An infinite-dimensional Luenberger-like observer for vibrating membranes

Yang Yu, Cheng-Zhong Xu

An infinite-dimensional Luenberger-like observer for vibrating membranes

Yang YU[1] and Cheng-Zhong XU[2],

1. App. A213, 20 Rue Colin, 69100, Villeurbanne, France
E-mail: yy139540@gmail.com

2. University of Lyon, LAGEP, Bâtiment CPE
University Lyon 1, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne Cedex, France
E-mail: xu@lagep.univ-lyon1.fr

Abstract: The main objective of this paper consists in studying the dynamic and observation of a wave equation [1] in a bounded domain in the plan. This work is inscribed in the field of control of systems governed by partial differential equations (PDE). We consider the wave equation system with Dirichlet boundary condition whose dynamic evolves in an infinite-dimensional Hilbert space. We assume that velocity is measured on some subdomain along the boundary. An infinite-dimensional exponentially convergent Luenberger-like observer is presented to estimate the system state: displacement and velocity on the whole domain. The main contribution of the work consists in building a reliable numerical simulator based on the finite element method (FEM). We examine the influence of the gain on the convergence rate of the observer.

Key Words: wave equation, observer, convergence, finite element method

1 INTRODUCTION

In this paper, we propose a Luengerger-like observer for a distributed parameter system, which is described by the wave equation in a bounded open set Ω in the plan $\mathbb{R}^2$ ($\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ the boundary of the domain). Hence the wave equation describes dynamic of the vibrating membrane. The Luenberger-like observer that we present can estimate the state evolution of the vibrating membrane by measuring the velocity on a sub-domain ω along the boundary. The dynamical model of system is written by:

$$\begin{cases} W_{tt}(x,t) = \Delta W(x,t),\ \forall\ (x,t) \in \Omega \times \mathbb{R} \\ W(x,t) = 0,\ \forall x \in \partial\Omega \\ W(x,0) = W_0(x),\ W_t(x,t)|_{t=0} = W_1(x) \\ y(x,t) = W_t(x,t),\ \forall x \in \omega \end{cases} \tag{1}$$

where Δ is the Laplace operator ($\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$) and $W_t(x,t)$ denotes the firth partial derivative of $W(x,t)$ with respect to t. It's important to note that the output measurement $y(x,t)$ is the velocity of membrane in the subdomain $\omega \subset \Omega$. We assume that the choice of subdomain ω with non empty interior satisfies the exactly observable property of the system (1).

Then the Luenberger-like observer that we propose is governed by the following partial differential equation (see [2] and [6]) :

$$\begin{cases} \widehat{W}_{tt}(x,t) = \Delta\widehat{W}(x,t) - k\ \chi_\omega(x)(\widehat{W}_t(x,t) - y(x,t)) \\ \widehat{W}(x,t) = 0,\ \forall x \in \partial\Omega \\ \widehat{W}(x,0) = \widehat{W}_0(x), \widehat{W}_t(x,t)|_{t=0} = \widehat{W}_1(x) \end{cases} \tag{2}$$

where $k > 0$ positive constant is the gain of correction and $\chi_\omega(x)$ is the indicator function of sub-domain ω, i.e., $\chi_\omega(x) = 1$ if $x \in \omega$ and $\chi_\omega(x) = 0$ if $x \notin \omega$. Exponential convergence of the observer can be shown by proving exponential stability of the error system $\varepsilon(x,t) = \hat{W}(x,t) - W(x,t)$. Similar proofs have been given in several literatures for vibration equations (see [4] and [7]).

2 Variational formulation

Actually the FEM is resulted from an application of the variational principe. We begin by transforming the PDE

978-1-4799-7016-2/15/$31.00

into integral equations. The aim of this step is to find the weak solutions of the wave equation. Then we utilize the FEM to calculate the basis function. This step has been studied for 40 years and several examples for the Poisson equation ca be found in [3] [5]. So we will not repeat it here in the paper.

We assume that the solution of system (1) is written by

$$W(x,t) = \sum_{j=1}^{n} A_j(t)\phi_j(x) \quad (3)$$

where $\phi_j(x)$, $j = 1, ..., n$, are basis functions. The state space X is given by $X = H_0^1(\Omega) \times L^2(\Omega)$ equipped with the inner product

$$< f, g >= \int_\Omega [\nabla f_1(x) \cdot \nabla g_1(x) + f_2(x)g_2(x)]\, dx.$$

Note that $\nabla f_1(x)$ denotes the gradien of f_1 at x, i.e., $\nabla f_1(x) = [f_{1x_1}\ \ f_{1x_2}]'$ and that $H_0^1(\Omega)$ is the Sobolev s-pace defined by

$$H_0^1(\Omega) = \{v \in L^2(\Omega) | v_{x_i} \in L^2(\Omega), i = 1, 2, v|_{\partial\Omega} = 0\}.$$

Let us develop the variational formulation from the wave equation (1). For each $v \in H_0^1(\Omega)$, we have

$$-\iint_\Omega (\Delta W - \frac{\partial^2 W}{\partial t^2}) v dx_1 dx_2 = 0$$

$$\Rightarrow \iint_\Omega (\frac{\partial W}{\partial x_1}\frac{\partial v}{\partial x_1} + \frac{\partial W}{\partial x_2}\frac{\partial v}{\partial x_2} + \frac{\partial^2 W}{\partial t^2} v) dx_1 dx_2$$

$$- \iint_\Omega \frac{\partial}{\partial x_1}(v\frac{\partial W}{\partial x_1}) + \frac{\partial}{\partial x_2}(v\frac{\partial W}{\partial x_2}) dx_1 dx_2 = 0.$$

We recall the **Green's theorem**

$$\iint_\Omega div F dx_1 dx_2 = \oint_\Omega F \cdot \vec{n} ds$$

where $div F = \nabla \cdot F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$ and $\vec{n}$ is exterior normal vector.

$$\Rightarrow \iint_\Omega \nabla W \cdot \nabla v + \frac{\partial^2 W}{\partial t^2} v dx_1 dx_2 - \oint_{\partial\Omega} v \frac{\partial W}{\partial n} ds = 0.$$

It's clear that $\oint_{\partial\Omega} v\frac{\partial W}{\partial n} ds = 0$ because $v|_{\partial\Omega} = 0$. Then it follows from the above variational formulation that

$$\iint_\Omega \nabla W \cdot \nabla v dx_1 dx_2 + \iint_\Omega \frac{\partial^2 W}{\partial t^2} v dx_1 dx_2 = 0 \quad (4)$$

We substitute (3) for W in equation (4), and we suppose $v_h = span\{\phi_1, \phi_2, \cdots, \phi_n\} \subset H_0^1(\Omega)$. Then we obtain the following dynamical integral equations instead of the wave equation :

$$\sum_{j=1}^{n} \left\{ \left(\iint_\Omega \nabla\phi_j \cdot \nabla\phi_i dx_1 dx_2 \right) A_j(t) + \right.$$

$$\left. + \left(\iint_\Omega \phi_j \phi_i dx_1 dx_2 \right) \frac{d^2 A_j(t)}{dt^2} \right\} = 0$$

where $K = [k_{ij}] = \iint_\Omega \nabla\phi_j \cdot \nabla\phi_i dx_1 dx_2$ denote the **rigidity matrix** and $M = [m_{ij}] = \iint_\Omega \phi_j \phi_i dx_1 dx_2$ denote the **mass matrix**. Then the **dynamic equation** of system (1) is approximated by

$$\boxed{M\frac{d^2 A}{dt^2} + KA = 0}$$

The initial condition of (2.3) could be obtained by the same method from the system (1).

3 Simulation and performance of the observer

We simulate the system (1) and observer (2), by Matlab programming, with a rectangular membrane 0.5×0.4, uniform density then regular unstructured grid, $R_1 + R_2$ subdomain of the output in red frame as follows:

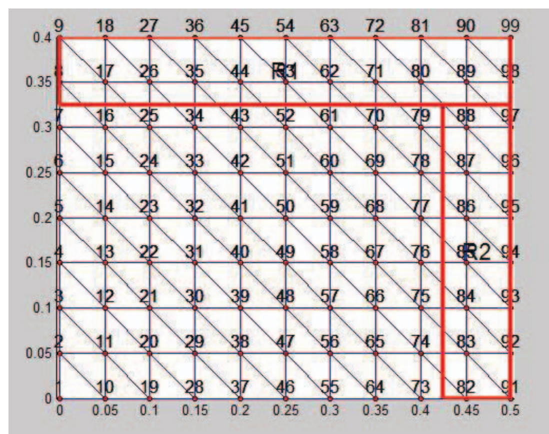


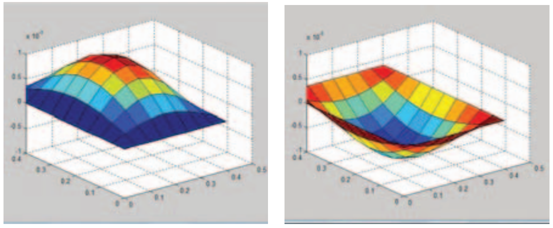
Figure 1: unstructured grid, 160 triangular elements, 99 nodes, $R_1 + R_2$ output subdomain

We have chosen a gain of correction $k = 1.01$ and vibrating amplitude $A = 0.001m$. The initial conditions of system (1) and observer (1) are defined as follows:

$$\begin{cases} W(x,0) = A sin(x_1\pi) sin(x_2\pi) \\ W_t(x,0) = 0 \end{cases} \quad (5)$$

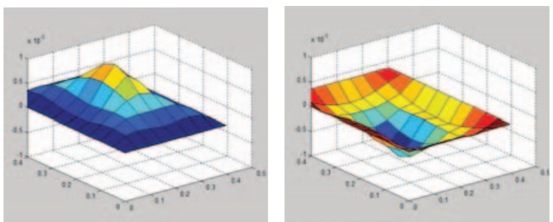
$$\begin{cases} \widehat{W}(x,0) = -A sin(x_1\pi) sin(x_2\pi) \\ \widehat{W}_t(x,0) = 0 \end{cases} \quad (6)$$

Then the evolution of vibrating membrane and observer are shown in several figures from 2 to 5 at $t = 0, t = 0.1s, t = 1s$ and $t = 30s$



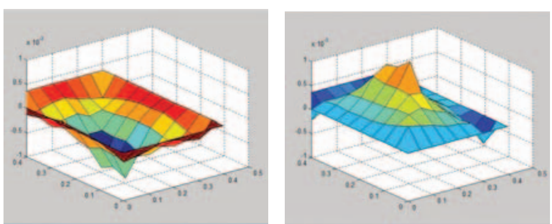
(a) vibrating system (b) observer

Figure 2: form at initial instant



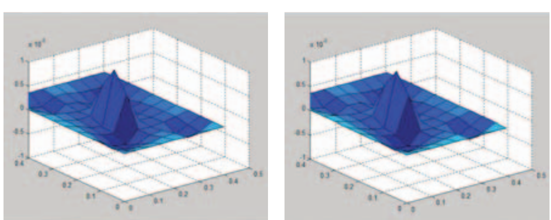
(a) vibrating system (b) observer

Figure 3: form at $t = 0.1s$



(a) vibrating system (b) observer

Figure 4: form at $t = 1s$



(a) vibrating system (b) observer

Figure 5: form at $t = 30s$

From these figures, we could conclude that the observer (2) is convergent to vibrating system (1) with domain measurement as ω if $t \geq 30s$. Therefore the estimated error is convergent to zero which is presented in next 2 figures, about displacements and velocities on the nodes from 40 to 60. Nevertheless there exist always some very low oscillations on nodes near the boundary.

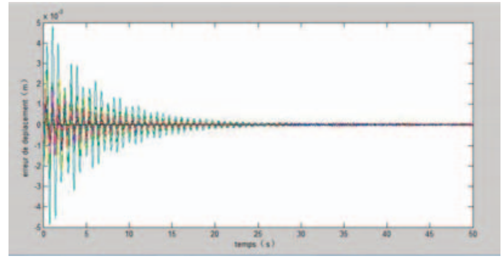


Figure 6: estimated errors of displacement for nodes from 40 to 60

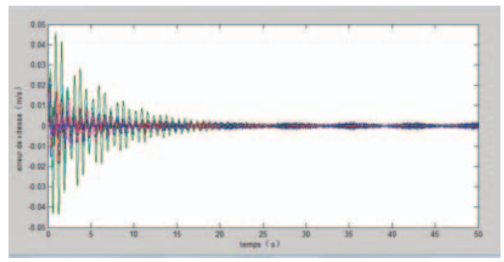


Figure 7: estimated error of velocity for nodes from 40 to 60

It's clear that the necessary time for the observer to catch the trajectory of vibrating system for node 50 is 25s. We present the estimated error at node 50 in 2 next figures.

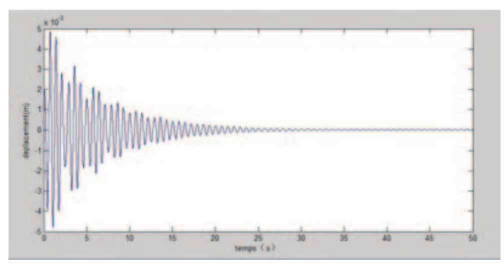


Figure 8: estimated error of displacement for node 50

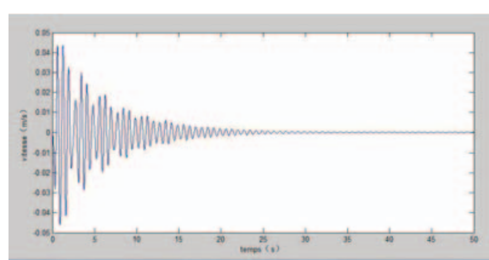


Figure 9: estimated error of velocity for node 50

The performance of the observer depends on the gain of correction. We enumerate the necessary time of catch for several gains in next table. We found that the best gain of correction is 0.5.

Gain	3	1.5	1.1	1.0011
Times	47.2	27.5	25.9	28

Gain	0.999	0.9	0.5	0.1
Times	29.5	30	25	47.3

And the performance of the observer depends also on the domain of observation. The larger the measured subdomain is, the faster the observer converges to the state. If the measurement subdomain is global, the time of catch will be 5s for node 50 in next figure.

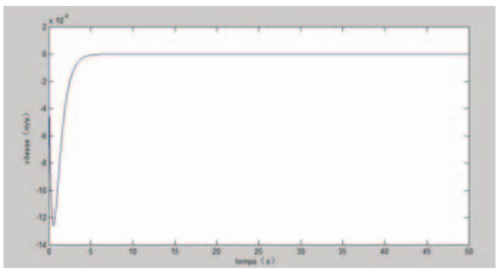


Figure 10: estimated error of velocity of node 50 with a global measurement

4 Conclusions and perspectives

We have proposed an infinite-dimensional Luenberger-like observer which allows to estimate the state of the wave system. And the numerical study has been carried out in order to develop a reliable simulator and test efficiency and performance of the observer. The main contribution of this work is to build a numerical simulator based on the finite element method. We have examined the influence of gain correction on the observer convergence rate.

For the simulation part, we have chosen the initial conditions well satisfying some compatibility condition to obtain very uniform and very smooth trajectories. We must emphasize that the choice is not mandatory. The result would be also good if we use an initial condition having no compatibility condition satisfied.

In the outlook, we shall complement the theoretical part of the study in a separate paper, and secondly take into account the non-linearity of the model. In the construction of the observers, we also wish to extend the study to other dynamic models in continuum mechanics as well as the wave equation in three dimensions. The stabilization's problem for 1D wave equation has been studied in [8], so we could extend the ideas for studying our membrane system.

REFERENCES

[1] C.H. GU, S.X. CHEN and Y.J. TAN, *Mathimatical and physical equation* (in chinese), ISBN 7-04-010701-5, chap.1, second approach, 2002.

[2] M. DAMETRIOU, Natural second-order observers for second-order distributed parameter system, *Systems & Control Letters*, 225-234, 2004.

[3] A. LE POUCHIET, *La résolution numérique des équations aux dérivées partielles*, ISBN 2-85428-175-6, une première approche, pages 356-363, 1988.

[4] K.S. LIU, Locally distributed control and damping for the conservative systems, *SIAM J. Control Optimization*, Vol. 35, 1574-1590, 1997.

[5] F. HARTMAN and C. KATZ, *Structural analysis with finite elements*, ISBN-13978-3-540-49698-4, 2003.

[6] A.J. DEGUENON, G. SALLET and C.Z. XU, Exponentially convergent observers of infinite-dimensional skew-adjoint systems with applications, *The proceedings of the Second International Symposium on Communications, Control and Signal Processing*, Marrakech, Morrocco, 2006.

[7] X.D. LI and C.Z. XU, Infinite-dimensional Luenberger-like observers for a rotating body-beam system, *Systems & Control Letters*, Vol.60, 138-145, 2011.

[8] B.Z. GUO and C.Z. XU, The stabilization of a one-dimensional wave equation by boundary feedback with non-collocated observation, *IEEE TAC*, Vol.52, 371-377, 2007.