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Identification of the Fractional-Order Systems: A Frequency Domain Approach

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The paper deals with a comparison of different optimization methods to identification of fractional order dynamical systems. The fractional models of the examples of physical systems - ultracapacitors - are established. Then different real frequency responses data from a laboratory setup of the processes are collected and the comparison of identification methods based on least squares and total least squares are presented. The accuracy of the methods is discussed using the frequency responses of the identified model and the theoretical one.

Keywords: *fractional calculus; frequency response; least squares method; total least squares method; ultracapacitor.*

Motivation

Energy efficiency is becoming a more and more important issue, these days. Currently, there is a lot of work and a huge effort both in industry and academia goong on in the area of energy savings and its effective use. Among many possible solutions, we can apply one of available options of the energy storage in the power distribution network. Some of them, which can be successfully used in electrical energy usage optimisation in mining and metallurgy industries can be found in [1]. In this paper we will concentrate only on one of the energy storage devices, namely ultracapacitor. The ultracapacitors began occupying more and more space in various areas of the applications, such as smart grid, hybrid and electric vehicles, backup sources to name just a few. This is the main reason, why we need more accurate models of such devices. In this paper we focus on identification of the new model parameters in the frequency domain.

Introduction

Ultracapacitors, also know as a supercapacitors, are electrical devices which are used to store electric energy and offer high electric power density that is not possible to achieve with traditional capacitors. Nowadays, ultracapacitors have many industrial applications and are used wherever a high current in a short time is needed. Thanks to a very complicated internal structure, they are able to store or yield a lot of energy in a short period of time. Many researchers started building more or less complicated models in order to explain the capability of ultracapacitors. Numerous articles have presented the RC model (e.g. [3, 4]), which is particularly accurate for low frequencies. Some authors describe ultracapacitors by the RC transmission line [2, 23]. Also, the dynamic behavior of ultracapacitors has been modeled using the technique based on impedance spectroscopy [3]. In the papers, [21], [22], [25] a very efficient approach using fractional order calculus was presented and in [5], [6] ultracapacitor frequency domain modeling was introduced. In this article, the fractional order modeling of the ultracapacitor is presented. The extension of modeling from frequency domain [7, 9] to time domain is presented in [8].

Fractional Calculus and Systems Frequency Response

Fractional order differential calculus used for the purpose of ultracapacitors modeling is but a generalization of integer order integral and differential calculus to real or even complex order. This idea has first emerged at the end of 17th century, and has been developed in the area of mathematics throughout 18th and 19th century in the works of e.g. Liouville, Riemann and others. More recently, by the end of 20th century, it turned out that some physical phenomena are modelled more accurately when fractional calculus is in use. There exist two (in fact three) main definitions of

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the fractional order integrals, derivatives and differences: Riemann-Liouville, Caputo, and Grünwald-Letnikov [20]. Some other, are also present in literature, but less comonly used in applications. In this paper the Caputo definition of the fractional order difference is used. To be precise, the Riemann-Liouville and Caputo definitions concern both fractional derivative and integral. Thus, the term differ-integral is used.

Definition of Gamma function

To define the fractional order differ-integral, the definition of the $\Gamma(x)$ function is needed. The $\Gamma(x)$ function is given in the following two ways [20]:

1. Integral definition (Euler):

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \tag{1}$$

   where $\Re(x) > 0$.

2. Limit definition:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\dots(x+n)} \tag{2}$$

   where $x \in \mathrm{C}$.

The $\Gamma(x)$ function satisfies the following relation

$$\Gamma(x+1) = x\Gamma(x), \tag{3}$$

which means that this function is a generalization of factorial function.

Definition of fractional order differ-integral

In this article the following definition of the fractional derivative, following the Caputo formulation will be used.
Caputo definition of fractional order differ-integral

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{4}$$

for $(n-1 < \alpha < n)$, where $\alpha \in \mathrm{R}$ is a fractional order of the differ-integral of the function $f(t)$. Usually we use a lower bound $a = 0$ in definitions (4).

When $\alpha > 0$ the result of this function is equivalent to the fractional order derivative, for $\alpha < 0$ to fractional order integral and for $\alpha = 0$ to the function itself. This is why tha above definition is called a differ-integral.

The Laplace transformation of the fractional order differ-integral in Caputo form is given as follows:

$$\int_0^\infty e^{-st} {}_0D_t^\alpha f(t)\, dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0),$$

for $(n-1 < \alpha \leq n)$, $n \in \mathrm{N}$, where $s \equiv j\omega$ denotes the Laplace operator.

For our purpose we will us the Caputo definition because of physical meaning of initial conditions, which can be observed in experiment. The initial conditions for the fractional order differential equations with the Caputo derivatives are in the same form as for the integer-order differential equations. In spite of this, the model presented in this paper is based on the fractional order transfer function, where initial conditions are equal to zero.

Definitions of fractional-order system

A general fractional-order linear system can be described by the transfer function of incommensurate real order of the following form [16, 19, 20]:

$$G(s) = \frac{b_m s^{\beta_m} + \dots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \dots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{5}$$

where a_k $(k = 0, \dots n)$, b_k $(k = 0, \dots m)$ are constant, and α_k $(k = 0, \dots n)$, β_k $(k = 0, \dots m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \dots > \alpha_1 > \alpha_0$, and $\beta_m > \dots > \beta_1 > \beta_0$.

The system transfer function (5) can be expressed as a complex number in the frequency domain as

$$G(j\omega) = P(\omega) + jQ(\omega) = A(\omega)e^{j\varphi(\omega)}, \tag{6}$$

where ω is an angular frequency in rad/sec unit.

Note that for complex number s raised to power α we can write

$$s^{\alpha} = (j\omega)^{\alpha} = \omega^{\alpha}(\cos\alpha\frac{\pi}{2} + j\sin\alpha\frac{\pi}{2}) = \omega^{\alpha}e^{\frac{j\pi\alpha}{2}}. \tag{7}$$

In this article we will use a theoretical model of the ultracapacitor with the following transfer function:

$$G_{uc}(s) = \frac{U_{uc}(s)}{I(s)} = R_c + \frac{(Ts+1)^{\alpha}}{Cs^{\beta}}, \ (\alpha, \beta \in \mathrm{R}) \tag{8}$$

where R_c is the resistance of the ultracapacitor, C is the capacitance and T is related to equivalent capacitance of the ultracapacitor.

Methods Applied to Parameters Identification

Least squares methods

The total least squares (TLS) method is a numerical linear algebra tool for finding approximate solutions to overdetermined systems of equations $Ax = b$. In the classical least squares (LS) approach the measurements A are assumed to be error-free and hence all errors are concerned in the observation vector b. In the TLS approach both, the vector b as well as the matrix A suffer from errors, due to the sampling errors, modeling errors, instrument errors and last but not least human errors. Some advantages of TLS can be found for example in [17, 18] (distance is not depending on choice of coordinate system, it is usable for n-dimensional fitting, etc.).

In 1805 Legendre published “Nouvelles methodes pour la determination des cometes”, in which he introduced the method of least squares and gave it this famous name. In 1809, Gauss published the book “Theoria motus corporum coelestium in sectionibus conicis solem ambientium”, translated in English [10], where he discussed the method of least squares and, mentioning Legendre’s work, stated that he himself had used the method since 1795.

Least squares method (LSM) uses the sum of squares of vertical offsets of data points from fitting line. The optimization criterion of LSM can be expressed as:

$$E = \sum_i [y_i - f(x_i, \alpha_1, \alpha_2, \ldots, \alpha_n)]^2, \tag{9}$$

where we look for the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Total least squares method

Although the name “total least squares” was first used in the literature only 30 years ago by Golub and van Loan [11] and later by Van Huffel [13], this fitting method is not new. It appeared first in statistics under the name “orthogonal regression”, “error-in-variables regression" or “measurement error modeling”.

Orthogonal distance fitting (ODF) uses the sum of squares of orthogonal (perpendicular) distances of data points from the fitting line. The optimization criterion of ODF can be expressed as:

$$E = \sum_i [d((x_i, y_i), f(x, \alpha_1, \alpha_2, \ldots, \alpha_n))]^2, \tag{10}$$

where $d((x_i, y_i), f)$ denotes the distance between the points (x_i, y_i) and the fitting line f.

Laboratory Setup and Experiments Description

The main aim of the experimental setup is to set the desired current value (in both directions) to the high capacity ultracapacitor and to measure the voltage on the ultracapacitor. The experimental setup contains:

- High Capacity Ultracapacitor 1500F/2.7V
- DS1104 Control Card
- Electronic System based on MOSFET Power Converter

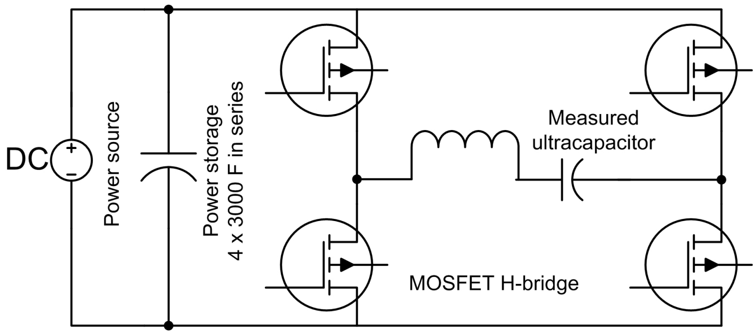

Fig. 1: Experimental setup scheme.

The general scheme of electronic system used is presented in Fig. 1. The MOSFET H-bridge allows to control the charge/discharge current on the measured ultracapacitor up to the value of 150A. For control and data acquisition purposes the DS1104 dSPACE Control Card is connected by electronic interface with opto-insulation of control signals. The DS1104 card is also used to control the current by PID controller.

The energy is exchanged between the measured ultracapacitor and energy storage built from four ultracapacitors of capacity 3000F/2.7V connected in series. The loss of energy (mainly due to heat losses) is compensated from the power source 100A/10V. The measured ultracapacitor is an ultracapacitor produced by Maxwell (BCAP1500 2.7V).

The ultracapacitor is an electrolytic capacitor and it can accept only positive voltages. In the case of using this type of setup with a current converter, to model high capacity ultracapacitor, the input signal was a current sine wave. The ultracapacitors had an initial voltage depending on a signal frequency (u_0). Capacitor voltage in this case was equal to $u_c(t) = u_0 + A_c(\omega)\sin(\omega t + \varphi_u)$, and input (capacitor) current was $i(t) = A_i(\omega)\sin(\omega t + \varphi_c)$.

The Bode diagram was obtained from the following relations:

$$M(\omega) = 20\log\left(\frac{A_c(\omega)}{A_i(\omega)}\right), \quad \varphi(\omega) = \varphi_i(\omega) - \varphi_u(\omega)$$

and the Nyquist diagram was obtained from the following relations:

$$P(\omega) = \Re\left(\frac{A_c(\omega)}{A_i(\omega)}\right)e^{i\varphi(\omega)}, \quad Q(\omega) = \Im\left(\frac{A_c(\omega)}{A_i(\omega)}\right)e^{i\varphi(\omega)},$$

for the angular frequency ω. The modeling of the ultracapacitor presented in this paper is based on the matching of Bode and Nyquist diagrams.

Discussion of Experimental Results

The first approach to the problem of ultracapacitor identification in frequency domain is to use the Bode diagram matching with the Least Squares error minimization. As ultracapacitor model we have used the one given by (8) with $\beta = 1$.

The results of using the Least Squares algorithm with cost function defined as

$$E = \sum_i \left[(A(\omega_i) - Am(\omega_i))^2 + (F(\omega_i) - Fm(\omega_i))^2\right] \tag{11}$$

is presented in Fig 2, where $A(\omega_i)$ and $F(\omega_i)$ are measured data and $Am(\omega_i)$ and $Fm(\omega_i)$ are modeled data. As it can be noticed the values for logarithmic magnitude have different range than the values of phase shift. This has an negative effect on the modeling result because the error of phase is minimized to a greater extent than the error of magnitude. This produces the model with less accuracy for magnitude. That problem can be compensated, however only in particular case, by modification of the cost function in such a way as to increase the values of magnitude error. For example we can assume the following cost function

$$E = \sum_i \left[4(A(\omega_i) - Am(\omega_i))^2 + (F(\omega_i) - Fm(\omega_i))^2\right], \tag{12}$$

the results of diagram matching is presented in Fig 3. As it can be seen the magnitude is much closer to the measured data than in Fig 2.

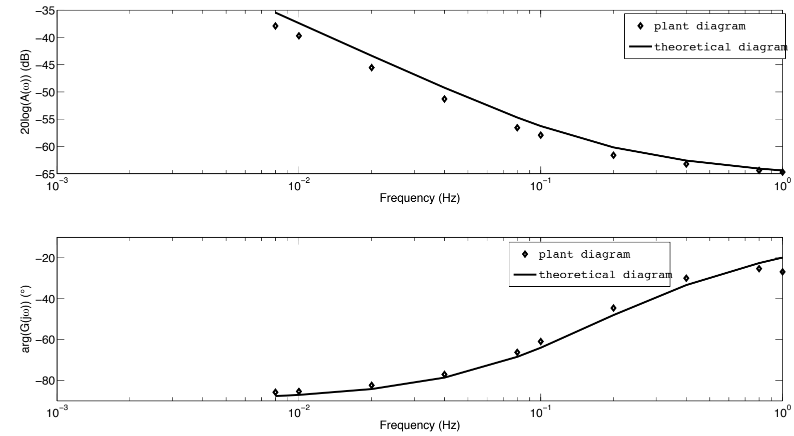

Fig. 2: Bode diagrams of 1500F ultracapacitor modeling with LS algorithm.

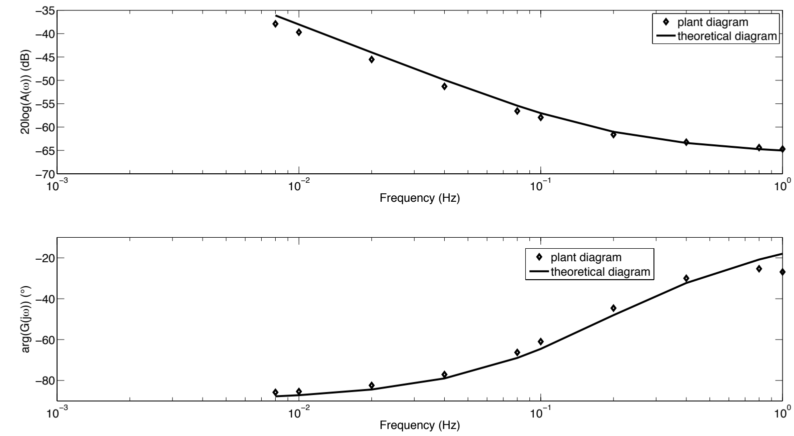

Fig. 3: Bode diagrams of 1500F ultracapacitor modeling with modified LS algorithm.

In order to compensate the effect of different range of values for magnitude and phase let us change the domain from the Bode diagram to the Nyquist (impedance) diagram domain. The cost function in this case has the form

$$E = \sum_i \left[(\Re(G(\omega_i)) - \Re(Gm(\omega_i)))^2 + (\Im(G(\omega_i)) - \Im(Gm(\omega_i)))^2 \right], \tag{13}$$

and both parts have the same physical meaning.

The results of modeling the ultracapacitor in impedance domain with using the least squares method is presented in Fig. 4. As it can be seen the model cannot properly describe such an impedance. For low frequencies the measured impedance has an argument different than -90° what is not included in the model. In Bode diagram values of phase for low frequencies are very close to the value of model argument i.e. -90° and at least produce small constant error. In the impedance (Nyquist) diagram case this effect is much stronger, and to compensate it needs more complicated model (than (8)). We use, therefore the model with fractional order integrator in the denominator. This model has better ability to model such an impedance. However, it has also some disadvantages. The main disadvantage is that the parameter C losses its physical meaning of capacitance. In this model the denominator Cs^β has a meaning of the impedance with a capacitance $\Im[C(j\omega)^\beta]$ and a frequency dependent resistance $\Re[C(j\omega)^\beta]$.

In impedance domain it is possible to use also the TLS method to parameters identification. The results of identification for all the methods: Least Squares in Bode domain, Least Squares in impedance domain and Total Least Squares in impedance domain are presented in Fig. 5 and 6. Fig. 5 presents these results in Bode diagram and Fig. 6 presents the same results in impedance diagram.

Fig. 6 presents final results of identification, as it can be seen the TLS method produces better accuracy than the other methods used for the identification.

Table 1 presents data obtained during identification process with different algorithms. The parameters R_c are

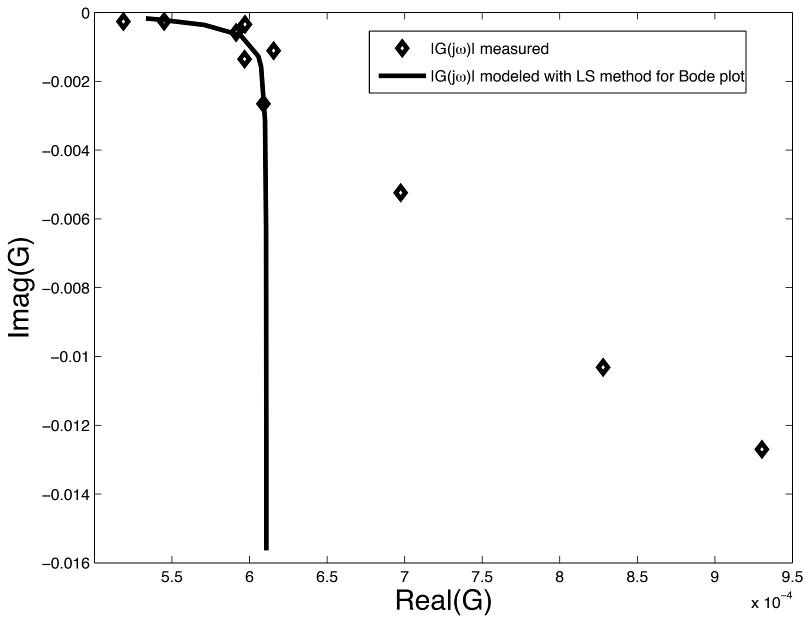

Fig. 4: Impedance diagrams of 1500F ultracapacitor modeling with LS algorithm.

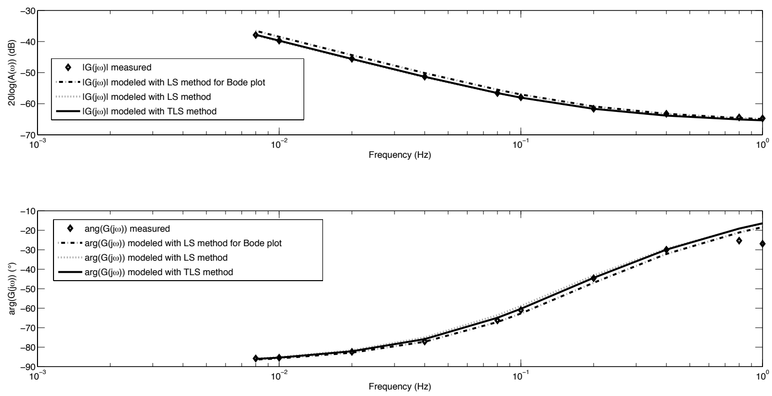

Fig. 5: Bode diagrams of 1500F ultracapacitor modeling with different identification algorithms for impedance domain.

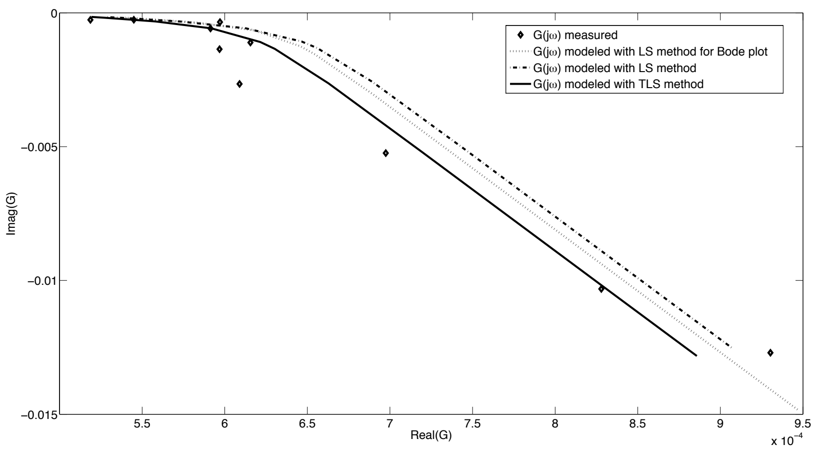

Fig. 6: Impedance diagrams of 1500F ultracapacitor modeling with different identification algorithms for impedance domain.

Tab. 1: Identified parameters of ultracapacitor

domain (method)	T	C	R_c	α	β
Bode (LS)	1.2782	1275	0.00047	0.3204	-
Bode (MLS)	1.4530	1402	0.00047	0.2604	-
Bode (LS)	1.1177	1235	0.00047	0.3112	0.986
Nyquist (LS)	0.9096	1491	0.00047	0.2339	0.986
Nyquist (TLS)	0.9870	1483	0.00047	0.2048	0.986

chosen from the datashet and parameter β is chosen manually because including these parameters into identification algorithms makes the identification problem highly nonlinear and the results were strongly dependent on the initial conditions.

Conclusions

In the paper some preliminary results of identification of fractional order dynamical systems basing on their frequency responses have been given. The frequency response in the form of both Bode plots and Nyquist diagrams for such systems have been used. The identification method was based on finding the model parameters for which the optimal matching of experimental frequency response with the one obtained for the model proposed was obtained. As optimization tools the Least Squares methods were used for Bode and Nyquist diagrams. For Nyquist diagrams only also the Total Least Squares optimization has been used. The results of identification of the parameters of a simple fractional order model for the high capacity ultracapactiors based on experimental data are presented. The experiments performed show that the frequency response based identification gives better results when Total Least Squares optimization is applied. However, the results are satisfactory only when experimental and model Bode diagrams are compared. The Nyquist diagrams obtained from model identification differ quite significantly from those obtained experimentally for low frequencies. After making some adjustments to the model (introducing the fractional order integrator instead of the integer order one) the results obtained have shown better matching.

The methods presented in the paper show good potential as far as their application to the identification of fractional order systems is concerned. However, for identification purposes of real systems, like the ultracapacitors used in this study the methods still need further refining. The identification problem is highly nonlinear and it may need the use of more advanced optimization methods suitable for this type of problems.

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