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Strong Admissibility Revisited (proofs)

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Abstract: The current document contains the proofs of the COMMA 2014 submission "Strong Admissibility Revisited" and should be read in conjunction with it.

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1 Introduction

The current document contains the proofs of the COMMA 2014 submission "Strong Admissibility revisited". It does not duplicate any of the definitions or existing proofs from the COMMA submission. Hence, this document should be read in conjunction with it.

One thing to keep in mind when reading the current document is that its structure is quite different from that of the COMMA submission. The COMMA submission starts with the extension-based definition of strong admissibility ("Strongly Admissible Sets") and then subsequently discusses the labelling-based version of strong admissibility ("Strongly Admissible Labellings"). This was done because the concept of strong admissibility already existed in its extension-based form (see the AIJ 2007 paper of Baroni and Giacomin) and we wanted to start with something people might already familiar with. In the current technical report, however, we start with the labelling-based version of strong admissibility, before going to the extension-based version of strong admissibility. This is done because we first need to establish some results for the labelling-based version of strong admissibility, which can then later be applied also for proving properties of the extension-based version of strong admissibility. For instance, the equivalence of Baroni and Giacomin's notion of a strongly admissible set and our own notion of a strongly admissible set (Theorem 1 in the COMMA submission) is proved using strongly admissible labellings as an intermediary.

2 Proofs

The idea of a partial min-max numbering is to have some form of min-max numberings that are not completely "ready", meaning that not every `in` or `out`-labelled argument is already numbered (some are still unnumbered) but those `in` and `out`-labelled arguments that *are* already numbered have a correct min-max number, as far as the current partial numbering is concerned. In this way, partial min-max numberings serve as intermediate results of the iterative numbering procedure sketched in the COMMA submission and made fully formal in the current technical report.

Definition 1 (partial min-max numbering)**.** *Let $\mathcal{L}ab$ be an admissible labelling of argumentation framework* (Ar, att)*. A* partial min-max numbering *is a partial function* $\mathcal{MM}_{\mathcal{L}ab} : \mathtt{in}(\mathcal{L}ab) \cup$

$\mathtt{out}(\mathcal{L}ab) \rightarrow \mathbb{N}$ *such that for each* $A \in Ar$ *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}$ *(that is, for which* $\mathcal{MM}_{\mathcal{L}ab}(A)$ *is defined) it holds that:*

- *if* $\mathcal{L}ab(A) = \mathtt{in}$ *then all* out*-labelled attackers of* A *are numbered by* $\mathcal{MM}_{\mathcal{L}ab}$ *and* $\mathcal{MM}_{\mathcal{L}ab}(A) = max(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ *is an* out*-labelled attacker of* $A\}) + 1$
- *if* $\mathcal{L}ab(A) = \mathtt{out}$ *then there is at least one* in*-labelled attacker of* A *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}$ *and* $\mathcal{MM}_{\mathcal{L}ab}(A) = min(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ *is an* in*-labelled attacker of* A *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}\}) + 1$

Definition 2 (ExtendIn/ExtendOut)**.** *Let* $\mathcal{L}ab$ *be an admissible labelling and* $\mathcal{MMS}_{\mathcal{L}ab}$ *be the set of all partial functions* $\mathcal{MM}_{\mathcal{L}ab} : \mathtt{in}(\mathcal{L}ab) \cup \mathtt{out}(\mathcal{L}ab) \rightarrow \mathbb{N}$.
We define the function ExtendIn $: \mathcal{MMS}_{\mathcal{L}ab} \rightarrow \mathcal{MMS}_{\mathcal{L}ab}$ *as follows:*
$\mathtt{ExtendIn}(\mathcal{MM}_{\mathcal{L}ab}) = \{(A, numberA) \mid A$ *is an* in*-labelled argument not numbered by* $\mathcal{MM}_{\mathcal{L}ab}$*, all* out*-labelled attackers of* A *are numbered by* $\mathcal{MM}_{\mathcal{L}ab}$ *and* $numberA = max(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ *is an* out*-labelled attacker of* A *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}\}) + 1\}$
We define the function ExtendOut $: \mathcal{MMS}_{\mathcal{L}ab} \rightarrow \mathcal{MMS}_{\mathcal{L}ab}$ *as follows:*
$\mathtt{ExtendOut}(\mathcal{MM}_{\mathcal{L}ab}) = \{(A, numberA) \mid A$ *is an* out*-labelled argument not numbered by* $\mathcal{MM}_{\mathcal{L}ab}$*, there exists an* in*-labelled attacker of* A *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}$ *and* $numberA = min(\{\mathcal{MM}_{\mathcal{L}ab}(B) \mid B$ *is an* in*-labelled attacker of* A *that is numbered by* $\mathcal{MM}_{\mathcal{L}ab}\}) + 1\}$

Definition 3 (numbering run)**.** *Given an admissible labelling* $\mathcal{L}ab$*, a* numbering run *is a sequence* $\mathcal{MM}^0_{\mathcal{L}ab}, \mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ *such that:*

- $\mathcal{MM}^0_{\mathcal{L}ab}$ *is the empty partial min-max numbering (that is, the partial min-max numbering where each argument is unnumbered)*
- *for each even* $i \geq 0$, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$
- *for each odd* $i \geq 1$, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$

To illustrate these definitions, consider again the argumentation framework of Figure 1 of the COMMA submission, and the grounded labelling thereof.
$\mathcal{MM}^0_{\mathcal{L}ab}$ is the empty numbering (so $\emptyset$)
$\mathcal{MM}^1_{\mathcal{L}ab} = \mathcal{MM}^0_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^0_{\mathcal{L}ab}) = \emptyset \cup \{(A,1),(D,1)\}$
$\mathcal{MM}^2_{\mathcal{L}ab} = \mathcal{MM}^1_{\mathcal{L}ab} \cup \mathtt{ExtendOut}(\mathcal{MM}^1_{\mathcal{L}ab}) = \{(A,1),(D,1)\} \cup \{(B,2),(E,2)\}$
$\mathcal{MM}^3_{\mathcal{L}ab} = \mathcal{MM}^2_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^2_{\mathcal{L}ab}) = \{(A,1),(D,1),(B,2),(E,2)\} \cup \{(C,3),(F,3)\}$
$\mathcal{MM}^4_{\mathcal{L}ab} = \mathcal{MM}^3_{\mathcal{L}ab} \cup \mathtt{ExtendOut}(\mathcal{MM}^3_{\mathcal{L}ab}) = \{(A,1),(D,1),(B,2),(E,2),(C,3),(F,3)\} \cup \emptyset$
$\mathcal{MM}^5_{\mathcal{L}ab} = \mathcal{MM}^4_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^4_{\mathcal{L}ab}) = \{(A,1),(D,1),(B,2),(E,2),(C,3),(F,3)\} \cup \emptyset$
It can be verified that for any $i \geq 3$, $\mathcal{MM}^i_{\mathcal{L}ab} = \mathcal{MM}^3_{\mathcal{L}ab}$.

We would like to prove that every $\mathcal{MM}^i_{\mathcal{L}ab}$ in the numbering run is a partial min-max numbering. A possible strategy for doing so would be to use induction. The basis would be the observation that $\mathcal{MM}^0_{\mathcal{L}ab}$ is a partial min-max numbering. The induction step would then have to handle two cases: one where i is even and one where i is odd. For even i, we would have to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ is a correct partial min-max numbering, whereas for odd i, we would have to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$ is a correct partial min-max numbering. In both cases, the induction hypothesis is that $\mathcal{MM}^i_{\mathcal{L}ab}$ is already a correct min-max numbering.

The problem of such an approach, however, is that the induction hypothesis is not strong enough. For instance, consider the grounded labelling of the argumentation framework of Figure 1 of the

COMMA submission. Here, $\mathcal{MM}_{\mathcal{L}ab} = \{(A,1),(B,2),(C,3),(E,4)\}$ is a correct partial min-max numbering, but $\mathcal{MM}_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}_{\mathcal{L}ab}) = \{(A,1),(B,2),(C,3),(E,4)\} \cup \{(D,1),(F,5)\}$ is *not* a correct min-max numbering, because out-labelled argument E is numbered with 4, whereas the minimal min-max number of its in-labelled attackers that are numbered is 1, so it should have been numbered with 2 instead! So the bare fact that some $\mathcal{MM}^n_{\mathcal{L}ab}$ is a partial min-max numbering is *not* sufficient to prove that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is also a partial min-max numbering. Clearly, we need a stronger induction hypothesis.

As a first observation towards such a stronger induction hypothesis, it can be observed that the above mentioned min-max numbering $\{(A,1),(B,2),(C,3),(E,4)\}$ cannot actually occur in a numbering run, as the same step that numbered A with 1 would also have numbered D with 1. If we look at the actual numbering run, we observe that each time we go from $\mathcal{MM}^i_{\mathcal{L}ab}$ to $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$, it holds that *all* possible $i+1$ numbers are generated, not just some of them. This leads to the concept of ***n**-completeness*. The idea is that each unnumbered argument that could be numbered given the existing min-max numbers, would have a correct min-max number of bigger than n. That is, up to n there are no missing numbers.

Definition 4 (n-complete)**.** *Let $\mathcal{L}ab$ be an admissible labelling of argumentation framework (Ar, att). A partial min-max numbering $\mathcal{MM}_{\mathcal{L}ab}$ is called* n-complete *iff*

- *for each unnumbered* in*-labelled argument of which all* out*-labelled attackers are already numbered, the MAX+1 value of its* out*-labelled attackers is bigger than n*
- *for each unnumbered* out*-labelled argument that has an* in*-labelled attacker that is already numbered, the MIN+1 value of its* in*-labelled attackers is bigger than n*

Given a set of arguments $\mathcal{A}rgs$, the MAX+1 value of $\mathcal{A}rgs$ is $max(\{\mathcal{MM}_{\mathcal{L}ab}(A) \mid A \in \mathcal{A}rgs\}) + 1$, whereas the MIN+1 value of $\mathcal{A}rgs$ is $min(\{\mathcal{MM}_{\mathcal{L}ab}(A) \mid A \in \mathcal{A}rgs\} + 1$.

So the idea of n-completeness is that the numbering is already "complete" for numbers up to n. It can be verified that in the earlier mentioned numbering run, each $\mathcal{MM}^i_{\mathcal{L}ab}$ is in fact i-complete. It can also be observed that, as a general property, $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$ for each $i \geq 0$. Moreover, in the earlier mentioned numbering run, it holds that each *additional* number generated by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ (so each j such that $(A,j) \in \mathcal{MM}^{i+1}_{\mathcal{L}ab} \setminus \mathcal{MM}^i_{\mathcal{L}ab}$ for some argument A) is $i+1$. This turns out to be a general property, as we will see.

We are now ready to sketch the structure of the induction proof. The idea is first, as a basis, to observe that $\mathcal{MM}^0_{\mathcal{L}ab}$ is a correct min-max numbering. Then, we need two different induction steps, one of ExtendIn where for some even i we go from $\mathcal{MM}^i_{\mathcal{L}ab}$ to $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, and one for ExtendOut where for some odd i we go from $\mathcal{MM}^i_{\mathcal{L}ab}$ to $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$. For both induction steps, we apply an induction hypothesis that for a given i it holds that:

1. $\mathcal{MM}^i_{\mathcal{L}ab}$ is a correct partial min-max numbering,
2. for each $j \in \{1,\ldots,i\}$, each "new" number in $\mathcal{MM}^j_{\mathcal{L}ab}$ is j, and
3. $\mathcal{MM}^i_{\mathcal{L}ab}$ is i-complete

Lemma 1. *Let $\mathcal{MM}^0_{\mathcal{L}ab}, \mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ be a numbering run of an admissible labelling $\mathcal{L}ab$ and let $i \geq 0$ be an even number. If*

(1) $\mathcal{MM}^i_{\mathcal{L}ab}$ *is a correct partial min-max numbering,*

(2) *for each $j \in \{1, \ldots, i\}$ it holds that for each $(A, k) \in \mathcal{MM}^{j}_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and*

(3) $\mathcal{MM}^{i}_{\mathcal{L}ab}$ *is i-complete*

then

(1') $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ *is a correct partial min-max numbering,*

(2') *for each $j \in \{1, \ldots, i+1\}$ it holds that for each $(A, k) \in \mathcal{MM}^{j}_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and*

(3') $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ *is $(i+1)$-complete*

Proof. We first observe that since i is even, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$.

(1') We need to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a correct partial min-max numbering. Let A be an arbitrary argument that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. We distinguish two cases:

- $\mathcal{L}ab(A) = \texttt{in}$. We then need to show that all `out`-labelled attackers of A are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A)$ is the MAX+1 value of its `out`-labelled attackers. Given that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$, we distinguish two subcases:
  1. A was already numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Since $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering (induction hypothesis (1)) it follows that all `out`-labelled attackers of A are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ and that $\mathcal{MM}^{i}_{\mathcal{L}ab}(A)$ is the MAX+1 value of these. As $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i}_{\mathcal{L}ab}$, it follows that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}^{i}_{\mathcal{L}ab}(A)$ and that the MAX+1 value of the `out`-labelled attackers of A in $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is the same as the MAX+1 value of the `out`-labelled attackers of A in $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Hence, A is correctly numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$.
  2. A as not numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ but became numbered by $\texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$. However, by definition of `ExtendIn`, this implies that A is correctly numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$.
- $\mathcal{L}ab(A) = \texttt{out}$. We then need to show that there is at least one `in`-labelled attacker of A that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A)$ is the MIN+1 value of all `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Given that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$, together with the fact that `ExtendIn` does not number any `out`-labelled arguments, it then follows that A is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Since $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering (induction hypothesis (1)) it follows that there is at least one `in`-labelled attacker of A that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ and that $\mathcal{MM}^{i}_{\mathcal{L}ab}(A)$ is the MIN+1 value of these `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. As $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, it holds that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}^{i}_{\mathcal{L}ab}(A)$, so it suffices to prove that the MIN+1 value of all `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is still the same as the MIN+1 value of all `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. For this, we ask ourselves two questions:
  - Can the MIN+1 value of the `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ be *bigger* than the MIN+1 value of the `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$?
    Since $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, it follows that the set of `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a *superset* of the set of `in`-labelled attackers of A that

are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$, so the minimal element of the former set can never be bigger than the minimal element of the latter set, so the answer is no.

– Can the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ be *smaller* than the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$?
Suppose, towards a contradiction, that this would be the case. Since $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, this means that $\mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ produced a number that is *lower* than that of each of the in-labelled attackers of A that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. Since the maximal number that can possibly occur in $\mathcal{MM}^i_{\mathcal{L}ab}$ is i (induction hypothesis (2)) this implies that $\mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ produced a number smaller than i. But then $\mathcal{MM}^i_{\mathcal{L}ab}$ is not i-complete, which is in contradiction with induction hypothesis (3). Therefore the answer is again no.

(2') It suffices to show that for each $(A, k) \in \mathcal{MM}^{i+1}_{\mathcal{L}ab} \setminus \mathcal{MM}^i_{\mathcal{L}ab}$ it holds that $k = i + 1$. That is, we need to show that for each $(A, k) \in \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ it holds that $k = i + 1$. Let $(A, k) \in \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$. We ask ourselves two questions:

- Can k be *smaller* than $i + 1$?
If this is the case, then (Definition 2) A is not numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$, all its out-labelled attackers are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$ and their MAX+1 value is smaller than $i + 1$, so smaller or equal to i. But this implies that $\mathcal{MM}^i_{\mathcal{L}ab}$ is not i-complete, which is in contradiction with induction hypothesis (3). So the answer is no.
- Can k be *bigger* than $i + 1$?
If this is the case, then (Definition 2) A is not numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$, all its out-labelled attackers are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$ and their MAX+1 value is bigger than $i+1$. From the definition of MAX+1, this implies that A has an out-labelled attacker that is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$ with a min-max number bigger than i. This is in contradiction with induction hypothesis (2), that implies that each min-max number in $\mathcal{MM}^i_{\mathcal{L}ab}$ is less or equal to i. So the answer is no.

(3') We need to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $(i + 1)$-complete. For this, we need to show two things (Definition 4).

- if A is an unnumbered in-labelled argument of which all its out-labelled attackers are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$, then the MAX+1 value of these out-labelled attackers is bigger than $i + 1$.
Let A be an unnumbered (by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$) argument of which all its out-labelled attackers are numbered. From the fact that $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$ it follows that A is also unnumbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, together with the fact that ExtendIn does not number any out-labelled arguments, it follows that all the out-labelled attackers of A are also numbered by $\mathcal{MM}^i_{\mathcal{A}rgs}$. But this would imply that A is numbered by $\mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, and therefore (since $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \mathtt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$) that A is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Contradiction.
- if A is an unnumbered out-labelled argument that has an in-labelled attacker that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$, then the MIN+1 value of all its in-labelled attackers that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is bigger than $i + 1$.

Let A be an unnumbered (by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$) out-labelled argument that has an in-labelled attacker that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, it follows that A is also unnumbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. We distinguish two cases:

1. A has an in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Then, from the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is i-complete (induction hypothesis (3)) it follows that the MIN+1 value of its in-labelled attackers that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is bigger than i. This means that the min-max number of the lowest numbered in-attacker of A (say B) is bigger than $i-1$, so bigger or equal to i. However, we recall that i is an even number, and that each numbered in-labelled argument in $\mathcal{MM}^{i}_{\mathcal{L}ab}$ has an odd mix-max number (this follows from induction hypothesis (2), together with the definition of the numbering run). This then implies that the min-max number of B is bigger or equal to $i+1$. Furthermore, any in-labelled attacker of A that became numbered by $\texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ (say C) will have a min-max number of $i+1$ (this is what we have just observed in (2')). This means that the lowest numbered in-labelled attacker of A in $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is still bigger or equal to $i+1$. This then implies that the MIN+1 value of all in-labelled attackers of A in $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is bigger or equal to $i+2$, so bigger than $i+1$, thus satisfying the requirement of i-completeness.
2. A does not have an in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Then, from the fact that A does have an in-labelled attacker that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$, it then follows that every in-labelled attacker of A that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is numbered by $\texttt{ExtendIn}(\mathcal{MM}^{i}_{\mathcal{L}ab})$. From the earlier obtained result (2') it then follows that every in-labelled attacker of A that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is numbered with the min-max number $i+1$. This implies that the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $i+2$, which is bigger than $i+1$, thus satisfying the requirement of i-completeness.

□

Lemma 2. *Let $\mathcal{MM}^{0}_{\mathcal{L}ab}, \mathcal{MM}^{1}_{\mathcal{L}ab}, \mathcal{MM}^{2}_{\mathcal{L}ab}, \ldots$ be a numbering run of an admissible labelling $\mathcal{L}ab$ and let $i \geq 0$ be an odd number. If*

(1) *$\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering,*

(2) *for each $j \in \{1, \ldots, i\}$ it holds that for each $(A,k) \in \mathcal{MM}^{j}_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and*

(3) *$\mathcal{MM}^{i}_{\mathcal{L}ab}$ is i-complete*

then

(1') *$\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a correct partial min-max numbering,*

(2') *for each $j \in \{1, \ldots, i+1\}$ it holds that for each $(A,k) \in \mathcal{MM}^{j}_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and*

(3') *$\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $(i+1)$-complete*

Proof. We first observe that since i is odd, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$.

(1') We need to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a correct partial min-max numbering. Let A be an arbitrary argument that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. We distinguish two cases.

- $\mathcal{L}ab(A) = \texttt{in}$. We then need to show that all out-labelled attackers of A are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A)$ is the MAX+1 value of its out-labelled attackers. Given the fact that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ and the fact that $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ does not number any in-labelled arguments, it follows that A is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering (induction hypothesis (1)), it follows that all out-labelled attackers of A are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$ it then follows that all the out-labelled attackers of A are also numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and that the MAX+1 value of these out-labelled attackers of A is the same under $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ as under $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Hence, from the fact that A is correctly numbered under $\mathcal{MM}^{i}_{\mathcal{L}ab}$ it follows that A is correctly numbered under $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$.
- $\mathcal{L}ab(A) = \texttt{out}$. We then need to show that A has at least one in-labelled attacker that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and that and that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A)$ is the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Given that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$, we distinguish two subcases.
  1. A was already numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering, it then follows that there is at least one in-labelled attacker of A that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$, and $\mathcal{MM}^{i}_{\mathcal{L}ab}(A)$ is the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. As $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, it follows that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}^{i}_{\mathcal{L}ab}(A)$. Moreover, the fact that $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ does not number any in-labelled arguments implies that the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is the same as the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Hence, A is still correctly numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$.
  2. A is numbered by $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$. From the definition of ExtendOut, it then follows that A has at least one in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$, and that $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})(A)$ is the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. Hence, A is correctly numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$.

(2') It suffices to show that for each $(A, k) \in \mathcal{MM}^{i+1}_{\mathcal{L}ab} \setminus \mathcal{MM}^{i}_{\mathcal{L}ab}$ it holds that $k = i + 1$. That is, we need to show that for each $(A, k) \in \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ it holds that $k = i + 1$. Let $(A, k) \in \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$. We ask ourselves two questions:

- Can k be *smaller* than $i + 1$?
  If this is the case, then (Definition 2) A is not numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$, and it has an in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$, and the MIN+1 value of its in-labelled attackers that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is smaller than $i + 1$, so smaller or equal to i. But this means that $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is not i-complete, which is in contradiction with induction hypothesis (3). So the answer is no.
- Can k be *bigger* than $i + 1$?
  If this is the case, then (Definition 2) A is not numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ and it has an in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ and the MIN+1 value of its in-labelled attackers that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is bigger than $i + 1$. From the definition of MIN+1, this then implies that A has an in-labelled attacker that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ with a min-max number that is bigger than i. This is in contradiction with induction hypothesis (2), that implies that each min-max number in $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is less or equal to i. So the answer is no.

(3') We need to show that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $(i+1)$-complete. For this, we need to show two things (Definition 4).

- If A is an unnumbered `in`-labelled argument of which all its `out`-labelled attackers are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$, then the MAX+1 value of these `out`-labelled attackers is bigger than $i+1$.

  Let A be an unnumbered `in`-labelled argument of which all its `out`-labelled attackers are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. We distinguish two subcases.

  1. All the `out`-labelled attackers of A were also numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$.
     Then, from the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is i-complete (induction hypothesis (3)) it follows that the MAX+1 value of these attackers is *bigger* than i. This (by definition of MAX+1) means that the min-max number of the highest numbered `out`-labelled attacker of A (say B) is bigger than $i-1$, so bigger or equal to i. However, we recall that i is an odd number, and that each `out`-labelled argument that is numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ has min-max number that is even (this follows from induction hypothesis (2), together with the definition of a numbering run). This then implies that the min-max number of B is *bigger* than i, so that the MAX+1 value of the `out`-labelled attackers of A that are numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is bigger than $i+1$. From the fact that $\mathcal{MM}^{i}_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$ it then follows that the MAX+1 value of the `out`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is also bigger or than $i+1$, which satisfies the requirement of i-completeness.
  2. At least one `out`-labelled attacker of A (say B) is numbered by $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$.
     Then, from (2') it follows that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(B) = i+1$. This then implies that the MAX+1 value of the `out`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is bigger or equal to $i+2$, so bigger than $i+1$, which satisfied the requirement of i-completeness.

- If A is an unnumbered `out`-labelled argument of which at least one `in`-labelled attacker is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$, then the MIN+1 value of the `in`-labelled attackers of A that are numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is bigger than $i+1$.

  Let A be an unnumbered `out`-labelled argument of which at least one `in`-labelled attacker (say B) is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Since $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^{i}_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ and $\texttt{ExtendOut}(\mathcal{MM}^{i}_{\mathcal{L}ab})$ does not number any `in`-labelled arguments, it follows that B was also numbered by $\mathcal{MM}^{i}_{\mathcal{L}ab}$. But from the definition of `ExtendOut` (Definition 2) it would then follow that A is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Contradiction.

□

Theorem 1. *Let $\mathcal{MM}^{0}_{\mathcal{L}ab}, \mathcal{MM}^{1}_{\mathcal{L}ab}, \mathcal{MM}^{2}_{\mathcal{L}ab}, \ldots$ be a numbering run of an admissible labelling $\mathcal{L}ab$. For every $i \geq 0$, $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a partial min-max numbering*

Proof. We prove this by induction over i.

basis We observe that for $i = 0$ it trivially holds that
(1) $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is a correct partial min-max numbering,
(2) for each $j \in \{1, \ldots, i\} = \emptyset$ it holds that for each $(A, k) \in \mathcal{MM}^{j}_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and
(3) $\mathcal{MM}^{i}_{\mathcal{L}ab}$ is i-complete.

step Suppose that for a given $i \geq 0$ it holds that
(1) $\mathcal{MM}^i_{\mathcal{L}ab}$ is a correct partial min-max numbering,
(2) for each $j \in \{1, \dots, i\}$ it holds that for each $(A, k) \in \mathcal{MM}^j_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and
(3) $\mathcal{MM}^i_{\mathcal{L}ab}$ is i-complete.
We have to prove that:
(1') $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a correct partial min-max numbering,
(2') for each $j \in \{1, \dots, i+1\}$ it holds that for each $(A, k) \in \mathcal{MM}^j_{\mathcal{L}ab} \setminus \mathcal{MM}^{j-1}_{\mathcal{L}ab}$ where A is an argument, $k = j$, and
(3') $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $(i+1)$-complete.
If i is even, this result follows from Lemma 1. If i is odd, this result follows from Lemma 2.

□

Definition 5. *Let $\mathcal{MM}^*_{\mathcal{L}ab}$ be a partial min-max numbering of admissible labelling $\mathcal{L}ab$. We define the function* `AddInf`$(\mathcal{MM}^*_{\mathcal{L}ab})$ *as $\{(A, \infty) \mid A$ is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}\}$.*

Theorem 2. *Let $\mathcal{L}ab$ be an admissible labelling of argumentation framework (Ar, att) and let $\mathcal{MM}^0_{\mathcal{L}ab}$, $\mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \dots$ be a numbering run of $\mathcal{L}ab$. Let $\mathcal{MM}^*_{\mathcal{L}ab} = \cup_{i=0}^{\infty} \mathcal{MM}^i_{\mathcal{L}ab}$ and $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}^*_{\mathcal{L}ab} \cup$ `AddInf`$(\mathcal{MM}^*_{\mathcal{L}ab})$. It holds that $\mathcal{MM}_{\mathcal{L}ab}$ is a min-max numbering of $\mathcal{L}ab$.*

Proof. We first observe that $\mathcal{MM}^*_{\mathcal{L}ab}$ is a partial min-max numbering with `ExtendIn`$(\mathcal{MM}^*_{\mathcal{L}ab}) =$ `ExtendOut`$(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$. Let $A \in$ `in`$(\mathcal{L}ab) \cup$ `out`$(\mathcal{L}ab)$. We distinguish two cases:

1. $\mathcal{L}ab(A) =$ `in`. We distinguish two subcases.

   (a) $\mathcal{MM}_{\mathcal{L}ab}(A) \in \mathbb{N}$. In that case, A is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^*_{\mathcal{L}ab}$ is a partial min-max numbering, it follows that $\mathcal{MM}^*_{\mathcal{L}ab}(A)$ is the MAX+1 value of all its `out`-labelled attackers. Since $\mathcal{MM}^*_{\mathcal{L}ab} \subseteq \mathcal{MM}_{\mathcal{L}ab}$, it follows that $\mathcal{MM}_{\mathcal{L}ab}(A) = \mathcal{MM}^*_{\mathcal{L}ab}(A)$ and the MAX+1 value of all the `out`-labelled attackers of A is the same in $\mathcal{MM}_{\mathcal{L}ab}$ as in $\mathcal{MM}^*_{\mathcal{L}ab}$. Therefore, $\mathcal{MM}_{\mathcal{L}ab}(A)$ is the MAX+1 value of all the `out`-labelled attackers of A (under $\mathcal{MM}_{\mathcal{L}ab}$). Hence, Definition 9 of the COMMA submission (first bullet) is satisfied.

   (b) $\mathcal{MM}_{\mathcal{L}ab}(A) = \infty$. In that case, A is numbered by `AddInf`$(\mathcal{MM}^*_{\mathcal{L}ab})$, so A is not numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. This, together with the fact that `ExtendIn`$(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$, implies that not all `out`-labelled attackers of A are numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Let B be an `out`-labelled attacker of A that is not numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Since $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}^*_{\mathcal{L}ab} \cup$ `AddInf`$(\mathcal{MM}^*_{\mathcal{L}ab})$, it then follows that $\mathcal{MM}_{\mathcal{L}ab}(B) = \infty$. Hence, the MAX+1 value of the `out`-labelled attackers of A is ∞ (under $\mathcal{MM}_{\mathcal{L}ab}$). Hence, Definition 9 of the COMMA submission (first bullet) is satisfied.

2. $\mathcal{L}ab(A) =$ `out`. We distinguish two subcases.

   (a) $\mathcal{MM}_{\mathcal{L}ab}(A) \in \mathbb{N}$. In that case, A is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\mathcal{MM}^*_{\mathcal{L}ab}$ is a partial min-max numbering, it follows that A has at least one `in`-labelled attacker that is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$ and $\mathcal{MM}^*_{\mathcal{L}ab}$ is the MIN+1 value of the `in`-labelled attackers of A that are numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Since $\mathcal{MM}^*_{\mathcal{L}ab} \subseteq \mathcal{MM}_{\mathcal{L}ab}$, it follows that $\mathcal{MM}_{\mathcal{L}ab}(A) = \mathcal{MM}^*_{\mathcal{L}ab}(A)$. Also, any `in`-labelled attacker of A that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$ will be numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$. Therefore, the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}_{\mathcal{L}ab}$ will be the same as the MIN+1 value of all numbered in-labelled attackers of A under $\mathcal{MM}^*_{\mathcal{L}ab}$. Hence, Definition 9 of the COMMA submission (second bullet) is satisfied.

(b) $\mathcal{MM}_{\mathcal{L}ab}(A) = \infty$. In that case, A is numbered by $\texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$, so A is not numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. This, together with the fact that $\texttt{ExtendOut}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$, implies that each in-labelled attacker of A is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. This also implies that each in-labelled attacker of A is numbered by $\texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$, so it is numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$. This means that the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}_{\mathcal{L}ab}$ is ∞. Hence, Definition 9 of the COMMA submission (second bullet) is satisfied.

□

Now that we have proved that the outcome of the numbering procedure is a correct min-max numbering, we proceed to prove that this min-max numbering is unique.

Theorem 3. *Let $\mathcal{L}ab$ be an admissible labelling of argumentation framework (Ar, att) and let $\mathcal{MM}^0_{\mathcal{L}ab}$, $\mathcal{MM}^1_{\mathcal{L}ab}$, $\mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ be a numbering run of $\mathcal{L}ab$. Let $\mathcal{MM}^*_{\mathcal{L}ab} = \cup_{i=0}^{\infty}\mathcal{MM}^i_{\mathcal{L}ab}$ and $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}^*_{\mathcal{L}ab} \cup \texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$. For any min-max numbering $\mathcal{MM}'_{\mathcal{L}ab}$ of $\mathcal{L}ab$, it holds that $\mathcal{MM}'_{\mathcal{L}ab} = \mathcal{MM}_{\mathcal{L}ab}$.*

Proof. Let $\mathcal{MM}'_{\mathcal{L}ab}$ be a min-max numbering of $\mathcal{L}ab$. We first show that for every $A \in Ar$, if $\mathcal{MM}_{\mathcal{L}ab}(A) \in \mathbb{N}$ then $\mathcal{MM}_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$. We do this by inductively proving that for each $i \geq 0$ it holds that:
(a) $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$, and
(b) for each argument B of which $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \{1, \ldots, i\}$, $\mathcal{MM}^i_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}(B)$

basis ($i = 0$) Since $\mathcal{MM}^0_{\mathcal{L}ab} = \emptyset$ it trivially holds that $\mathcal{MM}^0_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. Also, it trivially holds that for each argument B of which $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \emptyset$, $\mathcal{MM}^i_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}(B)$.

step (ExtendIn) (a) Suppose that for some even $i \geq 0$ it holds that $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. We now prove that also $\mathcal{MM}^{i+1}_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. Let A be an arbitrary argument that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Since i is even, it holds that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, so we can distinguish two cases:

1. A is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. Then from induction hypothesis (a) it directly follows that $\mathcal{MM}^i_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$, so (since $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$) $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$.
2. A is numbered by $\texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$. This implies that all its out-labelled attackers are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. From induction hypothesis (a) it then follows that the MAX+1 value of the out-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ is the same as under $\mathcal{MM}^i_{\mathcal{L}ab}$. Hence, the fact that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ and $\mathcal{MM}'_{\mathcal{L}ab}$ are (partial) min-max labellings, it follows that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$.

(b) Let B be an argument of which $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \{1, \ldots, i+1\}$. As induction hypothesis (b) tells us that for each B with $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \{1, \ldots, i\}$ it holds that $\mathcal{MM}^i_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}$ and $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, we only have to consider the case of $i + 1$.
So let B be an argument with $\mathcal{MM}'_{\mathcal{L}ab}(B) = i + 1$. Since i is even, it holds that B is labelled in by $\mathcal{L}ab$. So from the fact that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering with $\mathcal{MM}'_{\mathcal{L}ab}(B) = i + 1$,

it follows that the MAX+1 value of the out-labelled attackers of B (under $\mathcal{MM}'_{\mathcal{L}ab}$) is $i+1$. This implies that for each out-labelled attacker of B (say C), it holds that $\mathcal{MM}'_{\mathcal{L}ab}(C) \leq i$. Induction hypothesis (b) then implies that $\mathcal{MM}^i_{\mathcal{L}ab}(C) = \mathcal{MM}'_{\mathcal{L}ab}(C)$, so the MAX+1 value of the out-labelled attackers of B under $\mathcal{MM}^i_{\mathcal{L}ab}$ is the same as the MAX+1 value of the out-labelled attackers of B under $\mathcal{MM}'_{\mathcal{L}ab}$. Since $\mathcal{MM}'_{\mathcal{L}ab}(B) = i+1$ and $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering, it follows that the MAX+1 value of the out-labelled attackers of B under $\mathcal{MM}'_{\mathcal{L}ab}$ is $i+1$. Hence, the MAX+1 value of the out-labelled attackers of B under $\mathcal{MM}^i_{\mathcal{L}ab}$ is also $i+1$, so (since $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$) the MAX+1 value of the out-labelled attackers of B under $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is also $i+1$. Since $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is $(i+1)$-complete, it then follows that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(B) = i+1$.

step (`ExtendOut`**)** (a) Suppose that for some odd $i \geq 0$ it holds that $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. We now prove that also $\mathcal{MM}^{i+1}_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. Let A be an arbitrary argument that is numbered by $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$. Since i is odd, it holds that $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$, so we can distinguish two cases:

1. A is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. Then from induction hypothesis (a) it directly follows that $\mathcal{MM}^i_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$, so (since $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$) $\mathcal{MM}^{i+1}_{\mathcal{L}ab}(A) = \mathcal{MM}'_{\mathcal{L}ab}(A)$.
2. A is numbered by $\texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$. This implies that there exists an in-labelled attacker of A that is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$ and $\texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})(A)$ is the MIN+1 value of the in-labelled attackers of A that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. From induction hypothesis (a) it follows that the in-labelled attackers of A that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$ are numbered the same by $\mathcal{MM}'_{\mathcal{L}ab}$. We ask ourselves two questions.
   (a) Can the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ be *bigger* than the MIN+1 value of the in-labelled attackers of A (under $\mathcal{MM}^i_{\mathcal{L}ab}$) that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$?
   The answer is no, because the former MIN+1 value is based on a superset of arguments as what the latter MIN+1 value is based on.
   (b) Can the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ be *smaller* than the MIN+1 value of the in-labelled attackers of A (under $\mathcal{MM}^i_{\mathcal{L}ab}$) that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$?
   If this were the case, then there should be an in-labelled attacker of A (say B) for which $\mathcal{MM}'_{\mathcal{L}ab}(B)$ is smaller than the smallest in-out number (under $\mathcal{MM}^i_{\mathcal{L}ab}$) of the in-labelled attackers of A that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$, so smaller than i. But induction hypothesis (b) then implies that B is numbered with the same number by $\mathcal{MM}^i_{\mathcal{L}ab}$. Contradiction. So the answer is no.

   It then follows that the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ is *equal* to the MIN+1 value of the in-labelled attackers of A (under $\mathcal{MM}^i_{\mathcal{L}ab}$) that are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. From the fact that A is numbered by $\texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$, it follows that its min-max number (under $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$) is $i+1$, so the MIN+1 value of the in-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ is also $i+1$. From the fact that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering, it then follows that $\mathcal{MM}'_{\mathcal{L}ab}(A) = i+1$.

(b) Let B be an argument of which $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \{1, \ldots, i+1\}$. As induction hypothesis (b) tells us that for each B with $\mathcal{MM}'_{\mathcal{L}ab}(B) \in \{1, \ldots, i\}$ it holds that $\mathcal{MM}^i_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}$ and $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}^{i+1}_{\mathcal{L}ab}$, we only have to consider the case of $i+1$.

So let B be an argument with $\mathcal{MM}'_{\mathcal{L}ab}(B) = i{+}1$. Since i is odd, it holds that B is labelled `out` by $\mathcal{L}ab$. So from the fact that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering with $\mathcal{MM}'_{\mathcal{L}ab}(B) = i + 1$, it follows that the MIN+1 value of the `in`-labelled attackers of B (under $\mathcal{MM}'_{\mathcal{L}ab}$) is $i + 1$. This means that the smallest number (using $\mathcal{MM}'_{\mathcal{L}ab}$) among the `in`-labelled attackers of B is i. Let C be an `in`-labelled attacker of B that is numbered with i by $\mathcal{MM}'_{\mathcal{L}ab}$. From induction hypothesis (b) it then follows that $\mathcal{MM}^i_{\mathcal{L}ab}(C) = \mathcal{MM}'_{\mathcal{L}ab}(C) = i$, so the MIN+1 value of the numbered `in`-labelled attackers of B (using $\mathcal{MM}^i_{\mathcal{L}ab}$) is equal to the MIN+1 value of the `in`-labelled attackers of B (using $\mathcal{MM}'_{\mathcal{L}ab}$). The fact that $\mathcal{MM}^{i+1}_{\mathcal{L}ab}$ is a partial min-max numbering and that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering then implies that $\mathcal{MM}^i_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}(B)$.

From the thus proved fact for every $i \geq 0$ it holds that $\mathcal{MM}^i_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$ it follows that $\mathcal{MM}^*_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$. So for each $A \in Ar$, if $\mathcal{MM}_{\mathcal{L}ab}(A) \in \mathbb{N}$ then $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}'_{\mathcal{L}ab}$.

We proceed to show that also for every $A \in Ar$, if $\mathcal{MM}_{\mathcal{L}ab}(A) = \infty$ then $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}'_{\mathcal{L}ab}$. Suppose, towards a contradiction, that there exists an argument whose min-max number under $\mathcal{MM}_{\mathcal{L}ab}$ is ∞ but whose min-max number under $\mathcal{MM}'_{\mathcal{L}ab}$ is a natural number. This implies that the set $\{A \in Ar \mid \mathcal{MM}_{\mathcal{L}ab}(A) = \infty \wedge \mathcal{MM}'_{\mathcal{L}ab}(A) \in \mathbb{N}\}$ is non-empty. Let A be an argument from this set where $\mathcal{MM}'_{\mathcal{L}ab}(A)$ is minimal (say n). We distinguish two possibilities.

1. $\mathcal{L}ab(A) = \texttt{in}$
   In that case, from the fact that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering, it follows that n is the MAX+1 value of the `out`-labelled attackers of A. This implies that for each `out`-labelled attacker B of A, $\mathcal{MM}'_{\mathcal{L}ab}(B) < n$. Since n is the *smallest* number for which there exists an argument that is numbered with n by $\mathcal{MM}'_{\mathcal{L}ab}$ but with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$, it holds that B is numbered with a natural number by $\mathcal{MM}_{\mathcal{L}ab}$, meaning that it is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. However, since $\mathcal{MM}^*_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$, it then follows that $\mathcal{MM}^*_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}(B)$. Since this holds for *any* `out`-labelled attacker of A, it follows that the MAX+1 value of the `out`-labelled attackers of A under $\mathcal{MM}'_{\mathcal{L}ab}$ (n) is equal to the MAX+1 value of the `out`-labelled attackers of A under $\mathcal{MM}^*_{\mathcal{L}ab}$ (also n). But then $\mathcal{MM}_{\mathcal{L}ab}$ is not a correct min-max numbering, because A is numbered with ∞ whereas the MAX+1 value of its `out`-labelled attackers is n. Contradiction.

2. $\mathcal{L}ab(A) = \texttt{out}$
   In that case, from the fact that $\mathcal{MM}'_{\mathcal{L}ab}$ is a min-max numbering, it follows that n is the MIN+1 value of the `in`-labelled attackers of A. This implies that there exists an `in`-labelled attacker B of A with $\mathcal{MM}'_{\mathcal{L}ab}(B) = n - 1$. Since n is the *smallest* number for which there exists an argument that is numbered with n by $\mathcal{MM}'_{\mathcal{L}ab}$ but with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$, it follows that B is numbered with a natural number by $\mathcal{MM}_{\mathcal{L}ab}$, meaning that it is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. However, since $\mathcal{MM}^*_{\mathcal{L}ab} \subseteq \mathcal{MM}'_{\mathcal{L}ab}$, it then follows that $\mathcal{MM}^*_{\mathcal{L}ab}(B) = \mathcal{MM}'_{\mathcal{L}ab}(B)$. This then implies that the MIN+1 value of the `in`-labelled attackers of A (under $\mathcal{MM}^*_{\mathcal{L}ab}$) is at most n. But then $\mathcal{MM}_{\mathcal{L}ab}$ is not a correct min-max numbering, becaue A is numbered with ∞ whereas the $MIN + 1$ value of its `in`-labelled attackers is at most n (so a natural number). Contradiction.

□

Theorem 4 of the COMMA submission then directly follows from Theorem 2 and Theorem 3. That is, we just spent 12 pages just to prove one theorem from the COMMA submission (the topic of strong admissibility is far from trivial).

We now proceed to prove Theorem 5 from the COMMA paper. We do so in two parts.

Theorem 4. *Given an argumentation framework* (Ar, att)*. If* $\mathcal{L}ab$ *is a strongly admissible labelling, then* $\mathcal{A}rgs = \texttt{Lab2Args}(\mathcal{L}ab)$ *is a strongly admissible set.*

Proof. Let $\mathcal{MM}^0_{\mathcal{L}ab}, \mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ be the numbering run of $\mathcal{L}ab$. Let $\mathcal{MM}^*_{\mathcal{L}ab}$ be $\cup_{i=0}^{\infty}\mathcal{MM}^i_{\mathcal{L}ab}$ and let $\mathcal{MM}_{\mathcal{L}ab}$ be $\mathcal{MM}^*_{\mathcal{L}ab} \cup \texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$. For each $i \geq 0$ we define $\mathcal{A}rgs^i$ as $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{in}$ and A is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}\}$. We now show, by induction over i, that each $\mathcal{A}rgs^i$ is a strongly admissible set

basis (i=0) It holds that $\mathcal{A}rgs^0 = \emptyset$ and the empty set is trivially strongly admissible.

step (i is even) Suppose $\mathcal{A}rgs^i$ is a strongly admissible set, for some even $i \geq 0$. We need to prove that $\mathcal{A}rgs^{i+1}$ is a strongly admissible set as well. We first observe that, since i is even, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, so $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1}$. Let $A \in \mathcal{A}rgs^{i+1}$. We distinguish two possibilities.

1. $A \in \mathcal{A}rgs^i$. From the fact that $\mathcal{A}rgs^i$ is a strongly admissible set, it follows that A is defended by some $\mathcal{A}rgs' \subseteq \mathcal{A}rgs^i \setminus \{A\}$ which in its turn is again strongly admissible. Since $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1}$ it then follows that $\mathcal{A}rgs' \subseteq \mathcal{A}rgs^{i+1} \setminus \{A\}$, hence satisfying the requirement of strong admissibility.
2. $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$. We first show that A is defended by $\mathcal{A}rgs^i$. Let B be an argument that attacks A. From the fact that $\mathcal{L}ab$ is an admissible labelling, it follows that $\mathcal{L}ab(B) = \texttt{out}$. From the fact that A is numbered by $\texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ (this follows from $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$) it then follows that all out-labelled attackers of A are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. Since $\mathcal{MM}^i_{\mathcal{L}ab}$ is a partial min-max numbering, it holds that for every out-labelled argument that is numbered (for instance B), there exists an in-labelled attacker that is also numbered (say C). So $C \in \mathcal{A}rgs^i$. Hence, $\mathcal{A}rgs^i$ defends A. We also observe that $A \notin \mathcal{A}rgs^i$ (this follows from $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$). Furthermore, we recall that $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1}$. So, to sum up, A is defended by $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1} \setminus \{A\}$ which in its turn is again strongly admissible.

step (i is odd) Suppose $\mathcal{A}rgs^i$ is a strongly admissible set, for some odd $i \geq 0$. We need to prove that $\mathcal{A}rgs^{i+1}$ is a strongly admissible set as well. We first observe that, since i is odd, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$. However, as $\texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$ does not number any in-labelled arguments, it holds that $\mathcal{A}rgs^{i+1}_{\mathcal{L}ab} = \mathcal{A}rgs^i_{\mathcal{L}ab}$. From the fact that $\mathcal{A}rgs^i_{\mathcal{L}ab}$ is a strongly admissible set, it then trivially follows that $\mathcal{A}rgs^{i+1}$ is a strongly admissible set.

Let $\mathcal{A}rgs^*$ be $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{in}$ and A is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}\}$. From the fact that each $\mathcal{A}rgs^i$ is a strongly admissible set, it follows that $\mathcal{A}rgs^*$ is a strongly admissible set (after all, $\mathcal{MM}^*_{\mathcal{L}ab}$ is just some $\mathcal{MM}^j_{\mathcal{L}ab}$ for some j such that $\mathcal{MM}^j_{\mathcal{L}ab} = \mathcal{MM}^{j+1}_{\mathcal{L}ab}$). We now proceed to show that $\mathcal{A}rgs^* = \mathcal{A}rgs$.

"$\mathcal{A}rgs^* \subseteq \mathcal{A}rgs$" Let $A \in \mathcal{A}rgs^*$. Then A is labelled in by $\mathcal{L}ab$, so (by definition of Lab2Args) $A \in \mathcal{A}rgs$.

"$\mathcal{A}rgs \subseteq \mathcal{A}rgs^*$" Let $A \in \mathcal{A}rgs$. The fact that $\mathcal{L}ab$ is a strongly admissible labelling implies that no argument is numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab} = \mathcal{MM}^*_{\mathcal{L}ab} \cup \texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$, so no argument is numbered by $\texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$, which then implies that every in or out-labelled argument is numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. This means that A is also numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Therefore (by definition of $\mathcal{A}rgs^*$) $A \in \mathcal{A}rgs^*$.

$\square$

Theorem 5. *Given an argumentation framework* (Ar, att)*. If* $\mathcal{A}rgs \subseteq Ar$ *is a strongly admissible set then* $\mathcal{L}ab = \texttt{Args2Lab}(\mathcal{A}rgs)$ *is a strongly admissible labelling.*

Proof. Suppose, towards a contradiction, that $\mathcal{A}rgs$ is a strongly admissible set but that $\mathcal{L}ab$ is not a strongly admissible labelling. This implies that there exists an argument (say B) that is numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$, which means that B argument is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. We want to show that there exists at least one `in`-labelled argument that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$ (say A). For this, we distinguish two cases.

1. B is labelled `in` by $\mathcal{L}ab$. In that case, take A to be B.
2. B is labelled `out` by $\mathcal{L}ab$. In that case, from the fact that B is numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$, it follows that the MIN+1 value of its `in`-labelled attackers is ∞, so it has an `in`-labelled attacker (say A) that is numbered with ∞ by $\mathcal{MM}_{\mathcal{L}ab}$, so that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$.

From the fact that $\mathcal{A}rgs$ is a strongly admissible set, it follows that A is defended by some $\mathcal{A}rgs' \subseteq \mathcal{A}rgs \setminus \{A\}$ which in its turn is again strongly admissible. Can it be the case that all arguments in $\mathcal{A}rgs'$ are numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$? If this were the case, then the `out`-labelled attackers of A are also numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$ (this follows from the fact that $\texttt{ExtendOut}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$) so A itself would be numbered by $\mathcal{MM}^*_{\mathcal{L}ab}$ (this follows from the fact that $\texttt{ExtendIn}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$). Contradiction. Hence, there exists at least one argument in $\mathcal{A}rgs'$ (say A') that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$.

Since $\mathcal{A}rgs'$ is again a strongly admissible set, it follows that A' is defended by some $\mathcal{A}rgs'' \subseteq \mathcal{A}rgs' \setminus \{A'\}$ that in its turn is again strongly admissible. Using similar reasoning as above, we obtain that $\mathcal{A}rgs''$ contains some argument A'' that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$.

Since $\mathcal{A}rgs''$ is again a strongly admissible set, it follows that A'' is defended by some $\mathcal{A}rgs''' \subseteq \mathcal{A}rgs'' \setminus \{A''\}$ that in its turn is again strongly admissible. Using similar reasoning as above, we obtain that $\mathcal{A}rgs'''$ contains some argument A''' that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$, etc.

Can this line of reasoning go on forever? Since $\mathcal{A}rgs$ is a finite set of arguments, and every step we are essentially removing at least one argument, this means that at after some finite number of steps, we will encounter a strongly admissible set $\mathcal{A}rgs^{\#}$ which is equal to $\emptyset$. However, in line with the above reasoning, this $\mathcal{A}rgs^{\#}$ should still contain some $A^{\#}$ that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Contradiction. $\square$

Theorem 5 from the COMMA submission then follows from Theorem 4 and Theorem 5 of the current technical report.

We now proceed to prove Theorem 1 of the COMMA submission. The idea is first to prove equivalence to strongly admissible labellings.

Lemma 3. *Let* $\mathcal{A}rgs \subseteq \mathcal{A}rgs'$*. If* A *is strongly defended by* $\mathcal{A}rgs$ *then* A *is also strongly defended by* $\mathcal{A}rgs'$*.*

Proof. Let $\mathcal{A}rgs_1$ be $\mathcal{A}rgs$, $\mathcal{A}rgs'_1$ be $\mathcal{A}rgs'$, and A_1 be A. Assume towards a contradiction that A_1 is strongly defended by $\mathcal{A}rgs_1$ but not strongly defended by $\mathcal{A}rgs'_1$. By definition of strong defence, the latter means that not each attacker B_1 of A_1 is attacked by some $C_1 \in \mathcal{A}rgs'_1 \setminus \{A_1\}$ s.t. C_1 is strongly defended by $\mathcal{A}rgs'_1 \setminus \{A_1\}$. So there exists an attacker B_1 of A_1 s.t. any $C_1 \in \mathcal{A}rgs'_1 \setminus \{A_1\}$ that attacks it is not strongly defended by $\mathcal{A}rgs'_1 \setminus \{A_1\}$. However, the fact that A_1 is strongly defended by $\mathcal{A}rgs_1$ implies that each attacker B_1 of A_1 is attacked by some $C_1 \in \mathcal{A}rgs_1 \setminus \{A_1\}$ such that

C_1 is strongly defended by $\mathcal{A}rgs_1 \setminus \{A_1\}$. So C_1 is strongly defended by $\mathcal{A}rgs_1 \setminus \{A_1\}$ but not by $\mathcal{A}rgs'_1 \setminus \{A_1\}$. Notice that from $\mathcal{A}rgs \subseteq \mathcal{A}rgs'$ it follows that $\mathcal{A}rgs \setminus \{A_1\} \subseteq \mathcal{A}rgs' \setminus \{A_1\}$.

Let $\mathcal{A}rgs_2$ be $\mathcal{A}rgs_1 \setminus \{A_1\}$, $\mathcal{A}rgs'_2$ be $\mathcal{A}rgs'_1 \setminus \{A_1\}$, and A_2 be C_1. It then holds that A_2 is strongly defended by $\mathcal{A}rgs_2$ but not by $\mathcal{A}rgs'_2$. Using similar reasoning as above, we obtain that there exists a C_2 that is strongly defended by $\mathcal{A}rgs_2 \setminus \{A_2\}$ but not by $\mathcal{A}rgs'_2 \setminus \{A_2\}$.

Let $\mathcal{A}rgs_3$ be $\mathcal{A}rgs_2 \setminus \{A_2\}$, $\mathcal{A}rgs'_3$ be $\mathcal{A}rgs'_2 \setminus \{A_2\}$, and A_3 be C_2. It then holds that A_3 is strongly defended by $\mathcal{A}rgs_3$ but not by $\mathcal{A}rgs'_3$. Using similar reasoning as above, we obtain that there exists a C_3 that is strongly defended by $\mathcal{A}rgs_3 \setminus \{A_3\}$ but not by $\mathcal{A}rgs'_3 \setminus \{A_3\}$.

Can this line of reasoning go on infinitly? The answer is no, because at every step (perhaps with the exception of the first one) we are effectively removing an argument (A_j) from $\mathcal{A}rgs$. Since $\mathcal{A}rgs$ contains only a finite number of arguments (as we consider only finite argumentation frameworks) this means that at some moment we will encounter an i for which $\mathcal{A}rgs_i = \emptyset$. The fact that A_i is strongly defended by $\mathcal{A}rgs_i$ then implies that A_i does not have any attackers. But then A_i would also be strongly defended by $\mathcal{A}rgs'_i$. Contradiction. □

Theorem 6. *Given an argumentation framework (Ar, att). If $\mathcal{L}ab$ is a strongly admissible labelling, then $\mathcal{A}rgs = \texttt{Lab2Args}(\mathcal{L}ab)$ strongly defends each of its arguments.*

Proof. Let $\mathcal{MM}^0_{\mathcal{L}ab}, \mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ be the numbering run of $\mathcal{L}ab$, let $\mathcal{MM}^*_{\mathcal{L}ab}$ be $\cup_{i=0}^{\infty} \mathcal{MM}^i_{\mathcal{L}ab}$ and let $\mathcal{MM}_{\mathcal{L}ab}$ be $\mathcal{MM}^*_{\mathcal{L}ab} \cup \texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$. For each $i \geq 0$ we define $\mathcal{A}rgs^i$ as $\{A \in Ar \mid \mathcal{L}ab(A) = \texttt{in}$ and A is numbered by $\mathcal{MM}^i_{\mathcal{L}ab}\}$. We now show, by induction over i, that each $\mathcal{A}rgs^i$ strongly defends each of its arguments.

basis ($i = 0$) It holds that $\mathcal{A}rgs^0 = \emptyset$, which trivially defends each of its arguments.

step (i is even) Suppose $\mathcal{A}rgs^i$ strongly defends each of its arguments, for some even $i \geq 0$. We need to show that $\mathcal{A}rgs^{i+1}$ strongly defends each of its arguments as well. We first observe that, since i is even, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$, so $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1}$. Let $A \in \mathcal{A}rgs^{i+1}$. We distinguish two possibilities.

1. $A \in \mathcal{A}rgs^i$. From the fact that $\mathcal{A}rgs^i$ strongly defends each of its arguments (induction hypothesis) it follows that $\mathcal{A}rgs^i$ strongly defends A. Since $\mathcal{A}rgs^i \subseteq \mathcal{A}rgs^{i+1}$ it then follows from Lemma 3 that $\mathcal{A}rgs^{i+1}$ also strongly defends A.
2. $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$. We first show that each attacker B of A is attacked by some $C \in \mathcal{A}rgs^i$. Let B be an argument that attacks A. From the fact that $\mathcal{L}ab$ is an admissible labelling, it follows that $\mathcal{L}ab(B) = \texttt{out}$. From the fact that A is numbered by $\texttt{ExtendIn}(\mathcal{MM}^i_{\mathcal{L}ab})$ (this follows from $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$) it then follows that all $\texttt{out}$-labelled attackers of A (including B) are numbered by $\mathcal{MM}^i_{\mathcal{L}ab}$. Since $\mathcal{MM}^i_{\mathcal{L}ab}$ is a partial min-max numbering, it holds that for every $\texttt{out}$-labelled argument that is numbered (for instance B) there exists an $\texttt{in}$-labelled attacker that is also numbered (say C). So $C \in \mathcal{A}rgs^i$. We also observe that $A \notin \mathcal{A}rgs^i$ (this follows from $A \in \mathcal{A}rgs^{i+1} \setminus \mathcal{A}rgs^i$). To sum up, each attacker B of A is attacked by some $C \in \mathcal{A}rgs^i = \mathcal{A}rgs^i \setminus \{A\} \subseteq \mathcal{A}rgs^{i+1} \setminus \{A\}$. Furthermore, the induction hypothesis implies that C (by being member of $\mathcal{A}rgs^i$) is strongly defended by $\mathcal{A}rgs^i = \mathcal{A}rgs^i \setminus \{A\}$ so that (Lemma 3) C is strongly defended by $\mathcal{A}rgs^{i+1} \setminus \{A\}$, therefore satisfying the defintion of strong defence.

step (i is odd) Suppose $\mathcal{A}rgs^i$ strongly defends each of its arguments, for some odd $i \geq 0$. We need to show that $\mathcal{A}rgs^{i+1}$ strongly defends each of its arguments as well. We first observe that, since i is odd, $\mathcal{MM}^{i+1}_{\mathcal{L}ab} = \mathcal{MM}^i_{\mathcal{L}ab} \cup \texttt{ExtendOut}(\mathcal{MM}^i_{\mathcal{L}ab})$. Since $\texttt{ExtendOut}$ does not number any in-labelled arguments, it follows that $\mathcal{A}rgs^{i+1} = \mathcal{A}rgs^i$, so we can immediately apply the induction hypothesis and obtain the desired result.

□

Theorem 7. *Given an argumentation framework (Ar, att). If $\mathcal{A}rgs \subseteq Ar$ strongly defends each of its arguments then $\mathcal{L}ab = \texttt{Args2Lab}(\mathcal{A}rgs)$ is a strongly admissible labelling.*

Proof. Let $\mathcal{MM}^0_{\mathcal{L}ab}, \mathcal{MM}^1_{\mathcal{L}ab}, \mathcal{MM}^2_{\mathcal{L}ab}, \ldots$ be the labelling run of $\mathcal{L}ab$. Let $\mathcal{MM}^*_{\mathcal{L}ab}$ be $\cup_{i=0}^{\infty}\mathcal{MM}^i_{\mathcal{L}ab}$ and let $\mathcal{MM}_{\mathcal{L}ab}$ be $\texttt{AddInf}(\mathcal{MM}^*_{\mathcal{L}ab})$. Recall that $\mathcal{L}ab$ is a strongly admissible labelling iff $\mathcal{MM}_{\mathcal{L}ab}$ does not number any argument with ∞. Suppose, towards a contradiction, that $\mathcal{MM}_{\mathcal{L}ab}$ *does* number an argument (say A_1) with ∞. Since $\mathcal{MM}_{\mathcal{L}ab}$ only numbers arguments that are labelled in or out, we distinguish two cases.

1. $\mathcal{L}ab(A_1) = \texttt{in}$. The fact that $\mathcal{MM}_{\mathcal{L}ab}(A_1) = \infty$ implies that A_1 is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\texttt{ExtendIn}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$ it then follows that there is an out-labelled attacker (say B_1) of A_1 that is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\texttt{ExtendOut}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$ it follows that each in-labelled attacker of B_1 is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. The fact that $\mathcal{A}rgs$ strongly defends A_1 (the fact that $\mathcal{L}ab(A_1) = \texttt{in}$ implies that $A_1 \in \mathcal{A}rgs$, and $\mathcal{A}rgs$ strongly defends each of its arguments) then implies there is a $C_1 \in \mathcal{A}rgs \setminus \{A_1\}$ that attacks B_1 and is strongly defended by $\mathcal{A}rgs \setminus \{A_1\}$.

   Let A_2 be equal to C_1 (a different name for the same argument). A_2 is strongly defended by $\mathcal{A}rgs \setminus \{A_1\}$. However, it is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Using similar reasoning as above, we obtain that there is an unnumbered (by $\mathcal{MM}^*_{\mathcal{L}ab}$) $C_2 \in \mathcal{A}rgs \setminus \{A_1, A_2\}$ that is strongly defended by $\mathcal{A}rgs \setminus \{A_1, A_2\}$.

   Let A_3 be equal to C_2 (a different name for the same argument). A_3 is strongly defended by $\mathcal{A}rgs \setminus \{A_1, A_2\}$. However, it is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. Using similar reasoning as above, we obtain that there is an unnumbered (by $\mathcal{MM}^*_{\mathcal{L}ab}$) $C_3 \in \mathcal{A}rgs \setminus \{A_1, A_2, A_3\}$ that is strongly defended by $\mathcal{A}rgs \setminus \{A_1, A_2, A_3\}$.

   What happens if we continue to perform steps like the above? In essence, at every step we are removing some argument A_i from $\mathcal{A}rgs$. Since $\mathcal{A}rgs$ contains only a finite number of arguments (this is because we only consider finite argumentation frameworks) this can be done only a finite number of times (say n times). That is, after n steps, we obtain a set $\mathcal{A}rgs \setminus \{A_1, A_2, \ldots, A_n\} = \emptyset$ and some argument C_n that is strongly defended by this set, which implies that C_n does not have any attackers. From the fact that $\texttt{ExtendIn}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$ it then follows that C_n is numbered (with 1) by $\mathcal{MM}^*_{\mathcal{L}ab}$. Contradiction.

2. $\mathcal{L}ab(A_1) = \texttt{out}$. The fact that $\mathcal{MM}_{\mathcal{L}ab}(A_1) = \infty$ implies that A_1 is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\texttt{ExtendOut}(\mathcal{MM}^*_{\mathcal{L}ab}) = \emptyset$ it then follows that each in-labelled attacker of A_1 is unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. From the fact that $\mathcal{L}ab$ is an admissible labelling, it follows that there is at least one in-labelled attacker of A_1 (say A_1'), which then has to be unnumbered by $\mathcal{MM}^*_{\mathcal{L}ab}$. The rest of the proof then follows the same structure as the previous point.

□

We are now ready to prove Theorem 1 of the COMMA submission.

Theorem 8 (Theorem 1 of the COMMA submission)**.** *Let (Ar, att) be an argumentation framework and $\mathcal{A}rgs \subseteq Ar$. $\mathcal{A}rgs$ is a strongly admissible set iff each $A \in \mathcal{A}rgs$ is strongly defended by $\mathcal{A}rgs$.*

Proof. We prove this using strongly admissible labellings.

“⇒” Let $\mathcal{A}rgs$ be a strongly admissible set. Then (Theorem 5) $\mathcal{L}ab = \texttt{Args2Lab}(\mathcal{A}rgs)$ is a strongly admissible labelling. Therefore (Theorem 6) $\mathcal{A}rgs' = \texttt{Lab2Args}(\mathcal{L}ab)$ strongly defends each of its arguments. Also, from the definitions of `Args2Lab` and `Lab2Args` it follows that $\mathcal{A}rgs' = \mathcal{A}rgs$.

“⇐” Let $\mathcal{A}rgs$ strongly defend each of its arguments. Then (Theorem 7) $\mathcal{L}ab = \texttt{Args2Lab}(\mathcal{A}rgs)$ is a strongly admissible labelling. Therefore (Theorem 4) $\mathcal{A}rgs' = \texttt{Lab2Args}(\mathcal{L}ab)$ is a strongly admissible set. Also, from the definitions of `Args2Lab` and `Lab2Args` it follows that $\mathcal{A}rgs' = \mathcal{A}rgs$.

□