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ON SMALL VIBRATIONS OF A DAMPED STIELTJES STRING

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Abstract. Inverse problem of recovering masses, coefficients of damping and lengths of the intervals between the masses using two spectra of boundary value problems and the total length of the Stieltjes string (an elastic thread bearing point masses) is considered. For the case of point-wise damping at the first counting from the right end mass the problem of recovering the masses, the damping coefficient and the lengths of the subintervals by one spectrum and the total length of the string is solved.

Keywords: damping, Dirichlet boundary condition, point mass, Hermite-Biehler polynomial, continued fraction, eigenvalues.

Mathematics Subject Classification: 35Q99, 39A99.

1. INTRODUCTION

Investigation of spectral problems for damped systems was started in [8]. As far as we know first results on inverse problems for damped strings were obtain in [2] and [15,16]. In these papers the left end of the string was supposed to be free and the right end damped or in other words the right end of the string could move with viscous friction in the direction orthogonal to the equilibrium position of the string. In these papers very wide classes of strings were considered. In [2] the class of so-called regular strings was used, i.e. strings of finite mass and length, while in [15, 16] the class of so-called S-strings, i.e. strings of finite lengths and finite first momentum of mass distribution. Conditions necessary and sufficient for a sequence of complex numbers to be the spectrum of a damped string were given in [2] in implicit form and in [16] and [15] explicitly. If the string is smooth such that $\rho \in W_2^2(0,l)$ and $\rho(s) \geq \epsilon > 0$, where ρ is the density of the string and l is its length, then one can apply the Liouville transformation ([5, p. 202]) to reduce the equation of the string to the Sturm-Liouville equation. The corresponding boundary value problem with the spectral parameter in the boundary conditions was considered in many publications (see [7, 21, 22, 27] and references therein). In [10] and [23] conditions on a sequence of complex numbers were given necessary and sufficient to be the spectrum of a smooth inhomogeneous string damped at one end for any value of the damping parameter except for one crucial value.

The opposite case of extremely nonsmooth, so-called Stieltjes string (i.e. a thread bearing point masses) was considered in [9] and [14] assuming absence of damping. The inverse problem, i.e. the problem of recovering the parameters of the string by the spectrum of its vibrations and by the total length of the string, for Stieltjes strings of finite number of masses with point-wise (i.e. one dimensional) damping at the right end and the left end free was solved in [2]. Also this problem can be reduced to the problem of damped oscillators considered in [29, 30]. Another approach to the inverse problem for damped finite dimensional systems was developed in [17] where the given data included not only eigenvalues but also the so-called Jordan pairs. An inverse problem for a Stieltjes string with damping at the midpoint was solved in [4]. A nice review can be found in [6]. Inverse problems generated by the Stieltjes string recurrence relations on graph domains were considered in [24, 25] and [18].

The case of distributed damping is not investigated in detail. We should mention [31], where it was proved that two spectra of Dirichlet-Dirichlet boundary value problem (the problem with the Dirichlet boundary conditions at both ends) and Dirichlet-Neumann boundary problem (the problem with the Dirichlet boundary condition at the left end and the Neumann boundary condition at the right end) uniquely determine the density and the stiffness of the string if the damping is constant but the problem with constant damping considered in this paper can be reduced by a change of the spectral parameter to the case of an undamped string. In [1, 12] the inverse problems for a damped string were considered on the semi-axis and on the axis, respectively.

We consider the case of a Stieltjes string with finite number of point masses. In Section 2 we describe the spectra of a Stieltjes string small transversal vibrations with the both ends fixed (Dirichlet-Dirichlet boundary value problem) and with the left end fixed and the right end free (Dirichlet-Neumann boundary value problem). In Sections 2 and 3 we consider the case where only the mass neighboring the right end is damped. The Dirichlet-Neumann boundary value problem in this case can be reduced to the problem considered in [29, 30] and [3].

In Section 3 we solve the inverse Dirichlet-Dirichlet problem, i.e. the problem of recovering the values of masses, of subintervals and of the damping coefficient of the damped mass by given spectrum, the total length of the string and the length of the last subinterval at the right end.

It is well known that in the undamped case the eigenvalues of these two problems interlace. In Section 4 the analogues of the interlacing conditions for the case of one dimensional damping at the mass neighboring the right end are found. They appear to be the set of equalities and inequalities involving the Dirichlet-Dirichlet and Dirichlet-Neumann eigenvalues. In Section 5 the problem of vibrations of the Stieltjes string is considered for the case where all the masses are damped. The aim is to recover not only the values of the masses and the lengths of the subintervals but the coefficients of damping also. As the given data the spectra of the

Dirichlet-Dirichlet and Dirichlet-Neumann problems are used together with the total length of the string. The conditions on two sequences of complex numbers to be the spectra of the Dirichlet-Dirichlet and Dirichlet-Neumann boundary value problems are established. These conditions are given in implicit form: the ratio of the corresponding characteristic polynomials should admit decomposition in continued fraction of a special type. The decomposition is an analogue to that in [9] based on the results of [28].

2. DIRECT PROBLEM FOR A DAMPED STIELTJES STRING

Like in [9] we suppose the string to be a thread (i.e. a string of zero density) bearing a finite number of point masses. Let l_k $(k = 0, 1, \ldots, n)$ be the lengths of the intervals of zero density and let m_k $(k = 1, 2, \ldots, n)$ be the values of the masses separating the intervals (l_k lies between m_k and m_{k+1}), the last mass has only one thread at the left and $m_k > 0$ for $k = 1, 2, \ldots, n$. Let us denote $\alpha_k \geq 0$ the coefficient of damping (viscous friction) of the point mass m_k. Denote by $v_k(t)$ the transversal displacements of the point masses at the time t.

We impose a Dirichlet boundary condition on the left end, that is, the left end is fixed and consider the two cases: 1) the right end is fixed (Dirichlet-Dirichlet problem) and 2) right end is free to move in the direction orthogonal to the equilibrium position of the string (Dirichlet-Neumann problem).

We assume the thread to be stretched by the stretching force which is equal to 1. Taking into account that on the intervals of zero density the general solution of the differential equation is a linear function of s multiplied by a function of t we obtain

$$\frac{v_k(t) - v_{k+1}(t)}{l_k} + \frac{v_k(t) - v_{k-1}(t)}{l_{k-1}} + m_k v_k''(t) + \alpha_k v_k'(t) = 0 \quad (k = 1, 2, \ldots, n). \tag{2.1}$$

In this section we assume $\alpha_k = 0$ for $k = 1, 2, \ldots, n-1$ and $\alpha_n = \alpha > 0$. Substituting $v_k(t) = u_k e^{i\lambda t}$ we obtain

$$\frac{u_k - u_{k+1}}{l_k} + \frac{u_k - u_{k-1}}{l_{k-1}} - m_k \lambda^2 u_k = 0 \quad (k = 1, 2, \ldots, n-1), \tag{2.2}$$

$$\frac{u_n - u_{n+1}}{l_n} + \frac{u_n - u_{n-1}}{l_{n-1}} - m_n \lambda^2 u_n + i\alpha\lambda u_n = 0. \tag{2.3}$$

For the Dirichlet-Dirichlet problem we suppose the ends of the thread to be fixed, i.e. $v_0(t) = v_{n+1}(t) = 0$, what means that

$$u_0 = 0, \tag{2.4}$$

$$u_{n+1} = 0. \tag{2.5}$$

For the Dirichlet-Neumann problem we assume that the right end is free and bears no point mass. Then the boundary condition at the right looks as follows:

$$u_{n+1} = u_n. \tag{2.6}$$

According to [9] we have

$$u_k = R_{2k-2}(\lambda^2)u_1 \quad (k = 1, 2, \ldots, n-1), \tag{2.7}$$

where $R_{2k-2}(\lambda^2)$ is a polynomial of degree $2k-2$ obtained by (2.2) and by definition

$$R_{2k-1}(\lambda^2) = \frac{R_{2k}(\lambda^2) - R_{2k-2}(\lambda^2)}{l_k}.$$

Due to (2.2) the polynomials R_k satisfy the recurrence conditions:

$$R_{2k-1}(\lambda^2) = -\lambda^2 m_k R_{2k-2}(\lambda^2) + R_{2k-3}(\lambda^2), \tag{2.8}$$

$$R_{2k}(\lambda^2) = l_k R_{2k-1}(\lambda^2) + R_{2k-2}(\lambda^2), \tag{2.9}$$

$$(k = 1, 2, \ldots, n-1, \quad R_{-1}(\lambda^2) = \frac{1}{l_0}, \quad R_0(\lambda^2) = 1).$$

Using (2.7)–(2.9) and taking into account the boundary condition (2.5) we rewrite (2.3) as follows:

$$\phi(\lambda) := R_{2n-3}(\lambda^2) + (-m_n\lambda^2 + i\lambda\alpha + l_n^{-1})R_{2n-2}(\lambda^2) = 0. \tag{2.10}$$

The spectrum $\{\nu_k\}$ $(k = \pm 1, \pm 2, \ldots, \pm n)$ of problem (2.2)–(2.5) coincides with the set of zeros of $\phi(\lambda)$. We are also interested in problem (2.2)–(2.4), (2.6). Again using (2.7)–(2.9) and taking into account boundary condition (2.6) we obtain from (2.3):

$$\psi(\lambda) := R_{2n-3}(\lambda^2) + (-m_n\lambda^2 + i\lambda\alpha)R_{2n-2}(\lambda^2) = 0. \tag{2.11}$$

The spectrum $\{\mu_k\}$ $(k = \pm 1, \pm 2, \ldots, \pm n)$ of problem (2.2)–(2.4), (2.6) coincides with the set of zeros of $\psi(\lambda)$.

We call $\phi(\lambda)$ and $\psi(\lambda)$ characteristic polynomials of problems (2.2)–(2.5) and (2.2)–(2.4), (2.6), respectively.

It is known [9] that

$$\frac{R_{2n-2}(\lambda^2)}{R_{2n-3}(\lambda^2)} = l_{n-1} + \cfrac{1}{-m_{n-1}\lambda^2 + \cfrac{1}{l_{n-2} + \cfrac{1}{-m_{n-2}\lambda^2 + \ldots + \cfrac{1}{l_1 + \cfrac{1}{-m_1\lambda^2 + \frac{1}{l_0}}}}}}. \tag{2.12}$$

Definition 2.1. A function $\omega(\lambda)$ is said to be a Nevanlinna function (or R-function in terms of [13]) if:

1) it is analytic in the half-planes $\mathrm{Im}\lambda > 0$ and $\mathrm{Im}\lambda < 0$,
2) $\omega(\overline{\lambda}) = \overline{\omega(\lambda)}$ $(\mathrm{Im}\lambda \neq 0)$,
3) $\mathrm{Im}\lambda\ \mathrm{Im}\omega(\lambda) \geq 0$ for $\mathrm{Im}\lambda \neq 0$.

Definition 2.2 ([13])**.** A Nevanlinna function $\omega(\lambda)$ is said to be an S-function if it is defined and analytic in $C\backslash[0, \infty)$ and $\omega(\lambda) > 0$ for $\lambda < 0$. A meromorphic S-function is said to be an S_0-function if $\omega(0) < \infty$.

Lemma 2.3 ([13]). *The function* $\frac{R_{2n-2}(z)}{R_{2n-3}(z)}$ *is an* S_0*-function.*

Definition 2.4. A polynomial is said to be Hermite-Biehler (HB) if all its zeros lie in the open upper half-plane.

It should be mentioned that the transformation $\lambda \to i\lambda$ transforms a HB polynomial into a so-called Hurwitz polynomial.

Theorem 2.5 (Hermite-Biehler theorem, see [11,19]). *In order for the polynomial*

$$\omega(\lambda) = P(\lambda) + iQ(\lambda)$$

where $P(\lambda)$ *and* $Q(\lambda)$ *are real polynomials, to have no zeros in the closed lower half-plane* $\operatorname{Im}\lambda \leq 0$*, i.e. belong to HB, it is necessary and sufficient that the following conditions be satisfied:*

1) *the polynomials* $P(\lambda)$ *and* $Q(\lambda)$ *have only simple real zeros, while these zeros separate one another, i.e. between two successive zeros of one of these polynomials there lies exactly one zero of the other,*
2) *at some point* λ_0 *of the real axis*

$$Q'(\lambda_0)P(\lambda_0) - Q(\lambda_0)P'(\lambda_0) > 0.$$

The fact that the two polynomials satisfy condition 1) will be expressed by saying that "the zeros of the polynomials $P(\lambda)$ and $Q(\lambda)$ are interlaced".

Now Lemma 2.3 and Theorem 2.5 imply the following result.

Corollary 2.6. *The polynomial* $P(\lambda^2)+i\lambda Q(\lambda^2)$ *belongs to the Hermite-Biehler class.*

Definition 2.7. The polynomial $\omega(\lambda)$ is said to be symmetric if $\omega(-\overline{\lambda}) = \overline{\omega(\lambda)}$ for all $\lambda \in \mathbb{C}$. The polynomial $\omega(\lambda)$ is said to belong to the class SHB if it is symmetric and belongs to the Hermite-Biehler class.

For a symmetric polynomial $\omega(\lambda)$ the following is valid:

$$\omega(\lambda) = P(\lambda) + iQ(\lambda) = P(\lambda) + i\lambda\hat{Q}(\lambda) = \tilde{P}(\lambda^2) + i\lambda\tilde{\hat{Q}}(\lambda^2),$$

where $P(\lambda)$ and $\hat{Q}(\lambda)$ are real even functions. Here

$$\tilde{P}(\lambda^2) = P(\lambda), \quad \tilde{\hat{Q}}(\lambda^2) = \hat{Q}(\lambda).$$

3. INVERSE PROBLEMS I AND II

In this section we consider the following inverse problems.

Inverse problem I. Given the total length of the string $l > 0$, the length of the right subinterval $l_n \in (0, l)$ and the spectrum $\{\nu_k\}$ $(k = \pm1, \pm2, \ldots, \pm n)$ of problem (2.2)–(2.5). Find $\{m_k\}$ $(k = 1, 2, \ldots, n)$ and $\{l_k\}$ $(k = 0, 1, \ldots, n-1)$.

Inverse problem II. Given the total length of the string $l > 0$, the length of the right subinterval $l_n \in (0, l)$ and the spectrum $\{\mu_k\}$ $(k = \pm1, \pm2, \ldots, \pm n)$ of problem (2.2)–(2.4), (2.6). Find $\{m_k\}$ $(k = 1, 2, \ldots, n)$ and $\{l_k\}$ $(k = 0, 1, \ldots, n-1)$.

Theorem 3.1. *Let* $l > 0$ *and* $l_n \in (0, l)$ *be given together with the set of complex numbers* $\{\nu_k\}$ $(k = \pm 1, \pm 2, \ldots \pm n)$ *which satisfy the conditions:*

1) $\operatorname{Im} \nu_k > 0$ *for* $k = \pm 1, \pm 2, \ldots \pm n$,
2) $\nu_{-k} = -\overline{\nu_k}$ *for not pure imaginary* ν_{-k} *and the multiplicities of symmetrically located numbers are equal.*

Then there exists a unique Stieltjes string, i.e. a unique set of intervals $l_k > 0$ $(k = 0, 1, \ldots, n-1)$ *of total length* $\sum_{k=0}^{n-1} l_k = l - l_n$, *a unique set of masses* $m_k > 0$ $(k = 1, 2, \ldots, n)$ *and a unique positive number* α *which generate problem* (2.2)–(2.5) *with the spectrum coinciding with the set* $\{\nu_k\}$ $(k = \pm 1, \pm 2, \ldots, \pm n)$.

Proof. Let us construct the polynomial

$$\Phi(\lambda) = \prod_{-n,\ k\neq 0}^{n} \left(1 - \frac{\lambda}{\nu_k}\right). \tag{3.1}$$

Due to the symmetry of the zeros of this polynomial the following even polynomials are real:

$$P(\lambda^2) = \frac{\Phi(\lambda) + \Phi(-\lambda)}{2}$$

and

$$Q(\lambda^2) = \frac{\Phi(\lambda) - \Phi(-\lambda)}{2i\lambda}. \tag{3.2}$$

Set

$$\alpha = Q(0)\left(\frac{1}{l - l_n} + \frac{1}{l_n}\right). \tag{3.3}$$

Using (3.1) and (3.2) we obtain

$$\alpha = i\left(\frac{1}{l - l_n} + \frac{1}{l_n}\right) \sum_{k=-n, k\neq 0}^{n} \frac{1}{\nu_k} \tag{3.4}$$

and because of the symmetry in location of the zeros of the polynomial $\Phi(\lambda)$ and conditions 1) and 2) we conclude that $\alpha > 0$. Set

$$m_n = -\alpha \lim_{|\lambda| \to \infty} \frac{P(\lambda^2)}{\lambda^2 Q(\lambda^2)}. \tag{3.5}$$

The limit in the right-hand side of (3.5) exists because the degree of $P(\lambda^2)$ is $2n$ and the degree of $Q(\lambda^2)$ is $2n - 2$. Moreover,

$$\frac{P(\lambda^2)}{\lambda^2 Q(\lambda^2)} \underset{|\lambda| \to \infty}{=} \left(i \sum_{k=-n, k\neq 0}^{n} \nu_k\right)^{-1} + o(1).$$

Conditions 1) and 2) imply

$$i \sum_{k=-n, k\neq 0}^{n} \nu_k < 0,$$

and according to (3.5)

$$m_n = -\alpha \left(i \sum_{k=-n, k\neq 0}^{n} \nu_k \right)^{-1} > 0.$$

Since $P(\lambda^2) + i\lambda Q(\lambda^2) = \Phi(\lambda)$ belongs to SHB, $\phi(\lambda) := P(\lambda^2) + i\lambda\alpha^{-1}Q(\lambda^2)$ is also a SHB polynomial by the Hermite-Biehler theorem. We consider the polynomial

$$\phi(\lambda, m_n, l_n^{-1}) := P(\lambda^2) + \alpha^{-1}(m_n\lambda^2 - l_n^{-1})Q(\lambda^2) + i\lambda\alpha^{-1}Q(\lambda^2)$$

as a perturbation of $P(\lambda^2)+i\lambda\alpha^{-1}Q(\lambda^2)$. Since $\phi(\lambda, \eta, \zeta)$ is a polynomial with respect to the variables λ, η and ζ, the zeros of it in the λ-plane are piecewise analytic and continuous functions of η and of ζ ([20]). The zeros do not cross the real axis when η changes from 0 to m_n and ζ changes from 0 to l_n^{-1}. Otherwise, we would have $P(\lambda^2) = \lambda Q(\lambda^2) = 0$ for some $\eta > 0$, some $\zeta > 0$ and some real λ. If this $\lambda \neq 0$ then $Q(\lambda^2) = 0$ and consequently, $P(\lambda^2) = 0$ and $\Phi(\lambda) = 0$ for this real λ, what contradicts condition 1). If $\phi(0, \eta, \zeta) = 0$, then

$$P(0) - \alpha^{-1}\zeta Q(0) = 1 - \alpha^{-1}\zeta Q(0) = 0.$$

This is impossible for $\zeta \in (0, l_n^{-1})$ due to (3.3). The degree of the polynomial $\phi(\lambda, \eta, \zeta)$ is $2n$ for each $\eta \in [0, m_n)$ and each $\zeta \in [0, l_n^{-1}]$ and the degree is equal $2n - 1$ for $\eta = m_n$ and each $\zeta \in [0, l_n^{-1}]$. This means the zeros do not come from infinity. Therefore, $\phi(\lambda, \eta, \zeta) \in SHB$ for each $\zeta \in [0, l_n^{-1}]$ and each $\eta \in [0, m_n]$. This implies (see [19, p. 308]) that

$$\frac{\alpha^{-1}Q(z)}{P(z) + (m_n z - l_n^{-1})\alpha^{-1}Q(z)}$$

is an S-function. Then according to [9] we have

$$\frac{\alpha^{-1}Q(z)}{P(z) + (m_n z - l_n^{-1})\alpha^{-1}Q(z)} = a_{n-1} + \cfrac{1}{-b_{n-1}z + \cfrac{1}{a_{n-2} + \cfrac{1}{-b_{n-2}z + \ldots + \cfrac{1}{a_1 + \cfrac{1}{-b_1 z + \frac{1}{a_0}}}}}} \tag{3.6}$$

with $a_k > 0$ and $b_k > 0$ for each k.

We identify a_k with the length of k-th interval and b_k with the k-th mass of a Stieltjes string, i.e.

$$\frac{\alpha^{-1}Q(\lambda^2)}{P(\lambda^2) + (m_n\lambda^2 - l_n^{-1})\alpha^{-1}Q(\lambda^2)} = \frac{R_{2n-2}(\lambda^2)}{R_{2n-3}(\lambda^2)}, \tag{3.7}$$

where $R_{2n-2}(\lambda^2)$ and $R_{2n-3}(\lambda^2)$ are the corresponding polynomials for this Stieltjes string. Consequently,

$$\alpha^{-1}Q(\lambda^2) = TR_{2n-2}(\lambda^2),$$

$$P(\lambda^2) + (m_n\lambda^2 - l_n^{-1})\alpha^{-1}Q(\lambda^2) = TR_{2n-3}(\lambda^2),$$

where T is a positive constant. Therefore,

$$\Phi(\lambda) = P(\lambda^2) + i\lambda Q(\lambda^2) = T(R_{2n-3}(\lambda^2) + (-m_n\lambda^2 + l_n^{-1} + i\lambda\alpha)R_{2n-2}(\lambda^2)).$$

According to (2.10), this means that the set $\{\nu_k\}$ is the spectrum of problem (2.2)–(2.5) with the masses b_k ($k = 1, 2, \ldots, n-1$) and m_n and the lengths a_k ($k = 0, 1, \ldots, n-1$) and l_n damped at the mass m_n with the coefficient of damping α. The length of the interval between the left end of the string and the mass m_n according to [9] is equal to $\frac{R_{2n-2}(0)}{R_{2n-3}(0)}$. From (3.7) we obtain

$$\frac{R_{2n-2}(0)}{R_{2n-3}(0)} = \frac{\alpha^{-1}Q(0)}{P(0) - l_n^{-1}\alpha^{-1}Q(0)}. \tag{3.8}$$

Using (3.3) and the evident identity $P(0) = 1$ we obtain

$$\frac{R_{2n-2}(0)}{R_{2n-3}(0)} = l - l_n. \tag{3.9}$$

Let us prove uniqueness of the solution to our inverse problem. Suppose there exists another Stieljes string with the same total length l, the same length of the right interval l_n between the fixed end and the first mass from the right (which is damped) having the same spectrum $\{\nu_k\}_{k=-n,k\neq 0}^{n}$. In other words, we suppose that there exist sequences of positive numbers $\{\tilde{m}_k\}_{k=1}^{n}$ and $\{\tilde{l}_k\}_{k=0}^{n-1}$ ($\sum_{k=0}^{n-1}\tilde{l}_k = l - l_n$), not identical with sets $\{m_k\}_{k=1}^{n}$ and $\{l_k\}_{k=0}^{n-1}$, which together with $\tilde{l}_n = l_n$ generate problem (2.2)–(2.5) having the same spectrum $\{\nu_k\}_{k=-n,k\neq 0}^{n}$.

All the quantities related to problem (2.2)–(2.5) generated by $\{\tilde{m}_k\}_{k=1}^{n}$ and $\{\tilde{l}_k\}_{k=0}^{n-1}$ and $\tilde{l}_n = l_n$ will have the sign tilde. Then we have the following analogue of (2.12):

$$\frac{\tilde{R}_{2n-2}(\lambda^2)}{\tilde{R}_{2n-3}(\lambda^2)} = \tilde{l}_{n-1} + \cfrac{1}{-\tilde{m}_{n-1}\lambda^2 + \cfrac{1}{\tilde{l}_{n-2} + \cfrac{1}{-\tilde{m}_{n-2}\lambda^2 + \ldots + \cfrac{1}{\tilde{l}_1 + \cfrac{1}{-\tilde{m}_1\lambda^2 + \frac{1}{\tilde{l}_0}}}}}}. \tag{3.10}$$

The analogue of (2.10) is

$$\tilde{\phi}(\lambda) := \tilde{R}_{2n-3}(\lambda^2) + (-\tilde{m}_n\lambda^2 + i\lambda\tilde{\alpha} + l_n^{-1})\tilde{R}_{2n-2}(\lambda^2) = 0. \tag{3.11}$$

Since the sets of zeros of $\phi(\lambda)$ and $\tilde{\phi}(\lambda)$ coincide with the spectrum $\{\nu_k\}_{k=-n,k\neq 0}^{n}$, we conclude that

$$\tilde{\phi}(\lambda) = C\phi(\lambda) \tag{3.12}$$

with a constant C which is positive, because the following are positive

$$\tilde{\phi}(0) = \tilde{R}_{2n-3}(0) + l_n^{-1}\tilde{R}_{2n-2}(0) \quad \text{and} \quad \phi(0) = R_{2n-3}(0) + l_n^{-1}R_{2n-2}(0).$$

Using (2.10) and (3.11) we obtain from (3.12):

$$\tilde{R}_{2n-3}(\lambda^2) + (-\tilde{m}_n\lambda^2 + l_n^{-1})\tilde{R}_{2n-2}(\lambda^2) = C(R_{2n-3}(\lambda^2) + (-m_n\lambda^2 + l_n^{-1})R_{2n-2}(\lambda^2)) \tag{3.13}$$

and

$$\tilde{\alpha}\tilde{R}_{2n-2}(\lambda^2) = C\alpha R_{2n-2}(\lambda^2). \tag{3.14}$$

Equations (3.13) and (3.14) imply

$$\frac{\tilde{R}_{2n-3}(\lambda^2)}{\tilde{\alpha}\tilde{R}_{2n-2}(\lambda^2)} - \frac{\tilde{m}_n\lambda^2}{\tilde{\alpha}} + \frac{1}{\tilde{\alpha}l_n} = \frac{R_{2n-3}(\lambda^2)}{\alpha R_{2n-2}(\lambda^2)} - \frac{m_n\lambda^2}{\alpha} + \frac{1}{\alpha l_n}. \tag{3.15}$$

Since the polynomials $R_{2n-2}(\lambda^2)$, $R_{2n-3}(\lambda^2)$, $\tilde{R}_{2n-2}(\lambda^2)$ and $\tilde{R}_{2n-3}(\lambda^2)$ are of the same degree, (3.15) implies

$$\frac{\tilde{m}_n}{\tilde{\alpha}} = \frac{m_n}{\alpha} \tag{3.16}$$

and

$$\frac{\tilde{R}_{2n-3}(\lambda^2)}{\tilde{\alpha}\tilde{R}_{2n-2}(\lambda^2)} + \frac{1}{\tilde{\alpha}l_n} = \frac{R_{2n-3}(\lambda^2)}{\alpha R_{2n-2}(\lambda^2)} + \frac{1}{\alpha l_n} \tag{3.17}$$

and, consequently,

$$\frac{\tilde{R}_{2n-3}(0)}{\tilde{\alpha}\tilde{R}_{2n-2}(0)} + \frac{1}{\tilde{\alpha}l_n} = \frac{R_{2n-3}(0)}{\alpha R_{2n-2}(0)} + \frac{1}{\alpha l_n}. \tag{3.18}$$

Setting $\lambda = 0$ in (2.12) and (3.10) we obtain

$$\frac{R_{2n-2}(0)}{R_{2n-3}(0)} = \frac{\tilde{R}_{2n-2}(0)}{\tilde{R}_{2n-3}(0)} = \sum_{k=0}^{n-1} l_k = \sum_{k=0}^{n-1} \tilde{l}_k = l - l_n. \tag{3.19}$$

Combining (3.19) with (3.18) we arrive at $\tilde{\alpha} = \alpha$. Now (3.16) implies $m_n = \tilde{m}_n$ and it follows from (3.17) that

$$\frac{\tilde{R}_{2n-2}(\lambda^2)}{\tilde{R}_{2n-3}(\lambda^2)} = \frac{R_{2n-2}(\lambda^2)}{R_{2n-3}(\lambda^2)}.$$

Since the left-hand sides of (2.12) and (3.10) coincide, we conclude that $m_k = \tilde{m}_k$ for $k = 1, 2, \ldots, n-1$ and $l_k = \tilde{l}_k$ for $k = 0, 1, 2, \ldots, n-1$. The theorem is proved. □

Theorem 3.1 gives a solution of Inverse problem I. A solution of Inverse problem II is similar. It is necessary just to delete the summand l_n^{-1} in (3.3), (3.4), set $l_n^{-1} = 0$ in (3.6), (3.7), (3.8), (3.11), (3.13), (3.15), (3.17), (3.18).

4. COMPARISON OF PROBLEMS I AND II

In this section we compare the functions $\phi(\lambda)$ and $\psi(\lambda)$ defined by (2.10) and (2.11) and the sets of their zeros $\{\nu_k\}$ and $\{\mu_k\}$.

Comparison of (2.10) with (2.11) gives

$$\phi(\lambda) = \psi(\lambda) + \frac{\psi(\lambda) - \psi(-\lambda)}{2i\lambda\alpha l_n}$$

$$\begin{aligned}\frac{\phi(\lambda)}{\psi(\lambda)} &= 1 + \frac{l_n^{-1}}{-m_n\lambda^2 + i\alpha\lambda + \frac{1}{R_{2n-2}(\lambda^2)/R_{2n-3}(\lambda^2)}} \\ &= 1 + \frac{l_n^{-1}}{-m_n\lambda^2 + i\alpha\lambda + \frac{1}{l_{n-1}+\frac{1}{-m_{n-1}\lambda^2+\ldots+\frac{1}{l_1+\frac{1}{-m_1\lambda^2+\frac{1}{l_0}}}}}}.\end{aligned} \tag{4.1}$$

It follows from (4.1) that

$$\frac{\phi(0)}{\psi(0)} = \frac{l}{l_n} \tag{4.2}$$

and

$$\frac{R_{2n-2}(\lambda)}{R_{2n-3}(\lambda)} = \left(m_n\lambda - i\alpha\lambda^{1/2} + \left(l_n\left(\frac{\phi(\lambda^{1/2})}{\psi(\lambda^{1/2})} - 1\right)\right)^{-1}\right)^{-1} \tag{4.3}$$

is an S_0-function.

Theorem 4.1. *For the polynomials $\phi(\lambda)$ and $\psi(\lambda)$ of degree $2n$ each to be the characteristic polynomials of problems I and II, respectively, normalized by (4.2) it is necessary and sufficient that the function*

$$\left(m_n\lambda - i\alpha\lambda^{1/2} + \left(l_n\left(\frac{\phi(\lambda^{1/2})}{\psi(\lambda^{1/2})} - 1\right)\right)^{-1}\right)^{-1} \tag{4.4}$$

with

$$\alpha =: \left(\frac{1}{l-l_n} + \frac{1}{l_n}\right)\lim_{\lambda\to 0}\frac{\phi(\lambda)-\phi(-\lambda)}{2i\lambda} \tag{4.5}$$

and

$$m_n =: -i\alpha\lim_{|\lambda|\to\infty}\frac{\phi(\lambda)+\phi(-\lambda)}{\lambda(\phi(\lambda)-\phi(-\lambda)} \tag{4.6}$$

be a rational S_0-function.

Proof. Necessity follows from (4.3) and (2.12). Now let the function (4.4) be an S_0-function. Then it can be expanded into a continued fraction:

$$\begin{aligned}&\left(m_n\lambda - i\alpha\lambda^{1/2} + \left(l_n\left(\frac{\phi(\lambda^{1/2})}{\psi(\lambda^{1/2})} - 1\right)\right)^{-1}\right)^{-1} \\ &= a_{n-1} + \frac{1}{-b_{n-1}\lambda^2 + \frac{1}{a_{n-2}+\frac{1}{-b_{n-2}\lambda^2+\ldots+\frac{1}{a_1+\frac{1}{-b_1\lambda^2+\frac{1}{b_0}}}}}},\end{aligned}$$

where $a_k > 0$ for $k = 0, 1, \ldots, n-1$ and $b > 0$ for $k = 1, 2, \ldots, n-1$.

We identify b_k with masses on a Stieltjes string and a_k with the subintervals into which the masses divide the length $l-l_n$. This data together with the given l_n and α obtained from (4.5) and m_n obtained from (4.6) generate problems (2.2)–(2.5) and (2.2)–(2.4), (2.6) which have spectra $\{\nu_k\}$ and $\{\mu_k\}$, respectively. The proof is complete. □

Comparing (2.10) with (2.11), which can be rewritten as

$$\phi(\lambda)=(-1)^n m_n \prod_1^{n-1} m_k l_k \prod_{-n,k\neq 0}^{n} (\nu_k-\lambda) \tag{4.7}$$

and

$$\psi(\lambda)=(-1)^n m_n \prod_1^{n-1} m_k l_k \prod_{-n,k\neq 0}^{n} (\mu_k-\lambda), \tag{4.8}$$

we obtain

$$\phi(\lambda)-\psi(\lambda)=l_n^{-1}R_{2n-2}(\lambda^2).$$

Let us introduce the following notation:

$$M_p=\sum_{k=-n,\ k<k'<\ldots<k^{(p-1)},\ k\neq 0,\ k'\neq 0,\ldots,\ k^{(p-1)}\neq 0}^{k=n,\ k'=n,\ldots k^{(p-1)}=n} \sum\ldots\sum \nu_k\nu_{k'}\ldots\nu_{k^{(p-1)}}, \tag{4.9}$$

$$N_p=\sum_{k=-n,\ k<k'<\ldots<k^{(p-1)},\ k\neq 0,\ k'\neq 0,\ldots,\ k^{(p-1)}\neq 0}^{k=n,\ k'=n,\ldots k^{(p-1)}=n} \sum\ldots\sum \mu_k\mu_{k'}\ldots\mu_{k^{(p-1)}}. \tag{4.10}$$

Then

$$\begin{aligned}R_{2n-2}(\lambda^2)=l_n(-1)^n m_n \prod_1^{n-1} m_k l_k \left(\lambda^{2n-2}(N_2-M_2)+\lambda^{2n-4}(N_4-M_4)+\ldots\right.\\ \left.\ldots+N_{2n}-M_{2n}\right).\end{aligned}$$

$$\begin{aligned}R_{2n-2}(\lambda^2)&=\frac{\phi(\lambda)-\phi(-\lambda)}{2i\alpha\lambda}\\ &=i(-1)^n m_n \prod_1^{n-1} m_k l_k \alpha^{-1}\left(\lambda^{2n-2}N_1+\lambda^{2n-4}N_3+\ldots+N_{2n-1}\right).\end{aligned}$$

Comparing (4.9) with (4.10) we obtain

$$l_n(N_2-M_2)=\left(\frac{1}{l-l_n}+\frac{1}{l_n}\right)^{-1}\left(\sum_{k=-n,\ k\neq 0}^{n}\frac{1}{\nu_k}\right)^{-1}N_1,$$
$$l_n(N_4-M_4)=\left(\frac{1}{l-l_n}+\frac{1}{l_n}\right)^{-1}\left(\sum_{k=-n,\ k\neq 0}^{n}\frac{1}{\nu_k}\right)^{-1}N_3,$$
$$\dots$$
$$l_n(N_{2n}-M_{2n})=\left(\frac{1}{l-l_n}+\frac{1}{l_n}\right)^{-1}\left(\sum_{k=-n,\ k\neq 0}^{n}\frac{1}{\nu_k}\right)^{-1}N_{2n-1}.$$

Definitions (4.7) and (4.8) imply

$$\phi(\lambda)-\phi(-\lambda)=\psi(\lambda)-\psi(-\lambda). \tag{4.11}$$

Substituting (4.7) and (4.8) into (4.11) we obtain

$$M_{2k-1}=N_{2k-1},\quad k=1,2,\dots,n. \tag{4.12}$$

Using (2.11) we derive

$$R_{2n-3}(\lambda^2)=\frac{\psi(\lambda)+\psi(-\lambda)}{2}+m_n\lambda^2R_{2n-2}(\lambda^2). \tag{4.13}$$

Substituting (4.8) and (4.10) into (4.13) we obtain

$$\begin{aligned}R_{2n-3}(\lambda^2)=&(-1)^n m_n\prod_{k=1}^{n-1}m_kl_k(\lambda^{2n-2}(M_2-N_1^{-1}N_3)+\lambda^{2n-4}(M_4-N_1^{-1}N_5)+\dots\\&+\lambda^2(M_{2n-2}-N_1^{-1}N_{2n-1})+M_{2n}).\end{aligned} \tag{4.14}$$

Taking into account that $R_{2n-3}(z)$ has zeros only on the positive half-axis and comparing (4.14) with (2.8) and (2.9) we obtain

$$(-1)^k(M_{2k}-N_1^{-1}N_{2k+1})>0,\ k=1,2,\dots,n-1,\quad (-1)^nM_{2n}>0.$$

Using interlacing of the zeros of $R_{2n-3}(z)$ with the zeros of $R_{2n-2}(z)$ we obtain

$$(-1)^{k-1}\frac{M_{2k}-N_1^{-1}N_{2k+1}}{M_2-N_1^{-1}N_3}>(-1)^{k-1}\frac{N_{2k-1}}{N_1},\quad k=2,3,\dots,n-1,$$
$$(-1)^{n-1}\frac{M_{2n}}{M_2-N_1^{-1}N_3}>(-1)^{n-1}\frac{N_{2n-1}}{N_1}.$$

Equation (4.12) for $k=1$ is equivalent to

$$\sum_{k=-n,\ k\neq 0}^{n}\mathrm{Im}\mu_k=\sum_{k=-n,\ k\neq 0}^{n}\mathrm{Im}\nu_k.$$

5. INVERSE PROBLEM FOR A DAMPED STIELTJES STRING

Now we come back to the problems generated by equation (2.1) where all the masses are damped. By an inverse problem we mean recovering the parameters of problems generated by equation

$$\frac{u_k - u_{k+1}}{l_k} + \frac{u_k - u_{k-1}}{l_{k-1}} - m_k\lambda^2 u_k + i\alpha_k\lambda u_k = 0 \quad (k = 1, 2, \ldots, n) \tag{5.1}$$

and conditions (2.4), (2.5) and (2.4), (2.6), i.e. $\{m_k\}$, $\{\alpha_k\}$ $(k = 1, 2, \ldots, n)$, $\{l_k\}$ $(k = 0, 1, \ldots, n)$ using the spectra of these problems and the total length of the string $l = l_0 + l_1 + \ldots + l_n$.

Suppose we know $\{\mu_k\}$ $(k = \pm 1, \pm 2, \ldots \pm n)$ eigenvalues of problem (5.1), (2.4), (2.5), i.e. the zeros of the polynomial $R_{2n-1}(\lambda)$ which here are obtained from the recurrence relations

$$R_{2k-1}(\lambda^2) = (-\lambda^2 m_k + i\alpha\lambda)R_{2k-2}(\lambda^2) + R_{2k-3}(\lambda^2),$$

$$R_{2k}(\lambda^2) = l_k R_{2k-1}(\lambda^2) + R_{2k-2}(\lambda^2), \quad (k = 1, 2, \ldots, n,\ R_{-1}(\lambda^2) = \frac{1}{l_0},\ R_0(\lambda^2) = 1)$$

and $\{\nu_k\}$ $(k = \pm 1, \pm 2, \ldots \pm n)$ eigenvalues of problem (5.1), (2.4), (2.6), i.e. the zeros of the polynomial $R_{2n}(\lambda)$ and the total length of the string l. We construct the polynomials

$$p(\lambda) = \prod_{k=1}^{n}\left(1 - \frac{\lambda}{\mu_k}\right)\left(1 - \frac{\lambda}{\mu_{-k}}\right), \quad q(\lambda) = \prod_{k=1}^{n}\left(1 - \frac{\lambda}{\nu_k}\right)\left(1 - \frac{\lambda}{\nu_{-k}}\right).$$

These polynomials have the same sets of zeros as $R_{2n}(\lambda)$ and $R_{2n-1}(\lambda)$, respectively. Therefore,

$$R_{2n-1}(\lambda) = T_1 p(\lambda), \quad R_{2n}(\lambda) = T_2 q(\lambda)$$

and

$$\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)} = \frac{T_2}{T_1}\frac{q(\lambda)}{p(\lambda)}. \tag{5.2}$$

We construct $p(\lambda)$ and $q(\lambda)$ using $\{\mu_k\}$ $(k = \pm 1, \pm 2, \ldots \pm n)$ and $\{\nu_k\}$ $(k = \pm 1, \pm 2, \ldots \pm n)$. To find $\frac{T_2}{T_1}$ we substitute $\lambda = 0$ in (5.2):

$$\frac{R_{2n}(0)}{R_{2n-1}(0)} = \frac{T_2}{T_1}\frac{q(0)}{p(0)} = \frac{T_2}{T_1}.$$

Substituting $\lambda = 0$ into (2.8) we obtain

$$\frac{T_2}{T_1} = l_0 + l_1 + \ldots + l_n = l.$$

Thus the sets $\{\mu_k\}$ and $\{\nu_k\}$ together with given l uniquely determine the rational function

$$\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)} = l\frac{q(\lambda)}{p(\lambda)}.$$

By expanding $\frac{R_{2n}(\lambda)}{R_{2n-1}(\lambda)}$ into continued fraction according to (2.8) we can find $\{m_k\}$, $\{\alpha_k\}$ $(k = 1, 2, \dots, n)$ and $\{l_k\}$ $(k = 0, 1, \dots, n)$. Hence, we have proved the following theorem.

Theorem 5.1. *The eigenvalues* $\{\mu_k\}$ $(k = \pm 1, \pm 2, \dots \pm n)$ *and* $\{\nu_k\}$ $(k = \pm 1, \pm 2, \dots \pm n)$ *together with the given total length* l *uniquely determine* $\{m_k\}$, $\{\alpha_k\}$ $(k = 1, 2, \dots, n)$ *and* $\{l_k\}$ $(k = 0, 1, \dots, n)$.

Now let us find conditions which must be satisfied by sets of complex numbers $\{\nu_k\}$ and $\{\mu_k\}$ $(k = \pm 1, \pm 2, \dots, \pm n)$ to be the spectra of problems (5.1), (2.4), (2.5) and (5.1) (2.4), (2.6), respectively.

Theorem 5.2. *Let* $\{\nu_k\}$ *and* $\{\mu_k\}$ $(k = \pm 1, \pm 2, \dots, \pm n)$ *be two sequences of complex numbers and let* l *be a positive number. In order* $\{\nu_k\}$ *and* $\{\mu_k\}$ $(k = \pm 1, \pm 2, \dots, \pm n)$ *be the spectra of problems* (5.1), (2.4), (2.5) *and* (5.1) (2.4), (2.6), *respectively, with* $m_k > 0$, $\alpha_k > 0$ $(k = 1, 2, \dots, n)$, *it is necessary and sufficient that:*

1) $\{\mu_k\} \cap \{\nu_k\} = \emptyset$,
2) *the product* $l \prod\limits_{k=-n, k\neq 0}^{n} \left(1 - \frac{\lambda}{\nu_k}\right)\left(1 - \frac{\lambda}{\mu_k}\right)^{-1}$ *can be presented as a continued fraction of the form:*

$$\begin{aligned} & l \prod_{k=-n, k\neq 0}^{n} \left(1 - \frac{\lambda}{\nu_k}\right)\left(1 - \frac{\lambda}{\mu_k}\right)^{-1} \\ & = a_n + \cfrac{1}{b_n\lambda^2 + ic_n\lambda + \cfrac{1}{a_{n-1} + \cfrac{1}{b_{n-1}\lambda^2 + ic_{n-1}\lambda + \dots + \cfrac{1}{a_1 + \cfrac{1}{b_1\lambda^2 + ic_1\lambda + \frac{1}{a_0}}}}}} \end{aligned} \tag{5.3}$$

with $a_k > 0$ $(k = 0, 1, 2, \dots, n)$, $b_k < 0$, $c_k \geq 0$ $(k = 0, 1, 2, \dots, n)$.

Proof. Let us use the continued fraction (5.3) to construct another continued fraction:

$$a_n + \cfrac{1}{b_n\lambda^2 + \cfrac{1}{a_{n-1} + \cfrac{1}{b_{n-1}\lambda^2 + \dots \cfrac{1}{a_1 + \cfrac{1}{b_1\lambda^2 + \frac{1}{a_0}}}}}}.$$

It is clear from (5.3) that $\sum_{k=0}^{n} a_k = l$. According to [9] this fraction can be identified as the ratio of two polynomials the zeros of the numerator are the eigenvalues of an undamped Stieltjes string with fixed ends and the zeros of the denominator are the eigenvalues of the same string with the left end fixed and the right end free. The masses of this string are $|b_k|$ at distances a_k. The total length of the string is equal l. Now let us consider the same string (the same masses $|b_k|$ and the same lengths of intervals a_k) but with damping proportional to c_k at mass $|b_k|$. This new damped string generates the continued fraction (5.3). The theorem is proved. □

Remark 5.3.

a) Due to conditions 1), 2) $\mu_{-k} = -\overline{\mu}_k$ for each not pure imaginary μ_k and $\nu_{-k} = -\overline{\nu}_k$ for each not pure imaginary ν_k.

b) Condition 2) in explicit form consists of rather involved relations. The first, the second and the third of them, however, are

$$\sum_{k=-n,\ k\neq 0}^{n} \mathrm{Im}\mu_k = \sum_{k=-n,\ k\neq 0}^{n} \mathrm{Im}\nu_k,$$

$$\sum_{k=-n,k'=-n,k\neq 0,k'\neq 0}^{k=n,k'=n} \mu_k\mu_{k'} > \sum_{k=-n,k'=-n,k\neq 0,k'\neq 0}^{k=n,k'=n} \nu_k\nu_{k'},$$

$$\begin{aligned}&\sum_{-n,k\neq 0}^{n} \mathrm{Im}\mu_k + \\ &+ i \sum_{k=-n,k'=-n,k"=-n,k\neq 0,k'\neq 0,k"\neq 0}^{k=n,k'=n,k"=n} (\mu_k\mu_{k'}\mu_{k"} - \nu_k\nu_{k'}\nu_{k"})\cdot \\ &\cdot \left(\sum_{k=-n,k'=-n,k\neq 0,k'\neq 0,}^{k=n,k'=n,} (\mu_k\mu_{k'} - \nu_k\nu_{k'} \right)^{-1} > 0.\end{aligned}$$

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