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Evaluating the AdS dual of the critical $O(N)$ vector model

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Abstract: We argue that the AdS dual of the three dimensional critical $O(N)$ vector model can be evaluated using the Legendre transform that relates the generating functionals of the free UV and the interacting IR fixed points of the boundary theory. As an example, we use our proposal to evaluate the minimal bulk action of the scalar field that it is dual to the spin-zero "current" of the $O(N)$ vector model. We find that the cubic bulk self interaction coupling vanishes. We briefly discuss the implications of our results for higher spin theories and comment on the bulk-boundary duality for subleading N.

Keywords: Field Theories in Lower Dimensions, Sigma Models, Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The relation between gauge fields and strings has been significantly enlightened by the AdS/CFT correspondence [1]. The general picture emerging is that the large tension limit of string theory corresponds, holographically, to strongly coupled gauge theories. Nevertheless, one would ideally like to go further and understand the stringy picture of weakly coupled gauge theories. The small tension limit of string theory is an obvious candidate for this picture and a semiclassical description of it would be desirable. Higher spin gauge theories [2] might provide such a semiclassical description [3]. Moreover, the formulation of higher spin theories in AdS spaces [4] opens the possibility for an holographic interpretation of weakly coupled gauge theories. Recent work on higher spin theories includes [5].

Recently, it was suggested that an interesting laboratory for studying the relation between weakly coupled quantum field theories and higher spin theories is provided by the well-known three dimensional critical $O(N)$ vector model. The explicit proposal put forward in [6] is that both the free UV and the interacting IR fixed point of the $O(N)$ vector model are holographically described by the same AdS_4 higher spin theory. A manifestation of such a degeneracy in the holographic description is the fact that the UV and IR generating functionals of the critical $O(N)$ model are related by a Legendre transform for large N. This is one step further from the standard cases of AdS/CFT correspondence where the relation between the weakly and strongly coupled boundary CFTs, even for large N, is in general unknown. The apparent puzzle of having to describe both a free CFT (which does not have anomalous dimensions), as well as an interacting one, (which here has anomalous dimension of $O(1/N)$), by the same AdS theory was recently argued to be resolved by a Higgs mechanism for gauge fields with spin >2 in AdS_4 [7]. This mechanism is at work only when the bulk scalar is quantized with boundary conditions such that is corresponds to an operator of dimension $2+O(1/N)$ in the boundary. On the other hand, it is known [8, 9] that for subleading N the massless degrees of freedom coupled at the UV and IR fixed points of the critical $O(N)$ vector model are different, hence the relation between the corresponding UV and IR generating functionals is less clear.

The lagrangian for the AdS_4 higher spin theory is implicitly known through complicated field equations [2]. This may be reminiscent of the standard situation with the IIB SUGRA that is dual to $\mathcal{N}=4$ SYM, however there is an important difference: in the case at hand one knows explicitly both the weakly as well as the strongly coupled regimes of the boundary field theory. Therefore, one can work from bottom-to-top and evaluate the bulk theory using the knowledge of the boundary CFT. In this work we propose that the evaluation of the bulk AdS_4 theory dual to the critical $O(N)$ vector model can be done by a self-consistency procedure based on the Legendre transform that relates the generating functionals of the free UV and the interacting IR fixed points of the $O(N)$ vector model. To illustrate our idea, we consider here the minimal ansatz for the AdS_4 lagrangian and work out the tree level bulk couplings up to the quartic one. This is done by successively matching the correlation functions produced by the bulk AdS_4 lagrangian to the corresponding ones of the boundary CFT which can be explicitly calculated. Although we do not consider the coupling of the bulk scalar to higher spin fields in AdS_4, our result for the bulk cubic self interaction coupling can be carried over to the higher spin lagrangian. Our calculation of the bulk quartic self interaction coupling may be useful both as its stands i.e. for a possible pure gravity dual of the $O(N)$ vector model or as an intermediate result in future calculations of the higher spin dual of the model.

In section 2 we review the degeneracy in the holographic correspondence for scalar fields in AdS_{D+1} with mass m in the range $-D^2/4 < m^2 < 1-D^2/4$, and its manifestation in terms of the Legendre transform that relates the UV and the IR generating functionals of the boundary theory. In section 3 we apply our proposal to evaluate the bulk action up to the the quartic scalar self interaction term. In section 4 we briefly discuss the implications of our results for higher spin theories and comment on the nature of the bulk-boundary relation for subleading N. The appendix A and B is reserved for a compact presentation of the many technical details.

2. The degeneracy in the holographic description and the Legendre transform

It was noticed already in the early days of AdS/CFT that there is a potential ambiguity in the holographic description of a boundary theory [10]. Let $\phi(r,x)$ be a bulk scalar[1] with

[1] We use throughout the Euclidean version of the Poincaré patch of AdS_{D+1} with $ds^2 = (dr^2 + dx^2)/r^2$ where $x^i = (x^1, \ldots, x^D)$ and we set the AdS radius to 1 such that the cosmological constant $\Lambda = -D(D-1)/2$.

mass m. Its asymptotic behavior near the boundary of AdS_{D+1} is

$$\phi(r,x)|_{r\to 0} \approx r^{\Delta_-}\phi_0(x) + r^{\Delta_+}A(x)\,, \qquad \Delta_\pm = \frac{D}{2} \pm \nu\,, \qquad \nu = \frac{1}{2}\sqrt{D^2+4m^2} \geq 0\,. \tag{2.1}$$

The functions $\phi_0(x)$ and $A(x)$ are the two necessary boundary data to determine the solution of the second-order bulk equation of motion for $\phi(r,x)$. Quantizing then $\phi(r,x)$ with boundary condition $A(x)=0$ ($\phi_0(x)=0$) would give the generating functional of the boundary operator $\mathcal{O}(x)$ with dimension Δ_+ ($\tilde{\mathcal{O}}(x)$ with dimension Δ_-). The above ambiguity does not show up in most of the studied cases of AdS/CFT where the operator $\tilde{\mathcal{O}}(x)$ has dimension below the unitarity bound i.e. $\Delta_- < D/2-1$.

Nevertheless, there exist important cases where both $\Delta_\pm$ are above the unitarity bound. Then, the quantization ambiguity is present even when the asymptotic behavior (2.1) of $\phi(r,x)$ is determined by one arbitrary boundary data as when one requires that the bulk solution vanishes in the far interior ($r\to\infty$) of AdS. In such a case the two functions appearing in (2.1) are related by

$$A(x) = C_{\Delta_+}\int \mathrm{d}^D y \frac{1}{(x-y)^{2\Delta_+}}\phi_0(y)\,, \qquad C_{\Delta_+} = \frac{\Gamma(\Delta_+)}{\pi^{D/2}\Gamma(\nu)}\,. \tag{2.2}$$

Then, the application of AdS/CFT correspondence yields *either* a functional $W[\phi_0]$ of $\phi_0(x)$ *or* a functional $J[A]$ of $A(x)$. The first generates correlation functions of $\mathcal{O}(x)$ and the second of $\tilde{\mathcal{O}(x)}$. However, due to (2.2) the two functionals are not independent but one is the Legendre transform of the other as [10, 11]

$$W[\phi_0] + 2\nu\int \mathrm{d}^D x\,\phi_0(x)A(x) = J[A]\,, \qquad \frac{\delta W[\phi_0]}{\delta\phi_0(x)} = -2\nu A(x)\,. \tag{2.3}$$

An interesting observation regarding the relation between $W[\phi_0]$ and $J[A]$ was made in [12] and was further elaborated in [13, 14]. The interchange between the boundary conditions $\phi_0(x)=0$ and $A(x)=0$ is induced by a "double-trace" deformation.[2] One way to see this is to first choose the boundary condition $\phi_0(x)=0$, which would yield the theory for the operator $\tilde{\mathcal{O}}(x)$ with dimension Δ_-, and then perturb this theory by $\frac{f}{2}\tilde{\mathcal{O}}^2(x)$. Noting then that $D/2 > \Delta_- > D/2-1$, this perturbation is relevant and for $f\to\infty$ it leads to a possible IR fixed point of the $\tilde{\mathcal{O}}(x)$ theory, and at the same time to the boundary condition $A(x)=0$. Therefore, $J[A]$ may be viewed as the generating functional of the UV fixed point CFT while $W[\phi_0]$ as the one for the IR fixed point CFT, the two being connected by an RG flow.

An application of the above phenomenon can be found in the recently discussed case of the AdS dual of the critical three-dimensional $O(N)$ vector model. It has been suggested in [6] that this well-known three-dimensional CFT has a dual in AdS_4 which might be a higher spin theory. For example, one would expect that there exists an action for a massive scalar on AdS_4 that yields the generating functional for the spin-zero "current"[3] of the free UV

[2]This fact was also implicit in the OPE analysis of multi-trace deformations in [15].

[3]This name is used for the operator proportional to $\phi^a(x)\phi^a(x)$ where $\phi^a(x)$, $a=1,2,\ldots,N$ are the elementary fields.

$O(N)$ CFT. On the other hand, the generating functional of the interacting IR fixed point of the model gives the correlation functions for a composite operator of dimension $2+O(1/N)$. Then, by the arguments above, these two generating functional should be related by a Legendre transform as in (2.3). However, the IR generating functional is the one directly obtained from the bulk AdS_4 lagrangian by the standard AdS/CFT correspondence and involves the parameters of the bulk lagrangian. Moreover, the knowledge of higher-pt correlation functions in the IR CFT gives information about anomalous dimensions of various fields in the theory which can be directly translated to information about corrections to masses of the bulk fields. In the next section we describe explicitly the use of the Legendre transform as a self-consistency condition to evaluate the parameters of the AdS_4 lagrangian dual to the spin zero "current" of the critical $O(N)$ model up to quartic order.

3. Evaluation of the AdS dual of the critical $O(N)$ vector model

3.1 The cubic bulk coupling

The proposal of [6] for the AdS dual of the spin zero "current" of the critical $O(N)$ vector model is to consider a conformally coupled scalar on AdS_4. In this Subsection we perform the calculations for general D and keep in mind that at the end we want to set $D = 3$. The minimal gravity action that could reproduce the correlation functions of the spin-zero "current" of the critical $O(N)$ vector model is

$$S_{D+1} = \frac{1}{2\kappa_{D+1}^2} \int \mathrm{d}^{D+1}x\sqrt{g}\left[-R + 2\Lambda + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{D^2-1}{8}\phi^2 + \frac{g_3}{3!}\phi^3 + \frac{g_4}{4!}\phi^4 + \cdots\right]. \tag{3.1}$$

The overall normalization of the action can be fixed by requiring that the coefficient C_T of the energy momentum 2-pt function following from (3.1) coincides with the one of the $O(N)$ vector model. The latter is completely determined by the overall normalization of the $O(N)$ vector model action and does not depend on the normalization of the scalar fields. Following the general treatment of the $O(N)$ vector model in [8] we have

$$C_T^{O(N)} = N\frac{D}{(d-1)S_D^2}\,, \qquad S_D = \frac{2\pi^{D/2}}{\Gamma\left(D/2\right)}\,. \tag{3.2}$$

Then, from (3.2) and using the general result of [16] for C_T we obtain to leading order in $O(1/N)$

$$\frac{1}{2\kappa_{D+1}^2} = N\frac{\pi^D}{(D+1)\Gamma(D)}\frac{1}{S_D^3}\,, \qquad \frac{1}{2\kappa_4^2} = \frac{N}{2^9}\,. \tag{3.3}$$

Next we concentrate on correlation function of the operators dual to $\phi(r,x)$. We first use the "regular" boundary data $\phi_0(x)$, we set $\Delta \equiv \Delta_+$ and perform for simplicity the rescaling $\phi_0 \to (2\kappa_{D+1}^2)^{1/2}\phi_0$ to obtain the correlation functions of normalized [4] operators

$$W[\phi_0] = \sum_{n=2}^{\infty}\frac{1}{n!}\int \mathrm{d}^Dx_1\cdots\mathrm{d}^Dx_n\phi_0(x_1)\cdots\phi_0(x_n)\Pi_n(x_1,\ldots,x_n)\,. \tag{3.4}$$

[4] Operators whose 2-pt function is of order $O(1)$.

Up to 4-pt functions, the correlation functions that appear in (3.4) are given explicitly in (A.4)–(A.6) of appendix A. The Legendre transform (2.3) of (3.4) may be written as

$$J[A] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n A(x_1) \cdots A(x_n) P_n(x_1, \ldots, x_n)\,. \tag{3.5}$$

Up to 4-pt functions, the P-functions in (3.5) are related to the Π-functions in (3.4) as shown in (A.11)–(A.13) of appendix A.

Now the requirement that the P-correlation functions in (3.5) are correlation functions of the free UV $O(N)$ vector model comes into play. This provides the necessary dynamical principle for the evaluation of the P-correlation functions. At this point we need to make an assumption for the normalization of the 2-pt functions of the elementary N-component scalars of the model. We can choose to represent them as unit normalized 2-pt functions of free massless scalars in d dimensions, i.e.

$$\langle \phi^a(x_1) \phi^b(x_2) \rangle = \frac{\delta^{ab}}{(x_{12}^2)^{\frac{d}{2}-1}}\,, \qquad a = 1, 2, \ldots, N\,. \tag{3.6}$$

Notice that if $\tilde{\mathcal{O}}(x)$, whose correlation functions are given in (3.5), is required to be proportional to $\phi^2(x)$, the two parameters d and D are related as

$$D - 3 = 2(d-3)\,. \tag{3.7}$$

We are going to set $D = d = 3$ at the end, but one may view (3.7) as the relation between two different regularization methods based on the analytic continuation in the number of spacetime dimensions around 3. Using (A.4) and (A.11), the relation between $\tilde{\mathcal{O}}(x)$ and $\phi^2(x)$ can be found to be

$$\tilde{\mathcal{O}}(x) \equiv \frac{k}{\sqrt{N}} \phi^a(x) \phi^a(x)\,, \qquad k^2 = \frac{\Gamma\left(\frac{D-1}{2}\right)}{4\pi^{\frac{D+1}{2}}}\,. \tag{3.8}$$

Next, using (3.8) and (3.6) we obtain

$$P_3(x_1, x_3, x_4) \equiv \frac{8k^3}{\sqrt{N}} \frac{1}{(x_{12}^2 x_{13}^2 x_{23}^2)^{\frac{D-1}{4}}}\,. \tag{3.9}$$

Finally, from the relation

$$\Pi_3(x_1, x_2, x_3) = [P_3(x_1, x_2, x_3)]^{amp.}\,, \tag{3.10}$$

and using the D'EPP formula (A.14) to amputate by $[P_2]^{-1}$ the free 3pt function (3.9) we obtain after some algebra

$$g_3^2 = \frac{1}{N} \frac{1}{2\kappa_{D+1}^2} \frac{2^4 (D-3)^2 \pi^{\frac{D+3}{2}} \left[\Gamma\left(\frac{D-1}{2}\right)\right]^3}{\left[\Gamma\left(\frac{D-1}{4}\right)\right]^6 \left[\Gamma\left(\frac{D+3}{4}\right)\right]^2}\,. \tag{3.11}$$

The result (3.11) is consistent with the results of [8] where is was found that the 3-pt function of the scalar field with dimension $2 + O(1/N)$ vanishes at the interacting fixed point of the three dimensional $O(N)$ vector model. Moreover, (3.11) shows that for $D = 3$ the cubic coupling in the AdS action (3.1) vanishes. This result is independent of whether or not the bulk lagrangian (3.1) contains higher spins, hence we conclude that the cubic self interaction scalar coupling of the higher spin AdS$_4$ theory vanishes.

3.2 The quartic bulk coupling

To evaluate the quartic bulk coupling using the Legendre transform it is simpler to set $D = d = 3$. From (3.6) and (3.8) we obtain

$$P_4(x_1, x_2, x_3, x_4) \equiv \frac{16k^4}{N}\left[\frac{1}{(x_{12}^2 x_{24}^2 x_{43}^2 x_{31}^2)} + \text{crossed}\right]. \tag{3.12}$$

Then, the calculation we need to perform is

$$\begin{aligned}\Pi_4(x_1, x_2, x_3, x_4) =& \left[P_4(x_1, x_2, x_3, x_4)\right]^{\text{amp.}} + \\ &+\left\{\int \mathrm{d}^3x \mathrm{d}^3y \left[P_3(x_1, x_3, x)\right]^{\text{amp.}} \Pi_2(x, y) \left[P_3(y, x_3, x_4)\right]^{\text{amp.}} + \text{crossed}\right\},\end{aligned} \tag{3.13}$$

where the amputation is done with $[P_2]^{-1}$. Evaluating the integrals on the rhs of (3.13) and matching the results with the explicit expression for Π_4 given in (A.6), would give the value of the quartic coupling g_4.

Let us start from the rhs of (3.13). The integrals have been calculated in [17], for general dimension D, in terms of the invariant ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2}, \tag{3.14}$$

(see of [17, appendix C]), although the form of the results does not appear to be easily manageable. Nonetheless, our purpose here is to find g_4 and for that we only need the leading term in the short distance expansion of the rhs of (3.13) as $u, v \to 0$. In practice, we can simplify things further by considering the expansion of the integrals in terms of the variables v and $Y = 1 - v/u$ and consider the leading term in v for $Y = 0$.[5]

From the results in [17, appendix C] one can see that the rhs of (3.13) has an expansion of the form

$$[\text{RHS of (3.13)}] \propto \frac{1}{(x_{12}^2 x_{34}^2)^2}\left(v^2[-A \ln v + B] + \cdots\right), \tag{3.15}$$

where the dots stand for subleading terms. Now, the important point is that the coefficient A of the $\ln v$ term exactly vanishes. Therefore, we should not find a leading logarithmic term also in the lhs of (3.13). This condition determines g_4.

Before turning to the evaluation of the AdS integrals in Π_4, we comment on the vanishing of its leading logarithmic term. In order to obtain the full 4-pt function of the operator $\mathcal{O}(x)$ one should add to Π_4 the disconnected part. Once this is done, the OPE analysis of the 4-pt function can be performed. Notice now that the vanishing of the 3-pt function Π_3 implies that the field $\mathcal{O}(x)$ itself does not appear in the 4-pt function $\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_4)\rangle$. The next scalar that contributes to the OPE of this 4-pt function is

[5]This is inspired from OPE studies of conformal 4-pt functions where the leading term in v with $Y = 0$ corresponds to the leading contribution of conformal scalars [18].

a scalar field with dimension $\tilde{\Delta} = 4 + \tilde{\eta}$ where $\tilde{\eta} = O(1/N)$. Then, the 4-pt function is expected to have the form

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \sim \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta}}[1 + v^{\tilde{\Delta}/2} + \cdots]\,. \tag{3.16}$$

The vanishing of $\ln v$ term in (3.15) implies the following relation between the anomalous dimension η of $\mathcal{O}(x)$ and $\tilde{\eta}$

$$\frac{1}{2}\tilde{\eta} - \eta = 0\,. \tag{3.17}$$

Given the known values [17] $\eta = -2^5/3N\pi^2$ and $\tilde{\eta} = -2^6/3N\pi^2$ we see that (3.17) is satisfied.

Now turn to the calculation of the leading logarithm in the lhs of (3.13). The contribution to this from the AdS-star graph can be easily found using the general result (B.3) in appendix B

$$\Pi_4^{\rm star}\big|_{\rm lead.log} = \frac{g_4}{\pi^6}\frac{2^9}{N}\frac{1}{(x_{12}^2 x_{34}^2)^2}v^2\left[-\frac{1}{6}\ln v + \cdots\right]. \tag{3.18}$$

The contribution form the graviton exchanges is more complicated to evaluate, since the graphs do not reduce into finite sums of conformal integrals as in the four dimensional case. The direct channel graviton exchange has been computed in [20] for general dimensions, but here we also need the crossed channels. After some tedious algebra whose essentials are presented in appendix B the final result is

$$\Pi_4^{\rm grav}\big|_{\rm lead.log} = \frac{1}{2\pi^6}\frac{2^9}{N}\frac{1}{(x_{12}^2 x_{34}^2)^2}v^2[-\ln v]\,. \tag{3.19}$$

Requiring that $\left[\Pi_4^{\rm star} + \Pi_4^{\rm grav}\right]_{\rm lead.log} = 0$ we finally obtain

$$g_4 = -3\,. \tag{3.20}$$

4. Discussion

The critical three dimensional $O(N)$ vector model appears to be a very interesting laboratory for the study of the AdS/CFT correspondence. In contrast to most other cases of AdS/CFT, here it is the boundary CFT side that is well understood at strong coupling. This means that the correlation functions derived from AdS/CFT coincide with the well-known correlation functions of the interacting IR fixed point of the $O(N)$ vector model. Moreover, Legendre transforming the generating functional of the IR fixed point one gets, to leading order in the $1/N$ expansion, the generating functional for the free UV fixed point of the $O(N)$ vector model. Notice that the assumption that the UV and IR generating functionals are related via a Legendre transform is important dynamical information, in particular for the IR fixed point. In the present paper we initiate the evaluation of the AdS dual of the critical $O(N)$ vector model making use of its connection with the IR fixed point of the three dimensional CFT. Assuming a minimal form for the bulk action i.e. without higher spin or derivative couplings, we evaluate the cubic and quartic self interaction couplings of the bulk scalar that is dual to the spin-zero "current" of the $O(N)$ model.

The AdS dual of the $O(N)$ vector model is believed to correspond to a higher spin theory. Therefore it should be possible to check our results for the cubic couplings within the context of the equations of motion of the minimal bosonic higher spin theory $hs(4)$. In particular, the vanishing of the cubic coupling was conjectured in [8] to indicate a possible underlying discrete symmetry for the operator $\mathcal{O}(x)$. In the context of the higher spin theory this symmetry may be a manifestation that the dual operator of $\mathcal{O}(x)$ might actually be a fermion bilinear.[6]

Our result (3.20) for the quartic bulk self interaction coupling may not be directly applicable to finding the higher spin lagrangian Nevertheless, it is an intermediate result in this direction. For the full result one would have to take into account the couplings of the bulk scalar with higher spin fields. Since these couplings are believed to be fixed [19], by finding the leading logarithm of the higher spin exchange graphs in AdS_4 one should be able to unambiguously fix the quartic scalar self interaction coupling. We expect such a calculation to be complicated but straightforward.

Another interesting class of questions that one can ask is the extension of the bulk-boundary duality to higher orders in $1/N$. At the field theory side, there exist a number of results for the $O(1/N)$ corrections to anomalous dimensions. These results should somehow be reproduced by the bulk theory and this raises the intriguing possibility that we are dealing here with a quantum gravity theory in AdS_4 that yields sensible results. Another important quantity that has been calculated is the $1/N$ correction to C_T in (3.3) which was found to decrease as one goes from the UV to the IR fixed points of the $O(N)$ vector model [8]. Hence it appears to be a natural extension of the C-function to odd dimensions and is a measure of the degrees of freedom at the fixed point. Moreover, on the basis of the results in [8], it was argued in [9] that the interacting IR fixed point of the $O(N)$ vector model describes the symmetry breaking pattern $O(N) \to O(N-1)$. For this reason, if the degrees of freedom coupled to the UV free fixed point are N, the massless degrees of freedom coupled to the interacting fixed point are $N-1$ [9]. This raises a puzzle regarding the relation between the free UV and interacting IR fixed points of the $O(N)$ vector model for subleading N. Finally, it is also intriguing that the leading-N free energies at the free UV and interacting IR fixed point of the $O(N)$ vector model are different and related by a rational factor $4/5$ [21, 22]. This indicates that a holographic thermodynamical study of the model may hide interesting surprises. We hope to return to some of these issues in the near future.

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A. Basic AdS/CFT formulas

The scalar bulk-to-bulk and bulk-to-boundary propagators of a scalar corresponding to an

[6] I thank P. Sundell for discussions on this point.

operator with dimension $\Delta = D/2 + 1/2$ that we use are respectively

$$G(x,y) = c_\Delta \xi^{-\Delta} {}_2F_1\left(\frac{\Delta}{2}+\frac{1}{2}, \frac{\Delta}{2}; \Delta - 1; \xi^{-2}\right), \tag{A.1}$$

$$\xi^2 = \frac{r^2 + r'^2 + (x-y)^2}{2rr'}, \qquad c_\Delta = \frac{\Gamma(\Delta)}{2^\Delta \pi^\Delta}, \tag{A.2}$$

$$\hat{K}(r'; y, x) = C_\Delta \left[\frac{r'}{r'^2 + (y-x)^2}\right]^\Delta, \qquad C_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{D+1}{2}}}. \tag{A.3}$$

With the above, the explicit expressions for the Π-functions in (3.4) are

$$\Pi_2(x_1,x_2) = C_\Delta \frac{1}{x_{12}^{2\Delta}}, \tag{A.4}$$

$$\Pi_3(x_1,x_2,x_3) = -\frac{g_3}{2\pi^D}\sqrt{\frac{2^9}{N}} \frac{[\Gamma(\Delta/2)]^3\, \Gamma(3\Delta/2 - D/2)}{[\Gamma(1/2)]^3} \frac{1}{(x_{12}^2 x_{13}^2 x_{23}^2)^{\Delta/2}}, \tag{A.5}$$

$$\begin{aligned}\Pi_4(x_1,x_2,x_3,x_4) = & -g_4 C_\Delta^4 \frac{2^9}{N} \int_0^\infty \frac{\mathrm{d}r}{r^{D+1}} \int \mathrm{d}^D x \hat{K}(r;x,x_1)\hat{K}(r;x,x_2)\hat{K}(r;x,x_3)\hat{K}(r;x,x_4) - \\ & - \left\{ g_3^2 C_\Delta^4 \frac{2^9}{N} \int_0^\infty \frac{\mathrm{d}r \mathrm{d}r'}{(rr')^{D+1}} \int \mathrm{d}^D x \mathrm{d}^D y \Big[\hat{K}(r;x,x_1)\hat{K}(r;x,x_2) G(x,y) \times \right. \\ & \qquad \times \hat{K}(r';y,x_3)\hat{K}(r';y,x_4)\Big] + \\ & \left. + (x_2 \leftrightarrow x_3) + (x_2 \leftrightarrow x_4) \right\} + C_\Delta^4 \frac{2^9}{N}\left[\frac{1}{4} I_{\mathrm{grav}}^s + \frac{1}{4} I_{\mathrm{grav}}^t + \frac{1}{4} I_{\mathrm{grav}}^u\right].\end{aligned} \tag{A.6}$$

The s-channel graviton exchange amplitude can be read from the results of [23] in the general form given by [20]. Specializing to $D = 3$ and $\Delta = 2$ we have

$$\begin{aligned} I_{\mathrm{grav}}^s = & \frac{1}{(x_{12}^2 x_{13}^2 x_{14}^2)^2} \int \frac{\mathrm{d}^3 w \mathrm{d} w_0}{w_0^4} f(t') \times \\ & \times \left\{ 6 \left[\frac{w_0}{w_0^2 + (w - x'_{13})^2}\right]^2 \left[\frac{w_0}{w_0^2 + (w - x'_{14})^2}\right]^2 - \right. \\ & - 16\, w_0 \left(\left[\frac{w_0}{w_0^2 + (w - x'_{13})^2}\right]^3 \left[\frac{w_0}{w_0^2 + (w - x'_{14})^2}\right]^2 + \right. \\ & \qquad \left. + \left[\frac{w_0}{w_0^2 + (w - x'_{13})^2}\right]^2 \left[\frac{w_0}{w_0^2 + (w - x'_{14})^2}\right]^3 \right) \\ & \left. + 32\, w_0^2 \left[\frac{w_0}{w_0^2 + (w - x'_{13})^2}\right]^3 \left[\frac{w_0}{w_0^2 + (w - x'_{14})^2}\right]^3 \right\}, \end{aligned} \tag{A.7}$$

where

$$f(t') = -\frac{\pi}{2}\left[\frac{w_0^2}{(w - x'_{12})^2}\right]^{1/2} + \frac{2}{3} t'^2 {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; t'\right), \tag{A.8}$$

$$t' = \frac{w_0^2}{w_0^2 + (w - x'_{12})^2}, \qquad x'^i = \frac{x^i}{x^2}. \tag{A.9}$$

In (A.7), let us for simplicity denote the integrals that involve the first term in (A.8) with $\mathcal{I}^s$ and the ones that involve the hypergeometric function by I^s. Then, in an obvious notation we write

$$I^s_{\rm grav} = \frac{1}{(x^2_{12}x^2_{13}x^2_{14})^2}\left[6(I^s_1+\mathcal{I}^s_1) - 16(I^s_2+\mathcal{I}^s_2) + 32(I^s_3+\mathcal{I}^s_3)\right]. \tag{A.10}$$

The t and u-channels are obtained from (A.7) by the interchanges $x'_2 \leftrightarrow x'_3$ and $x'_2 \leftrightarrow x'_4$ respectively. A graphical representation of the 4-pt function is shown in Fig.1. The correlation functions in $W[\phi_0]$ and $J[A]$ are related as

$$\begin{aligned}
\Pi_1(x_1,x_2) &= -\left[P_2(x_1,x_2)\right]^{-1}, &\text{(A.11)}\\
\Pi_3(x_1,x_2,x_3) &= \left[P_3(x_1,x_2,x_3)\right]^{\rm amp.}, &\text{(A.12)}\\
\Pi_4(x_1,x_2,x_3,x_4) &= \left[P_4(x_1,x_2,x_3,x_4)\right]^{\rm amp.} - \\
&\quad -\left\{\int {\rm d}^3x{\rm d}^3y\,\left[P_3(x_1,x_2,x)\right]^{\rm amp.} P_2(x,y)\left[P_3(x_3,x_4,x)\right]^{\rm amp.} + \right.\\
&\qquad\left. + \,\text{crossed}\right\}. &\text{(A.13)}
\end{aligned}$$

The amputation is done with $[P_2(x_1,x_2)]^{-1}$ and with the help of the D'EPP formula

$$\begin{aligned}
\int {\rm d}^Dx \frac{1}{(x_1-x)^{2a_1}(x_2-x)^{2a_2}(x_3-x)^{2a_3}} &= \frac{U(a_1,a_2,a_2)}{(x^2_{12})^{D/2-a_3}(x^2_{13})^{D/2-a_2}(x^2_{23})^{D/2-a_1}}, \\
U(a_1,a_2,a_3) &= \pi^{D/2}\frac{\Gamma\left(D/2-a_1\right)\Gamma\left(D/2-a_2\right)\Gamma\left(D/2-a_3\right)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)},
\end{aligned} \tag{A.14}$$

which is valid for $a_1+a_2+a_3 = D$. To obtain the inverse 2-pt function of a scalar field we use the formula

$$\left[\frac{1}{x^{2A}}\right]^{-1} = \frac{1}{\pi^D}\frac{\Gamma\left(D-A\right)\Gamma(A)}{\Gamma\left(A-D/2\right)\Gamma\left(D/2-A\right)}\frac{1}{(x^2)^{D-A}}. \tag{A.15}$$

B. Leading logarithmic singularities of AdS integrals

Recall the definition of the cross ratios involved in the calculation of conformal 4-pt functions

$$u = \frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}, \qquad v = \frac{x^2_{12}x^2_{34}}{x^2_{14}x^2_{23}}, \qquad Y = 1-\frac{v}{u}. \tag{B.1}$$

Then, the general conformal 4-pt function can be expanded in the variables v and Y which makes it easier to read the contributions due to various conformal tensors. For example the leading (in the limit $x^2_{12}\,,\, x^2_{34} \to 0$), contribution due to a tensor of dimension Δ and rank k is of the form [18]

$$v^{\frac{\Delta-k}{2}}Y^k. \tag{B.2}$$

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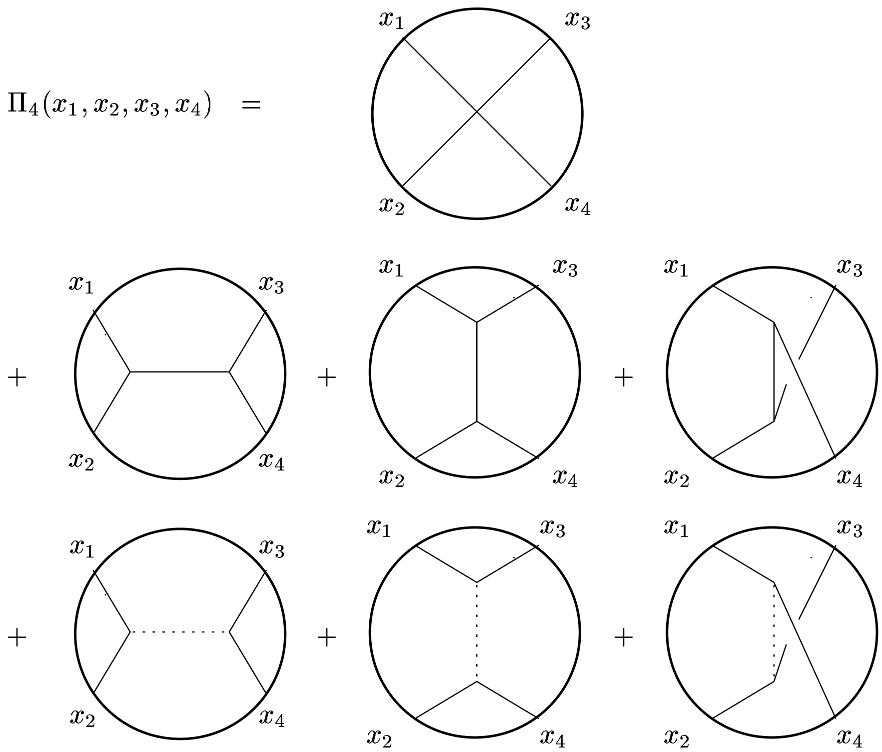

Figure 1: Graphical representation of $\Pi_4(x_1, x_2, x_3, x_4)$. The solid lines correspond to the scalar and the dotted lines to the graviton.

Using standard techniques for the calculation of AdS graphs we can easily evaluate the first term on the rhs of (A.6) that corresponds to the AdS-star graph in figure 1. We give for completeness the result

$$\frac{g_4}{\pi^6}\frac{2^9}{N}\frac{1}{(x_{12}^2 x_{34}^2)^2}v^2\left\{\sum_{n,m=0}^{\infty}\frac{v^n Y^m}{n!m!}\frac{\Gamma^2(2+n)\Gamma^2(2+2n+m)}{\Gamma(1+n)\Gamma(4+n+m)}\times\right.$$
$$\times\Bigg[-\ln v+2\psi(4+2n+m)+2\psi(1+n)-$$
$$\left.-2\psi(2+n)-2\psi(2+n+m)\Bigg]\right\}. \tag{B.3}$$

To calculate the leading logarithm of the graviton exchange graph we start with the integrals in (A.7) that come from the hypergeometric function in (A.8). For clarity we present here the integral in the first line of (A.7). Using the following representation [24]

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}}\mathrm{d}s\Gamma(-s)\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}(-z)^s\,, \tag{B.4}$$

with an appropriately chosen contour $\mathcal{C}$ parallel to the imaginary axis and a standard

Feynman parametrization we obtain

$$I_1^s = \frac{\pi^2}{4}\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \mathrm{d}s\Gamma(-s)\Gamma(1+s)\tilde{I}_1^s\,, \tag{B.5}$$

$$\tilde{I}_1^s = \int_0^\infty \mathrm{d}t_1 \ldots \mathrm{d}t_3\, t_1^{1+s}\, t_2 t_3 \left(\sum t\right)^{-4-s} \exp\left[\frac{-1}{\sum t}\left[t_1 t_2 A_1 + t_1 + t + 3A_2 + t_2 t_3 A_3\right]\right], \tag{B.6}$$

$$\sum t = t_1 + t_2 + t_3\,, \qquad A_1 = \frac{x_{23}^2}{x_{12}^2 x_{13}^2} \qquad A_2 = \frac{x_{24}^2}{x_{12}^2 x_{14}^2} \qquad A_1 = \frac{x_{34}^2}{x_{13}^2 x_{14}^2}\,. \tag{B.7}$$

Next we set in (B.6)

$$t_3 = \alpha_1\alpha_2\alpha_3\,, \qquad t_2 = \alpha_1\alpha_2(1-\alpha_3)\,, \qquad t_3 = \alpha_1(1-\alpha_2) \qquad 0 \le \alpha_1 < \infty\,, 0 \le \alpha_2, \alpha_3 \le 1\,, \tag{B.8}$$

and do successively the α_1 and α_2 integrations with result

$$\tilde{I}_1^s = A_1^{-2} B\left(2, 2+s\right) \int_0^1 \mathrm{d}\alpha_3 (1-\alpha_3) \left[1-\alpha_3 Y\right]^{-2} {}_2F_1\left(2, 2; 4+s; 1 - \frac{\alpha_3(1-\alpha_3)v}{1-\alpha_3 Y}\right). \tag{B.9}$$

To obtain the leading logarithm now is suffices to set $Y = 0$ in (B.9). Then, we may use the following representation for the hypergeometric function [24]

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}'} \mathrm{d}t\Gamma(-t)\Gamma(c-a-b-t)\Gamma(a+t)\Gamma(b+t)(1-z)^t = \\ = \Gamma(c-a)\Gamma(c-b)\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(a, b; c; z)\,, \tag{B.10}$$

where $\mathcal{C}'$ run parallel to the imaginary axis, and we do the α_3 integration to end up with a double Mellin-Barnes integral over t and s. The t integration is straightforward while there are double poles in the s integration. These are handled with the help of the general formula [18]

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \mathrm{d}s\Gamma^2(-s)g(s)v^s = \sum_{n=0}^\infty \frac{v^n}{(n!)^2}\left[2\psi(1+n)g(n) - g(n)\ln v - \frac{\mathrm{d}}{\mathrm{d}\xi}[g(\xi)]_{\xi=n}\right]. \tag{B.11}$$

Keeping only the leading term in v we obtain

$$I_1^s|_{\text{lead.log}} = \left(\frac{x_{12}^2 x_{13}^2}{x_{23}^2}\right)^2 [-\ln v]\frac{\pi^2}{24}\,. \tag{B.12}$$

Following the same procedure we can find the leading logarithmic terms in the direct channel as

$$I_2^s|_{\text{lead.log}} = \left(\frac{x_{12}^2 x_{13}^2}{x_{23}^2}\right)^2 [-\ln v]\frac{5\pi^2}{96}\,, \tag{B.13}$$

$$I_2^s|_{\text{lead.log}} = \left(\frac{x_{12}^2 x_{13}^2}{x_{23}^2}\right)^2 [-\ln v]\frac{7\pi^2}{384}\,. \tag{B.14}$$

It is also easy to see, either by direct calculation or from the results of [17] that the $\mathcal{I}^s$ integrals do not have any logarithmic terms. Then, from (B.12)–(B.14) and (A.10) we see that the leading logarithmic contribution in the direct channel vanishes.

In general, the calculation of the crossed t and u channels is considerably more complicated, but the extraction of the leading logarithms can be done relatively easy on the lines sketched above. For the I^t and I^u integrals we find that their leading logarithms are exactly the same as the ones of the corresponding I^s integrals. Therefore, their contribution to the leading logarithm of the graviton exchange graph vanishes. Hence, the only possible logarithms can come from the crossed channel integrals $\mathcal{I}^t$ and $\mathcal{I}^t$. Our calculation, done on the lines described above, yield for the terms that give a non vanishing contribution

$$\mathcal{I}_1^t\big|_{\text{lead.log}} = \mathcal{I}_1^u\big|_{\text{lead.log}} = -\left(\frac{x_{12}^2 x_{13}^2}{x_{23}^2}\right)^2 [-\ln v]\frac{\pi^2}{2}\,, \tag{B.15}$$

$$\mathcal{I}_2^t\big|_{\text{lead.log}} = \mathcal{I}_2^u\big|_{\text{lead.log}} = -\left(\frac{x_{12}^2 x_{13}^2}{x_{23}^2}\right)^2 [-\ln v]\frac{\pi^2}{4}\,. \tag{B.16}$$

Using these result we obtain (3.19) in the main text.

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