma"] \arrow[d, "p^{(2)}"] & \Gamma \arrow[r] \arrow[d, "p"] & 0 \\ & G^{(2)} \arrow[r, "="] & G^{(2)} \arrow[r, "m"] & G, & \end{tikzcd} \tag{10}$$

$$\begin{tikzcd} 0 \arrow[r] & (\ker dr)^{\perp} \arrow[r] \arrow[d, "p"] & \Gamma \arrow[r, "s_\Gamma"] \arrow[d, "p"] & \Gamma^{(0)} \arrow[r] \arrow[d, "p^{(0)}"] & 0 \\ & G \arrow[r, "="] & G \arrow[r, "s"] & G^{(0)}, & \end{tikzcd} \tag{11}$$

and

$$\begin{tikzcd} 0 \arrow[r] & (\ker ds)^{\perp} \arrow[r] \arrow[d, "p"] & \Gamma \arrow[r, "r_\Gamma"] \arrow[d, "p"] & \Gamma^{(0)} \arrow[r] \arrow[d, "p^{(0)}"] & 0 \\ & G \arrow[r, "="] & G \arrow[r, "r"] & G^{(0)}. & \end{tikzcd} \tag{12}$$

Using again the canonical map $\rho : T^*_{G^{(2)}} G^2 \to T^*G^{(2)}$, we get $\rho(\Gamma^{(2)}) = (\ker dm)^{\perp_{T^*(G^{(2)})}}$. Setting

$$\widetilde{\Gamma}^{(2)} = (\ker dm)^{\perp_{T^*(G^{(2)})}} \text{ and } \widetilde{m_\Gamma} = (m, ({}^t dm)^{-1}), \tag{13}$$

we get an additional exact sequence of vector bundles:

$$\begin{tikzcd} 0 \arrow[r] & \ker m_\Gamma \arrow[r] \arrow[d, "(m,0)"] & \Gamma^{(2)} \arrow[r, "\rho"] \arrow[d, "m_\Gamma"] & \widetilde{\Gamma}^{(2)} \arrow[r] \arrow[d, "\widetilde{m_\Gamma}"] & 0 \\ 0 \arrow[r] & G \times \{0\} \arrow[r, hook] & \Gamma \arrow[r, "="] & \Gamma \arrow[r] & 0. \end{tikzcd} \tag{14}$$

The kernels of the structural maps $m_\Gamma, r_\Gamma, s_\Gamma$ will play an important role later on.
Recall that any distribution on G with wave front set $\mathrm{WF}(u)$ satisfying

$$\mathrm{WF}(u) \cap \ker r_\Gamma = \mathrm{WF}(u) \cap \ker s_\Gamma = \emptyset \tag{15}$$

gives rise to a convolution operator $f \mapsto u * f$ mapping $C^\infty_c(G)$ into $C^\infty(G)$ and whose adjoint enjoys the same mapping property [36]. Therefore, Condition (15) is an analog for groupoids of the classical "no-zeros" condition:

$$\mathrm{WF}(u) \subset (T^*X \setminus 0) \times (T^*Y \setminus 0) \tag{16}$$

on a distibution $u \in \mathcal{D}'(X \times Y)$, which implies that u defines continuous linear operators $C^\infty_c(Y) \to C^\infty(X)$ and $C^\infty_c(X) \to C^\infty(Y)$ through the Schwartz kernel Theorem. Above, we have set, for any manifold M,

$$T^*M \setminus 0 = \{(x, \xi) \in T^*M \ ; \ \xi \neq 0\}.$$

This leads us to introduce the following open subset of T^*G:

$$\dot{T}^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma), \tag{17}$$

which is obviously a subgroupoid for $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \mathrm{pr}_1$, $s_\Gamma \circ m_\Gamma = s_\Gamma \circ \mathrm{pr}_2$ and $r_\Gamma \circ \iota_\Gamma = s_\Gamma$.

The previous analogy can be made more concrete as follows. Let W be any subset in G and consider the family of sets W_x parametrized by $x \in G^{(0)}$ and defined by pulling-back W by the maps m_x, that is:

(18)
$$W_x = m_x^* W = \{(\gamma_1, \xi_1, \gamma_2, \xi_2) \in T^*G_x \times T^*G^x \ ; \ \exists (\gamma, \xi) \in W, \ \gamma_1\gamma_2 = \gamma, \ {}^t(dm_x)_{\gamma_1,\gamma_2}(\xi) = (\xi_1, \xi_2)\}.$$

Alternatively, these sets are also given by:

$$W_x = \rho_x(m_\Gamma^{-1}(W) \cap T^*_{G_x \times G^x} G \times G) \subset T^*G_x \times T^*G^x, \tag{19}$$

where ρ_x denotes the restriction map $T^*_{G_x \times G^x} G \times G \to T^*G_x \times T^*G^x$. Then,

Proposition 1. *For any subset $W \subset T^*G$, the following assertions are equivalent:*

1. *W is included in $\dot{T}^*G$;*
2. *For any $x \in G^{(0)}$, $W_x \subset (T^*G_x \setminus 0) \times (T^*G^x \setminus 0)$.*

A subset of T^*G included in $\dot{T}^*G$ will be said to be *admissible*.

Proof. Differentiating m_x, we get

$$d(m_x)_{(\gamma_1,\gamma_2)}(t_1, t_2) = (dR_{\gamma_2})_{\gamma_1}(t_1) + (dL_{\gamma_1})_{\gamma_2}(t_2), \text{ for all } t_1 \in T_{\gamma_1}G_x, \ t_2 \in T_{\gamma_2}G^x$$

and then

$${}^t d(m_x)_{(\gamma_1,\gamma_2)}(\xi) = ({}^t dR_{\gamma_2}(\xi), {}^t dL_{\gamma_1}(\xi)) = ({}^t dR_{\gamma_1^{-1}}(\overline{r}(\xi)), {}^t dL_{\gamma_2^{-1}}(\overline{s}(\xi))) \in T^*_{\gamma_1}G_x \times T^*_{\gamma_2}G^x.$$

It follows that

$$W_x = \left\{ \left(\gamma_1, {}^t dR_{\gamma_1^{-1}}(\overline{r}(\xi)), \gamma_2, {}^t dL_{\gamma_2^{-1}}(\overline{s}(\xi))\right) \ ; \ (\gamma_1, \gamma_2) \in G_x \times G^x, \ (\gamma_1\gamma_2, \xi) \in W \right\}.$$

Since ${}^t dR_{\gamma_i^{-1}}$ and ${}^t dL_{\gamma_2^{-1}}$ are bijective, the result follows. □

Moreover, given any subset $W \subset T^*G$, the family $(m_x^*(W))_{x \in G^{(0)}}$ is *equivariant*, which means:

$$\forall x, y \in G^{(0)}, \ \forall (\gamma_1, \gamma_2, \xi_1, \xi_2) \in W_x, \ \forall \gamma \in G^x_y, \quad (\gamma_1\gamma, \gamma^{-1}\gamma_2, {}^t(dR_{\gamma^{-1}})(\xi_1), {}^t(dL_\gamma)(\xi_2)) \in W_y. \tag{20}$$

Indeed, if $\gamma \in G^x_y$ and $c_\gamma : G_x \times G^x \longrightarrow G_y \times G^y$ denotes the map defined by $c_\gamma(\gamma_1, \gamma_2) = (\gamma_1\gamma, \gamma^{-1}\gamma_2)$, then the commutative diagram

$$\begin{tikzcd} G_x \times G^x \arrow[r, "m_x"] \arrow[d, "c_\gamma"'] & G \\ G_y \times G^y \arrow[ur, "m_y"'] & \end{tikzcd} \tag{21}$$

yields the equality $(c_\gamma)^*(W_y) = W_x$ and then the property (20).

3. Lagrangian submanifolds

In the rest of this paper, all manifolds are C^∞, σ-compact and with connected components of the same dimension. All maps are C^∞ and submanifold means C^∞ submanifold.

3.1. **Local submanifolds, families of submanifolds.** As observed in [32, Chapter 21], immersed Lagrangrian submanifolds are the natural objects parametrized by nondegenerate phase functions. By definition, immersed submanifolds are ranges of (not necessarily injective) immersions, so they can be quite far from being submanifolds. Moreover, immersed submanifolds sometimes arise not as images of immersions, but as images of maps of constant rank.
As it is not always relevant to introduce the appropriate immersions to study immersed submanifolds, we propose the following elementary reformulation of the notion of immersed submanifold, which is, in our interpretation, suggested in [32, Proposition C.3.3].

Definition 1. *A* local submanifold *of a manifold is a subset consisting of a countable union of submanifolds all of the same dimension.*

The following companion terminology will be used:

Definition 2. *A* patch *in a p-dimensional local submanifold Z is a p-dimensional submanifold included in Z.*
A parametrization of a local submanifold is a diffeomorphism of a manifold onto a patch.

For instance, the union of the coordinates axes in $\mathbb{R}^2$ is a 1-dimensional local submanifold and each axis is a patch.

Local and immersed submanifolds refer to the same objects. Indeed, let Z be a local submanifold of X and let $(Z_j)_j$ be a countable family of patches such that $Z = \cup_j Z_j$. The disjoint union $Y = \sqcup_j Z_j$ is a manifold and the obvious map $f : Y \to X$ is an immersion satisfying $f(Y) = Z$.

Conversely, if $f : Y \to X$ is an immersion then one can cover Y by a countable family (U_j) of open subsets such that $f : U_j \to f(U_j)$ is a diffeomorphism. Thus the image $Z = f(Y) = \cup f(U_j)$ is a local submanifold of dimension $\dim Y$.

Actually, ranges of maps $f : Y \to X$ of constant rank are also local submanifolds. Indeed, by [32, Proposition C.3.3]) there exists countable families of local coordinate systems $(U_j)_j$ covering Y and $(V_j)_j$ covering X such that $f(U_j) \subset V_j$ and f is given in these coordinates by

$$f(y_1, \dots, y_m) = (y_1, \dots, y_p, 0, \dots, 0) \quad \forall (y_1, \dots, y_m) \in U_j. \tag{22}$$

Set $Y_j = \{y \in U_j \ ; \ y_{p+1} = \cdots = y_m = 0\}$ and consider the manifold $\widetilde{Y} = \sqcup_j Y_j$. The natural map

$$\widetilde{Y} \ni y \longmapsto f(y) \tag{23}$$

is an immersion with range Z.

We recall that two submanifolds Z_1, Z_2 of X have a clean intersection if $Z_1 \cap Z_2$ is a submanifold and at any point $z \in Z_1 \cap Z_2$,

$$T_z(Z_1 \cap Z_2) = T_z Z_1 \cap T_z Z_2. \tag{24}$$

The excess of the intersection is the number $e = \operatorname{codim}(T_z Z_1 + T_z Z_2)$. The intersection is transversal if $e = 0$ and the transversality of Z_1, Z_2 is denoted by $Z_1 \pitchfork Z_2$. This adapts to local submanifolds as follows.

Definition 3. *The intersection $Z_1 \cap Z_2$ of two local submanifolds Z_1, Z_2 of a manifold X is* clean (resp. transversal) *if there exists covers (Z_{1j}) and (Z_{2k}) of Z_1 and Z_2 by countably many patches such that $Z_{1j} \cap Z_{2k}$ is clean with the same excess (resp. transversal) for all j, k .*

In the following definitions, we consider a surjective submersion $\pi : X \longrightarrow B$ between manifolds. The fiber of π at the point b is noted X_b.

Definition 4. *A local submanifold Z of X is said to be* transverse to π *if it can be covered by a countable family $(Z_i)_{i\in I}$ of patches such that $\pi|_{Z_i} : Z_i \to B$ is a submersion for any $i \in I$.*

Definition 5. *A C^∞* family of local submanifolds subordinate to π *is a family $(Z_b)_{b\in U}$ where U is an open subset of B and Z_b is a subset of X_b for any b, such that $\cup_{b\in U} Z_b$ is a local submanifold of X transverse to π.*

Finally, patches and parametrizations of a family $\mathcal{Z} = (Z_b)_{b\in U}$ as above refer to the corresponding object for the local submanifold $Z = \cup_{b\in U} Z_b$; and a section of $\mathcal{Z}$ is a local section of π taking values in a patch of $\mathcal{Z}$.

3.2. **Phases, clean and non-degenerate phases.** Recall that a subset $\mathcal{U}$ of $\mathbb{R}^n \times \mathbb{R}^N$ is conic if $(x, \theta) \in \mathcal{U}$ implies that $(x, t\theta) \in \mathcal{U}$ for all $t > 0$. A map $\chi : \mathcal{U} \to \mathcal{V}$ between conic subsets is homogeneous if $\chi(x, t\theta) = t\chi(x, \theta)$ for all $t > 0$.

Definition 6. [31, p.86][32, 21.1.8] *A* cone bundle *consists of a surjective submersion $p : C \to X$ and an action of $\mathbb{R}_+^*$ on C which respects the fibers of p and such that:*
For all $v \in C$, there exists a conic neighborhood $\mathcal{U}$ of v in C and a homogeneous diffeomorphism $\chi : \mathcal{U} \to \mathcal{V} \subset \mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\})$ onto a conic open subset such that the following diagram commutes:

(25)
$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "p"'] & & \mathcal{V} \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd} .$$

The triple $(\mathcal{U}, \mathcal{V}, \chi)$ is then called a conic local trivialization of the cone bundle around v. When X is a point, C is called a conic manifold.

Example 1.
(1) *If X is a manifold, $T^*X \setminus 0$ is a conic manifold.*
(2) *Let $\pi : Z \to X$ be a submersion onto X, and set $C = Z \times \mathbb{R}^k \setminus 0$ and $p = \pi \circ \mathrm{pr}_1$. Then C, with the obvious $\mathbb{R}_+$-action, is a cone bundle over X. Conic local trivializations are built from local trivializations $\kappa : p^{-1}(U) \overset{\simeq}{\to} U \times Y \times \mathbb{R}^k \setminus 0$ composed with*
$$(x, y, \theta) \longmapsto (x, |\theta|.y, \theta) \in U \times (\mathbb{R}^{n_Z - n_X + k} \setminus 0).$$

Definition 7. [32, Def. 21.2.15],[30, p. 154]. *Let X be a manifold and $U \subset X$ an open subset.*

(1) *A* phase function *over U consists of a cone bundle (p, C, U) and a C^∞ homogeneous function $\phi : C \to \mathbb{R}$ without critical points.*
(2) *Let $\phi : C \to \mathbb{R}$ be a phase function over U. Let us denote by $\phi'_{\mathrm{vert}} : C \to (\ker dp)^*$ the restriction of the differential of ϕ to the fibers of $p : C \to U$. We say that ϕ is* clean *if the set*

(26)
$$C_\phi = \{v \in C;\ \phi'_{\mathrm{vert}}(v) = 0\} = (\phi')^{-1}(\ker dp^\perp)$$

is a submanifold of C with tangent space given by the equation $d\phi'_{\mathrm{vert}} = 0$. The excess of the clean phase *ϕ is the number $e = \dim C_\phi - \dim X = \dim \ker dp - \mathrm{rk}(d\phi'_{\mathrm{vert}})$.*
(3) *The phase function ϕ is* non degenerate *if ϕ'_{vert} is a submersion (that is, clean with $e = 0$).*

Using ${}^t dp^{-1} : (\ker dp)^\perp \to T^*X$, the "horizontal" part of $d\phi$ is then well defined on C_ϕ by $\phi'_{\mathrm{hor}}(v) = {}^t(dp_v)^{-1}(\phi'(v)) \in T^*_{p(v)}X$, that is

(27)
$$\phi'_{\mathrm{hor}}(v)(t) = \phi'(v)(u), \quad t \in p^*(TX)_v,\ dp(u) = t.$$

We introduce the map

$$T_\phi : C_\phi \longrightarrow T^*X \qquad v \longmapsto \big(p(v), \phi'_{\mathrm{hor}}(v)\big) \tag{28}$$

and we set

$$\Lambda_\phi = T_\phi(C_\phi) = \{(p(v), \phi'_{\mathrm{hor}}(v)) \ ; \ \phi'_{\mathrm{vert}}(v) = 0\}. \tag{29}$$

If ϕ is clean, then for any $v \in C_\phi$, there exists an open conic neighborhood V of v in C such that $T_\phi(V)$ is a C^∞ conic Lagrangian submanifold of $T^*X \setminus 0$ and $T_\phi : C_\phi \cap V \longrightarrow T_\phi(V)$ is a fibration with fibers of dimension e and therefore Λ_ϕ is a conic Lagrangian local submanifold of $T^*X \setminus 0$ ([32, 30], see also Remark 3.3 below). Moreover, if the fibers of T_ϕ are connected and compact then Λ_ϕ is a true submanifold and $T_\phi : C_\phi \longrightarrow \Lambda_\phi$ is a fibration. On the other hand, if ϕ is non-degenerate, then T_ϕ is an immersion, but nothing is gained in terms of the regularity of Λ_ϕ, which is still only a local submanifold.

Conversely, any conic Lagrangian local submanifold Λ of $T^*X \setminus 0$ can be parametrized by non-denegerate phase functions [32, 30]. This means that for any $(x, \xi) \in \Lambda$ there exist an open conic neighborhood W of (x, ξ) into T^*X, an open conic subset $V \subset X \times \mathbb{R}^N \setminus 0$ and a non-degenerate phase function $\phi : V \to \mathbb{R}$ with $\Lambda_\phi = \Lambda \cap W$.

3.3. **"Pushforward" of Lagrangian submanifolds.** The following statements are mainly reformulations of existing results [70, 32, 30]. Precisely, we discuss a procedure consisting firstly of taking the intersection of a given Lagrangian submanifold with a coisotropic submanifold and secondly of pushing forward this intersection to another symplectic manifold using a suitable map. This procedure, which is not a simple pushforward (the first part is rather a pull back), is the main step in the composition of Lagrangian submanifolds of cotangent groupoids.

Proposition 2. *Let $(S, \omega_S), (T, \omega_T)$ be symplectic manifolds, H be a submanifold of S and $\mu : H \to T$ be a surjective submersion such that*

$$\mu^*(\omega_T) = \omega_S|_H. \tag{30}$$

(1) *The following assertions are equivalent:*
   (a) *H is coisotropic;*
   (b) *$(\ker d\mu)^{\perp_{\omega_S}} = TH$;*
   (c) *the graph $\mathrm{Gr}_\mu = \{(x, \mu(x)) \ ; \ x \in H\}$ is a Lagrangian submanifold of $S \times (-T)$.*

(2) *Assume that the previous assertions are true. If $\widetilde{\Lambda}$ is a Lagrangian local submanifold of S with clean intersection with H then*

$$\Lambda := \mu(\widetilde{\Lambda} \cap H) \tag{31}$$

is a local Lagrangian submanifold of T. If moreover $\widetilde{\Lambda}$ is a submanifold and the map $\mu : \widetilde{\Lambda} \cap H \to \Lambda$ has compact and connected fibers, then Λ is a submanifold.

Proof. (1) The condition (30) implies that for any x, $\ker d\mu_x \subset (T_xH)^{\perp_{\omega_S}}$. Let us assume that H is coisotropic, that is, that $(T_xH)^{\perp_{\omega_S}} \subset T_xH$ for all x. Let $u \in (T_xH)^{\perp_{\omega_S}}$. Then by assumption

$$\omega_T(d\mu(u), d\mu(v)) = \omega_S(u, v) = 0 \text{ for all } v \in T_xH,$$

which by surjectivity of $d\mu$ proves that $u \in \ker d\mu_x$. This gives (a)⇒(b), and the converse implication is trivial.

Let $(s,t) \in (T\,\mathrm{Gr}_\mu)^{\perp_\omega}$ and choose $u \in T_xH$ with $d\mu(u) = t$. Then $\omega_T(d\mu(u), d\mu(s')) = \omega_S(s,s')$ for all $(s', d\mu(s')) \in T\,\mathrm{Gr}_\mu$. Using (30), this gives $u - s \in (TH)^{\perp_\omega}$, and assuming (b) this gives $t = d\mu(s)$, which proves that Gr_μ is coisotropic and thus (c) since (30) is obviously equivalent to the isotropy of Gr_μ.

For (c)⇒(a) we apply the following elementary lemma.

Lemma 3. *Let λ be a coisotropic linear subspace in a product of symplectic vector spaces $S_1 \times S_2$. Then $\mathrm{pr}_j(\lambda)$ is a coisotropic subspace of S_j, $j = 1,2$.*

(2) Using a decomposition of $\widetilde{\Lambda}$ into patches, it is sufficient to assume that $\widetilde{\Lambda}$ is a submanifold. The result now follows from a symplectic reduction procedure: see [32, Proposition 21.2.13, Theorem 21.2.14] or [70, page 12]. We outline the proof.

By assumption $\widetilde{\Lambda} \cap H$ is a C^∞ submanifold and at any point $x \in \widetilde{\Lambda} \cap H$,

$$T_x(\widetilde{\Lambda} \cap H) = T_x\widetilde{\Lambda} \cap T_xH. \tag{32}$$

Since $\ker d\mu_x \subset (\ker d\mu_x)^{\perp_\omega} = T_xH$, the symplectic reduction ([32, proposition 21.2.13]) applied to $\lambda = T_x\widetilde{\Lambda}$ asserts that

$$\lambda' = \Big(T_x\widetilde{\Lambda} \cap H\Big) \,/\, \Big(T_x(\widetilde{\Lambda}) \cap \ker d\mu_x\Big)$$

is a Lagrangian subspace of the symplectic vector space $S' = T_xH/\ker d\mu_x \simeq T_{\mu(x)}(T)$. Therefore, rank $d\mu_x = \dim T/2$ is independent of x and the image $\Lambda = \mu(\widetilde{\Lambda} \cap H)$ is a local submanifold of T of dimension $\dim T/2$. Assumption (30) implies that $d\mu_x(T_x\widetilde{\Lambda} \cap H)$ is Lagrangian, so Λ is a Lagrangian local submanifold. If the fibers of $\mu|_{\widetilde{\Lambda}\cap H}$ are moreover compact and connected, it follows by standard arguments of differential geometry that Λ is actually a submanifold of T.

□

We now give a generic example in which Proposition 2 applies. This example also shows how, and when, clean phase functions arise in the task of parametrizing Lagrangian submanifolds.

Proposition 4. *Let X, Y be manifolds, $Z \subset X$ a submanifold and $f : Z \to Y$ a submersion. Set $H = (\ker df)^\perp \subset T^*X$ and*

$$\mu : H \ni (x,\xi) \longmapsto (f(x), {}^t df_x^{-1}(\xi)) \in T^*Y.$$

The following assertions hold.

(1) *Gr_μ is a Lagrangian submanifold of $T^*X \times (-T^*Y)$.*

(2) *Let $\widetilde{\Lambda}$ be a conic Lagrangian local submanifold of $T^*X \setminus 0$ intersecting H cleanly with excess e and such that $\widetilde{\Lambda} \cap N^*Z = \emptyset$. Let $(x,\xi) \in \widetilde{\Lambda} \cap H$ and let*

$$\widetilde{\phi} : U \times \mathbb{R}^N \longrightarrow \mathbb{R} \qquad (U \text{ open subset of } X)$$

be a non-degenerate phase function parametrizing $\widetilde{\Lambda}$ around (x,ξ). Setting $V = U \cap Z$, the restriction ϕ of $\widetilde{\phi}$ to $V \times \mathbb{R}^N$ is a phase function on the cone bundle

$$f \circ \mathrm{pr}_1 : C = V \times (\mathbb{R}^N \setminus 0) \longrightarrow f(V) \subset Y. \tag{33}$$

This phase function is clean with excess e and parametrizes $\Lambda = \mu(\widetilde{\Lambda} \cap H)$ around $\mu(x,\xi) = (f(x), {}^t(df^{-1}(\xi)))$.

Proof. (1) This is immediately checked using local coordinates (x', x'', x''') for X such that Z is given by $x''' = 0$, x' gives local coordinates for Y and $f(x', x'') = x'$. Then one has $H = \{(x', x'', 0, \xi', 0, \xi''')\}$ so that H is coisotropic and one can apply the first part of Proposition 2.

(2) All the assertions being local, we may assume that $\widetilde{\Lambda}$ is C^∞. It is obvious that ϕ is C^∞ and homogeneous in the fibers of the given conic manifold. Assume that $d\phi$ vanishes at a point $(x, \theta) \in V \times \mathbb{R}^N$. This means that $\widetilde{\phi}$ satisfies

$$\widetilde{\phi}'_x(x, \theta)(t) = 0 \ \forall t \in T_x Z, \quad \widetilde{\phi}'_\theta(x, \theta) = 0.$$

This implies that $(x, \widetilde{\phi}'_x(x, \theta)) \in \widetilde{\Lambda} \cap N^*Z$, which contradicts the assumptions. Thus ϕ is a phase function.

To precise C_ϕ, we denote by $y \in Y$ the space coordinate of ϕ and

$$\omega = (z, \theta), \quad \text{where } z \in f^{-1}(y) \subset Z,$$

the parameters. Then

$$C_\phi = \{(x, \theta) \ ; \ \phi'_\omega(x, \theta) = 0\}. \tag{34}$$

Observe that $\phi'_\omega = (\widetilde{\phi}'_z, \widetilde{\phi}'_\theta) = (\phi'_z, \phi'_\theta)$. Thus

$$C_\phi = \{(x, \theta) \in C_{\widetilde{\phi}} \ ; \ \widetilde{\phi}'_z(x, \theta) = 0\}. \tag{35}$$

Observe that $\widetilde{\phi}'_z(x, \theta) = 0$ means exactly that $\widetilde{\phi}'_x(x, \theta) \in (\ker df)^\perp$. Therefore

$$(x, \theta) \in C_\phi \Leftrightarrow (x, \widetilde{\phi}'_x(x, \theta)) \in \widetilde{\Lambda} \cap H. \tag{36}$$

It follows that the local diffeomorphism $T_{\widetilde{\phi}} : C_{\widetilde{\phi}} \to \widetilde{\Lambda}$ maps C_ϕ onto $\widetilde{\Lambda} \cap H$:

$$C_\phi \ni (x, \theta) \xmapsto{T_{\widetilde{\phi}}} (x, \widetilde{\phi}'_x(x, \theta)) \in \widetilde{\Lambda} \cap H. \tag{37}$$

This proves that C_ϕ is a C^∞ submanifold of $C_{\widetilde{\phi}}$ since by assumption $\widetilde{\Lambda} \cap H$ is a C^∞ submanifold. Recall that C_ϕ is given by the equations

$$(x, \theta) \in V \times \mathbb{R}^N, \quad \widetilde{\phi}'_z(x, \theta) = 0, \quad \widetilde{\phi}'_\theta(x, \theta) = 0. \tag{38}$$

The first one means that $(x, \widetilde{\phi}'_x(x, \theta)) = T_{\widetilde{\phi}}(x, \theta) \in H$ and the second one that $(x, \widetilde{\phi}'_x(x, \theta)) = T_{\widetilde{\phi}}(x, \theta) \in \widetilde{\Lambda}$. Recall that H is given by the equation

$$\rho(x, \xi) = (x, 0), \tag{39}$$

where $\rho : T^*_Z X \to (\ker df)^*, (x, \xi) \mapsto (x, \xi_z)$ is the submersion given by the restriction of linear forms to $\ker df$. Since by assumption we have

$$T(\widetilde{\Lambda} \cap H) = T\widetilde{\Lambda} \cap TH \tag{40}$$

and since $T\widetilde{\Lambda}$ and TH are given respectively by the equations $d\phi'_\theta = 0$ and $d\rho = 0$, it follows that $(t, \zeta) \in TZ \times \mathbb{R}^N$ belongs to TC_ϕ if and only if $dT_{\widetilde{\phi}}(t, \zeta) \in T(\widetilde{\Lambda} \cap H)$. The latter condition is equivalent to

$$d\phi'_\theta(t, \zeta) = 0 \text{ and } d\rho dT_{\widetilde{\phi}}(t, \zeta) = 0. \tag{41}$$

Taking into account the definition of ρ and its linearity in the fibers, (41) is equivalent to

$$d\phi'_\theta(t, \zeta) = 0 \text{ and } d\phi'_z(t, \zeta) = 0, \tag{42}$$

that is, to $d\phi'_\omega(t, \zeta) = 0$ and this proves that ϕ is a clean phase function.

Remember that ϕ is a phase function on the cone bundle $V \times \mathbb{R}^N \overset{f\circ \mathrm{pr}_1}{\to} Y$, that is, the space variable is $y \in Y$ and the parameter variable is $\omega = (z,\theta)$ with $z \in f^{-1}(y)$. The differential $\phi'_h(y,\omega) \in T^*_yY$ of ϕ in the "horizontal direction y" is well defined if and only if the vertical differential ϕ'_ω vanishes, and then

$$\phi'_h(y,\omega)(v) = d_z\phi(z,\theta)(u) \quad \forall u \in T_zZ \text{ such that } df(u) = v. \tag{43}$$

Since $d\phi_{(z,\theta)}(u) = d\widetilde{\phi}_{(z,\theta)}(u)$, it follows that, around $\mu(x,\xi)$,

$$\begin{aligned} \Lambda_\phi &= \{(y,\phi'_h(y,\omega))\ ;\ (y,\omega)=(z,\theta)\in C_\phi\} \\ &= \{(f(z), {}^t df^{-1}(\phi'_z(z,\theta))\ ;\ (z,\theta)\in C_\phi\} \\ &= \Lambda. \end{aligned} \tag{44}$$

We have $\dim H = n_X + n_Y$ and $\dim C_\phi = \dim \widetilde{\Lambda}\cap H$. Then

$$\begin{aligned} e &= (2n_X - \dim\widetilde{\Lambda}) + (2n_X - \dim H) - (2n_X - \dim\widetilde{\Lambda}\cap H) \\ &= n_X + n_X - n_Y - 2n_X + \dim\widetilde{\Lambda}\cap H = \dim\widetilde{\Lambda}\cap H - \dim\Lambda \\ &= \dim C_\phi - \dim\Lambda, \end{aligned} \tag{45}$$

and the latter is by definition the excess e of ϕ.

□

Remark: Let ϕ be a phase function over Y, defined on the total space of a given cone bundle $p : C \to Y$. Then $\widetilde{\Lambda} = \mathrm{Gr}(\phi')$ is a Lagrangian submanifold of T^*C and in the notations of the previous proposition with $Z = X = C$, $f = p$, we get

$$\widetilde{\Lambda}\cap H \text{ is clean if and only if } \phi \text{ is a clean phase function,}$$

and since $\mu(v,\xi) = (p(v),\xi_{\mathrm{hor}})$ for all $(v,\xi)\in H = (\ker dp)^\perp$, we also have

$$\mu(\widetilde{\Lambda}\cap H) = \Lambda_\phi,$$

where Λ_ϕ is defined in (29).

3.4. **Families of Lagrangian submanifolds and submersions.** Let $\pi : M \to B$ be a surjective submersion, with fibers of dimension n and base of dimension q. The inclusion $M_b \hookrightarrow M$ is denoted by i_b. We consider the vector bundle $V^*M = (\ker d\pi)^* = \cup_{b\in B} T^*M_b$ over M and we denote by p both the projection maps $T^*M \to M$ and $V^*M \to M$. Similarly, the natural submersions maps $T^*M \to B$ and $V^*M \to B$ are both denoted by σ, while the natural restriction map $T^*M \to V^*M$ is denoted by ρ. The fibers of $V^*M \to B$ are exactly the cotangent spaces T^*M_b, $b \in B$. We have a short exact sequence of vector bundles over M,

$$0 \longrightarrow (\ker d\pi)^\perp \longrightarrow T^*M \overset{\rho}{\longrightarrow} V^*M \longrightarrow 0. \tag{46}$$

We are interested in C^∞ families $(\Lambda_b)_{b\in B}$ of (local, Lagrangian, conic) submanifolds subordinate to σ in the sense of Definition 5. By a slight abuse of vocabulary, we will say that they are subordinate to π. Similarly, we will say that $\Lambda \subset T^*M$ is transverse to π if it is transverse to $\sigma = \pi\circ p : T^*M \to B$ in the sense of Definition 5, which here is obviously equivalent to the condition

$$T_xM_b + dp(T_{x,\xi}\Lambda) = T_xM \quad \forall b\in B,\ \forall x \in M_b; \tag{47}$$

that is, equivalent to the transversality of the maps $i_b : M_b \to M$ and $p|_\Lambda : \Lambda \to M$ for any b. The next theorem is a straight adaptation of Theorem 21.2.16 in [32].

Theorem 5. *Let $\mathcal{L} = (\Lambda_b)_{b\in B}$ be a family of conic Lagrangian submanifolds subordinate to π and $L = \cup_{b\in B}\Lambda_b \subset V^*M \setminus 0$ the associated transversal submanifold.*

(1) *For any $(m_0, \xi_0) \in \Lambda_{b_0}$, there exist local trivializations of π around m_0 such that in the associated local coordinates (x, b, ξ) of V^*M, the map*

$$L \ni (x, b, \xi) \longmapsto (b, \xi) \tag{48}$$

is a local diffeomorphism. Such a local trivialization is called adapted to $\mathcal{L}$ (or L).

(2) *In local trivializations adapted to $\mathcal{L}$, there exist conic neighborhoods $\mathcal{W}$ of $(b_0, \xi_0) \in \mathbb{R}^q \times (\mathbb{R}^n \setminus 0)$ and $\mathcal{V}$ of $(m_0, \xi_0) \in V^*M \setminus 0$ and a unique C^∞ function $H : \mathcal{W} \to \mathbb{R}$ homogeneous of degree 1 such that*

$$L \cap \mathcal{V} = \{(H'_\xi(b, \xi), b, \xi)\ ;\ (b, \xi) \in \mathcal{W}\}. \tag{49}$$

In other words, the C^∞ function $\phi(x, b, \xi) = \langle x, \xi\rangle - H(b, \xi)$ provides a family labelled by b of non-degenerate phase functions $\phi(\cdot, b, \cdot)$ parametrizing Λ_b.

Using the notions of sections, transversality and parametrizations introduced in Paragraph 3.1, we see that the conclusions of the theorem hold for families of conic Lagrangian local submanifolds as well. We just need to replace L in (48) by a patch L'.

Thus, families of Lagrangian local submanifolds are parametrized by families of non-degenerate phase functions defined in open cones of $M \times (\mathbb{R}^n \setminus 0)$.

Proof. Let (y, z) be a local trivialisation around m_0. Here, $z = (z_1, \ldots, z_q)$ gives local coordinates of B at b_0 and for fixed b, $y = (y_1, \ldots, y_n)$ gives local coordinates of M_b. Following the proof of [32, Theorem 21.2.16], we can perform a change of variables $x = x(y)$ so that, as submanifolds of T^*M_b, the space Λ_{b_0} is transversal to the horizontal subspace $\xi = \xi_0$ at the point (m_0, ξ_0), and then so that the map $\Lambda_{b_0} \ni (x, b_0, \xi) \to \xi$ has a bijective differential at (x_0, b_0, ξ_0). Moreover, by assumption, the map $L \ni (x, b, \xi) \mapsto b$ has a surjective differential at (x_0, b_0, ξ_0). Therefore, the differential of $L \ni (x, b, \xi) \mapsto (b, \xi)$ is surjective at (x_0, b_0, ξ_0), hence bijective for $\dim L = n + q$.

We now turn to the second assertion which consists of routine computations (see for instance the end of the proof of [32, Theorem 21.2.16]). By 1., there exists a neighborhood $\mathcal{U} = (U \times W) \times C$ of $(x_0, b_0, \xi_0) \in \mathbb{R}^{n+q} \times (\mathbb{R}^n \setminus 0)$ and a C^∞ function $x(b, \xi)$ defined on $W \times C$ such that

$$L \cap \mathcal{U} = \{(x(b, \xi), b, \xi)\ ;\ b \in W, \xi \in C\}.$$

Since x is necessarily homogeneous of degree 0 in ξ, we can assume that C is a cone. Since the canonical 1-form of T^*M_b vanishes on Λ_b, we get

$$\sum_j \xi_j d_\xi x_j(b, \xi) = 0.$$

In other words, the linear form $u \mapsto \langle x'_\xi(b, \xi).u, \xi\rangle$ vanishes. It follows that

$$x(b, \xi) = H'_\xi(b, \xi), \qquad \text{with } H(b, \xi) = \langle x(b, \xi), \xi\rangle, \tag{50}$$

and that, by the Euler formula, this function H is unique among C^∞ functions $K(b, \xi)$ homogeneous of degree 1 in ξ and satisfying $K'_\xi = x$. Finally, it is clear that for fixed b, the function $\phi(x, b, \xi) := \langle x, \xi\rangle - H(b, \xi)$ is a non-degenerate phase function parametrizing Λ_b. $\square$

Theorem 6. *Let $(\Lambda_b)_{b\in B}$ be a family of conic Lagrangian local submanifolds subordinate to π. There exists a unique conic Lagrangian local submanifold $\Lambda\subset T^*M$ transverse to π such that*

$$i_b^*\Lambda=\Lambda_b,\quad b\in B. \tag{51}$$

One says that Λ is the gluing of the family $(\Lambda_b)_{b\in B}$.

Proof. We first assume that $(\Lambda_b)_{b\in B}$ is a family of submanifolds. Assume that Λ is a conic Lagrangian submanifold of T^*M satisfying (51). Let $\kappa:\mathcal{U}\to U\times W$, $\kappa(m)=(x,b)$, be an adapted local trivialisation and H the corresponding function constructed in Theorem 5. By assumption, we have in these coordinates

$$\kappa_*(\Lambda\cap T^*\mathcal{U})\subset\{(H'_\xi(b,\xi),b,\xi,\tau);\ \xi\in\mathcal{C},\ (b,\tau)\in T^*W\}. \tag{52}$$

The projection $(x,b,\xi,\tau)\to(b,\xi)$ restricted to $\kappa_*(\Lambda\cap T^*\mathcal{U})$ is still a local diffeomorphism since $\dim\Lambda=n+q$. Thus τ is a C^∞ function of (b,ξ). Since Λ is conic and Lagrangian, the fundamental one form of T^*M vanishes identically on Λ, which yields

$$\begin{aligned}
0 &= \sum_j \xi_j d(H'_{\xi_j})(b,\xi)+\sum_l \tau_l db_l\\
&= \sum_{i,j}\xi_j H''_{\xi_i\xi_j}(b,\xi)d\xi_i+\sum_{l,j}\xi_j H''_{b_l\xi_j}(b,\xi)db_l+\sum_l\tau_l db_l\\
&= \sum_{l,j}\xi_j H''_{b_l\xi_j}(b,\xi)db_l+\sum_l\tau_l db_l,\qquad\text{since } H'_{\xi_i} \text{ is homogeneous of degree 0 in } \xi,\\
&= \sum_{l,j} H'_{b_l}(b,\xi)db_l+\sum_l\tau_l db_l,\qquad\text{since } H'_{b_l} \text{ is homogeneous of degree 1 in } \xi.
\end{aligned}$$

This proves that $\tau(b,\xi)=-H'_b(b,\xi)$ and thus

$$\kappa_*(\Lambda\cap T^*\mathcal{U})=\{(H'_\xi(b,\xi),b,\xi,-H'_b(b,\xi));\ \xi\in\mathcal{C},\ b\in W\}\subset(T^*U\times T^*W)\setminus 0. \tag{53}$$

This proves uniqueness and the transversality of Λ with respect to π as well. It also proves existence in open subsets of the form $T^*\mathcal{U}$, $\mathcal{U}$ being the domain of an adapted local trivialisation. We note for future reference that given $(m,\xi)\in\Lambda_b$, there is a unique $(m,\zeta)\in\Lambda$ such that $\rho(m,\zeta)=(m,\xi)$.

The existence follows from the local existence and uniqueness . Indeed, let $(\kappa_j,\mathcal{U}_j)$, $j=1,2$, be two adapted local trivialisations such that $\mathcal{U}_1\cap\mathcal{U}_2\neq\emptyset$ and Λ_1,Λ_2 the submanifolds of $T^*\mathcal{U}_1$ and $T^*\mathcal{U}_2$ defined by (53). The previous argument of uniqueness proves that over $T^*\mathcal{U}_1\cap\mathcal{U}_2$ we have $\Lambda_1=\Lambda_2$. This allows us to define a solution Λ globally on T^*M using a cover by adapted trivialisations.

Now, let us consider the general case. Choose a countable cover of $\mathcal{L}=(\Lambda_b)_b$ by families $\mathcal{L}_j=(\Lambda_{bj})_{b\in U_j}$, $j\in J$, of conic Lagrangian submanifolds. By the first part of the proof, there exists for any j a unique $\Lambda_j\subset T^*M$ gluing $\mathcal{L}_j$. Then $\Lambda=\cup_J\Lambda_j$ is a gluing of $\mathcal{L}$ and this proves the existence. If Λ'_1 is a patch contained in another solution Λ', then $\rho(\Lambda'_1)$ is contained in $\rho(\Lambda')=L=\cup_B\Lambda_b$. For any $j\in J$, the set $L_j=\cup_b\Lambda_{bj}$ is a patch of L and by the remark made just after the proof of uniqueness in the submanifold case, we get that $\rho^{-1}(L_j)\cap\Lambda'_1$ is contained in the unique conic Lagrangian submanifold Λ_j gluing $\mathcal{L}_j$. Therefore, $\Lambda'_1\subset\cup_J\Lambda_j=\Lambda$ and uniqueness follows directly. □

Conversely, we have the following statement, which can be generalized to the local case.

Theorem 7. *Let $\Lambda\subset T^*M\setminus 0$ be a conic Lagrangian submanifold transverse to π. Then*

(1) $\Lambda \cap (\ker d\pi)^\perp = \emptyset$.
(2) $(i_b^*(\Lambda))_{b\in B}$ *is a family of conic Lagrangian local submanifolds of* $T^*M_b \setminus 0$. *In other words,* $\rho(\Lambda)$ *is a local submanifold of* V^*M *transverse to* π *and for any* b, *the fiber*

$$\Lambda_b = i_b^*(\Lambda) = \rho(\Lambda) \cap T^*M_b \tag{54}$$

is a conic Lagrangian local submanifold of $T^*M_b \setminus 0$.

Proof. (1) On one hand, by dualizing the transversality condition (47), we get

$$(\ker d\pi)^\perp \cap (dp(T\Lambda))^\perp = M \times \{0\} \subset T^*M. \tag{55}$$

On the other hand, the inclusion

$$\Lambda \subset (dp(T\Lambda))^\perp \tag{56}$$

holds. Indeed, by conicity of Λ, any (x,ξ) in Λ corresponds canonically to a vertical vector in $T_{(x,\xi)}\Lambda$, denoted by $v(\xi)$. Since Λ is Lagrangian, we have with these notations

$$\xi(dp(Z)) = \omega(v(\xi), Z) = 0, \quad \forall Z \in T_{(x,\xi)}\Lambda,$$

where ω denotes the symplectic form of T^*M. Therefore $\Lambda \cap (\ker d\pi)^\perp \subset \{0\}$ and since by assumption $\Lambda \subset T^*M \setminus 0$, the first assertion is proved.

(2) As observed in [30, Chap. 4, Par. 4], the transversality assumption (47) is actually equivalent to the transversality of the intersection of the canonical relation

$$\Lambda(i_b) = \{(m, -\xi, m, \zeta) \in T^*M_b \times T^*M \ ; \ m \in M_b, \zeta|_{T_mM_b} = \xi\}$$

with Λ, viewed as a canonical relation from T^*M to a point. Therefore, Hormander's product of canonical relations applies [32, Theorem 21.2.14], that is, the map

$$\rho_b : \Lambda \cap T^*_{M_b}M \longrightarrow T^*M_b \setminus 0 \ , \ (m,\zeta) \longmapsto (m, \zeta|_{T_mM_b})$$

is an immersion with range $\rho_b(\Lambda) = \Lambda_b = i_b^*(\Lambda)$ a conic Lagrangian local submanifold of $T^*M_b \setminus 0$, for any b. From now on, let $(m_0,\zeta_0) \in \Lambda$, $b_0 = \pi(m_0)$, $(m_0,\xi_0) = \rho(m_0,\zeta_0)$ and choose a local trivialization $\kappa(m) = (x,b)$ of π around m_0. After applying, if necessary, a diffeomorphism in the x variables independent of b, we can assume that κ is such that in a neighborhood of (m_0,ξ_0) in $T^*M_{b_0}$, the projection $\Lambda_{b_0} \ni (x,\xi) \to \xi$ has a bijective differential. Moreover, by assumption, the map $(x,b,\zeta) \to b$ has a surjective differential everywhere. It follows that the map

$$\Lambda \ni (x,b,\xi,\tau) \longmapsto (b,\xi) \in \mathbb{R}^q \times \mathbb{R}^n \tag{57}$$

has a surjective differential, and therefore is bijective for dimensional reasons. In particular, this proves that the map $\rho : \Lambda \to V^*M$ is an immersion, and thus $\rho(\Lambda)$ is a local submanifold. It is also obvious from the same argument that $\rho(\Lambda)$ is transverse to π, which proves that $(i^*(\Lambda_b))_{b\in B}$ is a C^∞ family of conic Lagrangian local submanifolds.

∎

4. Lagrangian distributions

4.1. **Lagrangian distributions on a manifold.** Unless otherwise stated, we use the definitions and notations of [32] for all the notions involved in the theory of Lagrangian distributions.

Let X be a C^∞ manifold of dimension n, E a complex vector bundle over X, Λ a conic Lagrangian submanifold of $T^*X\setminus 0$ and $m\in\mathbb{R}$. The set $I^m(X,\Lambda;E)$ consists of distributions belonging to $\mathcal{D}'(X,E)$ which, modulo $C^\infty(X,E)$, are locally finite sums of oscillatory integrals ([32, Section 25.1]):

$$u=\sum_{j\in J}(2\pi)^{-(n+2N_j)/4}\int e^{i\phi_j(x,\theta_j)}a_j(x,\theta_j)d\theta_j \mod C^\infty(X,E), \tag{58}$$

where for all j,

- $(x,\theta_j)\in\mathcal{V}_j\subset U_j\times\mathbb{R}^{N_j}$ with U_j a local coordinate patch of X and $\mathcal{V}_j$ an open conic subset;
- $\phi_j:\mathcal{V}_j\to\mathbb{R}$ is a non degenerate phase function providing a local parametrization of Λ;
- $a_j(x,\theta_j)\in S^{m+(n_X-2N_j)/4}(U_j\times\mathbb{R}^{N_j},E)$ has support in the interior of a cone with compact base and included in $\mathcal{V}_j$.

Such distributions are called Lagrangian distributions associated with Λ, with values in E. When Λ is the conormal bundle of a submanifold, they are called conormal distributions.

In the definition above, one can allow conic Lagrangian local submanifolds of $T^*X\setminus 0$, and thus the set $I^m(X,E)$ of all Lagrangian distributions with values in E is a vector space.

The principal symbol of an element in $I^m(X,\Lambda;E\otimes\Omega_X^{1/2})$ can be defined as an element of $S^{m+n/4}(\Lambda,I_\Lambda\otimes\hat{E})$, well defined modulo $S^{m+n/4-1}(\Lambda,I_\Lambda\otimes\hat{E})$. Here I_Λ is the tensor product of the Maslov bundle with half densities over Λ and $\hat{E}$ is the pull back of E onto Λ. The principal symbol map gives an isomorphism [32, Theorem 25.1.9]

$$\sigma: I^{[m]}(X,\Lambda;E\otimes\Omega_X^{1/2})\longrightarrow S^{[m+n/4]}(\Lambda,I_\Lambda\otimes\hat{E}), \tag{59}$$

with the conventions $I^{[*]}=I^*/I^{*-1}$, $S^{[*]}=S^*/S^{*-1}$.

Let X,Y,Z be C^∞ manifolds and $\Lambda_1\subset T^*X\setminus 0\times T^*Y\setminus 0$ and $\Lambda_2\subset(T^*Y\setminus 0)\times(T^*Z\setminus 0)$ be conic Lagrangian submanifolds closed in $T^*X\times(T^*Y\setminus 0)$ and $T^*Y\times(T^*Z\setminus 0)$ respectively. It is understood that the symplectic structures of $T^*X\times T^*Y$ and $T^*Y\times T^*Z$ are the product ones. Assume that the intersection of $\Lambda_1\times\Lambda_2$ with $T^*X\times N^*(\Delta_Y)\times T^*Z$ is clean with excess e, where $N^*(\Delta_Y)$ is the conormal space of the diagonal Δ_Y in Y^2. If $A_1\in I^{m_1}(X\times Y,\Lambda_1;\Omega^{1/2}_{X\times Y})$ and $A_2\in I^{m_2}(Y\times Z,\Lambda_2;\Omega^{1/2}_{Y\times Z})$ are properly supported, then [32, Theorem 25.2.3]

$$A=A_1\circ A_2\in I^{m_1+m_2+e/2}(X\times Z,\Lambda,\Omega^{1/2}_{X\times Z}). \tag{60}$$

Here $A_1\circ A_2$ is defined through the Schwartz kernel theorem and Λ is the conic Lagrangian local submanifold defined by the composition of Λ_1 and Λ_2:

$$\Lambda=\Lambda_1\circ\Lambda_2=\{(x,\xi,z,\zeta)\ ;\ \exists(y,\eta)\in T^*Y,(x,\xi,y,-\eta,y,\eta,z,\zeta)\in\Lambda_1\times\Lambda_2\}. \tag{61}$$

Under the same assumptions on Λ_i, $i=1,2$, there is thus a well defined product of principal symbols:

$$S^{[m_1+(n_X+n_Y)/4]}(\Lambda_1,I_{\Lambda_1})\times S^{[m_2+(n_Y+n_Z)/4]}(\Lambda_2,I_{\Lambda_2})\xrightarrow{\circ} S^{[m_1+m_2+e/2+(n_X+n_Z)/4]}(\Lambda,I_\Lambda) \tag{62}$$

which is defined abstractly by

$$a=a_1\circ a_2=\sigma(\sigma^{-1}(a_1)\circ\sigma^{-1}(a_2)) \tag{63}$$

and computed concretely through the integral

$$a(\gamma) = \int_{C_\gamma} a_1 \times a_2 \tag{64}$$

where a_1, a_2, a are representatives in S^* of the given classes in $S^{[*]}$, $\gamma \in \Lambda_1 \circ \Lambda_2$, the manifold C_γ is the fiber of the projection map

$$p : \widetilde{\Lambda} := \Lambda_1 \times \Lambda_2 \cap T^*X \times N^*(\Delta_Y) \times T^*Z \longrightarrow \Lambda_1 \circ \Lambda_2 \tag{65}$$

and $a_1 \times a_2$ is the density on C_γ with values in I_Λ resulting from the natural bundle homomorphism

$$\begin{tikzcd} I_{\Lambda_1} \otimes I_{\Lambda_2} \arrow[rr, "p"] \arrow[dr] & & p^*(I_\Lambda) \otimes \Omega(\ker dp) \otimes \Omega^{-1/2}(T^*Y) \arrow[dl] \\ & \widetilde{\Lambda} & \end{tikzcd} \tag{66}$$

and from the trivialization of $\Omega^{-1/2}(T^*Y)$ using the canonical density of T^*Y (see [32, Theorems 21.6.6, 25.2.3]).

4.2. **"Pushforward" of Lagrangian distributions.** In this paragraph, we describe a procedure of "restriction and pushforward" for Lagrangian distributions parallel to the one described for Lagrangian submanifolds in Paragraph 3.3. This is the main step in the composition of Fourier Integral Operators on groupoids.

Proposition 8. *We use the notations and assumptions of Proposition 4: let X, Y be manifolds, $Z \subset X$ a submanifold and $f : Z \to Y$ a submersion. Set $H = (\ker df)^\perp \subset T^*X$ and*

$$\mu : H \ni (x, \xi) \longmapsto (f(x), {}^t df_x^{-1}(\xi)) \in T^*Y.$$

Suppose that $\widetilde{\Lambda}$ is a conic Lagrangian local submanifold of $T^*X \setminus 0$ intersecting cleanly H with excess e and such that $\widetilde{\Lambda} \cap N^*Z = \emptyset$, and set $\Lambda = \mu(\widetilde{\Lambda} \cap H)$.
Let then Ω be any line bundle over X extending the density bundle Ω_Z over Z. We denote by

$$i^* : \mathcal{E}'_{\widetilde{\Lambda}}(X, \Omega) \longrightarrow \mathcal{E}'_{i^*\widetilde{\Lambda}}(Z, \Omega_Z)$$

the restriction to Z of distributions on X and by

$$f_* : \mathcal{E}'_{i^*\widetilde{\Lambda}}(Z, \Omega_Z) \longrightarrow \mathcal{E}'_\Lambda(Y, \Omega_Y),$$

the push-forward along f. The map

$$\begin{aligned} f_\# : I^m_c(X, \widetilde{\Lambda}; \Omega) &\longrightarrow I_c^{m+e/2+(n_X-2n_Z+n_Y)/4}(Y, \Lambda) \\ u &\longmapsto f_*(i^*u) \end{aligned}$$

is well defined.

For non compactly supported distributions, we get the same result by taking care of supports for the push-forward operation. For instance, given any $\varphi \in C^\infty(Z)$ such that $f : \mathrm{supp}(\varphi) \to Y$ is proper, the conclusion of the lemma holds true with

$$u \longmapsto f_*(\varphi i^* u).$$

Proposition 8 could be deduced from [32, Theorem 25.2.3], but the direct proof below is instructive.

Proof. Let

$$A(x) = \int e^{i\widetilde{\phi}(x,\theta)} a(x,\theta) d\theta \in I^m(X, \widetilde{\Lambda}).$$

Here $\widetilde{\phi} : U \times \mathbb{R}^N \to \mathbb{R}$ is a non-degenerate phase function parametrizing $\widetilde{\Lambda}$ and $a \in S^{m+(n_X-2N)/4}(U \times \mathbb{R}^N)$. Since $\mathrm{WF}(A) \subset \widetilde{\Lambda}$ and, by assumption, $\widetilde{\Lambda} \cap N^*Z = \emptyset$, the distribution $i^*(A)$ is well-defined ([30, Chp. 6, Section 1]) and given by the oscillatory integral

$$i^*(A)(z) = \int e^{i\phi(z,\theta)} a(z,\theta) d\theta \in I^{m+(n_X-n_Z)/4}(Z, i^*\widetilde{\Lambda}) \tag{67}$$

where we recall that $i^*\widetilde{\Lambda} = p(\widetilde{\Lambda} \cap T^*X|_Z)$ with $p : T^*X|_Z \to T^*Z$ being the canonical projection, since applying the previous proposition to the case when $Y = Z$ gives exactly that $\phi = \widetilde{\phi}|_{(U\cap Z)\times\mathbb{R}^N}$ is a non-degenerate phase function parametrizing $i^*\widetilde{\Lambda} = p(\widetilde{\Lambda} \cap T^*X|_Z)$.

The next step consists in pushing $i^*(A)$ forward by f. This amounts to integrating the Lagrangian distribution i^*A along the fibers of f, which gives:

$$f_\# A(y) = \int_{f^{-1}(y)\times\mathbb{R}^N} e^{i\phi(z,\theta)} a(z,\theta) dz d\theta, \tag{68}$$

where the integral is understood in the distributional sense. We already know by the previous proposition that ϕ is a clean phase function over $W = f(U \cap Z)$ subordinate to the cone bundle $(U \cap Z) \times \mathbb{R}^N \to W,\ (z,\theta) \mapsto f(z)$. To conclude, it just remains to pay attention to the fact that the fiber part of the variable z is not homogeneous and thus, strictly speaking, a is not a symbol on W. Working in local coordinates, we can write

$$a(x,\theta) = a(y, z', \theta) \in S^{m+(n_X-2N)/4}(\mathbb{R}^{n_Y} \times \mathbb{R}^{n_Z-n_Y} \times \mathbb{R}^N). \tag{69}$$

Setting $\omega(z',\theta) = (|\theta|.z', \theta)$; $\psi(y,\omega) = \phi(y,z',\theta)$ and $b(y,\omega) = a(y,z',\theta)|\det(\omega^{-1})|$, we get $|\det(\omega^{-1})| = |\theta|^{n_Y-n_Z}$ and thus $b \in S^{m+(n_X-2N)/4+n_Y-n_Z}(\mathbb{R}^{n_Y} \times \mathbb{R}^{n_Z-n_Y+N})$. It follows that

$$f_\# A(y) = \int_{f^{-1}(y)\times\mathbb{R}^N} e^{i\psi(y,\omega)} b(y,\omega) d\omega \tag{70}$$

belongs to $I^{m'}(Y, \Lambda)$ where

$$m' - e/2 + (n_Y - 2(n_Z - n_Y + N))/4 = m + (n_X - 2N)/4 + n_Y - n_Z$$

that is $m' = m + e/2 + (n_X - n_Y)/4 - (n_Z - n_Y)/2 = m + e/2 + (n_X - 2n_Z + n_Y)/4$. □

4.3. **Families of Lagrangian distributions and submersions.**

Definition 8. *Let $\pi : M \longrightarrow B$ be a C^∞ submersion of a manifold M of dimension n_M onto a manifold B of dimension n_B. A C^∞* family of Lagrangian distributions of order m relative to π *is a family $u_b \in I^m(\pi^{-1}(b), \Lambda_b, \Omega_\pi^{1/2})$, $b \in B$, such that $(\Lambda_b)_{b\in B}$ is a C^∞ family and in any local trivialization $\kappa : \mathcal{U} \to U \times W$ of π, we have*

$$\kappa_*(u|_{\mathcal{U}}) = \int e^{i\phi(x,b,\theta)} a(x,b,\theta) d\theta,$$

with $a \in S^{m+(n_M-n_B-2N)/4}(U \times W \times \mathbb{R}^N)$ and where $(x,b,\theta) \mapsto \phi(x,b,\theta)$ is C^∞ and a non-degenerate phase function in (x,θ) which parametrizes Λ_b locally, for all b.

Proposition 9. *Let $B \ni b \mapsto u_b \in I^m(\pi^{-1}(b), \Lambda_b, \Omega_\pi^{1/2})$ be a C^∞ family. The formula*

$$\langle \widetilde{u}, f \rangle = \int_B \langle u_b, f \rangle,\ \ f \in C_c^\infty(M, \Omega_\pi^{1/2} \otimes \pi^*(\Omega_B)) \tag{71}$$

defines a Lagrangian distribution

$$\widetilde{u} \in I^{m-n_B/4}(M,\Lambda), \tag{72}$$

where Λ is the gluing of the family $(\Lambda_b)_b$. The map $(u_b)_{b\in B} \mapsto \widetilde{u}$ is bijective.

Proof. Let $u_b \in I^m(\pi^{-1}(b), \Lambda_b, \Omega_\pi^{1/2})$ be a C^∞ family. In sufficently small local trivializations, we have

$$u_b(x) = \int e^{i\phi(b,x,\theta)} a(b,x,\theta) d\theta \tag{73}$$

for some C^∞ family of non-degenerate phases functions $(\phi(b,\cdot,\cdot))_b$ parametrizing the family $(\Lambda_b)_b$ and some symbol $a \in S^{m+(n_M-n_B-2N)/4}(U\times W\times\mathbb{R}^N)$.

Since $(x,\theta)\mapsto\phi(b,x,\theta)$ is a non-degenerate phase function for any b, the function $(b,x,\theta)\mapsto\phi(b,x,\theta)$ is actually a non-degenerate phase function. We have $\Lambda_\phi = \Lambda$ locally, because on one hand $i_b^*(\Lambda_\phi) = \Lambda_b$ for any b and on the other hand Λ is the unique Lagrangian satisfying this condition. It follows that $\widetilde{u}$ is given locally by the oscillatory integral:

$$\widetilde{u}(b,x) = \int e^{i\phi(b,x,\theta)} a(b,x,\theta) d\theta,$$

which proves that $\widetilde{u}\in I^{m-n_B/4}(M,\Lambda)$. Conversely, if $v \in I^{m-n_B/4}(M,\Lambda)$ then locally

$$v(x) = \int e^{i\phi(b,x,\theta)} a(b,x,\theta) d\theta \tag{74}$$

for some non-degenerate phase function ϕ parametrizing Λ and some symbol $S^{m-n_B/4+(n_M-2N)/4}(U\times W\times\mathbb{R}^N)$. Since Λ is transverse to π, the restriction v_b of v to M_b is allowed and given by the C^∞ family $b\mapsto v_b(x) = \int e^{i\phi(b,x,\theta)}a(b,x,\theta)d\theta$ where ϕ is regarded as a non degenerate phase function in (x,θ) for fixed b. This proves that $u\mapsto\widetilde{u}$ is bijective. □

5. Lagrangian submanifolds of symplectic groupoids

Let G be a Lie groupoid. The cotangent groupoid $T^*G \rightrightarrows A^*G$ plays a fundamental role in the convolution of distributions on G [36]. The point is that T^*G is a *symplectic* groupoid. We recall that a Lie groupoid Γ is symplectic if it is provided with a symplectic structure for which the graph of the multiplication

$$\mathrm{Gr}(m_\Gamma) = \{(\gamma_1,\gamma_2,\gamma)\in\Gamma^3 \ ;\ \gamma=\gamma_1\gamma_2\} \tag{75}$$

is a Lagrangian submanifold of $\Gamma\times\Gamma\times(-\Gamma)$ [19, 39, 59].

Since, in the sequel, we want to develop a calculus for G-operators associated with Lagrangian distributions on G, it is necessary to investigate the behavior of Lagrangian submanifolds of the symplectic groupoid T^*G. Actually, we can state most of the results in the case of an arbitrary symplectic groupoid.

5.1. **Product and adjunction.** Let Γ be a symplectic groupoid. We may apply Proposition 2 with $S=\Gamma^2$, $T=\Gamma$, $H=\Gamma^{(2)}$ and $\mu = m_\Gamma$. This gives:

Corollary 10. *Let Γ be a symplectic groupoid with multiplication map m_Γ. Let $\widetilde{\Lambda}$ be a Lagrangian local submanifold of Γ^2. If $\widetilde{\Lambda}\cap\Gamma^{(2)}$ is clean then*

$$\Lambda := m_\Gamma(\widetilde{\Lambda}\cap\Gamma^{(2)}) \tag{76}$$

is a Lagrangian local submanifold of Γ.

We have in mind, of course, the following example. If Λ_1, Λ_2 are two Lagrangian local submanifolds of Γ such that $\widetilde{\Lambda} = \Lambda_1 \times \Lambda_2$ satisifies the assumption of the previous corollary, then their product in Γ,

$$\Lambda_1.\Lambda_2 = m_\Gamma((\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}),$$

is a Lagrangian local submanifold.

Definition 9. *Two Lagrangian local submanifolds Λ_1, Λ_2 of the symplectic groupoid Γ are* cleanly (resp. transversally) composable *if the intersection $(\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}$ is clean (resp. transversal).*

When Γ is the cotangent groupoid of a given Lie groupoid G and Λ_1, Λ_2 are moreover conic, then the product $\Lambda = \Lambda_1\Lambda_2$ is also conic. Using Proposition 4 (applied with $X = G^2$, $Y = G$, $Z = G^{(2)}$ and $f = m_G$), we see how non-degenerate phase functions of Λ_1 and Λ_2 combine to produce clean phase functions parametrizing the product Λ.

Corollary 11. *Let Λ_1, Λ_2 be local conic Lagrangian submanifolds of $T^*G \setminus 0$. We assume that Λ_1 and Λ_2 are cleanly composable with excess e and satisfy*

$$(\Lambda_1 \times \Lambda_2) \cap N^*(G^{(2)}) = \emptyset. \tag{77}$$

Let $(\gamma_1^0, \xi_1^0, \gamma_2^0, \xi_2^0) \in (\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}$ and $\phi_j : U_j \times \mathbb{R}^{N_j} \longrightarrow \mathbb{R}$ be non-degenerate phase functions parametrizing Λ_j around (γ_j^0, ξ_j^0) for $j = 1, 2$. Then the function defined by

$$\phi(\gamma_1, \gamma_2, \theta_1, \theta_2) = \phi_1(\gamma_1, \theta_1) + \phi_2(\gamma_2, \theta_2) \tag{78}$$

is a phase function over the product open set $U = U_1.U_2 \subset G$. Its associated cone bundle (see Definition 7) is given by the map:

$$(U_1 \times U_2 \cap G^{(2)}) \times (\mathbb{R}^{N_1} \setminus 0) \times (\mathbb{R}^{N_2} \setminus 0) \longrightarrow U, \ (\gamma_1, \gamma_2, \theta_1, \theta_2) \longmapsto \gamma_1\gamma_2. \tag{79}$$

This phase function is clean with excess e and parametrizes $\Lambda_1.\Lambda_2$ around $(\gamma_1^0\gamma_2^0, \xi_1^0 \oplus \xi_2^0)$.

Another useful, although obvious, operation is adjunction. Precisely, if Λ is a local Lagrangian submanifold of a symplectic groupoid Γ, then its adjoint, that is, the subset of Γ defined by

$$\Lambda^\star = \iota_\Gamma(\Lambda)$$

is again a local Lagrangian submanifold.

5.2. **Invertibility.** Let Γ be a symplectic groupoid.

Definition 10. (1) *A Lagrangian submanifold $\Lambda \subset \Gamma$ is* invertible *if there exists a Lagrangian submanifold $\Lambda' \subset \Gamma$ cleanly composable with Λ and such that*

$$\Lambda.\Lambda' = r_\Gamma(\Lambda) \quad \text{and} \quad \Lambda'.\Lambda = s_\Gamma(\Lambda). \tag{80}$$

Λ' is in this case called an inverse *of Λ.*

(2) *A Lagrangian local submanifold Λ is* locally invertible *if it can be covered by invertible patches.*
In that case, any local Lagrangian submanifold consisting of inverses of the corresponding invertible patches is a local inverse of Λ.

Theorem 12. *Let Λ be a Lagrangian submanifold of Γ. Then Λ is locally invertible (resp. invertible) if and only if the maps*

$$r_\Gamma : \Lambda \longrightarrow \Gamma^{(0)} \text{ and } s_\Gamma : \Lambda \longrightarrow \Gamma^{(0)} \tag{81}$$

are local diffeomorphisms (resp. diffeomorphisms onto their ranges). In that case, Λ and $\Lambda^\star$ are transversally composable and $\Lambda^\star$ is a local inverse (resp. an inverse) of Λ.

Proof. Assume that Λ is locally inversible. By restricting our attention to a sufficiently small patch, we can assume that Λ is invertible. Let Λ' be an inverse. Firstly, note that $\Lambda.\Lambda'$ is a local submanifold of T^*G contained in Γ^0. Since $\Lambda.\Lambda'$ and $\Gamma^{(0)}$ are Lagrangian we have $\dim A^*G = \dim \Lambda.\Lambda'$ and thus each patch of $\Lambda.\Lambda'$ is an open subset of $\Gamma^{(0)}$. It follows that $\Lambda.\Lambda'$ itself is an open subset of $\Gamma^{(0)}$ and therefore a true submanifold. Now, by assumption,

$$m_\Gamma : (\Lambda \times \Lambda')^{(2)} \longrightarrow \Lambda.\Lambda' = r_\Gamma(\Lambda) \tag{82}$$

is a surjective submersion. Since the map r_Γ is equal to the identity map in restriction to $\Gamma^{(0)}$, we have the equality of maps

$$r_\Gamma \circ \mathrm{pr}_1 = r_\Gamma \circ m_\Gamma = m_\Gamma : (\Lambda \times \Lambda')^{(2)} \longrightarrow \Lambda.\Lambda' \subset \Gamma^{(0)}. \tag{83}$$

It follows that

$$r_\Gamma \circ \mathrm{pr}_1 : (\Lambda \times \Lambda')^{(2)} \longrightarrow \Lambda.\Lambda' = r_\Gamma(\Lambda) \tag{84}$$

is a submersion. Observe also that

$$\mathrm{pr}_1 : (\Lambda \times \Lambda')^{(2)} \to \Lambda \tag{85}$$

is surjective. Indeed, for any $\gamma \in \Lambda$, there exists, by surjectivity of the map (82), an element $(\gamma_1, \gamma_2) \in (\Lambda \times \Lambda')^{(2)}$ such that $\gamma_1\gamma_2 = r_\Gamma(\gamma)$. In particular $r(\gamma_1) = r(\gamma)$ and $\gamma_1^{-1} = \gamma_2 \in \Lambda'$. Thus $(\gamma_1^{-1}, \gamma) \in (\Lambda' \times \Lambda)^{(2)}$ and the assumption $\Lambda'.\Lambda \subset \Gamma^{(0)}$ implies $\gamma = \gamma_1 = \mathrm{pr}_1(\gamma_1, \gamma_2)$.

Since the map (85) is surjective, we deduce from the surjectivity of the differential of (84) at any point the surjectivity of the differential of $r_\Gamma : \Lambda \to \Gamma^{(0)}$ everywhere too. By equality of dimension, r_Γ is then a local diffeomorphism. The same holds for s_Γ.

Conversely, let us assume that $r_\Gamma, s_\Gamma : \Lambda \to \Gamma^{(0)}$ are local diffeomorphisms. Then the map

$$s_\Gamma \times r_\Gamma|_{\Lambda\times\Lambda^\star} : \Lambda \times \Lambda^\star \longrightarrow \Gamma^{(0)} \times \Gamma^{(0)}$$

is also a local diffeomorphism. It follows that

$$(\Lambda \times \Lambda^\star)^{(2)} = (s_\Gamma \times r_\Gamma)^{-1}_{|\Lambda\times\Lambda^\star}(\Delta_{\Gamma^{(0)}}) = (s_\Gamma \times r_\Gamma)^{-1}(\Delta_{\Gamma^{(0)}}) \cap \Lambda \times \Lambda^\star \tag{86}$$

is a submanifold of dimension n of Γ^2 with tangent space given by

$$T(\Lambda \times \Lambda^\star)^{(2)} = T\Gamma^{(2)} \cap T(\Lambda \times \Lambda^\star).$$

Therefore the intersection $\Lambda \times \Lambda^\star \cap \Gamma^{(2)}$ is clean with excess satisfying

$$e = \mathrm{codim}(\Lambda \times \Lambda^\star) + \mathrm{codim}(\Gamma^{(2)}) - \mathrm{codim}((\Lambda \times \Lambda^\star)^{(2)}) = 2n + n - 3n = 0.$$

In other words, we get $(\Lambda \times \Lambda^\star) \pitchfork \Gamma^{(2)}$. Moreover, for any $\delta \in \Lambda$, there exists an open conic neighborhood U of δ in Γ such that

$$r_\Gamma, s_\Gamma : \Lambda_U = \Lambda \cap U \longrightarrow \Gamma^{(0)}$$

are diffeomorphisms onto their respective images. By the previous arguments, $(\Lambda_U \times i_\Gamma(\Lambda_U) \pitchfork \Gamma^{(2)}$, and if $\eta \in i_\Gamma(\Lambda_U)$ is such that $(\delta, \eta) \in (\Lambda_U \times i_\Gamma(\Lambda_U))^{(2)}$ then by the injectivity of s_Γ we get

$\eta = \delta^{-1}$. It follows that $\Lambda_U.i_\Gamma(\Lambda_U) = r_\Gamma(\Lambda_U)$. This proves that Λ is locally invertible and since $i_\Gamma(\Lambda_U) = (\Lambda^\star)_{U^{-1}}$, we conclude that $\Lambda^\star$ is a local inverse.

Now assume that r_Γ, s_Γ are diffeomorphisms onto their ranges, that is, are injective local diffeomorphisms. If there exists $\delta \in \Lambda$ and $\eta \in \Lambda^\star$ such that $\delta\eta \notin \Gamma^{(0)}$ then $\delta, \eta^{-1} \in \Lambda$, $\delta \neq \eta^{-1}$ but $s_\Gamma(\delta) = s_\Gamma(\eta^{-1})$ which contradicts the injectivity of s_Γ. This gives the inclusion $\Lambda.\Lambda^\star \subset \Gamma^{(0)}$ and then the equality $\Lambda.\Lambda^\star = r_\Gamma(\Lambda)$ follows from the definition of $\Lambda^\star$. We get the equality $\Lambda^\star.\Lambda = s_\Gamma(\Lambda)$ using the injectivity of r_Γ.

Conversely, assume that Λ' is an inverse of Λ. Let $u \in r_\Gamma(\Lambda)$. Since $\Lambda.\Lambda' = r_\Gamma(\Lambda)$, there exists $(\delta_1, \delta_1') \in (\Lambda \times \Lambda')^{(2)}$ such that $\delta_1.\delta_1' = u$. Let $\delta \in \Lambda$ be such that $r(\delta) = u$. Then $(\delta_1', \delta) \in (\Lambda' \times \Lambda)^{(2)}$ and thus $\delta_1'.\delta \in \Gamma^{(0)}$. This gives

$$\delta_1 = {\delta_1'}^{-1} = \delta.$$

In other words, $r_\Gamma|_\Lambda : \Lambda \longrightarrow \Gamma^{(0)}$ is injective. The same holds for s_Γ. $\square$

Remark 13.

(1) *We have proved that the (local) invertible Lagrangian submanifolds of* Γ *are precisely the Lagrangian (local) bisections of* Γ. *Here we follow the terminology of* [3] *for bisections, while in* [19, see Paragraphs I.3 and II.1] *bisections are required to project onto* $\Gamma^{(0)}$*: this is a minor and technical distinction implying that the set* $\mathrm{Gr}(\Gamma)$ *is no longer a group here but a groupoid with unit space given by the collection of open subspaces of* $\Gamma^{(0)}$.

(2) *In particular we recover results from* [32] *in the case where* $G = M \times M$ *is the pair groupoid on a manifold* M. *In that example, a conic Lagrangian submanifold of* $\Gamma = T^*G$ *is (locally) invertible if and only if it coincides (locally) with the graph of a partially defined homogeneous canonical transformation* [32, Sections 25.3 and 21.2]*, that is, the graph of a homogeneous symplectomorphism from an open conic subset of* T^*M *to another one.*

5.3. **G-relations and family G-relations.** From now on, we consider a Lie groupoid G and its symplectic cotangent groupoid $\Gamma = T^*G$. As explained in the introduction (see also Paragraph 2.4), we need to strengthen the notion of conic Lagrangian submanifolds to allow a calculus for Lagrangian distributions on G. Also, if we consider a G-operator P ([36]) such that P_x is a classical Fourier operator for any $x \in G^{(0)}$, we get an equivariant family of Lagrangian conic submanifolds of $T^*G_x \times T^*G^x$, $x \in G^{(0)}$. We are going to clarify the relationship between these equivariant families and the conic Lagrangian submanifolds of T^*G. In view of Paragraph 2.4, see also [36], it is natural to begin with:

Definition 11. *A* (local) G-relation *is a conic Lagrangian (local) submanifold of* T^*G *contained in* $\dot{T}^*G = T^*G \setminus (\ker r_\Gamma \cup \ker s_\Gamma)$.

If Λ is a G-relation, then it is false in general that Λ_x is a Lagrangian submanifold for any $x \in G^{(0)}$ (see Example 4 below). In the following theorem, we characterize in various ways the G-relations enjoying this extra property. We will note $p : T^*G \to G$ the vector bundle projection map and $q : G^{(2)} \to G^{(0)}$ the submersion defined by $q(\gamma_1, \gamma_2) = s(\gamma_1) = r(\gamma_2)$. By construction we have $\widetilde{m_\Gamma}^{-1}(\Lambda) = m^*(\Lambda)$ (the map $\widetilde{m_\Gamma}$ is defined in (13)). Therefore, since the multiplication map m is a submersion, we get that the set $\widetilde{m_\Gamma}^{-1}(\Lambda)$ is always a Lagrangian submanifold of $T^*(G^{(2)})$ contained in $T^*(G^{(2)}) \setminus 0$ [30, Chp. 4, Proposition 4.1]. Then:

Theorem 14. *Let* Λ *be a* G*-relation. Then the following assertions are equivalent:*

(1) *The family* $(\Lambda_x)_{x\in G^{(0)}}$ *is a family of Lagrangian submanifolds and is subordinate to the submersion* q.
(2) *The submanifold* $\widetilde{m_\Gamma}^{-1}(\Lambda) \subset T^*(G^{(2)})$ *is Lagrangian and transverse to the submersion* q.
(3) *The transversality condition* $m_x \pitchfork p|_\Lambda$ *holds for all* $x \in G^{(0)}$.
(4) *The map* $p|_\Lambda : \Lambda \to G$ *is transversal to the foliation* $\mathcal{F}_G$.
(5) *For any leaf* $L \in \mathcal{F}_G$, *the transversality condition* $T^*_L G \pitchfork \Lambda$ *holds.*

A G relation satisfying the above assumption is called a *family* G-relation.

Proof. The equivalence between (1) and (2) is provided by Theorems 6 and 7.

Next, Condition (3) means that for all $x \in G^{(0)}$, we have

$$dm(T_{\gamma_1}G_x \times T_{\gamma_2}G^x) + dp(T_{(\gamma,\xi)}\Lambda) = T_\gamma G, \tag{87}$$

for all $(\gamma,\xi) \in \Lambda$ and $(\gamma_1,\gamma_2) \in G_x \times G^x$ such that $\gamma = \gamma_1\gamma_2$. Using the equalities $p \circ \widetilde{m_\Gamma} = m \circ p^{(2)}$ and $T\Lambda = d\widetilde{m_\Gamma}(T\widetilde{m_\Gamma}^{-1}(\Lambda))$, we get

$$dp(T_{(\gamma,\xi)}\Lambda) = dm(dp^{(2)}(T_{\rho(\gamma_1,\xi_1,\gamma_2,\xi_2)}\widetilde{m_\Gamma}^{-1}(\Lambda)))$$

for all $(\gamma,\xi) \in \Lambda$ and $(\gamma_1,\xi_1,\gamma_2,\xi_2) \in m_\Gamma^{-1}(\gamma,\xi)$. Moreover, we have $\ker d\widetilde{m_\Gamma} \subset T\widetilde{m_\Gamma}^{-1}(\Lambda)$ and $dp^{(2)}(\ker d\widetilde{m_\Gamma}) = \ker dm$. Therefore $\ker dm \subset dp^{(2)}(T\widetilde{m_\Gamma}^{-1}(\Lambda))$ and Condition (87) is equivalent to

$$T_{\gamma_1}G_x \times T_{\gamma_2}G^x + dp^{(2)}(T_{\rho(\gamma_1,\xi_1,\gamma_2,\xi_2)}\widetilde{m_\Gamma}^{-1}(\Lambda)) = T_{(\gamma_1,\gamma_2)}G^{(2)}, \tag{88}$$

where $\gamma,\xi,\gamma_i,\xi_i$ are as above. With the same notation, we have $\ker(dq)_{(\gamma_1,\gamma_2)} = T_{\gamma_1}G_x \times T_{\gamma_2}G^x$, and thus Condition (88) is equivalent to

$$dq dp^{(2)}(T_{\rho(\gamma_1,\xi_1,\gamma_2,\xi_2)}\widetilde{m_\Gamma}^{-1}(\Lambda)) = T_x G^{(0)} \tag{89}$$

for every γ_i,ξ_i as above, which is precisely Condition (2).

For the equivalence between (3) and (4), just observe that we have

$$(dm_x)_{(\gamma_1,\gamma_2)}(T_{\gamma_1}G_x \times T_{\gamma_2}G^x) = T_\gamma G_{s(\gamma)} + T_\gamma G^{r(\gamma)} = T_\gamma \mathcal{F}_G; \tag{90}$$

for any $(\gamma_1,\gamma_2) \in G_x \times G^x$ such that $\gamma_1\gamma_2 = \gamma$. Here $T_\gamma\mathcal{F}_G$ denotes the tangent space at γ of the leaf of $\mathcal{F}_G$ containing γ. With this observation, one gets that Condition (3) reads

$$T_\gamma\mathcal{F}_G + dp(T_{\gamma,\xi}\Lambda) = T_\gamma G, \quad \text{for all } (\gamma,\xi) \in \Lambda, \tag{91}$$

which is exactly Condition (4). Furthermore, (91) is clearly equivalent to

$$T_{(\gamma,\xi)}(T^*_L G) + T_{(\gamma,\xi)}\Lambda = T_{(\gamma,\xi)}T^*G, \quad \text{for all } (\gamma,\xi) \in \Lambda, \tag{92}$$

where L is the leaf of $\mathcal{F}$ containing γ. This gives the equivalence between (4) and (5). □

Remark 15. *It follows immediately that a G-relation* Λ *is a family G-relation if and only if*

$$\forall(\gamma,\xi) \in \Lambda,\ dr(T_\gamma G_{s(\gamma)}) + dr(d\pi T_{\gamma,\xi}\Lambda) = T_{r(\gamma)}G^{(0)} \text{ or } ds(T_\gamma G^{r(\gamma)}) + ds(d\pi T_{\gamma,\xi}\Lambda) = T_{s(\gamma)}G^{(0)}.$$

In particular, if Λ *is a G-relation such that* $r \circ \pi : \Lambda \to G^{(0)}$ *or* $s \circ \pi : \Lambda \to G^{(0)}$ *is a submersion then* Λ *is a family G-relation. The converse is false: consider* $G = X \times X \times Y$ *with its natural structure of groupoid (fibered pair groupoid) and* $\Lambda = N^*(\{(x_0,x_0)\} \times Y) \setminus 0$.

Example 2.
1) $\Lambda_0 = A^*G \setminus 0$ *is a family G-relation. It is of course self-composable and* $\Lambda_0.\Lambda_0 = \Lambda_0 = \Lambda_0^*$.
2) If Λ *is an invertible conic Lagrangian submanifold of* $T^*G \setminus 0$ *then it is a G-relation. Indeed, if there would exist* $(\gamma,\xi) \in \Lambda \cap \ker s_\Gamma$, *we would get from the homogeneity of* Λ *that the vertical vector associated with* (γ,ξ) *belongs to* $\ker ds_\Gamma$, *which would imply that* Λ *is not invertible.*

Remark 16. *Theorem 14 can be stated and proved exactly in the same way for local G-relations.*

We have seen that family G relations produce equivariant families of Lagrangian submanifolds subordinate to q. The converse statement is the following:

Theorem 17. *Let $(L_x)_x$ be a C^∞ equivariant family of local conic Lagrangian submanifolds contained in $(T^*G_x \setminus 0) \times (T^*G^x \setminus 0)$ and subordinate to q. Then there exists a unique local family G-relation Λ such that*

$$\Lambda_x = L_x, \quad \text{for all } x \in G^{(0)}.$$

Proof. To prove existence, we can decompose the family into patches and then we can assume that $(L_x)_x$ is a family of submanifolds. Let $\widetilde{L} \subset T^*(G^{(2)}) \setminus 0$ denote its gluing (Theorem 6). Recall that it is the unique Lagrangian submanifold such that $i_x^*\widetilde{L} = L_x$ for all x. In an appropriate local trivialization of $q : G^{(2)} \to G^{(0)}$, we have

$$\widetilde{L} = \{(\gamma_1, \gamma_2, \xi_1, \xi_2, \tau) \in T^*(G^{(2)}) \ ; \ (\gamma_1, \gamma_2, \xi_1, \xi_2) \in L_x\}, \tag{93}$$

where τ is a C^∞ function of ξ_1, ξ_2 and $x = s(\gamma_1) = r(\gamma_2)$. It is understood that $(\xi_1, \xi_2, \tau) \in T^*_{(\gamma_1,\gamma_2)}G^{(2)} \simeq T^*_{\gamma_1}G_x \times T^*_{\gamma_2}G^x \times T^*_x G^{(0)}$, where the decomposition comes from the local trivialisation of $q : G^{(2)} \to G^{(0)}$.

Let $\widetilde{\lambda} = (\delta, \xi) \in \widetilde{L}$ with $\delta = (\gamma_1, \gamma_2)$ and $\xi = (\xi_1, \xi_2, \tau)$. Let $u = (u_1, u_2) \in \ker dm_{\gamma_1,\gamma_2}$ and choose a C^∞ path $t \mapsto \gamma(t)$ in G such that $\gamma(0) = x$, $\frac{d}{dt}\gamma_1\gamma(t)|_{t=0} = u_1$, $\frac{d}{dt}\gamma(t)^{-1}\gamma_2|_{t=0} = u_2$. It gives rise to a C^∞ path in $(\ker dq)^*$ defined by

$$\lambda_t = (\gamma_1\gamma(t), \gamma(t)^{-1}\gamma_2, {}^t(dR_{\gamma(t)^{-1}})(\xi_1), {}^t(dL_{\gamma(t)})(\xi_2)). \tag{94}$$

Thanks to equivariance, we have

$$\lambda_t \in L_{s(\gamma(t))} \text{ for all } t. \tag{95}$$

Thus, we get a C^∞ path in $\widetilde{L}$ as well:

$$\widetilde{\lambda}(t) = (\gamma_1\gamma(t), \gamma(t)^{-1}\gamma_2, {}^t(dR_{\gamma(t)^{-1}})(\xi_1), {}^t(dL_{\gamma(t)})(\xi_2), \tau(t)) = (\delta(t), \xi(t)). \tag{96}$$

Since $\widetilde{L}$ is conic and Lagrangian, the canonical 1-form $\alpha = \xi d\delta$ vanishes identically on it, and in particular we get, for all t,

$$(\widetilde{\lambda})^*\alpha(t) = \langle \xi(t), \delta'(t) \rangle = 0. \tag{97}$$

For $t = 0$, this gives $\langle \xi, \ u \rangle = 0$, and therefore

$$\widetilde{L} \subset (\ker dm)^\perp = \rho(T^*G)^{(2)} \subset \rho(T^*G^2). \tag{98}$$

Note that, for every Lagrangian submanifold $\widetilde{L}$ of $T^*(G^{(2)})$, the set $L = \rho^{-1}(\widetilde{L})$ is a Lagrangian submanifold of T^*G^2 (it is the push-forward of $\widetilde{L}$ by the natural immersion $G^{(2)} \to G^2$, see [30, Prop 4.2]). We can then apply Corollary 10 to L. Indeed, by construction, $L \subset (T^*G)^{(2)}$ and thus the clean intersection assumption of Corollary 10 is trivially satisfied. It follows that the range $\Lambda = m_\Gamma(L)$ is a local G-relation such that $m^*(\Lambda) = \widetilde{L}$. Hence it is a local family G-relation such that $\Lambda_x = L_x$ for any x by Theorem 6.

Let Λ, Λ' be two local family G-relations such that $\Lambda_x = \Lambda'_x = L_x$, $\forall x \in G^{(0)}$ and set $\widetilde{\Lambda} = m^*(\Lambda)$ and $\widetilde{\Lambda}' = m^*(\Lambda')$. By Theorem 6 and the equalities

$$i_x^*\widetilde{\Lambda} = i_x^*\widetilde{\Lambda}', \ \forall x, \tag{99}$$

we get $\widetilde{\Lambda} = \widetilde{\Lambda}'$. Since $\widetilde{m_\Gamma}$ is surjective, we conclude

$$\Lambda = \widetilde{m_\Gamma}(\widetilde{\Lambda}) = \widetilde{m_\Gamma}(\widetilde{\Lambda}') = \Lambda'. \tag{100}$$

□

5.4. **Operations on (family) G-relations.** It is obvious that for any G-relations Λ_1, Λ_2 that are cleanly composable, their product is still a G-relation, for

$$r_\Gamma(\Lambda_1\Lambda_2) \subset r_\Gamma(\Lambda_1) \text{ and } s_\Gamma(\Lambda_1\Lambda_2) \subset s_\Gamma(\Lambda_2).$$

Unfortunately, it is not always true that the product of family G-relations is a family G-relation. Here is a conterexample.

Example 3. *Set $X = Z = \mathbb{R} \times \mathbb{R}$ and consider the fibred pair groupoid $G = X \times X \times Z \rightrightarrows X \times Z$. Let us define the maps $x_1, x_2 : \mathbb{R}^2 \to \mathbb{R}^2$ by:*

$$x_1(z) = z \text{ and } x_2(z', z'') = (z', -z'').$$

Introduce the submanifolds of G

$$V_j = \{(x_j(z), z, z) \ ; \ z \in Z\}, \quad j = 1, 2$$

and the conic Lagrangian submanifolds of $T^*G \setminus 0$

$$\Lambda_j = \{(x_j(z), \xi, z, \eta, z, -{}^t dx_j(\xi) - \eta) \ ; \ z \in Z, \ \xi, \eta \in \mathbb{R}^2 \setminus 0\} \subset N^*V_j.$$

Clearly, Λ_1 and Λ_2 are G-relations, which moreover satisfy Condition (4) of Theorem 14, and thus they are family G-relations. With the choices made, the intersection

$$\Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)} = \{(z', \xi_1, z', \eta_1, z', -\xi_1 - \eta_1, z', -\eta_1, z', \eta_2, z', {}^t dx_2(\eta_1) - \eta_2) \ ; \\ z' \in \mathbb{R} \times \{0\}, \ \xi_1, \eta_1, \eta_2 \in \mathbb{R}^2 \setminus 0\}$$

is clean. We obtain

$$\Lambda_1\Lambda_2 = \{(z', \xi, z', \eta, z', -\xi - \eta - \eta'') \ ; \ z' \in \mathbb{R} \times \{0\}, \ \xi, \eta \in \mathbb{R}^2 \setminus 0, \eta'' \in \{0\} \times \mathbb{R}\}.$$

Here, $\Lambda = \Lambda_1\Lambda_2$ is a G-relation but not a family G-relation. Indeed, $T\mathcal{F}_G = TX \times TX \times (Z \times \{0\})$ and $dp(T\Lambda) \subset TX \times TX \times (Z \times \mathbb{R} \times \{0\})$, which contradicts Condition (4) of Theorem 14.

Actually, starting with any composable G-relations Λ_1, Λ_2, whether the product $\Lambda_1.\Lambda_2$ is a family G-relation or not depends upon the position of the Cartesian product $\Lambda_1 \times \Lambda_2$ with respect to $\Gamma^{(2)}$ and not on extra transversality conditions added to each factor separately. Precisely, we obtain:

Theorem 18. *Let Λ_1, Λ_2 be composable G-relations. Then their product $\Lambda_1.\Lambda_2$ is a local family G-relation if and only if*

$$(T_{\gamma_1}\mathcal{F}_G \times T_{\gamma_2}\mathcal{F}_G) \cap TG^{(2)} + dp^2(T_{(\gamma_1,\xi_1,\gamma_2,\xi_2)}(\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}) = T_{(\gamma_1,\gamma_2)}G^{(2)} \tag{101}$$

for all $(\gamma_1, \xi_1, \gamma_2, \xi_2) \in (\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}$.

Remark 19. *The conclusions of the theorem are identical if we start with local submanifolds.*

The clean composability assumption together with Condition (101) will be called *complete composability.* The proof of the theorem uses an elementary fact about Lie groupoids.

Lemma 20. *For any $(\gamma_1, \gamma_2) \in G^{(2)}$, we have*

$$(T_{\gamma_1}\mathcal{F}_G \times T_{\gamma_2}\mathcal{F}_G) \cap TG^{(2)} = (dm_{(\gamma_1,\gamma_2)})^{-1}(T_{\gamma_1\gamma_2}\mathcal{F}_G). \tag{102}$$

Proof of the lemma. Let T be a Lie groupoid and $\mathcal{F}_T$ its canonical foliation. If $(\delta_1, \delta_2) \in T^{(2)}$ then δ_1, δ_2 and $\delta = \delta_1\delta_2$ are in the same leaf L. From the very definition of the leaves of $\mathcal{F}_T$, we get

$$(L \times L) \cap T^{(2)} = m_T^{-1}(L). \tag{103}$$

We apply this to $T = TG$, together with the observation $\mathcal{F}_{TG} = \{TL \ ; \ L \in \mathcal{F}_G\}$. The lemma follows. □

Proof of the theorem. Using the equalities

$$dm.dp^2(T((\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)})) = dp.dm_\Gamma(T((\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)})) = dp(T\Lambda) \tag{104}$$

as well as Lemma 20 and the fact that $\ker(dm) \subset (T\mathcal{F}_G \times T\mathcal{F}_G) \cap TG^{(2)}$, we get the equivalence

$$T\mathcal{F}_G + dp(T\Lambda) = TG \Leftrightarrow (T\mathcal{F}_G \times T\mathcal{F}_G) \cap TG^{(2)} + dp^2(T(\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}) = TG^{(2)},$$

where the suitable base points are understood. □

Remark 21. *Condition* (101) *means that the composable part of* $\Lambda_1 \times \Lambda_2$ *has a projection into* $G^{(0)}$ *transversal to the canonical foliation of* $\mathcal{F}_{G^{(0)}}$*. Indeed, we easily get that Condition* (101) *is equivalent to*

$$d(q \circ p^2)(T_{(\gamma_1,\xi_1,\gamma_2,\xi_2)}(\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}) + TO_x = T_xG^{(0)}, \tag{105}$$

for all $(\gamma_1, \xi_1, \gamma_2, \xi_2) \in (\Lambda_1 \times \Lambda_2) \cap \Gamma^{(2)}$ *and* $x = s(\gamma_1) = r(\gamma_2)$.

We finish by explaining the behavior of (family) G-relations with respect to restriction to suitable subgroupoids of Γ.

Proposition 22. *Let* Λ *be a* G*-relation and* $Y \subset G^{(0)}$ *be a saturated submanifold. We denote by* $H = G_Y^Y$ *the associated Lie subgroupoid and by* $i_H : H \hookrightarrow G$ *the inclusion map. Then,*

(1) *if* T_H^*G *and* Λ *are transversal, then* $i_H^*\Lambda$ *is a local* H*-relation.*
(2) *if* Λ *is a family* G*-relation, then* $T_H^*G \pitchfork \Lambda$ *and* $i_H^*\Lambda$ *is a local family* H*-relation.*

Proof. (1) The assumption already implies that the set $i_H^*\Lambda$ is a local Lagrangian submanifold. Let us consider the canonical map $\rho_H : T_H^*G \longrightarrow T^*H$. Obviously, the equality $i_H^*\Lambda = \rho_H(\Lambda \cap T_H^*G)$ holds. Since Y is saturated, we have $G_x = H_x$ and $G^x = H^x$ for all $x \in Y$, which implies $A_Y^*G = A^*H$ and the commutativity of the diagram

$$\begin{array}{ccc} T_H^*G & \xrightarrow{\rho_H} & T^*H \\ {\scriptstyle r_\Gamma}\downarrow\downarrow{\scriptstyle s_\Gamma} & & {\scriptstyle r_{T^*H}}\downarrow\downarrow{\scriptstyle s_{T^*H}} \\ A_Y^*G & \xrightarrow{=} & A^*H \end{array}.$$

In particular $\ker \sigma_{T^*H} = \rho_H(\ker \sigma_\Gamma \cap T_H^*G)$, $\sigma = s, r$ and

$$\rho_H(\Lambda \cap T_H^*G) \cap \ker \sigma_{T^*H} = \rho_H(\Lambda \cap \ker \sigma_\Gamma) = \emptyset,$$

where the first equality follows from $\ker \rho_H \subset \ker \sigma_\Gamma$ and the second from the assumption $\Lambda \cap \ker \sigma_\Gamma = \emptyset$.

(2) By assumption on Λ, we have

$$T_\gamma\mathcal{F}_G + dp(T_{(\gamma,\xi)}\Lambda) = T_\gamma G, \ \forall (\gamma, \xi) \in \Lambda \cap T_H^*G. \tag{106}$$

Since Y is saturated, we also have $\mathcal{F}_H = \{L \in \mathcal{F}_G \ ; \ L \cap H \neq \emptyset\}$, in particular $T_\gamma\mathcal{F}_G = T_\gamma\mathcal{F}_H \subset T_\gamma H$ for any $\gamma \in H$. Therefore we can replace $T_\gamma\mathcal{F}_G$ by $T_\gamma H$ in (106), which proves the transversality $i_H \pitchfork p|_\Lambda$, or equivalently the transversality of submanifolds: $T^*_H G \pitchfork \Lambda$. We also get from (106)

$$T_\gamma H = (T_\gamma \mathcal{F}_G + dp(T_{(\gamma,\xi)}\Lambda)) \cap T_\gamma H = T_\gamma \mathcal{F}_H + dp(T_{(\gamma,\xi)}\Lambda) \cap T_\gamma H,\ \forall (\gamma,\xi) \in \Lambda \cap T^*_H G. \tag{107}$$

Furthermore,

$$\begin{aligned} dp(T_{(\gamma,\xi)}\Lambda) \cap T_\gamma H &= dp(T_{(\gamma,\xi)}\Lambda \cap T_{(\gamma,\xi)} T^*_H G) = dp(T_{(\gamma,\xi)}(\Lambda \cap T^*_H G)) && \text{since } T^*_H G \pitchfork \Lambda \\ &= dp_H \circ d\rho_H (T_{(\gamma,\xi)}(\Lambda \cap T^*_H G)) && \text{since } p|_H = p_H \circ \rho_H \\ &= dp_H(T i^*_H \Lambda) && \text{since } \rho_H(\Lambda \cap T^*_H G) = i^*_H \Lambda. \end{aligned} \tag{108}$$

Using the result of this computation in (107) proves that $i^*_H\Lambda$ is a family H-relation.

□

6. Fourier integral operators on groupoids

6.1. Definitions.

Following [32], we are led to

Definition 12. *Let G be a Lie groupoid. Distributions belonging to $I(G,\Lambda;\Omega^{1/2})$, where Λ is any (family) local G-relation, are called* (family) Fourier integral G-operators.

We abbreviate Fourier integral G-operators as G-FIOs and family Fourier integral G-operators as G-FFIOs. If Λ is a G-relation then we get from [36] the inclusion:

$$I(G,\Lambda;\Omega^{1/2}) \subset \mathcal{D}'_{r,s}(G,\Omega^{1/2}). \tag{109}$$

In particular, any G-FIO u produces an equivariant C^∞ family of operators $u_x : C^\infty_c(G_x) \to C^\infty(G_x)$, $x \in G^{(0)}$, but each u_x is not necessarily a Fourier integral operator on G_x. It is worth giving an example.

Example 4. *Consider the fibred pair groupoid $G = X \times X \times Z \rightrightarrows X \times Z$ with $X = Z = \mathbb{R}$. Consider the open cone*

$$\mathcal{C} = \{(\gamma,\theta) \in G \times \mathbb{R}^2 \setminus 0\ ;\ \theta \in \mathcal{C}_{x_1}\}, \tag{110}$$

where $\gamma = (x_1,x_2,x_3)$ and $\theta \in \mathcal{C}_{x_1}$ means

$$2x_1\theta_2 + \theta_1 \neq 0,\ \ \theta_1 \neq 0. \tag{111}$$

The function

$$\phi : (\gamma,\theta) \longmapsto (x_1 - x_2).\theta_1 + (x_1^2 - z).\theta_2 \tag{112}$$

is a non-degenerate phase function with associated Lagrangian given by

$$\Lambda = \{(x,x,x^2,\theta_1 + 2x\theta_2, -\theta_1, -\theta_2)\ ;\ x \in \mathbb{R}, \theta \in \mathcal{C}_x\} \subset T^*G \setminus 0. \tag{113}$$

Note that Λ is a G-relation, but fails to be a family G-relation at the points where $x = 0$. Consider the closed cone

$$F = \{(\gamma,\theta) \in G \times \mathbb{R}^2\ ;\ |\gamma| \leq 1,\ 2|\theta_2| \leq |\theta_1|\} \subset \mathcal{C} \cup G \times \{0\} \tag{114}$$

and choose even functions $\chi \in C^\infty_c(\mathbb{R})$ and $b \in C^\infty_c(\mathbb{R}^3)$ such that $\chi(0) = \chi''(0) = 1$, $\chi(t) = 0$ if $|t| \geq \frac{1}{2}$, $\mathrm{supp}(b) \subset \{\gamma,\ |\gamma| \leq 1\}$ and $b(0) = 1$. Choose a symbol $a \in S^1(G \times \mathbb{R}^2)$, with support in F, such that $a(\gamma,\theta) = b(\gamma)\chi(\theta_2/\theta_1)\theta_1$ when $|\theta| \geq 1$. Then

$$u(\gamma) = \int e^{i\phi(\gamma,\theta)} a(\gamma,\theta) d\theta \in I^*(G,\Lambda), \tag{115}$$

and we consider the distribution $u_0 = m^*_{(0,0)}u$ *on* $G^{(0,0)} \times G_{(0,0)} \simeq \mathbb{R}^2$. *It is given by*

$$u_0(x_1, x_2) = \int e^{i((x_1-x_2).\theta_1 + x_1^2.\theta_2)} a_0(x_1, x_2, \theta_1, \theta_2) d\theta_1 d\theta_2 \in \mathcal{D}'(\mathbb{R}^2), \tag{116}$$

understood as a distribution where $a_0(x_1, x_2, \theta_1, \theta_2) = a(x_1, x_2, 0, \theta_1, \theta_2)$. *Indeed, observe that*

$$\phi_0 : (x, \theta) \longmapsto (x_1 - x_2).\theta_1 + x_1^2.\theta_2 \tag{117}$$

is a phase function on $\mathcal{C}_0 = \{(x, \theta) \in \mathbb{R}^2 \times \mathbb{R}^2 \ ; \ \theta \in \mathcal{C}_{x_1}\}$ *and that* $a_0 \in S^*(\mathbb{R}^2 \times \mathbb{R}^2)$ *is supported in*

$$F_0 = \{(x, \theta) \ ; \ |x| \leq 1, \ 2|\theta_2| \leq |\theta_1|\} \subset \mathcal{C}_0 \times \mathbb{R}^2 \times \{0\}. \tag{118}$$

It follows that (116) *is an oscillatory integral* [32, Paragraph 7.8] *and thus, by* [32, Theorem 8.1.9],

$$\mathrm{WF}(u_0) \subset \Lambda_0 = \{(0, 0, \theta_1, -\theta_1) \ ; \ \theta_1 \neq 0\}. \tag{119}$$

As expected, Λ_0 *fails to be a Lagrangian submanifold of* $T^*\mathbb{R}^2 \setminus 0$ *(actually, it is a one dimensional isotropic conic submanifold) and furthermore, there is no Lagrangian submanifold* Λ' *of* $T^*\mathbb{R}^2 \setminus 0$ *such that* $u_0 \in I^*(\mathbb{R}^2, \Lambda')$. *Before proving this assertion, observe that* (119) *implies* $u_0 \in C^\infty(\mathbb{R}^2 \setminus 0)$ *and that for any* x *with* $x_1 \neq 0$,

$$\begin{aligned} u_0(x) &= b_0(x) \int e^{i((x_1-x_2).\theta_1 + x_1^2.\theta_2)} \chi(\theta_2/\theta_1)\theta_1 d\theta_1 d\theta_2 \ \textit{modulo} \ C^\infty(\mathbb{R}^2) \\ &= b_0(x) \int e^{i((x_1-x_2).\theta_1 + x_1^2.\theta_1\theta_2)} \theta_1^2 \chi(\theta_2) d\theta_1 d\theta_2 = b_0(x) \int e^{i(x_1-x_2).\theta_1} \widehat{\chi}(-x_1^2\theta_1)\theta_1^2 d\theta_1 \\ &= b_0(x) x_1^{-6} \int e^{i\frac{x_2-x_1}{x_1^2}.\theta_1} \widehat{\chi}(\theta_1)\theta_1^2 d\theta_1 = x_1^{-6} b_0(x) \chi''(\frac{x_2 - x_1}{x_1^2}). \end{aligned}$$

Thus, u_0 *is not* C^∞ *at* $(0, 0)$ *and* $\mathrm{WF}(u_0)$ *contains at least a half line in* $T^*_{(0,0)}\mathbb{R}^2$. *Since* u_0 *is even,* $\mathrm{WF}(u_0)$ *also contains the opposite half line. This proves the equality in* (119). *Now assume that* $u_0 \in I^m(\mathbb{R}^2, \Lambda')$ *for some Lagrangian* Λ'. *If the principal symbol* $\sigma(u_0)$ *does not vanish at some point* $(x_0, \xi_0) \in \Lambda'$, *then* $(x_0, \xi_0) \in \mathrm{WF}(u_0)$. *Thus* $\sigma(u_0)$ *must vanish on* $\Lambda' \setminus \Lambda_0$. *Since* Λ_0 *is one dimensional, it has empty interior in* Λ' *and it follows that* $\sigma(u_0)$ *vanishes identically. Thus* $u_0 \in I^{m-1}(\mathbb{R}^2, \Lambda')$ *and repeating the argument proves that* u_0 *is* C^∞, *which is a contradiction.*

The phenomenon highlighted in this example disappears precisely for Fourier integral G-operators associated with family G-relations. Indeed,

Theorem 23. *Let* Λ *be a family* G*-relation and* $u \in \mathcal{D}'(G, \Omega^{1/2})$. *Then* $u \in I(G, \Lambda; \Omega^{1/2})$ *if and only if* u *is a* G*-operator and* $u_x = m_x^*(u) \in I(G_x \times G^x, m_x^*\Lambda; \Omega^{1/2}_{G_x \times G^x})$ *for all* $x \in G^{(0)}$.

Proof. Let us assume $u \in I(G, \Lambda; \Omega^{1/2})$. Then, as seen in (109), u is a G-operator and the pull-back distribution by the submersion m gives

$$m^*(u) \in I(G^{(2)}, m^*\Lambda; m^*\Omega^{1/2}).$$

Since $m^*\Lambda$ is transversal to $\pi : G^{(2)} \to G^{(0)}$, Proposition 9 gives the result for all the $m_x^*(u)$, $x \in G^{(0)}$.

Conversely, Proposition 9 gives rise to distribution $\widetilde{u} \in I(G^{(2)}, m^*\Lambda; m^*\Omega^{1/2})$ such that $\widetilde{u}|_{G_x \times G^x} = u_x$ and the result follows from Proposition 8 applied to $X = Z = G^{(2)}$, $Y = G$ and $f = m$, which yields $u = m_*\widetilde{u} \in I(G, \Lambda; \Omega^{1/2})$. $\square$

6.2. **Adjoint and composition.** Now, we can consider G-FFIOs equivalently either as families of ordinary Fourier integral operators acting on $G_x \times G^x$ or as single Lagrangian distributions on G whose underlying Lagrangian submanifold Λ has suitable properties. The second choice leads to simpler and more conceptual statements and also reveals the role played by the cotangent groupoid T^*G. Moreover, most of the statements hold true for the more general class of G-FIOs. The next two theorems support this point of view.

Theorem 24. *Let Λ be a G-relation and set $\Lambda^\star = i_\Gamma\Lambda$. If $A \in I^m(G,\Lambda)$ then $A^\star \in I^m(G,\Lambda^\star)$.*

Proof. It is enough to consider the case $A(\gamma) = \int e^{i\phi(\gamma,\theta)}a(\gamma,\theta)d\theta$ with ϕ a non-degenerate phase function parametrizing Λ locally. Then

$$A^\star(\gamma) = \int e^{-i\phi(\gamma^{-1},\theta)}\overline{a(\gamma^{-1},\theta)}d\theta. \tag{120}$$

The function $b(\gamma,\xi) = \overline{a(\gamma^{-1},\theta)}$ is a symbol of the same order as a. The function $\psi(\gamma,\theta) = -\phi(\gamma^{-1},\theta)$ is also a non-degenerate phase function and

$$\Lambda_\psi = \{(\gamma,\xi) \in T^*G \ ; \ (\gamma^{-1}, -{}^t(di_\gamma)(\xi)) \in \Lambda_\phi\}. \tag{121}$$

Since $i_\Gamma(\gamma,\xi) = (\gamma^{-1}, -{}^t(di_\gamma)(\xi))$, we get the result. $\square$

Note that if Λ is moreover a family, then so is $\Lambda^\star$, and the adjoint of a G-FFIO $u \in I(G,\Lambda)$ is given by the family of adjoints of each Fourier integral operator u_x on G_x.

Theorem 25. *Let Λ_1,Λ_2 be closed G-relations which are cleanly composable with excess e. If $A_1 \in I_c^{m_1}(G,\Lambda_1)$ and $A_2 \in I_c^{m_2}(G,\Lambda_2)$ then*

$$A_1.A_2 \in I^{m_1+m_2+e/2+n^{(0)}/2-n/4}(G,\Lambda_1.\Lambda_2). \tag{122}$$

Here n is the dimension of G and $n^{(0)}$ is the dimension of $G^{(0)}$.
If moreover, Λ_1,Λ_2 are family G-relations and are completely composable (i.e. condition (101) *is fulfilled), then $A_1.A_2$ is a family Fourier integral G-operator.*

Proof. We wish to apply Proposition 8 to the following data: $X = G^2$, $Y = G$, $Z = G^{(2)}$, $f = m_G$ $\widetilde{\Lambda} = \Lambda_1 \times \Lambda_2$ and $\widetilde{\phi} = \phi_1 + \phi_2$ where $\phi_j : U_j \times (\mathbb{R}^{N_j} \setminus 0) \to \mathbb{R}$ are non-degenerate phase functions parametrizing Λ_j in a conic neighborhood of points $(\gamma_j,\xi_j) \in \Lambda_j$, with the latter points satisfying $(\gamma_1,\xi_1,\gamma_2,\xi_2) \in \Lambda_1 \times \Lambda_2 \cap (T^*G)^{(2)}$. We may assume that

$$A_j(\gamma_j) = \int e^{i\phi_j(\gamma_j,\theta_j)}a_j(\gamma_j,\theta_j)d\theta_j, \tag{123}$$

where $a_j \in S^{m_j+(n-2N_j)/4}(U_j \times (\mathbb{R}^{N_j})$.

The only technical (and common) obstruction is that

$$a(\gamma_1,\gamma_2,\theta_1,\theta_2) = a_1(\gamma_1,\theta_1)a_2(\gamma_2,\theta_2) \tag{124}$$

is not a symbol in general. The condition of admissibility on Λ_j allows us to remove the regions in (θ_1,θ_2) where the symbolic estimates for a fail.

Indeed, thanks to the admissibility assumptions on Λ_1 and Λ_2, we can reduce the problem to the case where a_1,a_2 have support in compactly generated cones $\mathcal{C}_1,\mathcal{C}_2$ on which $\widetilde{s}(\phi'_{1\gamma})$ and $\widetilde{r}(\phi'_{2\gamma})$ never vanish. Using the degree one homogeneity of $\widetilde{s}(\phi'_{1\gamma})$, $\widetilde{r}(\phi'_{2\gamma})$ with respect to θ_1,θ_2, we can find constants C_1,C_2 such that

$$\text{if } (\gamma_j,\theta_j) \in \mathcal{C}_j \text{ and } \widetilde{s}(\phi'_{1\gamma}(\gamma_1,\theta_1)) = \widetilde{r}(\phi'_{2\gamma}(\gamma_2,\theta_2)) \text{ then } C_1|\theta_2| < |\theta_1| < C_2|\theta_2|. \tag{125}$$

We choose a homogeneous function $\chi(\theta_1,\theta_2)$ of degree 0 which is equal to 1 when $C_1|\theta_2|/2<|\theta_1|<2C_2|\theta_2|$ and supported in $C_1|\theta_2|/3<|\theta_1|<3C_2|\theta_2|$. We set

$$b(\gamma_1,\gamma_2,\theta_1,\theta_2)=\chi(\theta_1,\theta_2)a(\gamma_1,\gamma_2,\theta_1,\theta_2) \tag{126}$$

and

$$c(\gamma_1,\gamma_2,\theta_1,\theta_2)=(1-\chi(\theta_1,\theta_2))a(\gamma_1,\gamma_2,\theta_1,\theta_2). \tag{127}$$

We have, by construction of χ,

$$C_1|\theta_2|/3<|\theta_1|<3C_2|\theta_2| \text{ in } \operatorname{supp}(b). \tag{128}$$

which allows us to check that $b\in S^{m_1+m_2+(n-N_1-N_2)/2}$. Therefore we can apply the Proposition 8 to

$$\widetilde{B}=\int e^{i\widetilde{\phi}(\gamma_1,\gamma_2,\theta_1,\theta_2)}b(\gamma_1,\gamma_2,\theta_1,\theta_2)d\theta_1 d\theta_2,$$

and we get

$$B(\gamma)=\int_{m^{-1}(\gamma)\times\mathbb{R}^{N_1+N_2}} e^{i\phi(\gamma,\eta,\theta_1,\theta_2)}b(\gamma,\eta,\theta_1,\theta_2)d\eta d\theta_1 d\theta_2\in I^{m_1+m_2+e/2+n^{(0)}/2-n/4}(G,\Lambda_1*\Lambda_2). \tag{129}$$

Moreover, again using the degree one homogeneity of $\widetilde{\phi}$ with respect to (θ_1,θ_2) and using the expression of ϕ'_ω given in the proof of Proposition 4, we also get

$$|\theta_1|+|\theta_2|<C|\phi'_\eta(\gamma,\eta,\theta_1,\theta_2)| \text{ in } \operatorname{supp}(c), \tag{130}$$

where we have set $\gamma=\gamma_1\gamma_2$, $\eta=(\gamma_1,\gamma_2)\in m^{-1}(\gamma)$ and $\phi(\gamma,\eta,\theta_1,\theta_2)=\widetilde{\phi}(\gamma_1,\gamma_2,\theta_1,\theta_2)$. The previous estimates show that

$$C(\gamma)=\int_{m^{-1}(\gamma)\times\mathbb{R}^{N_1+N_2}} e^{i\phi(\gamma,\eta,\theta_1,\theta_2)}c(\gamma,\eta,\theta_1,\theta_2)d\eta d\theta_1 d\theta_2 \tag{131}$$

belongs to $C^\infty(G)$, and we conclude that

$$A_1*A_2(\gamma)=B(\gamma) \mod C^\infty(G), \tag{132}$$

which proves the theorem.

□

6.3. **Principal symbol.** By [32, Section 25.1], the principal symbol of $A\in I^m(G,\Lambda;\Omega^{1/2})$ belongs to $S^{[m+n/4]}(\Lambda,I_\Lambda\otimes\hat{\Omega}^{1/2}\otimes\hat{\Omega}_G^{-1/2})$ and the principal symbol map gives rise to an isomorphism

$$\sigma: I^{[m]}(G,\Lambda;\Omega^{1/2})\longrightarrow S^{[m+n/4]}(\Lambda,I_\Lambda\otimes\hat{\Omega}^{1/2}\otimes\hat{\Omega}_G^{-1/2}). \tag{133}$$

Here we have set $\hat{E}=(p|_\Lambda)^*(E|_\Lambda)$ for any bundle $E\to G$. To understand the product formula of symbols of G-FIOs, we analyse the auxiliary bundle

$$\Sigma^\alpha=\Omega_G^{-\alpha}\otimes\Omega^\alpha \tag{134}$$

involved in the right hand side of (133) with $\alpha=\frac{1}{2}$. As we shall see, $\hat{\Sigma}$ is strongly related to the groupoid structure of T^*G.

We need a simple statement about vector bundle epimorphisms.

Lemma 26. *Let $f : X \to Y$ be a submersion, $p : E \to X, q : F \to Y$ be C^∞ vector bundles and $g : E \to F$ a C^∞ epimorphism:*

$$\begin{tikzcd} 0 \arrow[r] & \ker g \arrow[r] \arrow[d, "p"] & E \arrow[r, "g"] \arrow[d, "p"] & F \arrow[r] \arrow[d, "q"] & 0 \\ & X \arrow[r, "="] & X \arrow[r, "f"] & Y & \end{tikzcd} \tag{135}$$

Then the sequence

$$0 \longrightarrow p^*(\ker g) \xrightarrow{v} \ker dg \xrightarrow{dp} p^* \ker df \longrightarrow 0. \tag{136}$$

is exact. The map v is defined by

$$p^*(\ker g)_{(x,e)} \ni \lambda \longmapsto v(\lambda) = \frac{d}{dt}(x, e + t\lambda)|_{t=0} \in \ker dg_{(x,e)} \cap \ker dp_{(x,e)}.$$

Proof. Thanks to the diagram (135), we have $dp(\ker g) \subset \ker df$ and the map

$$\ker dg \ni (e, u) \overset{dp}{\longmapsto} (e, dp_e(u)) \in p^* \ker df \tag{137}$$

is well defined. To prove its surjectivity, we work in local coordinates on TE associated with local coordinates on X and local trivializations of E, so that the map (137) corresponds to

$$(x, e, t, u) \longmapsto (x, e, t). \tag{138}$$

Writing $g(x, e) = (f(x), \widetilde{g}(x, e))$, we compute

$$(dg)_{(x,e)}(t, u) = (df_x(t), (d_x\widetilde{g})_{(x,e)}(t) + (d_e\widetilde{g})_{(x,e)}(u)) \tag{139}$$

$$= (df_x(t), (d_x\widetilde{g})_{(x,e)}(t) + \widetilde{g}(x, u)). \tag{140}$$

Since $\widetilde{g}$ is fiberwise linear and surjective, the linear equation $(d_x\widetilde{g})_{(x,e)}(t) + \widetilde{g}(x, u) = 0$ for fixed x, e and t has solutions in u. Let $u_{x,e,t}$ be such a solution. Then, for any $t \in \ker df_x$, the element $(t, u_{x,e,t})$ belongs to $\ker dg_{(x,e)}$.

Next, it follows from (137) and (139) that $(x, e, t, u) \in \ker dp \cap \ker dg$ if and only if $t = 0$ and $u \in \ker \widetilde{g}$. Since, in these coordinates

$$v : p^*(\ker g) \ni (x, e, u) \longmapsto (x, e, 0, u) \in \ker dp \cap \ker dg, \tag{141}$$

we get that $v(p^*(\ker g))$ is the kernel of the vector bundle epimorphism $\ker dg \xrightarrow{dp} p^* \ker df$. □

As claimed, we can interpret Σ in terms of density bundles associated with the groupoid structure of T^*G.

Proposition 27. *We have canonical identifications*

$$\hat{\Sigma}^{1/2} \simeq \Omega^{1/2}(\ker ds_\Gamma) \simeq \Omega^{1/2}(\ker dr_\Gamma), \tag{142}$$

$$\Omega(\ker dm_\Gamma) \simeq {\mathrm{pr}_{(1)}}^* \hat{\Sigma}^{1/2} \otimes {\mathrm{pr}_{(2)}}^* \hat{\Sigma}^{1/2} \simeq m_\Gamma^* \hat{\Sigma}, \tag{143}$$

$$\Omega(\ker dm_\Gamma) \simeq (p^2)^*(\Omega(\ker m_\Gamma) \otimes \Omega(\ker dm)). \tag{144}$$

Proof. Applying the lemma to (11), (12) and (10), one gets the exact sequence

$$0 \longrightarrow p^*((\ker dr)^\perp) \longrightarrow \ker ds_\Gamma \xrightarrow{dp} p^* \ker ds \longrightarrow 0, \tag{145}$$

$$0 \longrightarrow p^*((\ker ds)^\perp) \longrightarrow \ker dr_\Gamma \xrightarrow{dp} p^* \ker dr \longrightarrow 0. \tag{146}$$

and

$$0 \longrightarrow (p^2)^* \ker m_\Gamma \longrightarrow \ker dm_\Gamma \xrightarrow{dp^2} (p^2)^* \ker dm \longrightarrow 0. \tag{147}$$

To prove (142), observe that (145) gives

$$\Omega^{1/2}(\ker ds_\Gamma) \simeq p^*\Omega^{1/2}(\ker ds) \otimes p^*\Omega^{1/2}((\ker dr)^\perp)$$

and that $\Omega^{1/2}((\ker dr)^\perp) = \Omega^{-1/2}(TG/\ker dr) = \Omega_G^{-1/2} \otimes \Omega^{1/2}(\ker dr)$. Next, observe that for any Lie groupoid G, the maps

$$\ker dm \ni (\gamma_1, \gamma_2, X_1, X_2) \longmapsto (\gamma_1, \gamma_2, X_2) \in {\mathrm{pr}_{(2)}}^*(\ker ds)$$

and

$${\mathrm{pr}_{(1)}}^*(\ker ds) \ni (\gamma_1, \gamma_2, X_1) \longmapsto (\gamma_1, \gamma_2, (dR_{\gamma_2})_{\gamma_1}(X_1)) \in m^*(\ker ds)$$

are isomorphisms of vector bundles over $G^{(2)}$. Similarly, $\ker dm \simeq {\mathrm{pr}_{(1)}}^*(\ker dr)$ and ${\mathrm{pr}_{(2)}}^*(\ker dr) \simeq m^*(\ker dr)$. These facts applied to the groupoid T^*G give

$$\Omega(\ker dm_\Gamma) \simeq \Omega^{1/2}(\ker dm_\Gamma) \otimes \Omega^{1/2}(\ker dm_\Gamma) \tag{148}$$

$$\simeq {\mathrm{pr}_{(1)}}^* \,\Omega^{1/2}(\ker dr_\Gamma) \otimes {\mathrm{pr}_{(2)}}^* \,\Omega^{1/2}(\ker ds_\Gamma) \tag{149}$$

$$\simeq m_\Gamma^*(\Omega(\ker ds_\Gamma)) \simeq m_\Gamma^*(\Omega(\ker dr_\Gamma)), \tag{150}$$

where we have used (142) to pass from the second to the third line. This proves (143), and then (144) follows directly from (147). □

Proposition 28. *Let Λ_1, Λ_2 be closed G-relations which are cleanly composable. Let $\Lambda = \Lambda_1.\Lambda_2$. We have a natural homomorphism of vector bundles over $\Lambda_1 \times \Lambda_2 \cap \Gamma^{(2)}$:*

$$(\hat\Sigma^{1/2} \otimes I_{\Lambda_1}) \boxtimes (\hat\Sigma^{1/2} \otimes I_{\Lambda_2}) \longrightarrow m_\Gamma^*(I_\Lambda \otimes \hat\Sigma^{1/2}) \otimes \Omega(\ker dm_\Gamma \cap T(\Lambda_1 \times \Lambda_2)). \tag{151}$$

Proof. Applying [32, Theorem 21.6.6], we get

$$I_{\Lambda_1} \boxtimes I_{\Lambda_2} \longrightarrow m_\Gamma^* I_\Lambda \otimes \Omega^{-1/2}(\ker dm_\Gamma) \otimes \Omega(\ker dm_\Gamma \cap T(\Lambda_1 \times \Lambda_2)). \tag{152}$$

Contrary to what happens in the proof of [32, Theorem 21.6.7], the bundle $\Delta = \ker dm_\Gamma$ is not necessarily symplectic (actually, it may even be odd dimensional since the fibers are of dimension $n = \dim G$) and we cannot expect any natural trivialization of the corresponding density bundle. This is where the bundle Σ is useful.

Using (143) in Corollary 27 we get

$$(\hat\Sigma^{1/2} \otimes I_{\Lambda_1}) \boxtimes (\hat\Sigma^{1/2} \otimes I_{\Lambda_2}) \simeq \Omega(\ker dm_\Gamma) \otimes (I_{\Lambda_1} \boxtimes I_{\Lambda_2}) \simeq m_\Gamma^*(\hat\Sigma) \otimes (I_{\Lambda_1} \boxtimes I_{\Lambda_2}). \tag{153}$$

Using (143) again to get $\Omega(\ker dm_\Gamma)^{1/2} \simeq m_\Gamma^*(\hat\Sigma^{1/2})$ and combining (153) and (152), we obtain (151).

□

These identifications of Maslov and density bundles allow us to apply the formula for the product of principal symbols given in [32, Theorem 25.2.3]. In the present situation, it gives:

Corollary 29. *Let Λ_1, Λ_2 be closed G-relations which are cleanly composable with excess e and set $\Lambda = \Lambda_1.\Lambda_2$. Let $A_j \in I^{m_j}(G, \Lambda_j; \Omega^{1/2})$ be compactly supported G-FIOs and $a_j \in S^{m_j+n/4}(\Lambda_j, \hat\Sigma^{1/2} \otimes I_{\Lambda_j})$ be representatives of the principal symbol of A_j. Let $(a_1 \boxtimes a_2)_{\gamma,\xi}$ be the density on the compact manifold $m_\Gamma^{-1}(\gamma,\xi) \cap \Lambda_1 \times \Lambda_2$ with values in $\hat\Sigma^{1/2} \otimes I_\Lambda$, as given by (151). Then $a_1 * a_2$ defined by*

$$(\gamma, \xi) \in \Lambda, \quad a_1 * a_2(\gamma,\xi) = \int_{m_\Gamma^{-1}(\gamma,\xi)\cap\Lambda_1\times\Lambda_2} a_1 \boxtimes a_2 \tag{154}$$

belongs $S^{m_1+m_2+e/2+n^{(0)}/2}(\Lambda, \hat{\Sigma}^{1/2} \otimes I_\Lambda)$ *and represents the principal symbol of* $A = A_1 * A_2$.

We end with some direct consequences of the previous statements.

6.4. **Composition with pseudodifferential operators.**
We recall that the usual conventions for the order of conormal and Lagrangian distributions and for the order of pseudodifferential operators on groupoids yield

$$\Psi^m(G) = I^{m+(n-2n^{(0)})/4}(G, A^*G; \Omega^{1/2}). \tag{155}$$

Theorem 30. *Any closed* G*-relation* Λ *is transversally composable with the unit* G*-relation* A^*G *and the convolution product of distributions turns* $I(G, \Lambda; \Omega^{1/2})$ *into a* $\Psi_c(G, \Omega^{1/2})$*-bimodule:*

$$\Psi_c(G; \Omega^{1/2}) * I(G, \Lambda; \Omega^{1/2}) \subset I(G, \Lambda; \Omega^{1/2}) \ ; \ I(G, \Lambda; \Omega^{1/2}) * \Psi_c(G; \Omega^{1/2}) \subset I(G, \Lambda; \Omega^{1/2}).$$

When $\Lambda = A^*G$, we recover the fact that $\Psi_c(G)$ is an algebra.

Proof. The G-relations $\Lambda_0 = A^*G$ and Λ are transversally composable if $T(\Lambda_0 \times \Lambda) + T\Gamma^{(2)} = T\Gamma^2$ at any point $(\delta_1, \delta_2) \in \Lambda_0 \times \Lambda \cap \Gamma^{(2)}$. Passing to the symplectic orthocomplement, this is equivalent to

$$T_{(\delta_1,\delta_2)}(\Lambda_0 \times \Lambda) \cap \ker(dm_\Gamma)_{(\delta_1,\delta_2)} = 0, \tag{156}$$

and the latter follow from general properties of Lie groupoids. Indeed, let Γ be any Lie groupoid and consider $\gamma \in \Gamma$, $r(\gamma) = x$, $s(\gamma) = y \in \Gamma^{(0)}$, $(t_1, t_2) \in T_{x,\gamma}\Gamma^{(0)} \times \Gamma \cap \ker(dm_\Gamma)_{(x,\gamma)}$.

Since $r_\Gamma|_{\Gamma^{(0)}} = s_\Gamma|_{\Gamma^{(0)}} = \mathrm{Id}$, we get $ds_\Gamma(t_1) = dr_\Gamma(t_1) = t_1$ and since $r_\Gamma \circ m_\Gamma = r_\Gamma \circ \mathrm{pr}_1$, $s_\Gamma \circ m_\Gamma = s_\Gamma \circ \mathrm{pr}_2$ we get from the assumption on (t_1, t_2) that

$$t_1 = dr_\Gamma(t_1) = dr_\Gamma \circ dm_\Gamma(t_1, t_2) = 0 \text{ and } ds_\Gamma(t_2) = ds_\Gamma \circ dm_\Gamma(t_1, t_2) = 0.$$

Also, we get $0 = ds_\Gamma(t_1) = dr_\Gamma(t_2)$, therefore

$$(0, t_2) \in \ker(dm_\Gamma)_{(x,\gamma)}, \quad t_2 \in T_\gamma\Gamma^x_y.$$

Then

$$(0, d(R_{\gamma^{-1}})_\gamma(t_2)) \in \ker(dm_\Gamma)_{(x,x)}, \quad d(R_{\gamma^{-1}})_\gamma(t_2) \in T_x\Gamma^x_x.$$

Since $(dm_\Gamma)_{(x,x)}(u_1, u_2) = u_1 + u_2$ if $u_j \in T_x\Gamma^x_x$ we also conclude that $t_2 = 0$ and this proves that Λ_0 and Λ are transversally composable. This is obviously the same with Λ_0 on the right. In both cases, the fibers of the product $\Lambda_0.\Lambda = \Lambda.\Lambda_0 = \Lambda$ are just points, hence the product is proper and connected. Now Theorem 25 gives the conclusion. $\square$

Combining Theorems 30 and 25, we obtain:

Theorem 31. *(Egorov's Theorem for groupoids). Let* $\Lambda, \Lambda' \subset T^*G \setminus 0$ *be composable closed* G*-relations such that*

$$\Lambda.\Lambda' \subset A^*G \setminus 0 \textit{ and } \Lambda'.\Lambda \subset A^*G \setminus 0. \tag{157}$$

Then

$$I_c(G, \Lambda; \Omega^{1/2}) * \Psi(G; \Omega^{1/2}) * I_c(G, \Lambda'; \Omega^{1/2}) \subset \Psi(G; \Omega^{1/2}). \tag{158}$$

In [66], the second author introduced a class of generalized smoothing operators and a family of Sobolev spaces for general Lie groupoids. Recall that $C^*(G)$ denotes the C^*-algebra associated with the groupoid G. The set of generalized smoothing operators $\Psi^{-\infty}(G)$ (Definition 24, p77 in [66]) is defined as the subset of $C^*(G)$ of those elements R such that the closure of P_1RP_2 is again in $C^*(G)$ for any two compactly supported pseudodifferential G operators P_1 and P_2. For $s>0$ the Sobolev module H^s of rank s is defined as the $C^*(G)$-module $\mathrm{dom}(\overline{P})$ endowed with scalar product $\langle x\,,\,y\rangle_s=\langle Px\,,\,Py\rangle+\langle x\,,\,y\rangle$, where P is any elliptic operator of order s. The corresponding Sobolev module H^{-s} is defined by duality.
Next we can give, using techniques coming from [49] and [66], a result on the continuity in the spirit of Theorem 25.3.1 in [32].

Theorem 32. *Let Λ be a locally invertible local G-relation and $A\in I^m_c(G,\Lambda;\Omega^{1/2})$.*

1. *If $m=(n-2n^{(0)})/4$, then the associated Fourier integral G-operator, still denoted by A, extends to an operator which is a bounded multiplier of $C^*(G)$:*

$$A\in\mathcal{M}(C^*(G)). \tag{159}$$

2. *If $m<(n-2n^{(0)})/4$ then A extends to an element of $C^*(G)$.*
3. *In the general case, A can be extended to a morphism from H^s to $H^{s-m'}$ with*

$$m'=m-(n-2n^{(0)})/4.$$

Proof. By the definition of a G-FIO and the assumptions of the theorem, A can be decomposed into a finite sum $A=\sum A_i$ where for each i, $A_i\in I^m_c(G,\Lambda_i;\Omega^{1/2})$ and Λ_i is an invertible patch of Λ. Therefore, we may immediately assume that Λ is an invertible G-relation.

From [36], we know that A is an adjointable G-operator, and Theorem 24 gives $A^*\in I^m_c(G,\Lambda^\star;\Omega^{1/2})$. Recall that, with $\langle\,,\,\rangle$ denoting the Hilbertian product of $C^*(G)$ seen as a $C^*(G)$-Hilbert module, the adjoint A^* is characterized by

$$\langle Au\,,\,v\rangle=\langle u\,,\,A^*v\rangle\quad\forall u,v\in C^\infty_c(G,\Omega^{1/2}). \tag{160}$$

Observe that, by the Cauchy-Schwarz inequality for Hilbert modules, we have

$$\|Au\|^2=\|\langle Au\,,\,Au\rangle\|=\|\langle A^*Au\,,\,u\rangle\|\leq\|A^*Au\|\|u\|\quad\forall u\in C^\infty_c(G,\Omega^{1/2}), \tag{161}$$

and similarly for A^*. Since Λ is invertible, Theorems 25 and 12 give

$$A^*A\in\Psi_c^{2m-\frac{n-2n^{(0)}}{2}}(G).$$

Now a fundamental result [49, Theorem 18], [66, Proposition 39] says that

$$\Psi^0_c(G)\subset\mathcal{M}(C^*(G))\text{ and }\Psi^{m'}_c(G)\subset C^*(G)\text{ for any }m'<0, \tag{162}$$

and thus we can proceed as in [49],[66]:

1. Assume $m=(n-2n^{(0)})/4$. Then $A^*A\in\Psi^0_c(G)$ and there exists $C\geq 0$ such that, using (161),
$$\|Au\|^2\leq C\|u\|^2\quad\forall u\in C^\infty_c(G,\Omega^{1/2}),$$
and similarly for A^*. This allows us to extend the relation (160) by continuity to all $u,v\in C^*(G)$, which then proves that $A\in\mathcal{L}(C^*(G))\simeq\mathcal{M}(C^*(G))$.
2. Assume $m<(n-2n^{(0)})/4$. Then $A\in\mathcal{M}(C^*(G))$ as before and by (162), $A^*A\in C^*(G)$, which implies that $A\in C^*(G)$ as well, since $C^*(G)$ is an ideal of $\mathcal{M}(C^*(G))$ (see [55, Chapter 1], for instance).

(3) In the general case, we know from [66] that for all s there exists an invertible elliptic pseudodifferential G-operator $P(s)$ of order exactly s, of inverse $P(-s)$, and that this operator $P(s)$ is an isomorphism of Hilbert modules between H^s and $C^*(G)$.
Then $P(s-m)AP(-s)$ is an element of $I_c^{(n-2n^{(0)})/4}(G,\Lambda;\Omega^{1/2})$ and hence by the first result, a bounded morphism of Hilbert modules between $C^*(G)$ and itself, so the result follows by multiplying on the left by $P(m-s)$ an on the right by $P(s)$.

□

7. Example: manifolds with boundary

Manifolds with corners, stratified spaces (through their desingularisations into manifolds with iterated fibred corners [1, 24]) and foliations are examples of singular spaces for which there exist Lie groupoids delivering suitable pseudodifferential calculi [14, 47, 49, 54, 24, 66]. Therefore, in all these situations, our previous constructions provide a calculus of Fourier Integral Operators extending the one for pseudodifferential operators. We illustrate this with the basic example of manifolds with boundary. More sophisticated examples can be treated exactly in the same way but are postponed to a future work into which we will also apply our calculus of FIOs to investigate the properties of the unitary group (e^{itP}) (where P is a suitable first order elliptic operator) on singular spaces. The case of manifolds with boundary also allows us to compare our constructions with the existing ones [41].

7.1. **The b-stretched product.** Let X be a manifold with boundary $\partial X = Y$ and let x be a defining function for Y. The b-stretched product X^2_b of X is by definition the C^∞ manifold obtained by blowing up the submanifold $B = Y\times Y$ in X^2. As a set, X^2_b is given by the disjoint union $X^2_b = X^2\setminus B\cup S_+N(B)$ where $S_+N(B)$ denotes the inward pointing part of the spherical normal bundle of B [44, Sections 4.1 and 4.2]. There is a natural blow-down map $\beta : X^2_b \to X^2$ and several natural submanifolds are associated with X^2_b :

- the front face $\mathrm{ff} = S_+N(B)$,
- the left and right boundary faces, that is $\mathrm{lb} = \overline{\beta^{-1}(Y\times(X\setminus Y))}$ and $\mathrm{rb} = \overline{\beta^{-1}((X\setminus Y)\times Y)}$,
- the lifted diagonal $\overline{\beta^{-1}(\Delta_{\overset{\circ}{X}})} = \Delta_b$.

The space of vector fields tangent to the boundary is a finitely generated projective module over $C^\infty(X)$. The underlying vector bundle is called the b-tangent space and denoted by ${}^bTX$. There is a natural vector bundle homomorphism

$$a : {}^bTX \longrightarrow TX. \tag{163}$$

7.2. **The b-groupoid.**

7.2.1. *Definition.* The appropriate groupoid is then the following subgroupoid of $X^2\times\mathbb{R}^*_+ \rightrightarrows X$:

$$G_b = \{(p,q,t)\in X^2\times\mathbb{R}^*_+\ ;\ x(q) = tx(p)\}. \tag{164}$$

It is a Lie groupoid (G_b and X are manifolds with boundary) and we have canonical diffeomorphisms

$$\partial G_b \simeq Y^2\times\mathbb{R}^*_+ \text{ and } \overset{\circ}{G_b} \simeq \overset{\circ}{X}\times\overset{\circ}{X},$$

the second one being given by the inverse of the map $(p,q) \mapsto (p,q,x(p)/x(q))$. If X_b^2 denotes the b-stretched product described above then the map

$$\begin{array}{llll} \Phi: & G_b & \longrightarrow & X_b^2 \\ & (p,q,t) & \longmapsto & \begin{cases} (p,q) \text{ if } p \in \overset{\circ}{X} \\ [(s,ts,p,q)_{s\in[0,\epsilon]}] \text{ if } p \in Y. \end{cases} \end{array} \tag{165}$$

gives a diffeomophism $G_b \simeq X_b^2 \setminus (\mathrm{lb} \cup \mathrm{rb})$ where the left and right boundary faces lb, rb of X_b^2 are given as above. For the map Φ defined above, it is understood that a collar neighborhood $\mathcal{U}$ of Y is chosen, allowing us to write points in $\mathcal{U}$ in the form (x,p) with $x \in [0,+\infty)$ and $p \in Y$. For completeness, let us mention that Φ maps the unit space $G_b^{(0)}$ onto the lifted diagonal Δ_b.

7.2.2. *The Lie algebroid of the b-groupoid and the canonical foliation.* The map Φ also provides a vector bundle isomorphism between the Lie algebroid AG_b and the stretched tangent bundle ${}^bTX$. Through the latter isomorphism, the anchor map $dr : AG_b \longrightarrow TX$ coincides with (163).

The canonical foliation $\mathcal{F}_{G_b}$ has exactly two leaves: $Y^2 \times \mathbb{R}_+^*$ and $\overset{\circ}{X} \times \overset{\circ}{X}$.

7.2.3. *Local coordinates near the boundary.* Using a collar neighborhood identification $\mathcal{U} \simeq [0,\infty) \times Y$, we get a Lie groupoid isomorphism

$$\mathbb{R}_+ \rtimes \mathbb{R}_+^* \times Y^2 \simeq (\mathcal{G}_b)_{\mathcal{U}}^{\mathcal{U}} \tag{166}$$

given by

$$\begin{array}{llll} \Psi: & \mathbb{R}_+ \rtimes \mathbb{R}_+^* \times Y^2 & \longrightarrow & (G_b)_{\mathcal{U}}^{\mathcal{U}} \\ & (x,t,p,q) & \longmapsto & \begin{cases} (x,p,tx,q) \text{ if } x > 0 \\ (t,p,q) \text{ if } x = 0. \end{cases} \end{array} \tag{167}$$

The diffeomorphism Ψ is used to choose suitable local coordinates around ∂G_b.

7.3. **The cotangent groupoid of the b-groupoid.** The cotangent groupoid T^*G_b splits into two satured subgroupoids:

$$T^*G_b = T^*_{(\overset{\circ}{X}\times\overset{\circ}{X})} G_b \bigcup T^*_{\partial G_b} G_b.$$

The first one is the cotangent groupoid of the pair groupoid $\overset{\circ}{X} \times \overset{\circ}{X}$ and its structural maps are recalled in [36, Example 3]. The second one is the cotangent groupoid of $Y^2 \times \mathbb{R}_+^*$, so it is the cartesian product of the cotangent groupoid of $Y \times Y$ and of the cotangent groupoid of the Lie group $\mathbb{R}_+^*$, whose structural maps are recalled in [36, Example 2]. Concretely, in the coordinates induced by Ψ, the points in the unit space $A^*\mathcal{G}_b$ of T^*G_b are the following ones:

$$(x,1,y,y,0,\nu,-\xi,\xi) \in T^*_{G_b^{(0)}} G_b. \tag{168}$$

The source and target maps are given by:

$$s_\Gamma(x,t,p,q,\tau,\nu,\xi,\eta) = (xt,1,p,p,0,t\nu,-\eta,\eta) \tag{169}$$

$$r_\Gamma(x,t,p,q,\tau,\nu,\xi,\eta) = (x,1,p,p,0,t\nu - x\tau,\xi,-\xi). \tag{170}$$

To see that the multiplication is C^∞ around $T^*_{\partial G_b} G_b$, we express it in the coordinates above. It is easy to see that $\delta_1, \delta_2 \in T^*G_b$ are composable if and only if they are of the following form:

$$\delta_1 = (x,t_1,p_1,q,\tau_1,x\nu_1,\xi_1,\eta) \; ; \; \delta_2 = (xt_1,t_2,q,p_2,t_2t_1^{-1}\nu_2 - \nu_1, x\nu_2, -\eta, \xi_2),$$

and then

$$m_\Gamma(\delta_1, \delta_2) = \delta_1\delta_2 = (x, t_1t_2, p_1, p_2, \tau_1 + t_2\nu_2 - t_1\nu_1, t_1^{-1}x\nu_2, \xi_1, \xi_2).$$

Note that when $x > 0$, the expression of the product is much simpler in the natural coordinates of $X \times X$ and when $x = 0$, setting $\tau_2 = t_2t_1^{-1}\nu_2 - \nu_1$ in δ_2, we recover the more familiar law:

$$\delta_1\delta_2 = (0, t_1t_2, p_1, p_2, \tau_1 + t_1\tau_2, 0, \xi_1, \xi_2).$$

7.4. G_b **family relations.** Now, let $\Lambda \subset T^*G_b$ be a conic Lagrangian submanifold. Writing down the source and target maps s_Γ and r_Γ of $\Gamma = T^*G_b$, we get that Λ is a G_b-relation if and only if

$$\Lambda \cap T^*(\overset{\circ}{X} \times \overset{\circ}{X}) \subset (T^*\overset{\circ}{X} \setminus 0) \times (T^*\overset{\circ}{X} \setminus 0) \tag{171}$$

and

$$\Lambda \cap T^*_{Y^2\times\mathbb{R}_+^*} G_b \subset [T^*Y^2 \times (T^*\mathbb{R}_+^* \setminus 0)] \bigcup [(T^*Y \setminus 0) \times (T^*Y \setminus 0) \times (\mathbb{R}_+^* \times \{0\})]. \tag{172}$$

Next, a G_b-relation Λ is a family G_b-relation if and only if

$$\Lambda \pitchfork T^*_{Y^2\times\mathbb{R}_+^*} G_b. \tag{173}$$

This is obvious using condition (5) of Theorem 14 and the fact that $\mathcal{F}_{G_b}$ has exactly two leaves: $Y^2 \times \mathbb{R}_+^*$ and $\overset{\circ}{X} \times \overset{\circ}{X}$, the second one bringing in an empty transversality condition.

7.5. **Phase functions.** Let us interpret Condition (173) locally in terms of phase functions. Introduce the submersion

$$b : G_b \ni (p, q, t) \longmapsto x(p) \in \mathbb{R}_+.$$

Then Condition (173) is equivalent to the condition that $b|_\Lambda : \Lambda \to \mathbb{R}_+$ is a submersion near $x = 0$. Thus, given a family G_b-relation Λ, the results of Paragraph 3.4 show that Λ is locally parametrizable near the boundary by C^∞ families $(\phi_x)_{x\geq 0}$ of phase functions on $Y^2 \times \mathbb{R}_+^*$. More precisely, this means that for any $(\gamma^0, \xi^0) \in \Lambda$ with $\gamma^0 \in \partial G_b$, using the local coordinates provided by (167) and Y, we can find a phase function $\phi(x, t, p, q, \theta)$ such that $\phi_x(t, p, q, \theta) = \phi(x, t, p, q, \theta)$ is a non-degenerate phase function parametrizing $\Lambda_x = i_x^*(\Lambda)$ for any non-negative real number x close to 0 (see Paragraph 3.4 for the notation; in particular ϕ is also non-degenerate and parametrizes Λ).

7.6. **Indicial symbol and small b-calculus.** Next, given a family G_b-relation Λ, Proposition 22 can be applied with the groupoid G_b and the saturated subgroupoid $\partial G_b = (G_b)_Y^Y$. Together with Proposition 8, this yields an *indicial symbol* (or boundary symbol) map:

$$I^m(G_b, \Lambda) \ni u \longmapsto I(u) = i^*_{\partial G_b}(u) = i_0^*(u) \in I^{m+1/4}(Y^2 \times \mathbb{R}_+^*, \Lambda_0). \tag{174}$$

Concretely, in local coordinates and with phases as above, if the G_b-FFIO u is given by the oscillatory integral

$$u = \int e^{i\phi_x(t,p,p',\theta)} a(x, t, p, p', \theta) d\theta$$

for some symbol a and some family (ϕ_x) of phase functions as above, then the indicial symbol of u is the ∂G_b-FFIO given by the oscillatory integral

$$I(u) = \int e^{i\phi_0(t,p,p',\theta)} a(0, t, p, p', \theta) d\theta.$$

Let us give some mapping properties of G_b-FFIOs. Let $u \in I^m(G_b, \Lambda)$ be a G_b-FFIO and consider the equivariant family $(P_p)_{p\in X}$ of linear operator in the fibers. Using the equivariance condition and the obvious identifications $(G_b)_p \simeq (G_b)^{\,p} = \overset{\circ}{X}$ for any interior point p, we get that the distributions P_p, $p \in \overset{\circ}{X}$, do not depend on the base point and the common value $P_\circ$ satisfies

$$P_\circ \in I^m(\overset{\circ}{X} \times \overset{\circ}{X}, \overset{\circ}{\Lambda}),$$

where $\overset{\circ}{\Lambda} = \Lambda \cap T^*(\overset{\circ}{X} \times \overset{\circ}{X}) \subset (T^*\overset{\circ}{X} \setminus 0) \times (T^*\overset{\circ}{X} \setminus 0)$, as a set, is defined using Λ and (18). Therefore the distribution $P_\circ$ gives rise to a continuous linear operator denoted in the same way:

$$P_\circ : C_c^\infty(\overset{\circ}{X}) \longrightarrow C^\infty(\overset{\circ}{X}),$$

and whose formal adjoint enjoys the same mapping property (here and in what follows, the appropriate density bundles are suppressed in the notation; see [36] for details). Actually, if u is compactly supported, then $P_\circ$ can be extended into a continuous linear operator

$$u_\# : C^\infty(X) \longrightarrow C^\infty(X)$$

by the formula

$$u_\#(\varphi)(p) = u * (\varphi \circ r)(p), \quad \varphi \in C^\infty(X),\ p \in X. \tag{175}$$

This assertion is an easy consequence of the fact that $r : (G_b)_p \longrightarrow \overset{\circ}{X}$ is a diffeomorphism for any interior point p. Let $\dot{C}^\infty_{\mathrm{lb}\cup\mathrm{rb}}(X_b^2)$ be the space of C^∞ functions on X_b^2 vanishing to any order at the boundary faces lb, rb. It is well known [44] that the pushforward of distributions gives, for any $f \in \dot{C}^\infty_{\mathrm{lb}\cup\mathrm{rb}}(X_b^2)$, a continuous linear operator $f_\# : C^\infty(X) \longrightarrow C^\infty(X)$, which is again given by (175).

Definition 13. *Any linear operator $C^\infty(X) \longrightarrow C^\infty(X)$ equal to $u_\#$ for some u belonging to*

$$I_c^*(G_b, \Lambda) + \dot{C}^\infty_{\mathrm{lb}\cup\mathrm{rb}}(X_b^2)$$

for some family G_b-relation Λ is called a b-FIO on X.

Remark 33.
1. *Theorems 24, 25 and 32 extend immediately to the larger class of distributions above. Following the classical terminology of R. Melrose, we can then call this package of results a small calculus for b-FIOs.*
2. *Small calculi for FIOs with other behaviors near the boundary (for instance, the cusp-calculus, the ϕ-calculus, the fibred cusp calculus) are constructed in the same way using the appropriate blow-up spaces and groupoids.*

Obviously, the sub-family $(P_p)_{p\in Y}$ is the equivariant family associated with the ∂G_b-FFIO $i_0^* u$. Concretely, the operators P_p do not depend on $p \in Y$ and their common value is the Fourier integral operator

$$P_\partial : C_c^\infty(Y \times \mathbb{R}_+^*) \longrightarrow C^\infty(Y \times \mathbb{R}_+^*)$$

given by the natural action by convolution of $I(u)$. In particular, P_∂ commutes with dilations in $\mathbb{R}_+^*$.

7.7. **Comparison with the FIOs defined in** [41]**.** In [41], R. Melrose defined FIOs between manifolds with boundary N, M as Lagrangian distributions on the appropriate b-stretched $N \hat{\times} M$ and subordinate to *boundary canonical relations* [41, chap III, Definition 2.19]. In the case $N = M = X$, we are clearly dealing with the same space of distributions, and thus it just remains to compare *boundary canonical relations* with (family) G_b-relations.

Let Λ be a boundary canonical relation on X_b^2. This is a conic Lagrangian submanifold of $T^*X_b^2$ satisfying several additional conditions.

The first one, Condition (2.10) of [41, chap III, Definition 2.19], says that Λ does not intersect the left and right boundary of the cotangent space. This means that $\Lambda \subset T^*_{X_b^2\setminus(\mathrm{lb}\cup\mathrm{rb})}X_b^2$, and thus, Λ is a conic Lagrangian submanifold of T^*G_b through the identification given by Φ, which we fix in what follows.

The second one, Condition (2.15) of [41, chap III, Definition 2.19], coincides with our admissibility condition ("no zeros condition") and this proves that Λ is a $\mathcal{G}_b$-relation.

The third one, Condition (2.9) of [41, chap III, Definition 2.19], reads

$$T\Lambda + TT^*_{\partial G_b}G_b = TT^*G_b. \tag{176}$$

This is again the transversality Condition (173). Therefore, Λ is a family $\mathcal{G}_b$-relation.

We now analyse Condition (2.13) of [41, chap III]. We know that $\Lambda_\partial := i_0^*(\Lambda) \subset T^*\partial G_b$ is a family ∂G_b-relation. The fiber bundle considered in [41, chap III, (2.13)] is here

$$F = \{(p,q,t,\xi,\eta,\nu) \in T^*\partial G_b \ ; \ \nu = 0\} \longrightarrow T^*Y^2. \tag{177}$$

Thus the first part of Condition (2.13) consists in requiring that

$$TF + T\Lambda_0 = TT^*\partial G_b. \tag{178}$$

This is not a condition satisfied by arbitrary family G_b-relations. Let us give an example. We again identify $\mathbb{R}_+ \times \mathbb{R}_+^* \times Y^2$ and $(G_b)_{\mathcal{U}}^{\mathcal{U}}$ via Ψ. Let us fix a point $(p_1,p_2) \in Y^2$, and consider the set

$$V = \{(x,t,p_1,p_2) \ ; \ x \in \mathbb{R}_+, \ t \in \mathbb{R}_+^*\} \subset G_b. \tag{179}$$

Let $\mathcal{C}$ be the cone in $T^*G_b \setminus 0$ defined by

$$\mathcal{C} = \{(x,t,y_1,y_2,\tau,\nu,\xi_1,\xi_2) \in T^*(\mathbb{R}_+ \times \mathbb{R}_+^* \times Y^2) \ ; \ \epsilon|\xi_1| < |\xi_2| < \epsilon^{-1}|\xi_1|\} \tag{180}$$

for some $\epsilon > 0$. Then consider the intersection

$$\Lambda = N^*V \cap \mathcal{C} = \{(x,t,p_1,p_2,0,0,\xi_1,\xi_2) \ ; \ x \in \mathbb{R}_+, \ t \in \mathbb{R}_+^*, \ \epsilon|\xi_1| < |\xi_2| < \epsilon^{-1}|\xi_1|\}. \tag{181}$$

Let us check that Λ is a family $\mathcal{G}_b$-relation that does not satisfy (178). Firstly, Λ is obviously a conic Lagrangian submanifold. Then, using the expression of s_Γ and r_Γ in (169), one gets that Λ is a G_b-relation.

Next, for any boundary point $\gamma \in \partial G_b$, one has

$$T_\gamma \mathcal{F}_{G_b} = T_\gamma G_{b\,s(\gamma)} + T_\gamma G_b^{r(\gamma)} = \{0\} \times T(\mathbb{R}_+^* \times Y^2) \subset T_{\partial G_b}G_b. \tag{182}$$

Using the expression (181) we obtain immediately that

$$dp(T_{\gamma,\xi}\Lambda) + T_\gamma\mathcal{F}_{G_b} = T_\gamma G_b, \quad \forall(\gamma,\xi) \in \Lambda \text{ such that } \gamma \in \partial G_b. \tag{183}$$

Therefore, by Theorem 14 and the discussion following (173), we obtain that Λ is a family G_b-relation. Now consider

$$\Lambda_0 = i^*_{\partial G_b}\Lambda = \{(t,p_1,p_2,0,\xi_1,\xi_2) \ ; \ t \in \mathbb{R}_+^*, \ \epsilon|\xi_1| < |\xi_2| < \epsilon^{-1}|\xi_1|\} \subset T^*\mathrm{bf} \setminus 0. \tag{184}$$

We get

(185)
$$T\Lambda_0 = \{(t, p_1, p_2, 0, \xi_1, \xi_2; u, 0, 0, 0, \zeta_1, \zeta_2) \ ; \ t \in \mathbb{R}_+^*, \ \epsilon|\xi_1| < |\xi_2| < \epsilon^{-1}|\xi_1|, \ (\zeta_1, \zeta_2) \in T^*_{(p_1,p_2)} Y^2\}.$$

Since

(186)
$$TF = \{(t, p, 0, \xi; u, v, 0, \zeta) \ ; \ (t, u) \in T\mathbb{R}_+^*, \ (p, v) \in TY^2, \ \xi, \zeta \in T_p^* Y^2\},$$

we now see that $T\Lambda_0 + TF \neq T(T^*\partial G_b)$. Therefore Λ is not a boundary canonical relation in the sense of [41, chap III, Definition 2.19].

In conclusion, the class of Lagrangian submanifolds accepted here in the calculus of FIOs is slightly larger than the one obtained by boundary canonical relations. Moreover, the result on composition of FIOs given in [41] applies only to boundary canonical relations that are invertible in the terminology of the present paper. As we see in Theorem 25, this assumption can be ignored.

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