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THE TRANSVERSE IMPEDANCE OF A CYLINDRICAL PIPE WITH ARBITRARY SURFACE IMPEDANCE

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Abstract

The coupling between a charged particle beam and a perfectly conducting pipe is derived from basic physical principles. It leads to the expression of transverse impedance related to the image current and the image charge. A finite but arbitrary surface impedance of the pipe is introduced next and leads to a general expression of the total transverse impedance of a cylindrical pipe.

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1 Introduction

The derivation of the dispersion relation coefficients, which are related to the concept of transverse impedance, has been done before by B. Zotter by solving the wave equations in a cylindrical vacuum chamber [1,2]. The physical comprehension of this approach is not obvious. A more transparent physical approach was adopted by G. Nassibian and F. Sacherer [3]. However, some of the results that they obtained are inconsistent and have caused considerable confusion. The combination of large surface impedance, small aperture of the vacuum chamber and very low transverse oscillation frequencies due to sheer size of the LHC machine aroused renewed interest in these matters.

2 Transverse impedance of a circular pipe with arbitrary surface impedance

The geometry of the problem at hand is sketched in Figure 1.

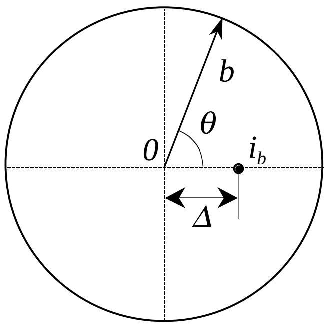

Figure 1 : Geometry of the problem. The radius of the pipe is b, the beam (black dot) is displaced from the center by an amount Δ.

The beam intensity is $i_b = \lambda\beta c$, where λ is the line density, c is the speed of light and βc is the speed of the particles of the beam. The radius b will occasionally be used as normalising factor such that $\delta = \Delta/b$. The length of the structure is unity.

2.1 Coupling of an off-centered beam on itself in a perfectly conducting pipe

The charge density on the wall of a cylindrical pipe induced by a beam with line density λ located at $\Delta = \delta\, b$ from the center is [4]:

$$D(\theta,\delta) = \frac{1-\delta^2}{1+\delta^2 - 2\delta\cos\theta}\frac{\lambda}{2\pi b}, \tag{1}$$

Likewise, the current density in the wall induced by a beam current i_b located at $\Delta = \delta\, b$ from the center is :

$$J(\theta,\delta) = \frac{1-\delta^2}{1+\delta^2 - 2\delta\cos\theta}\frac{i_b}{2\pi b}. \tag{2}$$

The local increment of the current density in the wall arising from the off-centered beam is :

$$dJ(\theta,\delta) = \frac{2\delta(\cos\theta - \delta)}{1+\delta^2 - 2\delta\cos\theta}\frac{i_b}{2\pi b}. \quad (3)$$

For $\delta << 1$ the expression reduces to:

$$dJ(\theta,\delta) = \frac{(i_b\delta)}{\pi b}\cos\theta. \quad (4)$$

The current density profile changes sign from one side of the pipe to the other. The magnitude of the current carried by one half of the pipe is simply:

$$i_d = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dJ(\theta,\delta) b d\theta = \frac{2}{\pi}\frac{i_b\Delta}{b}. \quad (5)$$

This expression defines the coupling between the dipole moment of the beam $i_b\Delta$ and the differential current i_d in the perfectly conducting pipe wall.
The differential wall current with its $cos\theta$ distribution will generate a constant magnetic field strength in the median plane perpendicular to it (think of the $cos\theta$ conductors of the dipoles in the *LHC*). The situation is shown in Figure 2. The current element on the wall that generates the magnetic field is $dJ(\theta,\delta)bd\theta$ (normalised $dJ(\theta,\delta)d\theta$), the point on the median plane where the field is evaluated is located at x from the center of the chamber. The distance between the wall current element and x is α (in normalised units).

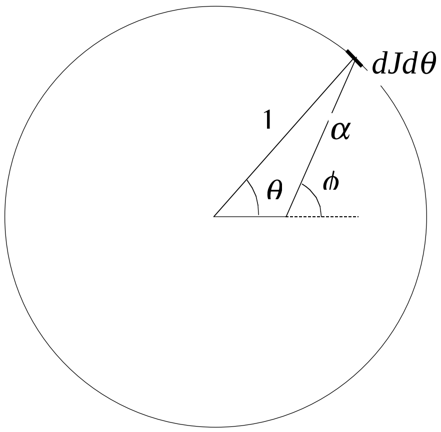

Figure 2 : Geometrical relation between wall current element $dJd\theta$ and location x on the median plane of the beam pipe.

The following relations can be established :

$$\alpha = \frac{\sin\theta}{\sin\phi}$$

$$\cos\theta = x + \alpha\cos\phi = x + \frac{\sin\theta}{tg\phi}. \quad (6)$$

$$tg\phi = \frac{\sin\theta}{\cos\theta - x}$$

The magnetic field strength normal to the median plane induced by the wall current element follows straightforwardly:

$$dH_n(\theta,\delta) = dJ(\theta,\delta)d\theta \frac{\cos\phi}{2\pi\alpha} = \frac{i_b\delta}{2\pi b} d\theta \frac{\sin 2\phi}{2\pi tg\theta} = \frac{i_b\delta}{2\pi^2 btg\theta} d\theta \frac{tg\phi}{1+tg\phi^2}$$
$$= \frac{i_b\delta}{2\pi^2 btg\theta} d\theta \frac{\sin\theta}{\cos\theta - x + \frac{\sin\theta^2}{(\cos\theta - x)}} = \frac{i_b\delta}{2\pi^2 b} d\theta \frac{\cos\theta(\cos\theta - x)}{1 - 2x\cos\theta + x^2} \quad . \tag{7}$$

The magnetic field strength follows after integration over variable θ:

$$H = \frac{i_b\delta}{2\pi b} \text{ and } B = \frac{\mu i_b \Delta}{2\pi b^2}, \tag{8}$$

independent of x as expected. The total magnetic flux in the pipe aperture is

$$\Phi = 2bB = \frac{\mu}{\pi} i_b \delta = \frac{\mu}{2} i_d = L i_d \ , \tag{9}$$

where L is the inductance that faces the differential current i_d.

The induced magnetic field produces a force in the median plane on the beam current i_b. This force can be expressed in terms of transverse impedance. Indeed the definition of transverse impedance in the median plane as in present case, per unit length is :

$$z_T = \frac{j}{i_b \Delta\beta}(E_T + B \times \beta c), \tag{10}$$

where E_T is the transverse component of the electric field. The magnetic field B is as defined before. The part of the transverse impedance generated by the inductance L is simply :

$$z_{Tm} = -\frac{j}{i_b\Delta}\frac{\mu c}{2\pi b^2} i_b \Delta = -j\frac{Z_0}{2\pi}\frac{1}{b^2}. \tag{11}$$

This is the well known expression for the magnetic *image* transverse impedance.

The electric counterpart can be found as well. The electric field induced on the wall by a charged dipole moment has the same field pattern as the magnetic field, see Eq. 1. Again the electric field generated in the median plane by the $cos\theta$ charge distribution on the wall is constant for the component in the median plane:

$$E_T = \frac{\lambda\delta}{2\pi\varepsilon b}. \tag{12}$$

The electric component of the *image* transverse impedance follows :

$$z_{Te} = \frac{j}{i_b\Delta}\frac{i_b\Delta}{2\pi\varepsilon b^2\beta^2 c} = \frac{j}{\beta^2}\frac{Z_0}{2\pi}\frac{1}{b^2}, \tag{13}$$

and the total *image* transverse impedance per unit length:

$$z_{Ti} = j\frac{Z_0}{2\pi}\frac{1}{b^2}\left(\frac{1}{\beta^2}-1\right) = j\frac{Z_0}{2\pi}\frac{1}{b^2(\beta\gamma)^2}, \qquad (14)$$

and the total *image* transverse impedance of a machine with length $2\pi R$ is :

$$Z_{Ti} = j\frac{Z_0 R}{b^2(\beta\gamma)^2}. \qquad (15)$$

The tune shift caused by this impedance (the image charge effect) is:

$$\Delta Q = -\frac{\langle\beta\rangle i_b}{4\pi(E/e)}\frac{Z_{Ti}}{j\beta} = -\frac{\langle\beta\rangle i_b}{4\pi(E_0/e)}\frac{Z_0 R}{b^2(\beta\gamma)^3} = -\frac{\langle\beta\rangle(\lambda/e)}{\gamma}\frac{Rr_0}{b^2(\beta\gamma)^2}, \qquad (16)$$

where r_0 is the classical particle radius :

$$r_0 = \frac{eZ_0 c}{4\pi(E_0/e)}. \qquad (17)$$

This tune shift is exactly the coherent Laslett image tune shift for a circular pipe with coefficient $\xi = 1/2$ as it should [5].
Based on Eq. 5, the relation between the beam dipole moment and the differential wall current, and on Eq. 9, the evaluation of the *image* inductance, an equivalent circuit can be derived, shown in Figure 3.

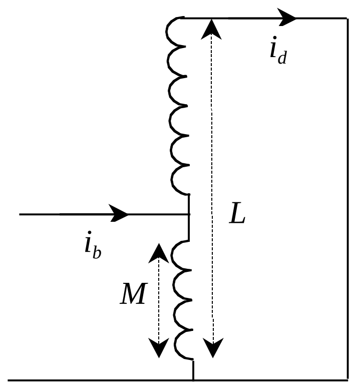

Figure 3: Equivalent circuit for a beam dipole moment in a perfectly conducting pipe.

The definition of the mutual inductance M follows directly from Eq. 5 :

$$M = \frac{2}{\pi}\frac{\Delta}{b}L. \qquad (18)$$

2.2 Coupling of an off-centered beam on itself in a pipe with impedance

A beam dipole moment in a pipe with impedance will again create a differential current i_d. This current will not only face the *image* inductance but also the impedance of the pipe. The equivalent circuit will be modified as shown in Figure 4.

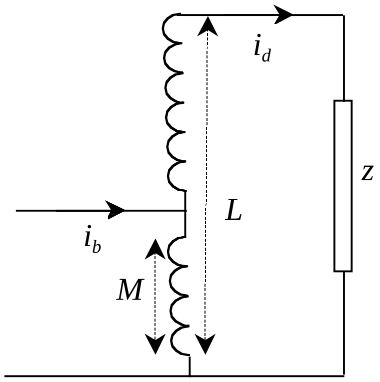

Figure 4 : Equivalent circuit of beam dipole in a pipe with impedance.

The relation between i_b and i_d as given by Eq. 5 is modified by the presence of z. Clearly z depends on the longitudinal impedance per unit length z_l. However, the impedance z is not equal to it since the differential current is not uniformly distributed over the circumference of the pipe. The basic distribution is again $\cos\theta$ such that:

$$J(\theta) = \frac{i_d}{2b}\cos\theta, \tag{19}$$

where i_d is still unknown. The voltage per unit length of pipe is then :

$$V(\theta) = 2Z_s J(\theta), \tag{20}$$

where Z_s is the surface impedance of the pipe. The peak voltage occurs in the plane of dipole moment and is given by:

$$\hat{V} = \frac{i_d}{b} Z_s = i_d 2\pi z_l, \tag{21}$$

while the average voltage is:

$$\langle V \rangle = 2\frac{i_d}{\pi b} Z_s = 4 i_d z_l. \tag{22}$$

Only the deflections in the plane of the dipole for small beam cross-sections are a concern. Therefore it is Eq. 21 that yields an equivalent wall impedance :

$$z = 2\pi z_l. \tag{23}$$

The peak longitudinal electric field in the wall using Eq. 19 is:

$$\hat{E}_l = \hat{J} Z_s = \frac{i_d}{2b} Z_s. \tag{24}$$

The deflecting magnetic field follows from Maxwell's law :

$$B = -\frac{\hat{E}_l}{j\omega b} = \frac{i_d}{2j\omega b^2} Z_s. \tag{25}$$

Plugging this expression in Eq. 10 gives the transverse impedance per unit length:

$$z_T = j\frac{Bc}{i_b\Delta} = \frac{ci_d}{i_b\Delta 2\omega b^2} Z_s = \frac{c}{\omega}\frac{\pi}{b}\frac{i_d}{i_b\Delta} z_l. \tag{26}$$

The relation between the beam current and the deflecting current is found from the equivalent circuit (Figure 4):

$$j\omega M(i_b - i_d) = i_d\left[j\omega(L-M) + 2\pi z_l\right]$$

$$i_d = i_b\frac{2}{\pi}\frac{\Delta}{b}\frac{j\omega L}{j\omega L + 2\pi z_l} = i_b\frac{2}{\pi}\frac{\Delta}{b}\frac{j\omega\frac{L}{2\pi}}{j\omega\frac{L}{2\pi} + z_l}. \tag{27}$$

Combining Eq. 26 and 27 then finally gives:

$$\boxed{z_T = \frac{2c}{\omega b^2}\frac{j\omega\frac{L}{2\pi}z_l}{j\omega\frac{L}{2\pi} + z_l} = \frac{2c}{\omega b^2}\frac{j\frac{\omega\mu}{4\pi}z_l}{j\frac{\omega\mu}{4\pi} + z_l}}. \tag{28}$$

For large ω the expression becomes:

$$\boxed{z_T = \frac{2c}{\omega b^2} z_l}, \tag{29}$$

which is the classical one. Hence for the purpose of computing the transverse impedance it is necessary to replace the standard longitudinal impedance z_l with a parallel circuit of the pipe impedance z_l and the *image* inductance L divided by 2π. Formally it is possible to include the *image* terms in the equivalent circuit by combining Eq. 14 and 29 yielding a *image* inductance of :

$$L_{im} = \frac{L}{2\pi(\beta\gamma)^2} = \frac{\mu}{4\pi(\beta\gamma)^2}. \tag{30}$$

The circuit shown in Figure 5 is not a longitudinal impedance of a beam pipe but it is an equivalent longitudinal impedance to be used in Eq 29 to obtain the correct transverse impedance. The inductances contain the geometric quantities while z_l contains the impedance properties of the wall, complex in general. It should be noted that direct space charge forces have not been considered.

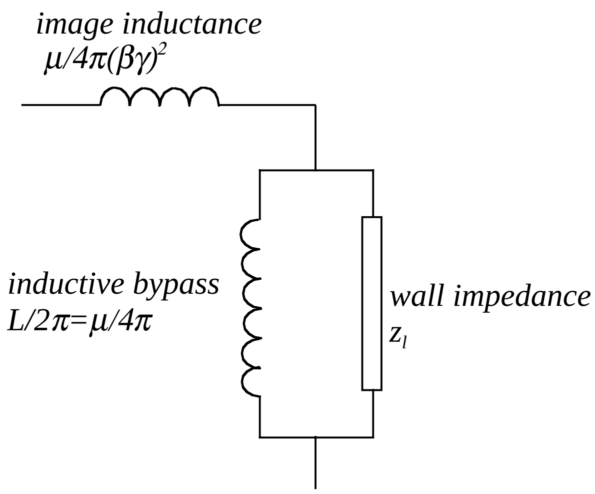

Figure 5 : Equivalent circuit to be used for computing transverse impedance of a pipe of unit length with longitudinal impedance z_l.

3 Discussion

The effect of the image inductance is only significant for low energy machines where $(\beta\gamma)^2$ is a small number. The opposite is true for the inductive bypass. The effect of the inductive by-pass $L/2\pi$ is more important at low frequencies, generally associated with large high energy machines. Its effect is further enhanced by smaller pipe apertures eventually combined with high surface impedance. It is important to note that the by-pass effect is beneficial in the sense that it reduces the effective transverse impedance, both the real and the imaginary part. In the following three examples the frequency environment of the LHC is assumed where the lowest betatron frequency is 8 *kHz*. The wall impedance was choosen to be purely resistive for simplicity. It is of course straightforward to include a more complicated frequency dependence of z_l.

Example 1
Consider a stainless steel pipe with b = 0.03*m*, wall thickness t = 0.001*m* and resistivity $\rho = 1*10^{-6}\Omega m$. The longitudinal impedance reduces to a constant resistance for frequency range below ~200 *kHz* :

$$R_l = \frac{\rho}{2\pi bt} = 0.0053\,\Omega.$$

From Figure 5 and using Eq. 26 yields :

$$f_{co} = \frac{\omega_{co}}{2\pi} = \frac{2R_l}{\mu} = 8400\,Hz$$

$$R_T = \frac{2c}{b^2}\frac{R_l\omega}{\omega^2+\omega_{co}^2}$$

$$X_T = \frac{2c}{b^2}\frac{\frac{\mu}{4\pi}\omega_{co}^2}{\omega^2+\omega_{co}^2} = \frac{Z_0}{2\pi}\frac{1}{b^2}\frac{\omega_{co}^2}{\omega^2+\omega_{co}^2}.$$

The resistive component of the transverse impedance at 8 *kHz* amounts to 33 $k\Omega/m^2$ and the inductive component is 80 $k\Omega/m^2$. A resistive transverse impedance of 70 $k\Omega/m^2$ is found when the inductive bypass is ignored.

Example 2
Consider the LHC beam screen. To simplify assume that it has a circular geometry with b = 0.02 *m* and a resistance per unit length of R_l = 27 $\mu\Omega$. The cut-off frequency is then f_{co} = 43 *Hz*, much smaller than the betatron frequency. Hence the inductive bypass has no influence on the impedance.

References

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