DIVISIBILITY OF THE CLASS NUMBERS OF IMAGINARY QUADRATIC FIELDS

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Abstract. For a given odd integer $n > 1$, we provide some families of imaginary quadratic number fields of the form $\mathbb{Q}(\sqrt{x^2 - t^n})$ whose ideal class group has a subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

1. Introduction

The divisibility properties of the class numbers of number fields are very important for understanding the structure of the ideal class groups of number fields. For a given integer $n > 1$, the Cohen-Lenstra heuristic [3] predicts that a positive proportion of imaginary quadratic number fields have class number divisible by n. Proving this heuristic seems out of reach with the current state of knowledge. On the other hand, many families of (infinitely many) imaginary quadratic fields with class number divisible by n are known. Most of such families are of the type $\mathbb{Q}(\sqrt{x^2 - t^n})$ or of the type $\mathbb{Q}(\sqrt{x^2 - 4t^n})$, where x and t are positive integers with some restrictions (for the former see [1, 10, 11, 13, 16, 17, 18, 19, 22, 24], and for the later see [5, 7, 9, 12, 15, 23]). Our focus in this article will be on the family $K_{t,x} = \mathbb{Q}(\sqrt{x^2 - t^n})$.

In 1922, T. Nagell [18] proved that for an odd integer n, the class number of imaginary quadratic field $K_{t,x}$ is divisible by n if t is odd, $(t, x) = 1$, and $q \mid x$, $q^2 \nmid x$ for all prime divisors q of n. Let b denote the square factor of $x^2 - t^n$, that is, $x^2 - t^n = b^2 d$, where $d < 0$ is the square-free part of $x^2 - t^n$. Under the condition $b = 1$, N. C. Ankeny and S. Chowla [1] (resp. M. R. Murty [16, Theorem 1], [17]) considered the family $K_{3,x}$ (resp. $K_{t,1}$, $K_{t,x}$). M. R. Murty also treated the family $K_{t,1}$ with $b < t^{n/4}/2^{3/2}$ ([16, Theorem 2]). Moreover, K. Soundararajan [22] (resp. A. Ito [10]) treated the family $K_{t,x}$ under the condition that $b < \sqrt{(t^n - x^2)/(t^{n/2} - 1)}$ holds (resp. all of prime divisors of b divide d). On the other hand, T. Nagell [19] (resp. Y. Kishi [13], A. Ito [11] and M. Zhu and T. Wang [24]) studied the family $K_{t,1}$ (resp. $K_{3,2^k}$, $K_{p,2^k}$ and $K_{t,2^k}$) unconditionally for b, where p is an odd prime. In

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the present paper, we consider the case when both t and x are odd primes and b is unconditional and prove the following:

Theorem 1.1. *Let $n \geq 3$ be an odd integer and p, q be distinct odd primes with $q^2 < p^n$. Let d be the square-free part of $q^2 - p^n$. Assume that $q \not\equiv \pm 1 \pmod{|d|}$. Moreover, we assume $p^{n/3} \neq (2q+1)/3, (q^2+2)/3$ whenever both $d \equiv 1 \pmod 4$ and $3 \mid n$. Then the class number of $K_{p,q} = \mathbb{Q}(\sqrt{d})$ is divisible by n.*

In Table 1 (respectively Table 2), we list $K_{p,q}$ for small values of p, q for $n = 3$ (respectively for $n = 5$). It is readily seen from these tables that the assumptions in Theorem 1.1 hold very often. We can easily prove, by reading modulo 4, that the condition "$p^{n/3} \neq (2q+1)/3, (q^2+2)/3$" in Theorem 1.1 holds whenever $p \equiv 3 \pmod 4$. Further, if we fix an odd prime q, then the condition "$q \not\equiv \pm 1 \pmod{|d|}$" in Theorem 1.1 holds almost always, and, this can be proved using the celebrated Siegel's theorem on integral points on affine curves. More precisely, we prove the following theorem in this direction.

Theorem 1.2. *Let $n \geq 3$ be an odd integer not divisible by 3. For each odd prime q the class number of $K_{p,q}$ is divisible by n for all but finitely many p's. Furthermore, for each q there are infinitely many fields $K_{p,q}$.*

2. Preliminaries

In this section we mention some results which are needed for the proof of the Theorem 1.1. First we state a basic result from algebraic number theory.

Proposition 2.1. *Let $d \equiv 5 \pmod 8$ be an integer and ℓ be a prime. For odd integers a, b we have*

$$\left(\frac{a + b\sqrt{d}}{2}\right)^{\ell} \in \mathbb{Z}[\sqrt{d}] \text{ if and only if } \ell = 3.$$

Proof. This can be easily proved by taking modulo some power of two. □

We now recall a result of Y. Bugeaud and T. N. Shorey [2] on Diophantine equations which is one of the main ingredient in the proof of Theorem 1.1. Before stating the result of Y. Bugeaud and T. N. Shorey, we need to introduce some definitions and notations.

Let F_k denote the kth term in the Fibonacci sequence defined by $F_0 = 0,\ F_1 = 1$ and $F_{k+2} = F_k + F_{k+1}$ for $k \geq 0$. Similarly L_k denotes the kth term in the Lucas sequence defined by $L_0 = 2,\ L_1 = 1$ and $L_{k+2} = L_k + L_{k+1}$ for $k \geq 0$. For

$\lambda \in \{1, \sqrt{2}, 2\}$, we define the subsets $\mathcal{F},\ \mathcal{G}_\lambda,\ \mathcal{H}_\lambda \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ by

$$\mathcal{F} := \{(F_{k-2\varepsilon}, L_{k+\varepsilon}, F_k) \,|\, k \geq 2, \varepsilon \in \{\pm 1\}\},$$

$$\mathcal{G}_\lambda := \{(1, 4p^r - 1, p) \,|\, p \text{ is an odd prime}, r \geq 1\},$$

$$\mathcal{H}_\lambda := \left\{ (D_1, D_2, p) \;\middle|\; \begin{array}{l} D_1,\ D_2 \text{ and } p \text{ are mutually coprime positive integers with} \\ p \text{ an odd prime and there exist positive integers } r,\ s \text{ such} \\ \text{that } D_1 s^2 + D_2 = \lambda^2 p^r \text{ and } 3D_1 s^2 - D_2 = \pm\lambda^2 \end{array} \right\},$$

except when $\lambda = 2$, in which case the condition "odd" on the prime p should be removed in the definitions of $\mathcal{G}_\lambda$ and $\mathcal{H}_\lambda$.

Theorem A. *Given $\lambda \in \{1, \sqrt{2}, 2\}$, a prime p and positive co-prime integers D_1 and D_2, the number of positive integer solutions (x, y) of the Diophantine equation*

$$D_1 x^2 + D_2 = \lambda^2 p^y \tag{1}$$

is at most one except for

$$(\lambda, D_1, D_2, p) \in \mathcal{E} := \left\{ \begin{array}{l} (2, 13, 3, 2), (\sqrt{2}, 7, 11, 3), (1, 2, 1, 3), (2, 7, 1, 2), \\ (\sqrt{2}, 1, 1, 5), (\sqrt{2}, 1, 1, 13), (2, 1, 3, 7) \end{array} \right\}$$

and $(D_1, D_2, p) \in \mathcal{F} \cup \mathcal{G}_\lambda \cup \mathcal{H}_\lambda$.

We recall the following result of J. H. E. Cohn [4] about appearance of squares in the Lucas sequence.

Theorem B. *The only perfect squares appearing in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$.*

3. Proofs

We begin with the following crucial proposition.

Proposition 3.1. *Let n, q, p, d be as in Theorem 1.1 and let m be the positive integer with $q^2 - p^n = m^2 d$. Then the element $\alpha = q + m\sqrt{d}$ is not an ℓ^{th} power of an element in the ring of integers of $K_{p,q}$ for any prime divisor ℓ of n.*

Proof. Let ℓ be a prime divisor of n. Since n is odd, so is ℓ.

We first consider the case when $d \equiv 2$ or $3 \pmod 4$. If α is an ℓ^{th} power, then there are integers a, b such that

$$q + m\sqrt{d} = \alpha = (a + b\sqrt{d})^\ell.$$

Comparing the real parts, we have

$$q = a^\ell + \sum_{i=1}^{(\ell-1)/2} \binom{\ell}{2i} a^{\ell-2i} b^{2i} d^i.$$

This gives $a \mid q$ and hence $a = \pm q$ or $a = \pm 1$.
Case (1A): $a = \pm q$.
We have $q + m\sqrt{d} = (\pm q + b\sqrt{d})^\ell$. Taking norm on both sides we obtain

$$p^n = (q^2 - b^2 d)^\ell.$$

Writing $D_1 = -d > 0$, we obtain

$$D_1 b^2 + q^2 = p^{n/\ell}.$$

Also, we have

$$D_1 m^2 + q^2 = p^n.$$

As ℓ is a prime divisor of n so $(x, y) = (|b|, n/\ell)$ and $(x, y) = (m, n)$ are distinct solutions of (1) in positive integers for $D_1 = -d > 0, D_2 = q^2, \lambda = 1$.

Now we verify that $(1, D_1, D_2, p) \notin \mathcal{E}$ and $(D_1, D_2, p) \notin \mathcal{F} \cup \mathcal{G}_\lambda \cup \mathcal{H}_\lambda$. This will give a contradiction. Clearly $(1, D_1, D_2, p) \notin \mathcal{E}$. Further, as $D_1 > 3$, we see that $(D_1, D_2, p) \notin \mathcal{G}_1$. From Theorem B, we see that $(D_1, D_2, p) \notin \mathcal{F}$. Finally, if $(D_1, D_2, p) \in \mathcal{H}_1$ then there are positive integers r, s such that

$$3D_1 s^2 - q^2 = \pm 1 \tag{2}$$

and

$$D_1 s^2 + q^2 = p^r. \tag{3}$$

By (2), we have $q \neq 3$, and hence we have $3D_1 s^2 - q^2 = -1$. From this together with (3), we obtain

$$4q^2 = 3p^r + 1,$$

that is

$$(2q - 1)(2q + 1) = 3p^r.$$

This leads to $2q - 1 = 1$ or $2q - 1 = 3$, but this is not possible as q is an odd prime. Thus $(D_1, D_2, p) \notin \mathcal{H}_1$.
Case (1B): $a = \pm 1$.
In this case we have $q + m\sqrt{d} = (\pm 1 + b\sqrt{d})^\ell$. Comparing the real parts on both sides, we get $q \equiv \pm 1 \pmod{|d|}$ which contradicts to the assumption "$q \not\equiv \pm 1 \pmod{|d|}$".

Next we consider the case when $d \equiv 1 \pmod 4$. If α is an ℓ^{th} power of some integer in $K_{p,q}$, then there are rational integers a, b such that

$$q + m\sqrt{d} = \left(\frac{a + b\sqrt{d}}{2}\right)^\ell, \quad a \equiv b \pmod 2.$$

In case both a and b are even, then we can proceed as in the case $d \equiv 2$ or $3 \pmod 4$ and obtain a contradiction under the assumption $q \not\equiv \pm 1 \pmod{|d|}$. Thus we can assume that both a and b are odd. Again, taking norm on both sides we obtain

$$4p^{n/\ell} = a^2 - b^2 d. \tag{4}$$

Since a, b are odd and $p \neq 2$, reading modulo 8 in (4) we get $d \equiv 5 \pmod 8$. As $\left(\frac{a+b\sqrt{d}}{2}\right)^\ell = q + m\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, by Proposition 2.1 we obtain $\ell = 3$. Thus we have

$$q + m\sqrt{d} = \left(\frac{a+b\sqrt{d}}{2}\right)^3.$$

Comparing the real parts, we have

$$8q = a(a^2 + 3b^2 d). \tag{5}$$

Since a is odd, therefore, we have $a \in \{\pm 1, \pm q\}$.
Case (2A): $a = q$.
By (5), we have $8 = q^2 + 3b^2 d$, and hence, $2 \equiv q^2 \pmod 3$. This is not possible.
Case (2B): $a = -q$.
By (4) and (5), we have

$$4p^{n/3} = q^2 - b^2 d \text{ and } 8 = -(q^2 + 3b^2 d).$$

From these, we have $3p^{n/3} = q^2 + 2$, which violates our assumption.
Case (2C): $a = 1$.
By (5) and $d < 0$, we have $8q = 1 + 3b^2 d < 0$. This is not possible.
Case (2D): $a = -1$.
By (4) and (5), we have

$$4p^{n/3} = 1 - b^2 d \text{ and } 8q = -(1 + 3b^2 d).$$

From these, we have $3p^{n/3} = 2q + 1$, which violates our assumption. This completes the proof. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1**.** Let m be the positive integer with $q^2 - p^n = m^2 d$ and put $\alpha = q + m\sqrt{d}$. We note that α and $\bar{\alpha}$ are co-prime and $N(\alpha) = \alpha\bar{\alpha} = p^n$. Thus we get $(\alpha) = \mathfrak{a}^n$ for some integral ideal $\mathfrak{a}$ of $K_{p,q}$. We claim that the order of $[\mathfrak{a}]$ in the ideal class group of $K_{p,q}$ is n. If this is not the case, then we obtain an odd prime divisor ℓ of n and an integer β in $K_{p,q}$ such that $(\alpha) = (\beta)^\ell$. As q and p are distinct odd primes, the condition "$q \not\equiv \pm 1 \pmod{|d|}$" ensures that $d < -3$. Also d is square-free, hence the only units in the ring of integers of $K_{p,q} = \mathbb{Q}(\sqrt{d})$ are ± 1.

Thus we have $\alpha = \pm\beta^\ell$. Since ℓ is odd, therefore, we obtain $\alpha = \gamma^\ell$ for some integer γ in $K_{p,q}$ which contradicts to Proposition 3.1. □

We now give a proof of Theorem 1.2. This is obtained as a consequence of a well known theorem of Siegel (see [8, 20]).

Proof of Theorem 1.2. Let $n > 1$ be as in Theorem 1.2 and q be an arbitrary odd prime. For each odd prime $p \neq q$, from Theorem 1.1, the class number of $K_{p,q}$ is divisible by n unless $q \equiv \pm 1 \pmod{|d|}$. If $q \equiv \pm 1 \pmod{|d|}$, then $|d| \leq q + 1$.

For any positive integer D, the curve

$$DX^2 + q^2 = Y^n \tag{6}$$

is an irreducible algebraic curve (see [21]) of genus bigger than 0. From Siegel's theorem (see [8, 20]) it follows that there are only finitely many integral points (X, Y) on the curve (6). Thus, for each $d < 0$ there are at most finitely many primes p such that

$$q^2 - p^n = m^2 d.$$

Since $K_{p,q} = \mathbb{Q}(\sqrt{d})$, it follows that there are infinitely many fields $K_{p,q}$ for each odd prime q. Further if p is large enough, then for $q^2 - p^n = m^2 d$, we have $|d| > q + 1$. Hence, by Theorem 1.1, the class number of $K_{p,q}$ is divisible by n for p sufficiently large. □

4. Concluding remarks

We remark that the strategy of the proof of Theorem 1.1 can be adopted, together with the following result of W. Ljunggren [14], to prove Theorem 4.1.

Theorem C. *For an odd integer n, the only solutions to the Diophantine equation*

$$\frac{x^n - 1}{x - 1} = y^2$$

in positive integers x, y, n with $x > 1$ is $n = 5, x = 3, y = 11$.

Theorem 4.1. *For any positive odd integer n and any odd prime p, the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{1 - p^n})$ is divisible by n except for the case $(p, n) = (3, 5)$.*

Theorem 4.1 alternatively follows from the work of T. Nagell ([19, Theorem 25]) which was elucidated by J. H. E. Cohn ([6, Corollary 1]). M. R. Murty gave a proof of Theorem 4.1 under condition either "$1 - p^n$ is square-free with $n > 5$" or "$m < p^{n/4}/2^{3/2}$ whenever $m^2 \mid 1 - p^n$ for some integer m with odd $n > 5$" ([16, Theorems 1 and 2]).

Now we give some demonstration for Theorem 1.1. All the computations in this paper were done using PARI/GP (version 2.7.6). Table 1 gives the list of imaginary quadratic fields $K_{p,q}$ corresponding to $n = 3$, $p \leq 19$ (and hence discriminant not exceeding 19^3). Note that the list does not exhaust all the imaginary quadratic fields $K_{p,q}$ of discriminant not exceeding 19^3. Table 2 is the list of $K_{p,q}$ for $n = 5$ and $p \leq 7$.

Table 1: Numerical examples of Theorem 1 for $n = 3$.

p	q	$q^2 - p^n$	d	$h(d)$	p	q	$q^2 - p^n$	d	$h(d)$
3	5	-2	-2	1*	5	3	-116	-29	6
5	7	-76	-19	1**	7	3	-334	-334	12
7	5	-318	-318	12	7	11	-222	-222	12
7	13	-174	-174	12	7	17	-54	-6	2*
11	3	-1322	-1322	42	11	5	-1306	-1306	18
11	7	-1282	-1282	12	11	13	-1162	-1162	12
11	17	-1042	-1042	12	11	19	-970	-970	12
11	23	-802	-802	12	11	29	-490	-10	2*
11	31	-370	-370	12	11	37	-38	-38	6*
13	3	-2188	-547	3	13	5	-2172	-543	12
13	7	-2148	-537	12	13	11	-2076	-519	18
13	17	-1908	-53	6	13	19	-1836	-51	2**
13	23	-1668	-417	12	13	29	-1356	-339	6
13	31	-1236	-309	12	13	37	-828	-23	3
13	41	-516	-129	12	13	43	-348	-87	6
13	47	-12	-3	1*	17	3	-4904	-1226	42
17	5	-4888	-1222	12	17	7	-4864	-19	1**
17	11	-4792	-1198	12	17	13	-4744	-1186	24
17	19	-4552	-1138	12	17	23	-4384	-274	12
17	29	-4072	-1018	18	17	31	-3952	-247	6
17	37	-3544	-886	18	17	41	-3232	-202	6
17	43	-3064	-766	24	17	47	2704	-1	1*
17	53	-2104	-526	12	17	59	-1432	-358	6
17	61	-1192	-298	6	17	67	-424	-106	6
19	3	-6850	-274	12	19	5	-6834	-6834	48
19	7	-6810	-6810	48	19	11	-6738	-6738	48
19	13	-6690	-6690	72	19	17	-6570	-730	12
19	23	-6330	-6330	48	19	29	-6018	-6018	48
19	31	-5898	-5898	48	19	37	-5490	-610	12
19	41	-5178	-5178	48	19	43	-5010	-5010	48
19	47	-4650	-186	12	19	53	-4050	-2	1*
19	59	-3378	-3378	24	19	61	-3138	-3138	24

Continued on next page

Table 1 – continued from previous page

p	q	$q^2 - p^n$	d	$h(d)$	p	q	$q^2 - p^n$	d	$h(d)$
19	67	-2370	-2370	24	19	71	-1818	-202	6
19	73	-1530	-170	12	19	79	-618	-618	12

Table 2: Numerical examples of Theorem 1 for $n = 5$.

p	q	$q^2 - p^n$	d	$h(d)$	p	q	$q^2 - p^n$	d	$h(d)$
3	5	-218	-218	10	3	7	-194	-194	20
3	11	-122	-122	10	3	13	-74	-74	10
5	3	-3116	-779	10	5	7	-3076	-769	20
5	11	-3004	-751	15	5	13	-2956	-739	5
5	17	-2836	-709	10	5	19	-2764	-691	5
5	23	-2596	-649	20	5	29	-2284	-571	5
5	31	-2164	-541	5	5	37	-1756	-439	15
5	41	-1444	-1	1*	5	43	-1276	-319	10
5	47	-916	-229	10	5	53	-316	-79	5
7	3	-16798	-16798	60	7	5	-16782	-16782	100
7	11	-16686	-206	20	7	13	-16638	-16638	80
7	17	-16518	-16518	60	7	19	-16446	-16446	100
7	23	-16278	-16278	80	7	29	-15966	-1774	20
7	31	-15846	-15846	160	7	37	-15438	-15438	80
7	41	-15126	-15126	120	7	43	-14958	-1662	20
7	47	-14598	-1622	30	7	53	-13998	-13998	100
7	59	-13326	-13326	100	7	61	-13086	-1454	60
7	67	-12318	-12318	60	7	71	-11766	-11766	120
7	73	-11478	-11478	60	7	79	-10566	-1174	30
7	83	-9918	-1102	20	7	89	-8886	-8886	60
7	97	-7398	-822	20	7	101	-6606	-734	40
7	103	-6198	-6198	40	7	107	-5358	-5358	40
7	109	-4926	-4926	40	7	113	-4038	-4038	60
7	127	-678	-678	20					

In both the tables we use $*$ in the column for class number to indicate the failure of condition "$q \not\equiv \pm 1 \pmod{|d|}$" of Theorem 1.1. Appearance of $**$ in the column for class number indicates that both the conditions "$q \not\equiv \pm 1 \pmod{|d|}$ and $p^{n/3} \neq (2q+1)/3, (q^2+3)/3$" fail to hold. For $n = 3$, the number of imaginary quadratic number fields obtained from the family provided by T. Nagell (namely $K_{t,1}$ with t any odd integer) with class number divisible by 3 and discriminant not exceeding 19^3 are at most 9, whereas, in Table 1 there are 59 imaginary quadratic fields $K_{p,q}$

with class number divisible by 3 and discriminant not exceeding 19^3 (Table 1 does not exhaust all such $K_{p,q}$). Out of these 59 fields in Table 1, the conditions of Theorem 1.1 hold for 58. This phenomenon holds for all values of n.

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