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POSITIVE SOLUTIONS WITH SPECIFIC ASYMPTOTIC BEHAVIOR FOR A POLYHARMONIC PROBLEM ON $\mathbb{R}^n$

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Abstract. This paper is concerned with positive solutions of the semilinear polyharmonic equation $(-\Delta)^m u = a(x)u^\alpha$ on $\mathbb{R}^n$, where m and n are positive integers with $n > 2m$, $\alpha \in (-1, 1)$. The coefficient a is assumed to satisfy

$$a(x) \approx (1 + |x|)^{-\lambda} L(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\lambda \in [2m, \infty)$ and $L \in C^1([1, \infty))$ is positive with $\frac{tL'(t)}{L(t)} \longrightarrow 0$ as $t \longrightarrow \infty$; if $\lambda = 2m$, one also assumes that $\int_1^\infty t^{-1}L(t)dt < \infty$. We prove the existence of a positive solution u such that

$$u(x) \approx (1 + |x|)^{-\widetilde{\lambda}} \widetilde{L}(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}^n,$$

with $\widetilde{\lambda} := \min(n - 2m, \frac{\lambda-2m}{1-\alpha})$ and a function $\widetilde{L}$, given explicitly in terms of L and satisfying the same condition at infinity. (Given positive functions f and g on $\mathbb{R}^n$, $f \approx g$ means that $c^{-1}g \leq f \leq cg$ for some constant $c > 1$.)

Keywords: asymptotic behavior, Dirichlet problem, Schauder fixed point theorem, positive bounded solutions.

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1. INTRODUCTION

This paper is concerned with positive solutions of the semilinear polyharmonic equation

$$(-\Delta)^m u = a(x)u^\alpha \text{ on } \mathbb{R}^n \ (n \geq 3) \text{ (in the sense of distributions)}, \tag{1.1}$$

where m and n are positive integers with $n > 2m$, $\alpha \in (-1, 1)$. The coefficient a is a positive measurable function on $\mathbb{R}^n$ assumed to satisfy

$$a(x) \approx (1 + |x|)^{-\lambda} L(1 + |x|) \quad \text{for} \quad x \in \mathbb{R}^n,$$

where $\lambda \in [2m, \infty)$ and $L \in C^1([1,\infty))$ is positive with $\frac{tL'(t)}{L(t)} \longrightarrow 0$ as $t \longrightarrow \infty$; if $\lambda = 2m$, one also assumes that $\int_1^\infty t^{-1} L(t)dt < \infty$. Here and throughout the paper, for positive functions f and g on a set S, the notation $f \approx g$ means that there exists a constant $c > 1$ such that $c^{-1} g \leq f \leq cg$ on S.

Recently, by applying Karamata regular variation theory, the authors in [7], studied equation (1.1) in the unit ball of $\mathbb{R}^n$ $(n \geq 2)$ with Dirichlet boundary conditions. They proved the existence of a continuous solution and gave an asymptotic behavior of such a solution.

For the case $m = 1$ and $\alpha < 1$, the pure elliptic equation

$$-\Delta u = a(x)u^\alpha, \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.2}$$

has been extensively studied for both bounded and unbounded domain Ω in $\mathbb{R}^n$ $(n \geq 2)$. We refer to [1, 2, 4–6, 8, 10–16, 18, 21] and the references therein, for various existence and uniqueness results related to solutions for the above equation with homogeneous Dirichlet boundary conditions. In particular, many authors studied the exact asymptotic behavior of solutions of equation (1.2), see for example [4–6, 10, 12, 13, 15, 16, 21]. For instance in [4], the authors studied (1.2) in $\mathbb{R}^n$ $(n \geq 3)$. Thanks to the sub and supersolution method, they showed that equation (1.2) has a unique positive classical solution which satisfies homogeneous Dirichlet boundary conditions. Moreover, by applying Karamata regular variation theory, they improved and extended the estimates established in [2,8,14]. On the other hand when $\alpha = 0$ and the equation (1.2) involves a degenerate operator p-Laplacian, Cavalheiro in [3] proved the existence and uniqueness solution under a suitable condition on the function a.

Also, the result of equation (1.2) is extended to a class of elliptic systems. We refer for example to [9] and [20] where the authors proved the existence and asymptotic behavior of continuous solutions.

In this work, we generalize the results of [4] to equation (1.1). Note that the sub and supersolution method is not available for (1.1). Then, we have to work around this obstacle and we shall use the Schauder fixed-point theorem which requires invariance of a convex set under a suitable integral operator. Hence, we are restricted to considering only the case $\alpha \in (-1, 1)$.

To simplify our statements, we refer to $B^+(\mathbb{R}^n)$ the set of Borel measurable nonnegative functions in $\mathbb{R}^n$ and $C_0(\mathbb{R}^n)$ the class of continuous functions in $\mathbb{R}^n$ vanishing continuously at infinity. Also, we use $\mathcal{K}$ to denote the set of functions L defined on $[1, \infty)$ by

$$L(t) := c \exp\left(\int_1^t \frac{z(s)}{s} ds \right),$$

where $z \in C([1,\infty))$ such that $\lim_{t \longrightarrow \infty} z(t) = 0$ and $c > 0$.

For the rest of the paper, we use the letter c to denote a generic positive constant which may vary from line to line.

Remark 1.1. It is obvious to see that $L \in \mathcal{K}$ if and only if L is a positive function in $C^1\big([1,\infty)\big)$ such that $\lim_{t\to\infty} \frac{tL'(t)}{L(t)} = 0$.

Example 1.2. Let $p \in \mathbb{N}^*$, $(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathbb{R}^p$ and ω be a positive real number sufficiently large such that the function

$$L(t) = \prod_{k=1}^{p} (\log_k(\omega t))^{-\lambda_k}$$

is defined and positive on $[1, \infty)$, where $\log_k(x) = (\log \circ \log \circ \ldots \circ \log)(x)$ (k times). Then $L \in \mathcal{K}$.

Throughout this paper, we denote by $G_{k,n}(x,y) = C_{k,n} \frac{1}{|x-y|^{n-2k}}$, the Green function of the operator $(-\Delta)^k$ in $\mathbb{R}^n$, where $C_{k,n} = \frac{\Gamma(\frac{n}{2}-k)}{4^k \pi^{\frac{n}{2}} (k-1)!}$, $1 \le k \le m < \frac{n}{2}$.

The function $G_{k,n}(x,y)$ satisfies, for $2 \le k \le m$,

$$G_{k,n}(x,y) = \int_{\mathbb{R}^n} G_{1,n}(x,z) G_{k-1,n}(z,y) dz.$$

We define the k-potential kernel $V_{k,n}$ on $B^+(\mathbb{R}^n)$ by

$$V_{k,n} f(x) = \int_{\mathbb{R}^n} G_{k,n}(x,y) f(y) dy.$$

Hence, for any $f \in B^+(\mathbb{R}^n)$ such that $f \in L^1_{loc}(\mathbb{R}^n)$ and $V_{k,n} f \in L^1_{loc}(\mathbb{R}^n)$, then we have $(-\Delta)^k V_{k,n} f = f$ in the sense of distributions.

Now we are ready to present our main result.

Theorem 1.3. *Equation* (1.1) *has a positive and continuous solution u satisfying for $x \in \mathbb{R}^n$*

$$u(x) \approx \theta(x),$$

where the function θ is defined on $\mathbb{R}^n$ by

$$\theta(x) := \begin{cases} \left(\int_{|x|+1}^{\infty} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = 2m, \\ (1+|x|)^{\frac{2m-\lambda}{1-\alpha}} (L(1+|x|))^{\frac{1}{1-\alpha}} & \text{if } 2m < \lambda < n-(n-2m)\alpha, \\ (1+|x|)^{2m-n} \left(\int_{1}^{2+|x|} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = n-(n-2m)\alpha, \\ (1+|x|)^{2m-n} & \text{if } \lambda > n-(n-2m)\alpha. \end{cases} \tag{1.3}$$

Our idea in Theorem 1.3 above is based on the Schauder fixed-point method and the convex set invariant under the integral operators. We note that (1.1) is formally equivalent to the integral equation

$$u = V_{m,n}(a u^{\alpha}), \tag{1.4}$$

and $u = V_{m,n}(a)$ is a solution in the linear case with $\alpha = 0$. The asymptotic behavior of $V_{m,n}(a)$ is similar to that of a itself, only with different λ and L (Proposition 2.4). Based on this observation, one constructs an asymptotic solution of (1.4), that is, a function θ, again with the same kind of asymptotic behavior, such that $\theta \approx V_{m,n}(a\theta^\alpha)$ (Proposition 2.5). One then finds a constant $c > 1$ such that the operator $u \longrightarrow V_{m,n}(au^\alpha)$ maps the order interval $[c^{-1}\theta, c\theta] \subset C_0(\mathbb{R}^n)$ into itself. The operator being compact, Schauder's fixed-point theorem yields a solution u of (1.4) with $u \approx \theta$, proving our main result. We also control the asymptotic behavior of the iterated Laplacians of u; in particular, u satisfies Navier boundary conditions at infinity.

The outline of the paper is as follows. In Section 2, we state some already known results on functions in $\mathcal{K}$, useful for our study and we give estimatises on some potential functions. Section 3 is reserved to the proof of our main result.

2. ESTIMATES AND PROPERTIES OF $\mathcal{K}$

2.1. TECHNICAL LEMMAS

In what follows, we collect some fundamental properties of functions belonging to the class $\mathcal{K}$.

First, we need the following elementary result.

Lemma 2.1 ([19, Chap. 2, pp. 86–87])**.** *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$. Then we have:*

- (i) *If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s)ds$ converges and $\int_t^\infty s^\gamma L(s)ds \sim \frac{-t^{\gamma+1}L(t)}{\gamma+1}$ as $t \longrightarrow \infty$,*
- (ii) *If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)ds$ diverges and $\int_1^t s^\gamma L(s)ds \sim \frac{t^{\gamma+1}L(t)}{\gamma+1}$ as $t \longrightarrow \infty$.*

Lemma 2.2 ([4])**.** (i) *Let L_1, $L_2 \in \mathcal{K}$, $p \in \mathbb{R}$. Then $L_1L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.*
(ii) *Let $L \in \mathcal{K}$ and $\eta > 0$. Then we have*

$$L(t) \approx L(t+\eta) \quad \text{for} \quad t \geq 1,$$

$$\lim_{t \longrightarrow +\infty} t^{-\eta}L(t) = 0, \tag{2.1}$$

$$\lim_{t \longrightarrow +\infty} t^{\eta}L(t) = \infty. \tag{2.2}$$

(iii) *Let $L \in \mathcal{K}$. Then $\lim_{t\longrightarrow\infty} \frac{L(t)}{\int_1^t \frac{L(s)}{s}ds} = 0$.*
In particular, the function

$$t \longrightarrow \int_1^{1+t} \frac{L(s)}{s}ds \quad \text{is in } \mathcal{K}. \tag{2.3}$$

Further, if $\int_1^\infty \frac{L(s)}{s}ds$ converges, then $\lim_{t\longrightarrow\infty} \frac{L(t)}{\int_t^\infty \frac{L(s)}{s}ds} = 0$.

In particular, the function

$$t \longrightarrow \int_t^\infty \frac{L(s)}{s} ds \quad \text{is in } \mathcal{K}. \tag{2.4}$$

The following behavior of the potential of radial functions on $\mathbb{R}^n$ is due to [17].

Lemma 2.3 ([17])**.** *Let $0 \le j \le m-1$ and let f be a nonnegative radial measurable function in $\mathbb{R}^n$ such that $\int_1^\infty r^{2(m-j)-1} f(r)dr < \infty$, then for $x \in \mathbb{R}^n$, we have*

$$V_{m-j,n} f(x) \approx \int_0^\infty \frac{r^{n-1}}{\max(|x|, r)^{n-2(m-j)}} f(r) dr. \tag{2.5}$$

2.2. ASYMPTOTIC BEHAVIOR OF SOME POTENTIAL FUNCTIONS

In what follows, we are going to give estimates on the potentials $V_{m-j,n}a$ and $V_{m-j,n}(a\theta^\alpha)$, for $0 \le j \le m-1$, where the function θ is given in (1.3).

Proposition 2.4. *For $0 \le j \le m-1$ and $x \in \mathbb{R}^n$*

$$V_{m-j,n}a(x) \approx \psi(|x|),$$

where ψ is the function defined on $[0,\infty)$ by

$$\psi(t) = \begin{cases} \int_{t+1}^\infty \frac{L(r)}{r} dr & \text{if } \lambda = 2(m-j), \\ (1+t)^{2(m-j)-\lambda} L(1+t) & \text{if } 2(m-j) < \lambda < n, \\ (1+t)^{2(m-j)-n} \int_1^{2+t} \frac{L(r)}{r} dr & \text{if } \lambda = n, \\ (1+t)^{2(m-j)-n} & \text{if } \lambda > n. \end{cases}$$

Proof. Let $\lambda \ge 2(m-j)$ and $L \in \mathcal{K}$ satisfying $\int_1^\infty t^{2(m-j)-1-\lambda} L(t) dt < \infty$ and such that

$$a(x) \approx \frac{L(1+|x|)}{(1+|x|)^\lambda}.$$

Thus, by (2.5), we have

$$V_{m-j,n}a(x) \approx \int_0^\infty \frac{r^{n-1}}{\max(|x|, r)^{n-2(m-j)}} \frac{L(1+r)}{(1+r)^\lambda} dr := I(|x|),$$

where I is the function defined on $[0,\infty)$ by

$$I(t) = \int_0^\infty \frac{r^{n-1}}{\max(t, r)^{n-2(m-j)}} \frac{L(1+r)}{(1+r)^\lambda} dr.$$

So to prove the result, it is sufficient to show that $I(t) \approx \psi(t)$ for $t \in [0, \infty)$.

Let $t \geq 1$. We have

$$\begin{aligned} I(t) &= \frac{1}{t^{n-2(m-j)}} \int_0^1 \frac{r^{n-1} L(1+r)}{(1+r)^\lambda} dr + \frac{1}{t^{n-2(m-j)}} \int_1^t \frac{r^{n-1} L(1+r)}{(1+r)^\lambda} dr \\ &+ \int_t^\infty \frac{r^{2(m-j)-1} L(1+r)}{(1+r)^\lambda} dr \\ &\approx \frac{1}{t^{n-2(m-j)}} \int_0^1 \frac{r^{n-1} L(1+r)}{(1+r)^\lambda} dr + \frac{1}{t^{n-2(m-j)}} \int_1^t r^{n-1-\lambda} L(r) dr \\ &+ \int_t^\infty r^{2(m-j)-1-\lambda} L(r) dr \\ &:= I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

It is clear that

$$I_1(t) \approx \frac{1}{t^{n-2(m-j)}}. \tag{2.6}$$

To estimate I_2 and I_3, we distinguish two cases.

Case 1. $\lambda > 2(m-j)$. Using Lemma 2.1 (i), we have for $t \geq 1$

$$I_3(t) \approx \frac{L(t)}{t^{\lambda-2(m-j)}}. \tag{2.7}$$

If $2(m-j) < \lambda < n$, then applying again Lemma 2.1 (ii), we have $\int_1^\infty r^{n-1-\lambda} L(r) dr = \infty$ and $\int_1^t r^{n-1-\lambda} L(r) dr \sim t^{n-\lambda} L(t)$, as $t \longrightarrow \infty$. So for $t \geq 1$ we obtain

$$I_2(t) \approx \frac{L(t)}{t^{\lambda-2(m-j)}}. \tag{2.8}$$

Then, by (2.6), (2.7), (2.8) and (2.1), for $t \geq 1$ we have

$$I(t) \approx \frac{1}{t^{n-2(m-j)}} + \frac{L(t)}{t^{\lambda-2(m-j)}} \approx \frac{L(t)}{t^{\lambda-2(m-j)}}.$$

Now, since the function $t \longrightarrow I(t)$ and $t \longrightarrow \frac{L(1+t)}{(1+t)^{\lambda-2(m-j)}}$ are positive and continuous in $[0, \infty)$, we obtain for $t \geq 0$

$$I(t) \approx \frac{L(1+t)}{(1+t)^{\lambda-2(m-j)}}.$$

If $\lambda > n$, then using Lemma 2.1 (i), we have $\int_1^\infty r^{n-1-\lambda} L(r) dr < \infty$ and for $t \geq 2$,

$$\int_1^t r^{n-1-\lambda} L(r) dr \approx 1.$$

So we obtain, for $t \geq 1$

$$I_2(t) \approx t^{2(m-j)-n}.$$

Moreover, using (2.1), we have for $t \geq 1$, $I_1(t)+I_2(t)+I_3(t) \approx t^{2(m-j)-n}(2+\frac{L(t)}{t^{\lambda-n}}) \approx t^{2(m-j)-n}$.

Then we deduce that for $t \geq 0$,

$$I(t) \approx (1+t)^{2(m-j)-n}.$$

If $\lambda = n$, we have $I_2(t) \approx \dfrac{1}{t^{n-2(m-j)}} \displaystyle\int_1^t \frac{L(r)}{r} dr$ and $I_3(t) \approx \frac{L(t)}{t^{n-2(m-j)}}$, then using (2.6) and (2.3), for $t \geq 1$, we have

$$I(t) \approx \frac{1}{t^{n-2(m-j)}}\big(1+\int_1^t \frac{L(r)}{r} dr + L(t)\big) \approx \frac{1}{t^{n-2(m-j)}} \int_1^t \frac{L(r)}{r} dr.$$

So we obtain for $t \geq 0$

$$I(t) \approx \frac{1}{(1+t)^{n-2(m-j)}} \int_1^{2+t} \frac{L(r)}{r} dr.$$

Case 2. $\lambda = 2(m-j)$. By Lemma 2.1 (ii), $\displaystyle\int_1^\infty r^{n-1-2(m-j)} L(r) dr = \infty$ and for $t \geq 2$,

$$\int_1^t r^{n-1-2(m-j)} L(r) dr \approx t^{n-2(m-j)} L(t).$$

Then we have for $t \geq 1$, $I_2(t) \approx L(t)$. So for $t \geq 1$, we have

$$I(t) \approx \frac{1}{t^{n-2(m-j)}} + L(t) + \int_t^\infty \frac{L(r)}{r} dr.$$

Hence using (2.2) and (2.4), for $t \geq 1$, we have

$$I(t) \approx \int_t^\infty \frac{L(r)}{r} dr.$$

So for $t \geq 0$, we obtain

$$I(t) \approx \int_{t+1}^\infty \frac{L(r)}{r} dr.$$

This completes the proof. □

The following proposition plays a crucial role in this paper.

Proposition 2.5. *Let θ be the function given in (1.3). Then we have for $x \in \mathbb{R}^n$ and for $0 \leq j \leq m-1$*

$$V_{m-j,n}(a\theta^\alpha)(x) \approx \widetilde{\theta}(x),$$

where $\widetilde{\theta}$ is the function defined on $\mathbb{R}^n$ by

$$\widetilde{\theta}(x) := \begin{cases} \left(\int_{|x|+1}^{\infty} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = 2m \text{ and } j = 0, \\ (1+|x|)^{-2j} L(1+|x|) \left(\int_{|x|+1}^{\infty} \frac{L(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} & \text{if } \lambda = 2m \text{ and } 0 < j \leq m-1, \\ (1+|x|)^{-\frac{\lambda - 2(m-j) - 2\alpha j}{1-\alpha}} \left(L(1+|x|) \right)^{\frac{1}{1-\alpha}} & \text{if } 2m < \lambda < n - (n-2m)\alpha, \\ (1+|x|)^{-(n-2(m-j))} \left(\int_{1}^{2+|x|} \frac{L(t)}{t} dt \right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = n - (n-2m)\alpha, \\ (1+|x|)^{-(n-2(m-j))} & \text{if } \lambda > n - (n-2m)\alpha. \end{cases}$$

Proof. Let $\lambda \geq 2m$ and $L \in \mathcal{K}$ satisfying $\int_1^\infty t^{2m-1-\lambda} L(t) dt < \infty$ and such that

$$a(x) \approx (1+|x|)^{-\lambda} L(1+|x|).$$

Then for every $x \in \mathbb{R}^n$, we have

$$a(x)\theta^\alpha(x)$$

$$\approx h(x) := \begin{cases} (1+|x|)^{-\lambda} L(1+|x|) \left(\int_{|x|+1}^{\infty} \frac{L(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} & \text{if } \lambda = 2m, \\ (1+|x|)^{-\frac{\lambda - 2m\alpha}{1-\alpha}} \left(L(1+|x|) \right)^{\frac{1}{1-\alpha}} & \text{if } 2m < \lambda < n - (n-2m)\alpha, \\ (1+|x|)^{-n} L(1+|x|) \left(\int_{1}^{2+|x|} \frac{L(t)}{t} dt \right)^{\frac{\alpha}{1-\alpha}} & \text{if } \lambda = n - (n-2m)\alpha, \\ (1+|x|)^{-(\lambda - (n-2m)\alpha)} L(1+|x|) & \text{if } \lambda > n - (n-2m)\alpha. \end{cases}$$

So, one can see that

$$h(x) := (1+|x|)^{-\mu} \widetilde{L}(1+|x|),$$

where $\mu \geq 2(m-j)$ for all $0 \leq j \leq m-1$ and by Lemma 2.2 (i) and (iii), we have $\widetilde{L} \in \mathcal{K}$.

The result follows from Proposition 2.4 by replacing L by $\widetilde{L}$ and λ by μ. □

Remark 2.6.

(i) Let θ be the function given in (1.3). Then from Proposition 2.5 for $j = 0$ and $x \in \mathbb{R}^n$, we have
$$V_{m,n}(a\theta^\alpha)(x) \approx \theta(x).$$

(ii) We mention that if $\alpha = 0$ and the function a satisfies an exact asymptotic behavior, the previous results in *i)* have been stated in Proposition 2.4.

Remark 2.7. By using (2.5) and Proposition 2.5, we see that for $0 \le j \le m-1$
$$\int_1^\infty \frac{\theta^\alpha(r)L(1+r)}{r^{\lambda-2(m-j)+1}}dr < \infty.$$

Proposition 2.8. *Let $0 \le j \le m-1$ and let θ the function given in* (1.3)*. Then the family of functions*
$$\Lambda = \Big\{x \longrightarrow Tu(x) := \int_{\mathbb{R}^n} \frac{a(y)u^\alpha(y)}{|x-y|^{n-2(m-j)}}dy;$$
$$\frac{1}{C}\theta \le u \le C\theta \text{ with } C > 0 \text{ is a fixed constant}\Big\}$$
is uniformly bounded and equicontinuous in $C_0(\mathbb{R}^n)$. Consequently, Λ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$.

Proof. Let $0 \le j \le m-1$, $x_0 \in \mathbb{R}^n$, $R > 0$ and let u be a positive function satisfying
$$\frac{1}{C}\theta \le u \le C\theta.$$
For $x, x' \in B(x_0, R)$, we have
$$\begin{aligned}
&|Tu(x) - Tu(x')| \\
&\le c\Bigg(\int\limits_{|x-y|\le 3R} \frac{a(y)\theta^\alpha(y)}{|x-y|^{n-2(m-j)}}dy + \int\limits_{|x'-y|\le 5R} \frac{a(y)\theta^\alpha(y)}{|x'-y|^{n-2(m-j)}}dy \\
&\quad + \int\limits_{|x-y|\ge 3R} \Big||x-y|^{2(m-j)-n} - |x'-y|^{2(m-j)-n}\Big| a(y)\theta^\alpha(y)dy\Bigg) \\
&\le c\Bigg(\int\limits_{(|x|-3R)^+}^{|x|+3R} \frac{r^{2(m-j)-1}L(1+r)\theta^\alpha(r)}{(1+r)^\lambda}dr \\
&\quad + \int\limits_{(|x'|-5R)^+}^{|x'|+5R} \frac{r^{2(m-j)-1}L(1+r)\theta^\alpha(r)}{(1+r)^\lambda}dr \\
&\quad + \int\limits_{|x-y|\ge 3R} \Big||x-y|^{2(m-j)-n} - |x'-y|^{2(m-j)-n}\Big| \frac{L(1+|y|)\theta^\alpha(|y|)}{(1+|y|)^\lambda}dy\Bigg).
\end{aligned}$$

We deduce from Remark 2.7, for $0 \leq j \leq m-1$, that the function

$$\varphi(t) := \int_0^t \frac{r^{2(m-j)-1}}{(1+r)^\lambda} L(1+r)\theta^\alpha(r)dr$$

is continuous in $[0, \infty)$. This implies that

$$\int_{(|x|-3R)^+}^{|x|+3R} \frac{r^{2(m-j)-1}L(1+r)\theta^\alpha(r)}{(1+r)^\lambda} dr = \varphi(|x|+3R) - \varphi((|x|-3R)^+) \longrightarrow 0 \text{ as } R \longrightarrow 0.$$

As in the above argument, we get

$$\lim_{R \longrightarrow 0} \int_{(|x'|-5R)^+}^{|x'|+5R} \frac{r^{2(m-j)-1}L(1+r)\theta^\alpha(r)}{(1+r)^\lambda} dr = 0.$$

If $|x-y| \geq 3R$ and since $|x-x'| < 2R$, then $|x'-y| \geq R$ and hence

$$\left| |x-y|^{2(m-j)-n} - |x'-y|^{2(m-j)-n} \right| \leq (3^{2(m-j)-n} + 1) R^{2(m-j)-n}.$$

We deduce by the dominated convergence theorem and Remark 2.7 that

$$\int_{|x-y| \geq 3R} \left| |x-y|^{2(m-j)-n} - |x'-y|^{2(m-j)-n} \right| \frac{L(1+|y|)\theta^\alpha(y)}{(1+|y|)^\lambda} dy \longrightarrow 0 \text{ as } |x-x'| \longrightarrow 0.$$

It follows that

$$|Tu(x) - Tu(x')| \longrightarrow 0 \quad \text{as} \quad |x-x'| \longrightarrow 0,$$

which implies that Tu is continuous in $\mathbb{R}^n$. Moreover, since

$$Tu(x) \approx V_{m-j,n}(a\theta^\alpha)(x) \approx \widetilde{\theta}(x),$$

where $\widetilde{\theta}$ is the function, given in Proposition 2.5.

From Lemma 2.2, we deduce that $\widetilde{\theta}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Then Λ is equicontinuous in $\mathbb{R}^n$. Moreover, the family $\{Tu(x); \frac{1}{C}\theta \leq u \leq C\theta\}$ is uniformly bounded in $\mathbb{R}^n$. It follows from Ascoli's theorem that Λ is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$. □

3. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. The aim of this section is to prove the existence of a positive solution of equation (1.1) and to give the asymptotic behavior of such a solution. Our idea is based on the Schauder fixed-point method and the convex set invariant under the integral operators. By Remark 2.6 (i), there exists $c_0 > 0$ such that for each $x \in \mathbb{R}^n$

$$\frac{1}{c_0}\theta(x) \le V_{m,n}(a\theta^\alpha)(x) \le c_0\theta(x).$$

Let $c > 1$ such that $c^{1-|\alpha|} \ge c_0$. In order to apply a fixed point argument, we consider the following convex set given by

$$Y := \left\{ u \in C_0(\mathbb{R}^n);\ \frac{1}{c}\theta \le u \le c\theta \right\}.$$

Then Y is a nonempty closed bounded in $C_0(\mathbb{R}^n)$. Let T be the integral operator defined on Y by

$$Tu(x) := V_{m,n}(au^\alpha)(x) = C_{m,n} \int\limits_{\mathbb{R}^n} \frac{a(y)u^\alpha(y)}{|x-y|^{n-2m}} dy, \quad x \in \mathbb{R}^n.$$

Since for every $u \in Y$ and $-1 < \alpha < 1$, $c^{-|\alpha|}\theta^\alpha \le u^\alpha \le c^{|\alpha|}\theta^\alpha$, then we get

$$\frac{1}{c}\theta \le c^{-|\alpha|}\frac{1}{c_0}\theta \le c^{-|\alpha|}V_{m,n}(a\theta^\alpha) \le Tu \le c^{|\alpha|}V_{m,n}(a\theta^\alpha) \le c^{|\alpha|}c_0\theta \le c\theta.$$

Thus $TY \subset Y$ and we conclude by Proposition 2.8 that TY is relatively compact in $C(\mathbb{R}^n \cup \{\infty\})$.

Next, let us prove the continuity of T in the uniform norm. Let (u_k) be a sequence in Y which converges uniformly to $u \in Y$ and let $x \in \mathbb{R}^n$, we have

$$|Tu_k(x) - Tu(x)| \le c \int\limits_{\mathbb{R}^n} |x-y|^{2m-n} a(y) |u_k^\alpha(y) - u^\alpha(y)| dy$$

and

$$|u_k^\alpha(y) - u^\alpha(y)| \le c\theta^\alpha(y).$$

Then, we deduce by the dominated convergence theorem, for $x \in \mathbb{R}^n$, $Tu_k(x) \longrightarrow Tu(x)$ as $k \longrightarrow \infty$. Finally, since TY is a relatively compact family in $C(\mathbb{R}^n \cup \{\infty\})$, then

$$\|Tu_k - Tu\|_\infty \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

We have proved that T is a compact mapping from Y to itself. So the Schauder fixed point theorem implies the existence of $u \in Y$ which satisfies the integral equation

$$u(x) = C_{m,n} \int\limits_{\mathbb{R}^n} \frac{a(y)u^\alpha(y)}{|x-y|^{n-2m}} dy = V_{m,n}(au^\alpha)(x).$$

Hence, applying $(-\Delta)^m$ on both sides of the equation above, we obtain in the sense of distributions

$$(-\Delta)^m u = au^\alpha.$$

Moreover, by iterated Green function, we have in the sense of distributions, for $0 \le j \le m-1$,

$$(-\Delta)^j u = V_{m-j,n}(au^\alpha).$$

It follows from Proposition 2.5 that

$$V_{m-j,n}(au^\alpha)(x) \approx V_{m-j,n}(a\theta^\alpha)(x) \approx \widetilde{\theta}(x),$$

where $\widetilde{\theta}$ is the function given in Proposition 2.5. Moreover, using (2.1) we have

$$\widetilde{\theta}(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty.$$

This ends the proof. □

We end this section by some examples and remarks.

Example 3.1. Let a be a nonnegative function in $\mathbb{R}^n$ satisfying for $x \in \mathbb{R}^n$,

$$a(x) \approx (1+|x|)^{-\lambda}(\log \omega(1+|x|))^{-\mu},$$

where $\lambda \ge 2m$, $\mu > 1$ and ω is a positive constant large enough.

Then using Theorem 1.3, equation (1.1) has a positive continuous solution u in $\mathbb{R}^n$ satisfying the following estimates:

(i) If $\lambda = 2m$, then for $x \in \mathbb{R}^n$

$$u(x) \approx \big(\log \omega(1+|x|)\big)^{\frac{1-\mu}{1-\alpha}}.$$

(ii) If $2m < \lambda < n - \alpha(n-2m)$, then for $x \in \mathbb{R}^n$

$$u(x) \approx (1+|x|)^{-\frac{\lambda-2m}{1-\alpha}}\big(\log \omega(1+|x|)\big)^{\frac{-\mu}{1-\alpha}}.$$

(iii) If $\lambda \ge n - \alpha(n-2m)$, then for $x \in \mathbb{R}^n$

$$u(x) \approx (1+|x|)^{2m-n}.$$

Remark 3.2. It is clear from Theorem 1.3 that the solution of equation (1.1) satisfies for $x \in \mathbb{R}^n$

$$u(x) \approx (1+|x|)^{-\min(n-2m,\frac{\lambda-2m}{1-\alpha})}\psi_{L,\alpha,\lambda,m}(1+|x|),$$

where $\psi_{L,\alpha,\lambda,m}$ is a function defined in $[1,\infty)$ by

$$\psi_{L,\alpha,\lambda,m}(t) := \begin{cases} \left(\displaystyle\int_t^\infty \frac{L(s)}{s}ds\right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = 2m, \\ (L(t))^{\frac{1}{1-\alpha}} & \text{if } 2m < \lambda < n-(n-2m)\alpha, \\ \left(\displaystyle\int_1^{1+t} \frac{L(s)}{s}ds\right)^{\frac{1}{1-\alpha}} & \text{if } \lambda = n-(n-2m)\alpha, \\ 1 & \text{if } \lambda > n-(n-2m)\alpha. \end{cases} \tag{3.1}$$

We conclude from Lemma 2.2 that the function $\psi_{L,\alpha,\lambda,m}$ is in $\mathcal{K}$.

Example 3.3. Let a_1 and a_2 be positive functions in $\mathbb{R}^n$ such that

$$a_1(x) \approx (1+|x|)^{-\lambda_1} L_1(1+|x|),$$

and

$$a_2(x) \approx (1+|x|)^{-\lambda_2} L_2(1+|x|),$$

where for $i \in \{1,2\}$, $\lambda_i \in \mathbb{R}$ and $L_i \in \mathcal{K}$. We consider the following system:

$$\begin{cases} (-\Delta)^{m_1} u_1 = a_1(x) u_1^{\alpha_{11}}, \\ (-\Delta)^{m_2} u_2 = a_2(x) u_1^{\alpha_{12}} u_2^{\alpha_{22}}, \end{cases} \tag{3.2}$$

where $n > 2\max(m_1, m_2)$, α_{11}, $\alpha_{22} \in (-1,1)$ and $\alpha_{12} \in \mathbb{R}$. We suppose that $\lambda_1 \in [2m_1, \infty)$ and if $\lambda_1 = 2m_1$, one also assumes that $\int_1^\infty t^{-1} L_1(t) dt < \infty$. By Theorem 1.3, there exists a positive continuous solution u_1 to the equation

$$(-\Delta)^{m_1} u_1 = a_1(x) u_1^{\alpha_{11}}.$$

Besides u_1 satisfies for $x \in \mathbb{R}^n$

$$u_1(x) \approx (1+|x|)^{-\gamma} \psi_{L_1,\alpha_{11},\lambda_1,m_1}(1+|x|), \tag{3.3}$$

where $\gamma := \min(n-2m_1, \frac{\lambda_1 - 2m_1}{1-\alpha_{11}})$ and $\psi_{L_1,\alpha_{11},\lambda_1,m_1}$ is the function defined in (3.1) by replacing L by L_1, α by α_{11}, λ by λ_1 and m by m_1.

Now, suppose that $\lambda_2 + \gamma^{\alpha_{12}} \in [2m_2, \infty)$ and if $\lambda_2 + \gamma^{\alpha_{12}} = 2m_2$, one also assumes that

$$\int_1^\infty t^{-1} L_2(t) \psi^{\alpha_{12}}_{L_1,\alpha_{11},\lambda_1,m_1}(t) dt < \infty.$$

Applying again Theorem 1.3, we deduce that equation

$$(-\Delta)^{m_2} u_2 = a_2(x) u_1^{\alpha_{12}} u_2^{\alpha_{22}}$$

has a positive continuous solution u_2 which satisfies for $x \in \mathbb{R}^n$

$$u_2(x) \approx (1+|x|)^{-\min(n-2m_2, \frac{\lambda_2+\gamma^{\alpha_{12}}-2m_2}{1-\alpha_{22}})} \psi_{\widetilde{L_2},\alpha_{22},\lambda,m_2}(1+|x|), \tag{3.4}$$

where $\widetilde{L_2} := L_2 \psi^{\alpha_{12}}_{L_1,\alpha_{11},\lambda_1,m_1}$ and $\lambda := \lambda_2 + \gamma^{\alpha_{12}}$. Hence, the system (3.2) has positive continuous solutions u_1, u_2 which satisfy (3.3) and (3.4), respectively.

Remark 3.4. We note that in the example above due to Remark 3.2 and Lemma 2.2, Theorem 1.3 can be applied recursively to systems of the form

$$(-\Delta)^{m_k} u_k = a_k(x) \prod_{j=1}^{k} u_j^{\alpha_{jk}} = \Big(a_k(x) \prod_{j=1}^{k-1} u_j^{\alpha_{jk}}\Big) u_k^{\alpha_{kk}},$$

$k \in \{1,2,\ldots,K\}$, $K \in \mathbb{N}^*$, under suitable assumptions on the coefficient and exponents.

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