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Periodically stimulated remodelling of a muscle fibre: perturbation analysis of a simple system of first-order ODEs

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Abstract

The paper deals with the dynamical analysis of the system of first-order ODE's describing the isometric stimulation of the muscle fibre. This system is considered to be a non-autonomous one having the periodical excitation. For the analysis of dynamical behaviour the system the multiple scale method (MSM) is employed. The main goal of this contribution is to show the application of MSM to the non-autonomous dynamical system using the first order approximation of the solution. The existence of the degenerated Hopf's bifurcation of the gained solution is presented.

Keywords: dynamical non-autonomous system, muscle fibre, multi-scale method, Hopf's bifurcation, chaos

1. Introduction

The paper follows the authors previous papers [6] dealing with the application of the non-irreversible thermodynamics and the growth and remodelling theory (GRT) [1] to the muscle fibre modelling. The approach allows taking into account also the change of the muscle fibre stiffness during time. The effect change of the muscle fibre stiffness during time was experimentally approved and modelled [2]. The same approach can be used to model the piezo-electric stack time evolution. The final simplified dimensionless formulation has the form of the dynamical system with two degrees of freedom. The numerical experiments have shown the interesting behaviour of this system, e.g. the existence of bifurcations. These effects correspond to some medical recognition like vasomotion or myogenic response. This contribution is devoted to the analysis of these properties using the multi-scale method (MSM) [3] which is kind of the perturbation method. Here, MSM is used to model the behaviour of mentioned system close to the Hopf's bifurcation point leading to the periodical or even chaotic motion.

2. Problem setting

In GRT the starting point is the initial configuration B_0 that "growths" and "remodels" , i.e. it changes its volume ("growth"), anisotropy ("geometrical remodelling") or material parameters ("material remodelling"). This process is expressed by the tensor $\mathbf{P}$ (further growth tensor) firstly that relates the initial configuration to the relaxed one B_r with zero inner stress to the real configuration B_t where the inner stress invoked by growth, geometrical remodelling and external loading can already exists. It is stated by the deformation tensor $\mathbf{F}$ (see Fig. 1).

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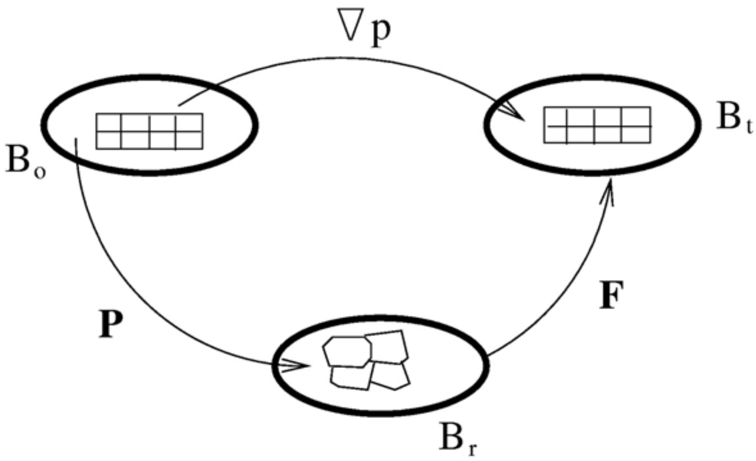

Fig. 1. Initial, relaxed and current configurations

The deformation gradient between the configurations B_0 a B_t is

$$\nabla p = \mathbf{F}\,\mathbf{P}. \tag{1}$$

Let the 1D continuum have the initial length l_0. Its actual length after growth, remodelling and loading will be l. The relaxed length (it means after growth and remodelling) is l_r. For the corresponding deformation gradients we can write

$$P = \frac{l_r}{l_0}, \qquad F = \frac{l}{l_r}, \qquad \nabla p = \frac{l}{l_0}. \tag{2}$$

In the isometric case, the actual length is constant, i.e. it holds $l = \mathrm{const.}$

In [6], the set of the ODE's describing in dimensionless form the behavior of the muscle fiber during the isometric excitation was derived according the work of DiCarlo and Quiligotti [1]

$$\dot{x} = -x\left\{C + \frac{y}{\lambda}\mathrm{e}^{\frac{\lambda}{2}(x-1)^2}\left[\lambda x(x-1) - 1\right] + \frac{y}{\lambda}\right\}, \tag{3}$$

$$\dot{y} = \operatorname{sgn} m\left[-\frac{1}{\lambda}\left(\mathrm{e}^{\frac{\lambda}{2}(x-1)^2} - 1\right)\right]. \tag{4}$$

The meaning of used variables is following

$$x = \frac{1}{l'_r}, \qquad y = k', \qquad k' = k\sqrt{\frac{|m|}{g}}, \qquad l'_r = \frac{l_r}{l} = \frac{1}{F}, \qquad t' = \frac{t}{\sqrt{g|m|}}, \tag{5}$$

where l, l_r are the lengths of the continuum in actual and relaxed configurations and k' is the dimensionless stiffness parameter. The parameters C, D, λ, m, g are the parameters.

Eqs. (3) and (4) are based on the Fung's form of the free energy

$$\psi = \frac{k}{\lambda}\left(\mathrm{e}^{\frac{\lambda}{2}(F-1)^2} - 1\right), \tag{6}$$

where k is the stiffness of the muscle fiber.

Setting $\lambda \to 0$ we obtain more simple form

$$\psi = \frac{1}{2}k(F-1)^2, \tag{7}$$

where C is the control parameter depending on the calcium concentration inside the muscle cell. Without using the complex model of the calcium concentration evolution we will suppose, that C is either constant or a periodical function of time and can be approximated in this form

$$C \to C + D\sin\omega t. \tag{8}$$

Using (7) and (8) we obtain the more simple non-autonomous system

$$\dot{x} = -x\left[C + D\sin\omega t + \frac{y}{2}(x^2-1)\right], \tag{9}$$

$$\dot{y} = \operatorname{sgn} m\left[r - \frac{1}{2}(x-1)^2\right]. \tag{10}$$

According to some numerical experiments we can see that both models have qualitatively the same properties. Therefore, we will further focus our attention on the simpler one (9) and (10).

3. Dynamical analysis for $D = 0$

This analysis was published in [5]. The existence of the degenerated Hopf's bifurcation was proved for $C = 0$ and $\operatorname{sgn} m = -1$. The situation is shown on Fig. 2 in the right corner. Depending on the sign of C there exists one stable and one unstable equilibrium point and a stable limit cycle around these points for $C = 0$.

4. Dynamical analysis using multiple scale method

We start with the dynamical system defined by the Eqs. (9) and (10). The steady solution of this dynamical system is

$$\begin{aligned} x_0 &= 1 \pm \Theta, \qquad \Theta = \sqrt{2r}, \\ y_0 &= 0, \\ C_0 &= D_0 = 0. \end{aligned} \tag{11}$$

Now we follow the procedure suggested by [4], but generalized for non-autonomous systems. Let us consider the solution

$$\begin{aligned} x &= x_0 + \varepsilon\xi, \\ y &= y_0 + \varepsilon\eta, \\ C + D\cos\omega t &= C_0 + D_0\cos\omega t + \varepsilon^2 C_2 + \varepsilon^3 D_3\cos\omega t. \end{aligned} \tag{12}$$

With this very specific approximation we restrict our analysis on the case of small C and even smaller D. We must not forget it when we will do some numerical experiments in order to validate this analysis.

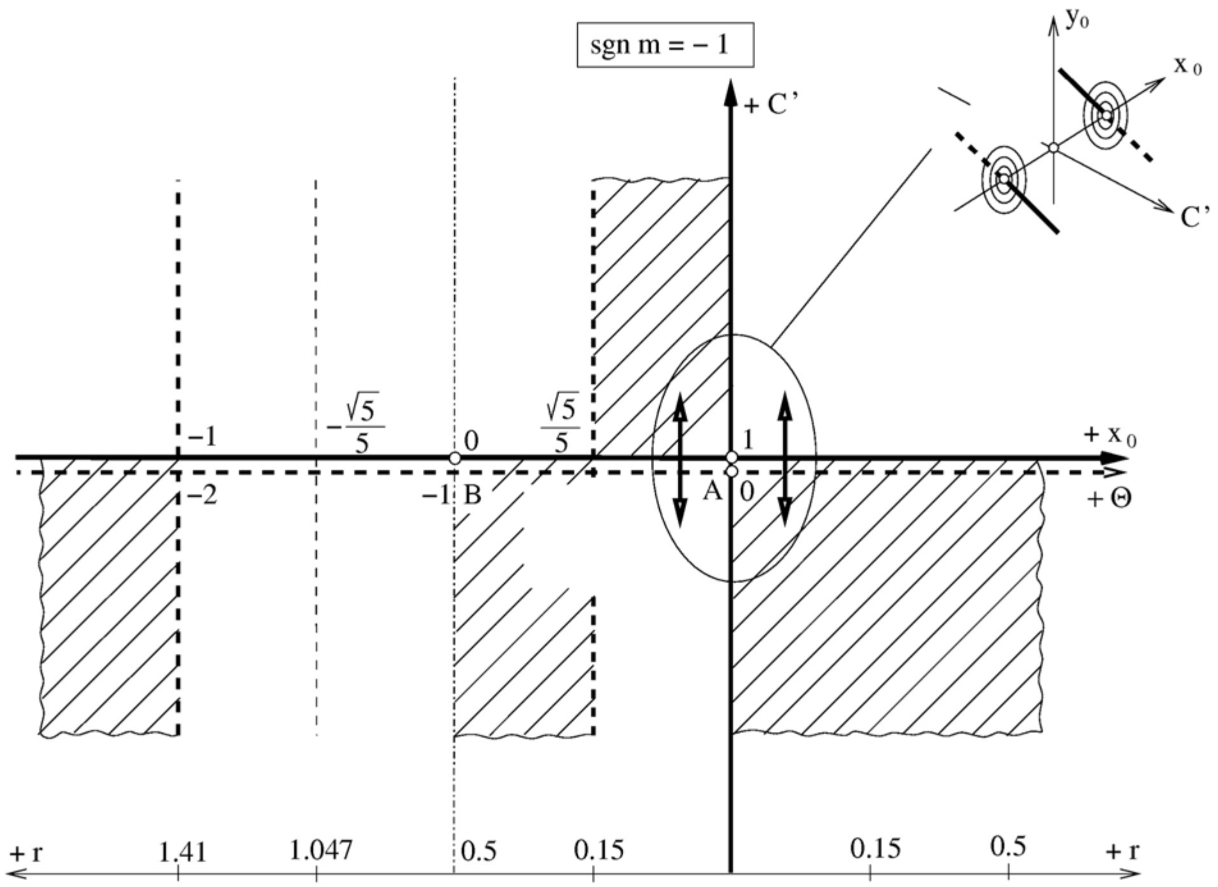

Fig. 2. Stability domains in the parameter space

We put (12) into (11) and while for x_0 and y_0 the RHSs are zero and neglecting the terms of order 4 and higher, we obtain the following non-autonomous dynamical system

$$\begin{aligned} \dot{\xi} &= -\frac{1}{2}x_0\left(x_0^2-1\right)\eta - \varepsilon\frac{1}{2}\left(3x_0^2-1\right)\xi\eta - \varepsilon^2\xi C_2 - \\ &\quad \varepsilon x_0 C_2 - \varepsilon^2\frac{3}{2}x_0\xi^2\eta - \frac{1}{2}\varepsilon^3\eta\xi^3 - \\ &\quad \varepsilon^3\xi D_3\cos\omega t - \varepsilon^2 x_0 D_3\cos\omega t, \\ \dot{\eta} &= (x_0-1)\xi + \frac{1}{2}\varepsilon\xi^2. \end{aligned} \tag{13}$$

This system will be solved using multiple-scale method. Let us assume

$$\begin{aligned} \xi &= \xi_1 + \varepsilon\xi_2 + \varepsilon^2\xi_3, \\ \eta &= \eta_1 + \varepsilon\eta_2 + \varepsilon^2\eta_3, \end{aligned} \tag{14}$$

where $\xi_i(T_0,T_2)$, $\eta_i(T_0,T_2)$; $i = 1,2,3$ and $T_0 = t$, $T_2 = \varepsilon^2 t$. Limitation of our analysis will be the restriction only on the first approximation of the solution ($x = x_0 + \varepsilon\xi_1$, $y = y_0 + \varepsilon\eta_1$).

Inserting this approximation into (13) we obtain (again by neglecting terms with the order higher than 3)

$$\frac{\partial \xi_1}{\partial T_0} + \varepsilon^2 \frac{\partial \xi_1}{\partial T_2} + \varepsilon \frac{\partial \xi_2}{\partial T_0} + \varepsilon^3 \frac{\partial \xi_2}{\partial T_2} + \varepsilon^2 \frac{\partial \xi_3}{\partial T_0} + \varepsilon^4 \frac{\partial \xi_3}{\partial T_2} =$$
$$= -\frac{x_0}{2}\left(x_0^2 - 1\right)\left(\eta_1 + \varepsilon\eta_2 + \varepsilon^2\eta_3\right) -$$
$$\frac{1}{2}\left(3x_0^2 - 1\right)\left(\varepsilon\xi_1\eta_1 + \varepsilon^2\eta_1\xi_2 + \varepsilon^3\eta_1\xi_3 + \varepsilon^2\eta_2\xi_1 + \varepsilon^3\eta_2\xi_2 + \varepsilon^3\eta_3\xi_1\right) -$$
$$\varepsilon x_0 C_2 - \frac{3}{2}x_0\left(\varepsilon^2\eta_1\xi_1^2 + 2\varepsilon^3\eta_1\xi_1\xi_2 + \varepsilon^3\eta_2\xi_1^2\right) - C_2\left(\varepsilon^2\xi_1 + \varepsilon^3\xi_2\right) -$$
$$\varepsilon^2 x_0 D_3 \cos\omega T_0 - \varepsilon^3\xi_1 D_3 \cos\omega t, \tag{15}$$
$$\frac{\partial \eta_1}{\partial T_0} + \varepsilon^2 \frac{\partial \eta_1}{\partial T_2} + \varepsilon \frac{\partial \eta_2}{\partial T_0} + \varepsilon^3 \frac{\partial \eta_2}{\partial T_2} + \varepsilon^2 \frac{\partial \eta_3}{\partial T_0} + \varepsilon^4 \frac{\partial \eta_3}{\partial T_2} =$$
$$= (x_0 - 1)\left(\xi_1 + \varepsilon\xi_2 + \varepsilon^2\xi_3\right) + \frac{1}{2}\left(\varepsilon\xi_1^2 + \varepsilon^3\xi_2^2 + 2\varepsilon^2\xi_1\xi_2 + 2\varepsilon^3\xi_1\xi_3\right). \tag{16}$$

If we compare the terms with the same order of ε, we obtain the following systems of equations

$$\varepsilon^0: \quad \begin{aligned} \frac{\partial \xi_1}{\partial T_0} &= -\frac{1}{2}x_0\left(x_0^2 - 1\right)\eta_1, \\ \frac{\partial \eta_1}{\partial T_0} &= (x_0 - 1)\xi_1, \end{aligned} \tag{17}$$

$$\varepsilon^1: \quad \begin{aligned} \frac{\partial \xi_2}{\partial T_0} &= -\frac{1}{2}x_0\left(x_0^2 - 1\right)\eta_2 - \frac{1}{2}\left(3x_0^2 - 1\right)\xi_1\eta_1 - x_0 C_2, \\ \frac{\partial \eta_2}{\partial T_0} &= (x_0 - 1)\xi_2 + \frac{1}{2}\xi_1^2, \end{aligned} \tag{18}$$

$$\varepsilon^2: \quad \begin{aligned} \frac{\partial \xi_1}{\partial T_2} + \frac{\partial \xi_3}{\partial T_0} &= -\frac{1}{2}x_0\left(x_0^2 - 1\right)\eta_3 - \frac{1}{2}\left(3x_0^2 - 1\right)\left(\eta_1\xi_2 + \eta_2\xi_1\right) - \\ &\quad \frac{3}{2}x_0\eta_1\xi_1^2 - C_2\xi_1 - x_0 D_3 \cos\omega T_0, \\ \frac{\partial \eta_1}{\partial T_2} + \frac{\partial \eta_3}{\partial T_0} &= (x_0 - 1)\xi_3 + \xi_1\xi_2. \end{aligned} \tag{19}$$

System (17) can be rewritten into form

$$\begin{aligned} \frac{\partial^2 \xi_1}{\partial T_0^2} + \Omega^2\xi_1 &= 0, \\ \eta_1 &= -\frac{2}{x_0\left(x_0^2 - 1\right)}\frac{\partial \xi_1}{\partial T_0}, \end{aligned} \tag{20}$$

where

$$\Omega^2 = \frac{1}{2}x_0\left(x_0^2 - 1\right)(x_0 - 1). \tag{21}$$

Solution of Eq. (20) can be written as

$$\begin{aligned} \xi_1 &= A(T_2)\mathrm{e}^{i\Omega T_0} + \bar{A}(T_2)\mathrm{e}^{-i\Omega T_0}, \\ \eta_1 &= -\Phi i\left(A\mathrm{e}^{i\Omega T_0} - \bar{A}\mathrm{e}^{-i\Omega T_0}\right), \end{aligned} \tag{22}$$

where

$$\Phi = \frac{2\Omega}{x_0\left(x_0^2-1\right)}. \tag{23}$$

Now we transform the system (18) into the form

$$\frac{\partial^2 \xi_2}{\partial T_0^2} + \Omega^2 \xi_2 = -\frac{1}{4}x_0\left(x_0^2-1\right)\xi_1^2 - \frac{1}{2}\left(3x_0^2-1\right)\frac{\partial \xi_1}{\partial T_O}\eta_1 - \frac{1}{2}\left(3x_0^2-1\right)\xi_1\frac{\partial \eta_1}{\partial T_O},$$
$$\eta_2 = -\frac{2}{x_0\left(x_0^2-1\right)}\left[\frac{\partial \xi_2}{\partial T_O} + \frac{1}{2}\left(3x_0^2-1\right)\xi_1\eta_1 + x_0 C_2\right]. \tag{24}$$

After inserting from (22) we obtain the equation

$$\frac{\partial^2 \xi_2}{\partial T_0^2} + \Omega^2 \xi_2 = -\frac{1}{2}x_0\left(x_0^2-1\right)A\bar{A} - Q\left[A^2 \mathrm{e}^{i2\Omega T_0} + \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right], \tag{25}$$

where

$$Q = \frac{1}{4}x_0\left(x_0^2-1\right) + \left(3x_0^2-1\right)\Phi\Omega \tag{26}$$

is constant. Particular solution of this equation with the frequency equal to 2Ω is

$$\xi_2 = -\frac{x_0}{x_0-1}A\bar{A} + \frac{Q}{3\Omega^2}\left(A^2\mathrm{e}^{i2\Omega T_0} + \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right),$$
$$\eta_2 = -\frac{2}{x_0\left(x_0^2-1\right)}\left[i\frac{2}{3}\frac{Q}{\Omega}\left(A^2\mathrm{e}^{i2\Omega T_0} - \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right) - \right.$$
$$\left. i\frac{1}{2}\left(3x_0^2-1\right)\Phi\left(A^2\mathrm{e}^{i2\Omega T_0} - \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right) + x_0 C_2\right]. \tag{27}$$

The last step is to transform the system of first order ODEs (19) into the second order ODE

$$\frac{\partial^2 \xi_3}{\partial T_0^2} + \Omega^2 \xi_3 = -\frac{1}{2}x_0\left(x_0^2-1\right)\xi_1\xi_2 + \frac{1}{2}x_0\left(x_0^2-1\right)\frac{\partial \eta_1}{\partial T_2} - \frac{\partial}{\partial T_0}\left(\frac{\partial \xi_1}{\partial T_2}\right) -$$
$$\frac{1}{2}\left(3x_0^2-1\right)\left(\frac{\partial \eta_1}{\partial T_0}\xi_2 + \eta_1\frac{\partial \xi_2}{\partial T_0} + \frac{\partial \eta_2}{\partial T_0}\xi_1 + \eta_2\frac{\partial \xi_1}{\partial T_0}\right) -$$
$$\frac{3}{2}x_0\frac{\partial \eta_1}{\partial T_0}\xi_1^2 - 3x_0\eta_1\xi_1\frac{\partial \xi_1}{\partial T_0} - C_2\frac{\partial \xi_1}{\partial T_0} - x_0 D_3\frac{\mathrm{d}}{\mathrm{d}T_0}(\cos\omega T_0). \tag{28}$$

Now we put the previous solutions into this equation. The result is

$$\frac{\partial^2 \xi_3}{\partial T_0^2} + \Omega^2 \xi_3 = -\frac{1}{2}x_0\left(x_0^2-1\right)\left(A\mathrm{e}^{i\Omega T_0} + \bar{A}\mathrm{e}^{-i\Omega T_0}\right)\cdot$$
$$\left[-\frac{x_0}{x_0-1}A\bar{A} + \frac{Q}{3\Omega^2}(A^2\mathrm{e}^{i2\Omega T_0} + \bar{A}^2\mathrm{e}^{-i2\Omega T_0})\right] -$$
$$\frac{1}{2}x_0\left(x_0^2-1\right)\Phi i\left(A'\mathrm{e}^{i\Omega T_0} - \bar{A}'\mathrm{e}^{-i\Omega T_0}\right) - i\Omega\left(A'\mathrm{e}^{i\Omega T_0} - \bar{A}'\mathrm{e}^{-i\Omega T_0}\right) -$$
$$\frac{1}{2}\left(3x_0^2-1\right)\left\{\begin{array}{l} \Phi\Omega\left(A\mathrm{e}^{i\Omega T_0} + \bar{A}\mathrm{e}^{-i\Omega T_0}\right)\left[-\frac{x_0}{x_0-1}A\bar{A} + \frac{Q}{3\Omega^2}\left(A^2\mathrm{e}^{i2\Omega T_0} + \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right)\right] + \\ \Phi\left(A\mathrm{e}^{i\Omega T_0} - \bar{A}\mathrm{e}^{-i\Omega T_0}\right)\frac{2Q}{3\Omega}\left(A^2\mathrm{e}^{i2\Omega T_0} - \bar{A}^2\mathrm{e}^{-i2\Omega T_0}\right) + ..NST.. - \\ \frac{2}{x_0\left(x_0^2-1\right)}(NST + x_0 C_2)i\Omega\left(A\mathrm{e}^{i\Omega T_0} - \bar{A}\mathrm{e}^{-i\Omega T_0}\right) \end{array}\right\} -$$
$$..NST.. - iC_2\Omega\left(A\mathrm{e}^{i\Omega T_0} - \bar{A}\mathrm{e}^{-i\Omega T_0}\right) - x_0 D_3\frac{\mathrm{d}}{\mathrm{d}T_0}(\cos\omega T_0), \tag{29}$$

where NST means “non-secular terms“ and A' designates derivative of A according to T_2.

Further, we will distinguish two cases:

1. Autonomous case – $D_3 = 0$,
2. Non-autonomous case corresponding to soft resonant stimulation – $D_3 \neq 0$; $\omega = \Omega + \varepsilon^2 \nu_2$.

4.1. *Autonomous case* – $D_3 = 0$

The conditions for secular terms (periodical with the frequency Ω and therefore leading to the resonance) from (29) is

$$\mathrm{e}^{i\Omega T_0}: \quad \left[\begin{array}{l} \frac{1}{2}\frac{x_0^2\left(x_0^2-1\right)}{x_0-1}A^2\bar{A} - \frac{1}{2}x_0\left(x_0^2-1\right)\Phi i A' - \\ i\Omega A' + \frac{1}{2}\left(3x_0^2-1\right)\frac{x_0}{x_0-1}\Phi\Omega A^2\bar{A} + \\ \frac{3x_0^2-1}{\left(x_0^2-1\right)}\Omega i C_2 A - iC_2 A\Omega \end{array} \right] = 0, \tag{30}$$

$$\mathrm{e}^{-i\Omega T_0}: \quad \left[\begin{array}{l} \frac{1}{2}\frac{x_0^2\left(x_0^2-1\right)}{x_0-1}\bar{A}^2 A + \frac{1}{2}x_0\left(x_0^2-1\right)\Phi i \bar{A}' + \\ i\Omega \bar{A}' + \frac{1}{2}\left(3x_0^2-1\right)\frac{x_0}{x_0-1}\Phi\Omega \bar{A}^2 A - \\ \frac{3x_0^2-1}{\left(x_0^2-1\right)}\Omega i C_2 \bar{A} - iC_2 \bar{A}\Omega \end{array} \right] = 0. \tag{31}$$

It can be shown easily that both conditions are identical and in the next we will work with the first one. This condition can be simplified into the form

$$\alpha A^2\bar{A} - A'\beta i - A\gamma i = 0, \tag{32}$$

where

$$\alpha = \frac{x_0}{2}\left(4x_0^2 + x_0 - 1\right), \qquad \beta = 2\Omega, \qquad \gamma = -C_2\Omega\frac{2x_0^2}{x_0^2-1}, \tag{33}$$

where α, β and γ are constants. Now we insert

$$A = a\mathrm{e}^{i\phi}, \qquad \bar{A} = a\mathrm{e}^{-i\phi} \tag{34}$$

and comparing the real and imaginary parts after reducing the equation by $\mathrm{e}^{-i\phi}$ and $\mathrm{e}^{i\phi}$ (they have unit absolute value) we obtain the following equations

$$\begin{aligned} \mathrm{Re}: &\quad \phi'\beta + \alpha a^2 = 0 \Rightarrow \phi' = -\tfrac{\alpha}{\beta}a^2, \\ \mathrm{Im}: &\quad a'\beta + \gamma a = 0 \Rightarrow a' = -\tfrac{\gamma}{\beta}a. \end{aligned} \tag{35}$$

Solution for a is

$$a = \mathrm{const.}\,\mathrm{e}^{-\frac{\gamma}{\beta}T_2}. \tag{36}$$

Only for $\gamma = 0 \Rightarrow C_2 = 0$ it exists $a = \mathrm{const.}$ corresponding to the periodical motion. For $C_2 \neq 0$ depending on its sign we can observe either convergence or divergence of the solution. This result fully corresponds with previous analysis [5] for $C = 0$ where the Hopf's degenerated bifurcation exists. On Figs. 3, 4 and 5 the results of the numerical experiments are shown. The limit cycle for $C = 0$ is presented on Fig. 3 and the convergence to the point with the coordinate $x_0 = 1 + \Theta$ for $C < 0$ and divergence from this point and convergence to the point with the coordinate $x_0 = 1 + \Theta$ for $C > 0$ on Figs. 4 and 5.

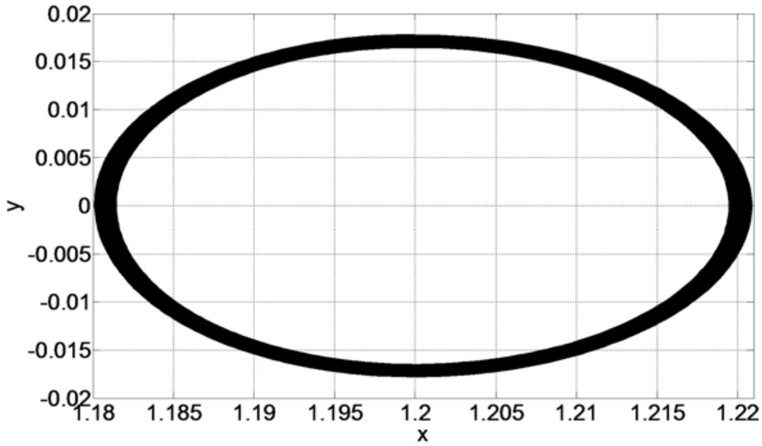

Fig. 3. Phase portrait for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = 0$, $D = 0$

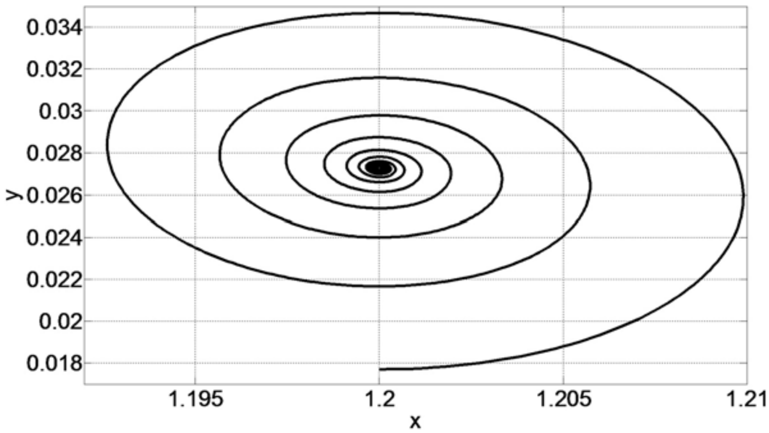

Fig. 4. Phase portrait for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.006$, $D = 0$

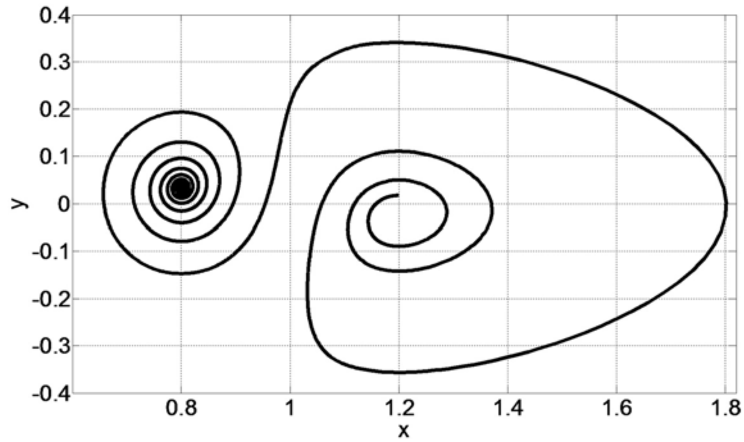

Fig. 5. Phase portrait for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = 0.006$, $D = 0$

4.2. Non-autonomous case – soft resonant stimulation – $D_3 \neq 0$*;* $\omega = \Omega + \varepsilon^2 \nu_2$

Here, after setting we obtain

$$\omega T_0 = \Omega T_0 + \varepsilon^2 \nu_2 T_0 = \Omega T_0 + \nu_2 T_2. \tag{37}$$

In this case the Eq. (29) will have the form

$$\begin{aligned}
&\frac{\partial^2 \xi_3}{\partial T_0^2} + \Omega^2 \xi_3 = -\frac{1}{2} x_0 \left(x_0^2 - 1\right) \cdot \\
&\left(A e^{i\Omega T_0} + \bar{A} e^{-i\Omega T_0}\right) \left[-\frac{x_0}{x_0 - 1} A\bar{A} + \frac{Q}{3\Omega^2} \left(A^2 e^{i2\Omega T_0} + \bar{A}^2 e^{-i2\Omega T_0}\right)\right] - \\
&\frac{1}{2} x_0 \left(x_0^2 - 1\right) \Phi i \left(A' e^{i\Omega T_0} - \bar{A}' e^{-i\Omega T_0}\right) - i\Omega \left(A' e^{i\Omega T_0} - \bar{A}' e^{-i\Omega T_0}\right) - \\
&\frac{1}{2}\left(3x_0^2 - 1\right) \left\{ \begin{array}{l} \Phi\Omega \left(A e^{i\Omega T_0} + \bar{A} e^{-i\Omega T_0}\right) \left[-\frac{x_0}{x_0-1} A\bar{A} + \frac{Q}{3\Omega^2}\left(A^2 e^{i2\Omega T_0} + \bar{A}^2 e^{-i2\Omega T_0}\right)\right] + \\ \Phi \left(A e^{i\Omega T_0} - \bar{A} e^{-i\Omega T_0}\right) \frac{2Q}{3\Omega} \left(A^2 e^{i2\Omega T_0} - \bar{A}^2 e^{-i2\Omega T_0}\right) + ..NST.. - \\ \frac{2}{x_0\left(x_0^2-1\right)} \left(NST + x_0 C_2\right) i\Omega \left(A e^{i\Omega T_0} - \bar{A} e^{-i\Omega T_0}\right) \end{array} \right\} - \\
&..NST.. - iC_2\Omega \left(A e^{i\Omega T_0} - \bar{A} e^{-i\Omega T_0}\right) - \frac{1}{2} x_0 D_3 \Omega i \left(e^{i(\Omega T_0 + \nu_2 T_2)} - e^{-i(\Omega T_0 + \nu_2 T_2)}\right). \qquad (38)
\end{aligned}$$

The condition for the secular terms (30) will have the form

$$e^{i\Omega T_0} : \quad \left[\begin{array}{l} \frac{1}{2} \frac{x_0^2\left(x_0^2-1\right)}{x_0-1} A^2 \bar{A} - \frac{1}{2} x_0 \left(x_0^2 - 1\right) \Phi i A' - \\ i\Omega A' + \frac{1}{2}\left(3x_0^2 - 1\right) \frac{x_0}{x_0-1} \Phi\Omega A^2 \bar{A} + \\ \frac{3x_0^2-1}{\left(x_0^2-1\right)} \Omega i C_2 A - iC_2 A\Omega - \frac{1}{2} i x_0 D_3 \Omega e^{i\nu_2 T_2} \end{array} \right] = 0 \qquad (39)$$

and further

$$\alpha A^2 \bar{A} - A'\beta i - A\gamma i - i\delta e^{i\nu_2 T_2} = 0, \qquad (40)$$

where

$$\delta = \frac{1}{2} x_0 D_3 \Omega. \qquad (41)$$

After inserting from (34) to (40) we obtain

$$\begin{array}{ll} \mathrm{Re}: & a\phi'\beta + \alpha a^3 + \delta \sin(\nu_2 T_2 - \phi) = 0, \\ \mathrm{Im}: & a'\beta + \gamma a + \delta \cos(\nu_2 T_2 - \phi) = 0. \end{array} \qquad (42)$$

To obtain the autonomous system we provide the substitution

$$\nu_2 T_2 - \phi = \psi(T_2). \qquad (43)$$

And then

$$\begin{aligned} \psi' &= \nu_2 + \tfrac{\alpha}{\beta} a^2 + \tfrac{\delta}{a\beta} \sin\psi, \\ a' &= -\tfrac{\gamma}{\beta} a - \tfrac{\delta}{\beta} \cos\psi. \end{aligned} \qquad (44)$$

Let the stationary solution of this dynamical system be $\tilde{a}, \tilde{\psi}$. If we eliminate $\tilde{\psi}$ from the corresponding equations, we obtain the following frequency response equation

$$\tilde{a}^2 \left[\left(\beta\nu_2 + \alpha\tilde{a}^2\right)^2 + \gamma^2\right] = \delta^2. \qquad (45)$$

Equations in variations of (44) are

$$\begin{aligned} \eta_a' &= -\frac{\gamma}{\beta} \eta_a - \tilde{a} \left(\nu_2 + \frac{\alpha}{\beta} \tilde{a}^2\right) \eta_\psi, \\ \eta_\psi' &= \left[2\frac{\alpha}{\beta} \tilde{a} + \frac{1}{\tilde{a}} \left(\nu_2 + \frac{\alpha}{\beta} \tilde{a}^2\right)\right] \eta_a - \frac{\gamma}{\beta} \eta_\psi. \end{aligned} \qquad (46)$$

The eigenvalues are

$$\lambda_{1,2} = -\frac{\gamma}{\beta} \pm \sqrt{\left(\frac{\gamma}{\beta}\right)^2 - \left(\nu_2 + \frac{\alpha}{\beta}\tilde{a}^2\right)\left(\nu_2 + 3\frac{\alpha}{\beta}\tilde{a}^2\right)}. \tag{47}$$

For the stability reason it is necessary that the square root is either zero or pure imaginary one

$$-\left(\nu_2 + \frac{\alpha}{\beta}\tilde{a}^2\right)\left(\nu_2 + 3\frac{\alpha}{\beta}\tilde{a}^2\right) \leq 0. \tag{48}$$

If this condition is fulfilled, the system is asymptotically stable for $\frac{-\gamma}{\beta} < 0$. Putting from (33) into this condition, we obtain the form

$$\frac{1}{2}C_2\frac{x_0^2}{x_0^2 - 1} < 0. \tag{49}$$

For $x_0 = 1 + \Theta$, the system is stable if $C_2 < 0$ and for $x_0 = 1 - \Theta$ it is stable if $C_2 > 0$. The first approximation of the solution is

$$\begin{aligned} x &= x_0 + \varepsilon 2\tilde{a}\cos\Omega t, \\ y &= \varepsilon\frac{4\Omega}{x_0\left(x_0^2 - 1\right)}\tilde{a}\sin\Omega t. \end{aligned} \tag{50}$$

The further approximations contain the multiple of Ω and the frequency of stimulation ω.

Again, now we will show some numerical examples. On Figs. 6 and 7, there is shown the solution of (46) together with the area of instability according (49) for different values of D.

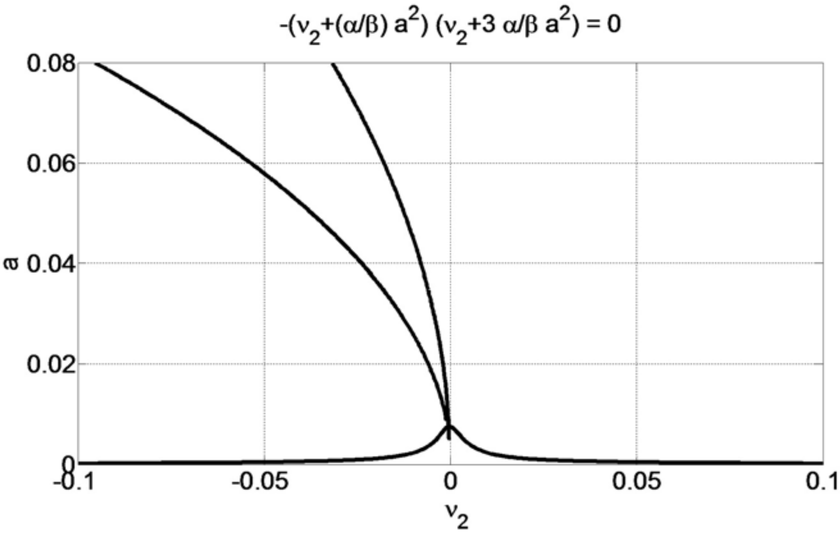

Fig. 6. Amplitude vs. frequency for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.0177$, $C = -0.004$, $D = 0.00006$

In Fig. 8, the solution of the original dynamical system for the corresponding data is stated. The correspondence for chosen small values of C and D is very good. The cause is the approximation (13).

Poincare mappings on Figs. 9 and 11 correspond to the quasi-periodical motion with two frequencies Ω and $\omega = \Omega + \varepsilon^2\nu_2$. These frequencies would occur in the second and further approximations that have not been solved in this contribution.

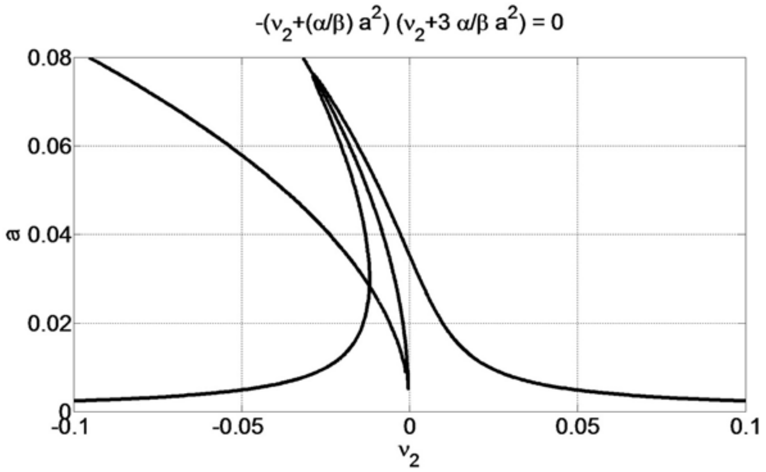

Fig. 7. Area of instability for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.004$, $D = 0.000\,6$

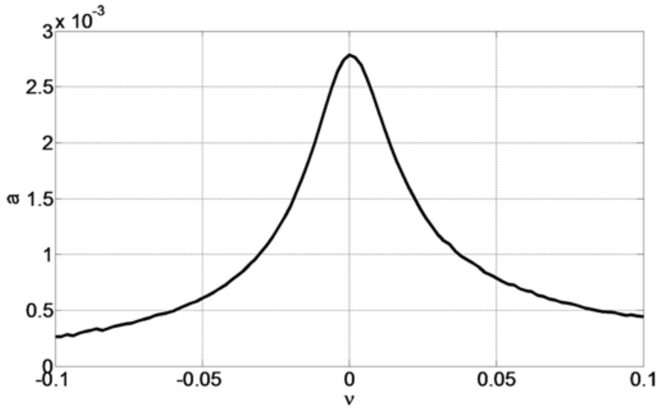

Fig. 8. Amplitude vs. frequency for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.004$, $D = 0.000\,06$

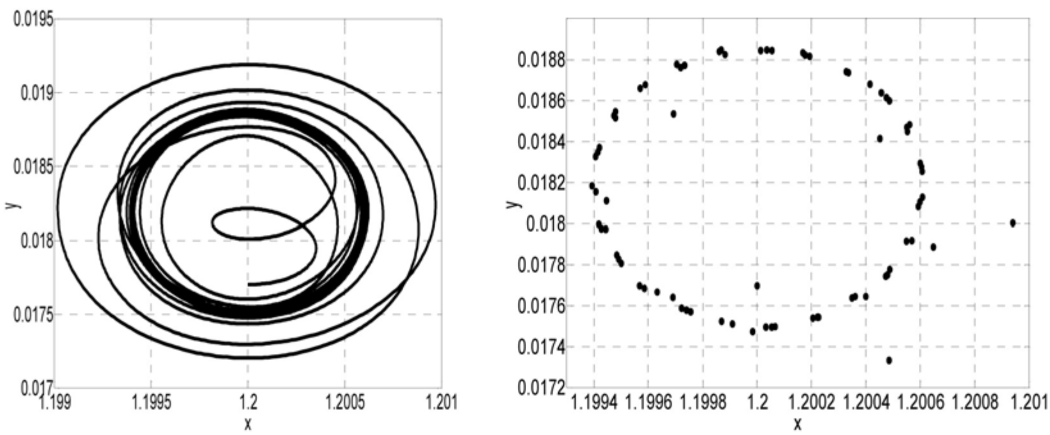

Fig. 9. Phase portrait and Poincare mapping with the period $T = \frac{2\pi}{\Omega}$ for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.004$, $D = 0.000\,06$, $\nu_2 = -0.05$

5. Conclusion

The paper shows the possibility of MSM usage for the analysis of non-autonomous dynamical system. This system is transformed into the autonomous one and the basic dynamical properties are examined using the common method. From given examples, it is obvious that the crucial point is the choice of the approximation (13) which influences the range where the approximated

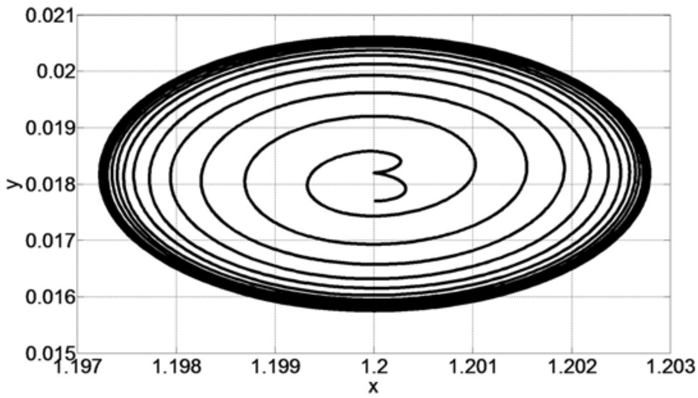

Fig. 10. Phase portrait for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.004$, $D = 0.000\,06$, $\nu_2 = 0$

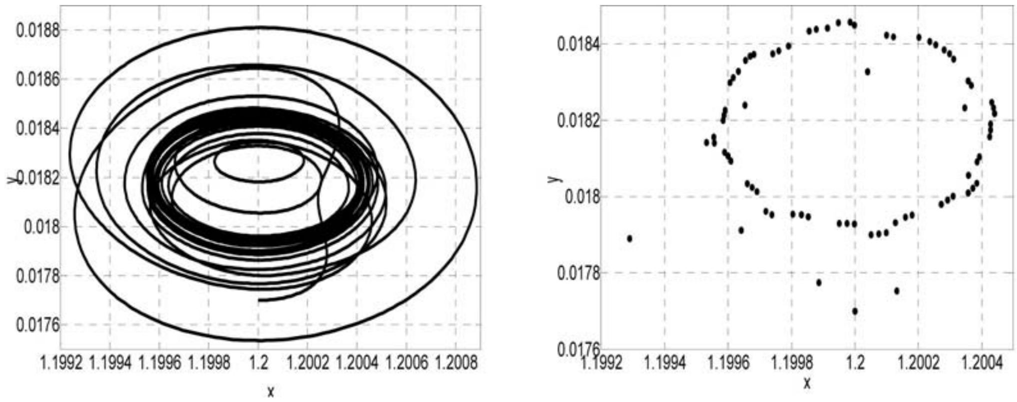

Fig. 11. Phase portrait and Poincare mapping with the period $T = \frac{2\pi}{\Omega}$ for $r = 0.02$, $x_0 = 1 + \Theta$, $y_0 = 0.017\,7$, $C = -0.004$, $D = 0.000\,06$, $\nu_2 = 0.1$

solution is valid. That is the reason why the transition to chaos slightly evident on Figs. 9 and 11 is not seen from the MSM analysis. The future analysis of this dependence effects seems to be necessary using the second order approximation.

Acknowledgements

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