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Natural relations among physical observables in the neutrino mass matrix

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Abstract: We find all possible relations among physical observables arising from neutrino mass matrices that describe in a natural way the currently observed pattern (θ_{23} and θ_{12} large, $\Delta m^2_{\odot}/\Delta m^2_{\rm Atm}$ and θ_{13} small) in terms of a minimum number of parameters. Natural here means due only to the relative smallness (vanishing) of some parameters in the relevant lagrangian, without special relations or accidental cancellations among them.

Keywords: Solar and Atmospheric Neutrinos, Beyond Standard Model, Neutrino Physics, CP violation.

Contents

1. Introduction

To date, 3 different kinds of experiments are being used or can be conceived to get informations on the physical parameters in the mass matrix of 3 Majorana neutrinos: i) oscillation experiments, which can make access to the 2 independent squared mass differences Δm^2_{32}, Δm^2_{21} ($|\Delta m^2_{32}| > \Delta m^2_{21} > 0$), to the 3 mixing angles θ_{23}, θ_{12}, θ_{13}, and to a CP violating phase δ; ii) β and $0\nu2\beta$ decay experiments, probing 2 appropriate combinations of masses and oscillation parameters, m_β and m_{ee}; iii) cosmological and/or astrophysical experiments, which might detect, e.g., the sum of the neutrino masses $\sum_i m_i$. An increasing level of difficulty is involved in the different experiments, however, to reach the needed sensitivity, crucially depending on the actual neutrino mass spectrum. For a hierarchical spectrum, either "normal" ($m_3 \gg m_2 > m_1$) or "inverted" ($m_1 \simeq m_2 \gg m_3$), as we shall consider in the following,[1], one can tentatively assume, at least for illustration purposes, that all the six oscillation observables will be measured, some of them with significant precision, perhaps together with the $0\nu2\beta$ mass m_{ee}. This makes a total of seven observables, which a theory of neutrino masses should be able to correlate among each other.

A simple minded conjecture is that some of these correlations could arise from the economy in the number of independent basic parameters. As the simplest example, one

[1]The degenerate spectrum is not considered since it does not satisfy the naturalness criterion as defined below.

may consider the possibility that the neutrino mass matrix M_ν, in the flavour basis, has a maximum number of negligibly small entries [1, 2]. Inspection shows that, out of the 6 independent elements of the 3×3 symmetric matrix M_ν, only 2 of them could at most vanish consistently with current observations [3, 4]

$$\begin{aligned} 0.02 < R \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{32}} < 0.04\,, \qquad 0.35 < \tan^2\theta_{12} < 0.55\,, \\ 0.6 < \tan^2\theta_{23} < 1.4\,, \qquad \sin^2\theta_{13} \lesssim 0.03\,. \end{aligned} \tag{1.1}$$

This leaves a total of 15 different mass matrices to be examined, all of which dependent on 4 real parameters and 1 phase that can affect the 6 oscillation observables and m_{ee}. One expects therefore 2 relations between these observables, a priori different in every case.

Although interesting, the limit of this approach is that the elements of the neutrino mass matrix in the flavor basis are, in general, only combinations of the basic parameters in the relevant piece of the lagrangian, $\mathcal{L}^\nu_m$. For sure M_ν is generally influenced also by the charged lepton mass matrix or, if the seesaw mechanism is operative, by the mass matrix of the right handed neutrinos. A natural question to ask therefore is what happens if the economy in the number of parameters is not in M_ν but rather in the basic lagrangian. This is the question we address in this paper.

To answer this question in full generality would require examining a huge number of different possibilities, of which the 15 cases for M_ν mentioned above are only a small subset. In the following we shall only consider the possibilities that describe the currently observed pattern of the data (1.1) in a "natural" way, i.e. only by the relative smallness (vanishing) of some parameters in $\mathcal{L}^\nu_m$, barring special relations or accidental cancellations among them. This should in particular be the case when accounting simultaneously for the smallness of $R \equiv \Delta m^2_{21}/\Delta m^2_{32}$ and for the largeness of θ_{23}, the most peculiar feature of the data so far, even though an accidental cancellation might also produce the same feature. As illustrated below, it turns out that none of the 15 cases for M_ν mentioned above satisfy this criterion in a strict sense. Note that naturality is not defined here in terms of any symmetry. In particular we do not require that the relative smallness (vanishing) of some parameters in the basic lagrangian be understood in terms of an explicit symmetry, exact or approximate. The coming into play of symmetries could add new cases to our list. Viceversa, it could prove hard to understand some of the cases we consider as due to a flavour symmetry of any sort.

As anticipated, we shall independently consider:

i) the direct non see-saw case (NSSC) for the 3 Majorana neutrinos $N_L^T = (n_1, n_2, n_3)$, described by the mass lagrangian (after electroweak symmetry breaking)

$$\mathcal{L}^\nu_m(\text{NSSC}) = \bar{E}_L M_E E_R + N_L^T M_N N_L + \text{h.c.}\ , \tag{1.2}$$

where $E_L^T = (e, \mu, \tau)_L$, $E_R^T = (e, \mu, \tau)_R$;

ii) the see-saw case (SSC) with the lagrangian mediated by 2 or 3 right handed neutrinos,[2] collectively denoted by N

$$\mathcal{L}^{\nu}_{m}(\mathrm{SSC}) = \bar{E}_L M_E E_R + N^T M_{RL} N_L + N^T M_R N + \mathrm{h.c.}\ . \tag{1.3}$$

The neutrino mass matrix in the flavour basis, M_ν, and the leptonic analog of the Cabibbo Kobayashi Maskawa matrix V_ν are given respectively by

$$M_\nu = U_l^T M_N U_l, \qquad V_\nu = U_l^\dagger U_\nu \tag{1.4}$$

in terms of the diagonal matrices

$$D_E = U_l^\dagger M_E V_l, \qquad D_N = U_\nu^T M_N U_\nu\,, \tag{1.5}$$

where, in the SSC, $-M_N = M_{RL}^T M_R^{-1} M_{RL}$. To count easily the number of effective parameters in M_N, in the SSC it is convenient to define

$$M_{RL} = mA\,, \tag{1.6}$$

where $m_{ij} = m_i \delta_{ij}$ and $(AA^\dagger)_{ii} = 1$ with $i = 1,2$ $(1,2,3)$ for 2 (3) right handed neutrinos. Note that this decomposition is unique, with all m_i's positive. It is then

$$-M_N = A^T \mu^{-1} A \qquad \text{with} \quad \mu^{-1} \equiv m M_R^{-1} m \tag{1.7}$$

(A is adimensional, while μ has the dimension of an inverse mass). Of course the naturalness requirement applies equivalently to M_R, M_{RL} or to μ, A. The number of effective parameters in M_N is the sum of the parameters in μ and A.

Within this framework, in the next section we describe all cases that i) are consistent with current data, as summarized in (1.1); ii) lead to some definite testable correlation between the different observables.[3] More precisely, to realize ii) we stick to the cases that give a neutrino mass matrix in the flavor basis involving at most 4 real parameters and one phase.

2. Summary of results

Our results for the NSSC and for the SSC with 2 and 3 right-handed neutrinos are summarized in table 1 and illustrated in figure 1 and figure 2. Table 1 is meant to be self explanatory. There we show the definite testable correlations between θ_{13}, m_{ee} and θ_{23}, θ_{12}, δ and R, that can arise from all possible cases, both in the NSSC and in the SSC, and are consistent with present data. There are only four cases (A, B, C, D) that allow to connect $\sin\theta_{13}$ and m_{ee} with θ_{23}, θ_{12} and R, whereas in case E the correlation also involves the CP-violating phase δ. Quite a few "natural" cases have been left out

[2] In a 3 neutrino context, the 2 neutrino case corresponds to the limit in which one or more entries of the heavy neutrino mass matrix become much larger than the others.

[3] A case which leads to $\sin\theta_{13} = 0$, $m_{ee} = 0$ is disregarded, since we do not see how it can receive experimental support.

	$\sin\theta_{13}$	$\lvert m_{ee}\rvert/m_{\rm atm}$	SSC	NSSC	U_l
A1	$\frac{1}{2}\tan\theta_{23}\sin 2\theta_{12}\sqrt{R}$	$\sin^2\theta_{12}\sqrt{R}$	2		$\mathbf{1}$
B1	$\frac{1}{2}\tan\theta_{23}\tan 2\theta_{12}\,(R\cos 2\theta_{12})^{1/2}$	0	3		$\mathbf{1}$
C	$\frac{1}{2}\tan 2\theta_{12}\,(R\cos 2\theta_{12})^{3/4}$	0	3	$\surd$	R_{23}
D	$\frac{1}{2}\frac{\tan 2\theta_{12}}{\lvert\tan 2\theta_{23}\rvert}(R\cos 2\theta_{12})^{1/2}$	$\left(\frac{\sin\theta_{13}}{\cos 2\theta_{23}}\right)^2$	3		$\mathbf{1}$
E1	$-\frac{\tan\theta_{23}}{\cos\delta}\frac{1-\tan\theta_{12}}{1+\tan\theta_{12}}$	$2\cot\theta_{23}\sin\theta_{13}$	2, 3	$\surd$	$R_{12}(R_{23})$

Table 1: Summary of the possible correlations between θ_{13}, m_{ee} and θ_{23}, θ_{12}, δ, R, to leading order of the expansion in R (with $m_{\rm atm} \equiv \sqrt{\lvert\Delta m^2_{32}\rvert}$). The column "SSC" gives the number of right-handed neutrinos involved in the see-saw realization of each case. An inverse hierarchy is obtained only in the case E. Cases A2, B2, E2, are obtained from A1, B1, E1, with the replacements $\tan\theta_{23} \to \cot\theta_{23}$ and $\cos\delta \to -\cos\delta$.

from table 1 because they are not compatible with current data at 90% confidence level: some give $\theta_{12} \geq 45°$, while two predict a too large value of $\sin\theta_{13}$, $\sin\theta_{13} = \sin\theta_{12}R^{1/4}$ or $\sin\theta_{13} = \tan\theta_{12}/2(R\cos\theta_{12})^{1/4}$. One case has been omitted because it is consistent, within the present uncertainty, with $\theta_{13} = 0$ (F in the following).

The relations quoted in table 1 are obtained through an expansion in R. The higher orders in the expansion are suppressed by $R^{1/2}$ in all cases except E, where the leading corrections to the relations in table 1 are of order R. In all cases, anyhow, the exact relations can be obtained from tables 2, 3, and 4, where the sets of parameters that originate these correlations are shown. They will be motivated and discussed in section 3. All cases can be obtained in the see-saw context. Table 1 specifies the number of right-handed neutrinos involved and whether the individual cases can also originate from the NSSC. It also gives the form of the rotation on the charged lepton sector. The form of the light neutrino mass matrix before the charged lepton rotation is given in the Appendix. E is the only case that leads to an "inverted" spectrum. For cases A1, B1, E1, the independent possibility exists where $\tan\theta_{23} \to \cot\theta_{23}$, $\cos\delta \to -\cos\delta$, denoted in the following by A2, B2, E2, respectively. Case A is discussed in ref. [5, 6].[4]

Given the present knowledge of θ_{23}, θ_{12} and Δm^2_{21}, including the recent Kamland result [3], the ranges of values for $\sin\theta_{13}$ are shown in figure 1 at 90% confidence level for the different cases. It is interesting that all the ranges for $\sin\theta_{13}$, except in case D, are above $\simeq 0.02$ and some can saturate the present limit. Long-baseline experiments of first or second generation should explore a significant portion of this range while reducing at the same time the uncertainties of the different predictions at about 10% level [4]. Note that, in cases E, although the determination of $\sin\theta_{13}$ requires the knowledge of the CP

[4] For recent studies of textures in neutrino masses, also in connection with leptogenesis, see also [7].

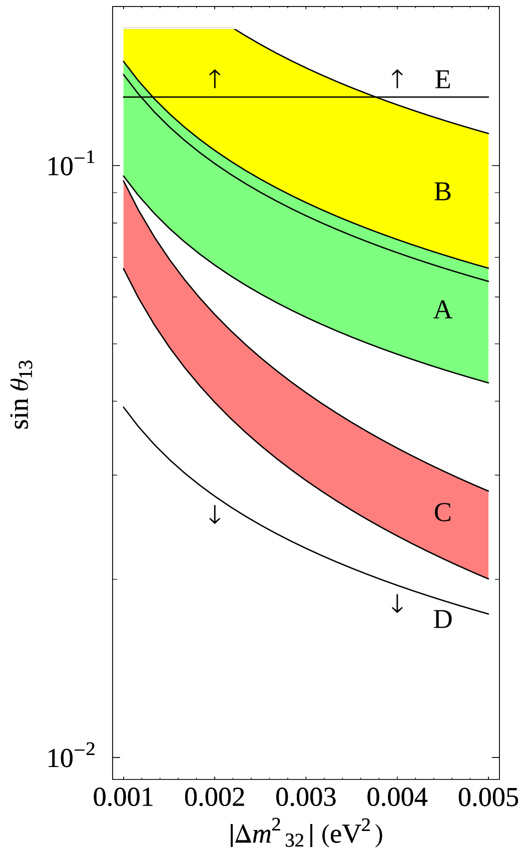

Figure 1: Ranges of values for $\sin\theta_{13}$ at 90% confidence level for the different cases, plotted as a function of Δm^2_{32}. Cases D, E, which only give a bound on $\sin\theta_{13}$, are shown with a double arrow.

violating phase as well, the allowed range is still limited, being $\sin\theta_{13} \gtrsim 0.10$. Furthermore, the requirement of not exceeding the present experimental bound on $\sin\theta_{13}$ gives a lower bound on $|\cos\delta|$ (and therefore an upper limit on CP-violation) that we can quantify as

$$|\cos\delta| > 0.8 \quad \text{at } 90\ \%\ \text{CL} \tag{2.1}$$

given the present uncertainties. Notice that $\cos\delta < 0$ (> 0) in case E1 (E2). Verifying the prediction for $\sin\theta_{13}$ in case D would require the measurement of $\theta_{23} \neq 45°$; a bound on $|1-\sin^2 2\theta_{23}|$ only sets an upper bound on $\sin\theta_{13}$, as shown in figure 1.[5]

While the prediction for $\sin\theta_{13}$ are in an experimentally interesting range, the expectations for the $0\nu2\beta$-decay effective mass are mostly on the low side, except, as expected [8], in the only inverted hierarchical case E. The ranges for each individual cases with non-vanishing m_{ee} are shown in figure 2. The challenge of detecting a non zero m_{ee}, when applicable, is therefore harder than for $\sin\theta_{13}$, with a better chance for the only inverted hierarchical case E.

[5] The inclusion of higher corrections in R does not change this conclusion in any significant way.

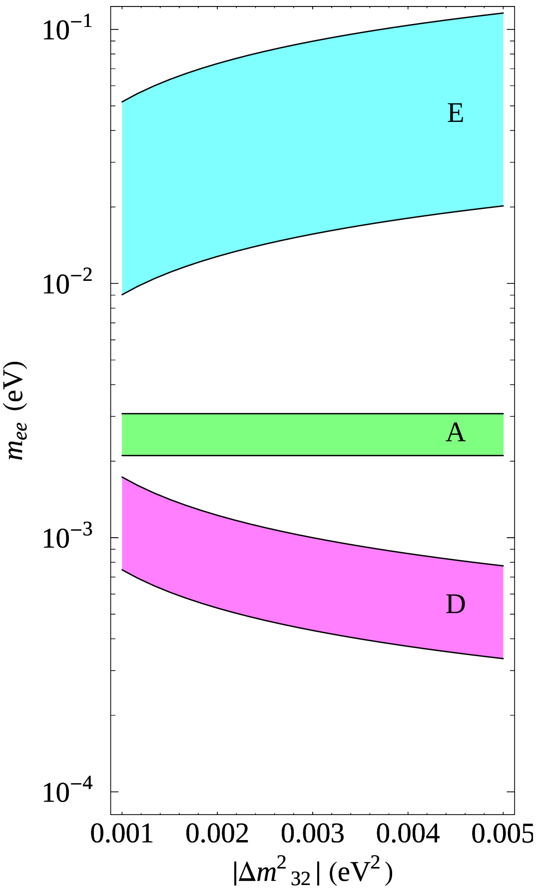

Figure 2: Ranges of values for m_{ee} at 90% confidence level.

3. Table 1 justified

In this section we describe how we arrive to select the relatively small number of cases enumerated in section 2. We build up the overall picture by commenting on the individual cases, which are summarized in tables 2, 3 and 4. It is non trivial to make sure that we are not missing any possibility. This is obtained partly by general considerations and partly by direct inspection of the individual cases.

3.1 Non see-saw case, M_E diagonal

As an illustration, before describing the positive cases, let us show why the NSSC with diagonal M_E does not lead to any acceptable example, according to our rules. The neutrino mass matrix in the flavour basis, M_ν, coincides with M_N in eq. (1.2). This justifies considering the matrices M_ν with 2 vanishing entries also in a natural sense. The only form of M_ν which gives the zeroth order pattern characterized by $R = 0$ and θ_{23} large without

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correlations between the different entries, is (PD = Pseudo-Dirac) [9]

$$M_\nu^{\mathrm{PD}} = m_0 \begin{pmatrix} 0 & c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix}, \tag{3.1}$$

where m_0 is the overall mass scale and c, s stand for the cosine and the sine of a mixing angle, without loss of generality. Perturbations to (3.1) can lead to a M_ν consistent with the data with an inverted hierarchical spectrum, except for one difficulty: the angle θ_{12} is too close to 45° degrees. To get inside the range (1.1) requires a fine-tuning. For this reason we have not considered these cases here. To get solutions in the NSSC without fine-tuning requires a non-trivial rotation from the charged lepton sector (see table 4).[6]

3.2 See-saw case (2 N's), M_E diagonal

Again because θ_{12} is too close to 45°, one does not get a natural solution with an inverted spectrum even in the SSC with a diagonal M_E. On the other hand, it is easy to obtain a normal hierarchical pattern. A 2×2 matrix μ defined in eq. (1.7) with one dominant diagonal element and a non zero determinant may lead, in fact, to a fully natural description of the pattern (1.1) [10, 11]. Taking μ_{22} as the dominant element, without loss of generality, one is readily convinced that the matrix A must have the form

$$A = \begin{pmatrix} 0 & s & c \\ c' & s'e^{i\phi} & 0 \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} 0 & s & c \\ c' & 0 & s'e^{i\phi} \end{pmatrix}, \tag{3.2}$$

where we explicitly included the only phases ineliminable by a redefinition of the neutrino fields. By counting the number of real free parameters, which should not exceed 4, the only ambiguity is where one places the single small but not vanishing entry in μ other than μ_{22}. There are 2 possibilities: $\mu_{11} \neq 0$, which leads to cases A in table 1, and $\mu_{12} = \mu_{21} \neq 0$, which leads to $\sin\theta_{13} = \sin\theta_{12} R^{1/4}$, outside the currently allowed region. Notice that case A splits in two phenomenologically similar cases A1 and A2 according to which of the two possibilities in eq. (3.2) is chosen. Clearly, case A2 can be obtained from case A1 by a $\nu_\mu \leftrightarrow \nu_\tau$ exchange in the mixing matrix, which corresponds to $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ in the predictions.

3.3 See-saw case (3 N's), M_E diagonal

In the SSC with 3 N's and a diagonal M_E, the atmospheric angle can originate either from the heavy neutrinos Majorana mass matrix M_R or from the Dirac mass matrix M_{RL}. In order to give rise to a large θ_{23}, the right-handed neutrino mass matrix must correspond to a μ in the form

$$\frac{\mu}{\mu_0} = \begin{pmatrix} a & c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix} + 1 \text{ small entry}, \tag{3.3}$$

[6] To account for the atmospheric neutrino observations, s/c in eq. (3.1) should be one within about 15%. This *is* a relation among parameters that we assume here and in the following without offering any explanation of it. We also assume that it is worth accounting for any other correlation among parameters in a natural way, as defined above.

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	μ/μ_0	A	U_l
A1	$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & s & c \\ c' & s'e^{i\phi} & 0 \end{pmatrix}$	$\mathbf{1}$
A2	$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & s & c \\ c' & 0 & s'e^{i\phi} \end{pmatrix}$	$\mathbf{1}$
E1 (E2)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ + 1 small entry	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \end{pmatrix}$	R_{12} (R_{13})

Table 2: Parameters for the see-saw cases with 2 right-handed neutrinos of table 1. A is the Dirac neutrino mass matrix with $(AA^\dagger)_{ii}$ normalized to unity, $M_{RL} = mA$, $m_{ij} = m_i\delta_{ij}$ and μ is related to the right handed neutrino mass matrix by $\mu = m^{-1}M_R m^{-1}$. ϵ and σ denote small entries relative to unity. c, s or c', s' denote the cosine and the sine of arbitrary angles, θ and θ'. U_l is the rotation of the left handed charged leptons.

where $a = \mathcal{O}(1)$ or $a = 0$. The large θ_{12} angle and a non-vanishing Δm^2_{21} can also be obtained from (3.3) by adding a small entry in the "23" submatrix, provided that $a \neq 0$. No special structure is demanded to the Dirac matrix, which has to be diagonal in this case ($A = \mathbf{1}$) since the four available parameters have been already used in μ.[7] On the other hand, if $a = 0$, the solar mixing angle must be provided by a non-diagonal Dirac matrix:

$$A = \begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

A non-vanishing entry in the "23" submatrix of μ is still necessary in order to have a non zero determinant[8].

As for the cases in which the θ_{23} rotation arises from the Dirac mass matrix, one needs

$$\frac{\mu}{\mu_0} = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \tag{3.5}$$

Again, the solar angle can originate from (3.5) if $a \neq 0$ [9] and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & c \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & s & c \end{pmatrix}. \tag{3.6}$$

[7] The texture in which the small entry in eq. (3.3) is in the 22 or 33 position appears also in [12].

[8] The small determinant limit is in principle as meaningful as the large determinant limit leading to the SSC with 2 N's. However, it does not give rise to any interesting case.

[9] Here, as above, we have conventionally chosen a labeling for the three right-handed neutrinos that corresponds to a given order of the rows in the matrix A.

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	μ/μ_0	A	U_l
B1 (B2)	$\begin{pmatrix} a & c & s \\ c & 0\,(\epsilon e^{i\phi}) & 0 \\ s & 0 & \epsilon e^{i\phi}\,(0) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & c & s \\ c & 0\,(\sigma) & 0 \\ s & 0 & \sigma\,(0) \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} ae^{i\phi} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}$	$\mathbf{1}$
4C	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	R_{23}
D	$\begin{pmatrix} a & c & s \\ c & 0 & \epsilon e^{i\phi} \\ s & \epsilon e^{i\phi} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & c & s \\ c & 0 & \sigma \\ s & \sigma & 0 \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
E1	$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	R_{12}
E2	same as E1		R_{13}

Table 3: Parameters for the see-saw cases with 3 right-handed neutrinos of table 1. ϵ and σ denote small positive quantities, while $a = \mathcal{O}(1)$. μ, A and U_l as in table 2.

Otherwise, if $a = 0$, the solar angle must also be provided by a matrix A in the form

$$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & s & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & s & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 0 & 1 \\ 0 & s & c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 0 & 1 \\ 0 & s & c \end{pmatrix}. \tag{3.7}$$

All the SSCs with 3 N's and a diagonal M_E are summarized in table 3, when they lead to a case consistent with current observations. Cases B2 can be obtained from cases B1 by exchanging the last two columns of A (and possibly a relabeling of N_1, N_2, N_3). The corresponding predictions are reported in table 1.

3.4 M_E non diagonal

With M_E non-diagonal, the mixing matrix $V_\nu = U_l^\dagger U_\nu$ receives a non trivial contribution

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	M_N/m_0	U_l
C	$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}$	R_{23}
E1 (E2)	$\begin{pmatrix} 0 & c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix}$ +1 small entry	R_{12} (R_{13})

Table 4: Parameters for the non see-saw case of table 1. ϵ and σ denote small entries relative to unity. M_N is the left handed neutrino mass matrix. U_l as in table 2.

also from the rotation matrix in the charged sector U_l. Within our hypotheses, none of the 4 general parameters in U_l can be correlated with the charged lepton masses. It is on the contrary possible to obtain an arbitrary U_l with the only limitation that a large rotation in the 23 sector, R^l_{23}, if present at all, should precede all the rotations in the other sectors R^l_{12} and R^l_{13}. Explicitly $U_l = R^l_{23} R^l_{13} R^l_{12}$, up to phases, is a natural form, with any of the R^l_{ij} possibly reduced to $\mathbf{1}$. This follows from the hierarchy of the charged lepton masses, ordered in the usual way, $(1,2,3) = (e, \mu, \tau)$. Note that R^l_{12} must be close to the identity since otherwise, when commuted with R^l_{23} to obtain the conventional order in V_ν, a large R_{13} rotation would also be generated.

3.4.1 $U_l = R^l_{23}$

$U_l = R^l_{23}$ (up to phases) must be considered in association with the NSSC and SSC cases for M_N. By inspection one shows that in both cases it is only in association with a hierarchical pattern of neutrino masses that interesting cases can be produced, if we disregard the further generation of cases with inverted spectrum and the $\theta_{12} \simeq 45°$ problem already encountered. The only predictive and viable form of M_N is

$$M_N = \begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix} \tag{3.8}$$

(σ, ϵ are small corrections), which leads to case C in table 1 and 4.

In the SSC with 2 N's, a R^l_{23} contribution from the charged lepton sector opens the possibility

$$\mu = \mu_0 \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 0 & 1 \\ c' & s' & 0 \end{pmatrix}. \tag{3.9}$$

This gives a variation of case A which predicts $\sin\theta_{13} = 0$ and m_{ee} as in the other cases A, probably too small to be detected and, as such, not included in the summary of table 1.

With 3 N's, the only viable possibility is

$$\frac{\mu}{\mu_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \qquad A = \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.10}$$

leading to case C again. The case

$$\frac{\mu}{\mu_0}=\begin{pmatrix}0 & \sigma & 0\\ \sigma & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad A=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ s' & c' & 0\end{pmatrix} \tag{3.11}$$

gives the same predictions as (3.9).

Since a $\nu_\mu \leftrightarrow \nu_\tau$ exchange can be reabsorbed in a redefinition of the θ_{23} mixing angle, the cases discussed in this subsection do not split in subcases. The experimental observation that $\tan^2\theta_{12}$ not only deviates from 1 but is lower than 1 serves to cut out a few otherwise acceptable cases.

3.4.2 R^l_{12} or $R^l_{13} \neq 0$

As already mentioned, the rotations in U_l other than R^l_{23}, if present at all, must be small. On the other hand, they introduce new parameters. Therefore, the only way in which they can give rise to a new realistic and predictive case is when they act in association with a M_N that depends at most on 3 effective parameters and is close enough to the fully realistic situation. On this basis, one is readily convinced that an interesting case is represented by any M_N, frequently encountered, which gives an inverted spectrum and $\theta^\nu_{12} \simeq 45°$. The role of the charged lepton contribution R^l_{12} or R^l_{13} is essential in providing the required correction to $\theta_{12} = 45°$ in a natural way.

In this way one is led to consider mixing matrices of the form

$$V_\nu = R^l_{12}(\alpha) R_{23} \begin{pmatrix} e^{i\phi} & & \\ & 1 & \\ & & 1\end{pmatrix} R_{12}(\theta^\nu_{12})\,, \tag{3.12}$$

where α is a free small mixing parameter. Note that the case of a R^l_{13} contribution from the lepton sector can be obtained from eq. (3.12) with a $\nu_\mu \leftrightarrow \nu_\tau$ exchange (and a redefinition of R_{23}), which corresponds to $\theta_{23} \leftrightarrow \pi/2 - \theta_{23}$ and $\delta \leftrightarrow \delta + \pi$ in the predictions. To obtain θ_{13}, it is necessary to bring V_ν to the standard form, i.e. to commute the R^l rotations, either R^l_{12} or R^l_{13}, with R_{23}. To first order in α, it is

$$R^l_{12}(\alpha) R_{23} \begin{pmatrix} e^{i\phi} & & \\ & 1 & \\ & & 1\end{pmatrix} R_{12}(\theta^\nu_{12}) = R_{23} R_{13}(\alpha s_{23}) \begin{pmatrix} e^{i\phi} & & \\ & 1 & \\ & & 1\end{pmatrix} R_{12}(\theta^\nu_{12} + \alpha c_{23}\cos\phi) \tag{3.13}$$

which leads to case E1 in table 1 and 3 (where the relations are exact in α). The corresponding case E2 is obtained for $U_l = R^l_{13}$.

The explicit realizations of the mechanism are obtained in the NSSC case by adding a single correction to M_N in eq. (3.1). In the SSC case with 2 N's, they correspond to adding a single correction to μ/μ_0 or A in

$$\mu = \mu_0 \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad A = \begin{pmatrix}1 & 0 & 0\\ 0 & c & s\end{pmatrix}, \tag{3.14}$$

while in the SCC with 3 N's they correspond to

$$\frac{\mu}{\mu_0} = \begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \text{or} \qquad \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.15}$$

Starting from similar configurations, $U_l = R^l_{12} R^l_{23}$ and $U_l = R^l_{13} R^l_{23}$ lead to the same cases as $U_l = R^l_{12}$ and $U_l = R^l_{13}$.

The leptonic rotations R^l_{12} and R^l_{13} can also play a role in hierarchical cases in which the neutrino contribution to V_ν gives $\theta_{12} > 45°$. The charged lepton contribution is then used to bring θ_{12} back to the allowed range below $45°$. This possibility is only realized in the SSC with 3 N's for

$$\frac{\mu}{\mu_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & c \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & s & c \end{pmatrix}, \tag{3.16}$$

giving respectively cases F1a, F1b for $U_l = R_{12}$ and cases F2a, F2b for $U_l = R_{13}$; or adding a single correction in the 2-3 submatrix of μ/μ_0 in

$$\frac{\mu}{\mu_0} = \begin{pmatrix} 0 & c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.17}$$

giving cases F1a, F1b, F1c for $U_l = R_{12}$ and cases F2a, F2b or F2c for $U_l = R_{13}$. As anticipated in section 2, cases F do not appear in table 1 because their predictions for $\sin\theta_{13}$ are either inconsistent with present data (Fa, Fc) or not conclusive, since it is at present compatible with zero (Fb). To get to this conclusion it is necessary to take into account not only the leading order prediction for $\sin\theta_{13}$, which is common to the three cases F1a, F1b, F1c,

$$-\cos\delta \sin\theta^{(0)}_{13} = \tan\theta_{23} \frac{1 - \tan\theta_{12}}{1 + \tan\theta_{12}} \tag{3.18}$$

but to include as well higher order corrections in R:

$$\begin{aligned} -\cos\delta \sin\theta_{13} &= \tan\theta_{23} \frac{t - \tan\theta_{12}}{1 + t\tan\theta_{12}} + q\left(\frac{R}{2}\right)^{1/3} \\ t &= 1 + \frac{p}{2}\left(\frac{R}{2}\right)^{1/3}, \qquad \text{with} \\ p &= (\tan\theta_{23})^{4/3},\ (\tan\theta_{23})^{-4/3},\ 1 \\ q &= (\tan\theta_{23})^{1/3},\ -(\tan\theta_{23})^{-1/3},\ (\tan 2\theta_{23})^{-1} \end{aligned} \tag{3.19}$$

in cases F1a, F1b, F1c respectively.

4. Conclusions

The economy in the number of basic parameters could be at the origin of some correlations between the physical observables in the neutrino mass matrix. At the present state of knowledge, the variety of the possibilities for the basic parameters themselves is large: M_N, M_E, M_R, M_{RL} are the matrices that might be involved. Finding the minimal cases that describe the present pattern of the data in a natural way could be a first step in the direction of discriminating the relevant $\mathcal{L}_m^\nu$. This we have done with the results summarized in tables 1–4 and illustrated in figure 1 and figure 2. It is remarkable that the number of possible correlations between the physical observables is limited (table 1), with a relatively larger number of possibilities for the basic parameters (tables 2–4). The relatively best chance for selecting experimentally one out of the few relevant cases is offered by $\sin\theta_{13}$. Its predictions, in table 1, should have a 10% uncertainty with the improved determination of the other parameters foreseen in long-baseline neutrino experiments [4]. Combining this with independent studies of leptogenesis or of lepton flavour violating effects could lead to the emergence of an overall coherent picture.

We have insisted on "naturalness" both in solving the "large θ_{23}-small R" problem for the normal hierarchy case and in obtaining a significant deviation of θ_{12} from 45°, with small R, in the inverted hierarchy case. Both features might be due to an accidental tuning of parameters. Nevertheless, explaining these features in a natural way offers a possible interesting guidance for model building. Note, in this respect, the use in cases E1, E2 of a relatively small charged lepton rotation in the 12 or 13 sectors to solve the second fine tuning problem mentioned above. This is analogous to the use of a large charged lepton rotation in the 23 sector to account for the "large θ_{23}-small R" problem in a natural way [13].

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A. General forms of the M_N matrices for the various cases

See table 5.

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	M_N	U_l
A1	$\begin{pmatrix} \rho^2 & \rho\gamma & 0 \\ \rho\gamma & s^2+\gamma^2 & sc \\ 0 & sc & c^2 \end{pmatrix}$	$\mathbf{1}$
A2	$\begin{pmatrix} \rho^2 & 0 & \rho\gamma \\ 0 & s^2 & sc \\ \rho\gamma & sc & c^2+\gamma^2 \end{pmatrix}$	$\mathbf{1}$
B1	$\begin{pmatrix} 0 & \rho & 0 \\ \rho & s^2+\gamma & sc \\ 0 & sc & c^2 \end{pmatrix}$	$\mathbf{1}$
B2	$\begin{pmatrix} 0 & 0 & \rho \\ 0 & s^2 & sc \\ \rho & sc & c^2+\gamma \end{pmatrix}$	$\mathbf{1}$
C	$\begin{pmatrix} 0 & \rho & 0 \\ \rho & 0 & \gamma \\ 0 & \gamma & 1 \end{pmatrix}$	R_{23}
D	$\begin{pmatrix} \rho^2 & s\rho & -c\rho \\ s\rho & s^2 & sc+\gamma \\ -c\rho & sc+\gamma & c^2 \end{pmatrix}$	$\mathbf{1}$
E1	$\begin{pmatrix} \rho & c & s \\ c & 0 & 0 \\ s & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ c & \rho & 0 \\ s & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ c & 0 & 0 \\ s & 0 & \rho \end{pmatrix}$ $\begin{pmatrix} 0 & c & s \\ c & 2\rho c & \rho s \\ s & \rho s & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ c & 0 & \rho c \\ s & \rho c & 2s\rho \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ c & \rho c^2 & \rho sc \\ s & \rho sc & \rho s^2 \end{pmatrix}$	R_{12}
E2	same as E1	R_{13}

Table 5: General forms of the M_N matrix (up to an overall factor) for the see-saw cases. For $U_l = \mathbf{1}$, this matrix coincides with the neutrino mass matrix M_ν in the flavor basis. For $U_l \neq \mathbf{1}$, both matrices are related by eq. (1.4), i.e. $M_\nu = U_l^T M_N U_l$. In these exact forms, up to a sign, the parameters s and c are approximately the same as those in tables 2, 3. ρ can be chosen positive, γ is in general complex and $|\gamma| \sim \rho \ll 1$.

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