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APPLICATION OF SOME GRAPH INVARIANTS TO THE ANALYSIS OF MULTIPROCESSOR INTERCONNECTION NETWORKS*

Dragoš CVETKOVIĆ, Tatjana DAVIDOVIĆ
Mathematical Institute, Serbian Academy of Science and Arts
Kneza Mihaila 36, P.O.Box 367, 11000 Belgrade, Serbia
ecvetkod@etf.bg.ac.yu, tanjad@mi.sanu.ac.yu

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Abstract: Let G be a graph with diameter D, maximum vertex degree Δ, the largest eigenvalue λ_1 and m distinct eigenvalues. The products $m\Delta$ and $(D+1)\ \lambda_1$ are called the tightness of G of the first and second type, respectively. In the recent literature it was suggested that graphs with a small tightness of the first type are good models for the multiprocessor interconnection networks. We study these and some other types of tightness and some related graph invariants and demonstrate their usefulness in the analysis of multiprocessor interconnection networks. Tightness values for graphs of some standard interconnection networks are determined. We also present some facts showing that the tightness of the second type is a relevant graph invariant. We prove that the number of connected graphs with a bounded tightness is finite.

Keywords: Multiprocessor systems, interconnection topologies, spectra of graphs, maximum vertex degree, diameter.

1. INTRODUCTION

Usually, a multiprocessor system consists of a number of processing units, each having its own local memory [10]. This type of multiprocessor system is known as a distributed memory system. Processing units may be identical or with different characteristics (as it is the case in heterogeneous computer networks).

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The processors within the distributed memory multiprocessor system communicate by sending/receiving messages through the communication links. The structure of communication links defines the topology of the multiprocessor system. The main drawback of multiprocessor systems is the communication overhead [3,18], the time required to exchange data between different processing units.

The easiest way to minimize the communication time within the multiprocessor system is to use completely connected topologies, the so called *parallel processors*. Unfortunately, it may not be feasible to design such topologies for many reasons [18]. Therefore, different interconnection networks have been proposed. These networks have to satisfy two contradictory properties: to minimize the "number of wires" and to maximize the data exchange rate. This means that the paths connecting each two processors have to be as short as possible while the average number of connections per processor has to be as small as possible.

Of course, multiprocessor interconnection networks can be modeled by (undirected, connected) graphs [15,16]. Vertices of these graphs represent the processors, while edges denote the connection links between adjacent processors. The two main parameters of the graph, which play an important role in the design of multiprocessor topologies are the maximum vertex degree Δ and the diameter D. In other words, Δ directly corresponds to the number of adjacent processors (vertices in the graph model), while D represent the length of the longest path in processor graph, i.e. the maximum distance between two processors.

Recently, the link between the design of multiprocessor topologies and the theory of graph spectra [8] has been recognized [9]. The main conclusion is that the product of the number m of distinct eigenvalues of a graph adjacency matrix and Δ has to be as small as possible. We call this product the *tightness of the first type* for a graph. Here we introduce the *tightness of the second type* as the product $(D+1)\lambda_1$, where λ_1 is the largest eigenvalue of a graph G.

In fact, we shall define various types of graph tightness, and investigate the relation between the tightness values and the suitability of the corresponding multiprocessor architecture. We show that the graphs with a small tightness of the second type are suitable for the design of multiprocessor topologies.

The paper is organized as follows. The next section contains basic definitions from graph theory, and especially from the theory of graph spectra. Definitions and properties of some new graph invariants, with common name *tightness* are given in Section 3. Some of the widely used interconnection topologies are described in Section 4 and their tightness values are discussed. In Section 5, we investigate quasi-regular trees and show that, according to the tightness of the second type, they can be suitable for the multiprocessor architectures. Graphs with few eigenvalues are reviewed in Section 6 with an indication that they can provide some sporadic examples of graphs with small tightness values.

2. PRELIMINARIES FROM GRAPH THEORY

The graph $G=(V,E)$ consists of a non-empty finite set V whose elements we call *vertices*, while $E \subset V \times V$ represents the connections between vertices and its elements are called *edges*. The number of elements in V we denote by n, i.e. $|V|=n$

and it represents the *order* of a graph G. Two vertices are called *adjacent* if they are connected by an edge. The *adjacency matrix A* is used to represent the adjacency relation between vertices. The element a_{ij} is equal to 1 if vertices i and j are connected by an edge, otherwise $a_{ij} = 0$.

The number of neighbors vertex i connected to is called *vertex degree* and it is denoted by d_i. Maximum vertex degree $\Delta = d_{\max}$ is the maximum over all $d_i, i = 1, 2..., n$. If there exists a path between any two vertices in G we say that graph G is *connected*. Otherwise, G is *disconnected*. Minimum number of edges along a path connecting two vertices is called *distance* between these two vertices. Maximum distance between two vertices in a connected graph is called *diameter* and it is denoted by D.

The characteristic polynomial of the adjacency matrix A of a graph G defined as $\det(xI - A)$ is called the *characteristic polynomial* of G and is denoted by $P_G(x)$. The eigenvalues of A (i.e. the zeros of $P_G(x)$) and the spectrum of A (which consists of the n eigenvalues) are also called the *eigenvalues* and the *spectrum* of G, respectively. The eigenvalues of a graph G with n vertices are usually denoted by $\lambda_1, ..., \lambda_n$, they are real because A is symmetric. Unless we indicate otherwise, we shall assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and the largest eigenvalue λ_1 is called the *index* of G. The number of distinct eigenvalues is denoted by m.

In graph theory and in the theory of graph spectra, some special types of graphs are studied in detail and their characteristics are well known and summarized in literature (see, for example, [8]). We emphasize *complete graphs* K_n of order n with each two vertices connected by an edge (the number of edges is equal to $n(n-1)/2$). The other extreme is a *path* P_n, containing n vertices and n-1 edges. Connecting two end vertices of a path P_n by an edge we obtain a *circuit* C_n. *Complete bipartite graphs* $K_{n_1 n_2}$ consist of n_1+n_2 vertices divided into two sets of the cardinalities n_1 and n_2 with the edges connecting each vertex from one set to all the vertices in the other set. This means that the number of edges is $n_1 n_2$. More generally, *bipartite* graphs consist of two sets of vertices with the edges connecting vertices from one set to vertices in the other set. A special case of complete bipartite graph is a *star*, more precisely, $S_n = K_{1,n-1}$ containing central vertex connected to all the others by n-1 edges.

These types of graphs are used in various combinations for multiprocessor system design. In Section 4 we shall analyze some widely used multiprocessor interconnection networks trying to describe their suitability by using tools from the graph theory.

3. VARIOUS TYPES OF TIGHTNESS OF A GRAPH

As we already pointed out, the graph invariant obtained as the product of the number of distinct eigenvalues m and the maximum vertex degree Δ of G has been investigated in [9] related to the design of multiprocessor topologies. The main conclusion of [9] with respect to the multiprocessor design and, particularly to the load balancing within given multiprocessor systems was the following: if $m\Delta$ is small for a given graph G, the corresponding multiprocessor topology was expected to have good communication properties and has been called *well-suited*. It has been pointed out that there exists an efficient algorithm which provides optimal load balancing within m-1 computational steps. The graphs with large $m\Delta$ were called *ill-suited* and were not considered suitable for design multiprocessor networks.

On the other hand there are many known widely used multiprocessor topologies based on graphs which appear to be ill-suited according to [9]. Such an example is the star graph $S_n = K_{1,n-1}$.

In order to extend the application of the theory of graph spectra to the design of multiprocessor topologies, in this paper we define some other types of tightness and investigate their suitability for describing the corresponding interconnection networks.

Definition 1. *The* tightness t_1 *of a graph G is defined as the product of the number of distinct eigenvalues m and the maximum vertex degree* Δ *of* G, *i.e.* $t_1(G) = m\Delta$.

As we can see, tightness is defined as the product of one spectral invariant m and one structural invariant Δ. Therefore, we will refer to this type of tightness as the *mixed tightness*. In the following, we introduce two alternative (homogeneous) definitions of tightness, the *structural* and the *spectral* one. Moreover, we introduce another mixed tightness, and therefore t_1 is called *type one mixed tightness*.

Definition 2. Structural tightness $stt(G)$ *is product* $(D+1)\Delta$ *where D is the diameter and* Δ *is the maximum vertex degree of a graph G*.

Definition 3. Spectral tightness $spt(G)$ *is product of the number of distinct eigenvalues m and the largest eigenvalue* λ_1 *of a graph G*.

Definition 4. Second type mixed tightness t_2(G) *is defined as a function of the diameter D of G and the largest eigenvalue* λ_1, i.e. $t_2(G) = (D+1)\lambda_1$.

If the type of tightness is not relevant for the discussion, all four types of tightness will be called, for short, *tightness*. In general discussions we shall use t_1, t_2, *stt*, *spt* independently of a graph to denote the corresponding tightness. An alternative term for tightness could be the word *reach*.

The use of the largest eigenvalue, i.e. the index, of a graph instead of the maximal vertex degree in description of multiprocessor topologies seems to be appropriate for several reasons. By Theorem 1.12 of [8] the index of a graph is equal to a kind of mean value of vertex degrees, i.e. to the so called dynamical mean value, which takes into account not only immediate neighbors of vertices but also neighbors of neighbors, etc. The index is also known to be a measure of the extent of branching of a graph, and in particular of a tree (see [6] for the application in chemical context and [5] for a treatment of directing branch and bound algorithms for the travelling salesman problem). The index, known also as the spectral radius, is a mathematically very important graph parameter as presented, for example, in a survey paper [7].

According to the well-known inequality ([8], p. 85)

$d_{\min} \leq \bar{d} \leq \lambda_1 \leq d_{\max} = \Delta,$ we have that $\mathrm{spt}(G) \leq t1(G)$.

Here $d_{\min}$ and $d_{\max}$ denote minimum and maximum vertex degree, respectively and $\overline{d}$ is used to denote the average value of vertex degrees.

The relation between $stt(G)$ and $t_1(G)$ is $t_1(G) \geq stt(G)$, since $m \geq 1+D$ (see Theorem 3.13. from [8]). For distance-regular graphs [2] $m = 1+D$ holds.

We also have $t_2(G) \leq spt(G)$ and $t_2(G) \leq stt(G)$.

The two homogeneous tightness values appear to be incomparable. To illustrate this, let us consider star graph with n=5 vertices (S_5=$K_{1,4}$) given on Figure 1a, and the graph $\overline{S}_5$ obtained if new edges are added to the star graph as it is shown on Figure 1b.

Figure 1: a) Star graph with n=5 vertices and b) extended star graph

From [8], pp. 272—275, Table 1, we can see that for S_5 it holds D=2, Δ=4, m=3 and λ_1=2 and hence $spt(S_5)=m\lambda_1=6 < 12=(D+1)\Delta=stt(S_5)$. On the other hand for the graph $\overline{S}_5$ we have D=2, Δ=4, m=4 and λ_1=3.2361 yielding to $spt(\overline{S}_5) > stt(\overline{S}_5)$.

The above mentioned table shows that this is not the only example. For n=5, there exist 21 different graphs. Only for 3 of them the two homogeneous tightness have the same value, while $stt(G)$ is smaller for 9 graphs, and for the remaining 9 $spt(G)$ has a smaller value.

For two graph invariants $\alpha(G)$ and $\beta(G)$ we shall say that the relation $\alpha(G) \prec \beta(G)$ holds if $\alpha(G) \leq \beta(G)$ holds for any graph G. With this definition we have the Hasse diagram for the relation $\prec$ between various types of tightness given on Figure 2.

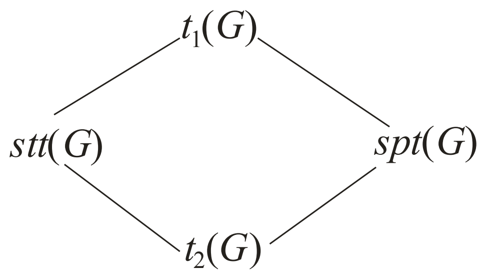


Figure 2: Partial order relation between different types of graph tightness

In order to study the behavior of a property or invariant of graphs when the number of vertices varies, it is important for that property (invariant) to be scalable.

Scalability means that for each n there exists a graph with n vertices having that property (invariant of certain value).

A family of graphs is called *scalable* if for any n there exists an n-vertex graph in this family. For example, in [9] the scalable families of sparse graphs (maximal vertex degree $O(\log n)$) with small number of distinct eigenvalues are considered. Obviously, sometimes it is difficult to construct scalable families of graphs for a given property.

We present a theorem which seems to be of fundamental importance in the study of the tightness of a graph.

Theorem 1. *For any kind of tightness, the number of connected graphs with a bounded tightness is finite.*

Proof: Let $t(G) \le a$ for a given positive integer a, where $t(G)$ stands for any kind of tightness. In all four cases, we shall prove that there exists a number b such that both diameter D and maximum vertex degree Δ are bounded by b. We need two auxiliary results from the theory of graph spectra.

By Theorem 3.13. from [8] we have $D \le m$-1 for the diameter D of G. For the largest eigenvalue λ_1 of a graph G the inequality $\lambda_1 \ge \sqrt{\Delta}$ holds (cf. [8], p. 112).

Now, $t(G) \le a$ implies

Case $t(G)=t_1(G)$. $m\Delta \le a$, $m \le a$ and $\Delta \le a$, $D \le a$-1, and we can adopt $b = a$;
Case $t(G) = stt(G)$. $(D+1)\Delta \le a$, $D \le a$-1 and $\Delta \le a$, here again $b = a$;
Case $t(G) = spt(G)$. $m\lambda_1 \le a$, $m \le a$ and $\lambda_1 \le a$, $D \le a$-1, and $\Delta \le \lambda_1^2 \le a^2$, and now $b = a^2$;
Case $t(G)=t_2(G)$. $(D+1)\lambda_1 \le a$, $D \le a$-1, and $\Delta \le a^2$, and again $b = a^2$.

It is well known that for the number of vertices n in G the following inequality holds

$$n \le 1 + \Delta + \Delta(\Delta-1) + \Delta(\Delta-1)^2 + \cdots + \Delta(\Delta-1)^{D-1}.$$

(Here, we have enumerated all vertices of G starting from a particular vertex and counting maximal numbers of neighbors at particular distances from that vertex.) Based on this relation and assuming that both D and Δ are bounded by a number b, we have

$$\begin{aligned} n \quad &< 1 + \Delta + \Delta^2 + \Delta^3 + \cdots + \Delta^D \le 1 + \Delta + \Delta^2 + \Delta^3 + \cdots + \Delta^b \\ &\le 1 + b + b^2 + b^3 + \cdots + b^b. \end{aligned}$$

In such a way we proved that the number of vertices of a connected graph with a bounded tightness is bounded. Therefore, it is obvious that it can be only finitely many such graphs and the theorem is proved. □

Corollary 1. The tightness of graphs in any scalable family of graphs is unbounded.

Corollary 2. Any scalable family of graphs contains a sequence of graphs, not necessarily scalable, with increasing tightness diverging to $+\infty$.

The asymptotic behavior of the tightness, when n tends towards $+\infty$, is of particular interest in the analysis of multiprocessor interconnection networks.

In a forthcoming paper [4] we shall present further results involving theoretical analysis of all four types of tightness. Here we reproduce just one theorem from that article.

Theorem. *Among connected graphs G on n ($n \le 10$) vertices the value $t_1(G)$ is minimal for following graphs*:

K_2, *for* $n = 2$,
K_3, *for* $n = 3$,
K_4, *for* $n = 4$,
C_5, *for* $n = 5$,
C_6, *for* $n = 6$,
C_7, *for* $n = 7$,
C_8, *for* $n = 8$,
C_9, *for* $n = 9$,
the Petersen graph, for $n = 10$.

The theorem is obtained by theoretical means without using a computer search.

4. A SURVEY OF FREQUENTLY USED INTERCONNECTION NETWORKS

In this section we survey the graphs that are often used to model multiprocessor interconnection networks and examine the corresponding tightness values. Since the tightness is product of two positive quantities, it is necessary for both of them to have small values to assure a small value of tightness.

1. An example of such a graph is the *d*-dimensional hypercube $Q(d)$. It consists of $n = 2^d$ vertices, each of them connected with d neighbors. The vertices are labeled starting from 0 to n-1 (considered as binary numbers). An edge connects two vertices with binary numbers differing in only one bit. For these graphs we have $m = d+1$, $D = d$, $\Delta = d$, $\lambda_1 = d$ and all four types of tightness are equal to $(d+1)d=O((\log n)^2)$.

2. Another example is *butterfly* graph $B(k)$ containing $n = (k+1)2^k$ vertices. The vertices of this graph are organized in k+1 levels (columns) each containing 2^k vertices. In each column, vertices are labeled in the same way (from 0 to 2^{k-1}). An edge connects two vertices if and only if they are in the consecutive columns i and i+1 and their numbers are the same or they differ only in the bit at the i-th position. The maximum vertex degree is $\Delta = 4$ (the vertices from two outer columns have degree 2 and the vertices in k-1 inner columns all have degree 4). Diameter D equals $2k$ while the spectrum is given in [9], Theorem 11. Therefrom, the largest eigenvalue is $\lambda_1 = 4 \cos (\pi/(k+1))$. However, it is not obvious how to determine parameter m. Therefore, we got only the values $stt = 4(2k+1) = O(\log n)$ and $t_2 = 4(2k+1) \cos (\pi/(k+1)) = O(k) = O(\log n)$.

Widely used interconnection topologies include some kinds of trees, meshes and toruses [14]. We shall describe these structures in some details.

3. *Stars* $S_n = K_{1,n-1}$ are considered as ill-suited topologies in [9], since the tightness $t_1(S_n)$ is large. However stars are widely used in multiprocessor system design:

the so-called master-slave concept is based on the star graph structure. This fact may be an indication that the classification of multiprocessor interconnection networks based on the value for t_1 is not always adequate.

For S_n: $m = 3$, $\Delta = n$-1, $D = 2$, $\lambda_1 = \sqrt{n-1}$ and we have

$t_1(S_n) = 3(n\text{-}1)$,

$stt\,(S_n) = 3(n\text{-}1)$, $spt\,(S_n) = 3\sqrt{n-1}$,

$t_2\,(S_n) = 3\sqrt{n-1}$.

Stars are only the special case in more general class of bipartite graphs. The main representatives of this class are complete bipartite graphs $K_{n_1n_2}$ having vertices divided into two sets and edges connecting each vertex from one set to all vertices in the other set. For $K_{n_1n_2}$ we have $m = 3$, $\Delta = \max\{n_1, n_2\}$, $D = 2$, $\lambda_1 = \sqrt{n_1n_2}$ and hence

$$t_1(K_{n_1n_2}) = stt(K_{n_1n_2}) = 3\max\{n_1, n_2\}, \qquad spt(K_{n_1n_2}) = t_2(K_{n_1n_2}) = 3\sqrt{n_1n_2}\,.$$

In the case $n_1 = n_2 = n/2$ all tightness values are of order $O(n)$. However, for the star S_n we have $t_2(S_n) = O(\sqrt{n}\,)$. This may be the indication that complete bipartite graphs are suitable for modelling multiprocessor interconnection networks only in some special cases. □

4. *Mesh* or *greed* (Figure. 3a) consists of $n = n_1n_2$ vertices organized within layers. We can enumerate vertices with two indices, like the elements of a $n_1 \times n_2$ matrix. Each vertex is connected to its neighbors (the ones whose one of the indices is differing from its own by one). Inner vertices have 4 neighbors, corner ones only 2, while the outer (but not corner ones) are of degree 3. Therefore, $\Delta = 4$, $D = n_1 + n_2 - 2$. Spectrum is given in [8], p. 74. In particular, the largest eigenvalue is $\lambda_1 = 2\cos\,(\pi/(n_1+1)) + 2\cos\,(\pi/(n_2+1))$ and for the tightness of the second type we obtain $t_2 = (n_1 + n_2 - 1)(2\cos\,(\pi/(n_1+1)) + 2\cos(\pi\,/(n_2+1)))$. Hence, $t_2 = O(\sqrt{n}\,)$ if $n_1 \approx n_2$. □

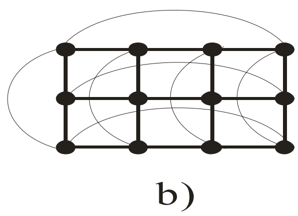


Figure 3: a) Mesh of order 3 × 4 and b) corresponding torus architecture

5. *Torus* (Figure 3b) is obtained if mesh architecture is "closed" with respect to both dimensions. We do not distinguish corner or outer vertices any more. The characteristics of a torus are $\Delta = 4$, $D = [n_1/2] + [n_2/2]$. Spectrum is given in [8], p. 75. In particular, the largest eigenvalue is $\lambda_1 = 2\cos(2\pi/n_1) + 2\cos(2\pi/n_2)$ and thus $t_2 = ([n_1/2] + [n_2/2] + 1)(2\cos(2\pi/n_1) + 2\cos(2\pi/n_2))$. As in the previous case (for mesh) we have $t_2 = O(\sqrt{n})$ if $n_1 \approx n_2$. □

All these architectures satisfy the both requests of designing the multiprocessor topologies (small distance between processors and small number of wires). Those of them which have a small value for t1 are called well-suited interconnection topologies in [9]. The other topologies are called ill-suited. Therefore, according to [9], well-suited and ill-suited topologies are distinguished by the value for the mixed tightness of the first type t1(G).

The star example suggests that t2(G) is more appropriate parameter for selecting well-suited interconnection topologies than t1(G). Namely, the classification based on the tightness t2 seems to be more adequate since it includes stars into the category of well-suited structures.

The obvious conclusion following from the Hasse diagram given on Figure 2, is that a well-suited interconnection network according to the value for t1 remains well-suited also when t2 is taken into consideration. In this way, some new graphs become suitable for modelling multiprocessor interconnection networks. Some of these "new" types of graphs have already been recognized by multiprocessor system designers (like stars and bipartite graphs). In the next section we propose a new family of t2-based well-suited trees.

5. COMPLETE QUASI-REGULAR TREES

In this section we shall study properties of some trees and show that they are suitable for our purposes.

The complete quasi-regular tree $T(d,k)$ (d=2,3,..., k=1,2,...) is a tree consisting of a central vertex and k layers of other vertices, adjacencies of vertices being defined in the following way.

The central vertex (the one on the layer 0) is adjacent to d vertices in the first layer.

1. For any i=1,2,...,k-1 each vertex in the i-th layer is adjacent to d-1 vertices in the (i+1)-th layer (and one in the (i-1)-th layer).

   The graph $T(3,3)$ is given in the Figure 4.

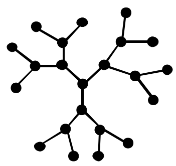


Figure 4: Quasi-regular tree $T(3,3)$

The graph $T(d,k)$ for $d > 2$, has $n = 1 + d((d-1)^{k-1})/(d-2)$ vertices, maximal vertex degree $\Delta = d$, diameter $D = 2k$ and the largest eigenvalue $\lambda_1 < d$. (The spectrum of

T(d,k) has been determined in [13]). We have $k = O(\log n)$ and, since $t_2(T(d,k)) = (D + 1)\lambda_1 < (D + 1)\Delta = stt(T(d,k)) = (2k + 1)d$, we obtain $t_2(T(d,k)) = O(\log n)$. This is asymptotically better than in the case of hypercube $Q(d)$, where $t\_2(Q(d)) = O((\log n)$ ^2)or in the case of a star graph where $t_2(K_{1,n-1}) = O(\sqrt{n})$ (see Section 4). Note that the path P_n with $t_2(P_n) = 2n$ cos(\π(n+1))=$O(n)$ also performs worse.

The coefficient of the main term in the expression for $t_2(T(d,k))$ is equal to $d/\log(d-1)$ with values 4.328, 3.641, 3.607, 3.728, 3.907, 4.111, 4.328 and 4.551 for d = 3,4,5,6,7,8,9,10 respectively. Further on, the coefficient is an increasing function of d. Therefore the small values of d are desirable and we shall discuss in details only the case $d = 3$ since it is suitable for resolving the stability issues. The other cases with small values of d can be analyzed analogously.

To examine the suitability of graphs $T(3,k)$, we compared its tightness values with the corresponding ones for two interesting classes of trees: paths P_n and stars $S_n = K_{1,n-1}$ containing the same number of vertices $n = 3 \cdot 2^{k-2}$. The results for small values of k are summarized in the Table 1.

Table 1: Tightness values for some trees

k	n	P_n $t_1(\geq t_2)$	$T(3,k)$ $stt(\geq t_2)$	S_n t_1 (t_2)
1	4	4·2	3·3	3·3 (3·$\sqrt{3}$)
2	10	10·2	5·3	3·9 (3·$\sqrt{9}$ = 3·3)
3	22	22·2	7·3	3·21 (3·$\sqrt{21}$ < 3·5)
4	46	46·2	9·3	3·45 (3·$\sqrt{45}$ < 3·7)
5	94	94·2	11·3	3·93 (3·$\sqrt{93}$ < 3·10)
6	190	190·2	13·3	3·189 (3·$\sqrt{189}$ > 3·13)
7	382	382·2	15·3	3·381 (3·$\sqrt{381}$ > 3·19)

Since for paths and quasi-regular trees mixed tightness of the second type has almost the same value as the mixed tightness of the first type, we put only the values for the first type mixed tightness for paths, while for $T(n,k)$ the structural tightness is given.

The last column (for stars) contains the values for two types of tightness, first for the mixed tightness of the first type and then the value for the mixed tightness of the second type in the parentheses.

As it can be seen from the Table 1, the tightness values for paths P_n are significantly larger than the values $stt(T(3,k))$. Star architecture seems to be better for small values of k, but starting from $k = 6$, we have $t_2(T(3,k)) < stt(T(3,k)) < t_2(S_n)$.

The intention when comparing complete quasi-regular trees $T(3,k)$ with paths P_n and stars S_n is to examine their place between two kinds of trees, extremal for many graph invariants. In particular, among all trees with a given number of vertices, the largest eigenvalue λ_1 and maximum vertex degree Δ have minimum values for the path and maximum for the star, while, just opposite, the number of distinct eigenvalues m and the diameter D have maximum values for the path and minimum for the star. Since the tightness (of any type) is the product of two graph invariants having, in the above sense, opposite behavior, it is expected that its extremal value is attained "somewhere in the middle". Therefore, for a tree of special structure (like quasi-regular trees) we expect the both tendencies to be in the equilibrium.

It is not difficult to extend the family of complete quasi-regular trees to a scalable family. A *quasi-regular tree* is a tree obtained from a complete quasi-regular tree by deleting some of its vertices of degree 1. If none or all vertices of degree 1 are deleted from a complete quasi-regular tree we obtain again a complete quasi-regular tree. Hence, a complete quasi-regular tree is also a quasi-regular tree.

While a complete quasi-regular tree is unique for the given number of vertices, there are several non-isomorphic quasi-regular trees with the same number of vertices which are not complete. Therefore, there are several ways to construct a scalable family of quasi-regular trees. The following way is a very natural one.

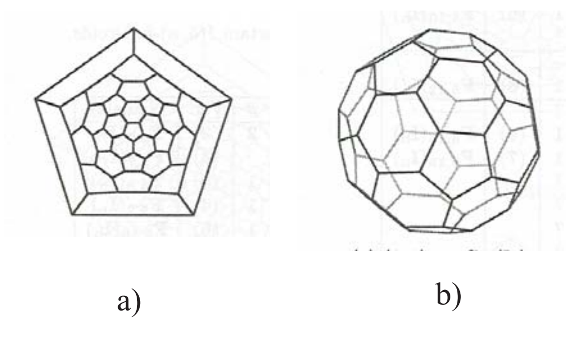
Consider a complete quasi-regular tree $T(d,k)$ and perform the breadth first search through the vertex set starting from the central vertex. Adding to $T(d,k\text{-}1)$ pendant vertices of $T(d,k)$ in the order they are traversed in the mentioned breadth first search defines the desired family of quasi-regular trees.

The constructed family has the property that its each member has the largest eigenvalue λ_1 among all quasi-regular trees with the same number of vertices [17]. At the first glance this property is something what we do not want since we are looking for graphs with the tightness t_2 as small as possible. Instead we would prefer, unlike the breadth first search, to keep adding pendant vertices to $T(d,k\text{-}1)$ in such a balanced way around that we always get a quasi-regular tree with largest eigenvalue as small as possible. Such a way of adding vertices is not known and its finding represents a difficult open problem in spectral graph theory.

A scalable family of trees with $O((\log\, n)^2)$ distinct eigenvalues have been studied in [9]. An open question remains to compare the performances of these two families.

In our context *fullerene graphs* are also interesting, corresponding to carbon compounds called *fullerenes*. Mathematically, fullerene graphs are planar regular graphs of degree 3 having as faces only pentagons and hexagons. It follows from the Euler theorem for planar graphs that the number of pentagons is exactly 12. Although being planar, fullerene graphs are represented (and this really corresponds to actual positions of carbon atoms in a fullerene) in 3-space with its vertices embedded in a quasi-spherical surface.

A typical fullerene C_{60} is given in Figure 5. It can be described also as a truncated icosahedron and has the shape of a football.



a) b)

Figure 5: a) Planar and b) 3D visualization of the icosahedral fullerene C_{60}

Without elaborating in details we indicate the relevance of fullerene graphs to our subject by comparing them with quasi-regular trees.

For a given number of vertices the largest eigenvalues of the two graphs are roughly equal (equal to 3 in fullerenes and close to 3 in quasi-regular trees) while the diameters are also comparable. This means that the tightness t_2 is approximately the same in both cases. In particular, values of relevant invariants for the fullerene graph C_{60} are $n = 60$, $D = 9$ (see [11]), $m = 15$ (see [12]), $\Delta = \lambda_1 = 3$. Hence, $stt = t_2 = 30$. A quasi-regular tree on 60 vertices has diameter $D = 9$ and we also get $stt = 30$.

Note that the tightness t_1 is not very small since it is known that fullerene graphs have a large number of distinct eigenvalues [12]. It is also interesting that fullerene graphs have a nice 3D-representation in which the coordinates of the positions of vertices can be calculated from the eigenvectors of the adjacency matrix (the so called *topological coordinates* which were also used in producing the atlas [12]).

6. GRAPHS WITH FEW DISTINCT EIGENVALUES

Graphs with a small number of distinct eigenvalues have attracted much attention in the research community. Since they are of interest in finding well-suited interconnection topologies, we shall survey some basic facts about them in this section.

The number of distinct eigenvalues of a graph is correlated with its symmetry property [1]: the graphs with a small number of distinct eigenvalues are (very frequently) highly symmetric. They also have a small diameter.

Let m be the number of distinct eigenvalues of a graph G.

Trivial cases are $m = 1$ and $m = 2$. If $m = 1$, all eigenvalues are equal to 0 and G consists of isolated vertices. Such a graph, of course, is not suitable for the multiprocessor topology since it does not contain communication links and the data exchange between processors is impossible. In the case $m = 2$, if G is connected it is a complete graph with $n \geq 2$ vertices and eigenvalues are $\lambda_1 = n$-1, and λ_i=-1, for $i = 2,3,\ldots,n$, i.e. it is the value of multiplicity n-1. Completely connected multiprocessor topologies were designed, but for a small n. When n is growing, it is hard not only to realize all the necessary wires, but also to control all required communications.

Further, we shall consider only connected graphs. If $m = 3$ and G is regular, then G is strongly regular (cf. [8], p. 108). For example, the well known Petersen graph is strongly regular with distinct eigenvalues 3, 1, -2 of multiplicities 1, 5, 4 respectively.

It is difficult to construct scalable families of strongly regular graphs and it is also not clear what would be the order of magnitude of the tightness in such families. It could be rather expected that one can find sporadic examples with small tightness compared with the number of vertices like it appears in the Petersen graph. Let us note that there are only finitely many strongly regular graphs and they cannot constitute a scalable family.

There are also some non-regular graphs with three distinct eigenvalues [19]. Although, such graphs usually have a vertex adjacent to all other vertices they may still be of interest for the design of multiprocessor systems (like star graphs are).

Several classes of regular graphs with four distinct eigenvalues are described in [19], but the whole set is not described yet.

Although graphs with few distinct eigenvalues allow a quick execution of load balancing algorithms [9], it is not expected that scalable families of such graphs with small tightness t_1 or t_2 can be constructed.

7. CONCLUSION

We have given several definitions of graph tightness in order to describe multiprocessor interconnection networks. It was shown that in scalable families the tightness of any kind is unbounded. Graphs which are expected to be good networks models have small tightness values.

The material from this paper shows that the tightness $t_2(G)$ is more suitable than the tightness $t_1(G)$ (previously used in the literature) for describing and classifying multiprocessor interconnection networks. According to the classification based on $t_2(G)$, quasi-regular trees perform better than hypercube graphs.

REFERENCES

[1] Biggs, N.L., *Algebraic Graph Theory*, (II edition), Cambridge University Press, Cambridge, 1993.

[2] Brouwer, A.E., Cohen, and A.M., Neumaier, A., *Distance-regular Graphs*, Springer, Berlin, 1989.

[3] Byrd, G.T., and Delagi, B.A., "Considerations for multiprocessor topologies", Technical report, STAN-CS-87-1144, Department of Computer Science, Stanford University, 1987.

[4] Cvetković, D., and Davidović, T., "Small multiprocessor interconnection networks", Submitted for publication, 2008.

[5] Cvetković, D., Dimitrijević, V., and Milosavljević, M., "A survey of some non-standard traveling salesman problems", *Yugoslav Journal of Operational Research*, 2(2) (1992) 163-185.

[6] Cvetković, D., and Gutman, I., "Note on branching", *Croat. Chem. Acta*, 49 (1977) 115-121.

[7] Cvetković, D., and Rowlinson, P., "The largest eigenvalue of a graph - a survey", *Linear and Multilinear Algebra*, 28 (1990) 3-33.

[8] Cvetković, D.M., Doob, M., and Sachs, H., *Spectra of Graphs: Theory and Applications,* (III edition), Johann Ambrosius Barth Verlag, Heidelberg-Leipzig, 1995.

[9] Elsässer, R., Královič, R., and Monien, B., "Sparse topologies with small spectrum size", *Theor. Comput. Sci.*, 307 (2003) 549-565.

[10] Ferreira, A., and Morvan, M., "Models for parallel algorithm design: An introduction", in: A., Migdalas, P., Pardalos, S., Storøy, (eds.), *Parallel Computing in Optimization*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1997, 1-26.

[11] Fowler, P.W., Caporossi, G., and Hansen, P., "Distance matrices, Wiener indices, and related invariants of fullerenes", *J. Chem. Phys. A*, 105 (2001) 6232-6242.

[12] Fowler, P.W., and Manolopoulos, D.E., *An Atlas of Fullerenes*, Clarendon Press, Oxford, 1995.

[13] Gutman, I., "Characteristic and matching polynomials of some compaund graphs", *Publ. Inst. Math.* (Beograd), 27(41) (1980) 61-66.

[14] Leighton, F.T., *Introduction to Parallel Algorithms and Architectures, Arrays, Trees, Hypercubes*, Morgan Kaufmann, San Mateo, 1992.

[15] Sarkar, V., *Partitioning and Scheduling Parallel Programs for Multiprocessors,* The M.I.T. Press, Cambridge, MA, 1989.

[16] Sih, G.C., Lee, E.A., "A compile-time scheduling heuristic for interconnection-constrained heterogeneous processor architectures", *IEEE Trans. Parallel and Distributed Systems*, 4(2) (1993) 175-187.

[17] Simić, S., Tošić, D., "The index of trees with specified maximum degree", *MATCH Commun. Math. Comput. Chem*, 54 (2005) 351-362.

[18] Sørevik, T., "A programmer's view of parallel computers", in: Migdalas, A., Pardalos, P., Storøy, S. (eds.), *Parallel Computing in Optimization*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1997, 57-72.

[19] Van Dam, E., "Graphs with few eigenvalues, An interply between combinatorics and algebra", PhD thesis, Center for Economic Research, Tilburg University, Tilburg, 1996.