Nonlin. Processes Geophys., 14, 351–359, 2007
www.nonlin-processes-geophys.net/14/351/2007/

Nonlinear Processes in Geophysics

Three-dimensional simulations of turbulent spectra in the local interstellar medium

D. Shaikh and G. P. Zank

Institute of Geophysics and Planetary Physics, University of California, Riverside CA 92521, USA

Received: 8 May 2007 – Revised: 8 May 2007 – Accepted: 26 June 2007 – Published: 6 July 2007

Abstract. Three-dimensional time dependent numerical simulations of compressible magnetohydrodynamic fluids describing super-Alfvénic, supersonic and strongly magnetized space and laboratory plasmas show a nonlinear relaxation towards a state of near incompressibility. The latter is characterized essentially by a subsonic turbulent Mach number. This transition is mediated dynamically by disparate spectral energy dissipation rates in compressible magnetosonic and shear Alfvénic modes. Nonlinear cascades lead to super-Alfvénic turbulent motions decaying to a sub-Alfvénic regime that couples weakly with (magneto)acoustic cascades. Consequently, the supersonic plasma motion is transformed into highly subsonic motion and density fluctuations experience a passive convection. This model provides a self-consistent explaination of the ubiquitous nature of incompressible magnetoplasma fluctuations in the solar wind and the interstellar medium.

1 Introduction

Density fluctuations in the interstellar medium (ISM) exhibit a Kolmogorov-like spectrum over an extraordinary range of scales (from an outer scale of a few parsecs to scales of 200 km or less) with a spectral index close to –5/3 (Armstrong et al., 1981, 1995). These fluctuations are detected with great sensitivity by Very Long Baseline Interferometer (VLBI) phase scintillation measurements (Armstrong et al., 1995; Spangler, 2001). In interstellar plasma turbulence, the plasma density fluctuates randomly in time and space. As the radio refractive index is proportional to the plasma density, there will be corresponding variations in the refractive index. The angular broadening measurements also reveal, more precisely, a Kolmogorov-like power spectrum for the density fluctuations in the interstellar medium with a spectral exponent slightly steeper than –5/3 (Spangler, 1999). Regardless of the exact spectral index, the density irregularities exhibit a definite power-law spectrum that is essentially characteristic of a fully developed isotropic and statistically homogeneous incompressible fluid turbulence, described by Kolmogorov (1941) for hydrodynamic and Kraichnan (1965) for magnetohydrodynamic fluids. This means that turbulence, manifested by interstellar plasma fluid motions, plays a major role in the evolution of the ISM plasma density, velocity, magnetic fields, and the pressure. Radio wave scintillation data indicates that the rms fluctuations in the ISM and interplanetary medium density, of possibly turbulent origin and exhibiting Kolmogorov-like behaviour, are only about 10% of the mean density (Matthaeus et al., 1991; Spangler, 2001). This suggests that ISM density fluctuations are only weakly compressible. Despite the weak compression in the ISM density fluctuations, they nevertheless admit a Kolmogorov-like power law, an ambiguity that is not yet completely resolved by any fluid/kinetic theory or computer simulations. That the Kolmogorov-like turbulent spectrum stems from purely incompressible fluid theories (Kolmogorov, 1941; Kraichnan, 1965) of hydrodynamics and magnetohydrodynamics offers the simplest possible turbulence description in an isotropic and statistically homogeneous fluid. However, since the observed electron density fluctuations in the ISM possess a weak degree of compression, the direct application of such simplistic turbulence models to understanding the ISM density spectrum is not entirely obvious. Moreover, the ISM is not a purely incompressible medium and can possess many instabilities because of gradients in the fluid velocity, density, magnetic field etc. where incompressibility is certainly not a good assumption. A fully self-consistent description of the ISM fluid, one that couples the incompressible modes with the compressible modes and deals with the strong nonlinear interactions amongst the ISM density, velocity and the magnetic field, is, therefore highly desirable.

Correspondence to: D. Shaikh
(dastgeer@ucr.edu)

Published by Copernicus Publications on behalf of the European Geosciences Union and the American Geophysical Union.

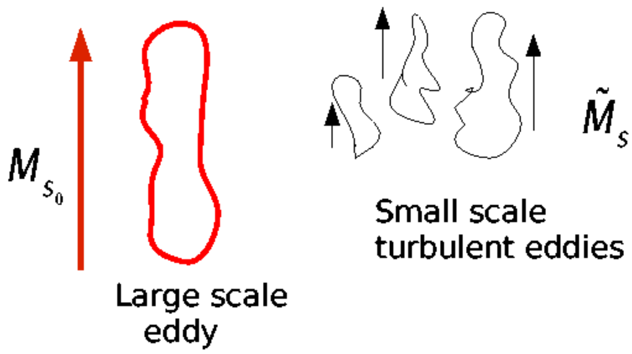

Fig. 1. Schematic of the Mach number as determined from the large-scale flows (left) and small-scale fluctuations (right). A large-scale flow or constant mean background flow leads typically to a constant Mach number, whereas local fluctuating eddies give rise to turbulent Mach numbers which depend upon local properties of high frequency and smaller-scale turbulent fluctuations.

Compressibility is therefore an intrinsic characteristic of interplanetary, interstellar, and laboratory magnetohydrodynamic (MHD) plasmas. A compressible fluid admits perturbative motions that have speeds comparable to the local sound speed. A fluctuation or turbulent Mach number defined locally as $M_s=U/C_s$, $C_s^2=\gamma p/\rho$, therefore expresses the compressibility of the magnetofluid. Since a compressible magnetofluid contains magnetoacoustic modes, an incompressible magnetofluid corresponds to a fluid in which fast-scale modes are absent. Understanding why magnetofluids observed in the solar wind or laboratory frequently behave as though they are incompressible has proved a major challenge to our understanding of small-scale dynamical processes in a plasma. The past 15 years have witnessed an effort to understand this apparent paradox of compressible MHD behaving as though it were incompressible in a variety of environments ranging from the solar wind (Matthaeus and Brown, 1988; Zank and Matthaeus, 1990; Bhattacharjee et al., 1998) to the interstellar medium (ISM). A significant motivation for all these studies, as mentioned above, is the observation of a Kolmogorov-like density spectrum over decades in wavenumber space that appears to pervade the ISM (Armstrong et al., 1981, 1995). Why an apparently compressive characteristic of ISM turbulence should behave as though it were a manifestation of incompressible MHD turbulence has yet to be answered conclusively although numerous attempts have been made (Higdon, 1984, 1986; Montgomery et al., 1987; Zank and Matthaeus, 1993; Lithwick and Goldreich , 2001; Bhattacharjee et al., 1998; Cho and Lazarian, 2003; Dastgeer and Zank, 2004a,b,c, 2006a). In this paper, we address the fundamental question of under what conditions fully compressible 3-D MHD turbulence can relax to an incompressible state. This question has to be answered if we are to address the outstanding question regarding the origin of the ISM density power law spectrum.

In this paper, we explore the dynamics of multiple scale coupling in a super-Alfvénic, supersonic, and a strongly magnetized compressible MHD plasma. Remarkably, we find a profound tendency of decaying compressible MHD turbulence to evolve towards a subsonic regime by segregating spectral energy in shear Alfvénic and fast/slow magnetosonic modes (MHD waves). In the subsonic regime, the compressibility of an MHD plasma weakens substantially and leads to a state of *near* incompressibility, in which density fluctuations advect only passively. The equations governing MHD are discussed in section II. Section III deals with the nonlinear three-dimensional MHD simulations of compressible magnetoplasmas, whereas section IV describes theoretical aspects of the simulation results in the context of nearly incompressible flows. A discussion concerning the development of NI theory is therefore outlined in the appendix A. Finally, conclusions are presented in Sect. V.

2 MHD model

Statistically homogeneous, isotropic and isothermal MHD plasma can be cast in terms of a single fluid density $\rho(\mathbf{r},t)$, magnetic $\mathbf{B}(\mathbf{r},t)$ and velocity $\mathbf{U}(\mathbf{r},t)$ fields and pressure $p(\mathbf{r},t)$ as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U})=0, \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{U}\times\mathbf{B})+\eta\nabla^2\mathbf{B}, \quad (2)$$

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\mathbf{U}=-\nabla p+\frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\mathbf{U}+\hat{\eta}\nabla(\nabla\cdot\mathbf{U}). \quad (3)$$

$$\nabla\cdot\mathbf{B}=0. \quad (4)$$

The equations are closed with an equation of state relating the perturbed density to the pressure variables. Here $\mathbf{r}=x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ is a three dimensional vector, η and ν are, respectively, magnetic and kinetic viscosities. The above equations can be normalized using a typical length scale (ℓ_0), density (ρ_0), pressure (p_0), magnetic field (B_0) and the velocity (U_0). With respect to these normalizing ambient quantities, one may define a constant sound speed $C_{s_\circ}=\sqrt{\gamma p_0/\rho_0}$, sonic Mach number $M_{s_\circ}=U_0/C_{s_\circ}$, Alfvén speed $V_{A_\circ}=B_0/\sqrt{4\pi\rho_0}$, and Alfvénic Mach number $M_{A_\circ}=U_0/V_{A_\circ}$. The magnetic and mechanical Reynolds numbers are $R_{m_\circ}\approx U_0\ell_0/\eta$ and $R_{e_\circ}\approx U_0\ell_0/\nu$, and the plasma beta $\beta_0=8\pi p_0/B_0^2$. While these quantities arise purely out of the normalizations and are associated with a bulk (large-scale) plasma motion, there can exist turbulent speeds, Mach and Reynolds numbers which depend locally on the small-scale and relatively high frequency fluctuations. This is illustrated schematically in Fig. 1. It is this component that

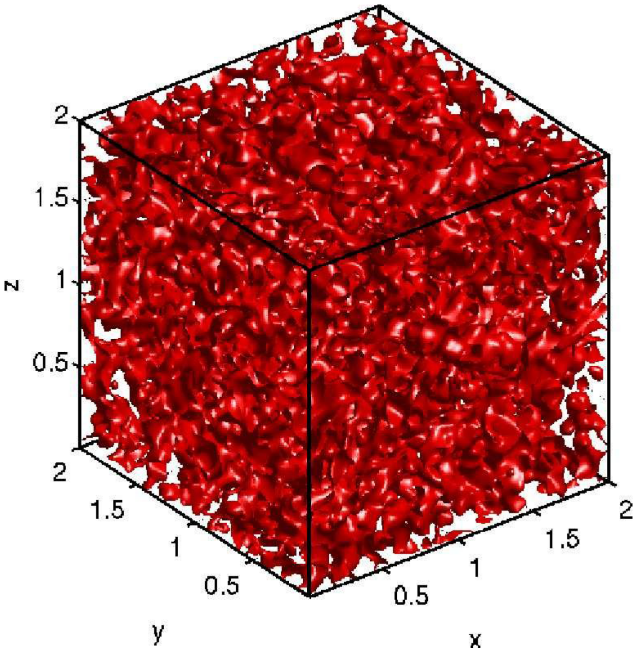

Fig. 2. Snap-shot of turbulent magnetic field in three dimensions. Shown are the iso-surfaces of $|\mathbf{B}|$ turbulent fluctuations at intermediate time step.

describes the high frequency contribution corresponding to the acoustic time-scales in the modified pseudosound relationship proposed in the Nearly Incompressible (NI) theory by Zank and Matthaeus (1990, 1991, 1993). A brief discussion of NI theory is outlined in the Appendix. We define the sound speed excited by the small-scale turbulent motion as

$$\tilde{C}_s(\mathbf{r}, t)=\sqrt{\gamma}\rho^{(\gamma-1)/2},$$

γ being the ratio of the specific heats, the sonic turbulent Mach number

$$\tilde{M}_s(\mathbf{r}, t) = \frac{\sqrt{\langle|\mathbf{U}|^2\rangle}}{\tilde{C}_s},$$

the fluctuating Alfvénic speed $\tilde{V}_A=\tilde{B}/\sqrt{4\pi\tilde{\rho}}$, and the turbulent Alfvénic Mach number

$$\tilde{M}_A(\mathbf{r}, t)=\frac{\sqrt{\langle|\mathbf{U}|^2\rangle}}{\tilde{V}_A}.$$

The turbulent Reynolds numbers and plasma beta $\tilde{\beta}$ can be defined correspondingly. Note that we follow the evolution of these local quantities to understand the predominance of Alfvénic fluctuations in the Solar wind and local interstellar medium which are believed to be responsible for turbulent cascades of energy and the origin of the density fluctuation spectrum. Furthermore, all the small-scale fluctuating parameters are measured in terms of their respective normalized quantities.

3 Nonlinear three-dimensional simulations

Nonlinear mode coupling interaction studies in three (3-D) dimensions are performed to investigate the multi-scale evolution of a decaying compressible MHD turbulence described by the closed set of Eqs. (1) and (4). All the fluctuations are initialized isotropically (no mean fields are assumed) with random phases and amplitudes in Fourier space and evolved further by integration of Eqs. (1) and (4) using a fully de-aliased pseudospectral numerical scheme (Gottlieb et al., 1977). This algorithm conserves energy in terms of the dynamical fluid variables rather than using a separate energy equation written in a conservative form (Ghosh et al., 1993). The evolution variables are discretized in Fourier space and we use periodic boundary conditions. The initial isotropic turbulent spectrum was chosen (for solenoidal as well as irrotational velocity components) to be close to k^{-2} with random phases in all three directions. The choice of such (or even a flatter than –2) spectrum treats the solenoidal as well as the irrotational components of the velocity field on an equal footing and avoids any influence on the dynamical evolution that may be due to the initial spectral non-symmetry. The equations are advanced in time using a second-order predictor-corrector scheme. The code is made stable by a proper de-aliasing of spurious Fourier modes and choosing a relatively small time step in the simulations. Our code is massively parallelized using Message Passing Interface (MPI) libraries to facilitate higher resolution in a 3-D volume. It satifies the condition of incompressibility associated with the magnetic field, i.e. $\nabla\cdot\mathbf{B}=0$, at each time step, and also conserves the total energy, i.e. $E=1/2\int(|\mathbf{U}|^2+|\mathbf{B}|^2)\,d\mathbf{v}$, and cross helicity $H=1/2\int|\mathbf{U}\cdot\mathbf{B}\,d\mathbf{v}$, in the absence of dissipation, throughout the simulation time. Kinetic and magnetic energies are equi-partitioned between the initial velocity and the magnetic fields. The latter helps treat the transverse or shear Alfvén and the fast/slow magnetosonic waves on an equal footing, at least during the early phase of the simulations. MHD turbulence evolves under the action of nonlinear interactions in which larger eddies transfer their energy to smaller ones through a forward cascade.

The energy in the smaller Fourier modes migrates towards the higher Fourier modes following essentially the vector triad interactions $\mathbf{k}+\mathbf{p}=\mathbf{q}$. These interactions involve the neighboring Fourier components $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ that are excited in the local inertial range turbulence. We have performed a number of simulations to verify the consistency of of our results by varying resolution, box size, small-dissipation parameter and other constants. A snap shot of turbulent magnetic field fluctuations at a resolution of 256^3 in a cube of volume 2^3 during the evolution is shown in Fig. 2. The corresponding density fluctuations are shown in Fig. 3. During this nonlinear spectral transfer process, MHD turbulent fluctuations are dissipated gradually due to the finite Reynolds number, thereby damping small scale motion as well. There also exists nonlinear damping that can occur due

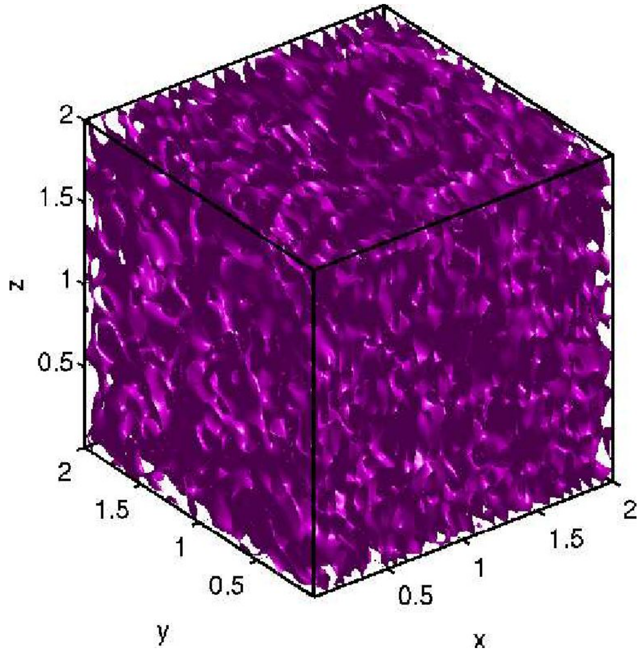

Fig. 3. Iso-surfaces of the density fluctuations at some intermediate time are shown in figure. Relatively small-scales are excited in the density fluctuations.

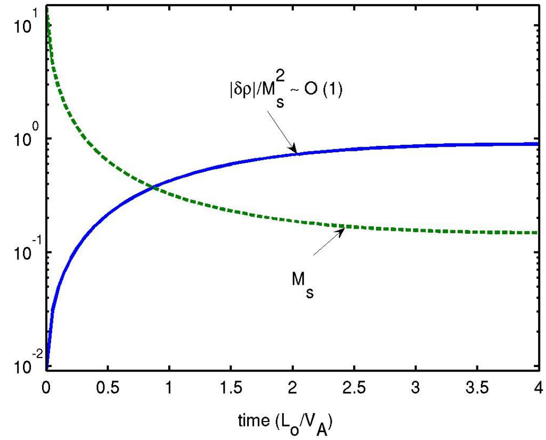

Fig. 4. Nonlinear compressible magnetofluid MHD simulations, in both three and two dimensions, show a decay of turbulent supersonic Mach number $\tilde{M}_s$ to below unity. A transition from a super-$\tilde{M}_s>1$ to a sub-$\tilde{M}_s<1$ sonic regime can be observed. In the latter, rms density fluctuations $\delta\rho$ tend to a steady-state and scale as $\mathcal{O}(\tilde{M}_s^2)$. The numerical resolution in a 3-D box of size π^3 is 150^3 or 200^3. Initial turbulent Reynolds numbers are $\tilde{R}_e=\tilde{R}_m\approx200$ (drop to about 20–25%). Other parameters are $\gamma=5/3$, $\beta_\circ=10^{-3}$, $M_{A_\circ}=1.-2.$, $M_{s_\circ}=5.-10.$, $\eta=\mu=10^{-4}-10^{-5}$, $dt=10^{-3}$. Different resolutions and initial conditions do not qualitatively change the evolution exhibited in the figure.

to the nonlinear mode coupling amongst various neighboring Fourier modes. This results in a net decay of turbulent sonic Mach number $\tilde{M}_s$ as shown in Fig. 4. The turbulent sonic Mach number continues to decay from a supersonic ($\tilde{M}_s>1$) to a subsonic ($\tilde{M}_s<1$) regime. This indicates that turbulent cascades associated with the nonlinear interactions, in combination with the dissipative effects at the small-scales, predominantly cause the supersonic MHD plasma fluctuations to damp strongly leaving primarily subsonic fluctuations in the MHD fluid. The most striking effect to emerge from the decay of the turbulent sonic Mach number is that the density fluctuations begin to scale quadratically with the subsonic turbulent Mach number as soon as the compressive plasma enters the subsonic regime, i.e. $\delta\rho\sim\mathcal{O}(\tilde{M}_s^2)$ when $\tilde{M}_s<1$. This signifies essentially a *weak* compressibility in the magnetoplasma, and can be referred to as a *nearly incompressible* state.

The transition of the compressible magnetoplasma from a supersonic to a subsonic or nearly incompressible regime is intriguing and warrants a detailed understanding of the nonlinear dynamics. While it is evident that the compressible (fast/slow magnetosonic) modes are being depleted in the subsonic regime, the mutual interplay between compressible and incompressible (Alfvén) MHD modes needs further clarification, as they are fundamentally responsible for the energy cascades. To identify the distinctive role of these MHD modes, we introduce diagnostics that distinguish energy cascades into Alfvénic and slow/fast magnetosonic fluctuations. Since the Alfvénic fluctuations are transverse, the propagation wave vector is orthogonal to the oscillations i.e. $\mathbf{k}\perp\mathbf{U}$, and the average spectral energy contained in these (shear Alfvénic modes) fluctuations can be computed as

$$\langle k_{\rm SAM}\rangle^2\simeq\frac{\sum_{\mathbf{k}}|i\mathbf{k}\times\mathbf{U}_{\mathbf{k}}|^2}{\sum_{\mathbf{k}}|\mathbf{U}_{\mathbf{k}}|^2}.$$

On the other hand, fast/slow magnetosonic modes propagate longitudinally along the fluctuations, i.e. $\mathbf{k}\parallel\mathbf{U}$, and thus carry

$$\langle k_{\rm SFM}\rangle^2\simeq\frac{\sum_{\mathbf{k}}|i\mathbf{k}\cdot\mathbf{U}_{\mathbf{k}}|^2}{\sum_{\mathbf{k}}|\mathbf{U}_{\mathbf{k}}|^2}$$

modal energy. Similarly, total energies in the respective modes can be quantified as

$$E_{\rm SAM}=\frac{1}{2}\sum_{\mathbf{k}}|\mathbf{U}_{\mathbf{k}}|^2\langle k_{\rm SAM}\rangle^2$$

and

$$E_{\rm SFM}=\frac{1}{2}\sum_{\mathbf{k}}|\mathbf{U}_{\mathbf{k}}|^2\langle k_{\rm SFM}\rangle^2.$$

The evolution of the modal and total energies is depicted in Fig. 5 (respectively the left and right y-axes). Although the modal energies in $k_{\rm SAM}$ and $k_{\rm SFM}$ modes are identical initially, the disparity in the cascade rate causes the energy in longitudinal fluctuations to decay far more rapidly than the energy in the Alfvénic modes. The Alfvénic modes, after a modest initial decay, sustain the energy cascade processes by actively transferring spectral power amongst various Fourier modes. By contrast, the fast/slow magnetosonic modes progressively weaken and suppress the energy cascades. The discrepancy in the cascades persists even at long times. The $k_{\rm SFM}$ mode represents collectively a dynamical evolution of small-scale fast plus slow magnetosonic cascades and does not necessarily distinguish the individual constituents (i.e. the fast and slow modes) due to their wave vector alignment relative to the magnetic field. The physical implication, however, that emerges from Fig. 5 is that the fast/slow magnetosonic modes *do not* contribute efficiently to the energy cascade process, and that the cascades are governed predominantly by non-dissipative Alfvénic modes that survive the collisional damping in compressible MHD turbulence. This immediately suggests that because of the decay of the fast/slow magnetosonic modes in compressible MHD plasmas, supersonic turbulent motions become dominated by subsonic motions and the nonlinear interactions are sustained primarily by Alfvénic modes thereafter; the latter being incompressible. The progressive dominance of the Alfvénic cascades can further be substantiated by plotting the ratio of the energies of the Alfvénic and fast/slow magnetosonic modes, i.e. $E_{\rm SAM}/E_{\rm SFM}$ (dashed curve in Fig. 5, right y-axis). The progressive weakening of the turbulent cascades in the compressive modes can be theoretically understood on the basis of a nearly incompressible model and is discussed in the subsequent section.

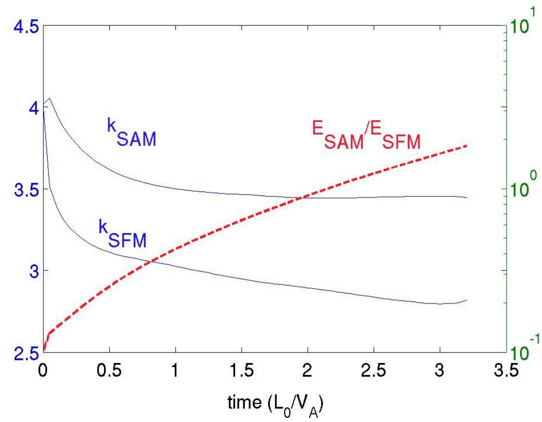

Fig. 5. Spectral energy transfer among shear Alfvén mode ($k_{\rm SAM}$) and slow/fast magnetosonic waves ($k_{\rm SFM}$) during a dissipative compressible MHD simulation is shown (left y-axis). These modes carry an identical energy initially so that $k_{\rm SAM}\approx k_{\rm SFM}(t{=}0)$. As time progresses, turbulence decay and spectral transfer due to $k_{\rm SFM}$ is suppressed significantly (left y-axis). While the latter decays, the former gradually increases indicating that the energy is being drained from the $k_{\rm SFM}$ modes. A monotonic enhancement of the ratio $E_{\rm SAM}/E_{\rm SFM}$ (right y-axis) confirms the progressive dominance of Alfvénic cascades over magnetosonic cascades.

4 Theoretical basis

The effect of inhibiting the fast/slow magnetosonic wave cascade is that the compressible magnetoplasma relaxes dynamically to a nearly incompressible (NI) state in the subsonic turbulent regime, and the solenoidal component of the fluid velocity makes a negligible contribution i.e. $\nabla\cdot\mathbf{U}\ll 1$, but not 0. The solenoidal velocity is associated essentially with the mode $k_{\rm SAM}$ whose eventual weakening can be understood by the NI theory in two ways. (1) In the subsonic regime, the turbulent sonic Mach is a small number, $\varepsilon^2{=}\gamma\tilde{M}_s^2\ll 1$. The normalized MHD momentum equation in the subsonic regime then reads

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\nabla\right)\mathbf{U}=-\frac{1}{\varepsilon^2}\nabla p+(\nabla\times\mathbf{B})\times\mathbf{B}+\bar{\nu}\nabla^2\mathbf{U}. \qquad (5)$$

It is evident that if $\mathbf{U}_t$ is to vary on slower time scales only (corresponding to incompressible motions), then it is necessary that several time derivatives of the solution be of order of $\mathcal{O}(1)$. The use of Kreiss' principle (Kreiss, 1982; Zank and Matthaeus, 1993) to eliminate fast-time and small-scale solutions (corresponding to the compressible motions) then yields a restriction on the pressure fluctuations as

$$p=1+\varepsilon^2 p_1,$$

where 1 stands for a leading order incompressible solution. On differentiating the momentum equation, and using the remaining equations, one obtains

$$\frac{\partial^2\mathbf{U}}{\partial t^2}\simeq\mathcal{O}(1)\ \text{terms}+\frac{1}{\varepsilon^2}\nabla(\gamma p\nabla\cdot\mathbf{U}). \qquad (6)$$

The *only* choice that leads to bounded solutions is

$$\nabla\cdot\mathbf{U}\rightarrow\mathcal{O}(\varepsilon^2).$$

This is consistent with the simulation results in that strongly compressible MHD plasmas tend to exhibit a nearly incompressible state by decreasing the solenoidal velocity component i.e. $\nabla\cdot\mathbf{U}\rightarrow\mathcal{O}(\varepsilon^2)$ (or $|i\mathbf{k}\cdot\mathbf{U_k}|\ll 1$ in Fourier space for the simulations). (2) The solenoidal component in the subsonic regime can be shown to be a smaller quantity, appearing at an order $\mathcal{O}(\varepsilon^2)$, by using a singular perturbative expansion series in $\mathbf{U}$, $\mathbf{B}$, ρ, as

$$\nabla_\eta\cdot\mathbf{U}_1\approx\frac{1}{\gamma}\frac{\partial p_1}{\partial\tau'}$$

where the slow and the fast time scales are decomposed according to

$$\frac{\partial}{\partial t}\simeq\frac{\partial}{\partial\tau}+\frac{1}{\varepsilon}\frac{\partial}{\partial\tau'}$$

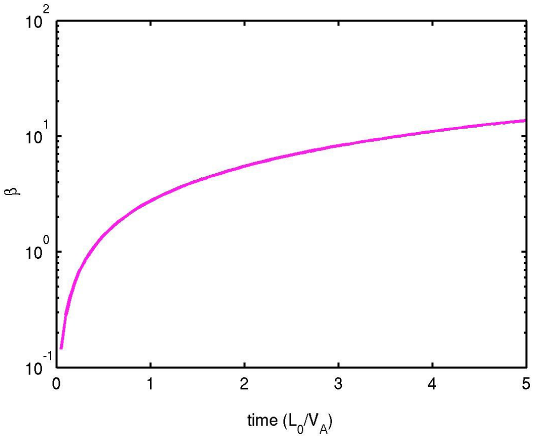

Fig. 6. Evolution of the turbulent plasma β in compressible MHD turbulence. Notice that the plasma is strongly magnetized initially with $\beta<1$. As the turbulence evolves, the plasma-β increases gradually to a higher values such that $\beta>1$. This indicates that the compressible magnetoplasma is dominated progressively by shear Alfvén waves that are largely incompressible. This is consistent with the results of Figs. 3 and 4.

and the small and large scales by

$$\nabla \simeq \nabla_\eta + \varepsilon \nabla_\xi .$$

The (small) solenoidal contribution estimated in this manner further suggests that it is indeed a manifestation of the small-scale, high-frequency turbulent motion. The two conjectures serve as the necessary and sufficient condition for the compressible motion to exhibit near incompressibility (Zank and Matthaeus, 1993). It is thus clear from both the arguments, made within the context of the simulation results, that compressible MHD turbulence tends asymptotically towards a state of nearly incompressibility through dissipation. Nearly incompressible MHD is a valid description of the magnetoplasma strictly in the subsonic regime in which the leading order plasma motion is incompressible, and weak compressibility enters at an order $\tilde{M}_s^2 \sim \mathcal{O}(\varepsilon^2) \ll 1$ such that the density fluctuations preserve the scaling $\delta\rho \sim \mathcal{O}(\tilde{M}_s^2)$.

One of the most exciting results of the transition to the subsonic regime in compressible MHD turbulence is that the density fluctuations advect *passively*. Within the context of our simulations, the passive convection of the density can be understood from the Fourier transformed fluid continuity equation,

$$\frac{\partial \sigma_{\mathbf{k}}}{\partial t} + i \sum_{\mathbf{k}} \delta(k+k') \mathbf{U}_{\mathbf{k}} \cdot \mathbf{k}' \sigma_{\mathbf{k}'} \simeq - \frac{i\mathbf{k} \cdot \mathbf{U}_{\mathbf{k}}}{\mathcal{N}_{\mathbf{k}}}, \quad (7)$$

where

$$\sigma_{\mathbf{k}} = \frac{\ln \rho}{\mathcal{N}_{\mathbf{k}}},$$

$$\mathcal{N}_{\mathbf{k}} = \sqrt{\sum_{\mathbf{k}} |\mathbf{U}_{\mathbf{k}}|^2}.$$

The Dirac delta function δ, resulting from a nonlinear deconvolution in Fourier space, is finite only for those interactions that obey the Fourier diad $k+k'=0$. It then follows from the simulations, the physical arguments advanced to justify the nearly incompressibility, and from the solenoidal component as estimated above, that the rhs of the continuity equation makes an insignificant contribution to the energy cascade. The entire dynamics is therefore dominated by convective transport that leads to a passive convection of density fluctuations in weakly compressible MHD turbulence as it evolves to a subsonic state.

The transition of magnetoplasma from a compressible to a nearly incompressible state not only transforms the characteristic supersonic motion into subsonic motion, but also *attenuates* plasma magnetization. This can be seen from the evolution of the turbulent plasma-$\tilde{\beta}$ as shown in Fig. 6. The plasma-$\tilde{\beta}$ is defined as the ratio of plasma thermal pressure and magnetic pressure. Figure 5 shows that the magnetic pressure exceeds the thermal pressure ($\tilde{\beta}<1$) initially in the fully compressible magnetoplasma. When the strongly magnetized compressible plasma fluctuations decay, (see Figs. 3 and 4), the magnetization decreases and the low beta plasma ($\tilde{\beta}<1$) evolves into a high beta ($\tilde{\beta}>1$) state. This implies that the plasma pressure evolves to exceed the magnetic energy dynamically by exciting perturbations that are populated largely by short turbulent length-scales

$$k\hat{\rho}_i \geq 1 > k_0 \lambda_{mpf}$$

(where $\hat{\rho}_i$ is the ion gyroradius, $k_0 \sim 1/\ell_0$ and λ_{mpf} is mean free path scale-length) in an inertial range. Physically this means that plasma particles tied to the magnetic field lines are expelled from their gyro orbits due to an increasingly dominant gas pressure. This leads eventually to a reduced plasma magnetization and hence plasma fluctuations, experiencing larger particle pressure than the magnetic pressure, transit into a $\beta>1$ regime where they exhibit near incompressibility. A direct consequence of the magnetoplasma evolving into a high beta regime is that super Alfvénic ($\tilde{M}_A>1$) plasma is damped into sub-Alfvénic motion ($\tilde{M}_A<1$). Mode conversion is a further factor that explains, in part, the transition of an MHD plasma from compressible to nearly incompressible. This can be illustrated by a simple argument. The turbulent plasma-$\tilde{\beta}$ is

$$\tilde{\beta} \simeq \frac{8\pi \tilde{p}}{\tilde{B}^2} \sim \frac{\tilde{C}_s^2}{\tilde{V}_A^2} \sim \frac{\tilde{M}_A^2}{\tilde{M}_s^2}.$$

Thus a monotonic decrease in $\tilde{M}_s$ corresponds to a higher value of $\tilde{\beta}$, i.e. as the magnetofluid becomes increasingly incompressible, $\tilde{M}_s$ becomes smaller, consistent with Fig. 4. This, however, places the stringent but justifiable criterion on the dynamical evolution of compressible magnetoplasmas, which is that the magnetosonic fluctuations must decay faster than the Alfvénic motions. This statement corresponds precisely to the theoretical justification for using an incompressible fluid description (Zank and Matthaeus, 1990, 1993). The latter, nevertheless, survive the nonlinear damping or dissipation.

The progressive enhancement of the turbulent plasma $\tilde{\beta}$ provides, additionally, a complimentary understanding of the discrepant cascade rates in the subsonic regime of compressible magnetoplasma. The high plasma $\tilde{\beta}$ regime implies that the shear Alfvén modes propagate more slowly than sound waves. Thus MHD perturbations in the steady state are ordered as $\tilde{\mathbf{U}} < \tilde{V}_A < \tilde{C}_s$. The nonlinear interaction time-scales associated with this ordering are

$$\hat{\tau}_s < \hat{\tau}_a < \tau_{NL},$$

where $\hat{\tau}_a$, $\hat{\tau}_s$ and τ_{NL} denote respectively the Alfvénic, the fast/slow magnetosonic and the nonlinear eddy turn over time-scales. This inequality indicates that the nonlinear interaction time for Alfvén modes increases compared to that of the magneto(acoustic) modes. Consequently, the plasma motion becomes increasingly incompressible on Alfvénic time scales. During this gradual transformation to incompressibility, the compressible fast/slow magnetosonic modes do not couple well with the Alfvén modes. The cascades are therefore progressively dominated by the shear Alfvén modes, while the compressible fast/slow magnetosonic waves suppress the nonlinear cascades by dissipating the longitudinal fluctuations.

It is interesting to compare our results with other work. For instance, a neutral (hydrodynamic) fluid evolves to a quasiequilibrium state, when forced externally, in which the rotational component of the velocity spectrum is very close to that of the incompressible case even for a large Mach number ($\simeq$0.9) case (Kida and Orszag , 1990). Similarly, decaying supersonic hydrodynamic turbulence develops a Kolmogorov-like (solenoidal) velocity spectrum (Porter et al., 1994). By contrast, our simulations deal with the magnetized compressible plasma and it is thus natural to ask what role the magnetic field plays on turbulent decay rates. It has been shown by Mac Low et al. (1998, 1999) that a magnetic field does not alter the decay rate and thus both isothermal, compressible MHD and hydrodynamical turbulence decay at an almost similar rate. In the context of our simulations, we believe that when a strongly magnetized compressible plasma decays nonlinearly, the magnetization decreases and the turbulent plasma pressure evolves to exceed the turbulent magnetic energy [i.e. the $\tilde{\beta}>1$ regime in Fig. 6] dynamically by exciting perturbations that are increasingly non-magnetized (i.e. hydrodynamic-like). The turbulent magnetic field thus becomes weak eventually and has almost no influence on the turbulent decay rates. Nonetheless, decaying supersonic, super Alfvenic MHD turbulence tends to follow a $\delta\rho \propto M_s^2$ scaling in a subsonic regime that correlates rms density fluctuations with the subsonic turbulent Mach number. Note that the latter serves as an important constraint on the passive scalar theory of density convection in our model.

5 Conclusions

The most *novel* and notable point to emerge from our simulations is the definite scaling between the turbulent Mach number and the rms density fluctuations, i.e. that $|\delta\rho|^2/\tilde{M}_s \sim \mathcal{O}(1)$ is obeyed in subsonic magnetofluid turbulence. Consequently, the density fluctuations behave as passively convected structures. A direct consqeuence of this result, as described above, is that it provides a possible explanation of the observed interplanetary (IPM) and possibly interstellar medium (ISM) density fluctuations which exhibit a Kolomogorov-like power spectrum. Our simulations are therefore *the first* results to identify that density fluctuations in the IPM and ISM possibly emerge as a result of weak compressibility in the gas and are convected passively in the background incompressible fluid flow field while preserving the constraint $|\delta\rho|^2/\tilde{M}_s \sim \mathcal{O}(1)$. We finally conclude that eventual nonlinear decoupling of Alfvénic and magnetosonic modes leads to near incompressibility in a nonlinearly decaying, compressible magnetoplasma turbulence. Nonlinear cascades play a catalytic role not only in weakening the energy cascades by the compressive fast/slow magnetosonic modes, but also damp the supersonic and the super Alfvénic plasma motions leaving only subsonic and Alfvénic regimes. A direct implication is that a nearly incompressible state develops naturally in a subsonic compressive ISM or IPM magnetoplasma and the density fluctuations, scaling quadratically with the subsonic turbulent Mach number, exhibit a characteristic spectrum that is determined typically by passive convection in the field of nearly incompressible velocity fluctuations. The super- to subsonic transformation seen in our simulations is realized explicitly because the fluctuating or turbulent Mach numbers depend on local small scale dynamics and not on bulk or mean motion. Note that we have not included forcing effects such as those due to solar flares or supernovae blasts etc. Our work thus assumes that the solar wind or interstellar plasma is evolving freely without experiencing external forcing. The spectral properties of the density, velocity and magnetic field fluctuations will be an objective of our furture investigations.

Appendix A

A physical description of NI theory

One of the earlier attempts to understand the ISM density fluctuations dates back to the analytic work of Higdon (1984, 1986) in which density fluctuations were shown to be advected by velocity and magnetic field fluctuations. Montgomery et al. (1987) related the density fluctuations to an incompressible fluid turbulence by assuming an equation of state to relate ISM density fluctuations to incompressible MHD. This approach, called a pseudosound approximation, assumes that density fluctuations are proportional to the pressure fluctuations through the square of sound speed i.e. $\delta\rho \sim C_s^2 \delta P^\infty$, where $\delta\rho$, C_s, δP^∞ are respectively the density, sound speed, and incompressible pressure. The density perturbations in Montgomery et al. (1987) are therefore "slaved" to the incompressible magnetic field and the velocity fluctuations. As described in Sect. II, this assumption ignores the contribution from relatively high frequency and short wavelength fluctuations. This hypothesis was further contrasted by Bayly et al. (1992) on the basis of their 2-D compressible hydrodynamic simulations who demonstrated that a spectrum for density fluctuations can arise purely as a result of abondoning a barotropic equation of state without even requiring a magnetic field. The pseudosound fluid description of compressibility, justifying the Montgomery et al. approach to the density-pressure relationship, was further extended by (Matthaeus and Brown, 1988) in the context of a compressible magnetofluid (MHD) plasma with a polytropic equation of state in the limit of a low plasma acoustic Mach number (Matthaeus and Brown, 1988). The theory, originally describing the generation of acoustic density fluctuations by incompressible hydrodynamics (Lighthill, 1952), is based on a generalization of Klainerman and Majda's work (Klainerman and Majda, 1981, 1982; Majda, 1984) and accounts for fluctuations associated with a low turbulent Mach number fluid, unlike purely incompressible MHD. Such a nontrivial finite departure from the incompressibility state is termed a "nearly incompressible" fluid description. The primary motivation behind NI fluid theory was to develop an understanding and explanation of the interstellar scintillation observations of weakly compressible ISM density fluctuations that exhibit a Kolmogorov-like power law. The NI theory is, essentially, an expansion of the compressible fluid or MHD equations in terms of weak fluctuations about a background of strong incompressible fluctuations. The expansion parameter is the turbulent Mach number. The leading order expansion satisfies the background incompressible hydrodynamic or magnetohydrodynamic equations (and therefore fully nonlinear) derived on the basis of Kreiss principle (Kreiss, 1982), while the higher order yields a high frequency weakly compressible set of nonlinear fluid equations that describe low turbulent Mach number compressive HD as well as MHD effects. Zank and Mathaeus derived the unified self-consistent theory of nearly incompressible fluid dynamics for non-magnetized hydrodynamics as well as magnetofluids, with the inclusion of the thermal conduction and energy effects, thereby identifying different and distinct routes to incompressibility (Zank and Matthaeus, 1990, 1991, 1993).

The theory of nearly incompressible (NI) fluid, developed by Matthaeus, Zank and Brown, based on a perturbative expansion technique is, perhaps the first rigorous theoretical attempt to understand the origin of weakly compressible density fluctuations in the interstellar medium, and one that provides formally a complete fluid description of ISM turbulence with the inclusion of thermal fluctuations and the full energy equation self-consistently, unlike the previous models described above (Zank and Matthaeus, 1990, 1991, 1993; Matthaeus and Brown, 1988). A central tenant of NI theory is that the ISM density fluctuations are of higher order, of higher frequency and possess smaller length-scales than their incompressible counterparts to which they are coupled through passive convection and the low frequency generation of sound. Thus, the NI fluid models, unlike fully incompressible or compressible fluid descriptions, allow us to address weakly compressible effects directly in a quasi-neutral ISM fluid. Furthermore, NI theory has enjoyed notable successes in describing fluctuations and turbulence in the supersonic solar wind.

Various nonlinear aspects of NI theory have lately been explored by Dastgeer and Zank (2004a,b,c, 2006a,b) in two as well as three dimensions using nonlinear fluid simulations aimed primarily at understanding the nonlinear cascades in the interstellar turbulence that lead to the observed density fluctuation spectrum. Some of these significant advancements are reported in this paper.

Acknowledgements. D. Shaikh and G. P. Zank have been supported in part by a NASA grant NNG-05GH38 and an NSF grant ATM-0317509.

Edited by: T. Hada
Reviewed by: two anonymous referees

References

Armstrong, J. W., Cordes, J. M., and Rickett, B. J.: Density power spectrum in the local interstellar medium, Nature, 291, 561–564, 1981.

Armstrong, J. W., Rickett, B. J, and Spangler, S.: Electron density power spectrum in the local interstellar medium, ApJ, 443, 209–221, 1995.

Bayly, B. J., Levermore, C. D., and Passot, T.: Density variations in weakly compressible flows, Phys. Fluids, A4, 945–954, 1992.

Bhattacharjee, A., Ng, C. S., and Spangler, S. R.: ApJ, 494, 409–418, 1998.

Cho, J. and Lazarian, A.: Compressible magnetohydrodynamic turbulence: mode coupling, scaling relations, anisotropy, viscosity-damped regime and astrophysical implications, MNRAS, 345, 325–339, 2003.

Dastgeer, S. and Zank, G. P.: Density Spectrum in the Diffuse Interstellar Medium and Solar Wind, ApJ, 602, L29–L32, 2004.

Dastgeer, S. and Zank, G. P.: Anisotropic Density Fluctuations in a Nearly Incompressible Hydrodynamic Fluid, ApJ, 604, L125–L128, 2004.

Dastgeer, S. and Zank, G. P.: Nonlinear flows in nearly incompressible hydrodynamic fluids, Phys. Rev. E., 69, 066309-1–066309-5, doi:10.1103/PhysRevE.69.066309, 2004.

Dastgeer, S. and Zank, G. P.: Nonlinear structures, spectral features, and correlations in a nearly incompressible hydrodynamic fluid, Phys. Fluids, 18, 045105, doi:10.1063/1.2191878, 2006.

Dastgeer, S. and Zank, G. P.: The Transition to Incompressibility from Compressible Turbulence, ApJ, 640, L195–L198, 2006.

Ghosh, S., Hossain, M., and Matthaeus, W. H.: The application of spectral methods in simulating compressible fluid and magnetofluid turbulence, Computer Phys. Comm. 74, 18–40, 1993.

Gottlieb, D. and Orszag, S. A.: Numerical Analysis of Spectral Methods: Theory and Applications (NSF-CBMS Monograph 26; Philadelphia: SIAM), 1977.

Higdon, J. C.: Density fluctuations in the interstellar medium: Evidence for anisotropic magnetogasdynamic turbulence. I – Model and astrophysical sites, 285, 109–123, 1984.

Higdon, J. C.: Density fluctuations in the interstellar medium: Evidence for anisotropic magnetogasdynamic turbulence. II – Stationary structures, 309, 342–361, 1986.

Klainerman, S. and Majda, A.: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, Commun. Pure Appl. Math. 34, 481–524, 1981.

Klainerman, S., and Majda, A.: Compressible and Incompressible Fluids, Commun. Pure Appl. Math. 35, 629–651, 1982.

Kida, S. and Orszag, A.: Energy and spectral dynamics in forced compressible turbulence, J. Sci. Comp., 5, 85, 1990.

Kolmogorov, A. N.: On degeneration of isotropic turbulence in an incompressible viscous liquid, Dokl. Akad. Nauk SSSR, 31, 538–541, 1941.

Kraichnan, R. H.: Inertial range spectrum in hydromagnetic turbulence, Phys. Fluids, 8, 1385–1387, 1965.

Kreiss, H. -O.: Problems with different time scales for partial differential equations, Commun. Pure Appl. Math, 33, 399–439, 1982.

Lighthill, M. J.: On sound generated aerodynamically I, General theory, Proc. R. Soc. London Ser., A 211, 564–587, 1952.

Lithwick, Y. and Goldreich, P.: Compressible Magnetohydrodynamic Turbulence in Interstellar Plasmas, ApJ, 562, 279–296, 2001.

Mac Low, M.-M., Klessen, R. S., Burkert, A., and Smith, M. D.: Kinetic energy decay rates of supersonic and super alfvenic turbulence in star-forming clouds, Phys. Rev. Lett., 80, 2754, 1998.

Mac Low, M.-M.: The energy dissipation rate of supersonic, magnetohydrodynamic turbulence in molecular clouds, Astrophys. J., 524, 169–178, 1999.

Majda, A.: Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables (Springer, New York), 1–172, 1984.

Matthaeus, W. H. and Brown, M.: Nearly incompressible magnetohydrodynamics at low Mach number, Phys. Fluids, 31, 3634–3644, 1988.

Matthaeus, W. H., Klein, L., Ghosh, S., and Brown, M.: Nearly incompressible magnetohydrodynamics, pseudosound, and solar wind fluctuations, J. Geophys. Res., 96, 5421–5435, 1991.

Montgomery, D. C., Brown, M. R., and Matthaeus, W. H.: Density fluctuation spectra in magnetohydrodynamic turbulence, J. Geophys. Res., 92, 282–284, 1987.

Porter, D. H., Pouquet, A., and Woodward, P. R.: Kolmogorov-like spectra in decaying three-dimensional supersonic flows, Phys. Fluids, 6, 2133–2142, 1994.

Spangler, S.: Two-dimensional Magnetohydrodynamics and Interstellar Plasma Turbulence, ApJ 522, 879–896, 1999.

Spangler, S. R.: Multi-Scale Plasma Turbulence in the Diffuse Interstellar Medium, Space Science Rev., 99, 261–270, 2001.

Zank, G. P. and Matthaeus, W. H.: Nearly incompressible hydrodynamics and heat conduction, Phys. Rev. Lett., 64, 1243–1246, 1990.

Zank, G. P. and Matthaeus, W. H.: The equations of nearly incompressible fluids. I – Hydrodynamics, turbulence, and waves, Phys. Fluids A, 3, 69–82, 1991.

Zank, G. P. and Matthaeus, W. H.: Nearly incompressible fluids. II – Magnetohydrodynamics, turbulence, and waves, Phys. Fluids, A5, 257–273, 1993.