Opuscula Mathematica • Vol. 28 • No. 4 • 2008

Józef Myjak

ANDRZEJ LASOTA'S SELECTED RESULTS

Abstract. In this article we recall Andrzej Lasota's selected results which either indicated new directions of research, or layed the foundations for new approaches, or solved interesting problems. The area of mathematical interests of Professor Andrzej Lasota was very large: ordinary differential equations, partial differential equations, dynamical systems, multifunctions, differential inclusions, functional differential equations, equations with retarded arguments, ergodic theory, invariant measures, chaos, stochastic differential equations, control theory, fixed point theory, theory of Markov operators, theory of fractals, theory of dimensions, biomathematics. In all these branches he obtained original and essential results.

1. INTRODUCTION

The area of mathematical interests of Prof. Andrzej Lasota was very large: ordinary differential equations, partial differential equations, dynamical systems, multifunctions, differential inclusions, functional differential equations, equations with retarded arguments, ergodic theory, invariant measures, chaos, stochastic differential equations, control theory, fixed point theory, theory of Markov operators, theory of fractals, theory of dimensions, biomathematics. It is surprising that in all these branches he obtained original and essential results. Moreover, his papers are very well written, inventive, smart and simple. Here we recall some selected results which either indicated new directions of research, or layed the foundations for new approaches, or solved interesting problems. We also recall some ideas of the proofs. For a deeper discussion of Lasota's contributions to and influence on the ergodic theory of stochastic operators, we refer the reader to article by W. Bartoszek [2], for an account of his contribution to engineering problems to article by P. Rusek [33] and for very important contributions to biomathematics to the nice and very personal article by Michael C. Mackey [29].

2. BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

Consider a system of ordinary differential equations

$$x' = f(t, x) \tag{1}$$

and a boundary value condition

$$Lx = r, \tag{2}$$

where $x(t) \in R^m$, $f : (a, b) \times R^m \to R^m$ is a continuous function and L is a linear operator from the space of all differential functions $C_1(a, b)$ into R^m. By a solution of (1), (2) we mean a continuously differentiable function on (a, b) satisfying (1) and (2) for every $t \in (a, b)$.

If for $t \in (a, b)$, $f(t, \cdot\,)$ is a linear transformation, then it follows from Fredholm's alternative that uniqueness of the solution of the boundary value problem implies the existence of the solution of this problem. In general, it is not true in the nonlinear case. Since uniqueness is usually easier to prove than existence, the question when the uniqueness implies the existence is of large importance and has been studied by several authors.

There are in general two approaches to this question. The first approach is to consider problem (1) with f from some family of functions and with fixed boundary condition (2). We assume that for any f of this family, the boundary value problem has at most one solution. Then, under appropriate conditions, we may show that for every f from this family the problem has a solution.

In the second approach the function f is fixed and the boundary value operator L belongs to a certain family. We assume that for any L of this family the problem has at most one solution. Then, under suitable condition, we show that for every L from this family the problem has a solution.

By using the Brouwer open mapping theorem, domain invariance theorem or some related technique, A. Lasota established several generalizations of Fredholm alternative. We recall some examples of such results (see [10, 10, 12–24]).

Theorem 1. *Let $f : (a, b) \times R^m \to R^m$ be a continuous function. Suppose that for every $(t_0, u) \in (a, b) \times R^m$ there exists the unique solution x of equation* (1) *satisfying initial condition $x(t_0) = u$. Let $\mathcal{U}$ be an open (in the norm topology) subset of the space of all bounded linear operators from the space $C^1(a, b)$ into R^m. If for every $L \in \mathcal{U}$ and every $r \in R^m$ problem* (1), (2) *has at most one solution, then for every $L \in \mathcal{U}$ and every $r \in R^m$ problem* (1), (2) *has exactly one solution.*

Proof. For $u \in R^m$, let $\varphi(\,\cdot\,; u)$ denote the solution of problem (1) such that $\varphi(t_0; u) = u$. It is sufficient to show that for every $L \in \mathcal{U}$ and $r \in R^m$ there exists $u \in R^m$ such that $L\varphi(\,\cdot\,; u) = r$.

Fix $L \in \mathcal{U}$ and define $\chi : R^m \to R^m$ by $\chi(u) = L\varphi(\,\cdot\,; u)$. Clearly, χ is a continuos injection. It is sufficient to show that $\chi(R^m) = R^m$. Suppose that $\chi(R^m) \neq R^m$. By Brouwer's open mapping theorem, $\chi(R^m)$ is open. Consequently, there exist a point $u \in R^m \setminus \chi(R^m)$ and a sequence $\{u_k\} \subset R^m$ such that $\chi(u_k) \to u$. Since the sequence

$\{u_k\}$ cannot be convergent, then, passing to the subsequence if necessary, we may assume that there is an $\varepsilon > 0$ such that for every $k \in \mathbb{N}$ there is $j = j(k)$ such that $|u_{k+j} - u_k| \geq \varepsilon$. Now put $\psi_k = \varphi(\,\cdot\,; u_{k+j}) - \varphi(\,\cdot\,; u_k)$ and define a linear operator $L_k : C^1 \to R^m$ such that $L_k\psi_k = -L\psi_k$, $\|L_k\| = |L\psi_k|/\|\psi_k\|_1$. (Here $\|x\|_1 = \sup\big\{|x(t)| + |x'(t)| : t \in (a,b)\big\}$). It is routine to see that $\|L_k\| \to 0$ as $k \to \infty$. Thus for k sufficiently large, $L_k + L \in \mathcal{U}$. Consequently, $(L_k + L)\big(\varphi(\,\cdot\,, \psi_k) - \varphi(\,\cdot\,, u)\big) = 0$, which means that equation (1) with the boundary value problem $(L_k + L)x = 0$ has two solutions: $\varphi(\,\cdot\,, u_{k+j})$ and $\varphi(\,\cdot\,, u_k)$. This contradiction completes the proof. □

In a similar way, one can prove the following

Theorem 2. *Let $f : (a,b) \times R^m \to R^m$ be a continuous function. Suppose that for every $(t_0, u) \in (a,b) \times R^m$ there exists the unique solution x of equation* (1) *satisfying the initial condition $x(t_0) = u$. Let $\mathcal{A}$ be an open subset of the space of $m\times m$-matrices. Assume that for every $A \in \mathcal{A}$ and $r \in R^m$ there exists at most one solution of equation* (1) *satisfying*

$$\sum_{j=1}^{m} a_{ij} x_j(t_i) = r_i, \qquad i = 1, \ldots, m. \tag{3}$$

Then problem (1), (3) *has exactly one solution.*

Now consider the iteration problem

$$x^{(n)} = f(t, x, x', \ldots, x^{(n-1)}), \tag{4}$$

$$x(t_i) = r_i, \qquad i = 1, \ldots, n, \tag{5}$$

where $t_1, \ldots, t_n \in (a,b)$ and $r_1, \ldots, r_n \in \mathbb{R}$.

Theorem 3. *Let $f : (a,b) \times R^n \to R^n$ satisfies the inequality*

$$|f(t, x_0, x_1, \ldots, x_{n-1})| \leq M + \sum_{i=0}^{n-1} L_i |x_i|, \qquad M \geq 0, \quad L_i > 0.$$

Suppose that for every sequence $t_1, \ldots, t_n \in (a,b)$ the function $x \equiv 0$ is the unique solution of the inequality

$$|x^{(n)}(t)| \leq \sum_{i=0}^{n-1} L_i |x^{(i)}(t)|$$

satisfying the condition

$$x(t_i) = 0, \qquad i = 1, \ldots, n.$$

Then for every sequence $t_1, \ldots, t_n \in (a,b)$ and $r_1, \ldots, r_n \in \mathbb{R}$, problem (4), (5) *has at least one solution.*

To give the idea of this result, we show the proof of Theorem 3 in the linear case, i.e., when the equation has the form

$$x^{(n)} = \sum_{i=1}^{n-1} p_i(t) x^{(i)} + q(t),$$

where $p_0, \ldots, p_{n-1}, q$ are continuous functions on (a, b) and $|p_i(t)| \leq L_i$, $t \in (a, b)$, $i = 1, \ldots, n-1$. Obviously, the solution of the last equation is of the form

$$x(t) = \sum_{i=1}^{n} C_i u_i(t) + w(t),$$

where $u_1, \ldots, u_n$ is the fundamental system of solutions of the associated homogeneous equation and w is a particular solution of the nonhomogeneous equation. Since the Jacobian of such solutions is different from zero, for every sequence $r_1, \ldots, r_n \in \mathbb{R}$ the system

$$r_i = \sum_{i=1}^{n} C_i u_i(t_i) + w(t_i), \qquad i = 1, \ldots, n$$

with respect to $C_1, \ldots, C_n$ has a unique solution. This completes the proof.

Theorem 4. *Suppose that a function $f : (a, b) \times R^n \to \mathbb{R}$ satisfies the condition*

$$|f(t, x_0, x_1, \ldots, x_{n-1})| \leq M + \sum_{i=0}^{n-1} P_i(t)|x_i|,$$

where $M \geq 0$ and $P_0, \ldots, P_{n-1}$ are continuous functions on (a, b).

Assume that there exists an $\varepsilon > 0$ such that for an arbitrary sequence of functions $p_0, \ldots, p_n$ defined on (a, b) and satisfying the conditions

$$|p_i(t)| \leq P_i(t) + \varepsilon, \quad t \in (a, b), \quad i = 0, \ldots, n-1,$$

the problem

$$x^{(n)} = p_0(t)x + \ldots + p_{n-1}(t)x^{n-1}, \qquad x(t_1) = \ldots = x(t_n) = 0,$$

where $a < t_1 < \ldots < t_n < b$, has the trivial solution only. Then problem (4), (5) admits at least one solution.

Theorem 5. *Suppose that for every $(t_0, x_0) \in [a, b] \times R^m$ there exists exactly one solution of equation (1) defined on $[a, b]$ such that $x(t_0) = x_0$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are subsets of the space of all $m \times m$- matrices, endowed with the supremum norm, and such that at least one of these sets is open. Assume that for each $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $r \in R^m$ there is at most one solution of equation (1) satysfying the boundary condition*

$$Ax(a) + Bx(b) = r. \tag{6}$$

Then for each $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $r \in R^m$, problem (1), (6) has exactly one solution.

3. METHOD OF MULTIVALUED FUNCTIONS

Let E be a Banach space and $c(E)$ the family of all convex nonempty subsets of E. For $u \in E$ and $A \subset E$, let $\rho(u, A)$ denote the distance from u to A. A map H from

E into $c(E)$ is called *homogeneous* if $H(\lambda u) = \lambda H(u)$ for $u \in E$, $\lambda \in \mathbb{R}$; *closed graph* if the set graph $F = \{(u,v) : v \in H(u), u \in E\}$ is closed in $E \times E$; *compact* if the set $F(B)$ is relativly compact for every bounded subset B of E and *completely continuous* if it is continous and compact.

Let $I \subset \mathbb{R}$ be an interval. As usual, $C(I,E)$ denotes the space of all continuous functions from I into E, endowed with the supremum norm.

Recall also that a multifunction $F : I \to E$ is called *measurable*, if for every open subset U of E, the set $F^{-}(U) = \{t \in I : F(t) \cap U \neq \emptyset\}$ is Borel measurable.

Theorem 6. *Let a map H from E into $c(E)$ be homogeneous and completely continous and let a function $h : E \to E$ be completely continuous. Suppose that*

$$\lim_{\|u\| \to \infty} \frac{\rho(h(u), H(u))}{\|u\|} = 0. \tag{7}$$

Then, if $u \in H(u)$ holds for $u = 0$ only, then h admits at least one fixed point.

Proof. Observe that there exists $K > 0$ such that

$$u + v \in H(u) \quad \text{implies} \quad \|u\| \leq K\|v\|.$$

Indeed, if the opposite is true, there are $\{u_n\}$, $\{v_n\}$ such that $u_n + v_n \in H(u_n)$ and $\|u_n\| > n\|v_n\|$. Put

$$w_n = u_n/\|u_n\|.$$

Obviously

$$w_n + v_n/\|u_n\| \in H(w_n), \quad \|w_n\| = 1, \quad \|v_n\|/\|u_n\| < 1/n.$$

Since H is compact and closed graph, without loss of generality, we may assume that $w_n \to w_* \in H(w_*)$, $\|w_*\| = 1$. A contradiction.

From (7) it follows that there exists $r > 0$ such that

$$\sup_{\|u\|=1} \rho(h(u), H(u)) < r/K.$$

We claim that h admits a fixed point in the ball $B(0,r)$. Indeed, by virtue of the antipodal theorem it suffices to show that

$$u - h(u) \neq \lambda(u - h(-u)) \qquad \text{for} \quad \lambda \in [0.1], \quad \|u\| = r.$$

On the contrary, suppose that that are u_0, $\|u_0\| = r$, and $\lambda_0 \in [0,1]$ such that

$$u_0 - h(u_0) = \lambda_0\big(- u_0 - h(-u_0)\big). \tag{8}$$

Since H is compact valued, there are $v_0 \in H(u_0)$ and $w_0 \in H(-u_0)$ such that

$$\rho(h(u_0), H(u_0)) = \|v_0 - h(u_0)\|, \qquad \rho(h(-u_0), H(-u_0)) = \|w_0 - h(-u_0)\|.$$

Now we can rewrite equality (8) in the form

$$u_0 + \frac{1}{1+\lambda_0}\big(v_0 - h(u_0)\big) - \frac{\lambda_0}{1+\lambda_0}\big(w_0 - h(-u_0)\big) = \frac{1}{1+\lambda_0}v_0 - \frac{\lambda_0}{1+\lambda_0}w_0. \tag{9}$$

Since H is homogeneous and convex valued,

$$\frac{1}{1+\lambda_0}v_0 - \frac{\lambda_0}{1+\lambda_0}w_0 \in H(u_0).$$

By (9) and the definition of the constant K, there is

$$\begin{aligned}\|u_0\| &\le \frac{K}{1+\lambda_0}\|\big(v_0 - h(u_0)\big) - \lambda_0\big(w_0 - h(-u_0)\big)\| \le \\ &\le \frac{K}{1+\lambda_0}\big[\|(v_0 - h(u_0)\| + \lambda_0\|w_0 - h(-u_0)\|\big] < \frac{K}{1+\lambda_0}\big(\frac{r}{K} + \lambda_0\frac{r}{K}\big) = r,\end{aligned}$$

a contradiction, since $\|u_0\| = r$. □

Theorem 7. *Let U be a neighbourhood of zero in the space E and let $H : U \to E$ be a completely continuous multifunction. Suppose that*

$$u \in H(u), \quad u \in U, \qquad \text{implies} \quad u = 0. \tag{10}$$

Then every continuous mapping $h : E \to E$ satisfying the condition

$$h(u) - h(v) \in H(u - v) \qquad \text{for} \quad u - v \in U,$$

has exactly one fixed point.

Proof. Put $U(u) = U + u$. Choose $\varepsilon > 0$ such that $B(u, 2\varepsilon) \subset U(u)$. From (10) it follows that $T = I - h$ is a one-to-one mapping on $U(u)$ for arbitrary $u \in E$. Let $S(u, \varepsilon)$ denote the boundary of the ball $B(u, \varepsilon)$. It is easy to see that there is a $\delta > 0$ such that

$$\|T(v) - T(u)\| \ge \delta \qquad \text{for every} \quad v \in S(u, \varepsilon).$$

From (10) it follows that h is completely continuous. Thus, by the last inequality, for arbitrary $u \in E$ we have

$$\big\{v : \|v - T(u)\| \le \delta\big\} \subset T\big(B(u, \varepsilon)\big).$$

This means that $T(E)$ is open in E. To complete the proof, it suffices o show that $T(E) = E$. □

Remark 1. *Observe that the continuity of the mapping h can be dropped if we assume that $H(0) = \{0\}$. Indeed, using the compactness of H, one can deduce that h must be continuous.*

Theorems 6 and 7 became useful and very popular tools in the existence results for boundary value problem for differential equations. The idea of such applications is described in the next theorem.

Theorem 8. *Let F be a multifunction defined on $I \times R^m$ with closed convex values and such that $F(t,\cdot\,)$ is continuous (with respect to the Hausdorff distance) for every $t \in I$ and $F(\,\cdot, x)$ is measurable for every $x \in R^m$. Moreover, assume that F is homogeneous with respect to the second variable and*

$$|F(t,x)| \le \varphi(t), \qquad (t,x) \in I \times R^m,$$

where φ is an integrable function on I.

Let $f : I \times R^m \to R^m$ be a Carathéodory function such that

$$\lim_{n\to\infty} \frac{1}{n} \int_I \sup_{|p|\le n} \rho\big(f(t,p), F(t,p)\big)dt = 0.$$

Suppose that L is a continuous and homogenous mapping from $C(I)$ into R^m.

If the problem

$$x' \in F(t,x), \qquad Lx = 0$$

admits only the trivial solution $x = 0$ in I, then for any $r \in R^m$ problem (1), (2) *amits at least one solution defined on I.*

Sketch of a proof. Let $E = C(I, R^m) \times R^m$. Consider the function h and the multifunction H defined on E by the formulas:

$$h(x,p) = \Big(\int_{t_0}^t f(s, x(s))ds,\ Lx + p - r \Big)$$

and

$$H(x,p) = \Big\{ \Big(\int_{t_0}^t g(s)ds + p,\ Lx + p \Big)\ :\ g(s) \in F(s, x(s)),\ g \text{ measurable} \Big\}.$$

To complete the proof, it suffices to verify that h and H satisfy the assumptions of Theorem 6.

To describe other simple applications of the above results, consider the boundary value problem for the differential inequality

$$|x'| \le p(t)|x|, \qquad Lx = 0, \tag{11}$$

where p is an integrable function and $L : C([0,h], R^m) \to R^m$ is a linear operator such that $x = 0$ is the unique solution of the problem

$$x' = 0, \qquad Lx = 0.$$

It can be proved that there exists a constant M such that if

$$\int_0^h p(t)dt < M, \tag{12}$$

then problem (11) has the trivial solution $x = 0$ only. □

Denote by M_L the best constant for which the above assertion holds. It is well known that for the initial value problem ($Lx = x(t_0)$), the constant $M_L = \infty$, for the Nicoletti problem ($Lx = (x(t_)), \ldots, x(t_n)))$, $M_L = \pi/2$ and for the Floquet problem ($Lx = x(0) + \lambda x(h)$), $M_L = \sqrt{\pi^2 + \ln^2 \lambda}$ if $\lambda > 0$, and $M_L = |\ln|\lambda||$ if $\lambda < 0$ (see [5, 16]). Below we recall a simple example of such a type of result.

Theorem 9. *Consider equation* (1) *with boundary value problem*

$$x(0) + \lambda x(h) = r \qquad (\lambda > 0), \tag{13}$$

where $f : [0, h] \to R^m$ *is a Carathéodory function and* $r \in R^m$. *Suppose that*

$$|f(t, x)| \leq p(t)|x| + q(t), \qquad (t, x) \in [0, h] \times R^m,$$

where p *and* q *are integrable functions on* $[0, h]$. *Assume that condition* (12) *holds with* $M_L = \sqrt{\pi^2 + \ln^2 \lambda}$. *Then problem* (1), (13) *admits at least one solution.*

If in addition

$$|f(t, x) - f(t, y)| \leq p(t)|x - y|,$$

then problem (1), (13) *has exactly one solution.*

Finally observe that presented approch of Lasota works by dint of the following general fact (see [3]).

Theorem 10. *Let* E *be a Banach space. Assume that a function* $h : E \to E$, *a multifunction* $H : E \to c(E)$, *a subset* D *of* E *and a number* $\alpha \in [0, 1]$ *are such that:*

(i) $H(\lambda u) = \lambda H(u)$ *for* $u \in E$, $\lambda \in \mathbb{R}$;
(ii) $H(u + v) \subset \overline{H(u) + H(v)}$ *for* $u, v \in E$;
(iii) $H(D) \subset \alpha D$, $\overline{B(0, 1)} \subset D$, $H(D)$ *is bounded;*
(iv) $h(u) - h(v) \in H(u - v)$, *for* $u, v \in E$.

Then there exists a norm $\|\cdot\|_H$ *equivalent to the norm* $\|\cdot\|$ *and such that* h *is a strict contraction with respect to* $\|\cdot\|_H$.

4. PERIODIC SOLUTIONS

A similar relationship between the existence and uniqueness of periodic solutions for ordinary differential equations was investigated by A. Lasota and Z. Opial in [19]. The authors established a general theorem which permits to obtain the existence of solutions for equations of various types. The main idea consits in comparing a given nonlinear system of differential equations with suitably chosen linear homogeneous system of equation. This approach has been repeatedly used by several people and this paper has frequently been cited.

To formulate this results we need some further notation. Let $\mathcal{L}_\omega$ denote the space of all ω-periodic functions from $\mathbb{R}$ into $\mathbb{R}$, integrable on $[0, \omega]$. We say that a sequence $\{a_n\} \subset \mathcal{L}_\omega$ converges weakly to a function $a \in \mathcal{L}_\omega$ if

$$\int_0^t a_n(s)ds \to \int_0^t a(s)ds$$

uniformly in $[0, \omega]$.

Let $\mathfrak{A}$ denote the set of all $m \times m$-matrices $A = (a_{ij})$, where $a_{ij} \in \mathcal{L}_\omega$, $i, j = 1, \ldots, m$. We say that a sequence of matrices $\{A_n\}$, $A_n = (a^n_{ij})$, converges weakly to a matrix $A = (a_{ij})$, if for every $i, j \in \{1, \ldots, m\}$ the sequence $\{a^n_{ij}\}$ converges weakly to a_{ij}.

Consider now the system of equation

$$x'_i = \sum_{j=1}^{m} a_{ij}(t, x_1, \ldots, x_m)\, x_j + b_i(t, x_1, \ldots, x_m), \quad i = 1, \ldots, m, \tag{14}$$

where functions a_{ij}, $b_i : R^{m+1} \to R^m$ satisfy the Carathéodory conditions, i.e. they are measurable with respect to t and continuous with recpect to $x_1, \ldots, x_m$.

Parallelly consider the homogeneous system

$$x'_i = \sum_{j=1}^{m} \alpha_{ij}(t)\, x_j, \quad i = 1, \ldots, m, \tag{15}$$

where α_{ij}, $i, j = 1, \ldots, m$, are integrable functions on $\mathbb{R}$.

Theorem 11. *Let $\mathcal{A}$ be a bounded weakly closed subset of $\mathfrak{A}$. Suppose that for every $A \in \mathcal{A}$, $A = (\alpha_{ij})$, the only ω-periodic solution of problem* (15) *is the trivial (null) ω-periodic solution.*

Assume that functions a_{ij}, $b_i : R^{m+1} \to R^m$ are ω-periodic with respect to the first variable, satisfy the Carathéodory conditions, and for every ω-periodic function $x : \mathbb{R} \to R^m$ the matrix $(\alpha_{ij}(t, x(t)) \in \mathcal{A}$. Moreover, assume that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{\sum_{i=1}^m |x_i| \le n} \sum_{i=1}^{m} |b_i(t, x_1, \ldots, x_m)| dt = 0.$$

Then problem (14) *admits at least one ω-periodic solution.*

Sketch of a proof. We begin with proving that for every matrix $A \in \mathcal{A}$, $A = (\alpha_{ij})$, and for every sequence $b_1, \ldots, b_m \in \mathcal{L}_\omega$, the system

$$x'_i = \sum_{j=1}^{m} \alpha_{ij}(t)\, x_j + b_i(t), \quad i = 1, \ldots, m, \tag{16}$$

has only one solution satisfying conditions $x_i(0) = x_i(\omega)$, $i = 1, \ldots, m$. Moreover, such solution satisfies the inequality

$$|x(t)| \le \lambda \int_0^\omega \sum_{i=1}^{m} |b_i(s)| ds, \qquad t \in [0, \omega],$$

where λ is independent of the choice of matrix $A \in \mathcal{A}$ and functions $b_1, \ldots, b_m$.

Now consider a map $T : C([0, \omega]) \to C([0, \omega])$ (here $C([0, \omega])$ stands for the space of continuous functions from $[0, \omega]$ to R^m), which maps a function $z \in C([0, \omega])$ to the (unique) solution of the problem

$$y'_i = \sum_{j=1}^{m} a_{ij}(t, z(t))\, y_j + b_i(t, z(t)), \quad y_i(0) = y_i(\omega), \quad i = 1, \ldots, m.$$

To complete the proof, it suffices to verify that all hypotheses of Schauder Theorem are satisfied. Clearly, the fixed point of T is the required solution. □

Now we recall some application of Theorem 11. Consider the equation of the second order

$$x'' + P(t, x, x')x = Q(t, x, x') \tag{17}$$

and the associated homgeneous equation

$$x'' + p(t)x = 0 \tag{18}$$

with the boundary condition

$$x(\omega) = x(0), \quad x'(\omega) = x'(0). \tag{19}$$

Assume that functions P and Q are ω-periodic with respect to t, satisfy the Carathéodory conditions and

$$p(t) \leq P(t, x, y), \quad |P(t, x, y)| \leq q(t) \qquad \text{for} \quad t \in [0, \omega], \quad x, y \in \mathbb{R}.$$

Moreover, assume that p and q are integrable on $[0, \omega]$, $p \neq 0$,

$$\int_0^\omega p(s)ds \geq 0, \quad \int_0^\omega |q(s)|ds \leq 16/\omega,$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_0^\omega \sup_{|x|+|y|\leq n} |Q(t, x, y)|dt = 0.$$

Under the above assumptions, equation (17) has at least one ω-periodic solution.

By virtue of Theorem 11, it suffices to verify that problem (18), (19) admits the trivial solution only, if ω-periodic function p satisfies conditions:

$$\int_0^\omega p(s)ds \geq 0, \quad \int_0^\omega |p(s)|ds \leq 16/\omega.$$

5. A TOPOLOGICAL APPROACH TO DIFFERENTIAL INCLUSIONS

The topological argument, in particular fixed point theorems of various type, for a long time been used in the theory of differential equations. In their pioneer paper [20], A. Lasota and Z. Opial proposed an analogous theory for differential inclusions. More precisely, they used Kakutani-Ky Fan Fixed Point Theorem to prove the existence of solutions of boundary value problem for differential inclusions. This idea has been largely developed by several authors and the above mentioned results is frequently still quoted.

Consider a boundary value problem for nonlinear differential inclusion

$$x' = A(t)x + F(t, x), \qquad Lx = r, \tag{20}$$

where $x : I \to R^m$, $A(t)$ is $m \times m$ integrable matrix, $r \in R^m$, F is a multifunction from $I \times R^m$ into R^m with nonempty convex closed values and L is a linear operator from $C(I, R^m)$ into R^m.

Apart from problem (20), consider the associated linear homogeneous problem

$$x' = A(t)x, \qquad Lx = r. \tag{21}$$

By a solution of problem (20) (resp. (21)) we mean an absolutely continuous function $x \in C(I, R^m)$ such that (20) (resp. (21)) holds for a.e. $t \in I$.

Recall that a multifunction $F : I \to R^m$ is called *measurable* if for every open subset U of R^m the set $F^-(U) = \{t \in I : F(t) \cap U \neq \emptyset\}$ is Borel measurable. A multifunction $F : R^m \to R^m$ is called *closed graph* if the graph of F is a close subset of $R^m \times R^m$.

Theorem 12. *Assume that $x = 0$ is the unique solution of problem* (21). *Then there exists β_0 (depending on $A(t)$ and L only) such that for every multifunction F satisfying the conditions:*

(i) $F(\cdot, x)$ *is measurable for every* $x \in R^m$*;*
(ii) $F(t, \cdot)$ *is closed graph for every* $t \in I$*;*
(iii) *There are integrable functions α, β such that*

$$|F(t,x)| \leq \alpha(t) + \beta(t)|x|, \qquad (t,x) \in I \times R^m$$

and

$$\int_I \beta(t)dt \leq \beta_0,$$

problem (20) *has at least one solution for every* $r \in R^m$*.*

Sketch of a proof. Observe that for $u \in L^1(I, R^m)$ the solution of the problem

$$x' = A(t)x + u, \qquad Lx = r$$

is given by the formula

$$x = \Gamma u + Hr,$$

where Γ is a linear compact continuous mapping from $L^1(I, R^m)$ into $C(I, R^m)$ and H is a linear mapping from R^m into $C(I, R^m)$.

Consider now a multivalued map $T : L^1(I, R^m) \to L^1(I, R^m)$ given by

$$T(x) = \Gamma \mathcal{F}(x) + H(x),$$

where, for a given $x \in L^1(I, R^m)$, $\mathcal{F}(x)$ denotes the set of all measurable selections of multifunction $F(\cdot\,, x(\cdot))$. It is routine to see that T satisfies the hypotheses of Kakutani-Ky Fan Fixed Point Theorem, whence the statement of Theorem 12 follows. □

In order to illustrate Theorem 12, consider Nicoletti's classical problem

$$x' \in F(t,x), \qquad x_i(t_i) = r_i, \quad i = 1, \dots, m. \tag{22}$$

Since the corresponding homogeneous problem

$$x' = 0, \qquad x_i(t_i) = r_i, \quad i = 1, \dots, m,$$

has a unique solution $x = 0$, from Theorem 12 it follows that for every F satisfying condition (i)-(iii) and for every $r \in R^m$, problem (22) has at least one solution. Note that to use Theorem 12 in this case it is sufficient to take $\beta_0 < 1$.

6. GENERIC PROPERTIES OF DIFFERENTIAL EQUATIONS

The study of generic properties of differential equations originats from an old paper by W. Orlicz [30], which apperaed in 1932. He proved that the set of all continuous functions $f : [0,a] \times R^m \to R^m$ for which the initial problem for a differential equation $x' = f(t,x)$ does not have the uniqueness property, is a set of the first category. Similar result for hyperbolic equations was proved by A. Aleksiewicz and W. Orlicz [1] in 1952. However, the real interest in these type of results started with paper by A. Lasota and J. Yorke [25], which apperaed in 1973.

Recall that a subset A of a complete metric space X is called a set of the *first Baire category*, if it is union of a countable family of nowhere dense sets and it is called *residual* if its complement is of the first Baire category. A space X is called a Baire space if the intersection of any countable family of open dense sets is dense in X. It is well known that every complete metric space is a Baire space. In a Baire space, the intersection of any countable family of residual sets is residual. If the set of all elements of X satisfying some property P is residual in X, then the property P is called *generic* or *typical*.

To elucidate the ideas contained in Lasota and Yorke's paper, we confine ourselves to the operator formulation of problem under consideration. For this purpose, denote by $\mathcal{C}(X)$ the space of all continuous mappings from X into itself, endowed with the usual supremum metric.

For $F \in \mathcal{C}(X)$, consider the equation

$$x = F(x). \tag{23}$$

As usual, $\{F^n(x)\}$ stands for the sequence of successive approximations starting from a point x.

We begin with the following simple but extremaly useful observation, indirectly contained in [25].

Proposition 1. *Let (X,d) be a complete metric space and let D be a dense subset of X. Let $\varphi : X \to [0,+\infty)$ be a function such that $\varphi(x_n) \to 0$ for every $\{x_n\} \subset X$, $x_n \to x \in D$. Then $X_0 = \{x \in X : \varphi(x) = 0\}$ is a residal subset of X.*

Proof. Since $X \setminus X_0 = \bigcup_{n=1}^{\infty} X_n$, where $X_n = \{x \in X : \varphi(x_n) > 1/n\}$ it suffices to show that for every $n \in \mathbb{N}$, $\overline{X}_n$ has the empty interior. On the contrary, suppose that for some n_0 the set $\overline{X}_{n_0}$ has a nonempty interior. Then there exists a ball $B(x,\delta) \subset \overline{X}_{n_0}$. Clearly $\varphi(x_0) = 0$ for some $x_0 \in B(x,\delta) \cap D$. Observe that there exists $\rho > 0$ such that $B(x_0,\rho) \subset B(x,\delta)$ and $\varphi(x) < 1/n_0$ for $x \in B(x_0,\rho)$. Otherwise, for $\rho = 1/k$ there is a point $x_k \in B(x_0, 1/k)$ such that $\varphi(x_k) \geq 1/n_0$. Obviously, $x_k \to x_0 \in D$, contrary to the assumption. Consequently, $x_0 \notin \overline{X}_{n_0}$, a contradiction. □

Theorem 13. *Let $\mathcal{X} \subset \mathcal{C}(X)$ be a complete metric space. Suppose that there is a dense subset $\mathcal{D} \subset \mathcal{X}$ such that:*

(i) *For every $H \in \mathcal{D}$, problem* (23) *has exactly one solution x_H;*
(ii) *For every $H \in \mathcal{D}$ and $x \in X$, $H^n x \to x_H$;*
(iii) *For every sequence $\{F_n\} \subset \mathcal{X}$ such that $F_n \to H \in \mathcal{D}$ and every sequence $\{x_n\}$ such that $x_n = F_n x_n$, we have $x_n \to x_h$.*

Then the set $\mathcal{X}_0$ of all $F \in \mathcal{X}$ such that equation (23) *has exactly one solution and this solution depends continuously upon the data (i.e. if the sequence $\{F_n\} \subset \mathcal{X}$ converges to F and x_n is a fixed point of F_n, then $\{x_n\}$ converges to x) is a residual subset of $\mathcal{X}$.*

Proof. Define $\varphi : \mathcal{X} \to [0, +\infty]$ by

$$\varphi(F) = \limsup_{\delta \to 0} \Big\{ d(x_1, x_2) \,:\, x_1 = F_1 x_1,\ x_2 = F_2 x_2,\ F_1, F_2 \in B(F,\delta) \Big\}.$$

One can see that $\varphi(F) = 0$ implies that equation (23) has exactly one solution and this solution depends on initial data. Since $\varphi(H) = 0$ for $H \in \mathcal{D}$, the statement of Theorem 13 follows from Proposition 1. □

Theorem 14. *Let $\mathcal{X} \subset \mathcal{C}(X)$ be a complete metric space. Suppose that the assumption of Theorem 13 are satisfied with condition (iii) replaced with:*

(iii') For every $H \in \mathcal{D}$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(H^n x, F^n x) < \varepsilon$ for $F \in B(H,\delta)$, $x \in X$, $n \in \mathbb{N}$.

Then the set $\mathcal{X}_0$ of all $F \in \mathcal{X}$ such that equation (23) *has exactly one solution, and moreover, for every $x \in X$ the sequence $\{F_n x\}$ converges to x, is a residual subset of $\mathcal{X}$.*

Sketch of a proof. Define

$$\widetilde{\mathcal{X}} = \bigcap_{k=1}^{\infty} \bigcup_{H \in \mathcal{D}} B\big(H, \delta_H(1/k)\big),$$

where $\delta_H(1/k)$ is taken according to assumption (iii'). Note that $\widetilde{\mathcal{X}}$, as a dense G_δ set, is residual in $\mathcal{C}(X)$. To complete the proof it suffices to show that for every $F \in \widetilde{\mathcal{X}}$, there exists a unique solution of equation (23) and for arbitrary $x \in X$ the sequence $\{F^n x\}$ converges to this solution. □

Now we recall the classical result due to A. Lasota and J. Yorke. Let $U \subset \mathbb{R} \times E$ be an open set with $(t_0, x_0) \in U$. Consider the problem

$$x' = f(t,x), \qquad x(t_0) = x_0, \tag{24}$$

where $f : U \to E$ is continuous. Note that examples of continuous f are known for which problem (24) has no solution in any neighborhood of t_0.

Theorem 15. *Let $U \subset \mathbb{R} \times E$ be a countable union of compact sets. Let $\mathcal{M}$ be the set of all continuous functions $f : U \to E$. Let $\mathcal{M}_0$ be the set of all $f \in \mathcal{M}$ for which there is $(t_0, x_0) \in U$ such that problem (24) has no solutions. Then $\mathcal{M}_0$ is a set of first Baire category in $\mathcal{M}$.*

7. INVARIANT MEASURE. ULAM'S CONJECTURE

Let $(X, \mathcal{A})$ be a measurable space and μ a measure defined on $\mathcal{A}$. The measure μ is called *invariant* under a measurable function $f : X \to X$ if

$$\mu(A) = \mu\big(f^{-1}(A)\big) \qquad \text{for} \quad A \in \mathcal{A}.$$

It is easy to see that for a function $f : [0,1] \to [0,1]$ there does not exist an absolutely continuous invariant measure if the graph of f is too flat (the shape is small). For example, for the mapping $f(x) = rx (\operatorname{mod} 1)$ with $|r| < 1$ an invariant measure does not exist.

In 1957 S. Ulam [34] posed the problem of the existence of an absolutely continuous invariant measure for the function defined by a sufficiently simple function (e.g., a broken line function or a polynomial) the graph of which does not cross the line $y = x$ with a slope of absolute value less than 1. The literal answer to this question is negative. For example, it is easy to see that an absolutely continuous measure invariant under transformation

$$f(x) = \begin{cases} 1 - 2x, & 0 \le x \le /12, \\ (2 - 2x)/7, & 5/12 < x \le 1, \end{cases}$$

does not exist. Note that this transformation crosses the line $y = x$ at the point $x = 1/3$ with slope $f'(1/3) = -2$.

Several results concerning the existence of absolutely continuous invariant measures for some classes of point transformations of the unite interval $[0,1]$ into itself have been proved (A. Rényi [32], Parry [31], Krzyżewski/Szlenk [6]). However, the best result in this direction was obtained by A. Lasota and J. Yorke [26].

Theorem 16. *Let $f : [0,1] \to [0,1]$ be a C^2 piecewise function satisfying the condition*

$$\inf_{x \in [0,1]} \left| \frac{d}{dx} f'(x) \right| > 1.$$

Then there exists an absolutely continuous measure invariant under f.

Recall that a function $f : [0,1] \to [0,1]$ is called C^2 *piecewise function* if there exist a partition $0 = x_0 < x_1 < \cdots < x_p = 1$ of the interval $[0,1]$ such that f restricted to each of the interval (x_{i-1}, x_i) is a C^2 function. Function f need to be continuous at the points x_i.

The Lasota/Yorke method of the proof is quite different from previous works. First they use the fact that the Frobenius–Perron operator corresponding to the point transformation under consideration has the property of shrinking the variation of the function. Then they prove the existence of invariant meaure using Kakutani–Yoshida Theorem.

We also recall another interesting result of A. Lasota and J. Yorke [27], which says that the existence of a periodic point of period three implies the existence of continuous invariant measure.

Let $f : [0,1] \to [0,1]$ be a continuous function. For any $x_0 \in X$, the set $\gamma_f(x_0) = \{(x_0, f(x_0), f^2(x_0), \ldots\}$ is called the *trajectory* starting from x_0. If $\gamma(x_0)$ is a finite set, the point x_0 is called *periodic*. For any trajectory γ, the set

$$L(\gamma) = \bigcap_{n=1}^{\infty} \mathrm{cl}\big(f^n(\gamma)\big) \qquad (\mathrm{cl} \ = \ \mathrm{closure})$$

is called the *limit set* of γ. A trajectory γ is called *strictly turbulent* if $L(\gamma)$ is a compact nonempty set which does not contain periodic points. From Krylov–Bogoluboff Theorem there easily follows

Proposition 2. *Let $f : [0,1] \to [0,1]$ be a continuous function and let $\gamma(x_0)$ be a strictly turbulent trajectory. Then there exists a notrivial measure μ supported on $L(\gamma)$ and invariant under f.*

Using the last proposition, one can prove the following:

Theorem 17. *Let $f : X \to X$ be a continous mapping. Assume that there exist compact disjoint subsets A and B of X such that $f(A) \cap F(B) \supset A \cup B$. Then there exists a strictly turbulent trajectory γ and, consequently, a nontrivial invariant measure supported on $L(\gamma)$.*

Using Theorem 17, it is possible to prove the existence of continuous invariant measures for transformations on the real line. Namely, the following theorem holds:

Theorem 18. *Let f be a continous mapping of an interval of the real line into itself. Then the existence of a periodic point of period $3n$ for some integer n implies the existence of a continuous invariant measure.*

8. INVARIANT MEASURE FOR MARKOV OPERATORS

Let (X, ρ) be a complete metric space. By $\mathcal{M}$ we denote the space of all finite Borel measures on X and by $\mathcal{M}_1$ we denote the space of all $\mu \in \mathcal{M}$ such that $\mu(X) = 1$. As usual, by $B(X)$ we denote the space of all bounded Borel measurable functions

$f : X \to \mathbb{R}$, by $C(X)$ the subspace of all continuous functions and by $C_0(X)$ the subspace of all continuous functions with compact support.

For $f \in B(X)$ and $\mu \in \mathcal{M}$, we write

$$\langle f, \mu \rangle = \int_X f(x)\mu(dx).$$

We say that a sequence $\{\mu_n\} \subset \mathcal{M}$ converges weakly to the measure $\mu \in \mathcal{M}$ if $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ for every $f \in \mathcal{C}_0(X)$.

The weak convergence can be metrized by Fortet–Mourier metric given by

$$d_{FM}(\mu, \nu) = \sup \big\{ |\langle f, \mu \rangle - \langle f, \nu \rangle| : f \in \mathcal{L} \big\},$$

where $\mathcal{L}$ is the set of all nonexpansive $f \in C(X)$ such tht $|f(x)| \leq 1$.

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if:

(i) $P(\lambda_1 \mu + P\lambda_2 \mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for λ_1, $\lambda_2 \in \mathbb{R}$, μ_1, $\mu_2 \in \mathcal{M}$;
(ii) $P\mu(X) = \mu(X)$ for $\mu \in \mathcal{M}$.

A Markov operator P is called *Markov-Feller operator* if there exists an operator $U : B(X) \to B(X)$ (called *dual*) such that $Uf \in C(X)$ for $f \in C(X)$ and

$$\langle Uf, \mu \rangle = \langle f, P\mu \rangle \quad \text{for} \quad f \in B(X) \quad \text{and} \quad \mu \in \mathcal{M}.$$

A Markov oparator P is called *nonexpansive* if

$$d_{FM}(P\mu, P\nu) \leq d_{FM}(\mu, \nu) \quad \textit{for} \quad \mu, \nu \in \mathcal{M}_1.$$

A measure μ is called *invariant* or *stationary* with respect to P if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that $P^n\mu \to \mu_*$ for every $\mu \in \mathcal{M}_1$.

The existence of an invariant measure was well known in the case of compact space and a proof goes as follows. First we construct a positive invariant functional defined on the space of all continuous and bounded functions $f : X \to \mathbb{R}$, and then using the Riesz representation theorem, one can define an invariant measure. The case of locally compact spaces requied some caution. The first existence result was established by J. Yorke and A. Lasota [28] by using the concept of nonepansiveness and lower bounde technique. When X is a Polish space, this approach fails, since tno measure may correspond to a positive functional. However, combining the above nonexpansivveness argument with the concept of tightness and suitable concentration properties of Markov operators, a student of A. Lasota, T. Szarek, established the existence results also in the case of Polish spaces.

Recall that a linear funcional $\varphi : C(X) \to \mathbb{R}$ is called positive if $\varphi(f) \geq 0$ for $f \geq 0$. According to Riesz Theorem, for every linear positive functional $\varphi : C_0(X) \to \mathbb{R}$ there is a unique measure $\mu \in \mathcal{M}_1$ such that $\varphi(f) = \langle f, \mu \rangle$.

Lemma 1. *Let P be a Feller operator. Assume that there exists a linear positive functional $\varphi : C(X) \to \mathbb{R}$ such that $\varphi(1_X) = 1$ and*

$$\varphi(Uf) = \varphi(f) \quad \text{for} \quad f \in C_0(X), \tag{25}$$

where U is dual to P. Further, let $\mu_* \in \mathcal{M}$ be the unique measure satisfying

$$\varphi(f) = \langle f, \mu_* \rangle \quad \text{for} \quad f \in C_0(X). \tag{26}$$

Then $\mu_* \in \mathcal{M}_*$ and $P\mu_* = \mu_*$.

Proof. Fix $x_0 \in X$ and for $n \in \mathbb{N}$ define $B_n = B(X_0, n)$. By (25), there is

$$\mu_*(B_n) = \langle 1_{B_n}, \mu_* \rangle \; = \; \varphi(1_{B_n}) \leq \varphi(1_X) \leq 1,$$

which implies $\mu_*(X) = \lim_{n\to\infty} \mu_*(B_n) \leq 1$. Thus $\mu_* \in \mathcal{M}$. Consider now the functional on $C(X)$ given by

$$\widetilde{\varphi}(f) = \langle f, \mu_* \rangle \quad \text{for} \quad f \in C(X).$$

We claim that $\widetilde{\varphi}(f) \leq \varphi(f)$ for $f \in C(X)$, $f \geq 0$. Indeed, let $f \in C(X)$, $f \geq 0$. Using the Tietze Extension Theorem, one can construct an increasing sequence of nonnegative functions $f_n \in C_0(X)$ such that $f_n(x) \to f(x)$ for $x \in X$. Since φ is positive,

$$\langle f_n, \mu_* \rangle \; = \; \widetilde{\varphi}(f_n) \; = \; \varphi(f_n) \leq \varphi(f).$$

Passing with n to ∞, by virtue of Monotone Convergence Theorem, we get

$$\langle f, \mu_* \rangle \; \leq \; \varphi(x),$$

which completes the proof of claim.

According to the last observation and condition (25), we infer

$$\widetilde{\varphi}(Uf) \leq \varphi(Uf) = \varphi(f) = \widetilde{\varphi}(f) \quad \text{for} \quad f \in C_0(X), \; f \geq 0,$$

which in turn implies

$$\langle f, P\mu_* \rangle \; = < Uf, \mu_*. \; = \; \widetilde{\varphi}(Uf) \leq \widetilde{\varphi}(f) \; = \langle f, \mu_* \rangle \quad \text{for} \quad f \in C_0(X), \; f \geq 0.$$

As above, we can verify that the last inequality holds for $f \in C(X)$, $f \geq 0$. Consequently, $P\mu_* \leq \mu_*$. Since P preserves the measure, $P\mu_* = \mu_*$. □

Theorem 19. *Let $P : \mathcal{M} \to \mathcal{M}$ be a Markov–Feller operator. Assume that there is a compact set $A \subset X$ and a distribution μ_0 such that*

$$\liminf_{n\to\infty} \Big(\frac{1}{n} \sum_{j=1}^{n} P^j \mu_0(A) \Big) > 0. \tag{27}$$

Then P has a stationary distribution.

Proof. Using (27) one can find a sequence $\{n_k\}$ and $\varepsilon > 0$ such that

$$\frac{1}{n_k} \sum_{j=1}^{n_k} P^j \mu_0(A) \geq \varepsilon, \qquad k = 1, 2, \ldots.$$

Let L_0 be a Banach limit. Then the operator L given by

$$L(\{a_j\}) = L_0\Big(\frac{1}{n_k}\sum_{j=1}^{n_k} a_j\Big) \qquad \text{for} \quad \{a_j\} \in l^\infty$$

is also a Banach limit and $L\big(P^n\mu_0(A)\big) \geq \varepsilon$. Define $\varphi : C(X) \to \mathbb{R}$ by

$$\varphi(f) = L\big(\langle f, P^n\mu_0\rangle\big).$$

Clearly φ is a positive linear functional. Observe that

$$\varphi(1_X) = L\big(\langle 1_X, P^n\mu_0\rangle\big) = L(1, 1, \ldots) = 1$$

and

$$\varphi(Uf) = L\big(\langle Uf, P^n\mu_0\rangle\big) = L\big(\langle f, P^{n+1}\mu_0\rangle\big) = L\big(\langle f, P^n\mu_0\rangle\big) = \varphi(f).$$

Thus hypotheses of Lemma 1 are satisfied, whence the statement of Theorem 19 follows immediately. □

Theorem 20. *Let $P : \mathcal{M}_1 \to \mathcal{M}_1$ be a nonexpansive Markov operator. Assume that for every $\varepsilon > 0$ there is a Borel set A with* diam$A \leq \varepsilon$ *and a number $\alpha > 0$ such that*

$$\liminf_{n\to\infty} P^n\mu(A) \geq \alpha \quad \text{for} \quad \mu \in \mathcal{M}_1.$$

Then P is asymptotically stable.

Sketch of a proof. First note that a nonexpansive Markov operator is a Feller operator. Moreove, since P satisfies condition (27) with Y equal to the closure of A, it has an invariant distribution μ_*. Thus, to prove that P is asymptotically stable it suffices to show that

$$\lim_{n\to\infty} \|P^n(\mu_1 - \mu_2)\| = 0 \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1.$$

Fix A and α according to hypotheses of Theorem 20 and choose $\sigma \in (0, \alpha)$. By induction argument define a sequenece of integers $\{n_k\}$ and sequences of distributions $\{\mu_i^k\}$, $\{\nu_i^k\}$, $i = 1, 2$, $k \in \mathbb{N}$, in the following way. We set $\nu_i^0 = \mu_i^0 = 0$. If $k \geq 1$ and n_{k-1}, ν_i^{k-1}, μ_i^{k-1} are given, we choose a number n_k such that

$$P^{n_k}\mu_i^{k-1}(A) \geq \sigma \quad \text{for} \quad i = 1, 2$$

and we define

$$\nu_i^k(B) = P^{n_k}\mu_i^{k-1}(B \cap A)/P^{n_k}\mu_i^{k-1}(A),$$
$$\mu_i^k(B) = 1/(1-\sigma)\big[P^{n_k}\mu_i^{k-1}(B) - \sigma\nu_i^k(B)\big].$$

Now, by an induction argument one can show that

$$\|P^{n_1+\ldots+n_k}(\mu_1 - \mu_2)\| \leq \varepsilon + 2(1-\sigma)^k \quad \text{for} \quad n \geq n_1 + \ldots + n_k.$$

Since $\varepsilon > 0$ and $k > 0$ are arbitrary, the statement of the theorem follows. □

9. FRACTALS AND SEMIFRACTALS

An Iterated Function System (briefly IFS) is given by a family of continuous functions

$$w_i : X \to X, \quad i \in I = \{1, \dots, N\}.$$

For $A \subset X$, set

$$F(A) = \bigcup_{i=1}^{N} w_i(A). \tag{28}$$

Obviously, F maps compact sets to compact sets. If all w_i are strictly contractive, then there exists a unique compact set such that

$$K = \bigcup_{i=1}^{N} w_i(K). \tag{29}$$

Moreover, for every compact set $A \subset X$, $F^n(A) \to K$ in Hausdorff distance. The set K is called the *attractor* or *fractal* corresponding to IFS $\{w_i : i \in I\}$.

The family $\{(w_i, p_i) : i \in I\}$, where $w_i : X \to X$, $p_i : X \to (0,1)$, $i \in I$, are continuous functions and $\sum_{i \in I} p_i(x) = 1$ for all $x \in X$, is called an IFS *with probabilities.*

Given an IFS $\{(w_i, p_i) : i \in I\}$, we can define a Markov operator $P : \mathcal{M}_1 \to \mathcal{M}_1$ by

$$P\mu(A) = \sum_{i \in I} \int_{w^{-1}(A)} p_i(x)\mu(dx), \quad A \in \mathcal{B}(X).$$

A measure $\mu_* \in \mathcal{M}_1(X)$ is called *invariand* with respect to operator P if $P\mu_* = \mu_*$. If in addition

$$\int_X f(x)P^n\mu(dx) \to \int_X f(x)\mu_*(dx) \quad \text{for every} \quad f \in C(X), \tag{30}$$

then operator P is called *asymptotically stable.*

We say that an IFS $\{(w_i, p_i) : i \in I\}$ is asymptotically stable if the corresponding Markov operator P is asymptotically stable.

Remark 2. *Assume that all functions w_i are Lipschitzean with corresponding Lipschitz constants L_i. If*

$$\sum_{i \in I} p_i L_i < 1,$$

then the IFS $\{(w_i, p_i) : i \in I\}$ is asymptotically stable.

Remark 3. *Let an IFS $\{(w_i, p_i) : i \in I\}$ be such that all functions w_i are strictly contractive. Then*

$$A_* = \operatorname{supp} \mu_*,$$

where A_* is the attractor of IFS $\{w_i : i \in I\}$ and μ_* is the invariant measure with respect to the IFS $\{(w_i, p_i) : i \in I\}$.

Now we will give a generalizaton of the above notion of attractor. For this purpose, we recall the notion of convergence of sequence of sets in Kuratowski's sense.

Let $\{A_n\}$ be a sequence of subsets of a metric space X. The *lower bound* $\mathrm{Li}A_n$ and the *upper bound* $\mathrm{Ls}A_n$ are defined by the following conditions. A point x belongs to $\mathrm{Li}A_n$ if there exists a sequence $\{x_n\}$, $x_n \in A_n$, such that $x_n \to x$. A point x belongs to $\mathrm{Ls}A_n$ if there exists a sequence $\{x_{n_k}\}$, $x_{n_k} \in A_{n_k}$, $\{n_k\} \subset \{n\}$, such that $x_{n_k} \to x$. Obviously $\mathrm{Li}A_n \subset \mathrm{Ls}A_n$. If $\mathrm{Li}A_n = \mathrm{Ls}A_n$, we say that the sequence $\{A_n\}$ is topologically convergent and we denote this common limit by $\mathrm{Lt}A_n$.

In the case when X is a compact set, $\mathrm{Lt}A_n = A$ if and only if $\{A_n\}$ converges to A in the Hausdorff distance.

Given an IFS $\{w_i : i \in I\}$, we define

$$H(A) = \overline{\bigcup_{i\in I} w_i(A)}.$$

A set A_0 such that $H(A_0) = A_0$ is called *invariant* with respect to the IFS $\{w_i : i \in I\}$. If in addition for every nonempty bounded subset A of X, $\mathrm{Lt}H^n(A) = A_0$, the IFS is called *asympotically stable* (on sets) and the set A_0 is called the attractor of IFS $\{w_i : i \in I\}$.

Note that if we consider H on the class of compact sets, this definition of attractor coincides with that used before.

We say that an IFS $\{w_i : i \in I\}$ is *regular* if there is a nonempty subset I_0 of I such that the IFS $\{w_i : i \in I_0\}$ is asymptotically stable. The attractor corresponding to the IFS $\{w_i : i \in I_0\}$ will be called a *nucleus*.

Theorem 21. *Let $\{w_i : i \in I\}$ be a regular IFS and A_0 be a nucleus of this system. Let*

$$A_* = \bigcup_{n=1}^{\infty} H^n(A_0).$$

Then:

(i) *A_* does not depend on the choice of the nucleus A_0;*
(ii) *A_* is the smallest nonempty set such that $H(A_*) = A_*$;*
(iii) *$\mathrm{Lt}H^n(A) = A_*$ for every nonempty set $A \subset A_*$.*

The set A_* is called the *semiattractor* or *semifractal* corresponding to the regular IFS $\{w_i : i \in I\}$.

Theorem 22. *Let X be a Polish space. Assume that an IFS $\{(w_i, p_i) : i \in I\}$ is asymptotically stable and $\{w_i : i \in I\}$ is regular. Then*

$$A_* = \operatorname{supp} \mu_*,$$

where A_* is the semiattractor of the IFS $\{w_i : i \in I\}$ and μ_* is the invariant measure with respect to the IFS $\{(w_i, p_i) : i \in I\}$.

The concept of semiattractor can be generalized to a large class of multifunctions. Given a multifunction $F : X \to X$, consider the set

$$C = \bigcap_{x \in X} \mathrm{Li} F^n(x).$$

If $C \neq \emptyset$, the multifunction F is called *asymptotically semistable* and the set C is called the *semiattractor* of F.

Recall that a multifunction $F : X \to X$ is called *lower semicontinuous* (briefly l.s.c.) if for every open subset U of X the set $F^-(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$ is open, and F is called *measurable* if $F^-(U)$ is measurable in X.

Theorem 23. *Assume that F is an asymptotically semistable, lower semicontinuous multifunction. Let C be the semiattractor of F. Then:*

(i) $C \subset \mathrm{Li} F^n(A)$ *for every* $A \subset X$, $A \neq \emptyset$*;*
(ii) $\overline{F(C)} = C$*;*
(iii) $\mathrm{Lt} F^n(A) = C$ *for every* $A \subset C$, $C \neq \emptyset$*;*
(iv) $C \subset A$ *for every nonempty closed subset* A *of* X *such that* $F(A) \subset A$.

Proof. Condition (i) is obvious. To see (ii), note that

$$F(C) \subset \bigcap_{x \in X} F\big(\mathrm{Li} F^n(x)\big) \subset \bigcap_{x \in X} \mathrm{Li} F^n(x) = C.$$

Since C is closed, $\overline{F(C)} \subset C$. To see the opposite inclusion, observe that $F^n(C) \subset F(C)$ for $n \geq 1$ which, in turn, implies $\mathrm{Li} F^n(C) \subset \overline{F(C)}$. Since $C \subset \mathrm{Li} F^n(C)$, condition (ii) follows.

To prove (iii), observe that inclusion $F(C) \subset C$ implies $\mathrm{Ls} F^n(C) \subset C$. Thus for an arbitrary set $A \subset C$, there is

$$C \subset \mathrm{Li} F^n(A) \subset \mathrm{Ls} F^n(A) \subset \mathrm{Ls} F^n(C) \subset C.$$

Condition (iv) can be verified as follows. Inclusion $F(A) \subset A$ implies $F^n(A) \subset A$. Consequently, $C \subset \mathrm{Li} F^n(A) \subset A$. □

Theorem 24. *Let $F : X \to X$ be a l.s.c. multifunction. Assume that there is an l.s.c. multifunction $F_0 : X \to X$ such that $F_0(x) \subset F(x)$, $x \in X$. Then F is asymptotically semistable and its semiattractor is $C = \mathrm{Lt} F^n(C_0)$, where C_0 is the semiattractor of F_0.*

For more details we refer the reader to [12, 13].

10. MARKOV MULTIFUNCTIONS

A mapping $\pi : X \times \mathcal{B} \to [0,1]$ is called a *transition function* if $\pi(x, \cdot)$ is a probability measure for every $x \in X$ and $\pi(\cdot, A)$ is a measurable function for every $A \in \mathcal{B}$.

Having a transition function π, we can define the Markov operator $P : \mathcal{M} \to \mathcal{M}$ by the formula

$$P\mu(A) = \int_X \pi(x, A)\mu(dx)$$

and having a Markov operator P, we may define a transition function setting

$$\pi(x, A) = P\delta_x(A).$$

Given a Markov operator P and the corresponding transition function π, we define the Markov multifunction $\Gamma : X \to X$ by

$$\Gamma(x) = \mathrm{supp}\pi(x, \cdot) = \mathrm{supp}P\delta_x.$$

It is easy to see that Γ is closed valued and measurable. Vice versa, the following theorem holds true.

Theorem 25. *Let $F : X \to X$ be a measurable, closed valued multifunction. Then there exists a transition function $\pi : X \times \mathcal{B} \to [0, 1]$ such that F coincides with the support of π.*

Proof. According to the Kuratowski-Ryll Nardzewski Theorem, there exists a sequence $\{f_n\}$ of measurable functions $f_n : X \to X$ such that

$$F(x) = \mathrm{cl}\{f_n(x) : n \in \mathbb{N}\}.$$

We define the function π by

$$\pi(x, A) = \sum_{n=1}^{\infty} p_n \delta_{f_n(x)}(A),$$

where $\{p_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} p_n = 1$ and δ_u stands for the δ-Dirac measure supported at u. A simple calculation shows that π is a transition function and that F is equal to the support of π. $\square$

Theorem 26. *Let $\pi : X \times \mathcal{B} \to [0, 1]$ be a Fellerian transition function.Then the corresponding Markov multifunction Γ is l.s.c..*

Proof. Fix an $x \in X$ and consider a sequence $\{x_n\} \subset X$ converging to x. Since π is Fellerian, the corresponding sequence of measures $\{\pi(x_n, \cdot)\}$ converges weakly to the measure $\pi(x, \cdot)$. This implies that $\Gamma(x) \subset \mathrm{Li}\Gamma(x_n)$, whence the statement of Theorem 26 follows. $\square$

Theorem 27. *Assume that $F : X \to X$ is a l.s.c. multifunction with closed values. Then there exists a Fellerian transition function $\pi : X \times \mathcal{B} \to [0, 1]$ such that F is equal to the support of π.*

Proof. Define a multifunction $\Phi : X \to \mathcal{M}_1$ by

$$\Phi(x) = \big\{\mu \in \mathcal{M}_1 : \mathrm{supp}\,\mu \subset F(x)\big\}.$$

Clearly Φ is convex and closed valued. It is easy to verify that Φ is l.s.c.. Observe that $\mathcal{M}_1$ is a convex subset of the space of signed Borel meaures. Recall also that $\mathcal{M}_1$ endowed with the Fortet–Mourier metric is complete. Thus, by Michael Selection Theorem there exists a sequence $\{\phi_n\}$ of continuous functions $\phi_n : X \to \mathcal{M}_1$ such that

$$\Phi(x) = \mathrm{cl}\big\{\phi_n(x) : n \in \mathbb{N}\big\}.$$

Now define $\pi : X \times \mathcal{B} \to [0, 1]$ by

$$\pi(x, A) = \sum_{n=1}^{\infty} p_n \phi_n(x)(A),$$

where $\{p_n\}$ is a sequence of positive numbers such that $\sum_{n=1}^{\infty} p_n = 1$. It remains to prove that π is a Fellerian transition function and F coincides with the support of π. □

The following properties of support of measures, proved in [11], are very useful (see [35]).

Proposition 3. *Let $P : \mathcal{M}_1 \to \mathcal{M}_1$ be a Fellerian Markov operator. If $\mu, \nu \in \mathcal{M}_1$ and* $\mathrm{supp}\mu \subset \mathrm{supp}\nu$*, then* $\mathrm{supp}P\mu \subset \mathrm{supp}P\nu$*.*

Proposition 4. *Let $P : \mathcal{M}_1 \to \mathcal{M}_1$ be the Markov operator corresponding to a Fellerian transition function $\pi : X \times \mathcal{B} \to [0, 1]$. Further, let Γ be the support of π. Then for every $\mu \in \mathcal{M}$ and $n \in \mathbb{N}$, there is*

$$\mathrm{supp}P^n\mu = \mathrm{cl}\,\Gamma^n(\mathrm{supp}\mu).$$

Theorem 28. *If a Fellerian Markov operator P is asymptotically stable, then the corresponding Markov multifunction Γ is asymptotically semistable and*

$$C = \mathrm{supp}\mu_*,$$

where C is the semiattractor of Γ and μ_* is the measure invariant with respect to operator P.

Proof. Fix an arbitrary $x \in X$. Since P is asymptotically stable, $\{P^n\delta_x\}$ converges weakly to μ_*. Consequently, $\mathrm{supp}\,\mu_* \subset \mathrm{Li}\;\mathrm{supp}P^n\delta_x = \mathrm{Li}\,\Gamma^n(x)$. This implies that $\mathrm{supp}\mu_* \subset C$.

To prove the opposite inclusion, fix $z \notin \mathrm{supp}\,\mu_*$ and choose $\varepsilon > 0$ such that $B(z, \varepsilon) \cap \mathrm{supp}\,\mu_* = \emptyset$. Let $x \in \mathrm{supp}\,\mu_*$. By Proposition 3 and 4,

$$\Gamma^n(x) \subset \mathrm{supp}P^n\delta_x \subset \mathrm{supp}P^n\mu_* = \mathrm{supp}\,\mu_*, \qquad n \in \mathbb{N}.$$

Thus $\Gamma^n(x) \cap B(z, \varepsilon) = \emptyset$. It follows that $z \notin \mathrm{Li}\Gamma^n(x)$ and consequently $z \notin C$. This completes the proof. □

Theorem 29. *Let P be a Feller-Markov operator and let Γ be the corresponding Markov set function. Assume that P has a unique invariant probability measure μ_*. Then*

$$\mu_*(D) = 0 \quad \text{or} \quad \mu_*(D) = 1$$

for every close set $D \subset X$ such that $\Gamma(D) \subset D$.

Proof. Let U be the operator dual to P and $\pi(x,\cdot) = P\delta_x$. Let $x \in D$ be arbitrary. Since $\operatorname{supp} \pi(x,\cdot) \subset D$, then $\pi(x, X \setminus D) = 0$. From this and the equality $U1_A = \pi(\cdot, A)$ it follows that $U1_{X\setminus D}(x) = 0$. Define

$$\mu_0(A) = \mu_*(A \cap D) \quad \text{for} \quad A \in \mathcal{B}.$$

A simple calculation shows that μ_0 is invariant with respect to P. If $\mu_*(D) = 0$, the assertion is obviously true. If $\mu_*(D) > 0$, it can be proved that $\mu_0 = \mu_*$ and consequently $\mu_*(D) = \mu_0(X) = 1$. □

For further results, see [11, 15].

11. CONCENTRATION AND THIN DIMENSION

In order to describe semifractals, A. Lasota proposed two new concepts of dimension: concentration and thin dimension (see [10,14]). These dimensions have the advantage, at least in the case of a fractal measure, of being relatively easy calculable. It is also important that the concentration dimension is strongly connected with the Hausdorff dimension, and the thin dimension with the fractal dimension.

Given a Borel measure $\mu \in \mathcal{M}_1(X)$, the *lower* and *upper concentration dimension* of μ are given by the formulae

$$\underline{\dim}_L \mu = \liminf_{r\to 0} \frac{\log Q_\mu(r)}{\log r}, \qquad \overline{\dim}_L \mu = \limsup_{r\to 0} \frac{\log Q_\mu(r)}{\log r},$$

where

$$Q_\mu(r) = \sup_{x\in X} \mu\big(B(x,r)\big). \tag{31}$$

If $\underline{\dim}_L \mu = \overline{\dim}_L \mu$ then this common value is called *concentration* or *Lasota* dimension of μ and denoted by $\dim_L \mu$.

The function Q_μ is called the Lévy concentration function and it is frequently used in the theory of stochastic processes.

It is easy to verify that for every $\mu \in \mathcal{M}_1(X)$,

$$\underline{\dim}_L \mu \le \underline{\dim}_C \mu \le 2\underline{\dim}_L \mu,$$

where $\underline{\dim}_C$ denotes the lower correlation dimension given by

$$\underline{\dim}_C \mu = \liminf_{r\to 0} \Big[\log\Big(\int_X \mu\big(B(x,r)\big)\mu(dx)\Big)\Big/\log r\Big].$$

Analogous inequalities hold for the corresponding upper dimensions.

Theorem 30. *Let $\mu \in \mathcal{M}_1(X)$ and let $A \in \mathcal{B}(X)$ be such that $\mu(A) > 0$. Then*

$$\dim_H A \geq \underline{\dim}_L \mu,$$

where $\dim_H A$ *denotes the Hausdorff dimension of the set A.*

Proof. Set $d = \underline{\dim}_L \mu$. Suppose $d > 0$. (If $d = 0$, the assertion is obvious). Choose $s \in (0, d)$. Then there exists $r_0 > 0$ such that

$$\mu\big(B(x,r)\big) \leq r^s \quad \text{for} \quad r \in (0, r_0) \quad \text{and} \quad x \in X.$$

According to Frostman Lemma, $\dim_H A \geq s$. Since $s < d$ was arbitrary, the statement of theorem follows. □

The concentration dimension of a closed set A is defined by the formula

$$\dim_L A = \sup\Big\{\underline{\dim}_L \mu \,:\, \mu \in \mathcal{M}_1(X), \quad \mathrm{supp}\mu \subset \mathrm{A}\Big\}.$$

From Theorem 30, there immediately follows that

$$\dim_H X \geq \dim_L X.$$

For $\mu \in \mathcal{M}_1(X)$, we define the *lower* and *upper thin* dimension of μ by

$$\underline{\dim}_T \mu = \liminf_{r \to 0} \frac{\log T_\mu(r)}{\log r}, \qquad \overline{\dim}_T \mu = \limsup_{r \to 0} \frac{\log T_\mu(r)}{\log r},$$

where

$$T_\mu(r) = \inf\Big\{\mu\big(B(x,r)\big) \,:\, x \in \mathrm{supp}\,\mu\Big\}.$$

If $\underline{\dim}_T \mu = \overline{\dim}_T \mu$, then this common value is called the *thin* dimension of the measure μ and denoted by $\dim_T \mu$ (see [10]).

The function $T_\mu : (0,\infty) \to [0,1]$ is called the *thin function* corresponding to the measure μ. Obviously, if $\mathrm{supp}\,\mu$ is a compact set, the values of T_μ are positive. In general, T_μ is only nonnegative. For convenience, we make $\log 0 = -\infty$.

The *lower* and *upper fractal* (or *box*) *dimension* of the set A is defined by

$$\underline{\dim}_F A = \liminf_{r \to 0} \frac{\log N(r)}{-\log r}, \quad \overline{\dim}_F A = \limsup_{r \to 0} \frac{\log N(r)}{-\log r},$$

where $N(r)$ denotes the smallest number of closed balls of radius r needed to cover A. If $\underline{\dim}_F A = \overline{\dim}_F A$, this common value is called the *fractal dimension.*

Theorem 31. *If $A \subset \mathrm{supp}\,\mu$, $A \in \mathcal{B}(X)$, then*

$$\underline{\dim}_F A \leq \underline{\dim}_T \mu \quad \text{and} \quad \overline{\dim}_F A \leq \overline{\dim}_T \mu.$$

Proof. Let $d = \underline{\dim}_T\mu$, $A \subset \operatorname{supp}\mu$, $A \neq \emptyset$, $d < \infty$. Choose $s \in (d, \infty)$. Obviously, there exists a sequence $\{r_n\}$ of positive numbers such that $r_n \to 0$ and

$$T_\mu(r_n) \geq r_n^s, \qquad n \in \mathbb{N}.$$

For a fixed $n \in \mathbb{N}$, let $I_n = N_A(r_n/2)$ be the smallest number of closed balls of radius $r_n/2$ needed to cover A. Let $\{B(x_i, r_n/2) : i \in I_n\}$ be the corresponding covering. Obviously, we can find $y_i \in A$ such that the family $\{B(y_i, r_n) : i \in I_n\}$ covers A. Let $J_n \subset I_n$ be such that $\{B(x_i, r_n) : i \in J_n\}$ are pairwise disjoint and the family $\{B(x_i, 4r_n) : i \in J_n\}$ covers A. Consequently, $N_A(4r_n) \leq \operatorname{card} J_n$. On the other hand,

$$\sum_{i \in J_n} \mu\big(B(y_i, r_n)\big) = \mu\Big(\bigcup_{i \in J_n} B(y_i, r_n)\Big) \leq 1.$$

Since $y_i \in A$, then $\mu\big(B(y_i, r_n)\big) \geq T_\mu(r_n)$, which implies that $T_\mu(r_n) \cdot \operatorname{card} J_n \leq 1$. Consequently, $N_A(4r_n) \leq r_n^{-s}$. From the last inequality it follows that $\underline{\dim}_F A \leq s$. Since $s \in (d, \infty)$ was arbitrary, the first inequality of Theorem 31 follows. The proof of the second one is similar. □

It is well known that the dimension of a measure allows us to estimate the dimension of its support. Moreover, the estimate for a set A can be obtained either as the greatest lower bound or as the least upper bound of the dimensions of measures supported on A. Such results are called variational principles. They are closely related with Frostman Lemma and Mass Distribution Principle.

In particular, variational principles for the Hausdorff and packing dimension of sets and the point dimension of measures were proved by C. Tricot and C.D. Cutler. The variational principles for the Hausdorff dimension and packing dimension of sets and Rényi dimension of measures were found by C.D. Cutler and L. Olsen. Here we recall the variational principles for Hausdorff and fractal dimensions of sets and concentration and thin dimension of measures (see [10, 14])

Theorem 32. *Let $K \subset X$ be a nonempty compact set. Then*

$$\dim_H K = \sup \underline{\dim}_L \mu,$$

where the supremum is taken over all $\mu \in \mathcal{M}_1(X)$ such that $\operatorname{supp}\mu \subset K$.

Proof. From Theorem 30 it follows that

$$\dim_H K \geq \sup \underline{\dim}_L \mu,$$

where the supremum is taken over all $\mu \in \mathcal{M}_1(X)$ such that $\operatorname{supp}\mu \subset K$.

We need to prove the opposite inequality. Let $d = \dim_H K$. Suppose $d > 0$ (for $d = 0$ the assertion is obvious). Choose $s \in (0, d)$. Obviously, $\mathcal{H}^s(K) > 0$ and by Frostman Lemma there are a measure $\mu \in \mathcal{M}_1(X)$ and constants $c > 0$ and $r_0 > 0$ such that

$$\mu\big(B(x, r)\big) \leq c r^s \quad \text{for} \quad 0 < r < r_0.$$

Consequently, $\underline{\dim}_L \mu \geq s$. Since $s \in (0, d)$ was arbitrary, $\dim_H K \geq d$. □

Theorem 33. *Let K be a nonempty compact subset of X. Then there exists a Borel probability measure μ such that $\operatorname{supp}\mu = K$ and*

$$\underline{\dim}_F K = \underline{\dim}_T \mu, \qquad \overline{\dim}_F K = \overline{\dim}_T \mu.$$

A proof can be found in [10].

12. UPPER AND LOWER ESTIMATES OF CONCENTRATION DIMENSION

Lemma 2. *Let α_i, β_i, $L_i \in (0,1)$ for $i \in J$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded increasing function. Suppose that*

$$\Phi(r) \geq \sup_{i\in J} \alpha_i \Phi(r/L_i) \quad \text{for} \quad r \in (0,a),\ a > 0. \tag{32}$$

Then there exists $c > 0$ such that

$$\Phi(r) \geq c r^s \quad \text{for} \quad r \in (0,a),$$

where

$$s = \min_{i\in J} \frac{\log \alpha_i}{\log L_i}.$$

Proof. For $i \in J$, define

$$c_i = a^{-s_i}\Phi(aL_i),$$

where

$$s_i = \log \alpha_i / (\log L_i).$$

We claim that for arbitrary $n \in \mathbb{N}$, there is

$$\Phi(r) \geq c_i r^{s_i} \quad \text{for} \quad r \in \big[L_i^n a, a\big). \tag{33}$$

Indeed, for $r \in \big[L_i a, a\big)$ by the definition of c_i, there follows

$$\Phi(r) \geq \Phi(L_i a) = c_i a^{s_i}.$$

Suppose that (33) holds for some $n \geq 1$. Since $r/L_i \in \big[L_i^n a, a\big)$ for $r \in \big[L_i^{n+1} a, L_i^n a\big)$, from (32), (33) and the definition of s_i, we infer

$$\Phi(r) \geq \alpha_i \Phi(r/L_i) \geq \alpha_i c_i \big(r/L_i\big)^{s_i} = c_i r^{s_i} \quad \text{for} \quad r \in \big[L_i^{n+1} a,\, L_i^n a\big).$$

By virtue of the induction principle, condition (33) holds for every $n \in \mathbb{N}$. Since $L_i < 1$, this implies

$$\Phi(r) \geq c_i a^{s_i} \quad \text{for} \quad r \in (0,a).$$

Since $i \in J$ is arbitrary, the statement of Lemma 2 follows. □

Theorem 34. *Suppose that an IFS* $\{(w_i, p_i) : i \in I\}$ *has an invariant measure* μ. *Assume that all functions* w_i *are Lipschitzean with Lipschitz constants* L_i *and the set* $J = \{i \in I : L_i < 1\}$ *is nonempty. Then*

$$\overline{\dim}_L \mu \leq \inf_{i \in J} \frac{\log \alpha_i}{\log L_i},$$

where

$$\alpha_i = \inf_{x \in X} p_i(x).$$

Proof. Since the measure μ is invariant, for arbitrary $i \in J$ there holds

$$\mu\big(B(x,r)\big) \geq \alpha_i \mu\Big(w_i^{-1}\big(B(x,r)\big)\Big) \quad \text{for} \quad x \in X,\ r > 0.$$

Substituting $x = w_i(y)$, we obtain

$$\mu\big(B(w_i(y),r)\big) \geq \alpha_i \mu\big(B(y, r/L_i)\big) \quad \text{for} \quad x \in X,\ r > 0.$$

This implies that

$$Q_\mu(r) \geq \alpha_i Q_\mu(r/L_i) \quad \text{for} \quad r > 0,\ i \in J,$$

where Q_μ is given by (31). Consequently, the function Q_μ satisfies the inequality

$$Q_\mu(r) \geq \sup_J \alpha_i Q_\mu(r/L_i) \quad \text{for} \quad r > 0.$$

From the last inequality and Lemma 2 it follows that

$$Q_\mu(r) \geq c r^s,$$

for some $c > 0$. Consequently

$$\overline{\dim}_L \mu \leq s,$$

which completes the proof of Theorem 34. □

To obtain a lower estimate of concentration dimension of measure, we need more restrictive assumptions on transformations w_i. Let $I_1, \ldots, I_m$ be a partition of I and let $K \subset X$ be a nonempty set. Define

$$K_j = \bigcup_{i \in I_j} w_i(K) \quad \text{for} \quad j = 1, \ldots, m. \tag{34}$$

We say that the family $\{w_i : i \in I\}$ satsfies the *mixed Moran condition* with respect to the set K and the partition $I_1, \ldots, I_m$, if $K_j \subset K$ for $j = 1, \ldots, m$ and

$$\operatorname{dist}\big(K_{j_1}, K_{j_2}\big) = \inf\Big\{\rho(x,y) : x \in K_{j_1}, y \in K_{j_2}\Big\} > 0$$

for arbitrary $j_1, j_2 \in \{1, \ldots, m\}$, $j_1 \neq j_2$.

Similarly as Lemma 2, one can prove the following lemma.

Lemma 3. *Let $m_j \in (0,1)$ and $\beta_j > 0$ for $j = 1,\dots,m$, be given. Suppose that $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded increasing function such that*

$$\Phi(r) \le \max_{1\le j\le m} \Phi(r/m_j) \quad \text{for} \quad r \in (0,a).$$

Then there is $c > 0$ such that

$$\Phi(r) \le cr^s \quad \text{for} \quad r \in (0,a),$$

where

$$s = \min_{1\le j\le m} \frac{\log \beta_j}{\log m_j}.$$

Theorem 35. *Suppose that an IFS $\{(w_i,p_i) : i \in I\}$ has an invariant measure μ. Assume that the family $\{w_i : i \in I\}$ satisfies the mixed Moran condition with respect to the set $K = \operatorname{supp}\mu$ and a partition $I_1,\dots,I_m$. Moreover, assume that the functions w_i satisfy the condition*

$$\rho\big(w_i(x), w_i(y)\big) \ge l_i \rho(x,y) \quad \text{for } x,y \in X,\ i \in I, \tag{35}$$

where l_i are constants such that

$$0 < \inf_{i\in I_j} l_i < 1 \qquad \text{for} \quad j = 1,\dots .m.$$

Then

$$\underline{\dim}_L \mu \ge \min_{1\le j\le m} \frac{\log \beta_j}{\log m_j},$$

where

$$\beta_j = \sum_{i\in I_j} \sup_{x\in X} p_i(x) \quad \text{and} \quad m_j = \inf_{i\in I_j} l_i.$$

Proof. Let

$$a = \min\Big\{\operatorname{dist}\big(K_{j_1}, K_{j_2}\big)\,;\, j_1,\, j_2 \in \{1,\dots,m\},\ j_1 \ne j_2\Big\},$$

where K_j are given by (35).

Obviously

$$w_i^{-1}(x) \cap K = \emptyset \quad \text{for} \quad i \notin I_j \quad \text{and} \quad x \in X \quad \text{such that} \quad \rho(x,K_j) < a. \tag{36}$$

Since μ is invariant, it follows that

$$\mu(A) \le \sum_{j=1}^{m} \sum_{i\in I_j} \gamma_i \mu\big(w_i^{-1}(A)\big) \quad \text{for} \quad A \in \mathcal{B}(X), \tag{37}$$

where $\gamma_i = \sup_{x\in X} p_i(x)$. Set

$$A_0 = X \setminus \bigcup_{j=1}^{m} \overline{K}_j.$$

From (36) it follows that $\mu\big(w_i^{-1}(A_0)\big) = 0$ for $i \in I$. Thus $A_0 \cap K = \emptyset$ and so $K \subset \bigcup_{j=1}^{m} \overline{K}_j$.

Let $A \in \mathcal{M}(X)$ be such that $\operatorname{diam} A \le r < a$ and $\mu(A) > 0$. Then $A \cap \overline{K}_j = \emptyset$ for some $j \in \{1,\dots,m\}$. Since $\operatorname{diam} A < a$, by virtue of (36), there is

$$w_i^{-1}(A) \cap K = \emptyset \qquad \text{for} \quad i \notin I_j.$$

Consequently, inequality (37) reduces to

$$\mu(A) \le \sum_{i\in I_j} \gamma_i \mu\big(w_i^{-1}(A)\big) \quad \text{for} \quad A \in \mathcal{B}(X),$$

From (35) it follows that $\operatorname{diam} w_i^{-1}(A) \le l_i \operatorname{diam} A \le r/m_j$ for $i \in I_j$. Thus for $A \in \mathcal{B}(X)$ with $\operatorname{diam} w_i^{-1}(A) \le r < a$ there is $j \in \{1,\dots,m\}$ such that

$$\mu(A) \le \sum_{i\in I_j} \gamma_i Q_\mu(r/m_j) = \beta_j Q_\mu(r/m_j).$$

Consequently

$$Q_\mu(r) \le \max_{1\le j\le m} \beta_j Q_\mu(r/m_j) \qquad \text{for} \quad r \in (0,a)$$

and by Lemma 3

$$Q_\mu(r) \le c r^s,$$

where

$$s = \min_{1\le j\le m} \frac{\log \beta_j}{\log m_j}.$$

From the last inequality the statement of Theorem 35 follows immediately. □

REFERENCES

[1] A. Alexiewicz, W. Orlicz, *Some remarks on the existence and uniqueness of solutions of hyperbolic equations* $z_{xy} = f(x,y,z_x,z_y)$, Studia Math. **15** (1956), 201–215.

[2] W. Bartoszek, *The work of Professor Andrzj Lasota on asymptotic stability and recent results*, Opusc. Math. **28** (2008) 4, 395–413.

[3] D. Bielawski, *Pewne zagadnienia brzegowe i ich wlasności generyczne dla rownań różniczkowych*, Praca doktorska, Uniwersytet Gdanski, (1994), 1–65.

[4] S.N. Chow, A. Lasota, *On boundary value problem for ordinary differential equations*, J. Diff. Equat. **14** (1973), 326–337.

[5] S. Kasprzyk, J. Myjak, *On the existence of solutions of the Floquet problem for ordinary differential equations*, Zeszyty Nauk. UJ, Prace Mat. **13** (1969), 35–39.

[6] K. Krzyżewski, W. Szlenk, *On invariant measure for expanding differentiable mappings*, Stud. Math., **33** (1969), 83–92.

[7] A. Lasota, *Sur l'existence des solutions d'un problème d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **10** (1962), 523–528.

[8] A. Lasota, *Une généralisation du premier théor eme de Fredholm et ses application á la théorie des équations différentielles ordinaires*, Ann. Polon. Math. **18** (1966), 65–77.

[9] A. Lasota, *On two-point boundary value problem for systems of ordinary non-linear, first order differential equations*, Ann. Polon. Math. **29** (1975), 391–396.

[10] A. Lasota, *A variational principle for fractal dimension*, Nonlinear Anal. TMA **5** (2005), 1181–1193.

[11] A. Lasota, J. Myjak, *Markov operators and fractals*, Bull. Pol. Acad. Sci. Math. **45** (1997), 197–210.

[12] A. Lasota, J. Myjak, *Semifractals on Polish spaces*, Bull. Pol. Acad. Sci. Math. **46** (1998), 179–196.

[13] A. Lasota, J. Myjak, *Attractors of multifunctions*, Bull. Pol. Acad. Sci. Math. **48** (2000), 319–334.

[14] A. Lasota, J. Myjak, *On a dimension of measures*, Bull. Pol. Acad. Sci. Math. **50** (2002), 221–235.

[15] A. Lasota, J. Myjak, T. Szarek, *Markov operators with a unique invariant measure*, J. Math. Anal. Appl. **276** (2002), 343–356.

[16] A. Lasota, C. Olech, *An optimal solution of Nicoletti's boundary value problemn*, Ann. Polon. Math. **18** (1966), 131–139.

[17] A. Lasota, Z. Opial, *Sur une probléme d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **9** (1961), 667–671.

[18] A. Lasota, Z. Opial, *L'existence et l'unicité des solutions du probléme d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Ann. Polon. Math. **15** (1964), 253–271.

[19] A. Lasota, Z. Opial, *Sur les solutions périodiques des équations différentielles ordinaires*, Ann. Polon. Math. **16** (1964), 69–80.

[20] A. Lasota, Z. Opial, *An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **13** (1965), 781–788.

[21] A. Lasota, Z. Opial, *On the existence of solutions of linear problems for ordinary differential equations*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **14** (1966), 371–376.

[22] A. Lasota, Z. Opial, *On the existence and uniqueness of solutions of nonlinear functional equations*, Coll. Math. **18** (1967), 1–5.

[23] A. Lasota, Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second order differential equation*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **15** (1967), 797–800.

[24] A. Lasota, J. Yorke, *Existence of solutions of two-point boundary value problems for nonlinear systems*, J. Diff. Equat. **11** (1973), 509–518.

[25] A. Lasota, J. Yorke, *The generic property of existence of solutions of differential equations in Banach spaces*, J. Diff. Equat. **13** (1973), 1–12.

[26] A. Lasota, J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. **186** (1973), 481–488.

[27] A. Lasota, J. Yorke, *On the existence of invariant measures for transformations with strictly turbulent trajectories*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Ast. Phys. **25** (1977), 233–238.

[28] A. Lasota, J. Yorke, *Lower bound technique for Markov operators and iterated functions systems*, Random Comp. Dynamics, **2** (1994), 41–77.

[29] M.C. Mackey, *Adventures in Poland: Having fun and doing research with Andrzj Lasota*, (2007).

[30] W. Orlicz, *Zur Theorie des Differenttialgleichung*, Bull. Pol. Acad. Sci. Math. (1932), 221–228.

[31] W. Parry, *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.

[32] A. Rényi, *Representation for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.

[33] P. Rusek, *Professor Andrzej Lasota's theories in engineering*, Opuscula Math. 28 (2008) 4, 415–432.

[34] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Appl. Math. **8** Wiley, New York, 1960.

[35] R. Zaharopol, *Invariant Probabilities of Markov–Feller Operators and Their Supports*, Birkhauser Verlag, Basel 2005.

Józef Myjak
myjak@univaq.it

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

Received: April 23, 2007.
Accepted: May 4, 2008.