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W. P. THURSTON AND FRENCH MATHEMATICS

FRANÇOIS LAUDENBACH AND ATHANASE PAPADOPOULOS,

WITH CONTRIBUTIONS BY WILLIAM ABIKOFF, NORBERT A'CAMPO, PIERRE ARNOUX, MICHEL BOILEAU, ALBERT FATHI, DAVID FRIED, GILBERT LEVITT, VALENTIN POÉNARU, HAROLD ROSENBERG, FRANCIS SERGERAERT, VLAD SERGIESCU AND DENNIS SULLIVAN

Abstract. We give a general overview of the influence of William Thurston on the French mathematical school and we show how some of the major problems he solved are rooted in the French mathematical tradition. At the same time, we survey some of Thurston's major results and their impact. The final version of this paper will appear in the Surveys of the European Mathematical Society.

AMS classification: 01A70; 01A60; 01A61; 57M50; 57R17; 57R30

Keywords: William P. Thurston: Geometric structures; Hyperbolic structures; Haefliger structures; low-dimensional topology; foliations; contact structures; history of French mathematics.

Part 1

1. Prologue

Seven years have passed since Bill Thurston left us, but his presence is felt every day in the minds of a whole community of mathematicians who were shaped by his ideas and his completely original way of thinking about mathematics.

In 2015-2016 a two-part celebration of Thurston and his work was published in the *Notices of the AMS*, edited by Dave Gabai and Steve Kerckhoff with contributions by several of Thurston's students and other mathematicians who were close to him [15]. Among the latter was our former colleague and friend Tan Lei, who passed away a few years later, also from cancer, at the age of 53. One of Tan Lei's last professional activities was the thesis defense of her student Jérôme Tomasini in Angers, which took place on December 5, 2014 and for which Dylan Thurston served on the committee. Tan Lei was suffering greatly at that time; her disease was diagnosed a few days later. She confessed to François Laudenbach that when she received the galley proofs of the article on Thurston which included her contribution, she was sad to realize that there was hardly any mention of Thurston's influence on the French school of mathematics, and in particular on the Orsay group. In several subsequent phone calls with Laudenbach she insisted that this story needed to be told. Our desire to fulfill her wish was the motivation for the present article. We asked for the help of a number of colleagues who were involved in the mathematical activity "in the tradition of Thurston" that took place (and continues to take place) in France. Most of these colleagues responded positively and sent us a contribution, either in the form of a short article or as an email, everyone in his personal style and drawing from his own memory. Sullivan, after we showed him our article, proposed to include a text of his which previously appeared in the *Notices*, considering that his text is complementary to ours and would naturally fit here. We happily accepted. All these contributions are collected in the second part of this article. We have included them as originally written, although there are a few minor discrepancies in some details pertaining to the description or the precise order of the events.

In the first part of this article we have tried to give a general overview of the influence of Thurston on the French mathematical community. At the same time we show how some of the major problems he solved have their roots in the French mathematical tradition.

Saying that Thurston's ideas radically changed the fields of low-dimensional geometry and topology and had a permanent effect on related fields such as geometric group theory and dynamics, is stating the obvious. On a personal level we remember vividly his generosity, his humility and his integrity.

We view this article as another general tribute to Thurston, besides being a chronicle of the connection of his work with that of French mathematicians. It is not a survey of Thurston's work—such a project would need several volumes, but it is a survey on the relation between Thurston and French mathematics. At the same time, we hope that it will give the reader who is unfamiliar with Thurston's œuvre an idea of its breadth.

2. *Vita*

It seemed natural to us to start with a short *Vita*. William Paul Thurston was born in Washington DC on Oct. 30, 1946 and he died on August 21, 2012. His father was an engineer and his mother a housewife. He went to Kindergarten in Holland and then entered the American educational system.

Thurston's parents were wise enough to let him choose the college he wanted to attend after he completed his secondary education. In 1964, he entered New College, a small private college in Sarasota (Florida). The college was newly founded. In a commencement address he gave there in 1987 (that was twenty years after his graduation), Thurston recalls that he landed in New College because, by chance, he read an advertisement for that college, amidst the large amount of literature which used to fill the family mail box every day. In that advertisement, two statements of educational principles attracted his attention. The first one was: "In the final analysis, each person is responsible for her or his education," and the second one: "The best education is the collaboration/conflict between two first-class minds." For him, he says, these statements looked like "a declaration of independence, of a sort of freedom from all the stupidities of all the schools that [he has] sort of rebelled against all [his] life." Thurston recalled that in grade school he was spending his time daydreaming and (maybe with a small amount of exaggeration) his grades were always Cs and Ds. In Junior high school, he became more rebellious with his teachers and was traumatised by the fact that these teachers, because they were in a position of authority, were supposed never to make mistakes. He kept fighting, as a child and then as a teenager, with the American educational system and he could not adapt to it until he went to college.

In New College, there was a focus on independent study, and writing a senior thesis was one of the important requirements. Thurston was interested in the foundations of mathematics and this motivated him to write a thesis on intuitionist topology. The title of the thesis, submitted in 1967, is: "A Constructive Foundation for Topology." He was particularly attracted by intuitionism, not only as a topic for a senior thesis, but he thought he might become an intuitionist logician. This is why, when he entered the University of California, Berkeley, he approached Alfred Tarski, the charismatic logician and mathematics teacher there, and asked him to be his advisor. Tarski told the young Thurston that Berkeley was not a good place for intuitionism. Thus, Thurston went instead to topology, the other theme of his senior thesis.

Thurston, especially in his last years, used to actively participate in discussions on mathematics and other science blogs, and reading his contributions gives us some hints on his way of thinking. The key word that is recurrent in his writings about his view on the goal of mathematics is "understanding". In a post dated May 17, 2010, he writes:

> In my high school yearbook, I put as my goal "to understand", and I've thought of that as summing up what drives me. My attitude toward the demarcation problem originated I think from childhood games my siblings and I used to play, where one of us would say something obviously implausible about the world, as if psychotic, and the others would try to trip up the fantasy and establish that it couldn't be right. We discovered how difficult it is to establish reality, and I started to think of these battles as futile. People of good will whose thinking is not confused and muddled or trapped in a rut can reach a common understanding. In the absence of good will or clarity, they do not, and an external criterion or external referee does not help.
>
> [...] I also used to think I would switch to biology when I reached the age of 35 or 40, because I was very drawn to the challenge of trying to understand life. It didn't happen.

Thurston received his PhD in 1972; his thesis is entitled *"Foliations of Three-Manifolds which are Circle Bundles"*. He was appointed full professor at Princeton in 1974 (at the age of 27), and he remained there for almost 20 years. He then returned to California where he joined the University of California, Berkeley (1992–1997), acting also as the director of MSRI, then the University of California, Davis (1996–2003). After that, he moved to Cornell where he spent the last 9 years of his life.

In the mid-1970s, Thurston formulated a conjecture on the geometry of 3-manifolds which is the analogue in that dimension of the fact that any surfaces (2-manifold) carries a metric of constant curvature. (The 2-dimensional statement is considered as a form of the uniformization theorem.) The conjecture, called Thurston's conjecture, says tat any 3-dimensional manifold can be decomposed in a canonical way into pieces such that each piece carries one of eight types of geometric structures that became known as Thurston geometries. The 3-dimensional geometrization conjecture is wider than the Poincaré conjecture (actually a question formulated by Poincaré in 1904 at the very end of the *Cinquième complément à l'Analysis Situs*), saying that a simply-connected 3-dimensional manifold without boundary is homeomorphic to the 3-sphere.[1] Thurston's geometrization conjecture was proved in 2002 by Grigory Perelman. At a symposium held in Paris in 2010 celebrating the proof of the Poincaré conjecture, Thurston recalls: "At a symposium on Poincaré in 1980, I felt emboldened to say that the geometrization conjecture put the Poincaré conjecture into a fuller and more constructive context." He then adds: "I expressed confidence that the geometrization conjecture is true, and I predicted it would be proven, but whether in one year or 100 years I could not say—I hoped it would be within my lifetime. I tried hard to prove it. I am truly gratified to see my hope finally become reality."

In 1976 Thurston was awarded the Oswald Veblen Geometry Prize of the American Mathematical Society for his work on foliations, in 1982 the Fields Medal and in 2012 the Leroy P. Steele Prize of the American Mathematical Society.

Thurston introduced a new way of communicating and writing mathematics. He had a personal and unconventional idea on what mathematics is about and why we do mathematics, and he tried to share it. On several occasions, he insisted that mathematics does not consist of definitions, theorems and proofs, but of ways of seeing forms and patterns, of internalizing and imagining the world, and of thinking and understanding certain kinds of phenomena. He was attached to the notion of mathematical community. After he finished college, he realized the existence of such a community, and this appeared to him like a revelation. In 2012, in his response to the Leroy P. Steele Prize, he declared: "I felt very lucky when I discovered the mathematical community—local, national and international—starting in graduate school." In a post on mathoverflow (October 30, 2010), he wrote:

> Mathematics only exists in a living community of mathematicians that spreads understanding and breathes life into ideas both old and new. The real satisfaction from mathematics is in learning from others and sharing with others. All of us have clear understanding of a few things and murky concepts of many more. There is no way to run out of ideas in need of clarification. The question of who is the first person to ever set foot on some square meter of land is really secondary. Revolutionary change does matter, but revolutions are few, and they are not self-sustaining—they depend very heavily on the community of mathematicians.

Thurston saw that a school of thought sharing his geometric vision was gradually growing. In the same response to the Leroy P. Steele Prize, he declared:

> I used to feel that there was certain knowledge and certain ways of thinking that were unique to me. It is very satisfying to have arrived at a stage where this is no longer true—lots of people have picked up on my ways of thought, and many people have proven theorems that I once tried and failed to prove."

[1]Poincaré made the following comment: "Cette question nous entraînerait trop loin", that is, *This question would take us too far.*

3. Foliations

Thurston made his first visit to the Orsay department of mathematics in 1972, the year he obtained his PhD. He was invited by Harold Rosenberg, who was a professor there. Orsay is a small city situated south of Paris, about 40 minutes drive from Porte d'Orléans (which is one of the main south entrances to Paris) with the usual traffic jam. The Orsay department of mathematics was very young; it was created in 1965 as part of a project to decentralize the University of Paris, and in 1971 it had become part of the newly founded Université de Paris-Sud. Thurston was working on foliations and France was, at that time, the world center for this topic. Specifically, the theory was born some two-and-a-half decades earlier in Strasbourg, where a strong group of topologists had formed around Charles Ehresmann, including René Thom, Georges Reeb, André Haefliger and Jean-Louis Koszul. Reeb's PhD thesis, entitled *Propriétés topologiques des variétés feuilletées* [38] which he defended in 1948, may safely be considered to be the birth certificate of foliation theory. Reeb described there the first example of a foliation of the 3-sphere, answering positively a question asked by Heinz Hopf in 1935. This question was communicated to Reeb by his mentor, Ehresmann (Reeb mentioned this several times). On the other hand, examples of the use of foliations of surfaces can be traced back to the early works on cartography by Ptolemy and others before him who searched for mappings of the 2-sphere onto a plane where the foliation of the sphere by parallels or by longitudes is sent to foliations of a planar surface satisfying certain *a priori* conditions (circles, ellipses, straight lines, lines that interpolate between circles near the North Pole and straight lines near the South Pole, etc.)

At the beginning of the 1970s the theory of foliations was a hot research topic among topologists and dynamicists at Orsay. The results that Thurston obtained during his graduate studies and the years immediately following (ca. 1970–1975) constituted a striking and unforeseeable breakthrough in the field. In this period of 4 or 5 years, he solved all the major open problems on foliations, a development which eventually led to the disappearance of the Orsay foliation group.

During his 1972 visit, Thurston lectured on his version of the *h*-principle—as it is now called—for foliations of codimension greater than 1, and in particular on his result saying that an arbitrary field of 2-planes on a manifold of dimension at least four is homotopic to a smooth integrable one, that is, a field tangent to a foliation. Several young researchers interested in foliations attended the lectures, including Robert Roussarie, Robert Moussu, Norbert A'Campo, Michel Herman and Francis Sergeraert.

The Godbillon-Vey invariant (GV) had been born the year before in Strasbourg,[2] but it was even unknown whether the invariant could be nonzero. Informed of this during a meeting at Oberwolfach which took place on May 23-29, 1971, Roussarie immediately found an example of a foliation with a non-zero GV invariant, namely the horocyclic foliation on a compact quotient of $SL_2(\mathbb{R})$. Shortly after, Thurston proved the much stronger result saying that there exists a family of foliations whose GV take all possible real

[2]Claude Godbillon, a former student of Reeb, was a professor there, and the discovery was made during a visit to him by Jacques Vey (1943–1979) who was a young post-doc.

values. He also gave a geometric interpretation of the GV class of a foliation F as a "helical wobble of the leaves of F". The paper was published in 1972 [48]. This was one of the first papers that Thurston published on foliations (in fact, it was his first paper on the subject after his thesis, the latter of which remains unpublished). Thurston sent the preprint of his paper to Milnor. On November 22, 1971, Milnor responded with a 5 page letter. We have reproduced here the first page of that correspondence. Thurston spent the next academic year at Princeton's Institute for Advanced Study, at the invitation of Milnor. It is interesting to note that the hyperbolic plane already appears in Thurston's paper as a central object. Orbifolds also appear in the background (Thurston calls them "surfaces having a number of isolated corners, with metrics of constant negative curvature everywhere else"). Shortly thereafter Sullivan, who was working at IHÉS in Bures-sur-Yvette (a few minutes walk from Orsay), gave another interpretation of the GV invariant using a notion of linking number for currents; this appeared in his 1976 paper *Cycles for the dynamical study of foliated manifolds and complex manifolds* [44].

During his 1972 stay in France, Thurston also visited Dijon, and he then went to Switzerland to participate in a conference on foliations at Plans-sur-Bex, a village in the Alps.[3] A'Campo recalls that at that time, Thurston was already thinking about hyperbolic geometry in dimension two. He asked Thurston how he came to know about this subject, and Thurston's answer was that his father first told him about it.

Thurston's most remarkable result during the period that followed his Orsay visit is probably the proof of the existence of a C^∞ codimension-one foliation on any closed manifold with zero Euler characteristic. The result was published in 1976 [52], and it solved one of the main conjectures in the field. Before that, there were only particular examples of foliations of spheres and some other particular manifolds. Sullivan recalls that the first new foliation of spheres after Reeb's was constructed by Lawson in Bahia in 1971, and Verjovsky helped in that. Other particular examples of foliations of special manifolds, sometimes restricted to a single dimension, were constructed by A'Campo, Durfee, Novikov and Tamura. On the other hand Haefliger had proved a beautiful and influential result saying that there is no codimension-one real-analytic foliation on a sphere of any dimension [19]. This emphasizes the fact that unlike manifolds, a smooth one-dimensional non-Hausdorff manifold—as is, in general, the space of leaves of a foliation—does not carry any analytic structure. It is interesting to note here that each of Haefliger and Reeb, at a conference in Strasbourg in 1944–1955, presented a paper on non-Hausdorff manifolds considered as quotient spaces of foliations [18, 39].

[3] In an obituary article on Michel Kervaire, written by S. Eliahou, P. de la Harpe, J.-C. Hausmann and M. Weber and published in the *Gazette des Mathématiciens* [12], the authors write that Haefliger knew that a young student from Berkeley had obtained remarkable results on foliations. The news that a week on foliations would be organized spread rapidly, and the organisers were almost refusing people because the housing possibilities were limited. There were not enough chairs in the classroom of the village school for all the participants but they managed to find one for him. It goes without saying that this student was William Thurston.

THE INSTITUTE FOR ADVANCE

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SCHOOL OF MATHEMATICS

Dr. William Thurston
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Dear Thurston:

Thanks for your preprint, which is very p
on this to the topology seminar here, I noticed se
I would like to write these out, in part just to get
mind.

1. It seems to me advantageous to define
Godbillon Vey invariant. Suppose for example th
codimension q foliation on M, which is defined
infinity" by a closed q-form ω_∞. Then one can
defining the foliation so that $\omega = \omega_\infty$ near infinity
g is a 1-form with compact support, and setting
gv now represents a well defined cohomology cla

This construction can of course be applied
be a bounded convex polygonal region in H^2 with
Q be the double of P, and T_1Q its unit tangent

In his paper [49] published in 1974, Thurston obtained a generalization of the so-called Reeb stability theorem. This theorem, proved by Reeb twenty-eight years earlier in his thesis [38], says that if a codimension-one foliation of a compact manifold has a two-sided compact leaf with finite fundamental group, then all the leaves of the foliation are diffeomorphic. Thurston showed that in the case of a C^1 foliation one can replace the hypothesis that the compact leaf has finite fundamental group by the much weaker one saying that the first real cohomology group of the leaf is zero. He also gave a counterexample in the case where the smoothness condition is not satisfied. As a corollary, he showed that there are many manifolds with boundary that do not admit foliations tangent to the boundary.

At the same time, Thurston proved in [51] a series of breakthrough results on codimension-k Haefliger structures when $k > 1$. Such a structure, introduced (without the name) by Haefliger in his thesis [20], is a generalization of a foliation: it is an $\mathbb{R}^k$-bundle over an n-dimensional manifold equipped with a codimenson-k foliation transverse to the fibers of the bundle. (The normal bundle to a foliation is naturally equipped with such a structure.)

Two years later, Thurston solved the tour de force case of codimension-one: every hyperplane field is homotopic to the tangent plane field of a C^∞-foliation [52]. In the same paper, he writes that "the theory of analytic foliations still has many unanswered questions."

Thurston gave a talk at the 1974 ICM (Vancouver) whose title is *On the construction and classification of foliations.* The proceedings of this congress contain a short paper (3 pages) [50] in which he states his major results. The definition he gives of a foliation is unusual, but it delivers the meaning of the object defined better than any formal definition: "A foliation is a manifold made out of a striped fabric—with infinitely thin stripes, having no space between them. The complete stripes, or 'leaves', of the foliation are submanifolds; if the leaves have codimension k, the foliation is called a codimension-k foliation".

After Thurston proved his series of results on foliations, the field stagnated. In his article *On proof and progress in mathematics* [53] Thurston wrote:

> Within a couple of years, a dramatic evacuation of the field started to take place. I heard from a number of mathematicians that they were giving or receiving advice not to go into foliations—they were saying that Thurston was cleaning it out. People told me (not as a complaint, but as a compliment) that I was killing the field. Graduate students stopped studying foliations.

Thurston was never proud of this, and it was not his intention to kill the subject. In the same article he explained that on the contrary he was sorry about the fact that many people abandoned the field and he thought the new situation arose out of a misunderstanding, by a whole group of mathematicians, of the state of the art of foliations. As a matter of fact, in the introduction to his 1976 paper [52] which supposedly killed the field, Thurston declares that further work on the subject is called for, especially using the geometrical methods of his predecessors and unlike his own method which, according to his words, is local and has the disadvantage that it is hard to picture the foliations constructed.

Thurston might be pleased to know that some problems on foliations that he was interested in have been revived recently. For instance, G. Meigniez recently revisited Thurston's ideas and obtained new unexpected results [33]. In particular, he showed that there exist minimal, C^∞, codimension-one foliations on every closed connected manifold of dimension at least 4 whose Euler characteristic is zero. Since by definition every leaf of a minimal foliation is dense, this proves that there is no generalization to higher dimensions of Novikov's 3-dimensional compact leaf theorem [36] (1965).

4. Contact geometry

At the same time he was working on foliations, Thurston obtained a number of important results on contact geometry. This subject is close to the theory of foliations even though, by definition, a contact structure is very different from a foliation: it is a hyperplane field that is maximally far from being integrable, that is, of being tangent to a foliation.

When Thurston started working on contact structures, the theory, like that of foliations, was already well developed in France. As in the case of the theory of foliations, a group of topologists in Strasbourg had been working on contact structures since the second half of the 1940s, under the guidance of Ehresmann. In particular Wu Wen Tsu, who was a young researcher in Strasbourg at the time, published two papers in 1948 in which he studied the existence of contact structures (as well as almost-complex structures) on spheres and sphere bundles. His motivation came from some problems on characteristic classes, a subject on which he was then competing with Thom.

In the 1970s, contact geometry was one of the favorite objects of study at the mathematics institute in Strasbourg. Robert Lutz, Jean Martinet and others were working on it under the guidance of Reeb. In his thesis defended in 1971, Lutz proved that every homotopy class of co-orientable plane fields on the 3-sphere contains a contact structure. In the same year, Lutz and Martinet, improving techniques used by Lutz in his thesis, showed that every closed orientable 3-manifold supports a contact structure. In 1975, Thurston came in. He published a paper with H. E. Winkelnkemper [59] which gives an amazingly short proof (less than one page) of the result of Lutz and Martinet by using the so-called *open-book decomposition* theorem of Alexander. Several years later, and still in dimension three, Emmanuel Giroux obtained a much more difficult result, namely, any contact structure is *carried* by an open-book decomposition.

Together with Eliashberg, Thurston later developed the notion of *confoliation* in dimension three and techniques of approximating smooth foliations by contact structures, thus further strengthening the links between the two subjects.

In higher dimensions, contact structures are much more complicated (hypotheses are needed for existence) and the complete picture is still not well understood. Nevertheless, Giroux over the course of several years obtained a generalization to higher dimensions of his theorem that we mentioned above in dimension three, using ideas originating in Donaldson's asymptotically holomorphic sequences of sections adapted to contact structures by Ibort, Martinez-Torres and Presas.

Thurston continued to think about contact structures. In contrast with the theory of foliations, the theory of contact structures is still extremely active in France.

A further close relation between contact structures, Thurston and French mathematics is given by the Bennequin–Thurston[4] invariant of a Legendrian knot, which describes its amount of coiling. This in turn gave rise to the Bennequin–Thurston number of a knot, which maximizes the Bennequin–Thurston invariant over all Legendrian representatives; these invariants were found independently by Thurston and by Daniel Bennequin. An inequality conjectured by Thurston and proved by Bennequin in his thesis (1982) has since become known as the Bennequin–Thurston inequality.

Thurston had a very personal way of explaining contact structures (and the same can be said regarding almost any topic that he talked about). The chapter titled "Geometric manifolds" of his monograph *Three-dimensional geometry and topology* [55] contains a section dedicated to contact structures. This comes between the section on bundles and connections and the one on the eight geometries. Thurston spends several pages trying to give an intuitive picture of contact structures, because, he says, "they give an interesting example of a widely occurring pattern for manifolds that is hard to see until your mind and eyes have been attuned." On p. 172 of this monograph, he writes:

> You can get a good physical sense for the contact structure on the tangent circle bundle of a surface by thinking about ice skating, or bicycling. A skate that is not scarping sideways describes a Legendrian curve in the tangent circle bundle to the ice. It can turn arbitrarily, but any change of position is in the direction it points. Likewise, as you cycle along, the direction of the bicycle defines a ray tangent to the earth at the point of contact of the rear wheel. Assuming you are not skidding, the rear wheel moves in the direction of this ray, and this motion describes a Legendrian curve in the tangent bundle of the earth.
>
> Young children are sometimes given bicycles with training wheels, some distance off to the side of the rear wheel. The training wheel also traces out a Legendrian curve—in fact, for any real number t, the diffeomorphism ϕ_t of $\mathbb{R}^2 \times S^1$ that takes a tangent ray a signed distance t to the left of itself is a contact automorphism. The training wheel path is the image of the rear wheel path under such a transformation. Note that this transformation applied to curves in the plane often creates or removes cusps. The same thing happens when you mow a lawn, if you start by making a big circuit around the edge of the lawn and move inward [...]

We have quoted this long passage because it is characteristic of Thurston's style, that of providing mental images with analogies borrowed from the real

[4]The story of this discovery (which is not joint work, but work in parallel, by Thurston and Bennequin) is intricate, and we were not able to reconstruct the exact chronology. Thus, we decided to follow the alphabetical order.

world. At the same time, it may be appropriate to recall Thurston's warning that "one person's clear mental image is another person's intimidation" [53].

5. Hyperbolic geometry, surfaces and 3-manifolds

Among the important classical subjects that Thurston revived one finds the study of hyperbolic structures on surfaces and 3-manifolds. This brings us to the second sensational piece of Thurston's work that had a long-lasting impact on the Orsay group of geometry, namely, his work on surface mapping class groups, Teichmüller spaces, and the geometry and topology of 3-manifolds.

The idea to launch the Orsay seminar known as *Travaux de Thurston sur les surfaces* came from Valentin Poénaru. At the 1976 "Autumn cocktail" of IHÉS, he showed up with a set of notes by Thurston, whose first page is shown in Figure 2. The notes contained the outline (definitions, pictures, and the statements of results) of what became known later as Thurston's theory of surfaces.[5] These notes were published several years later in the Bulletin of the AMS under the title *On the geometry and dynamics of diffeomorphisms of surfaces* [54]. Figure 3 shows the drawings by Thurston at the end of his preprint.

The Orsay seminar on the works of Thurston on surfaces took place in the academic year 1976-77. Thurston never attended this seminar. At some point the seminar members were stuck with the problem of gluing the space of projective measured foliations to Teichmüller space as a boundary, and they asked him for assistance. At that time there was no email, only postal mail. They finally managed to work out a complete answer only when Thurston attended another session of the seminar at Plans-sur-Bex in 1978.[6]

The final result of this seminar was published in book form in 1979 in *Astérisque*, and inspired many young topologists and geometers. One feature of the book was a first course in hyperbolic geometry, a topic for which, at that time, there were very few reference books. Some important notions that were introduced by Thurston shortly after the seminar took place were missing, in particular *train tracks* and *geodesic laminations*. We learned about them after Thurston's manuscript on surfaces reached Orsay, in the mimeographed notes of his Princeton 1978 course on the topology and geometry of 3-manifolds. These were certainly new and fundamental concepts, but the theory of surfaces worked quite well without them.

Thurston's notes on 3-manifolds arrived in France in installments and their importance was immediately realized by those who had been following his work. They were xeroxed and bound chapter by chapter, in several dozen copies. They were made available in the secretarial office of the Orsay topology research group, to all the members of the group, but also to the

[5]In reality, Thurston had been thinking about simple closed curves on surfaces ever since he was a student in Berkeley; see the Second Story in the recollections by Sullivan in Part II of the present paper.

[6]In the obituary article on Kervaire cited previously [12], the authors list the foreign participants at that meeting as A. Connes, D. Epstein, M. Herman, D. McDuff, J. Milnor, V. Poénaru, L. Siebenmann, D. Sullivan and W. Thurston.

On the Geometry and Dynamics of Diffeomorphisms of Surfaces

William P. Thurston

§1. This paper is a description of some results about diffeomorphis
surfaces, and the topology of surfaces. Proofs are deferred to the
If M^2 is a surface, we will denote by $\mathcal{S}(M^2)$ = the set of al
closed curves on M^2, up to isotopy. It is easy to see that, for mo
elements of $\mathcal{S}(M^2)$ can be quite complicated (cf. Fig. 1). In worki
simple closed curves, one gets the sense of some geometric concep
"nearness" among them, not closely related to the homotopy class,
to do with how many strands pass around in a certain direction. We
shortly formalize such a concept, defining a completion of $\mathcal{S}(M^2)$,
denoted $\mathcal{PS}(M^2)$. It turns out that $\mathcal{PS}(M^2)$ is homeomorphic to a
and $\mathcal{S}(M^2)$ consists of a dense set of "rational" points on this sph
the coordinates of such a point, the corresponding simple closed cu
drawn, rather mechanically, and quickly by a computer.

$\mathcal{PS}(M^2)$ has two other interpretations. First, it is the sp
certain type of foliation of M^2, up to an equivalence relation. Sec
forms a boundary for the Teichmuller space of M^2, to which the a
group of diffeomorphisms of M^2 extends.

With the aid of this tool, a canonical representative is fou
isotopy class of diffeomorphisms of M^2, well-defined up to conju

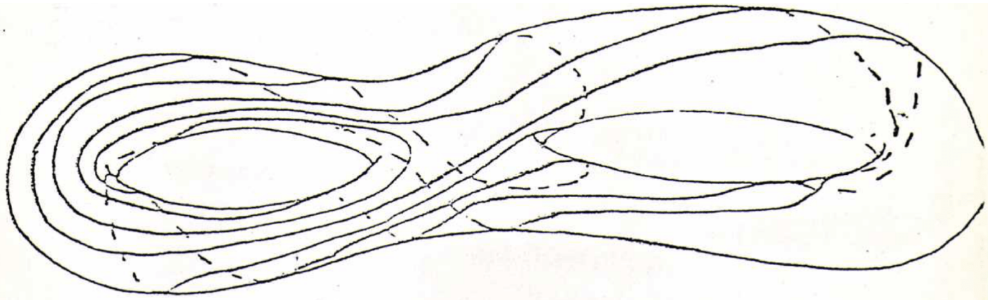
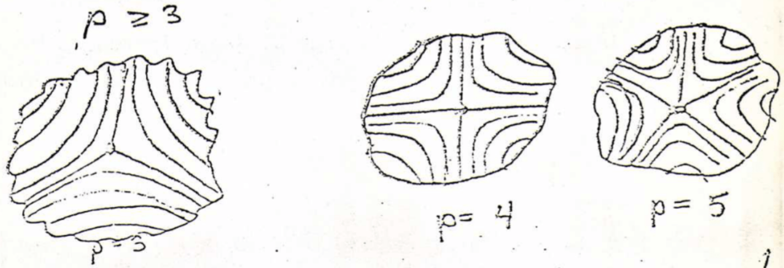


Fig. 1. a typical simple closed curve

Fig 2. Interior Singularities of measured foliations are p-pronged saddles, $p \geq 3$



Singularities of a measured foliation on ∂M^2:

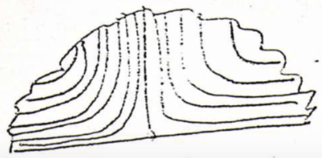
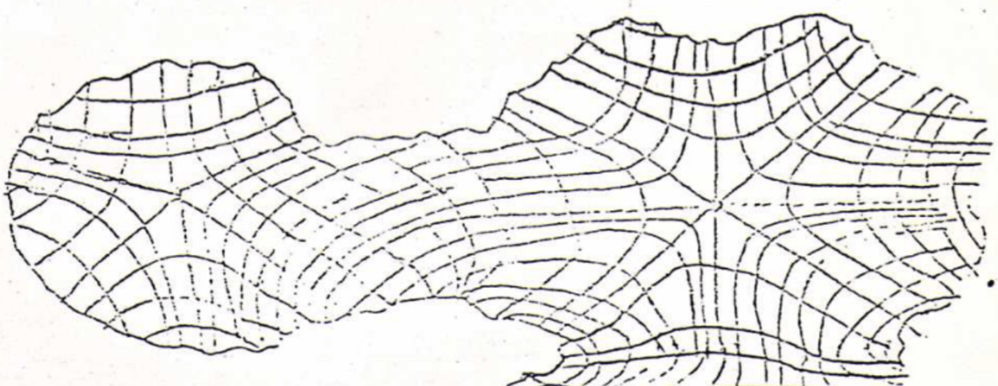


Fig 3. A pair of transverse measured foliations near singular points:



other mathematicians who were curious about the theory. In those days the Orsay topology seminar attracted a large number of mathematicians from all over France, and for many of them, the secretarial office was a necessary passage; they had to sign papers there. In this way the whole community of French topologists became aware of these notes.

During the year that the *Orsay seminar* took place and during several years after that, a significant number of foreign mathematicians visited Orsay and gave lectures and courses on topics related to Thurston's ideas. These included Bill Abikoff, Lipman Bers, David Epstein, David Fried, John Morgan, Peter Shalen, Mike Shub, and many others. Joan Birman came with two students, John McCarthy and Józef Przytycki. Bob Penner also visited as a student. Most of these mathematicians continued to maintain strong relations with their French colleagues. A'Campo and Poénaru gave graduate courses that were attended by many students and colleagues. Starting in the early 1980s, several doctoral dissertations were defended at Orsay and Paris 7 in which the authors had benefitted from the book issued from the Orsay seminar, the mimeographed notes and the courses given at Orsay. Among the early graduates were Gilbert Levitt (Thèse d'État 1983), Claude Danthony (PhD thesis 1986) and Athanase Papadopoulos (Thèse d'État 1989), who worked on surfaces. Francis Bonahon (Thèse d'État 1985), Michel Boileau (Thèse d'État 1986) and Jean-Pierre Otal (Thèse d'État 1989) worked on 3-manifolds. Otal later wrote a book which became a standard reference for Thurston's hyperbolization theorem for fibered 3-manifolds.

After obtaining their doctoral degrees, these young geometers obtained jobs in various places in France—it was a period of "decentralization" for mathematics appointments in France, especially at CNRS. One seminar was organized in Strasbourg by Morin and Papadopoulos, under the name GT3 (in honor of Thurston's notes on the Geometry and Topology of 3-manifolds). Wolpert, Floyd, Mosher, Epstein, Bowditch, Fried, Gabai and others visited this seminar. McCarthy, Oertel and Penner were long-term visitors in Strasbourg. All of them were involved in Thurston type geometry, and Thurston's results on surface geometry and 3-manifolds were discussed extensively. The seminar still runs today. A series of results on Thurston's asymmetric metric on Teichmüller space, after those of Thurston's foundational paper *Minimal Stretch maps between hyperbolic surfaces* [56] (1985), were obtained by Strasbourg researchers. They concern the boundary behavior of stretch lines, the action of the mapping class group on this metric, the introduction of an analogous metric on Teichmüller spaces of surfaces with boundary (the so-called arc metric), and there are several other results. Thurston's paper [56] was (and is still) considered as being difficult to read, although it uses only material from classical geometry and first principles. This shows —if proof is needed— that profound and difficult mathematics remains the one that is based on simple ideas. Other groups of topologists influenced by Thurston were formed in Marseille (Lustig, Short), Toulouse (Boileau, Otal), and various other places in France. Bonahon moved to the US after he proved a major result that was conjectured by Thurston in his Princeton lecture notes, namely, that the ends of a hyperbolic 3-manifold whose fundamental group is isomorphic to that of a closed surface are geometrically tame [5]. (The notion of tameness was introduced by Thurston.) A'Campo went to teach in Basel and introduced several young mathematicians to Thurston-type geometry and topology. His student Walter Brägger gave a new proof of Thurston's version of Andreev's theorem, see [6].

6. Holomorphic dynamics

Dennis Sullivan was a major promoter of Thurston's ideas in France, and he was probably the person who best understood their originality and implications. For more than twenty years Sullivan ran a seminar at IHÉS on topology and dynamics. Recurrent themes at that seminar were Kleinian groups (discrete isometry groups of hyperbolic 3-space), a subject whose foundations were essentially set by Poincaré, and holomorphic dynamics, another subject rooted in French mathematics, namely in the works of Fatou and Julia, revived 60 years later, by Adrien Douady and John Hubbard, in the early 1980s, but preceded sometime in the late 1970s by Milnor and Thurston who developed their so-called kneading theory for the family of maps $x \mapsto x^2 + c$. The topics discussed in Sullivan's seminar also included the geometry of 3-manifolds, deformations of Kleinian groups and their limit sets, pleated surfaces, positive eigenfunctions of the Laplacian, quasiconformal mappings, and one-dimensional dynamics. Thurston's ideas were at the forefront, and Sullivan spent years explaining them.

In 1982, while Sullivan was running his seminar on holomorphic dynamics, Douady gave a course on the same subject at Orsay. At the same time, Sullivan established his dictionary between the iteration theory of rational maps and the dynamics of Kleinian groups.

In the same year, Sullivan was the first to learn from Thurston about his theorem characterizing postcritically finite rational maps of the sphere, that is, rational maps whose forward orbits of critical points are eventually periodic. The proof of this theorem, like the proofs of several of Thurston's big theorems, uses a fixed point argument for an action on a Teichmüller space. Specifically, Thurston associated to a self-mapping of the sphere which is postcritically finite a self-map of the Teichmüller space of the sphere with some points deleted (the postcritical set). The rational map in the theorem is then obtained through an iterative process as a fixed point of the map on Teichmüller space.

In addition to the map on Teichmüller space, the proof of Thurston's theorem involves hyperbolic geometry, the action of the mapping class group of the punctured sphere on essential closed curves and the notions of invariant laminations. All of these notions form the basis for a beautiful analogy between the ideas and techniques used in the proof of this theorem and those used in the proof of Thurston's classification of mapping classes of surfaces, and this correspondence is an illustration of the fact that mathematics, for Thurston, was a single unified field.

Thurston circulated several versions of a manuscript in which he gave all the ingredients of the proof of his theorem, but the manuscript was never finished.[7] A proof of this theorem following Thurston's outline was written by Douady and Hubbard. A first version was circulated in preprint form in 1984 and the paper was eventually published in 1993 [11].

Three years later, Sullivan published a paper in which he gave the proof of a long-standing question formulated by Fatou and Julia [46] (1985). The result became known as the No-wandering-domain Theorem. It says that every component of the Fatou set of a rational function is eventually periodic. A fundamental tool that was introduced by Sullivan in his proof is that of quasiconformal mappings, one of the main concepts in classical Teichmüller theory. These mappings became a powerful tool in the theory of iteration of rational maps. It is interesting that Sullivan, in his paper [46], starts by noting that the perturbation of the analytic dynamical system $z \mapsto z^2$ to $z \mapsto z^2 + az$ for small a strongly reminds one of Poincaré's perturbations of Fuchsian groups $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ into quasi-Fuchsian groups in $\mathrm{PSL}(2, \mathbb{C})$ where the Poincaré limit set changes from a round circle to a non-differentiable Jordan curve, and that Fatou and Julia, the two founders of the theory of iteration of analytic mappings, were well aware of the analogy with Poincaré's work. He then writes: "We continue this analogy by injecting the modern theory of quasiconformal mappings into the dynamical theory of iteration of complex analytical mappings."

Thurston's theorem, together with Sullivan's dictionary, now constitute the two most fundamental results in the theory of iterations of rational maps.

In his PhD thesis, defended under Thurston in Princeton in 1985, Silvio Levy obtained several applications of Thurston's theorem, including a condition for the existence of a *mating* of two degree-two polynomials that are postcritically finite [30]. The notion of mating of two polynomials of the same degree was introduced in 1982 by Douady and Hubbard. The idea was to search for a rational self-map of the sphere that combines the dynamical behavior of the two polynomials. Levy, in his thesis, formulated the question of mating in a more combinatorial way, and using Thurston's characterization of rational maps was able to give a necessary and sufficient condition for the existence of a mating of two postcritically finite degree-two polynomials in terms of their associated laminations. This result solved a question formulated in several precise forms by Douady in his Bourbaki seminar [10, Questions 11 and 12]. At the same time, also in his thesis, Levy established connections between Thurston's geometric approach and Douady-Hubbard's more analytical approach to the subject of iteration of rational maps. In particular, he established the relation between Thurston's invariant laminations and the so-called Hubbard trees that were introduced by Douady and Hubbard in the context of degree-two polynomials. Both notions arise from identifications that arise on the boundary of the unit disc when it is sent by a Riemann mapping to the complement (in the Riemann sphere) of the so-called filled Julia set of a polynomial, in the case where this set is connected.

[7]The first version, circulated in 1983, carries the title *The combinatorics of iterated rational maps*; the subsequent versions known to the authors of the present article do not carry any title.

Tan Lei's thesis, which she defended in 1986 at Orsay under the supervision of Douady, is in some sense the French counterpart of Levy's thesis. It uses Thurston's theorem in an essential way, but instead of laminations Tan Lei works with Hubbard trees. A criterion that Douady and Hubbard formulated in [10, III. 3] gives a necessary condition for the existence of a rational function realizing the mating two degree-two polynomials. Tan Lei provided sufficient conditions for this to happen, at the same time giving a more precise form of the criterion found by Levy.

In her paper *Branched coverings and cubic Newton maps* [29] Tan Lei applies Thurston's theory of postcritically finite branched coverings of the sphere to a new family of maps. Specifically, she studies the dynamics of a class of degree-3 rational maps that arise in Newton's method for approximating the roots of a cubic polynomial. She introduces the notion of a postcritically finite cubic Newton map and investigates the question of whether branched coverings of the sphere are equivalent (in the sense of Thurston) to such a map. The problem of understanding and giving precise information about the roots of a complex polynomial is one of those basic mathematical questions which Thurston was always interested in.

In 2011, about a year and a half before his death, Thurston posted a thread on math overflow concerning the intersection of the convex hull of level sets $\{z|Q(z) = w\}$ for a polynomial Q. He writes: "By chance, I've discussed this question a bit with Tan Lei; she made some nice movies of how the convex hulls of level sets vary with w. (Also, it's fun to look at their diagrams interactively manipulated in Mathematica). If I get my thoughts organized I'll post an answer." Thurston never had a chance to post the answer.

Motivated by this question, Tan Lei wrote an article with Arnaud Chéritat in the French electronic journal *Images des mathématiques* dedicated to the popularization of mathematics. In this article, they first present a classical result known as the Gauss–Lucas Theorem, saying that the convex hull of the roots of any polynomial P of degree at least one contains the roots of its derivative P'.[8] Note that the roots of the derivative are the critical points of the original polynomial. Tan Lei and Chéritat present a result of Thurston which gives a complete geometric picture of the situation: *Let P be a non-constant polynomial. Let F be a half-plane bounded by a support line of the convex hull of the roots of the derivative P' of P and not containing this convex hull and let c be a root of P' contained on this support line. Then there is a connected region contained in F on which P is bijective and whose interior is sent by P onto a plane with a slit along a ray starting at $P(c)$.* Tan Lei and Chéritat gave the details of Thurston's proof that avoids computations, and they provided the computer movies that Thurston talked

[8]Gauss implicitly used this result in 1836, while he formulated the problem of locating the zeros of the derivative of a polynomial in a mechanical way: he showed that these zeros (provided they are distinct from the multiple roots of the polynomial), are the equilibrium positions of the field of force generated by identical particles placed at the roots of the polynomial itself and where each particle generates a force of attraction which satisfies the inverse proportional distance law [17]. Lucas, in 1874, published a mechanical proof of the same theorem, while he was unaware of Gauss's work [31]. (Gauss's notes, published later in his *Collected Works*, were still poorly known.)

about in his post. Their article constitutes a tribute to Bill Thurston; it was published less than 3 months after his death. A more detailed version, including two more authors, Yan Gao and Yafei Ou, was later published in the Comptes Rendus [8].

Over the years, Douady's courses on holomorphic dynamics at Orsay were attended by a number of students and also by more senior mathematicians, including John Hubbard, Pierette Sentenac, Marguerite Flexor, Tan Lei, Pierre Lavaurs, Jean Ecalle, Sébastien Godillon, Arnaud Chéritat, Ricardo Pérez-Marco, Xavier Buff, and Jean-Christophe Yoccoz.[9] Tan Lei, in her tribute to Thurston in [15], writes that he never stopped thinking about iterations of rational maps. She gives a lively description of her conversations and email exchanges with him on this subject in 2011 and 2012, the last two years of his short life.

In the realm of conformal geometry, Thurston introduced the subject of discrete conformal mappings, and in particular the idea of discrete Riemann mappings. In 1987, Sullivan, together with Burton Rodin, proved an important conjecture of Thurston on approximating a Riemann mapping using circle packings [47]. Colin de Verdière, motivated by Thurston's work, proved the first variational principle for circle packings [9].

7. Corrugations

After foliations and contact structures, let us say a word about corrugations.

From the very beginning of his research activity, Thurston was interested in immersion theory. This was shortly after the birth of the *h-principle* in Gromov's 1969 thesis.[10] Probably, Thurston had noticed a precursor of this principle in Smale's 1957 announcement which includes the *sphere eversion*. Very likely he also read the written version of Thom's lecture at the Bourbaki seminar on this topic (December 1957). That report contained the very first figure illustrating immersion theory; this was a *corrugation*.

We remember Thurston explaining to us on a napkin in a Parisian bistro how to create an immersed curve in the plane out of a singular plane curve equipped with a non-vanishing vector field: just make corrugations (a kind of waves) along the curve in the direction of the vector field. The beautiful 1992 pamphlet by Silvio Levy, "Making waves, A guide to the ideas behind *Outside In*", contains an expository paper by Thurston on corrugations with application to the classification of immersed plane curves (the Whitney-Graustein Theorem) and above all, a few steps of a sphere eversion.

Vincent Borrelli (from Claude-Bernard University in Lyon) applied the idea of corrugation in a geometric context to the problem of finding isometric embeddings in the C^1 category. After the work of Nash, as generalized by Kuiper, this problem had a theoretical solution: such isometric embeddings exist. However there was no concrete method of constructing them. Borrelli

[9]It is probably under Douady's influence that the topology research unit called *Équipe de topologie* at the University of Orsay was replaced by a unit called *Équipe de topologie et dynamique* which until today forms one of the five research units at the mathematics department there.

[10]The name *h-principle* was chosen a few years later.

FIGURE 4. Bill Thurston and Tan Lei, Banff, February 2011.
Photo courtesy of A. Chéritat and H. Rugh

used corrugations together with the so-called *convex integration* method of Gromov to find an algorithm consisting of a succession of corrugations and convex integrations for building a C^1-embedding of the flat torus into 3-space. Unlike the Nash–Kuiper existence result, Borrelli's algorithm can be implemented on (big!) computers and produces pictures of such a flat torus. This was done in collaboration with computer scientists.

After this initial success, Borrelli obtained a C^1-embedding of the unit sphere into a ball of radius 1/2 (see Figure 5). In a recent paper, his student Mélanie Theilliere considerably simplified the convex integration method to obtain a "lighter" algorithm.

FIGURE 5. First corrugating step for an isometric embedding of the unit sphere into the ball of radius 1/2. Courtesy of the Hevea Project.

8. One-dimensional dynamics

There is a topic in dynamics which we still have not talked about, that Thurston started working on with Milnor in the year (1972–1973) he spent at the Institute for Advanced Study, namely, 1-dimensional dynamics. Thurston and Milnor studied the quadratic family of maps $x \mapsto x^2 + c$ and showed that it is universal in the sense that it displays the dynamical properties of any unimodal map.[11] Several important concepts emerged from this work, including the notion of universality, kneading sequence, and kneading determinant. This work became one of the themes of the seminars that Sullivan conducted at IHÉS, where he became a permanent member in 1974.

Sullivan's seminar had a considerable impact on French mathematical physicists, and activity on this topic continues today. In a paper published in 1996, Viviane Baladi (Paris-Sorbonne) and David Ruelle (IHÉS) revisited

[11] A paper by Milnor and Thurston on this subject was published in 1988 [34]. Leo Jonker, in reviewing this paper in Mathscinet writes: "If there were a prize for the paper most widely circulated and cited before its publication, this would surely be a strong contender. An early handwritten version of parts of it was in the reviewer's possession as long ago as 1977."

the Milnor–Thurston determinants (as they are called today) in more general 1-dimensional settings [2]. These give the entropy of a piecewise monotone map of the interval in terms of the smallest zero of an analytic function, keeping track of the relative positions of the forward orbits of critical points. Baladi worked out a generalization in higher dimensions, which is the subject of her recent book *Dynamical zeta functions and dynamical determinants for hyperbolic maps. A functional approach* [1].

Among the other works in France that further develop Thurston's work on 1-dimensional dynamics, we mention that of Tan Lei and Hans Henrik Rugh (Orsay), *Kneading with weights*, in which they generalize Milnor-Thurston's kneading theory to the setting of piecewise continuous and monotone interval maps with weights [42]. Rugh's paper *The Milnor-Thurston determinant and the Ruelle transfer operator* [41] gives a new point of view on the Milnor–Thurston determinant.

By the end of his life, Thurston returned to the study of the dynamics of maps of the interval. This was the topic of his last paper (published posthumously) *Entropy in dimension one* [58]. The paper uses techniques from a number of other fields on which he worked: train tracks, zippers, automorphisms of free groups, PL and Lipschitz maps, postcritically finite maps, mapping class groups and a generalization of the notion of pseudo-Anosov mapping, Perron-Frobenius matrices, and Pisot and Salem numbers (two classes of numbers that appear in Thurston's theory of pseudo-Anosov mapping classes of surfaces).

9. Geometric structures

Foliations and contact structures are closely related to the notion of *locally homogeneous geometric structures* introduced by Ehresmann (in the paper [23], these structures are called Ehresmann structures). This notion, with its associated developing map and holonomy homomorphism constitutes one of the key ideas revived by Thurston and that we find throughout his work on low-dimensional manifolds. It originates in the work of Ehresmann from 1935, which is based on earlier contributions by Élie Cartan and Henri Poincaré. We refer the interested reader to the recent geometrico-historical article by Goldman [23]. Thurston framed the Geometrization Conjecture in the context of locally homogeneous geometric structures, thereby rejuvenating interest in this field of mathematics. He also developed the theory of geometric structures with orbifold singularities.

Singular flat structures on surfaces with conical singularities whose angles are rational multiples of right angles provide examples of orbifold geometric structures. Thurston had been developing that theory, including the relation with interval exchange transformations, billiards and Teichmüller spaces, since his early work on surfaces, in the mid-seventies, and he motivated the work of Veech and others on the subject.

In a preprint circulated in 1987 titled *Shapes of polyhedra*,[12] Thurston studied moduli spaces of singular flat structures on the sphere, establishing relations between these spaces and number theory. At the same time, he

[12] The paper was published in 1998 under the title *Shapes of polyhedra and triangulations of the sphere* [57].

introduced the notion of (X, G)-cone-structure on a space, extending to the orbifold case the notion of (X, G)-structure. In particular he proved that the space of Euclidean cone structures on the 2-sphere with n cone points of fixed cone angles less than 2π and with area 1 has a natural Kähler metric which makes this space locally isometric to the complex hyperbolic space $\mathbb{C}H^{n-1}$. He also proved that the metric completion of that space is itself a hyperbolic cone manifold. This work provides geometric versions of results of Picard, Deligne and Mostow on discrete subgroups of $\mathrm{PU}(n, 1)$, interpreting them in terms of flat structures with conical singularities on the sphere. Over the years since, the subject of flat structures on surfaces with conical singularities and their moduli has become an active field of research in France (there are works by Pascal Hubert, Erwan Lanneau, Samuel Lelièvre, Arthur Avila and many others.)

Influenced by Thurston's ideas, the study of foliations with transverse geometric structures emerged as another research topic among geometers in France. One should add that in the 1970s, and independently of Thurston's work, several PhD dissertations on this topic were defended in Strasbourg, under the guidance of Reeb and Godbillon, including those of Edmond Fedida on Lie foliations (1973), Bobo Seke on transversely affine foliations (1977), and Slaheddine Chihi on transversely homographic foliations (1979). After Thurston gave several new interesting examples of transverse structures of foliations (the typical one is the class of singular foliations of surfaces equipped with transverse measures) the theory became much more widely studied, and some geometers started working on foliations equipped with a variety of transverse structures; these included Isabelle Liousse on transversely affine foliations, Gaël Meigniez and Thierry Barbot on transversely projective foliations, Étienne Ghys and Aziz El Kacimi-Alaoui on transversely holomorphic foliations and Yves Carrière on transversely Riemannian and transversely Lie foliations. Several other mathematicians (such as Abdelghani Zeghib and Cyril Lecuire) began working on laminations in various settings. The notion of complex surface lamination also emerged from Thurston's ideas and was studied by Ghys, Bertrand Deroin, François Labourie and others.

Thurston was also the first to highlight the importance of the representation variety $\mathrm{Hom}(\pi_1(S), G)$, where S is a surface and G a Lie group, in the setting of geometric structures, a point of view which eventually gave rise to the growing activity on higher Teichmüller theory. He was the first to realize explicitly that holonomy of geometric structures provides a map from the deformation space of Ehresmann structures into the representation variety, which tries to be a local homeomorphism. Although many examples in specific cases of this were known previously, Thurston realized that this was a very general guiding principle for the classification of locally homogeneous geometric structures. We refer the reader to Goldman's article [23] in which he talks about what he calls the *Ehresmann-Weil-Thurston holonomy principle*. Labourie, McShane, Vlamis and Yarmola in their papers used the expression "Higher Teichmüller-Thurston theory", and this is likely to become a generally accepted name.

From a philosophical point of view, Thurston was an intuitionist, a constructive and an experimental mathematician. He was also among the first to use computers in geometry, in combinatorial group theory and in other topics, and to talk about the rapidity of convergence of geometric construction algorithms. During a visit to Orsay in November 1987, he gave three talks in which computing played a central role.[13] The book *Word processing in groups* [13], written by Cannon, Epstein, Holt, Levy, Peterson and Thurston, is the result of Thurston's ideas on cellular automata and automatic groups. These ideas formed the basis of the work of several researchers in France (Coornaert in Strasbourg, Short and Lustig in Marseille, etc.)

10. Grothendieck

We cannot speak of Thurston's influence in France without mentioning Alexander Grothendieck, the emblematic figure who worked at IHÉS for a dozen years and then resigned in 1970 on the pretext that the institute was partially run by military funds. One may note here that Thurston was similarly involved in the US in a campaign against military funding of science. In the 1980s, the *Notices of the AMS* published several letters from him on this matter. In an attempt to obtain a position at the French CNRS in the years that followed,[14] Grothendieck wrote his famous research program called *Esquisse d'un programme* [26] (released in 1984), in which he introduced his theory of *dessins d'enfants* and where he set out the basis for an extensive generalization of Galois theory and for what later became known as Grothendieck–Teichmüller theory. At several places of his manuscript Grothendiek expresses his fascination for Thurston-type geometry, drawing a parallel between his own algebraic constructions in the field $\mathbb{Q}$ of rational numbers and what he calls Thurston's "hyperbolic geodesic surgery" of a surface by pairs of pants decompositions. He also outlined a principle which today bears the name *Grothendieck reconstruction principle.* This principle had already been used (without the name) in the 1980 paper by Hatcher and Thurston *A presentation for the mapping class group of a closed orientable surface* [22] in the following form: there is a hierarchical structure on the set of surfaces of negative Euler characteristic ordered by inclusion in which "generators" are 1-holed tori and 4-holed spheres and "relators" are 2-holed tori and 5-holed spheres. The analogy between Grothendieck's and Thurston's theories was expanded in a paper by Feng Luo *Grothendieck's reconstruction principle and 2-dimensional topology and geometry* [32]. Incidentally, the result of Hatcher and Thurston in the paper mentioned above is based on Cerf theory. We mention that Jean Cerf was a professor at Orsay. He was appointed there in the first years of existence of that department, and he created there the topology group (at the request of Jean-Pierre Kahane).

Grothendieck's ideas on the action of the absolute Galois group and on profinite constructions in Teichmüller's theory that are based on Thurston-type geometry are also developed in his *Longue marche à travers la théorie de*

[13]The titles were *Automatic groups with applications to the braid group*, *Conway's tilings and graphs of groups* and *Shapes of polyhedra.*

[14]The application was unsuccessful.

Galois, a 1600-page manuscript completed in 1981 which is still unpublished [25]. At the university of Montpellier, where he worked for the last 15 years of his career, Grothendieck conducted a seminar on Thurston's theory on surfaces, and directed Yves Ladegaillerie's PhD thesis on curves on surfaces.

Grothendieck again mentions Thurston's work on surfaces in his mathematical autobiography, *Récoltes et semailles* [27, §6.1]. In that manuscript he singles out twelve themes that dominate his work and which he describes as "great ideas" (grandes idées). Among the two themes he considers as being the most important is what he calls the "Galois–Teichmüller yoga", that is, the topic now called Grothendieck–Teichmüller theory [27, §2.8, Note 23].

11. Bourbaki seminars

Thurston's work was the subject of several reports at the *Séminaire Bourbaki.* This seminar is held three times a year in Paris (over a week-end). It is probably still the most attended regular mathematical seminar in the world.

In the three academic years 1976/1977, 1978/1979 and 1979/1980, a total of five Bourbaki seminars were dedicated to Thurston's work. In the first one, titled *Construction de feuilletages, d'après Thurston* [40], Roussarie reports on Thurston's result saying that any compact manifold without boundary whose Euler characteristic vanishes admits a C^∞ foliation of codimension one. In the second seminar, titled $B\Gamma$ *(d'après John N. Mather et William Thurston)* [43], Sergeraert reviews one of Thurston's deep theorems: the homology of the group of C^r diffeomorphisms of $\mathbb{R}^q$ with compact support (as a discrete group) is closely related to the homology of $\Omega^q(B\Gamma^q)$, the q-th loop space of the classifying space of codimension-q Haefliger structures of class C^r.[15] In the third seminar, titled *Travaux de Thurston sur les difféomorphismes des surfaces et l'espace de Teichmüller* [37], Poénaru gives an outline of Thurston's theory of surfaces, which appeared later in [14]. In the fourth seminar, titled *Hyperbolic manifolds (according to Thurston and Jørgensen)* [24], Gromov reports on some of the powerful techniques contained in Thurston's 1997/98 Princeton notes, including his work on limits of hyperbolic 3-manifolds, his rigidity theorems, and the result stating that the set of values of volumes of hyperbolic 3-manifolds of finite volume is a closed non-discrete subset of the real line. As a matter of fact, Gromov arrived to France and lectured at Orsay at the end of the 1970s. His notion of simplicial volume played, via the techniques of smearing out and straightening, a key role in the (so-called Gromov–Thurston) version of Mostow rigidity theorem for 3-dimensional hyperbolic manifolds contained in Chapter 6 of Thurston's Princeton notes [55]. In the fifth Bourbaki seminar, titled *Travaux de Thurston sur les groupes quasi-fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur* S^1 [45], Sullivan gives an outline of Thurston's results on hyperbolic structures on irreducible 3-manifolds which

[15] The notion of Haefliger structures translates an idea of singular foliation equipped with a desingularization. This leads to a homotopy functor which has a classifying space, analogous to $BO(n)$ for vector bundles of rank n. The case $q = 1$ was already known to J. Mather in 1970. For this reason, one speaks today of the Mather-Thurston homology equivalence theorem. Let us mention that Takashi Tsuboi circulated a pamphlet with pictures explaining a map that induces this isomorphism. It is available on Tsuboi's homepage.

fiber over the circle and which contain no essential tori. At the same time, using a limiting procedure in the space of quasi-Fuchsian groups, he gives a new proof of Thurston's result saying that the mapping torus of a homeomorphism of a closed surface of genus ≥ 2 with pseudo-Anosov monodromy carries a hyperbolic structure.

Thurston's work has been the subject of several other Bourbaki seminars over the years; we mention in particular seminars by Morgan on finite group actions on the sphere [35], by Ghys on the Godbillon-Vey invariant [16], by Boileau [4] on uniformization in dimension three, by Lecuire on ending laminations [28], and, finally, by Besson on the proof of the geometrization of 3-manifolds and the Poincaré conjecture [3].

12. The last visits to Paris

Speaking of the proof of the Poincaré conjecture—another problem rooted in French mathematics, a problem that haunted Thurston during all his mathematical life— we are led to the last time we saw Thurston in Paris. This was in June 2010 at the Clay research conference, where he gave two beautiful talks at the magnificent lecture hall of the Oceanographic Institute. The first talk was titled "The mystery of three-manifolds." The second one, shorter, was a *Laudatio* on Grigory Perelman. Thurston recounted his personal experience with the Poincaré conjecture. In a few minutes, he expressed his deep admiration and appreciation for Perelman and he said in a few moving words how much he was gratified to see that the geometrization conjecture became a reality during his lifetime. With an amazing humbleness, he declared that when he read the proof he realized that it is a proof that he could not have done ("some of Perelman's strengths are my weaknesses"). He concluded with these words:

> Perelman's aversion to public spectacle and to riches is mystifying to many. I have not talked to him about it and I can certainly not speak for him, but I want to say I have complete empathy and admiration for his inner strength and clarity, to be able to know and hold true to himself. Our true needs are deeper—yet in our modern society most of us reflexively and relentlessly pursue wealth, consumer goods and admiration. We have learned from Perelman's mathematics. Perhaps we should also pause to reflect on ourselves and learn from Perelman's attitude towards life.

Paris is also the capital of fashion design. A few weeks before the Clay conference Thurston was there for the fashion week, which takes place every year in March at the Carrousel du Louvre, which sits between the Jardin des Tuileries and the Louvre museum. The fashion designer Dai Fujiwara presented a beautiful collection of pieces made for the Issey Miyake brand, inspired by Thurston's eight geometries. A journalist covering the event wrote: "Two decades ago, in the same venue, Romeo Gigli transfixed Paris with a show so rich and romantic that it moved its audience to tears. Maybe that didn't happen today, but, at the very least, Fujiwara used his inspiration to blend art and science in a manner so rich and romantic, it stirred the emotions in a way that reminded us of Gigli."[16] Among the other comments

[16]Tim Blanks, *Vogue*, March 5, 2010.

on this event, we note: “Fashion scaled the ivory tower at Miyake, where complicated mathematical theorems found expression in fabric” (Associated Press); “Mathematics and fashion would seem to be worlds apart, but not so, says Dai Fujiwara” (The Independent); “you did not need a top grade in maths to understand the fundamentals of this thought-provoking Issey Miyake show: clean geometric lines with imaginative embellishment” (International Herald Tribune); “Fujiwara used his inspiration to blend art and science in a manner so rich and romantic, it stirred the emotions” (Style).

Thurston wrote a brief essay, distributed during the Miyake fashion show, on beauty, mathematics and creativity. Here is an excerpt:

> Many people think of mathematics as austere and self-contained. To the contrary, mathematics is a very rich and very human subject, an art that enables us to see and understand deep interconnections in the world. The best mathematics uses the whole mind, embraces human sensibility, and is not at all limited to the small portion of our brains that calculates and manipulates with symbols. Through pursuing beauty we find truth, and where we find truth, we discover incredible beauty.

In another article written on that occasion for the fashion magazine *Idoménée*, Thurston made the following comment about the collection:

> The design team took these drawings as their starting theme and developed from there with their own vision and imagination. Of course it would have been foolish to attempt to literally illustrate the mathematical theory— in this setting, it’s neither possible nor desirable. What they attempted was to capture the underlying spirit and beauty. All I can say is that it resonated with me.

In an interview released on that occasion Thurston recounted how he came to contribute to the collection, and he declares there: “Mathematics and design are both expressions of human creative spirit.” One of the comments on the video posted after this interview says: “I can’t believe this mathematics guy. He’s so ... not like what I expected.”

Part 2

13. Valentin Poénaru

Bill Thurston went through the mathematical sky like an immensely bright, shiny, meteor. My short contribution certainly does not have the ambition of trying to describe his fantastic trajectory. Rather, more modestly, we want to tell something about Bill’s impact on the mathematical life of the Orsay Department of Mathematics (Université de Paris-Sud) in the seventies.

We heard for the first time about Thurston’s activity from our friend and then colleague Harold Rosenberg, who invited Bill to talk about his work on foliations. The first contact already triggered the mathematical career of Michel Herman.

What we had heard so far was Bill’s work on foliations of codimension higher than one. Next, we still vividly remember the lectures which Bob Edwards gave us on Thurston’s codimension-one theorem. Several other mathematicians had already tried to explain this to us, but they always got bogged down before managing to get to the main point. Bob's lectures were not only very illuminating, they also had the aesthetic quality of a drama stage, with suspense and "coup de théâtre."

Then, via Dennis Sullivan who was visiting Orsay for a year, before settling for the next twenty years or so at the neighboring IHÉS, we were introduced to the hyperbolic world, where Bill was now one of the brightest stars. We vividly remember how this field where Bill, Misha Gromov, Dennis and others did such big things, was then looked down upon by many. A distinguished colleague told us that hyperbolic geometry was just "a gadget." In his panoramic books on the mathematics of our time, Dieudonné does not even mention the topic, obviously thinking that it was too marginal and parochial.

It is also Dennis who, via some very convincing drawings, conveyed to us Thurston's discovery that, by infinite iteration via a pseudo-Anosov diffeomorphism, curves can turn into measured foliations. And when we tried to make that quantitatively precise, a very nice mixing property popped up; ergodic theory was there, big.

Somebody brought us some notes written after some lectures of Thurston on his theory of surfaces. This triggered the Orsay seminar on this topic, organized by Fathi, Laudenbach and myself, where we tried to provide full details for the theory. A lot of distinguished visitors joined us. But then, there was a big problem where we got bogged down. We did not manage to glue the space of measured foliations to the infinity of the Teichmüller space. The Teichmüller specialists offered us several suggestions, but none of them was good, since we needed a *natural* compactification of the Teichmüller space where the automorphisms of the surface should extend continuously.

Then, during a very hectic week at Plans-sur-Bex, in the Swiss Alps, we were together with Bill who during a memorable and intense half-hour gave us the correct hint on how to proceed. Thus, we could both finish our seminar and write the corresponding book. The seminar in question, and in particular the influence of Albert Fathi, triggered the mathematical activity of Jean-Christophe Yoccoz.

Next, we were introduced by Sullivan, and others, to Thurston's program of introducing hyperbolic geometry into the field of 3-manifolds. With Larry Siebenmann as the main organizer, this created a lot of activity. This is how Francis Bonahon and Jean-Pierre Otal started their mathematical trajectories. Our colleagues and friends François Labourie and Pierre Pansu were also quite influenced by all these activities.

Very naturally, our Department of Mathematics proposed to our University that Bill be awarded the title of Doctor Honoris Causa. The ceremony was held in the Fall of 1986. It so happened that at about the same time, the Computer Science Department did the same for Donald Knuth. Thus, in the largest lecture room on our campus there were two big public lectures by the two laureates. Bill gave us a brilliant and amazing lecture on how geometry (hyperbolic or SOL) can optimize computer construction, providing maximum connectivity with a minimum of parasitical interferences, in a given volume.

In the beginning of the Summer of 2010 Bill came for a last time to Paris and lectured on some future plans of his at the Institut Henri Poincaré. But those were not to be since Bill passed away soon afterwards.

Bill Thurston's impact on the mathematical life of our department was immense and quite a number of people here owe a lot to him. We will never forget him.

14. Harold Rosenberg

(Excerpt from an email dated March 22, 2016, addressed to François Laudenbach)

In 1971, I visited Berkeley for 6 months. I taught a class on foliations, and Bill Thurston was a student in the class. By the end of the term, Bill and I were working together, and we wrote a paper which was published in the proceedings of a meeting on dynamical systems which took place in the city of Salvador, on July 26-August 14, 1971. (This was my first visit to Brazil.) The book was edited by M. Peixoto. Upon returning to Paris, I invited Bill to visit. He came to Orsay. Haefliger organized a meeting (I believe it was in Plans-sur-Bex), and Bill and I went and Haefliger met Bill. Also, Bill met Dennis in Paris at that time. Bill was working on the geometry of foliations in 3-manifolds at the time. Shortly thereafter, he managed to construct a family of foliations whose Godbillon-Vey invariants varied continously onto an interval. Based on this, Milnor offered Bill a visiting position at Princeton. The following year (Bill was in Princeton), he started proving his important integrability theorems in higher dimensions, and thinking about diffeomorphisms of surfaces.

15. Francis Sergeraert

(Excerpt from an email dated April 4, 2016, addressed to François Laudenbach, translated from the French)

I remember very well Thurston's visit to Orsay in 1971. It was the time of his proof of $\mathrm{Im(GV)} = \mathbb{R}$. Someone (I don't remember who, maybe Rosenberg) was trying to find a mistake in that result.

At that time, Thurston already knew how to connect the homological aspect of diffeomorphism groups and that of $B\Gamma$. Michel Herman's note on the simplicity of $\mathrm{Diff}_+(T^n)$ dates from the same year. I don't know what was the real influence of Thurston on that work, but it was non-negligible. The spaces $B\Gamma$, since their discovery by Haefliger, were still very mysterious. Mather had started to unblock the problem by first establishing the connection with the homology of $\mathrm{Diff}_c(\mathbb{R})$. He had difficulties publishing his result, which eventually appeared in 1975, but starting in 1973, Thurston, in Princeton, was already explaining its generalization in any dimension.

Regarding the Poincaré conjecture, when Thurston started working on it, everybody considered it as a problem in algebraic topology. Thurston's work on foliations was probably the first indication that there were connections with differential analysis. It is tempting to see there the germ of the idea of geometrization.

16. Norbert A'Campo

(Excerpt from an email dated February 1, 2019, addressed to Athanase Papadopoulos, translated from the French)

In the beginning of the seventies, an offer was made to Morris Hirsch for a position at Orsay. He accepted, paperwork was done, but Hirsch eventually resigned, the reason being that he had a very good student which he did not want to leave. After a moment of consternation, Rosenberg and Siebenmann decided to invite the student. Thurston came to Orsay. He also visited Dijon and Plans-sur-Bex. We did a large portion of these trips was together, by car.

The conference at Plans-sur-Bex was organized by Haefliger, one week in March. The wonderful inspiring wine was provided by Kervaire. Local organisation with excellent cooking was done by the family Amiguet from Geneva.

During the travel, and at the conference, Thurston explained during his talks and in informal discussions some vivid and beautiful new mathematics. In particular he proved that the Godbillon-Vey map

$$GV : \{\textit{codimension 1 foliations on}\, S^3\} \to \mathbb{R}$$

is surjective. He used the theorem stating that planar hyperbolic polygons of equal area are scissor equivalent. For me, this was the first time I saw hyperbolic geometry at work.

A few years later, after many other celebrated uses of and contributions to hyperbolic geometry by Thurson, Larry Siebenmann asked me to give a graduate course on hyperbolic geometry at Orsay. I knew nothing about that subject. Fortunately, I was planning a visit the Mittag-Leffler Institute, and in one of the attics there I found several old documents on this topic. I took some notes and I came back to Orsay ready to give my course and (at the same time) learn hyperbolic geometry.

17. Gilbert Levitt

I first heard the name Thurston in 1976 (or maybe 1975). David Epstein gave a graduate course on foliations in Orsay. He explained Thurston's stability theorem for foliations of codimension 1, and a large part of the course was devoted to trying to understand the proof of Thurston's theorem on foliations of codimension greater than 1.

The following year I started working towards a PhD under the guidance of Harold Rosenberg. He made me study Thurston's thesis, about foliations of 3-manifolds which are circle bundles, and my first published paper may be viewed as a write-up of that thesis.

Following the advice of Harold, I then turned to (singular) foliations on surfaces. Almost all my papers on that subject mention Thurston in the bibliography; when compactifying Teichmüller space and classifying homeomorphisms of surfaces, he introduced measured foliations (and laminations), which may be viewed as a building block for constructing general foliations on surfaces.

He also defined train tracks, which I encountered later while working on automorphisms of free groups. Following the seminal paper by Bestvina-Handel, train tracks were carried over from surfaces to free groups and became an extremely important tool in geometric group theory.

One of Thurston's last contributions is a paper posted on the arxiv in 2014, where (among other things) he completely characterizes which numbers may appear as growth rates of automorphisms of free groups.

My first meeting with Thurston was in the fall of 1978. At that time Harold Rosenberg visited Santa Cruz for several months and he arranged for Rémi Langevin and me to spend some time in Berkeley. On the way home we stopped in New York and he secured an appointment with Thurston for me.

Rémi and I drove out to Princeton and I spent quite some time with Thurston discussing foliations on surfaces and related topics. For some reason we didn't use a blackboard, and I still have a notepad covered with his drawings. I was 23 at the time, I hadn't yet proved a real theorem, and thinking back I am really grateful and honored that he devoted so much of his time to me.

18. Vlad Sergiescu

I would like to say a few words on Thurston's influence on French foliation theorists in (and between) Lille and Orsay, in a specific situation: the geometry of the Godbillon-Vey class of a codimension-one foliation.

Around 1970 a big excitement arose in foliation theory, due to the discoveries of the Bott vanishing theorem, Haefliger's classifying space, Fuchs-Gelfand cohomology and the Godbillon-Vey class.

For a foliation given by a 1-form ω, let η be a 1-form such that $d\omega = \omega \wedge \eta$. The 3-cohomology class of the closed form $\eta \wedge d\eta$ is the Godbillon-Vey class GV.

A short time after its discovery, Harold Rosenberg, who introduced foliations at Orsay and was an outstanding advisor there, wrote a paper with Thurston, published in the proceedings of a conference in Bahia, in which they asked whether the Godbillon-Vey class vanishes for a foliation without holonomy.

In 1973, Sullivan began a course at Orsay on his new rational homotopy of differential forms, with (among others) the following question: What does the Godbillon-Vey class measure?

Many people around Orsay contributed in some way or another to make advances on this question. Let me mention Roussarie, Herman, Moussu, Sergeraert, and Roger as well as Haefliger, Epstein and Sullivan as regular visitors. Roussarie proved that the GV class is non-zero on the unit tangent bundle of a hyperbolic surface. Thurston showed this as well (as a graduate student), and later, he proved that it varies continuously! In an influential paper, Herman proved (and Guy Wallet as well) a vanishing theorem on the torus T^3.

During those years, Thurston was involved in foliations, until 1976–77, when he switched to 3-manifolds where he made a strong use of measured foliations for diffeomorphisms of surfaces. He revolutionized both subjects.

Some of his landmark theorems, besides the continuous variation, are the existence of a codimension-one foliation on any manifold with vanishing Euler characteristic and the $(q+1)$-connectedness of the classifying space $B\Gamma_q$. Let me add his inspiring picture of the helical wobble.

In 1976, I joined a seminar in Lille organized by Gilbert Hector. Among the participants were Duminy, Ghys, El-Kacimi Alaoui, Lehmann, and soon, first and foremost, Alberto Verjovski. All of us were sensible to the GV world. There, we learned that the absence of Lamoureux' resilient leaves (which are self-spiraling) is a good context to attack the problem of the vanishing of GV. It contained the non-exponential setting that Moussu and Sullivan had already suggested (thus also the vanishing of the holonomy).

Shortly after a joint note for the S^1-foliated case, Gérard Duminy found a brilliant proof of the general vanishing theorem. One major innovation was the exploration of a decomposition of GV as the product of a "Godbillon" measure and a "Vey" class. Several other new ideas paved the way for the introduction of techniques of ergodic theory and dynamical systems in foliation theory. An example is the connection between the entropy defined by Ghys–Langevin–Walczak and the GV class. Hurder and Katok proved that GV is invariant under absolutely continuous homeomorphisms, while the topological invariance is still open—Ghys obtained several results in this direction. Hurder and Langevin present a modern view of the above topics in a recent article on GV and C^1 dynamics.

Thurston introduced a 2-cocycle t_{gv} on $\mathrm{Diff}(S^1)$ (the names of Bott or Virasoro are sometimes linked to this) closely related to GV:

$$(f,g) \to \int \log g'(\log(f \circ g)')'.$$

A useful observation made later was that t_{gv} can be extended below the C^2 class, to the Denjoy P class (maps with bounded log-derivative variation)[17] and to the class $C^{1+\alpha}$, $\alpha > \frac{1}{2}$. Takashi Tsuboi made a thorough study of such extensions leading to his beautiful GV-cobordism characterization.

In his Berkeley years, Thurston met a fellow student, Richard Thompson who was working in algebraic logic and discovered in this context three groups with wonderful properties. One of them turned out to be isomorphic to the group T of PL dyadic homeomorphisms of the circle.

Thurston talked about this to several people around him. Ghys and myself learned about T from Epstein and Sullivan. We then found a PL version of Thurston's cocycle called $\overline{t_{gv}}$. It is intriguing and remarkable that together with the Euler class, it generates the cohomology of T, and this turned out to be exactly the Fuchs-Gelfand cohomology of $\mathrm{Diff}(S^1)$.

To conclude, let me point out that the classes GV, t_{gv} and $\overline{t_{gv}}$ appear to be ubiquitous, with connections to braids, mapping class groups and Teichmüller spaces, loop groups, Chern-Simons invariants, Virasoro algebra and groups, $\mathbb{C}^*$-algebras, index theory, strings, solitons and hydrodynamics.

I never had the privilege to meet Thurston, but my friend and collaborator Peter Greenberg did. He told me once that an important thing he learned from him was how to play with mathematics. I vividly remember a day of

[17] This terminology was used by M. Herman.

1987 in Mexico when both of us were playing with David Epstein to connect Thompson groups with braid groups. At that time, it was not a success. Suddenly David exclaimed: I am sure that Thurston would find it on the spot!

19. Michel Boileau

(Excerpt from an email dated February 20, 2017, addressed to François Laudenbach)

The first time I had the occasion to listen to a lecture of Thurston was at the conference held in Bangor (G. B.) in July 1979. He gave four lectures. The first three were about the geometrization conjecture of 3-manifolds and its proof for Haken manifolds. The last lecture was about the Smith conjecture whose recent proof relied upon the geometrization of Haken manifolds. Thurston motivated his conjecture by the fact that it dealt with all 3-manifolds. One could hope that it would be solved within the next thirty years; history showed that this was right.

The second occasion on which I followed lectures by Thurston was at the 1984 Durham conference. Thurston gave a series of lectures on the geometrization of orbifolds, in particular on hyperbolic conical structures and their geometric limits. Again, objects and methods presented in dimension three were completely new. It was only thirteen years later, when I started with Joan Porti and Bernhard Leeb to write a complete proof of this theorem, that I understood the ideas that Thurston had tried to communicate in these lectures.

In the Fall of 1986, William Thurston was awarded the degree of Doctor Honoris Causa from the University of Paris-Sud (Orsay). On that occasion, he gave two lectures. In the first one, for a large audience, he explained how to apply methods from hyperbolic geometry to computer science. The second one, more specialized, was on the deformation space of polygons in the plane. On that occasion, Gromov, from the audience, challenged him with an objection. Thurston answered with the smile he has always had when he tried to communicate his extraordinary vision of geometry.

His style and his manner of communicating mathematical ideas, though very generous, have frequently raised criticism. In my opinion, they rather reveal his highly demanding commitment to the quality of writing. In a talk he gave at a conference in Tokyo in 1998, concerning the proof of his orbifold theorem, Thurston declared: “I am reproached for not writing enough but what I have in mind is much more beautiful than what I am able to put on paper.”

At the same conference, I had the opportunity to ask him whether every 3-manifold could have a conical hyperbolic structure with angles arbitrarily close to 2π. This seemed *a priori* impossible for the 3-sphere and many colleagues to whom I had asked the question were of the same opinion. Thurston immediately answered: “Yes”, and he showed me a simulation on his computer, precisely for the 3-sphere. This result was proved later by Juan Souto.

20. Pierre Arnoux

I started as a mathematics graduate student at Orsay at 1979, by following the lectures of Michael Shub on dynamical systems.[18], This was two years after the seminar on *Thurston's work on surfaces*. At that time at Orsay, one was immersed, without even realising it, in a particular mathematical culture. I became quickly aware of the classification of surface automorphisms, foliations etc., even if I was far from understanding the proofs.

Michel Herman suggested me to work on interval exchange maps for my PhD. thesis. I always thought of them geometrically, associated with surface foliations. At that time, very few explicit examples of surface automorphisms were known; most of them were related to coverings of automorphisms of the torus. We came across a paper by William Veech in which he started the parametrization of holomorphic forms[19] (hence, particular strata of the cotangent bundle of Teichmüller space) using interval exchange maps; that gave a way of building *self-similar foliations*. I remember a night trip back from England (possibly Durham) with Jean-Christophe Yoccoz and Albert Fathi during which they built such an example with a cubic coefficient (I was rather a spectator than anything else). This was the first example of a pseudo-Anosov diffeomorphism which does not arise from a torus automorphism. This was also the kind of things that I enjoyed: to build explicit and concrete, yet slightly strange objects. Twenty years later, Maki Furukado, a Japanese mathematician, gave me a model of this foliated surface constructed by sewing rectangles of striped material; this is not a trivial thing to do, because the singularities of the foliation impose heavy constraints. But you can easily see why the suspension has to be on a surface of genus 2.

A few years later, around 1984, I came across a paper by Gérard Rauzy in which he had built a self-similar fractal set associated with a substitution whose similarity coefficient is the same as the one of the pseudo-Anosov example. I thought that this was more than a coincidence. By using the symbolic models associated with the two systems, I was able to show that the interval exchange which had made possible this example was measurably conjugate to a rotation of the 2-torus by a continuous map $\mathbb{T}^1 \to \mathbb{T}^2$ whose image is a Peano curve filling the 2-torus. It followed easily that, by taking suspensions, the pseudo-Anosov at hand was measurably conjugate by a continuous map to a hyperbolic automorphism of the 3-dimensional torus $\mathbb{T}^3$.

In Orsay, people were also familiar with the work of Adler and Weiss yielding an explicit Markov partition of the hyperbolic automorphisms of $\mathbb{T}^2$ and showing that these automorphisms are classified up to measurable

[18]This was called a DEA (Diplôme d'Études Approfondies) course, usually attended by graduate students, the year before they choose a subject for their PhD dissertation. But the courses were also sometimes followed by confirmed researchers.

[19]The history is quite complicated: Veech was interested in results of ergodic theory; he had already worked on particular types of interval exchange maps given by skew-product of rotations, and found the general idea of induction in a paper of Rauzy, as generalized continued fractions; he then learnt, apparently from Thurston, of the link between interval exchange maps and measured foliations, and this led him to the proof of almost everywhere unique ergodicity. I think his papers have not been fully read, and they still contain a number of unnoticed results.

conjugacy by their largest eigenvalue (their logarithm is the topological entropy, as well as the measure theoretic entropy for the Lebesgue measure). We also knew that any hyperbolic automorphisms of $\mathbb{T}^n$ has a Markov partition. But, these partitions are not at all trivial; for instance, it follows from a very short paper by Rufus Bowen that for $n > 2$ their boundary is a fractal set. It is indeed difficult to exhibit explicit examples of Markov partitions, except in the case of surface automorphisms, where they are made of explicit rectangles. The particular example we had found, by giving a measurable conjugacy between an automorphism of a surface with an explicit Markov partition and a toric automorphism, showed an explicit partition, with fractal boundary of known dimension, for the toral automorphism.

In this theory, there is a basic example easy to understand, that is, $\mathbb{T}^2$ with its Teichmüller space which is the hyperbolic plane, and its moduli space, which is the classical modular surface equipped with its geodesic flow. There are two simple ways for generalizing that case: taking higher dimensional tori, or taking surfaces of higher genus. Sometimes, one has the feeling that the two ways are but the same: a hyperbolic automorphism of the n-dimensional torus $\mathbb{T}^n$ can be often unfolded to an automorphism of a surface of the corresponding genus, a little like these *kirigami*, Japanese paper flowers which unfold when you put them in water.

Now I would like to talk about the period 1992–2019 spent in Luminy-Marseille.

I was more and more attracted by the arithmetic and combinatoric constructions of Rauzy and I went to work in the laboratory he had founded in Luminy. But I was still interested in the geometric side of these constructions. Thurston had also written a paper, not published but available as a xeroxed preprint, about tiles associated with algebraic numbers; this was parallel to what we were doing with substitutions.

In Luminy, we worked on symbolic sequences with low (sublinear) complexity, in particular the Sturmian sequences which appear in many different settings: dynamics of rotations on the circle, Farey sequences, dynamics of continuous fractions, and more curiously, dynamics of the Mandelbrot set.[20] Here, the basic lemma, attributed to Thurston, states the following: an orbit of the map $x \mapsto 2x \bmod 1$ is cyclically ordered if and only if its binary encoding is Sturmian. In all the papers I have worked on since, there is the influence of the geometry I learnt at Orsay at that time, mixed with the discrete mathematics and the number theory which was the mark of Rauzy.

In Marseille, there was another group of mathematicians who were studying outer automorphisms of free groups. They included Arnaud Hilion and Martin Lustig, collaborating with Gilbert Levitt who was in Caen. Outer automorphisms of free groups have a lot of analogies with mapping class groups of surfaces. The substitutions that we studied in Luminy were simpler cases of these automorphisms, in the same manner as matrices with positive coefficients are simpler than general matrices (Perron-Frobenius). The two groups started to collaborate.

In the articles that I write or read today, I continue to feel what happened at Orsay around 1976, with Thurston, Douady, and Hubbard (who is now

[20] S. Bullett and P. Sentenac wrote a beautiful paper on this subject [7].

regularly in Marseille): tiles associated with automorphisms of free groups, generalized Teichmüller spaces, explicit conjugations between bifurcations for families of continuous fractions or for families of quadratic polynomials, etc.

21. Albert Fathi

Bill Thurston's impact on French topologists is certainly one of the best influences on the group.

I first heard of Bill when I started my Graduate Studies in 1971. At that time he was already a legend for his work on foliations.

I think I first met Bill at the CIME school on Differential Topology in 1976 at Varenna. Thurston was, along with André Haefliger and John Mather, one of the three people delivering the courses. It is unfortunate that he never delivered the manuscript of his lectures for publication. My most vivid impression of this meeting was the private explanation by André Haefliger on Thurston's beautiful geometric argument on how to obtain that the (connected component of the) diffeomorphism group of a compact manifold is perfect from the case of the n-dimensional torus that was previously done by Michel Herman. This was an Aha! moment: how a deep insight in geometry can circumvent the impossible adaptation to general manifolds of Herman's work on the torus. It used KAM methods and hard implicit function theorems in neighborhoods of irrational translations on the torus.

The work of Bill on diffeomorphisms of surfaces led to the monograph that we edited with François Laudenbach and Valentin Poénaru.

It was Valentin Poénaru who drew us to this subject. He came one day from IHÉS with a set of hand-written notes that Mike Handel produced while listening to Bill's course in Princeton. He convinced François and myself to run a seminar on the subject. This seminar took place in 1976-77 in Université Paris-Sud (Orsay).

The group of diffeomorphisms of a surface up to isotopy is called the mapping class group (of the surface). Bill's work essentially produced a "best" representative in each element of the mapping class group.

Valentin Poénaru gave us the Grand Tour on the subject in the first lectures of the seminar. I started to work almost everyday with François to be able to understand the details. It took us a couple of years to produce a usable manuscript. We benefited from advice of Francis Sergeraert who served as a referee. At this time, Bill used measured foliations rather than measured laminations which appeared after most of our manuscript was finished. This is why measured laminations are not in the monograph. Anyway, I find it very rewarding that 30 years later, it was still found useful to have an English version of our monograph. The mathematical world is smaller than we think it is: one of the two editors of the English version is Dan Margalit who now is my colleague at Georgia Tech.

Bill's work was a revolution in the old subject of classification up to isotopy of surface diffeomorphisms. Before him, there was a remarkable work of Nielsen in the 1930s which pointed out the elements of finite order of the mapping class group. However, nobody really realized the existence and irreducibility of what Bill called pseudo-Anosov diffeomorphisms. Of course, the fact that Anosov diffeomorphisms were, by that time, extensively studied, in particular, through the properties of the stable and unstable foliations, is certainly what motivated Bill to introduce these pseudo-Anosov diffeomorphisms. Obviously, Nielsen could not have benefited from such a knowledge. What was also remarkable in Bill's approach was that he also made strong connections with objects, besides pseudo-Anosov maps, that were subjects of intensive research in Dynamical Systems like interval exchange and entropy. For me, who was turning from Topology to Dynamical Systems, it was another Aha! moment.

Bill's main tool is of course the compactification of Teichmüller space by the projectified space of measured foliations, yielding a space homeomorphic to a ball, on which the mapping class group acts naturally. Therefore by Brouwer's fixed point theorem, each element has a fixed point in this compactification. The underlying geometrical nature of the fixed point gives the classification.

The Orsay seminar on Thurston's work was very lively. The number of attendants was large. Jean-Christophe Yoccoz who was just starting graduate school told me that he attended it (I do not remember that, I really did not meet him till the end of that academic year) and it left on him a lasting impression.

One of the main challenges during the lectures was the discussions with complex analysts who had a compactification of Teichmüller space as a Euclidean ball by quadratic differentials. The discussions were driven by the belief that these two compactifications were the same. It was a surprise when, sometime during the year, we learned that Steve Kerckhoff, then a PhD student of Bill, showed that the two compactifications were distinct. Of course, both compactifications are nowadays important, and they can be used to prove the classification of elements in the mapping class group.

After that, the lamination point of view pervaded the subject. It was quite remarkable that Bill Thurston and Mike Handel were able to show using laminations that the ideas of Nielsen that dated back to the 1930's potentially contained the classification of elements of the mapping class group. At that time, I was already getting back to dynamics problems and lost track of the subject.

Twenty years later laminations (not necessarily geodesic) came back to haunt me. There is hardly a day in my mathematical life without thinking about laminations.

In fact, about 1982, John Mather and Serge Aubry established the now so-called Aubry-Mather theory for twist maps of the annulus. Although not usually expressed that way, the Aubry-Mather set (or rather its suspension) is a lamination (not geodesic). When Mather generalized these results to higher dimensions in the setting of Lagrangian systems, the connection became much clearer. Aubry-Mather sets are foliated by 1-dimensional trajectories. They are therefore laminations. Mather's graph theorem is in fact a proof that these laminations are Lipschitz (the speed of the trajectory is a Lipschitz function of the point). This is a crucial property for geodesic laminations in dimension two, which follows in that case from a simple (hyperbolic) planar argument.

In 1996, I discovered (like Weinan E and Craig Evans-Diogo Gomes) the relation between the Aubry-Mather theory and the viscosity solutions of the Hamilton-Jacobi equation. The fact that I knew the lamination theory set up by Bill was definitely instrumental going deeper in this relationship that keeps me still mathematically busy today.

I do not mention the work of Bill on Poincaré's conjecture and on holomorphic dynamics both of which had immense influence on several French mathematicians. I personally had not been involved in these parts.

In the Fall of 1986, Université Paris-Sud (Orsay) gave a Doctorat Honoris Causa to both Bill Thurston and Don Knuth. At that time a plane ticket was a physical piece of paper that you actually needed to have to take the plane. Of course, the University President's staff bought an expensive ticket and they were worried to send it by (regular mail): UPS, Fedex etc. with their overnight delivery were not operating in France or at least not thought of. Anyway, I was planning to spend the 1986-87 academic year at IAS in Princeton, so one day Jean Cerf came to my office and asked me to deliver the ticket to Bill as soon as I would arrive, and to ask Bill to notify by fax the staff of the University Presidency that the plane ticket has been delivered. It was very stressful: I seem to remember that the price of the ticket was more than my monthly salary. I arrived in Princeton late in the afternoon, hardly slept that night, first thing next day I ran to Fine Hall, found Bill, delivered the ticket and followed him to the secretary's office to make sure that the fax was sent. I felt much better afterwards.

I would like to end by mentioning New College in Sarasota (Florida) where Bill did his undergraduate studies (John Smillie was also an undergraduate there). I think that the informal and congenial atmosphere at this wonderful institution was instrumental in Bill's mathematical formation. The (apocryphal?) story I heard is that Bill spent four years at New College essentially reading Fricke and Klein's book (in German!). I discovered New College during my years at the University of Florida. After I returned to France to work at ENS Lyon, I visited several times New College trying (unsuccessfully) to attract some of the students to do their graduate studies in France. I was hoping that French mathematics would return to the next Bill Thurston, at least a small part of what Bill gave us.

22. Bill Abikoff

I spent the academic year 1976-77 as a Sloan Fellow at IHÉS. I was hoping it would be a quiet place to work on Kleinian groups and indeed it was so quiet that I tiptoed in the halls so as not to disturb anyone.

That all changed in the spring semester. Sullivan came back from the US and Gromov also arrived. Almost immediately, there were informal seminars in the hall with participants seated on the floor and sometimes shouting at each other. While in the States, Sullivan had proved that a compact metric manifold, of dimension unequal to four, admits a compatible Lipschitz structure. The theorem was unknown at IHÉS except to the Director, Nico Kuiper, who had heard of it during a visit to the US. Kuiper asked Sullivan to lecture on that theorem.

A seminar at Orsay had already started trying to understand Thurston's work on 2-manifolds. It was related to Teichmüller theory, and I was asked to lecture on Teichmüller's theorems. Bers arrived and he lectured on his complex-analytic proof of Thurston's result on the classification of mapping classes.

I mentioned in the seminar the question of whether a change of basepoint in Bers' embedding of the Teichmüller space induces a map of Teichmüller space which extends to the Bers boundary. My interest in the problem was its consequence that the whole mapping class group extends to the boundary. Neither of these results is true and we later learned that Thurston used a geometry on the Teichmüller space which is quite different from that of Bers.

By the spring, there were several seminars related to hyperbolic geometry at both IHÉS and Orsay. Bill Harvey lectured on the *curve complex* he had introduced; it is currently of great interest.

Even by the end of the Orsay seminar, we didn't really understand Thurston's 2-manifold work. The issue was how to attach the boundary to Teichmüller space in a fashion that the mapping class group action extends. People like me were still thinking in terms of classical Teichmüller theory, and not in terms of hyperbolic geometry.

Thurston had already moved on from surfaces to 3-manifolds. Siebenmann, who commuted several times that year between Orsay and Princeton came back with news about Thurston's *bounded image theorem*, and told us that Thurston had announced a proof of the hyperbolization theorem for 3-manifolds.

Marden had already shown that hyperbolic 3-manifolds, which arise in the context of classical Kleinian groups, are sufficiently large in the sense of Waldhausen; the hyperbolization theorem is a geometrization of the construction algorithm of Haken using, in the non-fibered case, the combination theorems due to Maskit. People decided that we should forget about the planned lectures and concentrate on that. I outlined the proof for non-fibered Haken manifolds in a four hour marathon session. The details of Thurston's ideas didn't even start to appear, in the notes prepared by Floyd and Kerckhoff, until two years later.

23. David Fried

I came to IHÉS in the spring of 1977 to meet and work with Dennis Sullivan. I soon learned that many mathematicians in Paris, including Dennis, were obsessed with the new results of Thurston and that there was an active seminar devoted to his remarkable work on surfaces and 3-manifolds. This was a learning opportunity for me and I was pleased to play a small part in this seminar.

It began when Dennis spoke on a novel invariant, the Thurston norm N of an oriented closed 3-manifold M. N is a geometrically defined seminorm on the first cohomology of M that takes integer values at each integral class u. Roughly speaking $N(u)$ is the minimal value of $|e(S)|$, where S is a closed aspherical surface in M associated to u and $e(S)$ is its Euler characteristic. Dennis vigorously explained why N was a seminorm. Using foliation theory, especially Thurston's thesis, he showed that $N(u) = |e(F)|$

when u corresponds to a fibration of M over the circle with fiber F. Dennis described Thurston's theorem that a seminorm in finite dimensions with integer values at integer points must have a finite-sided unit ball but he did not remember the proof and the results were not yet in preprint form.

I found a proof, however, and soon found myself in Orsay presenting it to the seminar. This was the first talk I ever gave in France and I recall one incident from it fondly. Someone in the audience inquired whether a seminorm with rational values at rational points must also have a finite-sided unit ball. I admitted that I didn't know the answer and I returned to the blackboard. Another participant rose, pondered this delightful question, stroked his beard, wandered the room, chatted to himself, and began to use the far end of my blackboard for his scratchwork. No one seemed to find this odd and I happily carried on with my talk.

I learned subsequently that the bearded thinker was Adrien Douady. At the next meeting of the seminar he presented his elegant counterexample: the norm N on the Cartesian plane whose unit ball is the convex hull of the union of two unit discs with centers $(-1, 0)$ and $(1, 0)$. The unit sphere of N meets each line through $(0, 0)$ with rational slope in a rational point, so N takes rational values at rational points.

I hope this suggests the fresh and open character of the Orsay seminar, which gave Thurston's work the close attention it deserved.

24. Dennis Sullivan

(A Decade of Thurston Stories)

First story.[21] In December of 1971, a dynamics seminar ended at Berkeley with the solution to a thorny problem in the plane which had a nice application in dynamics. The solution purported to move N distinct points to a second set of "epsilon near" N distinct points by a motion which kept the points distinct and only moved while staying always "epsilon prime near". The senior dynamicists in the front row were upbeat because the dynamics application up to then had only been possible in dimensions at least three where this matching problem is obvious by general position. But now the dynamics theorem also worked in dimension two.

A heavily bearded long haired graduate student in the back of the room stood up and said he thought the algorithm of the proof didn't work. He went shyly to the blackboard and drew two configurations of about seven points each and started applying to these the method of the end of the lecture. Little paths started emerging and getting in the way of other emerging paths which to avoid collision had to get longer and longer. The algorithm didn't work at all for this quite involved diagrammatic reason. I had never seen such comprehension and creative construction of a counterexample done so quickly. This combined with my awe at the sheer complexity of the geometry that emerged.

Second story. A couple of days later the grad students invited me (I was also heavily bearded with long hair) to paint math frescoes on the corridor

[21]Editor's note: From an email Sullivan sent to Athanase Papadopoulos, on April 27, 2019: "I wrote these stories at one sitting soon after Bill passed away."

wall separating their offices from the elevator foyer. While milling around before painting that same grad student came up to ask “Do you think this is interesting to paint?” It was a complicated smooth one-dimensional object encircling three points in the plane. I asked “What is it?” and was astonished to hear “It is a simple closed curve.” I said “You bet it’s interesting!”. So we proceeded to spend several hours painting this curve on the wall. It was a great learning and bonding experience. For such a curve to look good it has to be drawn in sections of short parallel slightly curved strands (like the flow boxes of a foliation) which are subsequently smoothly spliced together. When I asked how he got such curves, he said by successively applying to a given simple curve a pair of Dehn twists along intersecting curves. The “wall curve painting”, two meters high and four meters wide dated and signed, lasted on that Berkeley wall with periodic restoration for almost four decades before finally being painted over a few years ago (see Figure 7).

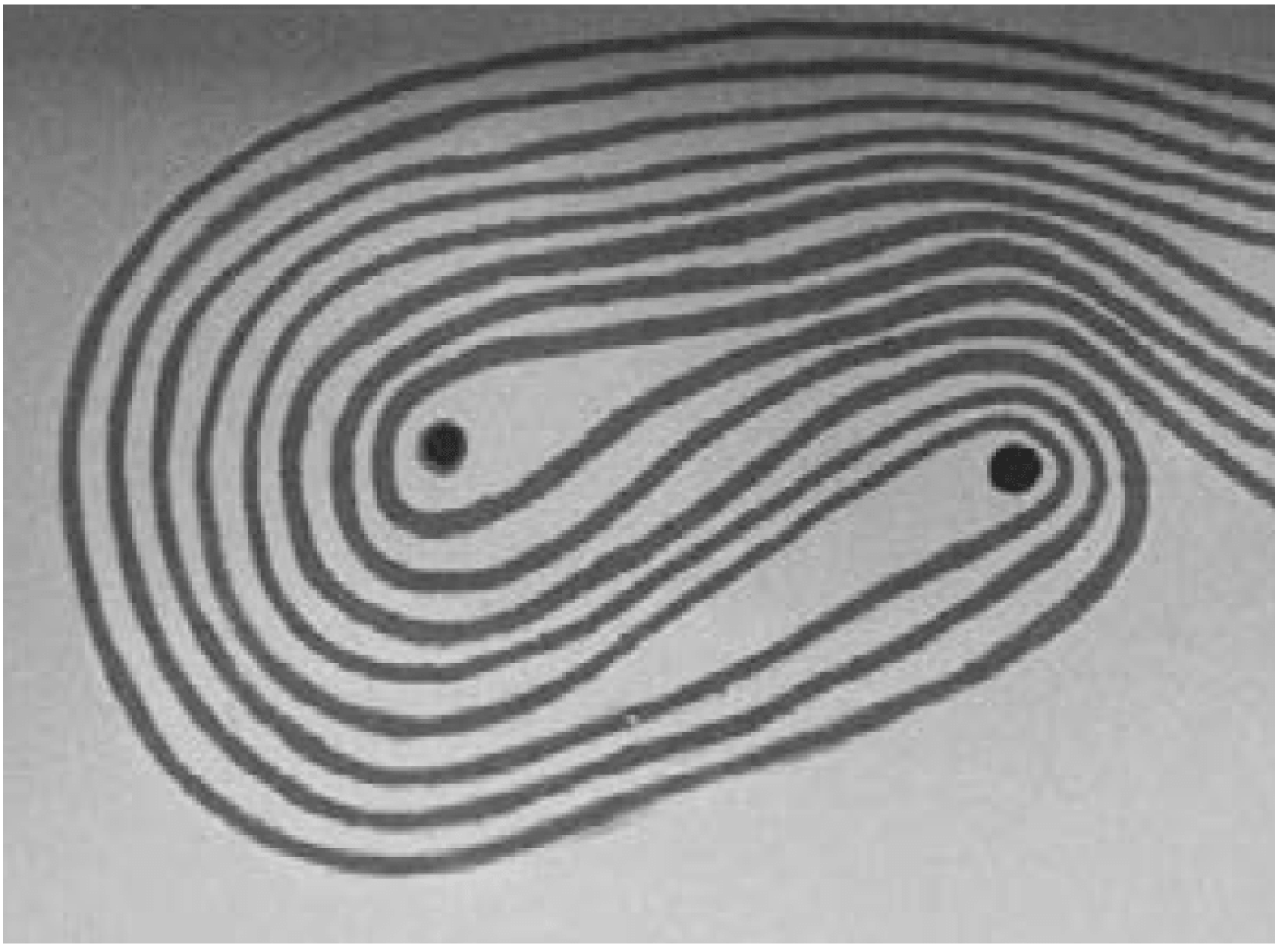


FIGURE 7. The Berkeley wall curve painting by Thurston and Sullivan

Third story. That week in December 1971 I was visiting Berkeley from MIT to give a series of lectures on differential forms and the homotopy theory of manifolds. Since foliations and differential forms were appearing everywhere, I thought to use the one-forms that emerged in my story describing the lower central series of the fundamental group to construct foliations. Leaves of these foliations would cover graphs of maps of the manifold to the nilmanifolds associated to all the higher nilpotent quotients of the fundamental group. These would generalize Abel's map to the torus associated with the first homology torus. Being uninitiated in Lie theory I had asked all the differential geometers at MIT and Harvard about this possibility but couldn't make myself understood. It was too vague/too algebraic. I presented the discussion in my first lecture at Berkeley and to Bill privately without much hope because of the weird algebra/geometry mixture. However the next day Bill came with a complete solution and a full explanation. For him it was elementary and really only involved actually understanding the basic geometric meaning of the Jacobi relation in the Elie Cartan $d \circ d = 0$ dual form.

In between the times of the first two stories above I had spoken to my old friend Mo Hirsch about Bill Thurston who was working with Mo and was finishing in his fifth year after an apparently slow start. Mo or someone else told how Bill's oral exam was a slight problem because when asked for an example of the universal cover of a space Bill chose the surface of genus two and started drawing awkward octogons with many [eight] coming together at each vertex. This exposition quickly became an unconvincing mess on the blackboard. I think Bill was the only one in the room who had thought about this nontrivial kind of universal cover. Mo then said "Lately, Bill has started solving thesis level problems at the rate of about one every month." Some years later I heard from Bill that his first child Nathaniel didn't like to sleep at night so Bill was sleep deprived "walking the floor with Nathaniel" for about a year of grad school.

That week of math at Berkeley was life changing for me. I was very grateful to be able to seriously appreciate the Mozart-like phenomenon I had been observing; and I had a new friend.

Upon returning to MIT after the week in Berkeley I related my news to the colleagues there, but I think my enthusiasm was too intense to be believed: "I have just met the best graduate student I have ever seen or ever expect to see." It was arranged for Bill to give a talk at MIT which evolved into a plan for him to come to MIT after going first to IAS in Princeton. It turned out he did come to MIT for just one year 1973-74.[22]

Fourth story, IAS Princeton 1972–73. When I visited the environs of Princeton from MIT in 72-73 I had a chance to interact more with Bill. One day walking outside towards lunch at IAS, I asked Bill what a horocyle was. He said "you stay here" and he started walking away into the Institute meadow.

After some distance he turned and stood still saying "You are on the circumference of a circle with me as center." Then he turned, walked much further away, turned back and said something which I couldn't hear because of the distance. After shouting back and forth to the amusement of the

[22]That year I visited IHES where I ultimately stayed for twenty odd years while Bill was invited back to Princeton, to the University.

members we realized he was saying the same thing "You are on the circumference of a circle with me as center". Then he walked even further away, just a small figure in the distance and certainly out of hearing, whereupon he turned and started shouting presumably the same thing again and again. We got the idea what a horocycle was.

The next day, Atiyah asked some of us topologists if we knew if flat vector bundles had a classifying space. (He had constructed some new characteristic classes for such.) We knew it existed from Brown's theorem but didn't know how to construct it explicitly. The next day, Atiyah said he asked Thurston this question who did it by what was then a shocking construction: take the Lie structure group of the vector bundle as an abstract group with the discrete topology and form its classifying space.

Later, I heard about Thurston drawing Jack Milnor a picture proving any dynamical pattern for any unimodal map appears in the quadratic family $x \mapsto x^2 + c$. Since I was studying dynamics, I planned to spend a semester with Bill at Princeton to learn about the celebrated Milnor–Thurston universality paper that resulted from this drawing.

Fifth story, Princeton University fall 1976. I expected to learn about one-dimensional dynamics upon arriving in Princeton in September 1976, but Thurston had already developed a new theory of surface transformations. The first few days, he expounded on this in a wonderful three hour extemporaneous lecture at the Institute. Luckily for me, the main theorem about limiting foliations was intuitively clear because of the painstaking Berkeley wall curve painting described above.

At the end of that semester Bill told me he believed the mapping torus of these carried hyperbolic metrics. When I asked why, he told me he couldn't explain it to me because I didn't understand enough differential geometry.

A few weeks after, I left Princeton, with more time to work and without my distractions. Bill essentially understood the proof of the hyperbolic metric for appropriate Haken manifolds. The mapping torus case took two more years as discussed below. During the semester grad course that Bill gave, the grad students and I learned several key ideas:

1) The quasi-analogue of "hyperbolic geometry at infinity becomes conformal geometry on the sphere at infinity"." (A notable memory here is the feeling that Bill conveyed about really being inside hyperbolic space rather than being outside and looking at a particular model. For me this made a psychological difference.)

2) We learned about the intrinsic geometry of convex surfaces outside the extreme points: Bill came into class one day, and, for many minutes, he rolled a paper contraption he had made around and around in the lecturer's table without saying a word until we felt the flatness.

3) We learned about the thick-thin decomposition of hyperbolic surfaces. I remember how Bill drew a 50 meter long thin part winding all around the blackboard near the common room, and suddenly everything was clear. Including geometric convergence to the points of the celebrated DM compactification of the space of Riemann surfaces.

During that fall '76 semester stay at Princeton, Bill and I discussed understanding the Poincaré conjecture by trying to prove a general theorem about all closed three manifolds based on the idea that three is a relatively small dimension. We included in our little paper on "canonical coordinates" the sufficient for Poincaré Conjecture possibility that all closed three manifolds carried conformally flat coordinate atlases.[23] However, an undergrad, Bill Goldman, who was often around, disproved this a few years later for the nilmanifold prime.[24] We decided to try to spend a year together in the future.

Meeting in the Alps, spring 1978. In the next period Bill developed limits of quasi-Fuchsian Kleinian groups and pursued the mapping torus hyperbolic structure in Princeton while I pursued the Ahlfors limit set measure problem in Paris. After about a year Bill had made substantial positive progress (e.g., closing the cusp) and I had made substantial negative progress (showing all known ergodic methods coupled with all known Kleinian group information were inadequate: there was too much potential nonlinearity). We met in the Swiss Alps at the Plans-sur-Bex conference and compared notes. His mapping torus program was positively finished but very complicated while my negative information actually revealed a rigidity result extending Mostow's, which allowed a simplification of Bill's proof.[25]

Sixth story, The Stony Brook meeting summer 1978. There was a big conference on Kleinian Groups at Stony Brook and Bill was in attendance but not as a speaker. Gromov and I got him to give a lengthy impromptu talk outside the schedule. It was a wonderful trip out into the end of a hyperbolic 3-manifold, combined with convex hulls, pleated surfaces and ending laminations . . . During the lecture Gromov leaned over and said watching Bill made him feel like "this field hadn't officially started yet."

Seventh story, Colorado June 1980 to August 1981. Bill and I shared the Stanislaw Ulam visiting chair at Boulder and ran two seminars, a big one drawing together all the threads for the full hyperbolic theorem and a smaller one on the dynamics of Kleinian groups and dynamics in general.

All aspects of the hyperbolic proof passed in review with many grad students in attendance.

One day in the other seminar Bill was late and Dan Rudolph was very energetically explaining in just one hour a new shorter version of an extremely complicated proof. The theorem promoted an orbit equivalence to a conjugacy between two ergodic transformations if the discrepancy of the orbit equivalence was controlled. The new proof was due to a subset of the triumvirate Katznelson, Weiss and Ornstein and was notable because it could be explained in one hour whereas the first proof took a mini-course to explain. Thurston at last came in and asked me to bring him up to speed, which I did. The lecture continued to the end with Bill wondering in loud whispers what the difficulty was and with me shushing him out of respect for the context. Finally, at the end, Bill said just imagine a bi-infinite string of

[23] This class is closed under connected sum and contains many prime three manifolds.

[24] When looking for Mo Hirsch's current email, I noticed he had over over 200 descendants with a dozen coming from all but two students, Bill Goldman with about 30 descendants and Bill Thurston with the rest.

[25] See my Bourbaki report on Bill's mapping torus theorem [45].

beads on a wire with finitely many missing spaces and just slide them all to the left say. Up to some standard bookkeeping this gave a new proof. Later that day an awestruck Dan Rudolph said to me he never realized before then just how smart Bill Thurston really was.

Eighth story, La Jolla and Paris end of summer 1981. The Colorado experience was very good, relaxing in the Thurston seminar with geometry (one day we worked out the eight geometries and another day we voted on terminology "manifolded" or "orbifold") and writing several papers of my own on Hausdorff dimension, dynamics and measures on dynamical limit sets.

Later closer to Labor day I was flying from Paris to La Jolla to give a series of AMS lectures on the dynamics stuff when I changed the plan and decided instead to try to expose the entire Hyperbolic Theorem "for the greater good" and as a self imposed Colorado final exam. I managed to come up with a one-page sketch while on the plane. There were to be two lectures a day for four or five days. The first day would be okay, I thought: just survey things and then try to improvise for the rest, but I needed a stroke of luck. It came big time.

There is a nine hour time difference between California and Paris and the first day I awoke around midnight local time and went to my assigned office to prepare. After a few hours I had generated many questions and fewer answers about the hyperbolic argument. I noticed a phone on the desk that miraculously allowed long distance calls and by then it was around 4:00 a.m. California time and 7:00 a.m. in Princeton. I called Bill's house, and he answered. I posed my questions. He gave quick responses, I took notes, and he said call back after he dropped kids at school and got to his office. I gave my objections to his answers around 9:30 a.m. his time and he responded more fully. We ended up with various alternate routes that all in all covered every point. By 8:00 a.m. my time I had a pair of lectures prepared. The first day went well: lecture/lunch/beach/swim, second lecture, dinner then goodby to colleagues and back to bed. This took some discipline but as viewers of the videos will see the audience was formidable (Ahlfors, Bott, Chern, Kirby, Siebenmann, Edwards, Rosenberg, Freedman, Yau, Maskit, Kra, Keen, Dodziuk,. . .) and I was motivated.

Bill and I repeated this each day, perfecting the back and forth so that by 8:00 a.m. California time each day, I had my two lectures prepared and they were getting the job done. The climax came when presenting Bill's delicious argument that controls the length of a geodesic representing the branching locus of a branched pleated surface by the dynamical rate of chaos or entropy created by the geodesic flow on the intrinsic surface. One knows that this is controlled by the area growth of the universal cover of the branched surface which by negative curvature is controlled by the volume growth of the containing hyperbolic three space QED. There was in addition Bill's beautiful example showing the estimate was qualitatively sharp. This splendid level of lecturing was too much for Harold Rosenberg, my astute friend from Paris, who was in the audience. He came to me afterwards and asked frustratingly "Dennis, do you keep Thurston locked up in your office upstairs?" The lectures were taped by Michael Freedman and I have kept my lips sealed until now. The taped (Thurston)-Sullivan lectures are available online.

Ninth story, Thurston in Paris fall 1981. Bill visited me in Paris and I bought a comfy sofa bed for my home office where he could sleep. He politely asked what would I have talked about had I not changed plans for the AMS lectures, and in particular what had I been doing in detail in Colorado beyond the hyperbolic seminar. There were about six papers to tell him about. One of the most appealing ideas I had learned from him. Namely the visual Hausdorff f-dimensional measure of an appropriate set on the sphere at infinity, as viewed from a point inside, defines a positive eigenfunction for the hyperbolic 3-space Laplacian with eigenvalue $f(2-f)$.

I started going through the ideas and statements. I made a statement and he either immediately gave the proof or I gave the idea of my proof. We went through all the theorems in the six papers in one session with either him or me giving the proof. There was one missing result that the bottom eigenfunction when f was > 1 would be represented by a normalized eigenfunction whose square integral norm was estimated by the volume of the convex core. Bill lay back for a moment on his sofa bed, his eyes closed, and immediately proved the missing theorem. He produced the estimate by diffusing geodesics transversally and averaging.

Then we went out to walk through Paris from Porte d'Orléans to Porte de Clignancourt. Of course we spoke so much about mathematics that Paris was essentially forgotten, except maybe the simultaneous view of Notre-Dame and the Conciergerie as we crossed over the Seine.

Tenth story, Princeton-Manhattan 1982-83. I began splitting time between IHES and the CUNY grad center where I started a thirteen year long Einstein chair seminar on dynamics and quasiconformal homeomorphisms (which changed then to quantum objects in topology) while Bill continued developing a cadre of young geometers to spread the beautiful ideas of negatively curved space. Bill delayed writing a definitive text on the hyperbolic proof in lieu of letting things develop along many opening avenues[26] by his increasingly informed cadre of younger/older geometers. He wanted to avoid what happened when his basic papers on foliations "tsunamied" the field in the early 1970s.

Once we planned to meet in Manhattan to discuss holomorphic dynamics in one variable and its analogies with hyperbolic geometry and Kleinian groups that I had been preoccupied with. We were not disciplined and began talking about other things at the apartment, and finally got around to our agenda about thirty minutes before he had to leave for his train back to Princeton. I sketched the general analogy: Poincaré limit set, domain of discontinuity, deformations, rigidity, classification, Ahlfors finiteness theorem, the work of Ahlfors–Bers,... to be compared with Julia set, Fatou set, deformations, rigidity, classification, non-wandering domain theorem, the

[26]I watched recently with great pleasure the unfolding of the ingenious proof by Kahn and Markovic of the hyperbolic subsurface conjecture. As each step was revealed I remembered when some key/crucial aspect of an analogous device was introduced by Bill more than thirty years before and then later taught to his proteges at Princeton.

work of Douady–Hubbard,... which he perfectly and quickly absorbed until he had to leave for the train. Two weeks later we heard about his reformulation of a holomorphic dynamical system as a fixed point on Teichmüller space analogous to part of his hyperbolic theorem. There were many new results including those of Curt McMullen some years later and the subject of holomorphic dynamics was raised to another higher level.

Postscript. Thurston and I met again at Milnor's 80th fest at Banff after essentially thirty years and picked up where we had left off. I admired his checked green shirt the second time it appeared and he presented it to me the next day. We promised to try to attack together a remaining big hole in the Kleinian group/holomorphic dynamics dictionary: "the invariant line field conjecture". It was a good idea but unfortunately turned out to be impossible. At the same conference, I recall a comment whispered by Bill who sat next to me during a talk by Jeremy Kahn about the Kahn–Markovich proof of the Subsurface Conjecture from decades before. Bill whispered : "I missed the 'offset' step."[27]

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[27]In the Kahn–Markovich proof, one glues up all possible ideal triangles building many immersed surfaces, and uses ergodic theory of the natural actions on this space. The offset idea is to glue the triangles not at their midpoints but after offsetting by a fixed large amount. This prevents missing everything in the limit. This step was the one that was missed before by Bill.

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