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arXiv:hep-ph/0301162v1 20 Jan 2003

CERN-TH/2003/008

# Anisotropic Color Superconductivity

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**Abstract**

We discuss the possibility that in finite density QCD an anisotropic phase is realized. This case might arise for quarks with different chemical potential and/or different masses. In this phase crystalline structures may be formed. We consider this possibility and we describe, in the context of an effective lagrangian, the corresponding phonons as the Nambu-Goldstone bosons associated to the breaking of the space symmetries.

## 1 Introduction

The study of color superconductivity goes back to the first days of QCD [1], but only recently this phenomenon has received a lot of attention (for recent reviews see refs. [2, 3]). Naively one would expects that, due to asymptotic freedom, quarks at very high density would form a Fermi sphere of almost free fermions. However, Bardeen, Cooper and Schrieffer [4] proved that the Fermi surface of free fermions is unstable in presence of an attractive, arbitrary small, interaction. Since in QCD the gluon exchange in the $\bar{3}$ channel is attractive one expects the formation of a coherent state of particle/hole pairs (Cooper pairs). An easy way to understand the origin of this instability is to remember that, for free fermions, the Fermi energy distribution at zero temperature is given by $f(E) = \theta(\mu - E)$ and therefore the maximum value of the energy (Fermi energy) is $E_F = \mu$. Then consider the corresponding grand-potential, $\Omega = E - \mu N$, with $\mu = E_F$. Adding or subtracting a particle (or adding a hole) to the Fermi surface does not change $\Omega$, since

[1] On leave from the Department of Physics of the University of Florence.

[2] Contributed paper to the International Workshop SCGT 02 "Strong Coupling Gauge Theories and Effective Field Theories", 10-13 December 2002, Nagoya, Japan.

$\Omega \to (E \pm E_F) - \mu(N \pm 1) = \Omega$. We see that the Fermi sphere of free fermions is highly degenerate. This is the origin of the instability, because if we compare the grand-potential for adding two free particles or two particles bounded with a binding energy $E_B$, we find that the difference is given by $\Omega_B - \Omega = -E_B < 0$. Since a bound state at the Fermi surface can be formed by an arbitrary small attractive interaction [5], it is energetically more favorable for fermions to pair and form condensates.

From the previous considerations it is easy to understand why to realize superconductivity in ordinary matter is a difficult job. In fact, one needs an attractive interaction to overcome the repulsive Coulomb interaction among electrons, as for instance the one originating from phonon exchange for electrons in metals. On the other hand the interaction among two quarks in the channel $\bar{3}$ is attractive, making color superconductivity a very robust phenomenon. Notice also that once taken into account the condensation effects, at very high density one can use asymptotic freedom to get exact results. For instance, it is possible to get an analytical expression for the gap [6]. In the asymptotic regime it is also possible to understand the structure of the condensates. In fact, consider the matrix element

$$\langle 0|\psi^{\alpha}_{ia}\psi^{\beta}_{jb}|0\rangle \tag{1}$$

where $\alpha, \beta = 1, 2, 3$ are color indices, $a, b = 1, 2$ are spin indices and $i, j = 1, \cdots, N$ are flavor indices. Its color, spin and flavor structure is completely fixed by the following considerations:

- antisymmetry in color indices $(\alpha, \beta)$ in order to have attraction;
- antisymmetry in spin indices $(a, b)$ in order to get a spin zero condensate. The isotropic structure of the condensate is favored since it allows a better use of the Fermi surface;
- given the structure in color and spin, Pauli principles requires antisymmetry in flavor indices.

Since the momenta in a Cooper pair are opposite, as the spins of the quarks (the condensate has spin 0), it follows that the left(right)-handed quarks can pair only with left(right)-handed quarks. In the case of 3 flavors the favored condensate is

$$\langle 0|\psi^{\alpha}_{iL}\psi^{\beta}_{jL}|0\rangle = -\langle 0|\psi^{\alpha}_{iR}\psi^{\beta}_{jR}|0\rangle = \Delta \sum_{C=1}^{3} \epsilon^{\alpha\beta C}\epsilon_{ijC} \tag{2}$$

This gives rise to the so-called color–flavor–locked (CFL) phase [7, 8]. The reason for the name is that simultaneous transformations in color and in flavor leave the condensate invariant. In fact, the symmetry breaking pattern turns out to be

$$SU(3)_c \otimes SU(3)_L \otimes SU(3)_R \otimes U(1)_B \to SU(3)_{c+L+R}$$

where $SU(3)_{c+L+R}$ is the diagonal subgroup of the three $SU(3)$ groups. This is the typical situation when the chemical potential is much bigger than the quark masses $m_u$, $m_d$ and $m_s$ (here the masses to be considered are in principle density depending). However we may ask what happens decreasing the chemical potential. At intermediate densities we have no more the support of asymptotic freedom, but all the model calculations show that one still has a sizeable color condensation. In particular if the chemical potential $\mu$ is much less than the strange quark mass one expects that the strange quark decouples, and the corresponding condensate should be

$$\langle 0|\psi^{\alpha}_{iL}\psi^{\beta}_{jL}|0\rangle = \Delta\epsilon^{\alpha\beta 3}\epsilon_{ij} \tag{3}$$

In fact, due to the antisymmetry in color the condensate must necessarily choose a direction in color space. Notice that now the symmetry breaking pattern is completely different from the three-flavor case. In fact, we have

$$SU(3)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B \to SU(2)_c \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_B$$

It is natural to ask what happens in the intermediate region of $\mu$. It turns out that the interesting case is for $\mu \approx M_s^2/\Delta$. To understand this point let us consider the case of two fermions: one massive, $m_1 = M_s$ and the other one massless, $m_2 = 0$, at the same chemical potential $\mu$. The Fermi momenta are of course different

$$p_{F_1} = \sqrt{\mu^2 - M_s^2}, \qquad p_{F_2} = \mu \tag{4}$$

The grand potential for the two unpaired fermions is (factor 2 from the spin degrees of freedom)

$$\Omega_{\text{unpair.}} = 2\int_0^{p_{F_1}} \frac{d^3p}{(2\pi)^3}\left(\sqrt{\vec{p}^{\,2} + M_s^2} - \mu\right) + 2\int_0^{p_{F_2}} \frac{d^3p}{(2\pi)^3}\left(|\vec{p}| - \mu\right) \tag{5}$$

In order to pair the two fermions must reach some common momentum $p^F_{\rm comm}$, and the corresponding grand potential is

$$\begin{aligned}\Omega_{\rm pair.} &= 2\int_0^{p^F_{\rm comm}} \frac{d^3p}{(2\pi)^3}\left(\sqrt{\vec{p}^{\,2}+M_s^2}-\mu\right)+2\int_0^{p^F_{\rm comm}}\frac{d^3p}{(2\pi)^3}\left(|\vec{p}|-\mu\right)\\ &- \frac{\mu^2\Delta^2}{4\pi^2}\end{aligned} \tag{6}$$

where the last term is the energy necessary for the condensation of a fermion pair [9]. The common momentum $p^F_{\rm comm}$ can be determined by minimizing $\Omega_{\rm pair.}$ with respect to $p^F_{\rm comm}$, with the result

$$p^F_{\rm comm} = \mu - \frac{M_s^2}{4\mu} \tag{7}$$

It is now easy to evaluate the difference $\Omega_{\rm unpair.}-\Omega_{\rm pair.}$ at the order $M_s^4$, with the result

$$\Omega_{\rm unpair.}-\Omega_{\rm pair.} \approx \frac{1}{16\pi^2}\left(M_s^4-4\Delta^2\mu^2\right) \tag{8}$$

We see that in order to have condensation the condition

$$\mu > \frac{M_s^2}{2\Delta} \tag{9}$$

must be realized. The problem of one massless and one massive flavor has been studied in ref. [10]. However, one can simulate this situation by taking two massless quarks with different chemical potentials, which is equivalent to have two different Fermi spheres. The big advantage here is that one can use a study made by Larkin and Ovchinnikov [11] and Fulde and Ferrel [12]. These authors studied the case of a ferromagnetic alloy with paramagnetic impurities. The impurities produce a magnetic field which, acting upon the electron spins, gives rise to a different chemical potential for the two populations of electrons. It turns out that it might be energetically favorable to pair fermions which are close to their respective Fermi surface (LOFF phase). However, since the Fermi momenta are different, the Cooper pair cannot have zero momentum and there is a breaking of translational and rotational invariance. Therefore, a crystalline phase can be formed. The previous situation is very difficult to be realized experimentally, but there have been claims of observation of this phase in heavy-fermion superconductors [13] and in quasi-two dimensional layered organic superconductors [14]. The authors of

ref. [15] have extended the calculation of ref. [11, 12] to the case of two-flavor QCD and we will review here their results. Also, since the LOFF phase can give rise to crystalline structures, phonons are expected. We will also discuss the effective lagrangians for the phonons in different crystalline phases and show how to evaluate the parameters characterizing them, as the velocity of propagation. Finally we will consider an astrophysical application.

# 2 The LOFF phase

According to the authors of ref. [11, 12] when fermions belong to two different Fermi spheres, they may prefer to pair staying as much as possible close to their own Fermi surface. When they are sitting exactly at the surface, the pairing is as shown in Fig. 1. We see that the total momentum of the pair

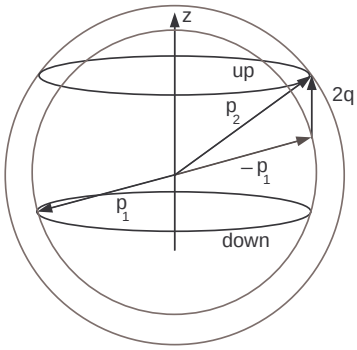


Figure 1: Pairing of fermions belonging to two Fermi spheres of different radii according to LOFF.

is $\vec{p}_1 + \vec{p}_2 = 2\vec{q}$ and, as we shall see, $|\vec{q}|$ is fixed variationally whereas the direction of $\vec{q}$ is chosen spontaneously. Since the total momentum of the pair is not zero the condensate breaks rotational and translational invariance. The simplest form of the condensate compatible with this breaking is just a simple plane wave (more complicated functions will be considered later)

$$\langle \psi(x)\psi(x) \rangle \approx \Delta\, e^{2i\vec{q}\cdot\vec{x}} \tag{10}$$

It should also be noticed that the pairs use much less of the Fermi surface than they do in the BCS case. In fact, in the case considered in Fig. 1 the fermions can pair only if they belong to the circles depicted there. More generally there

is a quite large region in momentum space (the so called blocking region) which is excluded from the pairing. This leads to a condensate smaller than the BCS one.

Before discussing the LOFF case let us review the gap equation for the BCS condensate. We have said that the condensation phenomenon is the key feature of a degenerate Fermi gas with attractive interactions. Once one takes into account the condensation the physics can be described using the Landau's idea of quasi-particles. In this context quasi-particles are nothing but fermionic excitations around the Fermi surface described by the following dispersion relation

$$\epsilon(\vec{p}, \Delta_{BCS}) = \sqrt{\xi^2 + \Delta_{BCS}^2} \tag{11}$$

with

$$\xi = E(\vec{p}) - \mu \approx \frac{\partial E(\vec{p})}{\partial \vec{p}}\Big|_{\vec{p}=\vec{p}_F} \cdot (\vec{p} - \vec{p}_F) = \vec{v}_F \cdot (\vec{p} - \vec{p}_F) \tag{12}$$

The quantities $\vec{v}_F$ and $(\vec{p} - \vec{p}_F)$ are called the Fermi velocity and the residual momentum respectively. A easy way to understand how the concept of quasi-particles comes about in this context is to study the gap equation at finite temperature. For simplicity let us consider the case of a four-fermi interaction. The euclidean gap equation is given by

$$1 = -ig \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p_4 - i\mu)^2 + |\vec{p}|^2 + \Delta_{BCS}^2} \tag{13}$$

From this expression is easy to get the gap equation at finite temperature. We need only to convert the integral over $p_4$ into a sum over the Matsubara frequencies

$$1 = gT \int \frac{d^3p}{(2\pi)^3} \sum_{n=-\infty}^{+\infty} \frac{1}{((2n+1)\pi T)^2 + \epsilon^2(\vec{p}, \Delta_{BCS})} \tag{14}$$

Performing the sum we get

$$1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1 - n_u - n_d}{\epsilon(\vec{p}, \Delta_{BCS})} \tag{15}$$

Here $n_u$ and $n_d$ are the finite-temperature distribution functions for the excitations (quasi-particles) corresponding to the original pairing fermions

$$n_u = n_d = \frac{1}{e^{\epsilon(\vec{p}, \Delta_{BCS})/T} + 1} \tag{16}$$

At zero temperature ($n_u = n_d \to 0$) we find (restricting the integration to a shell around the Fermi surface)

$$1 = \frac{g}{2} \int \frac{d\Omega_p \, p_F^2 \, d\xi}{(2\pi)^3} \frac{1}{\sqrt{\xi^2(\vec{p}) + \Delta_{BCS}^2}} \tag{17}$$

In the limit of weak coupling we get

$$\Delta_{BCS} \approx 2 \, \bar{\xi} \, e^{-2/(g\rho)} \tag{18}$$

where $\bar{\xi}$ is a cutoff and

$$\rho = \frac{p_F^2}{\pi^2 v_F} \tag{19}$$

is the density of states at the Fermi surface. This shows that decreasing the density of the states the condensate decreases exponentially.

Let us now consider the LOFF case. For two fermions at different densities we have an extra term in the hamiltonian which can be written as

$$H_I = -\delta\mu\sigma_3 \tag{20}$$

where, in the original LOFF papers [11, 12] $\delta\mu$ is proportional to the magnetic field due to the impurities, whereas in the actual case $\delta\mu = (\mu_1 - \mu_2)/2$ and $\sigma_3$ is a Pauli matrix acting on the space of the two fermions. According to ref. [11, 12] this favors the formation of pairs with momenta

$$\vec{p}_1 = \vec{k} + \vec{q}, \quad \vec{p}_2 = -\vec{k} + \vec{q} \tag{21}$$

We will discuss in detail the case of a single plane wave (see eq. (10)). The interaction term of eq. (20) gives rise to a shift in $\xi$ (see eq. (12)) due both to the non-zero momentum of the pair and to the different chemical potential

$$\xi = E(\vec{p}) - \mu \to E(\pm\vec{k} + \vec{q}) - \mu \mp \delta\mu \approx \xi \mp \bar{\mu} \tag{22}$$

with

$$\bar{\mu} = \delta\mu - \vec{v}_F \cdot \vec{q} \tag{23}$$

Here we have assumed $\delta\mu \ll \mu$ (with $\mu = (\mu_1 + \mu_2)/2$) allowing us to expand $E$ at the first order in $\vec{q}$ (see Fig. 1). The gap equation has the same formal expression as in eq. (15) for the BCS case

$$1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1 - n_u - n_d}{\epsilon(\vec{p}, \Delta)} \tag{24}$$

but now $n_u \neq n_d$

$$n_{u,d} = \frac{1}{e^{(\epsilon(\vec{p},\Delta)\pm\bar{\mu})/T}+1} \tag{25}$$

where $\Delta$ is the LOFF gap. In the limit of zero temperature we obtain

$$1 = \frac{g}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\epsilon(\vec{p},\Delta)}\left(1-\theta(-\epsilon-\bar{\mu})-\theta(-\epsilon+\bar{\mu})\right) \tag{26}$$

The two step functions can be interpreted saying that at zero temperature there is no pairing when $\epsilon(\vec{p},\Delta) < |\bar{\mu}|$. This inequality defines the so called blocking region. The effect is to inhibit part of the Fermi surface to the pairing giving rise a to a smaller condensate with respect to the BCS case where all the surface is used.

We are now in the position to show that increasing $\delta\mu$ from zero we have first the BCS phase. Then at $\delta\mu = \delta\mu_1$ there is a first order transition to the LOFF phase [11, 15], and at $\delta\mu = \delta\mu_2 > \delta\mu_1$ there is a second order phase transition to the normal phase (with zero gap) [11, 15]. We start comparing the grand potential in the BCS phase to the one in the normal phase. Their difference is given by

$$\Omega_{\rm BCS} - \Omega_{\rm normal} = -\frac{p_F^2}{4\pi^2 v_F}\left(\Delta_{BCS}^2 - 2\delta\mu^2\right) \tag{27}$$

where the first term comes from the energy necessary to the BCS condensation (compare with eq. (6)), whereas the last term arises from the grand potential of two free fermions with different chemical potential. We recall also that for massless fermions $p_F = \mu$ and $v_F = 1$. We have again assumed $\delta\mu \ll \mu$. This implies that there should be a first order phase transition from the BCS to the normal phase at $\delta\mu = \Delta_{BCS}/\sqrt{2}$, since the BCS gap does not depend on $\delta\mu$. The situation is depicted in Fig. 2. In order to compare with the LOFF phase we will now expand the gap equation around the point $\Delta = 0$ (Ginzburg-Landau expansion) in order to explore the possibility of a second order phase transition. Using the gap equation for the BCS phase in the first term on the right-hand side of eq. (26) and integrating the other two terms in $\xi$ we get

$$\frac{gp_F^2}{2\pi^2 v_F}\log\frac{\Delta_{BCS}}{\Delta} = \frac{gp_F^2}{2\pi^2 v_F}\int\frac{d\Omega}{4\pi}\,\mathrm{arcsinh}\frac{C(\theta)}{\Delta} \tag{28}$$

where

$$C(\theta) = \sqrt{(\delta\mu - qv_F\cos\theta)^2 - \Delta^2} \tag{29}$$

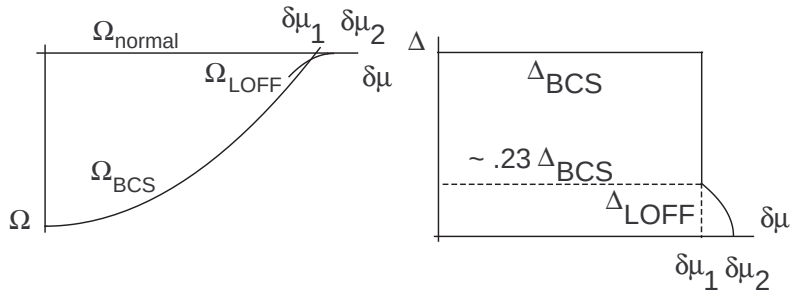


Figure 2: The grand potential and the condensate of the BCS and LOFF phases vs. $\delta\mu$.

For $\Delta \to 0$ we get easily

$$\log \frac{\Delta_{BCS}}{2\delta\mu} = \frac{1}{2}\int_{-1}^{+1} \log\left(1 - \frac{u}{z}\right), \quad z = \frac{\delta\mu}{qv_F} \tag{30}$$

This expression is valid for $\delta\mu$ smaller than the value $\delta\mu_2$ at which $\Delta = 0$, therefore the right-hand side must reach a minimum at $\delta\mu = \delta\mu_2$. The minimum is fixed by the condition

$$\frac{1}{z}\tanh\frac{1}{z} = 1 \tag{31}$$

implying

$$qv_F \approx 1.2\,\delta\mu \tag{32}$$

Putting this value back in eq. (30) we obtain

$$\delta\mu_2 \approx 0.754\,\Delta_{BCS} \tag{33}$$

From the expansion of the gap equation around $\delta\mu_2$ it is easy to obtain

$$\Delta^2 \approx 1.76\,\delta\mu_2(\delta\mu_2 - \delta\mu) \tag{34}$$

According to ref. [9] the difference between the grand potential in the superconducting state and in the normal state is given by

$$\Omega_{\rm LOFF} - \Omega_{\rm normal} = -\int_0^g \frac{dg}{g^2}\Delta^2 \tag{35}$$

Using eq. (18) and eq.(33) we can write

$$\frac{dg}{g^2} = \frac{\rho}{2}\frac{d\Delta_{BCS}}{\Delta_{BCS}} = \frac{\rho}{2}\frac{d\delta\mu_2}{\delta\mu_2} \tag{36}$$

Noticing that $\Delta$ is zero for $\delta\mu_2 = \delta\mu$ we are now able to perform the integral (35) obtaining

$$\Omega_{\rm LOFF} - \Omega_{\rm normal} \approx -0.44\,\rho(\delta\mu - \delta\mu_2)^2 \tag{37}$$

We see that in the window between the intersection of the BCS curve and the LOFF curve in Fig. 2 and $\delta\mu_2$ the LOFF phase is favored. Furthermore at the intersection there is a first order transition between the LOFF and the BCS phase. Notice that since $\delta\mu_2$ is very close to $\delta\mu_1$ the intersection point is practically given by $\delta\mu_1$. In Fig. 2 we show also the behaviour of the condensates. Altough the window $(\delta\mu_1, \delta\mu_2) \simeq (0.707, 0.754)\Delta_{BCS}$ is rather narrow, there are indications that considering the realistic case of QCD [16] the windows may open up. Also, for different structures than the single plane wave there is the possibility that the windows opens up [16].

# 3 Crystalline structures

The ground state in the LOFF phase is a superposition of states with different occupation numbers ($N$ even)

$$|0\rangle_{LOFF} = \sum_N c_N |N\rangle \tag{38}$$

Therefore the general structure of the condensate in the LOFF ground state will be

$$\langle\psi(x)\psi(x)\rangle = \sum_N c_N^* c_{N+2} e^{2i\vec{q}_N\cdot\vec{x}} \langle N|\psi(x)\psi(x)|N+2\rangle = \sum_N \Delta_N e^{2i\vec{q}_N\cdot\vec{x}} \tag{39}$$

The case considered previously corresponds to all the Cooper pairs having the same total momentum $2\vec{q}$. A more general situation, although not the most general, is when the vectors $\vec{q}_N$ reduce to a set $\vec{q}_i$ defining a regular crystalline structure. The corresponding coefficients $\Delta_{\vec{q}_i}$ (linear combinations of subsets of the $\Delta_N$'s) do not depend on the vectors $\vec{q}_i$ since all the vectors belong to

the same orbit of the group. Furthermore all the vectors $\vec{q}_i$ have the same lenght [17] given by eq. (32). In this case

$$\langle 0|\psi(x)\psi(x)|0\rangle = \Delta_q \sum_i e^{2i\vec{q}_i\cdot\vec{x}} \tag{40}$$

This more general case has been considered in [11, 17] by evaluating the grand-potential of various crystalline structures through a Ginzburg-Landau expansion, up to sixth order in the gap [17]

$$\Omega = \alpha\Delta^2 + \frac{\beta}{2}\Delta^4 + \frac{\gamma}{3}\Delta^6 \tag{41}$$

These coefficients can be evaluated microscopically for each given crystalline structure. The procedure that the authors of ref. [17] have followed is to start from the gap equation represented graphically in Fig. 3. Then, they

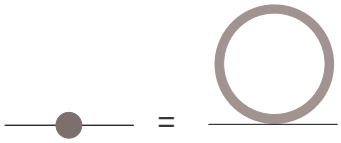


Figure 3: Gap equation in graphical form. The thick line is the exact propagator. The black dot the gap insertion.

expand the exact propagator in a series of the gap insertions as given in Fig. 4. Inserting this expression back into the gap equation one gets the


Figure 4: The expansion of the propagator in graphical form. The darker boxes represent a $\Delta^*$ insertion, the lighter ones a $\Delta$ insertion.

expansion illustrated in Fig. 5. On the other hand the gap equation is obtained minimizing the grand-potential (41), i.e.

$$\alpha\Delta + \beta\Delta^3 + \gamma\Delta^5 + \cdots = 0 \tag{42}$$

Comparing this expression with the result of Fig. 5 one is able to derive the coefficients $\alpha$, $\beta$ and $\gamma$.

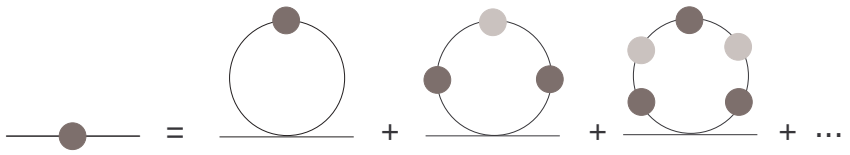

Figure 5: The expansion of the gap equation in graphical form. Notations as in Fig. 4.

In ref. [17] more than 20 crystalline structures have been considered, evaluating for each of them the coefficients of Eq. (41). The result of this analysis is that the face-centered cube appears to be the favored structure among the ones considered (for more details see ref. [17]).

# 4 Phonons

Since in the LOFF phase translational and rotational symmetries are broken, we expect the corresponding Nambu-Goldstone bosons (phonons) to appear in the theory. The number and the features of the phonons depend on the particular crystalline structure. We will consider here the case of the single plane-wave [18] and of the face-centered cube [19]. We will introduce the phonons as it is usual for NG bosons [18], that is as the phases of the condensate. Considering the case of a single plane-wave we introduce a scalar field $\Phi(x)$ through the replacement

$$\Delta(\vec{x}) = \exp^{2i\vec{q}\cdot\vec{x}}\,\Delta \to e^{i\Phi(x)}\Delta \tag{43}$$

We require that the scalar field $\Phi(x)$ acquires the following expectation value in the ground state

$$\langle\Phi(x)\rangle = 2\,\vec{q}\cdot\vec{x} \tag{44}$$

The phonon field is defined as

$$\frac{1}{f}\phi(x) = \Phi(x) - 2\vec{q}\cdot\vec{x} \tag{45}$$

Notice that the phonon field transforms nontrivially under rotations and translations. From this it follows that non derivative terms for $\phi(x)$ are not allowed. One starts with the most general invariant lagrangian for the field

$\Phi(x)$ in the low-energy limit. This cuts the expansion of $\Phi$ to the second order in the time derivative. However one may have an arbitrary number of space derivative, since from eq. (44) it follows that the space derivatives do not need to be small. Therefore

$$L_{\text{phonon}} = \frac{f^2}{2}\left(\dot{\Phi}^2 + \sum_k c_k \Phi(\vec{\nabla}^2)^k \Phi\right) \tag{46}$$

Using the definition (45) and keeping the space derivative up to the second order (we can make this assumption for the phonon field) we find

$$L_{\text{phonon}} = \frac{1}{2}\left(\dot{\phi}^2 - v_\perp^2 \vec{\nabla}_\perp \phi \cdot \vec{\nabla}_\perp \phi - v_\parallel^2 \vec{\nabla}_\parallel \phi \cdot \vec{\nabla}_\parallel \phi\right) \tag{47}$$

where

$$\vec{\nabla}_\parallel = \vec{n}(\vec{n}\cdot\vec{\nabla}), \qquad \vec{\nabla}_\perp = \vec{\nabla} - \vec{\nabla}_\parallel, \qquad \vec{n} = \frac{\vec{q}}{|\vec{q}|} \tag{48}$$

We see that the propagation of the phonon in the crystalline medium is anisotropic.

The same kind of considerations can be made in the case of the cube. The cube is defined by 6 vectors $\vec{q}_i$ pointing from the origin of the coordinates to the vertices of the cube parameterized as in Fig. 6.

The condensate is given by [17]

$$\Delta(x) = \Delta \sum_{k=1}^{8} e^{2i\vec{q}_k \cdot \vec{x}} = \Delta \sum_{i=1,(\epsilon_i=\pm)}^{3} e^{2i|\vec{q}|\epsilon_i x_i} \tag{49}$$

We introduce now three scalar fields such that

$$\langle \Phi^{(i)}(x) \rangle = 2|\vec{q}|x_i \tag{50}$$

through the substitution

$$\Delta(x) \to \Delta \sum_{i=1,(\epsilon_i=\pm)}^{3} e^{i\epsilon_i \Phi^{(i)}(x)} \tag{51}$$

and the phonon fields

$$\frac{1}{f}\phi^{(i)}(x) = \Phi^{(i)}(x) - 2|\vec{q}|x_i \tag{52}$$

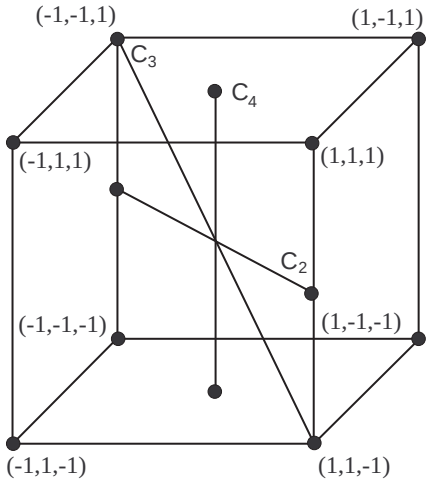


Figure 6: The figure shows the vertices and corresponding coordinates of the cube described in the text. Also shown are the symmetry axis.

Notice that the expression (51) is invariant under the symmetry group of the cube acting upon the scalar fields $\Phi^{(i)}(x)$. This group has three invariants for the vector representation

$$I_2(\vec{X}) = |\vec{X}\,|^2,\ I_4(\vec{X}) = X_1^2X_2^2 + X_2^2X_3^3 + X_3^2X_1^2,\ I_6(\vec{X}) = X_1^2X_2^2X_3^2 \quad (53)$$

Therefore the most general invariant lagrangian invariant under rotations, translations and the symmetry group of the cube, at the lowest order in the time derivative, is

$$L_{\rm phonon} = \frac{f^2}{2}\sum_{i=1,2,3}(\dot{\Phi}^{(i)})^2 + L_{\rm s}(I_2(\vec{\nabla}\Phi^{(i)}), I_4(\vec{\nabla}\Phi^{(i)}), I_6(\vec{\nabla}\Phi^{(i)})) \quad (54)$$

Expanding this expression at the lowest order in the space derivatives of the phonon fields one finds [19]

$$\begin{aligned} L_{\rm phonos} &= \frac{1}{2}\sum_{i=1,2,3}(\dot{\phi}^{(i)})^2 - \frac{a}{2}\sum_{i=1,2,3}|\vec{\nabla}\phi^{(i)}|^2 - \frac{b}{2}\sum_{i=1,2,3}(\partial_i\phi^{(i)})^2 \\ &- c\sum_{i<j=1,2,3}\partial_i\phi^{(i)}\partial_j\phi^{(j)} \end{aligned} \quad (55)$$

The parameters appearing in the phonon lagrangian can be evaluated following the strategy outlined in [20, 21]. One starts from the QCD lagrangian and derives an effective lagrangian describing fermions close to the

Fermi surface, that is at momenta such that $p \approx p_F$ but $p - p_F \gg \Delta$. The relevant degrees of freedom are the fermions dressed by the interaction, the so-called quasi-particles [22]. Going closer to the Fermi surface the gapped quasi-particles decouple and one is left with the light modes as NG bosons, phonons and un-gapped fermions. It is possible to derive the parameters of the last description by the one in terms of quasi-particles evaluating the self-energy of the phonons (or the NG bosons) through one-loop diagrams due to fermion pairs. The couplings of the phonons to the fermions are obtained noticing that the gap acts as a Majorana mass for the quasi-particles. Therefore the couplings originate from the substitutions (43) and (51). In this way one finds the following results: for the single plane-wave

$$v_\perp^2 = \frac{1}{2}\left(1 - \left(\frac{\delta\mu}{|\vec{q}|}\right)^2\right), \quad v_\parallel^2 = \left(\frac{\delta\mu}{|\vec{q}|}\right)^2 \tag{56}$$

and for the cube

$$a = \frac{1}{12}, \quad b = 0, \quad c = \frac{1}{12}\left(3\left(\frac{\delta\mu}{|\vec{q}|}\right)^2 - 1\right) \tag{57}$$

# 5 Astrophysical consequences

A typical phenomenon of the pulsars are the glitches (for a review see [23]), that is sudden jumps in the period of the star. If pulsars are neutron stars with a dense metallic crust, the effect is explained assuming that some angular momentum is stored in the vortices present in the inner neutron superfluid. When the period of the star slows down due to the gravitational radiation, the vortices, which are pinned to the crystalline crust, do not participate in the slowing-down until they become unstable releasing suddenly the angular momentum. Since the density in the inner of a star is a function of the radius, it results that one has a sort of laboratory to study the phase diagram of QCD at zero temperature, at least in the corresponding range of densities. A possibility is that one has a CFL state as a core of the star, then a shell in the LOFF state and eventually the exterior part made up of neutrons. Since in the CFL state the baryionic number is broken there is superfluidity. Therefore the same mechanism explained above might work with vortices in the CFL state pinned to the LOFF crystal. This could reinforce the ideas about the existence of strange stars (made of up, down and strange quarks).

# Acknowledgments

I am grateful to R. Gatto, M. Mannarelli and G. Nardulli for the very pleasant scientific collaboration on the subjects discussed in this talk.

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