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OPEN TRAILS IN DIGRAPHS

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Abstract. It has been shown in [S. Cichacz, A. Görlich, *Decomposition of complete bipartite graphs into open trails*, Preprint MD **022**, (2006)] that any bipartite graph $K_{a,b}$, is decomposable into open trails of prescribed even lengths. In this article we consider the corresponding question for directed graphs. We show that the complete directed graphs $\overleftrightarrow{K}_n$ and $\overleftrightarrow{K}_{a,b}$ are arbitrarily decomposable into directed open trails.

Keywords: trail, graph decomposition, bipartite graph.

Mathematics Subject Classification: 05C70.

1. INTRODUCTION

Consider a simple graph G whose size we denote by $\|G\|$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. If G is a graph, $\overleftrightarrow{G}$ will denote the digraph obtained from G by replacing each edge $xy \in E(G)$ by the pair of arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$.

Here and subsequently, a directed trail $\overrightarrow{T}$ of length n will be identified with a sequence $(v_1, v_2, \ldots, v_{n+1})$ of vertices of $\overrightarrow{T}$ such that $\overrightarrow{v_i v_{i+1}}$ are distinct arcs of $\overrightarrow{T}$ for $i = 1, 2, \ldots, n$. Notice that we do not require the v_i to be distinct. A trail $\overrightarrow{T}$ is closed if $v_1 = v_{n+1}$ and $\overrightarrow{T}$ is open if $v_1 \neq v_{n+1}$.

A sequence of positive integers $\tau = (t_1, t_2, \ldots, t_p)$ is called *admissible for a digraph* $\overleftrightarrow{G}$ if it adds up to $\|\overleftrightarrow{G}\|$ and for each $i \in \{1, \ldots, p\}$ there exists an open directed trail of length t_i in $\overleftrightarrow{G}$. Let $\tau = (t_1, t_2, \ldots, t_p)$ be an admissible sequence for $\overleftrightarrow{G}$. If $\overleftrightarrow{G}$ is arc-disjointly decomposed into directed open trails $\overrightarrow{T}_1, \overrightarrow{T}_2, \ldots, \overrightarrow{T}_p$ of lengths $t_1, t_2, \ldots, t_p$ respectively, then τ is called *realizable in* $\overleftrightarrow{G}$ and the sequence $(\overrightarrow{T}_1, \ldots, \overrightarrow{T}_p)$ is said to be *a* $\overleftrightarrow{G}$*-realization of* τ or *a realization of* τ *in* $\overleftrightarrow{G}$. If for each admissible sequence τ for $\overleftrightarrow{G}$ there exits a $\overleftrightarrow{G}$-realization of τ, then the digraph $\overleftrightarrow{G}$ is called *arbitrarily decomposable into open trails.*

Let $\overleftrightarrow{K}_n$ be a complete digraph and $\overleftrightarrow{K}_{a,b}$ be a complete bipartite digraph with two sets of vertices A and B such that $|A| = a$ and $|B| = b$. In our paper we prove a necessary and sufficient condition for digraphs $\overleftrightarrow{K}_n$ and $\overleftrightarrow{K}_{a,b}$ to be decomposable into arc-disjoint open trails of positive lengths $t_1, t_2, ...t_p$ for any admissible sequence $\tau = (t_1, t_2, ...t_p)$.

Observe that in $G = \overleftrightarrow{K}_n$ or $G = \overleftrightarrow{K}_{a,b}$ there does not exist an open trail of length $\|G\|$, so we can assume that $p > 1$.

Such problems were investigated in [4]:

Theorem 1.1 ([4])**.** *A complete bipartite graph $K_{a,b}$ is arbitrarily decomposable into open trails if and only if one of the following conditions holds:*

1^0 $a = 1$ or
2^0 a and b are both even.

For oriented graphs the similar problem of decomposition was considered by Meszka and Skupień ([6]). They showed that complete multidigraphs are arbitrarily decomposable into nonhamiltonian paths.

The first result on the topic of the arbitrary decomposition of graphs into trails is due to Balister, who proved that if $G = K_n$ for n odd or $G = K_n - I$, where I is a 1-factor in K_n, for n even, then G is arbitrarily decomposable into closed trails ([1]). Horňák and Woźniak ([5]) showed that complete bipartite graphs $K_{a,b}$ for a, b even are also arbitrarily decomposable into closed trails. The notion of arbitrarily decomposable graphs into closed trails were generalized to oriented graphs (see Balister [2] and Cichacz [3]). Balister proved a necessary and sufficient condition for a complete digraph $\overleftrightarrow{K}_n$.

Theorem 1.2 ([2])**.** *If $\sum_{i=1}^{p} t_i = 2\binom{n}{2}$ and $t_i \geqslant 2$ for $i = 1, \ldots, p$, then $\overleftrightarrow{K}_n$ can be decomposed as an arc-disjoint union of directed closed trails of lengths $t_1, t_2, \ldots, t_p$, except in the case when $n = 6$ and all $t_i = 3$.*

Whereas Cichacz ([3]) showed that complete directed graphs $\overleftrightarrow{K}_{a,b}$ are arbitrarily decomposable into closed directed trails.

In this article we consider the corresponding question for open directed trails.

2. DECOMPOSITION OF COMPLETE BIPARTITE DIGRAPHS

There is no loss of generality in assuming that $a \leqslant b$ for each complete bipartite digraph $\overleftrightarrow{K}_{a,b}$. For simplicity of notation let $s_i = t_1 + \ldots + t_i$.

Before we prove the main result in that section we will need the following lemma.

Lemma 2.1. *Let $t_1, \ldots, t_p$ be positive odd integers and let $\overrightarrow{G}$ be a bipartite directed trail of size $\|G\| = t_1 + \ldots + t_p$. Then, $\overrightarrow{G}$ can be decomposed into p open directed trails of lengths $t_1, \ldots, t_p$.*

Proof. Write $\overrightarrow{G} = (x_0, x_1, \ldots, x_{s_p})$ and define the decomposition of $\overrightarrow{G}$ as follows: $\overrightarrow{T}_1 = (x_0, x_1, \ldots, x_{t_1})$, $\overrightarrow{T}_2 = (x_{t_1}, x_{t_1+1}, \ldots, x_{t_1+t_2}), \ldots, \overrightarrow{T}_p = (x_{t_1+\ldots+t_{p-1}}, \ldots, x_{s_p})$.

Because in a bipartite digraph there is no closed directed trail of odd size, all trails defined above are open. □

Theorem 2.2. *Let* $G = \overleftrightarrow{K}_{a,b}$ *(*$b \geq a \geq 1$*) and let* A *and* B *be the partition sets of* G *with* $|A| = a$ *and* $|B| = b$*. Let* $\tau = (t_1, \ldots, t_p)$ *be an admissible sequence for* G*. Then, unless* $a = b = 2$ *and* $\tau = (2,2,4)$*,* $(2,4,2)$ *or* $(4,2,2)$*,* G *has a realization of* τ *such that each* T_i *has at leat one endvertex in* B*.*

Proof. We will argue by induction on a. For $a = 1$ Theorem 2.2 is true. Thus let $a \geq 2$. If $a = b = 2$ then one can prove the theorem by inspection. From now on assume that $b \geq 3$ (even if $a = 2$). Let s denote the sum of odd terms in τ. Suppose first that $s > 0$. Notice that any trail of odd length has an endvertex in B. Thus, if $s = 2ab$ we obtain a proper $\overleftrightarrow{K}_{a,b}$-realization of τ by Lemma 2.1. We may assume $s < 2ab$. Note that s is even, since the size of G is even. Let $(t'_1, \ldots, t'_q)$ be a sequence obtained form τ by deleting all odd terms. If we can find a realization $(S, T'_1, \ldots, T'_q)$ of a sequence $(s, t'_1, \ldots, t'_q)$ such that each trail has endvertices in B, then we obtain a desired realization of τ in G by applying Lemma 2.1 to S.

In the view of the above paragraph, we may assume that all terms of τ are even. Suppose first that there exists $t_j > 2b$. We may assume without loss of generality that $j = p$. Let $t'_p = t_p - 2b$. By induction we can find a realization $(\overrightarrow{T}_1, \overrightarrow{T}_2, \ldots, \overrightarrow{T}_{p-1}, \overrightarrow{T}'_p)$ of the sequence $(t_1, t_2, \ldots, t_{p-1}, t'_p)$ in $\overleftrightarrow{K}_{a-1,b}$. Putting $\overrightarrow{T}_p = \overrightarrow{T}'_p \cup \overleftrightarrow{K}_{1,b}$ we obtain a proper $\overleftrightarrow{K}_{a,b}$-realization of τ.

From now on suppose that $t_i \leqslant 2b$ for all i. We assume first that there exists $t_i < 2b$. In this case at least two terms of the sequence τ satisfy $t_i < 2b$. We may assume that $t_1 < 2b$ and $t_p < ab$. We consider G as an arc-disjoint union of $H = \overleftrightarrow{K}_{a-1,b}$ and $F = \overleftrightarrow{K}_{1,b}$ of sizes $2(a-1)b$ and $2b$, respectively. Let j be the index such that $s_{j-1} \leqslant 2b$ and $s_j > 2b$. Therefore $2 \leqslant j \leqslant p-1$. If $s_{j-1} = 2b$ then $j \geqslant 3$ and we obtain a realization by applying the induction assumption to F with $(t_1, t_2, \ldots, t_{j-1})$ and to H with $(t_j, t_{j+1}, \ldots, t_p)$. Thus we may assume $s_j < 2b$. Let us introduce $t'_j = 2b - s_{j-1}$, $t''_j = t_j - t'_j$. By induction we can find a realization $(\overrightarrow{T}_1, \ldots, \overrightarrow{T}_{j-1}, \overrightarrow{T}'_j)$ of $(t_1, \ldots, t_{j-1}, t'_j)$ in F and a realization $(\overrightarrow{T}''_j, \overrightarrow{T}_{j+1}, \ldots, \overrightarrow{T}_p)$ of $(t''_j, t_{j+1}, \ldots, t_p)$, such that each trail has both endvertices in B (since all trails have even length). Since $b \geqslant 3$ we can permute vertices of B in F in such a way that the trail $\overrightarrow{T}'_j$ has precisely one endvertex common with $\overrightarrow{T}''_j$ forming an open trail $\overrightarrow{T}_j$ of length t_j.

Hence, we are left with the case $t_1 = t_2 = \ldots = t_p = 2b$. Let $t''_{p-1} = 2$, $t''_p = 2b - 2$ and $t'_{p-1} = t_{p-1} - 2$, $t'_p = 2$. By induction we can find a realization $(\overrightarrow{T}_1, \overrightarrow{T}_2, \ldots, \overrightarrow{T}_{p-2}, \overrightarrow{T}'_{p-1}, \overrightarrow{T}'_p)$ of the sequence $(t_1, t_2, \ldots, t_{p-2}, t'_{p-1}, t'_p)$ in $\overleftrightarrow{K}_{a-1,b}$. Putting $\overrightarrow{T}_{p-1} = \overrightarrow{T}'_{p-1} \cup \overleftrightarrow{K}_{1,1}$, $\overrightarrow{T}_p = \overrightarrow{T}'_p \cup \overleftrightarrow{K}_{1,b-1}$ we obtain a desired $\overleftrightarrow{K}_{a,b}$-realization of τ. □

3. DECOMPOSITION OF COMPLETE DIGRAPHS

Let $\tau = (t_1, \ldots, t_p)$ be an admissible sequence for G. We can assume that $t_1 \leq \ldots \leq t_p$. We shall write $(t_1^{r_1}, \ldots, t_l^{r_l})$ for the sequence $(\underbrace{t_1, \ldots, t_1}_{r_1}, \ldots, \underbrace{t_l, \ldots, t_l}_{r_l})$.

Using Theorem 2.2 we prove the analogous result for complete digraphs.

Theorem 3.1. *If $\tau = (t_1, \ldots, t_p)$ is an admissible sequence for $G = \overleftrightarrow{K}_n$ and $n \leq 4$, then τ is realizable in G, except in the case when $n = 3$ and all $t_i = 2$.*

Proof. Let $\tau = (t_1, \ldots, t_p)$ be an admissible sequence for $G = \overleftrightarrow{K}_n$ $(p > 1)$. We will argue by induction on n. The basic idea of the proof is to consider $\overleftrightarrow{K}_n$ as an arc-disjoint union of $\overleftrightarrow{K}_{1,n-1}$ and $\overleftrightarrow{K}_{n-1}$, each of which have sizes $2(n-1)$ and $(n-1)(n-2)$, respectively. Let $A = \{x\}$ and B be the partition sets of $\overleftrightarrow{K}_{1,n-1}$ such that $|A| = 1$ and $|B| = n-1$.

We start our analysis by dealing with $n \leqslant 3$. The digraph $\overleftrightarrow{K}_2$ is trivially arbitrarily decomposable into open trails. One may check that every admissible sequence τ for the digraph $\overleftrightarrow{K}_3$, except the sequence $\tau = (2^3)$ is $\overleftrightarrow{K}_3$-realizable.

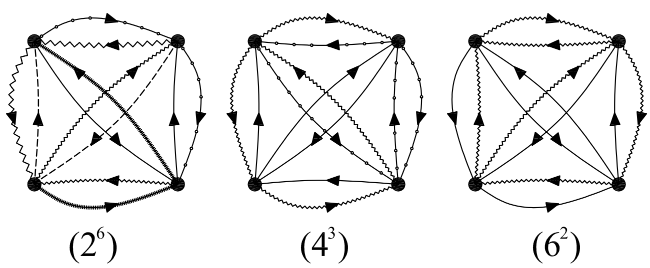

Fig. 1. $\overleftrightarrow{K}_4$-realizations of three sequences

Suppose now that $n \geqslant 4$.

The sequences (2^6), (4^3) and (6^2) are $\overleftrightarrow{K}_4$-realizable, see Figure 1. Obviously every admissible sequence τ such that $t_p = 6$ is realizable in $\overleftrightarrow{K}_4$ (see the realization of (6^2) in Figure 1). The same is with any admissible sequence for K_4 such that $t_p = 2$ (see the realization of (2^6) in Figure 1)

Let $\tau = (t_1, \ldots, t_p)$ be an admissible sequence for $\overleftrightarrow{K}_n$ different than described above. Notice that, since an admissible sequence τ is non-decreasing, $t_p \geq 3$ in any admissible sequence for K_4. We will consider now two cases.

Case 1: For some $i \in \{1, \ldots, p\}$, $s_{i-1} < 2(n-1)$ and $s_i > 2(n-1)$ $(s_0 = 0)$.
Let $\tau' = (t_1, \ldots, t_{i-1}, t_i')$ and $\tau'' = (t_i'', t_{i+1}, \ldots, t_p)$, where $t_i' = 2(n-1) - s_{i-1} \geqslant 1$ and $t_i'' = t_i - t_i' \geqslant 1$.

Assume first that $i = 1$, then $t_i' = 2(n-1)$. By induction we can find a $\overleftrightarrow{K}_{n-1}$-realization $(\overrightarrow{T}_1'', \overrightarrow{T}_2 \ldots, \overrightarrow{T}_p)$ of τ' (notice that $n > 4$). Let $\overrightarrow{T}_1 = \overrightarrow{T}_1'' \cup \overleftrightarrow{K}_{1,n-1}$ and we obtain a $\overleftrightarrow{K}_n$-realization of τ.

Similarly if $i = p$ then we can find a realization $(\overrightarrow{T}_1, \overrightarrow{T}_2 \dots, \overrightarrow{T}_{p-1}, \overrightarrow{T}'_p)$ of τ' in $\overleftrightarrow{K}_{1,n-1}$ by Theorem 2.2 and we put $\overrightarrow{T}_p = \overrightarrow{T}'_p \cup \overleftrightarrow{K}_{n-1}$ obtaining a $\overleftrightarrow{K}_n$-realization of τ.

For $1 < i < p$ we can find a $\overleftrightarrow{K}_{1,n-1}$-realization $(\overrightarrow{T}_1, \dots, \overrightarrow{T}_{i-1}, \overrightarrow{T}'_i)$ of τ' by Theorem 2.2 and a $\overleftrightarrow{K}_{n-1}$-realization $(\overrightarrow{T}''_i, \overrightarrow{T}_{i+1} \dots, \overrightarrow{T}_p)$ of τ'' by induction (notice that $\tau'' \neq (2^3)$). We define the open directed trail $\overrightarrow{T}'_i$ as $(v_1, \dots, v_{t'_i+1})$ and $\overrightarrow{T}''_i$ as $(w_1, \dots, w_{t''_i+1})$. Observe that if t'_i is odd, then $v_1 \in B$ or $v_{t'_i+1} \in B$, whereas if t'_i is even, then $v_1, v_{t'_i+1} \in B$. Assume first that $v_1 \in B$. Since $n - 1 \geqslant 3$, we may choose the realization of τ' in such a way that $v_1 = w_{t''_i+1}$ and $v_{t'_i+1} \neq w_1$. In such a case, if we denote $\overrightarrow{T}_i$ as $(w_1, \dots, w_{t''_i+1}, v_2, \dots, v_{t'_i+1})$, then $\overrightarrow{T}_i$ is an open directed trail of length t_i and the sequence $(\overrightarrow{T}_1, \dots, \overrightarrow{T}_p)$ is a $\overleftrightarrow{K}_n$-realization of τ. Hence, let $v_1 \notin B$. It implies that $v_{t'_i+1} \in B$ and we can assume that $v_{t'_i+1} = w_1$ and $\overrightarrow{T}_i = (v_1, \dots, v_{t'_i+1}, w_2, \dots, w_{t''_i+1}))$. The sequence $(\overrightarrow{T}_1, \dots, \overrightarrow{T}_p)$ is a $\overleftrightarrow{K}_n$-realization of τ.

Case 2: For some $i \in \{1, \dots, p\}$, $s_i = 2(n-1)$. Let us define $\tau' = (t_1, \dots, t_i)$ and $\tau'' = (t_{i+1}, \dots, t_p)$.

Suppose first that $n = 4$. Recall that $3 \leqslant t_p < 6$. Therefore the sequences τ' and τ'' are realizable in $\overleftrightarrow{K}_{1,3}$ and $\overleftrightarrow{K}_3$, respectively.

From now on let $n \geqslant 5$. Assume first that $i > 1$. If $p > i + 1$, then by Theorem 2.2 we can find a realization $(\overrightarrow{T}_1, \dots, \overrightarrow{T}_i)$ of τ' in $\overleftrightarrow{K}_{1,n-1}$ and $(\overrightarrow{T}_{i+1} \dots, \overrightarrow{T}_p)$ is a $\overleftrightarrow{K}_{n-1}$-realization of τ'' by induction. If $p = i+1$ then $t_p > 2(n-1) = \| \overleftrightarrow{K}_{1,n-1} \|$ (because $n \geq 5$). So we proceed in the same way as in Case 1 (splitting t_p into two parts this time).

Suppose now that $i = 1$. It implies $t_1 = 2(n-1)$. If additionally $t_1 < t_p$, we have $s_i - t_1 + t_p < 2(n-1)$ and we again proceed in the same way as in Case 1 (splitting t_p into two parts). Hence, we are left with the case $t_1 = t_p = 2n - 2 \geq 8$.

By induction we can find a $\overleftrightarrow{K}_{n-1}$-realization $(\overrightarrow{T}_2, \overrightarrow{T}_3, \dots, \overrightarrow{T}_p)$ of $(t_2, t_3, \dots, t_p)$ such that $\overrightarrow{T}_2$ ends with an arc $\overrightarrow{yz}$. Let $\overrightarrow{T}_1$ be a closed trail $\overleftrightarrow{K}_{1,n-1}$ and assume (without loss of generality) that the trail finishes with $\overrightarrow{yx}$. Changing $\overrightarrow{yx}$ to $\overrightarrow{yz}$ in $\overrightarrow{T}_1$ and $\overrightarrow{yz}$ to $\overrightarrow{yx}$ in $\overrightarrow{T}_2$ we complete the proof. $\square$

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