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THE GENERALIZED SINE FUNCTION AND GEOMETRICAL PROPERTIES OF NORMED SPACES

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Abstract. Let $(X, \|\cdot\|)$ be a normed space. We deal here with a function $s : X \times X \to \mathbb{R}$ given by the formula

$$s(x,y) := \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|}$$

(for $x = 0$ we must define it separately). Then we take two unit vectors x and y such that y is orthogonal to x in the Birkhoff-James sense. Using these vectors we construct new functions $\phi_{x,y}$ which are defined on $\mathbb{R}$. If X is an inner product space, then $\phi_{x,y} = \sin$ and, therefore, one may call this function a generalization of the sine function. We show that the properties of this function are connected with geometrical properties of the normed space X.

Keywords: geometry of normed spaces, smoothness, strict convexity, Birkhoff-James orthogonality, conditional functional equations.

Mathematics Subject Classification: 46B20, 39B55, 39B52.

1. INTRODUCTION

The purpose of the present paper is to study the properties of a function which may be viewed as a generalization of the sine function. However, the idea to deal with this function appeared in the theory of functional equations. In the whole paper we assume that $(X, \|\cdot\|)$ is a real normed linear space of dimension greater than or equal to two. Let $f : X \to \mathbb{R}$ and $g : [0,\infty) \to \mathbb{R}$ be some functions. We consider the equation

$$f\left(\frac{x+y}{2}\right) = g\left(\frac{\|x-y\|}{\|x+y\|}\right)[f(x)+f(y)], \quad x,y \in X,\ x \neq -y. \tag{1.1}$$

In [6] solutions f of this equation were determined. We say that function f is a solution of (1.1) if there exists a function $g : [0,\infty) \to \mathbb{R}$ such that the pair (f, g) satisfies (1.1).

Elements x, y of X are called James orthogonal if $\|x+y\| = \|x-y\|$. In such a case we write $x \perp_J y$.

Taking in (1.1) x, y such that $x \perp_J y$ and $x \neq 0$ or $y \neq 0$, we obtain the following equation:

$$f\left(\frac{x+y}{2}\right) = g(1)[f(x)+f(y)].$$

That means that we obtain a conditional modified Jensen equation as a special case of equation (1.1). Therefore equations of this kind are closely connected with conditional equations and, using some results concerning orthogonal equations (see [3], for example), we are able to solve equation (1.1) and other equations of this type. In the case where X is a normed space in which the norm does not come from an inner product we cannot use the natural orthogonality but we still may consider equation (1.1). In such a case the papers dealing with James orthogonal equations are useful (see [4] and [5]).

Moreover, it was proved in Theorem 2 from the paper [6] that in an inner product space if f satisfies (1.1) (with some g), then either $f(x) = a(x) + b$ for some additive function a and some constant b or $f(x) = k\|x\|^2$ for some constant k. Since it is well known that solutions of the orthogonal Cauchy equation

$$x \perp y \Longrightarrow f(x+y) = f(x) + f(y) \tag{1.2}$$

are of the form

$$f(x) = a(\|x\|^2) + b(x),$$

we can see that continuous solutions of (1.2) may be expressed as a sum of two solutions of the equation (1.1) (up to a constant). Furthermore, in Lemma 1 from the same paper it is shown that in a normed space (with dimension greater than or equal to three) in which the norm does not come from an inner product the odd part of a solution of (1.1) must be additive and the even part is constant. These results mean that (1.1) is an unconditional equation which behaves similarly to the equation of orthogonal additivity in the sense of James.

A question may be asked whether similar equations may be found for other kinds of orthogonal equations. Consider, for example, the Birkhoff-James orthogonality. We say that x and y are orthogonal in the sense of Birkhoff-James if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all} \quad \lambda \in \mathbb{R}.$$

If that is the case we write $x \perp_{BJ} y$. The following conditional equation may then be considered:

$$x \perp_{BJ} y \Longrightarrow f(x+y) = f(x) + f(y).$$

If a function $f : X \to \mathbb{R}$ satisfies this equation, then we say that f is Birkhoff-James orthogonally additive. Have a closer look at the equation (1.1). The quotient $\frac{\|x-y\|}{\|x+y\|}$ appearing in this equation characterizes the James orthogonality. More precisely, if we take x, y such that $x + y \neq 0$, then

$$\frac{\|x-y\|}{\|x+y\|} = 1 \Longleftrightarrow x \perp_J y.$$

Now we shall construct a function which is supposed to play the same role for the Birkhoff-James orthogonality.

Definition 1.1. Let $(X, \|\cdot\|)$ be a normed space. The function $s : X \times X \to \mathbb{R}$ is defined by the following formula

$$s(x, y) := \begin{cases} \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

This function characterizes the Birkhoff-James orthogonality in the sense that

$$s(x, y) = 1 \iff x\perp_{BJ}y.$$

Using function s, we are able to consider an equation of the type (1.1) which would be connected with the Birkhoff-James orthogonality. Such equations were considered in the paper [7]. However, in the present paper we are going to study some properties of the function s itself.

2. RESULTS

If X is a Euclidean space, then $s(x, y)$ is equal to the absolute value of the sine of the angle between x and y. Some further properties of this function are pointed out in [8]; let us only note that it can easily be proved that for all $x, y \in X$ and for all $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ we have

$$s(x, y) = s(\alpha x, \beta y). \tag{2.1}$$

Now we shall define a function $\phi_{x,y} : \mathbb{R} \to \mathbb{R}$ which will establish an analogue of the sine function for a given real normed space.

Definition 2.1. Let $(X, \|\cdot\|)$ be normed space and let $x, y \in X$ be unit vectors such that $x\perp_{BJ}y$. Given number $t \in \mathbb{R}$, take any $a, b \in \mathbb{R}$ such that $\angle((1, 0), (a, b)) = t$. Function $\phi_{x,y} : \mathbb{R} \to \mathbb{R}$ is then defined by the following formula:

$$\phi_{x,y}(t) := \mathrm{sgn} b \cdot s(x, ax + by).$$

Note that the above definition is correct, since equality (2.1) holds true. Similarly as before, a lot of properties of this function are to be found in [8]. The most important result contained in that paper is the following theorem.

Theorem 2.2. *Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ be normed spaces. Further, let $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$ be such that $\|x_1\|_1 = \|y_1\|_1 = \|x_2\|_2 = \|y_2\|_2 = 1$ and $x_1\perp_{BJ}y_1$ $x_2\perp_{BJ}y_2$. Then the equality*

$$\phi^1_{x_1,y_1} = \phi^2_{x_2,y_2}$$

implies that

$$\|ax_1 + by_1\|_1 = \|ax_2 + by_2\|_2$$

for all real numbers a, b.

Functions ϕ^1 and ϕ^2 occuring in this theorem are defined as in the Definition 2.1 and are connected with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Using this theorem, we are going to present a characterization of inner product spaces. But first let us quote another result from [8].

Corollary 2.3. *Let $(X, \|\cdot\|)$ be a normed space. Then X is an inner product space if and only if for all $x, y \in X, x \perp_{BJ} y$ with norm equal to 1, and for every $t \in \mathbb{R}$ we have $\phi_{x,y}(t) = \sin t$.*

As we can see, the function $\phi_{x,y}$ may really be called a generalization of the sine function. One can also observe that in the case of X being an inner product space the form of function $\phi_{x,y}$ in fact does not depend on the choice of vectors x and y. A natural question may be asked whether this property characterizes inner product spaces. The following remark will provide us with a positive answer to this question.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a normed space. If for all pairs (x, y) of unit vectors such that $x \perp_{BJ} y$ all $t \in \mathbb{R}$ and some function $f : \mathbb{R} \to \mathbb{R}$ we have $\phi_{x,y}(t) = f(t)$, then the norm $\|\cdot\|$ comes from an inner product and, consequently, $f = \sin$.*

Proof. If X is an inner product space, then our assertion follows directly from Corollary 2.3. Thus we only need to prove the opposite implication.

In the monograph [1] (p. 79) one can find the following conditions. Each of them separately is equivalent for the space $(X, \|\cdot\|)$ to be an inner product space:

(i) $(\|x\| = \|y\| = 1, x \perp_{BJ} y) \Longrightarrow \|x + y\| = \sqrt{2}$,
(ii) $(\|x\| = \|y\| = 1, x \perp_{BJ} y) \Longrightarrow \|x + y\| \leq \sqrt{2}$,
(iii) $(\|x\| = \|y\| = 1, x \perp_{BJ} y) \Longrightarrow \|x + y\| \geq \sqrt{2}$.

Let now $(X, \|\cdot\|)$ be a normed space which is not an inner product space. From the above conditions we know that there exist unit vectors x_1, y_1, x_2, y_2 such that $x_1 \perp_{BJ} y_1$, $x_2 \perp_{BJ} y_2$ and

$$\|x_1 + y_1\| < \sqrt{2} \quad \text{and} \quad \|x_2 + y_2\| > \sqrt{2}.$$

In view of Theorem 2.2, this means that $\phi_{x_1,y_1} \neq \phi_{x_2,y_2}$. □

Till now we have shown that the function s is convenient if we want to characterize inner product spaces. Now we are going to deal with some further properties of normed spaces. However first we need the following result.

Theorem 2.5. *Let $(X, \|\cdot\|)$ be a real normed space. If $x, y \in X$ are unit vectors such that $x \perp_{BJ} y$, then the function $\phi_{x,y}\,_{|\,[0,\frac{\pi}{2}]} : \left[0, \frac{\pi}{2}\right] \to [0, 1]$ is increasing.*
Furthermore, if $\phi_{x,y}(t_1) = \phi_{x,y}(t_2)$ for some $t_1, t_2 \in \left[0, \frac{\pi}{2}\right]$, $t_1 \neq t_2$, then $\phi_{x,y}(t_1) = 1$.

Proof. Let x and y satisfy the assumptions of the theorem. Fix also $t_1, t_2 \in \left(0, \frac{\pi}{2}\right)$, $t_2 > t_1$. We are going to show that $\phi_{x,y}(t_1) \leq \phi_{x,y}(t_2)$. To this end define

$$z_1 := x + (\tan t_1) y \quad \text{and} \quad z_2 := x + (\tan t_2) y.$$

Directly from the definition of the function $\phi_{x,y}$ we have

$$\phi_{x,y}(t_1) = s(x, z_1) \quad \text{and} \quad \phi_{x,y}(t_2) = s(x, z_2).$$

Let us also denote $b_1 := \tan t_1,\ b_2 := \tan t_2$; we clearly have $b_2 > b_1$. Further

$$s(x, z_2) = \min_{\lambda\in\mathbb{R}} \frac{\|x + \lambda z_2\|}{\|x\|} = \min_{\lambda\in\mathbb{R}} \|x + \lambda x + \lambda b_2 y\| = \|(1 + \lambda_2)x + \lambda_2 b_2 y\|$$

with some $\lambda_2 \in \mathbb{R}$. Moreover, we fix λ_2 as a maximal number satisfying the above equality.

Note that $s(x, z_2) \leq 1$ and $\|(1 + \lambda_2)x + \lambda_2 b_2 y\| \geq |1 + \lambda_2|$. The first of these inequalities is a consequence of the properties of the function s and the latter may be obtained from the Birkhoff-James orthogonality of the vectors x, y. It follows that $\lambda_2 \leq 0$ and

$$\lambda_2 = 0 \iff x \perp_{BJ} z_2. \tag{2.2}$$

Putting

$$\alpha := \frac{b_1}{b_1 - \lambda_2(b_2 - b_1)} \quad \text{and} \quad \lambda_1 = \frac{\lambda_2 b_2}{b_1 - \lambda_2(b_2 - b_1)},$$

we obtain

$$\alpha(x + \lambda_2 z_2) = x + \lambda_1 z_1.$$

Hence

$$s(x, z_1) \leq \|x + \lambda_1 z_1\| = \alpha\|x + \lambda_2 z_2\| = \alpha s(x, z_2). \tag{2.3}$$

Let us have a closer look at the number α. Since $\lambda_2 \leq 0$ and $b_2 > b_1$, we have $\alpha \leq 1$. Further, in view of (2.3) we obtain $s(x, z_1) \leq s(x, z_2)$. Thus we have proved the first assertion of the theorem.

Now assume that $s(x, z_1) = s(x, z_2)$; then $\alpha = 1$ and, consequently, $\lambda_2 = 0$. In such a case from (2.2) we have $x \perp_{BJ} z_2$ i.e. $s(x, z_2) = 1$. Finally, from the assumption that $s(x, z_1) = s(x, z_2)$ we infer that $s(x, z_1)$ is also equal to 1. □

Theorem 2.6. *Let $(X, \|\cdot\|)$ be a real normed space, then X is a smooth space if and only if the function $\phi_{x,y}\,|\,[0, \frac{\pi}{2}]$ is strictly increasing for all $x, y \in X$, such that $\|x\| = \|y\| = 1$ and $x \perp_{BJ} y$.*

Proof. Assume that X is not a smooth space. Then there exists a two dimensional subspace Y of the space X such that Y is not smooth (see for example [2, p. 144]). Fix $x \in Y$, $\|x\| = 1$ such that x is not a point of smoothness of the sphere $S(0, 1)$. Now take $y \in Y$ with $\|y\| = 1$ such that $x \perp_{BJ} y$. This means that $\{x + \lambda y : \lambda \in \mathbb{R}\}$ is a supporting hyperplane at x. Since x is not a point of smoothness of $S(0, 1)$, there exists another supporting hyperplane at this point. Thus there exists $z \in Y, \|z\| = 1$, $z \neq y, z \neq -y$ satisfying $x \perp_{BJ} z$. Then we have $z = \alpha x + \beta y$ for some real numbers α and β both different from zero. Without loss of generality we may assume that $\beta > 0$.

Now we shall consider the case where $\alpha > 0$. Then the angle $t := \angle((1, 0), (\alpha, \beta))$ lies in the interval $\left(0, \frac{\pi}{2}\right)$ and we have $\phi_{x,y}(t) = 1$, which means that the function $\phi_{x,y}\,|\,[0, \frac{\pi}{2}]$ is not injective (because we obviously have $\phi_{x,y}\left(\frac{\pi}{2}\right) = 1$).

In the case $\alpha < 0$ we are going to consider the function $\phi_{x,z}$. Let us represent y in the form $\gamma x + \delta z$ for some positive real numbers γ, δ. Then, similarly as before, we get

$$\phi_{x,z}(\angle((1,0),(\gamma,\delta)) = 1.$$

Thus the first part of the proof has been finished.

Now assume that X is a smooth space. Fix any $x, y \in X$, $\|x\| = \|y\| = 1$ such that $x \perp_{BJ} y$. Since X is smooth, so is every two dimensional subspace of X. In particular the space $\mathrm{lin}(x,y)$ is a smooth space.

We are going to show that $\phi_{x,y|(0,\pi/2)}$ is injective. Using Theorem 2.5, we know that it sufficies to show that $\phi_{x,y}(t) \neq 1$ for all $t \in \left(0, \frac{\pi}{2}\right)$. However if we had $\phi_{x,y}(t) = 1$ for some $t \in \left(0, \frac{\pi}{2}\right)$, then we would have $z = \alpha x + \beta y$, such that $\alpha > 0, \beta > 0$ and $x \perp_{BJ} z$. Then $\{x + \lambda z : \lambda \in \mathbb{R}\}$ is a supporting hyperplane at x which is different from $\{x + \lambda y : \lambda \in \mathbb{R}\}$. This contradicts the fact that $\mathrm{lin}(x,y)$ is smooth. □

This finishes the studying of connections between the smoothness of the considered spaces and the properties of the function $\phi_{x,y}$. Now we are going to deal with strict convexity. However, first we need the following lemma which may be viewed as a reformulation of Lemma 2.16 from [8].

Lemma 2.7. *Let $(X, \|\cdot\|)$ be a real normed space and let $x, y \in X$ be unit vectors such that $x \perp_{BJ} y$. Define a function $f : (-1,1) \to \mathbb{R}$ such that the set*

$$S := \{(a,b) \in \mathbb{R}^2 : \|ax + by\| = 1, |a| < 1, b > 0\}$$

forms the graph of f, extend this function continuously to the closed interval $[-1,1]$. Now for a given $\alpha \in (0,1)$ define the function $\psi_\alpha : [1-\alpha, 1+\alpha] \to \mathbb{R}$ by the formula

$$\psi_\alpha(a) := \alpha f\left(\frac{1}{\alpha}(a-1)\right).$$

Then for every $t \in (0, \frac{\pi}{2})$, the value $\phi_{x,y}(t)$ is equal to the minimal number α such that the straight line $l := \{(a,b) : b = (\tan t)\, a\}$ has a nonempty intersection with the graph of ψ_α.

Remark 2.8. Let $t \in \left(0, \frac{\pi}{2}\right)$ and α_0 be such that $\phi_{x,y}(t) = \alpha_0$. In the paper [8] (Lemma 2.13) it was proved that the function ψ_{α_0} is concave. Now from Lemma 2.7 we know that the line $l = \{(a,b) : b = (\tan t)\, a\}$ supports the graph of ψ_{α_0}. In view of the mentioned concavity of the function ψ_{α_0}, this means that l lies (weakly) above the graph of ψ_{α_0}.

Lemma 2.9. *Let $(X_1, \|\cdot\|_1), (X_2, \|\cdot\|_2)$ be real normed spaces and let $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$ be unit vectors satisfying $x_i \perp_{BJ} y_i$, $i \in \{1,2\}$. Define functions $f_i : (-1,1) \to \mathbb{R}$, $i \in \{1,2\}$, in such a way that the following sets*

$$S_i := \{(a,b) \in \mathbb{R}^2 : \|ax_i + by_i\|_i = 1, |a| < 1, b > 0\}$$

form graphs of these functions. If $f_1(r) \leq f_2(r)$ for all $r \in (-1,1)$, then $\phi_{x_1,y_1}(t) \geq \phi_{x_2,y_2}(t)$ for all $t \in \left(0, \frac{\pi}{2}\right)$.

Proof. Fix a $t \in \left(0, \frac{\pi}{2}\right)$ and define the line $l := \{(a,b) : b = (\tan t)\, a\}$. Now continuously extend functions f_i to the interval $[-1,1]$. Observe that these extended functions still satisfy the assumed inequality. For given $\alpha \in (0,1)$ we define functions $\psi^i_\alpha : [1-\alpha, 1+\alpha] \to \mathbb{R}$ by the formula

$$\psi^i_\alpha(a) := \alpha f_i\left(\frac{1}{\alpha}(a-1)\right).$$

Note that for every $\alpha \in (0,1)$ we have $\psi^1_\alpha \leq \psi^2_\alpha$, since the functions f_i satisfy the analogous inequality.

Put

$$\alpha_i := \phi_{x_i,y_i}(t) = s(x_i, x_i + (\tan t)\, y_i) = \min_{\lambda\in\mathbb{R}} \|x_i + \lambda x_i + \lambda(\tan t)\, y_i\|_i.$$

From Lemma 2.7 we infer that α_i is equal to the minimal number $\alpha \in (0,1)$ for which the graph of the function ψ^i_α has a nonempty intersection with the line l.

Using Remark 2.8 and the fact that the graph of the function ψ^1_α lies below the graph of ψ^2_α, we can see that such α for the function ψ^1_α is greater than or equal to the corresponding number for the function ψ^2_α. □

Theorem 2.10. *Let $(X, \|\cdot\|)$ be a real normed space. If for all unit vectors $x, y \in X$ satisfying $x \perp_{BJ} y$ the function $\phi_{x,y}$ is differentiable at the point $\frac{\pi}{2}$, then X is a strictly convex space.*

Proof. Let X be a real normed linear space which is not strictly convex. We are going to find vectors $x, y \in X$ satisfying the assumptions of the theorem such that the function $\phi_{x,y}$ will not be differentiable at $\frac{\pi}{2}$. Since X is not strictly convex, there exist unit vectors x_1, x_2 different from each other such that

$$\|\lambda x_1 + (1-\lambda)x_2\| = 1 \quad \text{for } \lambda \in [0,1]. \tag{2.4}$$

Put $x := \frac{x_1+x_2}{2}$ and $y := \frac{x_1-x_2}{\|x_1-x_2\|}$. It is easy to see that the norms of the vectors x, y are both equal to 1. We shall prove that $x \perp_{BJ} y$. Supposing on the contrary, we obtain $\|x + \lambda y\| < \|x\| = 1$ for some $\lambda \in \mathbb{R}$. Further, we may write

$$1 > \left\|\frac{x_1+x_2}{2} + \lambda_0(x_1 - x_2)\right\| = \left\|x_1\left(\frac{1}{2} + \lambda_0\right) + x_2\left(\frac{1}{2} - \lambda_0\right)\right\|$$

where $\lambda_0 := \frac{\lambda}{\|x_1-x_2\|}$. If now $\lambda_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then we obtain a contradiction with (2.4).

Define $\gamma := \frac{1}{2} + \lambda_0$ if $\lambda_0 > \frac{1}{2}$ and $\gamma := \frac{1}{2} - \lambda_0$ if $\lambda_0 < -\frac{1}{2}$. Then

$$0 < \|\gamma x_1 + (1-\gamma)x_2\| < 1$$

and moreover $\gamma > 1$. Now consider the vector

$$z := \frac{\gamma x_1 + (1-\gamma)x_2}{\|\gamma x_1 + (1-\gamma)x_2\|}.$$

We have $\|z\| = 1$ which implies

$$\|\alpha z + (1-\alpha)x_2\| \le 1 \quad \text{for} \quad \alpha \in (0,1). \tag{2.5}$$

For the sake of simplicity, we put $\beta := \|\gamma x_1 + (1-\gamma)x_2\|$; note that $\beta < 1$. Take $\alpha := \frac{\beta}{\gamma+\beta-1}$ and write

$$\begin{aligned}\alpha z + (1-\alpha)x_2 &= \frac{\beta}{\gamma+\beta-1}\frac{\gamma x_1 + (1-\gamma)x_2}{\beta} + \frac{\gamma-1}{\gamma+\beta-1}x_2 \\ &= \frac{\gamma}{\gamma+\beta-1}x_1 + \left(\frac{(1-\gamma)\beta}{\beta(\gamma+\beta-1)} + \frac{\gamma-1}{\gamma+\beta-1}\right)x_2 = \frac{\gamma}{\gamma+\beta-1}x_1.\end{aligned}$$

Thus we have

$$\|\alpha z + (1-\alpha)x_2\| = \frac{\gamma}{\gamma+\beta-1} > 1,$$

since $\beta < 1$. Further since $\alpha < 1$, the above inequality yields a contradiction with (2.5), i.e. we have shown that x and y defined above are unit vectors, which are orthogonal in the Birkhoff-James sense.

Now our aim is to show that the function $\phi_{-x,y}$ is not differentiable at $\frac{\pi}{2}$. To this end consider a function $f : (-1,1) \to \mathbb{R}$ such that the set

$$S := \{(a,b) \in \mathbb{R}^2 : \|a(-x) + by\| = 1, |a| < 1, b > 0\}$$

is a graph of this function. Extend f in a continuous way to a function which will be defined on the closed interval $[-1,1]$.

Now define the function

$$g(r) := \begin{cases} [1-f(-1)]r + 1 & \text{for} \quad r \in [-1,0], \\ [-1+f(1)]r + 1 & \text{for} \quad r \in (0,1]. \end{cases}$$

Since $f(0) = 1$, from the concavity of f it results that $g(r)$ is less than or equal to $f(r)$ for every $r \in [-1,1]$. Let $G_1, \ldots, G_4 \subset \mathbb{R}^2$ be defined in the following way:

$$\begin{aligned} &G_1 := \{(r, g(r) : r \in (-1,1)\}, && G_2 := \{(r, -g(-r) : r \in (-1,1)\}, \\ &G_3 := \{(-1,s) : s \in [-f(1), f(-1)]\}, && G_4 := \{(1,s) : s \in [-f(-1), f(1)]\}, \end{aligned}$$

and let G be the set bounded by $G_1 \cup G_2 \cup G_3 \cup G_4$. Then the Minkowski functional associated with G yields a norm $\|\cdot\|_0 : \mathbb{R}^2 \to [0,\infty)$, for which the graph of the function $g_{|(-1,1)}$ coincides with the set

$$\{(a,b) \in \mathbb{R}^2 : \|ax_0 + by_0\|_0 = 1, |a| < 1, b > 0\}$$

where $x_0 = (1,0), y_0 = (0,1)$ satisfy $x_0 \perp_{BJ} y_0$. Similarly as before, we define the function $\psi_\alpha : [1-\alpha, 1+\alpha] \to \mathbb{R}$ by the formula $\psi_\alpha(a) := \alpha g\left(\frac{1}{\alpha}(a-1)\right)$. Now fix $t \in \left(0, \frac{\pi}{2}\right)$. We want to evaluate $\phi_{x_0,y_0}(t)$. To this end consider the line $l := \{(a,b) : b = (\tan t)\ a\}$. We are looking for the minimal number α such that the graph of the function ψ_α has a nonempty intersection with the line l. Note that if t is chosen sufficiently close to $\frac{\pi}{2}$ then the only common point of the line l and the graph of the function ψ_α is the point $(1-\alpha, \alpha f(-1))$. Since this point belongs to the mentioned line, we are able to write

$$\alpha f(-1) = (\tan t)\,(1-\alpha), \tag{2.6}$$

i.e.

$$\alpha(f(-1) + \tan t) = \tan t$$

and finally

$$\alpha = \phi_{x_0,y_0}(t) = \frac{\tan t}{f(-1) + \tan t}.$$

On the other hand, from Lemma 2.9 we infer that for every $t \in \left(0, \frac{\pi}{2}\right)$ we have

$$\phi_{-x,y}(t) \le \phi_{x_0,y_0}(t).$$

This means that, in particular,

$$\phi_{-x,y}(t) \le \frac{\tan t}{f(-1) + \tan t} =: h(t) \tag{2.7}$$

for all t from a certain left-hand neighbourhood of $\frac{\pi}{2}$. Now let us assume that $\phi_{-x,y}$ is differentiable at $\frac{\pi}{2}$. We also know that this function has a maximum at the point $\frac{\pi}{2}$ since its value at this point is equal to 1. Therefore, we have $\phi'_{-x,y}\left(\frac{\pi}{2}\right) = 0$. It is a contradiction with (2.7). Indeed, note that from (2.6) we get the inequality $f(-1) > 0$. Thus we have

$$\lim_{t \to \frac{\pi}{2}} h'(t) = f(-1) > 0. \qquad \square$$

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