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Saturation of optimality limits in hadron-hadron scatterings.

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Abstract

In this paper the optimal unitarity lower bound on logarithmic slope of the diffraction peak is investigated. It is shown that the unitarity lower bound is just the *optimal logarithmic slopes* predicted by the *principle of least distance in space of states.* A systematic tendency towards the saturation of *the most forward-peaked limit* is observed from all the available experimental data of all principal hadron-hadron (e.g., PP, $\overline{P}P$, $K^{\pm}P$, $\pi^{\pm}P$) scatterings practically at all laboratory momenta.

1. Introduction

Recently, in Ref. [1], by using *reproducing kernel Hilbert space* (RKHS} methods [2-4], we described the quantum scattering of the spinless particles by a *principle of minimum distance in the space of the scattering states (PMD-SS).* Some preliminary experimental tests of the *PMD-SS, even* in the crude form [1] when the complications due to the particle spins are neglected, showed that the actual experimental data for the differential cross sections of all PP, $\overline{P}P$, $K^{\pm}P$, $\pi^{\pm}P$, scatterings at all energies higher than 2 GeV, can be well systematized by *PMD-SS* predictions. Moreover, connections between the *optimal states [1], the PMD-SS in the space of quantum states and* the *maximum entropy principle for the statistics of the scattering channels* was also recently established by introducing *quantum scattering entropies* [5-7]. Then, it was shown that the experimental pion-nucleon as well as pion-nucleus scattering entropies are well described by optimal entropies and that the experimental data are consistent with the *principle of minimum distance in the space of scattering states (PMD-SS)* [1].

An important model independent result obtained Ref. [1], via the description of quantum scattering by the *principle of minimum distance in space of states (PMD-SS)*, is the following *optimal lower bound on logarithmic slope of the forward diffraction peak* in hadron-hadron elastic scattering:

$$b \equiv \frac{d}{dt} \ln \left[\frac{d\sigma}{d\Omega}(s,t) \right]_{|t=0} \geq b_o \equiv \frac{\bar{\lambda}^2}{4} \left[\frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(1) - 1 \right] \tag{1}$$

where$\sqrt{s}$ and $\sqrt{|t|}$ are c.m. energy and transfer momentum, respectively, while $\bar{\lambda}$ is c.m. d' Broglie wave length, $\frac{d\sigma}{d\Omega}(x)$ is the differential cross section, $x \equiv \cos\theta$, being the c.m. scattering angle, and σ_{el} is the integrated elastic cross section. So, the quantum scattering state which saturate the optimal bound (1) is *the most forward-peaked quantum state*. The optimal bound (1) improves in a more general and exact form not only the unitarity bounds derived by Martin [8], MacDowell and Martin [9] (see also Ref. [10]) for the logarithmic slope b_A of absorptive contribution $\frac{d\sigma_A}{d\Omega}(s,t)$ to the elastic differential cross sections but also the unitarity lower bound derived in Ref. [11] for the slope b of the entire $\frac{d\sigma}{d\Omega}(s,t)$ differential cross section. Therefore, it would be very important to clarify the *optimal status of the bound* (1) and to make an experimental detailed investigation of its saturation in the hadron-hadron scattering especially in the low energy region.

The aim of this paper is twofold: to show that the optimal bound (1) is the "*singular*" solution of the problem *to find an unitarity lower bound on the logarithmic slope b when: σ_{el} and $\frac{d\sigma}{d\Omega}(1)$ are given*, and, to obtain experimental tests of the bound (1) at all available energies for the principal hadron-hadron elastic scatterings: $\pi^{\pm}P \to \pi^{\pm}P$, $K^{\pm}P \to K^{\pm}P$, $PP \to PP$, and $\overline{P}P \to \overline{P}P$. So, in Sect. 2 the problem of the optimal status of the bound (1) is completely solved by showing that this bound is the singular solution ($\lambda_0 = 0$) in the unconstrained Lagrangean minimization problem. The relation of the optimal bound (1) with the Martin-MacDowell bound as well as its saturations in different particular cases (e. g. pure absorptive cases, etc.) are discussed. Some details about the PMD-SS-history as well as about other predictions of the principle of minimum distance in the space of states are presented in Sect. 3. Then, it is shown that the *optimal state* obtained with the simplified version of the *PMD-SS* have not only the property that is *the most forward-peaked quantum state* but also possesses many other peculiar properties such as: (i) scaling in the variable $\tau_o = 2\sqrt{|t|b_o}$, (ii) *maximum Tsallis-like scattering entropies* [6,7] $S_\theta(q)$, $S_L(q)$, $S_{\theta L}(q)$, $\tilde{S}_{\theta L}(q)$ for the all positive nonextensivity parameters q, etc., that make it a good candidate for the description of the quantum scattering. The results on the experimental test of the *saturation of the most forward-peaked state limit* are presented in Sect. 4 while the Sect. 5 is reserved for summary and conclusions.

2. The problem of the unitarity lower bound on logarithmic slope

Let us start with the hadron-hadron elastic scattering process of spinless particles for which the *description of the scattering amplitude* $f(x)$ *of the system* is given in terms of *partial amplitudes* $f_l,\ l = 0, 1, ...$as

$$f(x) = \sum (2l+1) f_l P_l(x), \qquad x \in [-1, 1],\ f_l \in C \tag{2}$$

where $P_l(x)$, $l = 0, 1, ...$, are Legendre polynomials, $f_l, l = 0, 1, ...$ being the partial amplitudes. If the normalization is chosen such that $\frac{d\sigma}{d\Omega}(x) =\mid f(x) \mid^2$, *the elastic integrated cross section* σ_{el}, is expressed in terms of partial amplitudes by

$$\frac{\sigma_{el}}{2\pi} = 2 \sum (2l+1) \mid f_l \mid^2 = \|f\|^2 \tag{3}$$

Now, a rigorous *unitarity lower bound* on the slope parameter b can be obtained by solving completely the following *constrained minimization* problem:

$$\min\{b\},\ when\ \sigma_{el} = fixed,\ \frac{d\sigma}{d\Omega}(1) = fixed,\ \mid f_l \mid^2 \leq Im f_l,\ \ l = 0, 1, .. \tag{4}$$

which is equivalently to the following *unconstrained minimization problem*:

$$L \equiv \lambda_0 b + \lambda_1 \left[\frac{\sigma_{el}}{4\pi} - \sum (2l+1) \mid f_l \mid^2\right] + \lambda_2 \left[\frac{d\sigma}{d\Omega}(1) - \mid \sum (2l+1) f_l \mid^2\right] + \sum \xi_l g_l \rightarrow \min \tag{5}$$

where $g_l \equiv Im f_l - \mid f_l \mid^2$, $\lambda_i,\ i = 0, 1, 2$, and ξ_l , $l = 0, 1, ...$, are the corresponding Lagrange multipliers, and $b = (R\ \overset{\circ}{R} + I\ \overset{\circ}{I})/(R^2 + I^2)$, where $R \equiv \sum (2l+1) Re f_l$, $I \equiv \sum (2l+1) Im f_l$, and $\overset{\circ}{R} \equiv \sum (2l+1) \frac{l(l+1)}{2} Re f_l$, $\overset{\circ}{I} \equiv \sum (2l+1) \frac{l(l+1)}{2} Im f_l$.

The non singular solution ($\lambda_0 \neq 0$) of the minimization problem (5) was completely treated in Ref. [11]. Here we must underline that, according to the rules from Ref. [12], the singular solution $\lambda_0 = 0$, if it exists, can be essential in obtaining a rigorous unitarity lower bound on the logarithmic slope parameter b. In order to prove this statement, let us consider as an simple example the minimization problem: $x \rightarrow \min$, subject to $x^2 + y^2 = 0$. The solution of this problem is evident: $x_{\min} = y_{\min} = 0$. Now, let use the Lagrange multiplier's problem with $\lambda_0 \neq 0$: $L = x + \lambda(x^2 + y^2)$. Then, we obtain the following incompatible equation: $1 + 2\lambda x = 0$, $2\lambda y = 0$, and $x^2 + y^2 = 0$. Hence, we must consider the case $\lambda_0 = 0$ from which we get the correct solution. Such a singular solution of the Lagrange function (5) can be obtained by solving the *minimization problem*

$$\left\{\sum (2l+1) \mid f_l \mid^2 + \alpha \left[\frac{d\sigma}{d\Omega}(1) - \mid \sum (2l+1) f_l P_l(1) \mid^2\right]\right\} \rightarrow \min \tag{6}$$

The problem (6) is just the problem of *minimum norm* (*minimum distance in the space of states*) when $\frac{d\sigma}{d\Omega}(1)$ is fixed. Thus, the unique solution of (6) exists and is given the following optimal state (see Ref. [1])

$$f_{o1}(x) = f(1)\frac{K(x,1)}{K(1,1)} = f(1)\frac{\dot{P}_{L_o+1}\ (x) + \dot{P}_{L_o}\ (x)}{(L_o+1)^2} \tag{7}$$

where $\dot{P}_l\ (x) \equiv dP_l(x)/dx$ while the *reproducing kernel* $K(x,1)$ is given as follows

$K(x,1) = \frac{1}{2}\sum_{l=0}^{L_o}(2l+1)P_l(x) = \frac{1}{2}\{\dot{P}_{L_o+1}\ (x) + \dot{P}_{L_o}\ (x)\}$, $K(1,1) = (L_o+1)^2/2$,

and

$$L_o = integer\left\{\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1)\right]^{1/2} - 1\right\} \tag{8}$$

Hence, the singular solution ($\lambda_0 = 0,\ \chi_l = 0,\ l = 0, 1, ...$) of the minimization problem (5) (or (4)) is just the *optimal slope*

$$b_o = \frac{d}{dt}\ln\left[\mid f_{o1}(x)\mid^2\right]_{|t=0} = \overline{\lambda}^2\frac{L_o(L_o+2)}{4} = \frac{\overline{\lambda}^2}{4}\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1) - 1\right] \tag{9}$$

This result is in agreement with that obtained in Ref. [1] directly from the fundamental inequality $\mid f(y)\mid^2 \leq \frac{\sigma_{el}}{2\pi}K(y,y), y\in[-1,+1]$. Then, by developing the scattering amplitude $f(y)$ and the *reproducing kernel function* $K(y,y)$ in Taylor's series around the point $y = 1$ and $L = L_o$, it was shown that

$$\frac{d}{dy}\left[\frac{d\sigma}{d\Omega}(y)\right]_{|\ y=1} \geq \frac{\sigma_{el}}{2\pi}\cdot\frac{d}{dy}\left[K(y,y)\right]_{|\ y=1,L=L_o} = \frac{d\sigma}{d\Omega}(1)\frac{L_o(L_o+2)}{2} \tag{10}$$

From (10) and (8) we get the optimal lower bound (1) since, according to the definition of the logarithmic slope b combined with the kinematical relations $t = -2q^2(1-y)$, we have $dt = 2dy/\overline{\lambda}^2$ and

$$b = \frac{\overline{\lambda}^2}{2}\frac{d}{dy}\left[\frac{d\sigma}{d\Omega}(y)\right]_{|\ y=1}/\frac{d\sigma}{d\Omega}(1) \tag{11}$$

Now, by comparing the singular solution (1) with that $b_{\min}(\lambda_0 \neq 0)$ obtained in Ref. [11] for the non singular case $\lambda_0 \neq 0$, we see that: $b_o > b_{\min}(\lambda_0 \neq 0) \simeq \frac{8}{9}b_o$. Consequently, since we get the chain of bounds: $b \geq b_o > b_{\min}(\lambda_0 \neq 0)$, the unique solution of the problem (5) is that given by the optimal bound (1) since $\max\{b_{\min}(\lambda_0 = 0), b_{\min}(\lambda_0 \neq 0)\} = b_o$

Remark 1: The solution to the minimization problem (5) is of relevance since in the pure absorptive case ($Ref(x) = 0$) the singular solution $b \geq b_o$ as well as that non singular $b > b_{\min}(\lambda_0 \neq 0) \simeq \frac{8}{9}b_o$ (see Ref.[11]) are going in

$$b_A \geq \frac{\overline{\lambda}^2}{4}\left[\frac{\sigma_T^2}{4\pi\overline{\lambda}^2\sigma_{el}} - 1\right] \equiv b_o^A \tag{12}$$

and, respectively, in the MacDowell-Martin bound

$$b_A > \frac{8}{9} b_o^A = \frac{2\overline{\lambda}^2}{9}\left[\frac{\sigma_T^2}{4\pi\overline{\lambda}^2\sigma_{el}} - 1\right] \equiv b_{MD-M} \tag{13}$$

since

$$\frac{d\sigma^A}{d\Omega}(1) = [\operatorname{Im} f(x)]^2 = \sigma_T^2/(4\pi\overline{\lambda})^2, \text{ and} db = b_A \tag{14}$$

Remark 2: Let σ_{el} and σ_T be fixed from experiment let us consider the minimization problem

$$\min\{b\}, \text{when} \sigma_{el} = fixed, \text{and} \sigma_T = fixed, \mid f_l \mid^2 \leq Imf_l, \ l = 0, 1, . \tag{15}$$

which is equivalent to the minimization of the Lagrangean

$$L \equiv \lambda_0 b + \lambda_1 \left[\frac{\sigma_{el}}{4\pi} - \sum (2l+1) \mid f_l \mid^2\right] + \lambda_2 \left[\sigma_T - \sum (2l+1) Imf_l\right] + \sum \xi_l g_l \to \min \tag{16}$$

Then, the unique solution of Eq. (15) is given by

$$b \geq \max\{b_{\min}(\lambda_0 = 0), b_{\min}(\lambda_0 \neq 0)\} = \max\{b_o^A, b_{MD-M}\} = b_o^A \tag{17}$$

The quantum scattering state which saturate the optimal bound (23) is *the most forward-peaked quantum state* which is obtained as the absorptive limit of the optimal state (7).

Indeed, the singular solution $\lambda_0 = 0$ of the problem (16), can be obtained by solving the *minimization problem*

$$\left\{\sum (2l+1) \mid f_l \mid^2 + \alpha \left[\sigma_T - \sum (2l+1) Imf_l\right]\right\} \to \min \tag{18}$$

The unique solution of this problem is given by

$$\begin{gathered} Ref_l^o = 0, \text{ forall} l, \\ Imf_l^o = 4\pi\overline{\lambda} \cdot \tfrac{\sigma_{el}}{\sigma_T}, \text{for } 0 \leq l \leq L_o^A, \ Imf_l = 0, \text{for} l \geq L_o^A + 1 \end{gathered} \tag{19}$$

where

$$L_o^A = \text{integer}\left\{\left[\frac{\sigma_T^2}{4\pi\overline{\lambda}^2\sigma_{el}}\right]^{1/2} - 1\right\} = \text{integer}\left\{\left[\frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}^A(1)\right]^{1/2} - 1\right\} \tag{20}$$

By introducing the results (19) and (20) in Eq. (2) we obtain

$$\begin{gathered} Ref_{o1}^A(x) = 0 \\ Imf_{o1}^A(x) = Imf(1)\tfrac{\sum (2l+1)P_l(x)}{(L_o^A+1)^2} = Imf(1)\tfrac{\dot{P}_{L_o+1}(x) + \dot{P}_{L_oT}(x)}{(L_o^A+1)^2} \end{gathered} \tag{21}$$

Hence, for purely absorptive amplitudes the optimal solution to the minimization problem (19) exists and is given by Eq. (21) which in fact is the pure absorptive limit of the optimal state (7) but with $L_o = L_o^A$. Hence, in the pure absorptive case from (20) we get

$$b_{\min}(\lambda = 0) = \overline{\lambda}^2 \frac{L_o^A(L_o^A + 2)}{4} = \frac{\overline{\lambda}^2}{4}\left[\frac{\sigma_T^2}{4\pi\overline{\lambda}^2\sigma_{el}} - 1\right] \equiv b_o^A \qquad (22)$$

Therefore, the unique solution of the problem (15) is given by

$$b \geq \max\{b_o^A, b_{MD-M}^A\} = b_o^A \qquad (23)$$

The MacDowell-Martin bound $b^A > b_{MD-M}^A$ (13) emerge as the non singular solution in a special case of the minimization problem (5) when $Ref(x) = 0$ and due the fact that is $\frac{8}{9}b_o^A$ is eliminated in the final solution (23) of the problem (15). Hence, the bound (1) include in a more general and exact form the MacDowell-Martin bound since the validity of the bound $b \geq b_o$ includes the validity of any inequality obtained as a consequence of the inequality $b_o > a$, e.g., $a \equiv b_{MD-M}^A$(13).

Finally, we note that the fundamental inequality (16) from Ref. [1] tell us that if σ_{el} and$\frac{d\sigma}{d\Omega}(1)$ (or if σ_{el} and σ_T) are fixed from experiment, then the number $(L+1)$ of partial amplitudes in any phase shift analysis (PSA) must obey the optimal bounds

$$(L+1)^2 \geq \frac{4\pi}{\sigma_{el}}\frac{d\sigma}{d\Omega}(1), \;\; (L+1)^2 \geq \frac{\sigma_T^2}{4\pi\overline{\lambda}^2\sigma_{el}} \qquad (24)$$

respectively, or equivalently $L \geq L_o \geq L_o^A$.

3. Why principle of minimum distance in space of states?

As we already underlined in introduction, in Ref. [1] are investigated the essential features of the hadron-hadron scattering by using the optimization theory [1] in the Hilbert space of the partial scattering amplitudes. Then, knowledge about the hadron-hadron scattering system (or more concretely, about partial amplitudes) are deduced by assuming that it behaves as to optimize some given *measure of its effectiveness*, and thus the *behavior of the system* is completely specified by identifying the criterion of effectiveness and applying constrained optimization to it. This approach is in fact known as describing the system in terms of an *optimum principle.*

The earliest *optimum principle* was proposed by Hero of Alexandria (125 B.C.) in his *Catoptrics* in connection with the behavior of light. Thus, Hero of Alexandria (125 B.C.) proved mathematically the following first genuine scientific *minimum principle* of physics.

Hero's principle of minimum distance (PMD) tell us that when a ray of light is reflected by a mirror, the path actually taken from the object to the observer's eye is the shortest path from all possible paths. If we extend the PMD-idea to the behavior of light in gravitational fields, then, we obtain immediately that according to the PMD-optimum principle, modified to include the interaction of light with the gravitational field, *the light must move on a specific shortest path which is the geodesic.* Having in mind this successful result, the *principle of minimum distance* was recently [15] extended to the quantum physics by choosing the "*partial transition amplitudes*" as fundamental physical quantities since they are labelled with good quantum numbers such as charge, angular momentum, isospin, etc. These physical quantities are chosen as the *system variational variables* while the *distance in the Hilbert space of the quantum states* is taken as measure of the system effectiveness expressed in terms of the system variables. The *principle of minimum distance in the space of states* (PMD-SS) is chosen as variational optimum principle by which one should obtain those values of the "*partial amplitudes*" yielding minimum *effectiveness.* Of course this new optimum principle can be formulated in a more general mathematical form by using the S-matrix theory of the strong interacting systems.

The Hilbert space $\mathbf{H}$ of the scattering states, with the inner product $< . , . >$ and the norm $\|\cdot\|$, defined on the interval $S \equiv (-1, +1)$, can be given by

$$< f, g >= \int_{-1}^{+1} f(x)\overline{g(x)}dx\,, \quad f, g \in \mathbf{H}, \tag{25}$$

$$\| f \|^2 =< f, f >= \int_{-1}^{+1} \mid f(x) \mid^2 dx, \quad f \in \mathbf{H} \tag{26}$$

If the *quantum distance* $D(f, g)$ between two quantum scattering states f and g is defined [1, 13] by

$$D(f, g) = \min_{\Phi} \|f - g\exp(-i\Phi)\| = [\|f\|^2 + \|g\|^2 - 2\,|< f, g >|]^{\frac{1}{2}} \tag{27}$$

then the PMD-SS can be formulated in the following form.

PMD-SS: *If h is the quantum state of the system when the interaction is missing, then the true interacting quantum state f of the interacting system is that state which have the shortest distance $D(f, h)$ in the space of interacting states compatible with the constraints imposed by the interaction. (Of course, if $h = 0$, then $D(f, 0) = \|f\|$.)*

Therefore, let M_{eff} be the measure of the scattering effectiveness and let us consider the problems of optimization of few M_{eff} defined as follows:

(a) $M_{eff} \equiv D^2(f, 0)$, or equivalently $M_{eff} = \sigma_{el}/4\pi$, where σ_{el} is given in terms of partial amplitudes by Eq. (3);

(b) $M_{eff} \equiv b$, where the *logarithmic slope b of the diffraction peak* is expressed in terms of the partial amplitudes as in

Sect. II;

(c) $M_{eff} \equiv S_L = -\sum(2l+1)p_l \ln p_l$, where: $p_l = 4\pi|f_l|^2/\sigma_{el}$, and $\sum(2l+1)p_l = 1$;

(d) $M_{eff} \equiv S_\theta = -\int_{-1}^{+1} dx\, P(x) \ln P(x)$, where $P(x) = \frac{2\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(x)$, with $\int_{-1}^{+1} dx\, P(x) = 1$;

Now, let us consider the constrained optimization problems:

$$\text{extremum}\{M_{eff}\} \text{ when } \sigma_{el} \text{ and } \frac{d\sigma}{d\Omega}(1) \text{ are fixed.} \tag{28}$$

which, via Lagrange multipliers method [12], are reduced to the problems:

$$\pounds_F \equiv \lambda_0 M_{eff} + \lambda_1 \left[\frac{\sigma_{el}}{4\pi} - \sum(2l+1) \mid f_l \mid^2\right] + \lambda_2 \left[\frac{d\sigma}{d\Omega}(1) - \mid \sum(2l+1) f_l \mid^2\right] \to \text{extremum} \tag{29}$$

Hence, according to the general Lagrange multiplier rules [12], for all these optimization problems (28)-(29) we are obligated to analyze in detail the *singular solution* $\lambda_0 = 0$. Such a singular solution of the Lagrangean function (29) can be obtained by solving the minimization problem (6) which is just the *problem of minimum norm* (minimum distance in the space of states)

$$\min[D(f,0)] \text{ when } \frac{d\sigma}{d\Omega}(1) \text{ is fixed } fixed \tag{30}$$

Thus, the unique solution of the problem (30) exists and is given by the *PMD-SS-optimal state* (7) (see details in Ref. [1]). Therefore, the *optimal effectiveness* are as follows (see e.g. Ref. [7] for the (c), (d) cases):

$$\begin{array}{c} M^{o1}_{eff} = b_o, \text{ in case(b)} \\ M^{o1}_{eff} = S^{o1}_L = \ln[2K(1,1) = \ln[(L_o+1)^2], \text{ in case (c)} \\ M^{o1}_{eff} = S^{o1}_\theta = -\int_{-1}^{+1} dx \left[\frac{[K(x,1)]^2}{K(1,1)}\right] \ln\left[\frac{[K(x,1)]^2}{K(1,1)}\right], \text{ in case (d)} \end{array} \tag{31}$$

while the *optimal angular momentum* is given by Eq. (8).

Now, we recall briefly some peculiar properties of the PMD-SS-optimal state (7) obtained in Refs. [1], [4]-[7] that can illustrate the importance of the PMD-SS for an unified description of all quantum phenomena. These characteristic features are as follows:

I. The *PMD-SS-optimal scaling* property (see the experimental tests in Fig. 1 in Ref.[1])

$$\frac{1}{\frac{d\sigma}{d\Omega}(s,1)} \frac{d\sigma^{o1}}{d\Omega}(s,x) = \left[\frac{K(x,1)}{K(1,1)}\right]^2 \approx \left[\frac{2\, J_1(\tau_o)}{\tau_o}\right]^2, \text{ for small } \tau_o \tag{32}$$

as a function of the *PMD-SS-optimal scaling variable* τ_o

$$\tau_o \equiv 2\ [|t|\, b_{o1}]^{1/2} = \left\{\overline{\lambda}^2\ |t| \left[\frac{4\pi}{\sigma_{el}} \frac{d\sigma}{d\Omega}(1) - 1\right]\right\}^{1/2} \tag{33}$$

II. The *PMD-SS-optimal state* (7) is the *most forward peaked scattering state* in the sense given by inequality (1) (see Figs. 1-3 in this paper);

III. The *PMD-SS-optimal state* (7) is the state with *maximum scattering entropies* (see proofs and the experimental illustrations in Ref. [7]) in the sense that the entropies S_L and S_θ of any scattering state must fulfill the inequalities

$$S_\theta \leq S^{o1}_\theta, \ \ S_L \leq S^{o1}_L \tag{34}$$

IV. *PMD-SS-optimal state* (7) is the state of *equilibrium* of the angular momenta channels considered as a *quantum statistical ensemble.* Indeed, using Eqs. (31) and (34) we observe that the entropy S_L is similar to the Boltzmann entropy with a maximum value given by the *logarithm of number of the optimal states:* $S_L \leq S_L^{o1} = \ln[(L_o+1)^2]$. The results like (34) are also proved in Ref. [7] for the nonextensive statistics [14] of the angular momenta channels.

V. From mathematical point of view, the *PMD-SS-optimal states,* are functions of *minimum constrained norm* and consequently can be completely described by *reproducing kernel functions (*see Ref. [1,3-4]. So, with this respect the *PMD*-SS-*optimal states* from the *reproducing kernel Hilbert space (RKHS)* of the scattering amplitudes are analogous to the *coherent states* from the RKHS of the *wave functions.*

VI. For the scattering of particle with spins, the *PMD-SS-optimal states* are *s-channel helicity conserving* states (see Refs. [4]).

Therefore, the *PMD-SS-optimal state* (7) have not only the property that is *the most forward-peaked quantum state* but also possesses many other peculiar properties such as I-V that make it a good candidate for the description of the quantum scattering via an optimum principle. So, the systematic experimental investigation of the *optimal bound* (1) as well as of the bounds (34) can be of great interest since, as a direct signature of the *PMD-SS-optimal state dominance* in the hadron-hadron scattering phenomena, the s*saturation of the most forward-peaked limit* as well as the *saturation of the maximum entropy limits* are expected to be experimentally observed.

4. Experimental tests

Now, in order to obtain a consistent experimental test of the optimal unitarity lower bound (1) we have compiled from the literature [7]-[18] the experimental data on the diffraction slope parameter b for $\pi^{\pm}P \to \pi^{\pm}P$, $K^{\pm}P \to K^{\pm}P$, $PP \to PP$, and $\overline{P}P \to \overline{P}P$ at all available energies.

$\pi^{\pm}P$-*scattering*: In this case, for $P_{LAB} \geq 3GeV/c$, the experimental data on b, $\frac{d\sigma}{d\Omega}(1)$ and σ_{el} are collected mainly from the original fit and data of Refs.[15]-[19]. To these data we added some values of b from the linear fit of Lasinski et al. [24] and also calculated directly from the *phase shifts analysis* (PSA) of Holer et al. [20]. In the low laboratory momenta $P_{LAB} \leq 2$ GeV/c, the slope parameters b as well as the optimal slope b_o are determined in pairs (b, b_o), at each laboratory momenta, from the PSA [20]. Unfortunately, the values of b_o corresponding to the Lasinski's data [24] was impossible to be calculated since the values of $\frac{d\sigma}{d\Omega}(1)$ from their original fit are not given.

$K^{\pm}P-$*scattering.* The experimental data on b, $\frac{d\sigma}{d\Omega}(1)$ and σ_{el} , in the case of K^-P, are collected from the original fit and data of Refs. [15], [17], [19], [21]. For K^+P-scattering, to these data from Refs. [15], [17], [19], [22] we added some values of b from the linear fit of Lasinski et al. [24] and also those pairs (b, b_o) calculated directly from the *experimental* (PSA) solutions of Arndt et al. [23].

$\overline{P}P$ and $PP-$*scatterings.* The experimental data on b, $\frac{d\sigma}{d\Omega}(1)$ and σ_{el}, in these cases, are obtained from Refs. [19]-[21]. The data from original fits where $\frac{d\sigma}{d\Omega}(1)$ is given numerically was first selected for to obtain the pairs (b, b_o).

Now, for comparison of the experimental values of the slope parameter b with the *optimal PMD-SS-optimal slope* b_o we used a (b, b_o)-plot of the elastic diffraction slopes as in Figs. 1a,b, where we plotted a total number $N_p = 420$ of experimental pairs (b, b_o). From these pairs a number $N_p = 125$ pairs shown in Fig. 1b are coming from the experiments at $P_{LAB} \geq 2$ GeV/c. The dependence of the experimental slopes b and of the *optimal PMD-SS-slope* b_o (see Eq. (9)) on laboratory momenta (P_{LAB}) is presented in Figs. 2-3. The values of the $\chi^2 = \sum_j (b_j - b_{oj})^2/(\epsilon_{bj}^2 + \epsilon_{b_oj}^2)$, (where ϵ_{bj} and ϵ_{b_oj} are the experimental errors corresponding to b and b_o respectively) are used for the estimation of departure from the *optimal PMD-SS-slope* b_o, and then, we obtain the statistical parameters presented in Table 1.

5. Summary and Conclusion

The main results obtained in this paper can be summarized as follows:

(i) In this paper we proved that the optimal bound (1) is the singular solution ($\lambda_0 = 0$) of the problem *to find the unitarity lower bound on the logarithmic slope b with the constraints imposed by unitarity when* σ_{el} *and* $\frac{d\sigma}{d\Omega}(1)$ *are fixed.* This result is similar with that obtained recently in Ref. [7] for the problem *to find an upper bound for the scattering entropies when* σ_{el} *and* $\frac{d\sigma}{d\Omega}(1)$ *are fixed from the experimental data.* So, the *PMD-SS-optimal state* is not only the most peaked state but also a state which saturates the maximum entropy limit;

(ii) The optimal bound (1) is verified experimentally with high accuracy (see the region $b \geq b_o$ in Fig. 1a) at all available energies for all the principal hadron-hadron scatterings.

(iii) A systematic tendency toward the saturation of the *most forward-peaked optimal state limit* is observed from the experimental data on the logarithmic slope of all the $PP \to PP$, $\overline{P}P \to \overline{P}P$, $\pi^{\pm}P \to \pi^{\pm}P$ and $K^{\pm}P \to K^{\pm}P$ scatterings at all available laboratory momenta. The *proton-proton* ($\bar{\chi}^2 = 5.06$) and *antiproton-proton* ($\bar{\chi}^2 = 1.86$) and K^+P ($\bar{\chi}^2 = 3.05$) scatterings are examples where this saturation of the most peaked optimal state limit is observed at all available laboratory momenta. In fact the validity *of the principle of least distance in space of states in hadron-hadron scattering* for $p_{LAB} \geq 2$ GeV/c is well illustrated in Fig. 1b and Table 1. A similar tendency towards the saturation of the *most forward peaked PMD-SS-optimal state limit* is observed (see Figs. 2-3) even in the low energy regions at the energies between the resonances positions or/and in the region corresponding to the "*diffractive resonances*".

(iv) In order to see why the experimental logarithmic slopes $b_{\exp}$(as well as the experimental entropies S_L and S_θ) are well described by the *PMD-SS-optimal state (7)* we observe that the entropy S_L is similar to the Boltzmann entropy with a maximum value given by the *logarithm of number of the optimal states.* Indeed, from (31) and (34) we can conclude ($S_L \leq S_L^{o1} = \ln[(L_o +$

$1)^2$]) that the *PMD-SS-optimal state* is the state of *equilibrium* of the angular momenta channels considered as a *quantum statistical ensemble.*

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Table 1 : $\bar{\chi}^2$ −statistical parameters of the principal hadron-hadron scattering. In these estimations for $P_{LAB} \leq 2$ GeV/c the errors $\epsilon_{b_o}^{PSA}(\pi^{\pm}P) = 0.5$ GeV^{-2} and $\epsilon_{b_o}^{PSA}(K^{+}P) = 0.3$ GeV^{-2} are taken into account for the optimal slopes b_o calculated from phase shifts analysis [20] and [23], respectively.

		$N_p(P_{LAB} \geq 2GeV/c)$		N_p(all P_{LAB})
Statistical parameters	N_p	χ^2/N_p	N_p	χ^2/N_p
$P\ P \to P\ P$	29	5.01	32	5.06
$\bar{P}\ P \to \bar{P}\ P$	27	0.56	45	1.86
$\pi^+P \to \pi^+P$	12	2.90	43	9.70
$\pi^-P \to \pi^-P$	14	2.61	68	14.9
$K^+P \to K^+P$	20	0.75	55	3.05
$K^-P \to K^-P$	23	2.18	97	6.40
All experiments	125	2.38	340	6.83

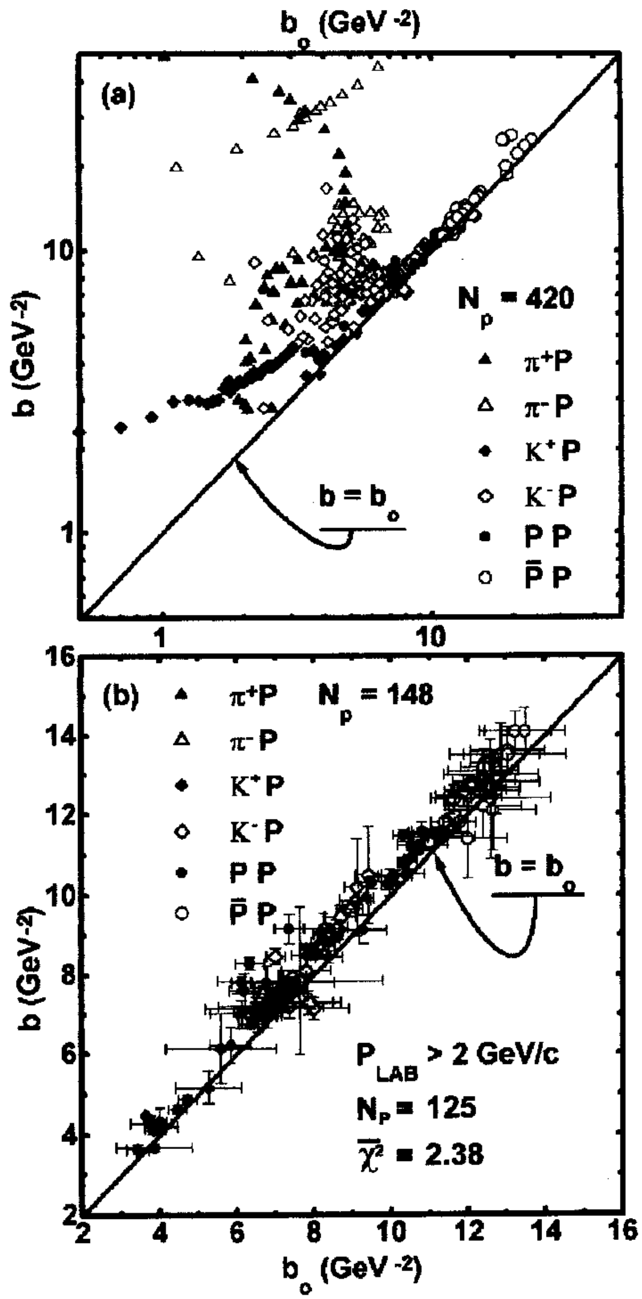

Fig. 1: The experimental values b versus optimal values b_o of the logarithmic slope for all principal hadron-hadron scatterings: (a) at all available momenta, and (b) only for $P_{LAB} \geq 2$ GeV/c. The experimental data for $b, \frac{d\sigma}{d\Omega}(1)$ and σ_{el}, are taken from Refs. [15]-[26].

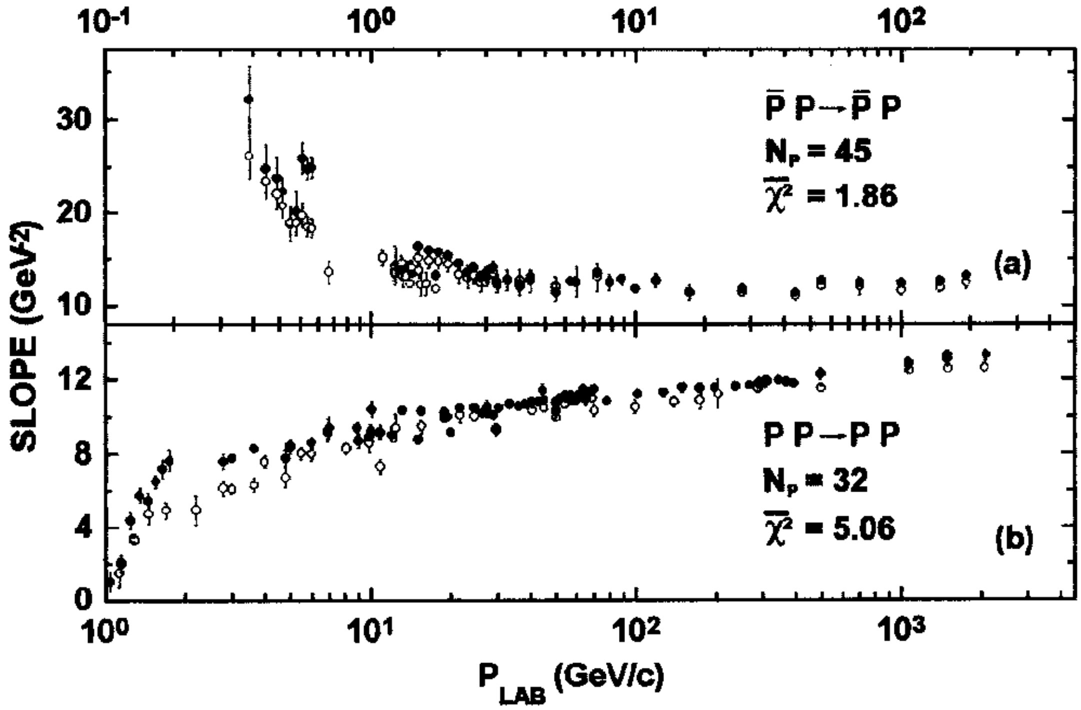

Fig. 2: The experimental values (black circles) of the logarithmic slope b for the proton-proton and antiproton-proton scatterings are compared with the *optimal PMD-SS-predictions* b_o (white circles). The experimental data for b, $\frac{d\sigma}{d\Omega}(1)$ and σ_{el}, are taken from Refs. [24]-[26] (see the tex).

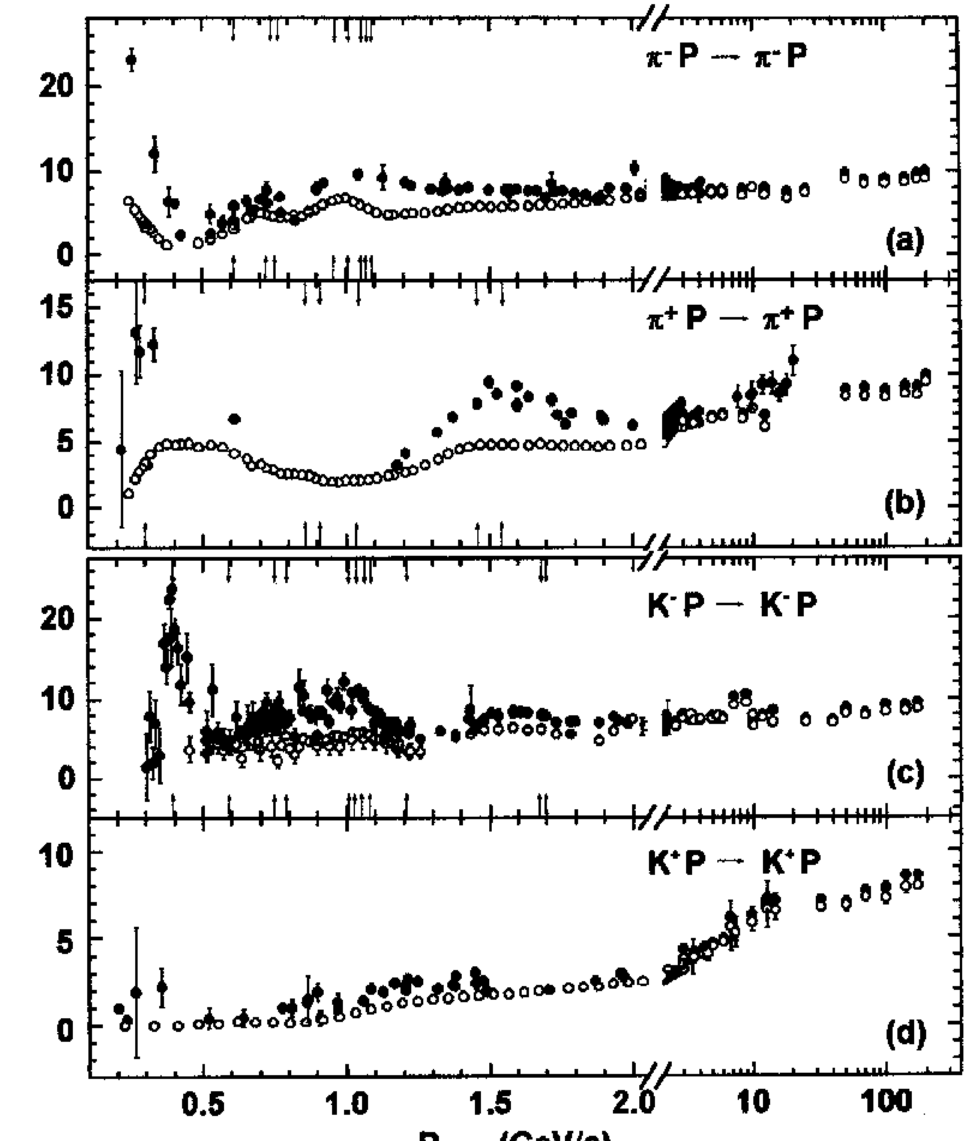

Fig. 3: The experimental values (black circles) of the logarithmic slope b for the principal meson-nucleon scatterings are compared with the *optimal PMD-SS-predictions* b_o (white circles). The experimental data for b, $\frac{d\sigma}{d\Omega}(1)$ and σ_{el}, are taken from Refs. [15]-[23]. The solid curves shows the values of b calculated from the experimental PSA [20,23] (see the text).