Opuscula Math. 33, no. 1 (2013), 5–17
http://dx.doi.org/10.7494/OpMath.2013.33.1.5 Opuscula Mathematica

A NOTE ON BLASIUS TYPE BOUNDARY VALUE PROBLEMS

Grzegorz Andrzejczak, Magdalena Nockowska-Rosiak, and Bogdan Przeradzki

Communicated by Zdzisław Kamont

Abstract. The existence and uniqueness of a solution to a generalized Blasius equation with asymptotic boundary conditions is proved. A new numerical approximation method is proposed.

Keywords: Blasius equation, shooting method.

Mathematics Subject Classification: 34B15, 34B40, 65L10, 34D05.

1. INTRODUCTION

We study the BVP of the form:

$$x''' + cx^p \cdot x'' = 0, \qquad x(0) = 0 = x'(0), \quad \lim_{t\to\infty} x'(t) = \beta, \tag{1.1}$$

where $p \geq 1$, c and β are positive constants. The problem is motivated by the classical Blasius equation describing the velocity profile of a fluid in a boundary layer where $c = \frac{1}{2}$, $p = \beta = 1$. The Blasius equation is a basic equation in fluid mechanics which appears in the study of the flow of an incompressible viscous fluid over a semi infinite plane. Blasius ([2]) used a similarity transform technique to convert the partial differential equation into his famous ordinary differential equation

$$x''' + \tfrac{1}{2}x \cdot x'' = 0, \qquad x(0) = 0 = x'(0), \quad \lim_{t\to\infty} x'(t) = 1, \tag{1.2}$$

where x is the stream function $x = \frac{\Psi}{\sqrt{2U\nu y_1}}$, U is the fluid velocity, ν is the fluid kinematic viscosity and t is the similarity variable defined as $t = y_2\sqrt{\frac{U}{2\nu y_1}}$, where y_1, y_2 are Cartesian coordinates with y_1 pointing along the free stream direction and y_2 perpendicular to y_1. We refer to [3,4] for an excellent introduction to the problem.

A series expansions method was used to solve (1.2) by Blasius. There has been many analytical and numerical methods handling this problem since Blasius's work, [1,6,8, 9,11–15] for instance.

The existence and the uniqueness for the Blasius problem was proved by Weyl in 1942 ([14]) by using an elegant method based on the symmetry of the equation (1.2). If $\eta \to w(\eta)$ is a solution of (1.2) then $\eta \to kw(k\eta)$ is as well for any k. It enabled him to work with the initial condition

$$x(0) = x'(0) = 0, \quad x''(0) = 1.$$

This problem where the second-order derivative is given at 0 is equivalent to the fixed point problem with an operator possessing some monotonicity property. Weyl solved it as a limit of odd and even successive iterates. For our problem this method cannot work so simply and we shall consider the third initial condition $x''(0) = a$ and the dependence of the solution with respect to $a > 0$. Moreover, our method gives estimates for errors. It enables us to propose a combined numerical method to solve the problem. For a given $\varepsilon > 0$, we can find an interval $[0, T]$ such that outside of it the derivative of a solution cannot change more than ε.

In the first part of this paper, the existence and uniqueness of (1.1) will be analytically proved by changing the boundary value problem to an initial problem. Using the obtained estimates we will be able to find the value of $a = x''(0)$ which guarantees that the solution x_a on an initial problem

$$x''' + cx^p \cdot x'' = 0, \qquad x(0) = 0 = x'(0), \quad x''(0) = a, \tag{1.3}$$

is the solution of (1.1) we are looking for.

In the second part of the article, we shall propose a numerical approximation method and some examples that compare it with many others in the literature.

2. AUXILIARY LEMMAS

Let x_a stand for the unique solution of $x''' + cx^p \cdot x'' = 0$ satisfying initial conditions

$$x(0) = 0 = x'(0), \qquad x''(0) = a.$$

If $p \in \mathbb{N}$ and $a < 0$, then x_a and x_a'' are negative for small t's thus the solution is concave and negative for all arguments and it cannot solve (1.1). Using the analogous argument about negative values of x_a'' we get that the negative value of a for $p > 1$, $p \notin \mathbb{N}$ contradicts the assumption of nonnegative x_a. For $a = 0$ we have a trivial solution $x_a \equiv 0$ and the seeking solution can be obtained for $a > 0$.

Lemma 2.1. *x_a is defined for all $t \geq 0$.*

Proof. Notice that x_a is an analytic function. Moreover, x_a'' cannot vanish at any point. If there exists $t_0 > 0$ such that $x_a''(t_0) = 0$ then from the equation $x''' + cx^p \cdot x'' = 0$ we get that $x_a^{(k)}(t_0) = 0$ for $k \geq 2$. So we get that the solution of the ODE is the form

$x_a(t) = c_1 t + c_2$ which contradicts the initial conditions $x_a(0) = x_a'(0) = 0$, $x_a''(0) = a$. Hence dividing the equation by x_a'' and integrating on $[0, t]$ we have

$$x_a''(t) = a \exp\Big(-c\int_0^t x_a(s)^p\, ds\Big), \tag{2.1}$$

which implies

$$x_a'(t) = a\int_0^t \exp\Big(-c\int_0^s x_a(\tau)^p d\tau\Big)ds. \tag{2.2}$$

Integrating once more and applying the Fubini Theorem we get

$$x_a(t) = a\int_0^t (t-s)\exp\Big(-c\int_0^s x_a(\tau)^p d\tau\Big)ds. \tag{2.3}$$

By (2.1),(2.2) and (2.3) we have apriori estimates:

$$0 < x_a(t) < \frac{1}{2}at^2, \qquad 0 < x_a'(t) < at, \qquad 0 < x_a''(t) < a$$

for any $t > 0$. It follows ([10, p. 146]) that x_a is extendable to $[0, \infty)$. □

Lemma 2.2. *For any $a > 0$, $\lim_{t\to\infty} x_a''(t) = 0$ and there exists a finite and positive limit*

$$h(a) := \lim_{t\to\infty} x_a'(t). \tag{2.4}$$

Proof. Since $x_a'' > 0$ and $x_a' > 0$, then $\lim_{t\to+\infty} x_a(t) = +\infty$. Moreover, since $x_a'' > 0$, $x_a > 0$ and $x_a''' = -cx_a^p x_a''$, then x_a'' is a decreasing function so $\lim_{t\to\infty} x_a''(t) = g_a \in [0, a)$. Suppose $g_a > 0$. From $x_a''' = -cx_a^p x_a''$ we get that

$$\lim_{t\to+\infty} x_a'''(t) = -\infty. \tag{2.5}$$

On the other hand

$$\forall\, t \geq 0\ \exists\, s_t \in (t, t+1) \quad x_a''(t+1) - x_a''(t) = x_a'''(s_t)$$

and hence $\lim_{t\to+\infty} x_a'''(t) = 0$, which contradicts (2.5).

Since $x_a'' > 0$, then x_a' is an increasing function (so the limit defining $h(a)$ exists, possibly infinite). From $\lim_{t\to+\infty} x_a(t) = +\infty$ we get there exists $t_a > 0$ such that $cx_a(t)^p > 1$ for $t > t_a$. From (2.2), we obtain for $t > t_a$,

$$x_a'(t) \leq a\int_0^t \exp\Big(-c\int_0^{t_a} x_a(\tau)^p\, d\tau\Big)\exp\Big(-\int_{t_a}^s d\tau\Big)ds \leq$$

$$\leq a\int_0^{t_a} ds + a\int_{t_a}^{t} \mathrm{e}^{-(s-t_a)}\, ds \leq at_a + a\mathrm{e}^{t_a}\int_{t_a}^{\infty} \mathrm{e}^{-s}\, ds = a(t_a+1).$$

Thus

$$h(a) \leq a(t_a+1) \tag{2.6}$$

for any $a > 0$. □

Lemma 2.3. *For any $a > 0$, there exists a finite and positive limit*

$$\mu(a) := \lim_{t\to\infty} (h(a)t - x_a(t)), \tag{2.7}$$

where the constant $h(a)$ is defined by (2.4). It means that the graph of x_a has a slant asymptote and the following estimates hold:

$$\max(0, h(a)t - \mu(a)) \leq x_a(t) \leq h(a)t. \tag{2.8}$$

Proof. The function $t \mapsto h(a)t - x_a(t)$ is increasing, hence the limit from the assertion exists but it can be infinite. Suppose it equals $+\infty$. By the arguments from the proof of Lemma 2.2, we have $x_a'''(t) \leq -x_a''(t)$ for $t \geq t_a$. Integrating this inequality from s to $+\infty$ and using the fact $x_a''(+\infty) = 0$, we get

$$-x_a''(s) \leq -h(a) + x_a'(s).$$

Next integration from t_a to t leads to the following inequality

$$x_a'(t_a) - x_a'(t) \leq -h(a)(t - t_a) + x_a(t) - x_a(t_a)$$

or equivalently

$$h(a)t - x_a(t) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a).$$

Thus

$$0 < \mu(a) \leq h(a)t_a - x_a(t_a) + h(a) - x_a'(t_a). \tag{2.9}$$

The last part of the assertion is a simple consequence. □

For an upper bound on $h(a)$, $\mu(a)$ depending explicitly on a we use $x_a(t) \leq at^2/2$ to (2.3). Hence,

$$x_a(t) \geq a\int_0^t (t-s)\exp\left(-\int_0^s c\frac{a^p}{2^p}\tau^{2p}d\tau\right)ds = a\int_0^t (t-s)\exp\left(-c\frac{a^p}{(2p+1)2^p}s^{2p+1}\right)ds.$$

One can easily show that the function

$$\varphi(t) := \int_0^t (t-s)\exp\left(-ks^{\alpha}\right)\, ds,\ k = \frac{ca^p}{2^p(2p+1)},\ \alpha = 2p+1$$

has a similar behaviour as x_a in the sense that its graph has an asymptote $x = h^* t - \mu^*$, where

$$h^* = \int_0^\infty \exp\left(-ks^\alpha\right) ds = \tfrac{\Gamma(1/\alpha)}{\alpha k^{1/\alpha}},\ \mu^* = \int_0^\infty s \exp\left(-ks^\alpha\right) ds = \tfrac{\Gamma(2/\alpha)}{\alpha k^{2/\alpha}} \tag{2.10}$$

and its graph sits above this line. Hence,

$$x_a(\tau) \geq a^{1-\frac{p}{2p+1}} \cdot \Gamma(\tfrac{1}{2p+1})(\tfrac{2^p}{c\cdot(2p+1)^{2p}})^{\frac{1}{2p+1}} \cdot \tau - a^{1-\frac{2p}{2p+1}} \cdot \Gamma(\tfrac{2}{2p+1})(2p+1)^{\frac{1-2p}{2p+1}}(\tfrac{2^p}{c})^{\frac{2}{2p+1}}.$$

$$x_a(\tau) \geq c_2 \cdot a^{\frac{p+1}{2p+1}} \cdot \tau - c_3 \cdot a^{\frac{1}{2p+1}}, \tag{2.11}$$

where

$$c_2 := \Gamma(\tfrac{1}{2p+1})(\tfrac{2^p}{c\cdot(2p+1)^{2p}})^{\frac{1}{2p+1}},\ c_3 := \Gamma(\tfrac{2}{2p+1})(2p+1)^{\frac{1-2p}{2p+1}}(\tfrac{2^p}{c})^{\frac{2}{2p+1}}. \tag{2.12}$$

Now, we are able to get appropriate estimates for $h(a)$.

Lemma 2.4. *For any $a > 0$,*

$$c_2 \cdot a^{(p+1)/(2p+1)} \leq h(a) \leq c_1 \cdot a^{(p+1)/(2p+1)}, \tag{2.13}$$

where

$$c_1 := \frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c_2(p+1))^{p/(p+1)}}.$$

Proof. For a lower bound we apply the estimate $x_a(\tau) \leq \frac{1}{2}a\tau^2$ to the equality

$$h(a) = a\int_0^\infty \exp\left(-c\int_0^s x_a(\tau)^p d\tau\right) ds. \tag{2.14}$$

This leads to the inequality

$$h(a) \geq a\int_0^\infty \exp\left(-\frac{ca^p}{2^p(2p+1)}s^{2p+1}\right) ds.$$

(2.10) for $k = \frac{ca^p}{2^p(2p+1)}$, $\alpha = 2p+1$ gives the lower bound on h.

For an upper bound on h we use the lower estimate of x_a – (2.11) to the equality (2.14) and we get

$$\begin{aligned} h(a) \le a &\int_0^{c_3/c_2 \cdot a^{-p/(2p+1)}} ds + \\ &+ a \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^{\infty} \exp\left(-c \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^{s} \left(c_2 a^{\frac{p+1}{2p+1}} \tau - c_3 a^{\frac{1}{2p+1}} \right)^p d\tau \right) ds \le \\ &\le \frac{c_3}{c_2} a^{\frac{p+1}{2p+1}} + \frac{1}{c_2} a^{\frac{p}{2p+1}} \int_0^{\infty} \exp\left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt, \end{aligned}$$

where we used linear substitutions twice. At last, using (2.10) for $k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}}$, $\alpha = p+1$ we have for any $a > 0$,

$$h(a) \le \left(\frac{c_3}{c_2} + \frac{\Gamma(1/(p+1))}{c^{1/(p+1)} \cdot (c_2(p+1))^{p/(p+1)}} \right) \cdot a^{\frac{p+1}{2p+1}}.$$

□

The next lemma presents estimates for $\mu(a)$.

Lemma 2.5. *For any $a > 0$, the constant $\mu(a)$, which is defined by (2.7), satisfies the following estimates:*

$$c_4 \cdot a^{1/(2p+1)} \le \mu(a) \le c_5 \cdot a^{1/(2p+1)},$$

where

$$\begin{aligned} c_4 &:= \frac{2^{2p/(2p+1)} \Gamma(2/(2p+1))}{(2p+1)^{(2p-1)/(2p+1)} c^{2/(2p+1)}}, \\ c_5 &:= \tfrac{1}{c_2^2} \left[\tfrac{c_3^2}{2} + (\tfrac{c_2}{c})^{2/(p+1)} (p+1)^{(1-p)/(1+p)} \Gamma(\tfrac{2}{p+1}) + c_3 (\tfrac{c_2}{c \cdot (p+1)^p})^{1/(p+1)} \Gamma(\tfrac{1}{p+1}) \right]. \end{aligned}$$

Proof. From (2.3) and (2.14) we get

$$\mu(a) = a \int_0^{\infty} s \exp\left(-c \int_0^s x_a(\tau)^p \, d\tau \right) ds. \tag{2.15}$$

Using the estimate $x_a(\tau) \le \frac{1}{2} a \tau^2$ we get

$$\mu(a) \ge a \int_0^{\infty} s \cdot \exp\left(-\frac{c a^p}{2^p (2p+1)} s^{2p+1} \right) ds.$$

Using (2.10) for $k = \frac{ca^p}{2^p(2p+1)}$, $\alpha = 2p+1$ we obtain the lower bound. On the other hand, from (2.11), (2.15) we have

$$\mu(a) \leq a \int_0^{c_3/c_2 \cdot a^{-p/(2p+1)}} s\, ds +$$

$$+ a \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^{\infty} s \exp\left(-c \int_{c_3/c_2 \cdot a^{-p/(2p+1)}}^{s} \left(c_2 a^{\frac{p+1}{2p+1}} \tau - c_3 a^{\frac{1}{2p+1}} \right)^p d\tau \right) ds \leq$$

$$\leq \frac{c_3^2}{2c_2^2} a^{\frac{1}{2p+1}} + \frac{1}{c_2^2} a^{\frac{-1}{2p+1}} \int_0^{\infty} t \exp\left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt +$$

$$+ \frac{c_3}{c_2^2} \int_0^{\infty} \exp\left(-\frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}} t^{p+1} \right) dt,$$

where we used linear substitutions twice. At last, using (2.10) for $k = \frac{c}{(p+1)c_2} a^{-\frac{p+1}{2p+1}}$, $\alpha = p+1$ we have for any $a > 0$ the upper bound. □

3. MAIN RESULTS

Now, we are able to prove the existence of a solution to (1.1).

Theorem 3.1. *Let $p \geq 1$, $c > 0$. The BVP*

$$x''' + cx^p \cdot x'' = 0, \qquad x(0) = 0 = x'(0), \quad \lim_{t\to\infty} x'(t) = \beta,$$

has a solution for any $\beta \geq 0$.

Proof. The function $h : [0, \infty) \to \mathbb{R}$ is continuous on $(0, +\infty)$ by the continuous dependence of solutions of ODEs on initial conditions and locally uniform convergence of the integral

$$\int_0^{\infty} \exp\left(-c \int_0^s x_a(\tau)^p \, d\tau \right) ds.$$

By the estimates from Lemma 2.4, we have

$$\lim_{a\to 0^+} h(a) = 0, \qquad \lim_{a\to\infty} h(a) = +\infty.$$

Thus, for any $\beta > 0$, there exists $a > 0$ such that $h(a) = \beta$. For $\beta = 0$, it is obvious. □

Finally, the uniqueness of the solution of (1.1) will be proved by using ideas from [5]. For any $a > 0$ consider the one-to-one function $v_a : [0, h(a)^2) \to [0, \infty)$

such that $v_a(x_a'(t)^2) = x_a(t)$ for each $t \geq 0$. It is well defined since x_a and x_a' are increasing functions and it belongs to $C^2(0, h(a)^2)$. Substituting $y = x'(t)^2$, we shall find an ODE satisfied by v (we omit subscript a for simplicity).

$$x(t) = v(y), \qquad x'(t) = v'(y)2x'(t)x''(t)$$

hence,

$$x''(t) = \frac{1}{2v'(y)}, \qquad x'''(t) = -\frac{v''(y)}{2v'(y)^2} \cdot 2x'(t)x''(t) = -\frac{v''(y)\sqrt{y}}{2v'(y)^3}.$$

Put x and its derivatives in our ODE and find

$$v''(y) = \frac{cv(y)^p v'(y)^2}{\sqrt{y}}. \tag{3.1}$$

From boundary conditions on x we get

$$v(0) = 0, \qquad v'(0) = \frac{1}{2a}, \qquad \lim_{y \to h(a)^2-} v(y) = +\infty. \tag{3.2}$$

Now, we are in a position to prove the following theorem.

Theorem 3.2. *Let $p \geq 1$, $c > 0$. For any $\beta \geq 0$ the solution of*

$$x''' + cx^p \cdot x'' = 0, \qquad x(0) = 0 = x'(0), \quad \lim_{t \to \infty} x'(t) = \beta,$$

is unique.

Proof. We need to show that the function h is one-to-one. Suppose that $h(a_1) = h(a_2)$, $a_2 > a_1$ and take v_1 and v_2 obtained by x_{a_1} and x_{a_2}, respectively, that is v_i satisfies (3.1) with boundary conditions (3.2) (for $a = a_i$, $i = 1, 2$). Put $w = v_1 - v_2$. Then $w(0) = 0$, $w'(0) = \frac{a_2 - a_1}{2a_1 a_2} > 0$. Notice that $w' > 0$ on the whole interval $(0, h(a_1)^2)$ – both functions are defined on the same interval.

In fact, if it is not true, then there exists s in this interval such that $w' > 0$ on $(0, s)$ and $w'(s) = 0$. Hence $w(s) > w(0) = 0$ and

$$w''(s) = \lim_{\xi \to 0+} \frac{w'(s - \xi) - w'(s)}{-\xi} \leq 0.$$

On the other hand,

$$w''(s) = v_1''(s) - v_2''(s) = cs^{-1/2}(v_1(s)^p - v_2(s)^p)v_{a_i}'(s)^2,$$

since $w'(s) = 0$ implies $v_{a_1}'(s) = v_{a_2}'(s)$. But $v_1(s)^p > v_2(s)^p$ from $w(s) > 0$ and this gives $w''(s) > 0$ – a contradiction. Thus, we have $w > 0$ and $w' > 0$ on $(0, h(a_1)^2)$.

Set $V_i = 1/v_i'$, $i = 1, 2$ and $W = V_1 - V_2$. We have, for any $y \in (0, h(a_1)^2)$,

$$W'(y) = V_1'(y) - V_2'(y) = -\frac{v_1''(y)}{v_1'(y)^2} + \frac{v_2''(y)}{v_2'(y)^2} = c\frac{v_2(y)^p - v_1(y)^p}{\sqrt{y}} < 0$$

from $w(y) > 0$. Hence,

$$W(y) < W(0) = \frac{1}{v_1'(0)} - \frac{1}{v_2'(0)} = 2(a_1 - a_2)$$

and

$$\lim_{y \to h(a_1)^2-} W(y) \le 2(a_1 - a_2) < 0.$$

On the other hand, $V_i(x_{a_i}'(t)^2) = 2x_{a_i}''(t)$ implies

$$\lim_{y \to h(a_1)^2-} V_i(y) = \lim_{t\to\infty} V_i(x_{a_i}'(t)^2) = 2 \lim_{t\to\infty} x_{a_i}''(t) = 0$$

and, therefore,

$$\lim_{y \to h(a_1)^2-} W(y) = 0,$$

which contradicts the previous inequality. □

4. NUMERICAL APPROACH

All numerical methods cannot work on the infinite interval $[0, \infty)$ and we do not know the exact value of $a = x''(0)$ for the solution. Our earlier results make it possible to find a finite interval $[0, T]$ for any positive value ϵ of the error control tolerance such that

$$x''(T) < \epsilon, \qquad h(a) - x'(T) < \epsilon, \qquad x(T) - (h(a)T - \mu(a)) < \epsilon.$$

Since all these functions decrease, all three inequalities hold for any $t > T$. First, by using estimates (2.13), we can find an interval $[a_{min}, a_{max}]$ such that $h(a_{min}) < \beta < h(a_{max})$. Next, by (2.1), we need

$$a_{max} \exp\left(-c \int_0^T x_{a_{min}}(\tau)^p \, d\tau\right) < \epsilon,$$

by (2.14), we should have

$$a_{max} \int_T^\infty \exp\left(-c \int_0^s x_{a_{min}}(\tau)^p \, d\tau\right) ds < \epsilon,$$

and by (2.3) and (2.15), we get

$$-a_{max} \int_T^\infty (T - s) \exp\left(-c \int_0^s x_{a_{min}}(\tau)^p \, d\tau\right) ds < \epsilon.$$

We do not know the function $x_{a_{min}}$ but we can use estimate (2.11) to get

$$a_{max} \exp\Bigg(- c \int\limits_0^T \left(c_2 a_{min}^{\frac{p+1}{2p+1}} \tau - c_3 a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \Bigg) < \epsilon, \tag{4.1}$$

$$a_{max} \int\limits_T^{\infty} \exp\Bigg(- c \int\limits_0^s \left(c_2 a_{min}^{\frac{p+1}{2p+1}} \tau - c_3 a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \Bigg) ds < \epsilon, \tag{4.2}$$

$$-a_{max} \int\limits_T^{\infty} (T-s) \exp\Bigg(- c \int\limits_0^s \left(c_2 a_{min}^{\frac{p+1}{2p+1}} \tau - c_3 a_{min}^{\frac{1}{2p+1}} \right)^p d\tau \Bigg) ds < \epsilon. \tag{4.3}$$

We start with a family of initial value problems

$$x_a(t), t \in [0,T], x_a(0) = 0 = x_a'(0), x_a''(0) = a, a \in [a_{min}, a_{max}]. \tag{4.4}$$

If we approximate this solution in $[0,T]$ with an error less than ϵ, then the best approximation of x in $[T, \infty)$ is

$$x(t) = \beta t + (x_a(T) - \beta T)\,.$$

The lower and upper bounds for the second derivative describe the shooting window - for each a the only solution in this direction exists at $t = T$, and the computed value of $x'(T)$ is more and more close to the expected limit β. As long as β is contained between the computed values $x'(T)$ of the best two shots, we apply the classical bisection method:

If y and z are solutions such that $y''(0) < z''(0)$, and there is $y'(T) < \beta < z'(T)$, then the next problem to solve is (4.4) with $a = (y''(0) + z''(0))/2$.

For any value $\epsilon > 0$ we determine T by solving the inequality (4.1). After that we check if inequalities (4.2), (4.3) hold for the obtained T. Otherwise, we increase T to the moment where (4.2), (4.3) hold.

4.1. EXAMPLES

While solving the initial value problems we apply an adaptive Runge-Kutta-Fehlberg method RK45 [7], in which a tolerance parameter ϵ controls the local error of the method. The values of ϵ range from 10^{-8} to 10^{-14}. For representing real values we use the standard $16-17$-digits double data type.

Numerical results for the classical Blasius equation $p = 1, c = 1/2$, **and** $\beta = 1$.

Here $a_{min} = 0.2\,694\,860\,459$, $a_{max} = 0.3\,420\,953\,216$. For $T = 14$ we get the all three inequalities (4.1), (4.2) and (4.3) for $\epsilon = 1.0e - 14$. Below N stands for the number of steps in RK45 (average) (Tab. 1).

Table 1.

ϵ	N	a	$\lvert x''(0) - a\rvert$	$\lvert 1 - x'(T)\rvert$	$x(T)$
$1.0e-08$	112	0.332 057 330 068 201	$6.15e-09$	$1.2e-08$	12.279 212 180 321
$1.0e-09$	197	0.332 057 335 646 357	$5.69e-10$	$1.1e-09$	12.279 212 327 474
$1.0e-10$	338	0.332 057 336 149 237	$6.60e-11$	$1.3e-10$	12.279 212 340 740
$1.0e-11$	611	0.332 057 336 210 248	$4.95e-12$	$5.7e-13$	12.279 212 342 472
$1.0e-13$	1831	0.332 057 336 215 154	$4.19e-14$	$6.5e-14$	12.279 212 342 479
$1.0e-14$	3346	0.332 057 336 215 186	$1.06e-14$	$5.5e-16$	12.279 212 342 480

The last value of a differs in the two last digits from the one cited in [3]:

$$a = 0.332\,057\,336\,215\,196\,30.$$

The last two columns of Table 1 have been computed for $\epsilon = 10^{-14}$. As there is $x''(T) = 7.68e-13$, for $t > T$ the straight line approximation of the solution is the most effective.

Numerical results for the equation with $p = 7$, $c = 1/2$ and $\beta = 1$.

Here, $a_{min} = 0.373\,397\,838\,8$, $a_{max} = 0.380\,548\,242\,7$. As above for the tolerance $\epsilon = 1.0e-14$, the interval $[0, 4]$ is sufficiently large and we get the following results by the RK45 method (Tab. 2).

Table 2.

ϵ	N	a	$\lvert x''(0) - a\rvert$	$\lvert 1 - x'(T)\rvert$	$x(T)$
$1.0e-08$	189	0.379 398 164 451 122	$2.47e-08$	$3.5e-08$	2.673 055 448 977
$1.0e-09$	316	0.379 398 187 063 634	$2.04e-09$	$2.9e-09$	2.673 055 570 853
$1.0e-10$	549	0.379 398 189 005 442	$1.03e-10$	$1.4e-10$	2.673 055 581 319
$1.0e-11$	961	0.379 398 189 086 642	$2.20e-11$	$3.1e-11$	2.673 055 581 757
$1.0e-12$	1688	0.379 398 189 106 905	$1.69e-12$	$2.3e-12$	2.673 055 581 866
$1.0e-13$	2827	0.379 398 189 108 438	$1.62e-13$	$1.9e-13$	2.673 055 581 874
$1.0e-14$	4634	0.379 398 189 108 571	$2.91e-14$	$0.0e-14$	2.673 055 581 875

Remarks – as above. Here $x''(T) = 9.03e-18$. The value of $a = 0.3793981891086$ – here, all digits are true.

Numerical experiment for the equation with $p = 0.1$, $c = 1/2$, $\beta = 1$.

Taking $\epsilon = 10^{-14}$ we get $T = 50$. The proof of the existence and uniqueness result for $p < 1$ fails, since we cannot claim that the initial value problem (1.3) has a unique solution and that it depends continuously on a. Hence, function h can be multivalued. If one will prove the uniqueness, then, due to [10, p. 172], h will be continuous and all results of this paper will be true also for $p < 1$. The stability of numerical experiments cited below suggests it is the fact. Numerical results presented below in Table 3 for this case suggest that the method works also wihout the uniqueness of solution to initial value problems.

Table 3.

ϵ	N	a	$\lvert x''(0)-a\rvert$	$\lvert 1-x'(T)\rvert$	$x(T)$
$1.0e-08$	142	0.443 643 205 985 427	$2.16e-07$	$4.5e-07$	48.054 260 863 24
$1.0e-09$	256	0.443 643 403 844 908	$1.78e-08$	$3.7e-08$	48.054 280 581 20
$1.0e-10$	466	0.443 643 420 192 529	$1.49e-09$	$3.1e-09$	48.054 282 210 34
$1.0e-11$	839	0.443 643 421 402 885	$2.80e-10$	$5.8e-10$	48.054 282 330 96
$1.0e-12$	1505	0.443 643 421 660 499	$2.25e-11$	$4.7e-11$	48.054 282 356 64
$1.0e-13$	2669	0.443 643 421 681 506	$1.49e-12$	$3.6e-12$	48.054 282 358 73
$1.0e-14$	4922	0.443 643 421 683 245	$2.45e-13$	$2.2e-16$	48.054 282 358 90

Remarks – as above. Here $x''(T) = 1.02e - 15$. The value of $a = 0.443\,643\,421\,683$ – here, all digits are true.

5. CONCLUSIONS

The authors know that our computation of the value of the second derivative of the solution are not more exact than others. However, the proposed method gives a possibility of controlling errors and it is very simple. We hope a similar approach can be applied for more general equations as $x''' + f(x) \cdot g(x'') = 0$ with qualitative assumptions on functions f and g.

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Grzegorz Andrzejczak
grzegorz.andrzejczak@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

Magdalena Nockowska-Rosiak
magdalena.nockowska@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

Bogdan Przeradzki
bogdan.przeradzki@p.lodz.pl

Lodz University of Technology
Institute of Mathematics
Wolczanska 215, 90-924 Lodz, Poland

Received: June 7, 2012.
Accepted: June 23, 2012.