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Lieb-Liniger gas in a constant-force potential

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We use Gaudin's Fermi-Bose mapping operator to calculate exact solutions for the Lieb-Liniger model in a linear (constant-force) potential (the constructed exact stationary solutions are referred to as the Lieb-Liniger-Airy wave functions). The ground-state properties of the gas in the wedgelike trapping potential are calculated in the strongly interacting regime by using Girardeau's Fermi-Bose mapping and the pseudopotential approach in the $1/c$ approximation (c denotes the strength of the interaction). We point out that quantum dynamics of Lieb-Liniger wave packets in the linear potential can be calculated by employing an N-dimensional Fourier transform as in the case of free expansion.

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I. INTRODUCTION

One-dimensional (1D) Bose gases hold great potential for studying the physics of interacting quantum many-body systems. In recent years, a growing interest in studies of theoretical 1D models, first introduced by Lieb and Liniger [1] and Girardeau [2], has largely been inspired by experimental progress in realizing these models with ultracold atomic gases [3–7]. In experiments, an effectively one-dimensional system is achieved in elongated and transversely tight atomic wave guides, loaded with ultracold atoms, where transverse excitations are strongly suppressed [3–7]. These atomic gases are well described by the Lieb-Liniger model [1]—a system of identical Bose particles in 1D which interact via δ-function interactions of strength c. In the limit of infinite interaction strength ($c \to \infty$), the Bose particles can be described by the Tonks-Girardeau model [2], describing an "impenetrable" Bose gas. This regime occurs when effective interactions are strong, whereas temperatures and linear densities are low [8–10]. In passing, we note that the Lieb-Liniger model can also be realized in the field of quantum optics [11]. An ubiquitous part of the ultracold atomic experiments is the gravitational force, which can be nullified in a precise experiment, where 1D atomic wave guides are horizontal. However, to the best of our knowledge, exact solutions for the Lieb-Liniger gas (consisting of an arbitrary number of particles) in the presence of a constant external force have not been constructed yet.

An important breakthrough in the context of interacting Bose gases was achieved in 1963 when Lieb and Liniger found exact eigenstates for a 1D Bose gas with δ-function (repulsive) interactions of arbitrary strength c [1]. The underlying structure of the eigenstates reveals the Bethe ansatz, and each eigenstate is determined by a set of N quasimomenta, where N denotes the number of particles. If the quasimomenta obey a particular set of transcendental equations [1], the wave functions will obey periodic boundary conditions. This breakthrough was followed by studies of the model with attractive interactions [12]. A superposition of Lieb-Liniger eigenstates, where again quasimomenta obey a particular set of transcendental equations, can be used to construct the ground state in an infinitely deep box [13]. More recent studies of the Lieb-Liniger model (e.g., [14–21]) are mainly motivated by the growing experimental progress [3–7]. It should be noted that all of the aforementioned studies focus on a system where an external potential is zero in a given region of the x space, and boundary conditions are imposed on the wave function at the border(s) of this region (e.g., periodic [1], semi-infinite line [13], and infinitely deep box [13]). For other external potentials, there are studies of few-body systems [22,23]. Exact solutions were presented for $N = 2$ particles in a harmonic trap in Ref. [23]; solutions for $N = 2$ and $N = 3$ particles in a linear potential were presented in [22].

One approach in attempting to broaden the set of known exact solutions is by using Gaudin's Fermi-Bose mapping operator (call it $\hat{O}_c$) [24], which is applicable also to find time-dependent Lieb-Liniger wave functions [24–28]. In the limit of the Tonks-Girardeau gas $c \to \infty$, the Fermi-Bose mapping (which was discovered in 1960 [2]) can be utilized in any external potential [2] and for time-dependent problems [29] (for a review, see also [30]). For the finite coupling Lieb-Liniger gas (c finite), the method of Gaudin has been shown to be valid in the absence of any external potential (i.e., on an infinite line [25]), and has been used to study free expansion from localized initial conditions [26,27]; in this case the time-dependent wave function can be calculated via an N-dimensional Fourier transform. Interestingly, such a transform can be also utilized for a Lieb-Liniger gas reflecting from the wall [31]. However, Gaudin's method (at least in its current form) is not applicable to find eigenstates of a Lieb-Liniger gas in generic trapping potentials $V(x)$ (such as the harmonic oscillator); technically, this arises because the differential operator $\hat{O}_c$ does not generally commute with such potentials.

Here, we study the Lieb-Liniger model in the constant-force (linear) potential. Exact stationary solutions for this system are constructed (we call these wave functions the Lieb-Liniger-Airy states) by employing Gaudin's operator $\hat{O}_c$. The construction is enabled by the fact that this operator commutes with the linear (constant-force) potential. We calculate the ground-state properties of the Lieb-Liniger gas in the wedgelike potential [$V(x) = \alpha x$ for $x > 0$ ($\alpha > 0$), and ∞ otherwise] in the strongly interacting regime. This is achieved in the Tonks-Girardeau regime and below that regime in $1/c$ approximation

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by employing the pseudopotential approach [32]. Finally, we point out that the time-dependent Lieb-Liniger wave packets in the linear potential can be calculated via an N-dimensional Fourier transform.

II. LIEB-LINIGER-AIRY STATES

The Lieb-Liniger model describes N identical bosons in one spatial dimension which interact via (delta-function) contact potential [1]. In this section we consider this system placed in a linear external potential. The stationary Schrödinger equation for the many-body wave function $\psi_B(x_1,\ldots,x_N)$ in such a system is

$$E\psi_B = -\sum_{i=1}^{N}\frac{\partial^2\psi_B}{\partial x_i^2} + \sum_{1\leqslant i<j\leqslant N} 2c\,\delta(x_i-x_j)\psi_B + \alpha\sum_{i=1}^{N} x_i\psi_B, \tag{1}$$

where $c>0$ denotes the strength of the interaction, and $\alpha>0$ is the constant external force. Solutions of Eq. (1) for a single particle ($N=1$) are the Airy functions. For this reason, in what follows, we will call the solutions of Eq. (1) for $N>1$ the Lieb-Liniger-Airy (LLA) states (we are interested only in those solutions which decay to zero when $x\to\infty$). The constant force in ultracold atomic experiments can arise from the gravity force (e.g., if the one-dimensional atomic wave guides are tilted with respect to gravity).

In what follows, we will demonstrate that LLA states can be constructed via Gaudin's Fermi-Bose transformation [24]. Because of the bosonic symmetry of the wave functions, one can consider only the fundamental permutation sector of the coordinate space $R_1: x_1<x_2<\cdots<x_N$. Within this sector, the Schrödinger equation (1) reads

$$E\psi_B = -\sum_{i=1}^{N}\frac{\partial^2\psi_B}{\partial x_i^2} + \alpha\sum_{i=1}^{N} x_i\psi_B. \tag{2}$$

The interaction term is taken into account as a boundary condition (the so-called cusp condition), which is imposed upon ψ_B at the borders of R_1 (i.e., when two particles touch [1]):

$$\left[1-\frac{1}{c}\left(\frac{\partial}{\partial x_{j+1}}-\frac{\partial}{\partial x_j}\right)\right]_{x_{j+1}=x_j}\psi_B = 0. \tag{3}$$

Equation (2) holds in all other permutation sectors, whereas the interaction cusp (3) can be re-expressed on the borders of other sectors as well. To construct the LLA states we utilize Gaudin's Fermi-Bose mapping operator [24],

$$\hat{O}_c = \prod_{1\leqslant i<j\leqslant N}\left[\mathrm{sgn}(x_j-x_i)+\frac{1}{c}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\right], \tag{4}$$

which acts upon an antisymmetric (fermionic) wave function ψ_F. The wave function ψ_F must obey the Schrödinger equation for noninteracting spinless fermions in the linear potential:

$$E\psi_F = -\sum_{i=1}^{N}\frac{\partial^2\psi_F}{\partial x_i^2} + \alpha\sum_{i=1}^{N} x_i\psi_F. \tag{5}$$

The wave function ψ_F can be written in the form of Slater determinant with Airy functions as entries:

$$\psi_F = \alpha^{-\frac{N}{6}}\frac{1}{\sqrt{N!}}\det_{i,j=1}^{N}\mathrm{Ai}\left(\alpha^{\frac{1}{3}}x_j-\alpha^{-\frac{2}{3}}E_i\right), \tag{6}$$

where $E=\sum_{i=1}^{N}E_i$.

The LLA states [i.e., solutions of the Schrödinger Eq. (2), together with the cusp condition (3)], are given by

$$\psi_{B,c} = \mathcal{N}_c\hat{O}_c\psi_F, \tag{7}$$

where $\mathcal{N}_c$ is the normalization constant. It is known that all wave functions of the form (7) obey the cusp conditions throughout the configuration space [24–26]. To show that $\psi_{B,c}$ is also a solution of Eq. (2), it is sufficient to prove that the following commutators are zero: $[\sum_i \partial^2/\partial x_i^2,\hat{O}_c]=0$ and $[\sum_i x_i,\hat{O}_c]=0$; this is sufficient because ψ_F obeys Eq. (5). The first commutator is trivially satisfied, and therefore we are left to verify that

$$\left[\sum_i x_i,\hat{O}_c\right]=0. \tag{8}$$

As a first step, we restrict ourselves to the case of two particles, $N=2$. By using $[x_j,\partial/\partial x_i]=-\delta_{j,i}$, we have

$$\left[x_1+x_2,\mathrm{sgn}(x_2-x_1)+\frac{1}{c}\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_1}\right)\right] = \frac{1}{c}\left[x_2,\frac{\partial}{\partial x_2}\right]-\frac{1}{c}\left[x_1,\frac{\partial}{\partial x_1}\right]=0. \tag{9}$$

Now we generalize this for any number of particles N. Let us write the differential operator as $\hat{O}_c=\prod_{1\leqslant i<j\leqslant N}\hat{B}_{i,j}$, where

$$\hat{B}_{i,j}=\left[\mathrm{sgn}(x_j-x_i)+\frac{1}{c}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)\right]. \tag{10}$$

A general expression, $[\hat{V},\prod_{l=1}^{M}\hat{W}_l]=\sum_{l=1}^{M}\hat{W}_1\cdots\hat{W}_{l-1}[\hat{V},\hat{W}_l]\hat{W}_{l+1}\cdots\hat{W}_M$, valid for operators $\hat{V}$ and $\hat{W}_l$, $l=1,\ldots,M$, enables us to write the required commutator for the case of N particles:

$$\left[\sum_k x_k,\hat{O}_c\right]=\sum_{i<j}\hat{B}_{N-1,N}\cdots\left[\sum_k x_k,\hat{B}_{i,j}\right]\cdots\hat{B}_{1,2}. \tag{11}$$

Now Eq. (8) follows immediately because for any $\hat{B}_{i,j}$ we have $[\sum_k x_k,\hat{B}_{i,j}]=[x_i+x_j,\hat{B}_{i,j}]=0$, as is verified for the $N=2$ case. This completes the proof that the wave function $\psi_{B,c}$ defined in (7) is a solution of Eq. (1).

In this section we have found exact closed form solutions of Eq. (1). We point out that the eigenstates (7) with total energy E are degenerate, because the choice of single particle energies E_i for which $E=\sum_{i=1}^{N}E_i$ is not unique. By superposition of degenerate eigenstates (7), one can construct eigenstates which are of different mathematical form. In [22] the authors study Eq. (1) for $N=2$ and $N=3$ particles. They constructed solutions by introducing a new set of coordinates and separating Eq. (1). For $N=2$ they separate the center of mass and relative motion. Their solution for a given energy can be written as a superposition of eigenstates (7). For $N=3$ the procedure in [22] becomes more cumbersome, which clearly

points out the advantage of using Fermi-Bose transformation for solving Eq. (1).

III. THE LIEB-LINIGER GAS IN A WEDGELIKE POTENTIAL: STRONGLY INTERACTING LIMIT

In this section, we consider the Lieb-Liniger gas in the wedgelike potential defined as

$$V(x) = \begin{cases} x & \text{if } x \geqslant 0; \\ \infty & \text{if } x < 0. \end{cases} \tag{12}$$

For simplicity, we have fixed the value of the constant force to $\alpha = 1$. Solutions for any other value can be obtained by simple rescaling: $x \to \alpha^{1/3}x$ and $E \to \alpha^{-2/3}E$.

In order to find the ground state in such a potential, one should find solutions of Eqs. (2) and (3) (assuming we work in the fundamental sector R_1), together with the following boundary condition: $\psi_{B,c}(x_1 = 0, x_2, \ldots, x_N) = 0$. The first idea that may come to mind in attempting to find such a ground state is to utilize Eq. (7) as an ansatz, since it apparently obeys (2) and (3), and try to adjust the N free parameters E_i such that $\psi_{B,c}(x_1 = 0, x_2, \ldots, x_N) = 0$. Namely, such a procedure leads to the solutions for the ground states of a Lieb-Liniger gas on the ring [1], where instead of the ansatz (7) with Airy functions, one utilizes an ansatz with plane waves, $\psi_{B,c} = \mathcal{N}_c \hat{O}_c \det_{m,j=1}^{N} e^{ik_j x_m}$ (e.g., see [33]), and instead of E_j, one adjusts the quasimomenta k_j (which have to obey Bethe's equations) to acquire the proper boundary conditions. However, for this wedgelike potential such a line of reasoning fails. Mathematically, this occurs because the first derivative of the Airy function is not simply related to the Airy function itself (whereas a derivative of a plane wave is proportional to the plane wave itself).

Nevertheless, we can find solutions in the form (7) in the Tonks-Girardeau limit ($c \to \infty$), and we can utilize some form of $1/c$ approximation to find deviations from the Tonks-Girardeau ground state for large but finite c. The Tonks-Girardeau ground state is constructed by symmetrizing the Slater determinant of N lowest single-particle eigenstates [2]:

$$\psi_{\mathrm{TG}} = \Pi_{k<m}\mathrm{sgn}(x_m - x_k)\frac{1}{\sqrt{N!}}\det_{i,j=1}^{N}\phi_i(x_j), \tag{13}$$

where

$$\phi_i(x) = \frac{\mathrm{Ai}(x - E_i)}{\mathrm{Ai}'(-E_i)}. \tag{14}$$

The single-particle energies E_i are such that $\mathrm{Ai}(-E_i) = 0$ [i.e., $\phi_i(0) = 0$], and the eigenstates form an orthonormal set: $\int_0^\infty \phi_i^*(x)\phi_j(x)dx = \delta_{i,j}$. The ground-state energy is simply $E_{\mathrm{TG}} = \sum_{i=1}^{N} E_i$. As an illustration, in Fig. 1 we display the single-particle density for the Tonks-Girardeau ground state (dashed blue line) comprising $N = 10$ particles.

An approximative perturbative approach for calculating the properties of a Lieb-Liniger gas in the strongly interacting regime has been suggested by Sen [32]. It can be shown that

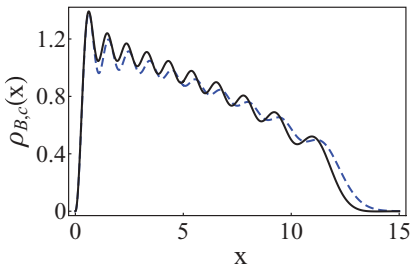

FIG. 1. (Color online) The single-particle density $\rho_{B,c}(x)$ (solid black line) of $N = 10$ Lieb-Liniger bosons in a wedgelike potential ($c = 40$, $\alpha = 1$). Dashed blue line shows the density in the Tonks-Girardeau limit.

the perturbation around $c = \infty$ (the Tonks-Girardeau limit) is correctly described by a pseudopotential [32]:

$$\hat{V}_{\mathrm{pp}} = -\frac{4}{c}\sum_{i<j}\delta''(x_i - x_j), \tag{15}$$

that is, the pseudopotential (15) is utilized as a small perturbation around the Tonks-Girardeau ground state (unperturbed state) for large c. It gives the correct first-order correction to the ground-state energy and wave function when plugged into the standard perturbation expressions with $1/c$ as a small parameter.

In the $1/c$ approximation, the ground-state energy of the Lieb-Liniger system is

$$\begin{aligned} E_{B,c} &= E_{\mathrm{TG}} + \langle\psi_{\mathrm{TG}}|\hat{V}_{\mathrm{pp}}|\psi_{\mathrm{TG}}\rangle \\ &= E_{\mathrm{TG}} - \frac{1}{c}N(N-1). \end{aligned} \tag{16}$$

Result (16) is obtained by a direct calculation of the expectation value of the pseudopotential $\hat{V}_{\mathrm{pp}}$ for the Tonks-Girardeau ground state. Such matrix elements are readily evaluated by using Slater-Condon rules:

$$\begin{aligned} &\langle\psi_{\mathrm{TG}}|\hat{V}_{\mathrm{pp}}|\psi_{\mathrm{TG}}\rangle \\ &\quad = -\frac{4}{c}\sum_{i<j}\int_0^\infty dx\left\{\phi_i^*(x)\phi_i(x)\frac{d^2}{dy^2}[\phi_j^*(y)\phi_j(y)]_{y=x}\right. \\ &\qquad \left. - \phi_i^*(x)\phi_j(x)\frac{d^2}{dy^2}[\phi_j^*(y)\phi_i(y)]_{y=x}\right\}. \end{aligned} \tag{17}$$

We have verified (16) numerically (by employing MATHEMATICA) up to $N = 20$ particles, and we conjecture that the expression is valid for any number of particles trapped by the potential (12).

To first order in $1/c$, the Lieb-Liniger wave function is given by [32]

$$\begin{aligned} \psi_{B,c} \approx \psi_{\mathrm{TG}} &+ \sum_{n\leqslant N, m>N}\frac{\langle\psi_{\mathrm{TG}}^{(m;n)}|\hat{V}_{\mathrm{pp}}|\psi_{\mathrm{TG}}\rangle}{E_n - E_m}\psi_{\mathrm{TG}}^{(m;n)} \\ &+ \sum_{\substack{n<n'\leqslant N \\ m'>m>N}}\frac{\langle\psi_{\mathrm{TG}}^{(m,m';n,n')}|\hat{V}_{\mathrm{pp}}|\psi_{\mathrm{TG}}\rangle}{E_n + E_{n'} - E_m - E_{m'}}\psi_{\mathrm{TG}}^{(m,m';n,n')}, \end{aligned} \tag{18}$$

where $\psi_{\mathrm{TG}}^{(m;n)}$ labels an excited Tonks-Girardeau state; this state is obtained from the ground state ψ_{TG} by replacing the

single-particle state ϕ_n, where $n \leqslant N$, with the single-particle state ϕ_m of higher energy, $m > N \geqslant n$. Analogously, $\psi_{\rm TG}^{(m,m';n,n')}$ labels two-particle excitation of the TG gas state. The expression for the single-particle density $\rho_{B,c}(x) = N\int dx_2 \cdots dx_N |\psi_{B,c}|^2$ can be calculated straightforwardly by employing the wave function from Eq. (18), by keeping the terms up to $1/c$:

$$\rho_{B,c}(x) \approx \rho_{\rm TG}(x) + N \sum_{n\leqslant N, m>N} \left(\frac{\langle \psi_{\rm TG}^{(m;n)} | \hat{V}_{\rm pp} | \psi_{\rm TG} \rangle}{E_n - E_m} \times \int dx_2 \cdots dx_N \, \psi_{\rm TG}^* \psi_{\rm TG}^{(m;n)} + {\rm c.c.} \right)$$
$$\approx \rho_{\rm TG}(x) + \frac{1}{c} \sum_{n\leqslant N, m>N} \frac{V_{\rm pp}^{(m;n)}}{E_n - E_m} \phi_n(x)\phi_m(x). \quad (19)$$

Here, the matrix element $V_{\rm pp}^{(m;n)} \equiv -8\langle \psi_{\rm TG}^{(m;n)} | \sum_{i<j} \delta''(x_i - x_j) | \psi_{\rm TG} \rangle$ of the single-particle excitation from the level $n \leqslant N$ with energy E_n, to the level $m > N$ with energy E_m is given by

$$V_{\rm pp}^{(m;n)} = -8 \sum_{i=1, i\neq n}^{N} \int_0^\infty dx \left\{ \phi_m^*(x)\phi_n(x) \frac{d^2}{dy^2} [\phi_i^*(y)\phi_i(y)]_{y=x} - \phi_m^*(x)\phi_i(x) \frac{d^2}{dy^2} [\phi_i^*(y)\phi_n(y)]_{y=x} \right\}. \quad (20)$$

In Fig. 1 we illustrate the single-particle density $\rho_{B,c}(x)$ in $1/c$ approximation (solid black line), which is obtained by using Eq. (19) for $N = 10$ and $c = 40$. It should be mentioned that the two-particle excitations [second sum in Eq. (18)] do not yield any contribution to the first-order single particle density $\rho_{B,c}(x)$, due to the vanishing of the overlap of the wave functions in calculation of the density (in the same way as demonstrated for the case of bosons confined in an infinitely deep box [32]). In our calculation of the density $\rho_{B,c}$ via (19), we have included only a finite number of terms, where the cutoff is chosen to be sufficiently large, such that the contribution of the remaining terms is negligible [for the calculation illustrated in Fig. 1, we kept 150 terms in Eq. (19) with the highest contribution].

IV. EXACT QUANTUM DYNAMICS VIA A FOURIER TRANSFORM

In this section we discuss the time-dependent solutions of the Lieb-Liniger system in a linear potential. Before proceeding, we note that dynamics in the strongly interacting regime (i.e., dynamics of a Tonks-Girardeau gas in a linear potential) was studied in Ref. [34]. Here, we assume that the bosons are initially localized by some external trapping potential. At time $t = 0$, this potential is suddenly turned off, and bosons are released to evolve in the linear potential. This problem can be related to free expansion of the Lieb-Liniger wave packet by simple rescaling of the coordinates. If the wave function $\psi_{\rm free}(x_1, \ldots, x_N, t)$ obeys the equation,

$$i\frac{\partial \psi_{\rm free}}{\partial t} = -\sum_{i=1}^{N} \frac{\partial^2 \psi_{\rm free}}{\partial x_i^2} + \sum_{1\leqslant i<j\leqslant N} 2c\,\delta(x_i - x_j)\psi_{\rm free}, \quad (21)$$

(i.e., $\psi_{\rm free}$ describes free expansion [26,27]), then the wave function,

$$\psi_{B,c}(x_1, \ldots, x_N, t) = e^{-i\alpha t \sum_{i=1}^{N}(x_i + \alpha t^2/3)} \psi_{\rm free}(x_1 + \alpha t^2, \ldots, x_N + \alpha t^2, t) \quad (22)$$

is the solution of the time-dependent problem in the constant-force potential,

$$i\frac{\partial \psi_{B,c}}{\partial t} = -\sum_{i=1}^{N} \frac{\partial^2 \psi_{B,c}}{\partial x_i^2} + \sum_{1\leqslant i<j\leqslant N} 2c\,\delta(x_i - x_j)\psi_{B,c} + \alpha \sum_{i=1}^{N} x_i \psi_{B,c}. \quad (23)$$

The initial conditions coincide (i.e., at $t = 0$ we have $\psi_{B,c} = \psi_{\rm free}$). Note that the phase factor in Eq. (22) accounts for the momentum per particle αt, which is acquired in time in the field of constant force α (in units used here, $m = 1/2$, and therefore the classical acceleration is 2α). Transformation (22) can be verified by direct substitution in Eq. (23), from which it becomes evident that it is valid for any two-particle interaction $V(x_i - x_j)$. Namely, transformation $x_i \to x_i + \alpha t^2$ does not affect the two-particle interaction term $V(x_i - x_j)$ [in fact, because of this, Eq. (22) can be deduced from the well-known solution for a single-particle wave packet in a linear potential].

It is known that freely expanding Lieb-Liniger wave packets can be calculated by solving an N-dimensional Fourier transform [26]:

$$\psi_{\rm free}(x_1, \ldots, x_N, t) = \int dk_1 \cdots dk_N G(k_1, \ldots, k_N)\, e^{i\sum_{i=1}^{N}(k_i x_i - k_i^2 t)}. \quad (24)$$

We note that the function G is *not* the Fourier transform of the wave function $\psi_{\rm free}$ because it depends on the coordinates x_j through the $\mathrm{sgn}(x_j - x_i)$ terms (see Refs. [26,27] for details), that is, it differs from one permutation sector in x space to the next. Nevertheless, by calculating the integral in Eq. (24) in one sector (say R_1), we obtain $\psi_{\rm free}$ in that sector, which is sufficient due to bosonic symmetry. The function G contains all information on initial conditions and it can be expressed in terms of the projections of the initial wave function on the Lieb-Liniger free space eigenstates (e.g., see Refs. [26,27]). By using Eqs. (24) and (22) we can express $\psi_{B,c}$ in terms of an N-dimensional Fourier transform:

$$\psi_{B,c}(x_1, \ldots, x_N, t) = \int dk_1 \cdots dk_N G(k_1, \ldots, k_N) \times \exp\left\{ i \sum_{i=1}^{N} \left[(k_i - \alpha t)x_i + \frac{(k_i - \alpha t)^3 - k_i^3}{3\alpha} \right] \right\}. \quad (25)$$

We would like to note that result (25) can be obtained by straightforward use of Fermi-Bose transformation. The time-dependent wave function ψ_F which describes the system

of N noninteracting fermions in a linear potential $V(x) = \alpha x$ can be written via its Airy transform:

$$\psi_F(x_1,\ldots,x_N,t) = \int dE_1 \cdots dE_N \bar{\psi}_F(E_1,\ldots,E_N) \times e^{-it\sum_{i=1}^{N} E_i} \prod_{i=1}^{N} \mathrm{Ai}(\alpha^{-2/3}(\alpha x_i - E_i)). \tag{26}$$

Here, $\bar{\psi}_F(E_1,\ldots,E_N)$ contains information on initial conditions,

$$\bar{\psi}_F(E_1,\ldots,E_N) = \left(\frac{1}{\alpha^{1/3}}\right)^N \int dx_1 \cdots dx_N \psi_F(x_1,\ldots,x_N,0) \times \prod_{i=1}^{N} \mathrm{Ai}(\alpha^{-2/3}(\alpha x_i - E_i)). \tag{27}$$

By using the well-known relation between the Airy and Fourier transform $\tilde{\psi}_F$ [35],

$$\bar{\psi}_F(E_1,\ldots,E_N) = \left(\frac{1}{\alpha^{2/3}}\right)^N \int dk_1 \cdots dk_N \tilde{\psi}_F\, e^{i\sum_{i=1}^{N}(k_i E_i - k_i^3/3)/\alpha}, \tag{28}$$

we find

$$\psi_F(x_1,\ldots,x_N,t) = \int dk_1 \cdots dk_N \tilde{\psi}_F \exp\left\{ i \sum_{i=1}^{N} \left[(k_i - \alpha t)x_i + \frac{(k_i - \alpha t)^3 - k_i^3}{3\alpha} \right] \right\}. \tag{29}$$

The time-dependent solution of the Lieb-Liniger model [i.e., Eq. (25)] can now be found directly from the expression above by applying the Fermi-Bose transformation operator $\hat{O}_c$ onto Eq. (29).

Our discussion in this section adds upon the previous studies of Lieb-Liniger wave-packet dynamics on an infinite line [24–27], and in the presence of the hard-wall potential [31]; in all these cases the motion of an interacting Lieb-Liniger wave packet can be calculated by using an N-dimensional Fourier transform.

In order to illustrate the connection between (21) and (22), we present the following numerical example. The system of three Lieb-Liniger bosons are trapped in the ground state of an infinitely deep box of length $L = \pi$; at $t = 0$, the trap is turned off and the bosons start to experience the constant force $\alpha = 3$. The exact initial wave function is constructed as a superposition of free space eigenstates [13]. From this state we can find the function $G(k_1,\ldots,k_N)$ which keeps all information on initial conditions [26,27]. By numerically calculating the integral in (25), we obtain the time-dependent wave function $\psi_{B,c}(x_1,\ldots,x_N,t)$ describing the system. Here,

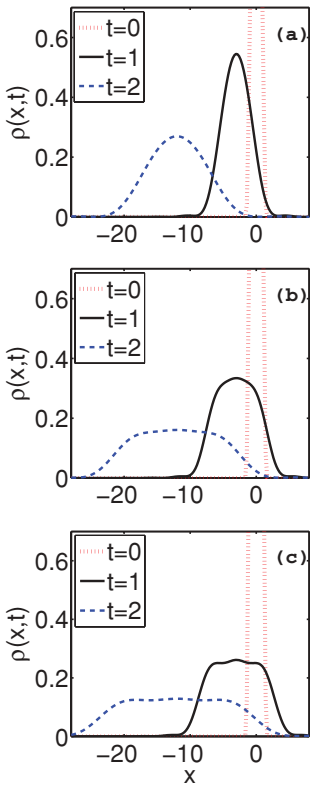

FIG. 2. (Color online) Evolution of $N = 3$ Lieb-Liniger bosons in the linear potential αx ($\alpha = 3$) from the ground state of a box with infinitely high walls. Single-particle density in time for various interaction strengths c: (a) $c = 0.25$, (b) $c = 3$, and (c) $c = 10$. Red dotted lines are for $t = 0$, solid black lines are for $t = 1$, and blue dashed lines are for $t = 2$.

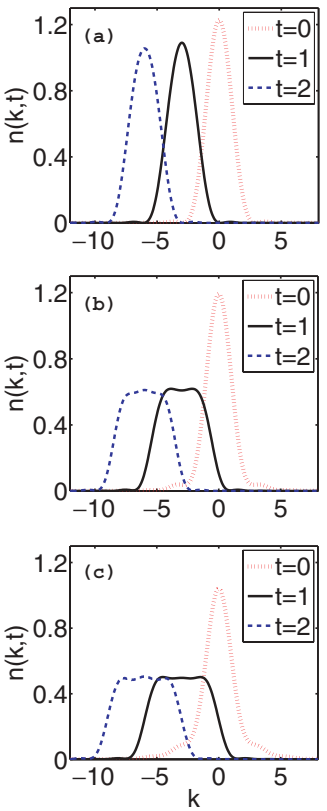

FIG. 3. (Color online) Evolution of the momentum distribution. The colors and lines for different c and t are identical as in Fig. 2.

we plot two relevant physical quantities, the single-particle density, $\rho_{B,c}(x,t) = N\int dx_2 \cdots dx_N |\psi_{B,c}(x,\ldots,x_N,t)|^2$, and the momentum distribution $n(k,t)$ (density in k space).

From Eq. (22) it follows that the density in coordinate space will be the same as in the case of free expansion ($\alpha = 0$), with mere translation of the coordinates [$\rho_{B,c}(x,t) = \rho_{\text{free}}(x + \alpha t^2,t)$]. The momentum distribution will be equivalent also up to the simple transformation $k \to k - \alpha t$. The density profile and momentum distribution of the wave packet are plotted in Figs. 2 and 3, respectively, for three various interaction strengths c: (a) $c = 0.25$, (b) $c = 3$, and (c) $c = 10$. Starting from $t = 0$, the wave packet evolves to the left in x space with the center of mass motion αt^2, while at the same time it spreads in width independently. For large c, the spread is more pronounced, as can also be conjectured from the initial momentum distribution. For very large c, the wave packet will asymptotically experience fermionization of the momentum distribution [36,37].

V. CONCLUSION

We have studied the Lieb-Liniger model in the constant-force (linear) potential. Exact stationary solutions for this system, referred to as the Lieb-Liniger-Airy states, were constructed by employing Gaudin's Fermi-Bose mapping operator $\hat{O}_c$. This was enabled by the fact that the operator commutes with the linear potential: $[\hat{O}_c, \sum_j \alpha x_j] = 0$. We have calculated the ground-state properties of the Lieb-Liniger gas, in the strongly interacting regime, in the wedgelike potential: $V(x) = \alpha x$ for $x > 0$ ($\alpha > 0$), and $V(x) = \infty$ for $x < 0$. This was achieved in the Tonks-Girardeau regime and in $1/c$ approximation by employing the pseudopotential approach [32]. Finally, we have pointed out that the time-dependent Lieb-Liniger wave packets in the linear potential can be found by employing an N-dimensional Fourier transform.

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